Cancellative Superposition Decides the Theory of Divisible Torsion-Free Abelian Groups

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Abstract

In divisible torsion-free abelian groups, the efficiency of the cancellative superposition calculus can be greatly increased by combining it with a variable elimination algorithm that transforms every clause into an equivalent clause without unshielded variables. We show that the resulting calculus is a decision procedure for the theory of divisible torsion-free abelian groups.

Keywords

Automated Theorem Proving, First-Order Logic, Superposition, Cancellative Abelian Monoids, Associativity, Commutativity, Variable Elimination, Term Rewriting, Divisible Torsion-free Abelian Groups, Decision Problem.

1 Introduction

Equational reasoning in the presence of the associativity and commutativity axioms is known to be difficult – theoretically [4, 8], as well as practically [1, 9, 10, 11, 12, 13, 17]. Using AC-unification and extended clauses the worst inefficiencies of a naïve approach can be avoided, but still the extended clauses lead to numerous variable overlaps – one of the most prolific types of inferences in resolution or superposition style calculi. Besides, minimal complete set of AC-unifiers may have doubly exponential size. If the theory contains also the identity law

$$x + 0 \approx x\,, \tag{U}$$

then AC-unification can be replaced by ACU-unification, but the minimal complete set is still simply exponential.

A substantial improvement can be observed when we consider structures that satisfy also the cancellation axiom

$$x + y \approx x + z \;\Rightarrow\; y \approx z\,, \tag{K}$$

or the inverse axiom

$$x + (-x) \approx 0\,, \tag{Inv}$$

(which implies (K)), that is, when we switch over from abelian semigroups or monoids to abelian groups (ACUInv) or at least cancellative abelian monoids (ACUK). The cancellative superposition calculus (Ganzinger and Waldmann [6, 14]) is a refined superposition calculus for cancellative abelian monoids which requires neither explicit inferences with the theory clauses nor extended equations or clauses. Strengthened ordering constraints lead to a significant reduction of the number of variable overlaps, compared with traditional AC-calculi. Some variable overlaps remain necessary, however.

In (non-trivial) divisible torsion-free abelian groups, e. g., the rational numbers and rational vector spaces, the abelian group axioms ACUInv are extended by the torsion-freeness axioms

$$kx \approx ky \;\Rightarrow\; x \approx y \tag{T}$$

(for all $k \in \mathbf{N}^{>0}$), the divisibility axioms[1]

$$k\ \mathit{div\text{-}by}_k(x) \approx x \tag{Div}$$

(for all $k \in \mathbf{N}^{>0}$), and the non-triviality axiom[2]

$$a \not\approx 0\,. \tag{Nt}$$

[1]In non-skolemized form: $\forall x\ \exists y\colon ky \approx x$ for all $k \in \mathbf{N}^{>0}$.

[2]In non-skolemized form: $\exists y\colon y \not\approx 0$.

Divisible torsion-free abelian groups (DTAGs) allow quantifier elimination: For every quantified formula over 0, $+$, and $\approx$ there exists a quantifier-free formula that is equivalent modulo the theory axioms. In particular, every closed formula over this vocabulary is provably true or false: the theory of DTAGs is complete and decidable. Superposition calculi, however, work on formulae that do not contain any existential quantifiers, but that may contain free function symbols – possibly introduced by skolemization, possibly given initially. In the presence of free function symbols, there is of course no way to eliminate all variables from a formula – not even all universally quantified ones – but we can at least give an effective method to eliminate all *unshielded* variables, that is, all variables not occurring below any free function symbol. This elimination algorithm has been integrated into the cancellative superposition calculus in (Waldmann [16]). The resulting calculus is refutationally complete with respect to the axioms of divisible torsion-free abelian groups and allows us to dispense with variable overlaps completely.

Starting with Joyner [7], several resolution or superposition calculi have been shown to be decision procedures for certain classes of formulae (e.g., Bachmair, Ganzinger, and Waldmann [3], Fermüller et al. [5]). As the theory of DTAGs is decidable, it is now a natural question to ask whether the combination of cancellative superposition and variable elimination for unshielded universally quantified variables is powerful enough to be usable as a decision procedure for the theory of DTAGs. We show in this paper that this is indeed the case: The combined calculus is refutationally complete in the presence of arbitrary free function symbols; and it is a decision procedure, if all free function are the result of skolemization.

2 Preliminaries

We will first give a short overview over the cancellative superposition calculus and its specialization for DTAGs. The reader is referred to (Waldmann [14, 16]) for more technical details.

Throughout this paper we assume that our signature[3] contains a binary function symbol $+$ and a constant 0. If t is a term and $n \in \mathbf{N}$, then nt is an abbreviation for the n-fold sum $t + \cdots + t$; in particular, $0t = 0$ and $1t = t$.

A function symbol is called free, if it is different from 0 and $+$. A term is called atomic, if it is not a variable and its top symbol is different from $+$. We say that a term t occurs at the top of s, if there is a position $o \in \mathrm{pos}(s)$ such that $s|_o = t$ and for every proper prefix o' of o, $s(o')$ equals $+$; the term t occurs in s below a free function symbol, if there is an $o \in \mathrm{pos}(s)$ such that

[3]The cancellative superposition calculus as described in (Waldmann [14, 16]) works in a many-sorted framework. For the purposes of this paper, it is sufficient to restrict to the one-sorted case.

$s|_o = t$ and $s(o')$ is a free function symbol for some proper prefix o' of o. A variable x is called shielded in a clause C, if it occurs at least once below a free function symbol in C. Otherwise, x is called unshielded.

We say that an ACU-compatible ordering $\succ$ has the multiset property, if whenever a ground atomic term u is greater than v_i for every i in a finite non-empty index set I, then $u \succ \sum_{i \in I} v_i$.

From now on we will work *only* with ACU-congruence classes, rather than with terms. So *all* terms, equations, substitutions, inference rules, etc., are to be taken modulo ACU, i.e., as representatives of their congruence classes. The symbol $\succ$ will always denote an ACU-compatible ordering that has the multiset property and is total on ground ACU-congruence classes.[4]

Without loss of generality we assume that the equality symbol $\approx$ is the only predicate of our language. Hence a literal is either an equation $t \approx t'$ or a negated equation $t \not\approx t'$. The symbol $\dot{\approx}$ denotes either $\approx$ or $\not\approx$. A clause is a finite multiset of literals, usually written as a disjunction.

Let A be a ground literal $nu + \sum_{i \in I} s_i \mathrel{\dot{\approx}} mu + \sum_{j \in J} t_j$, where u, s_i, and t_j are atomic terms, $n \geq m \geq 0$, $n \geq 1$, and $u \succ s_i$ and $u \succ t_j$ for all $i \in I$, $j \in J$. Then u is called the maximal atomic term of A, denoted by $\mathrm{mt}(A)$.

The ordering $\succ_{\mathrm{L}}$ on literals compares lexicographically first the maximal atomic terms of the literals, then the polarities (negative $\succ$ positive), then the multisets of all non-zero terms occurring at the top of the literals, and finally the multisets consisting of the left and right hand sides of the literals. The ordering $\succ_{\mathrm{C}}$ on clauses is the multiset extension of the literal ordering $\succ_{\mathrm{L}}$. Both $\succ_{\mathrm{L}}$ and $\succ_{\mathrm{C}}$ are noetherian and total on ground literals/clauses.

We denote the entailment relation modulo equality and ACUKT by $\models_{\mathrm{ACUKT}}$. In other words, $\{C_1, \ldots, C_n\} \models_{\mathrm{ACUKT}} C_0$ if and only if $\mathrm{ACUKT} \cup \{C_1, \ldots, C_n\} \models C_0$.

3 Cancellative Superposition

The cancellative superposition calculus (Waldmann [14]) is a refutationally complete variant of the standard superposition calculus (Bachmair and Ganzinger [2]) for sets of clauses that contain the axioms ACUK and (optionally) T. It requires neither extended clauses, nor explicit inferences with the axioms ACUKT, nor symmetrizations. Compared with standard superposition or AC superposition calculi, the ordering restrictions of its inference rules are strengthened: Inferences are not only limited to maximal sides of maximal literals, but also to maximal summands thereof. As shielded vari-

[4] For ground terms, such an ordering can be obtained for instance from the recursive path ordering with precedence $f_n \succ \ldots \succ f_1 \succ + \succ 0$ and multiset status for $+$ by comparing normal forms w.r.t. $x + 0 \to x$ and $0 + x \to x$. If clauses are fully abstracted eagerly (cf. Sect. 4), the compatibility requirement becomes void.

ables are non-maximal, this implies in particular that there are no overlaps with such variables.

The inference system $\mathfrak{K}$ of the cancellative superposition calculus[5] consists of the inference rules *cancellation*, *equality resolution*, *standard superposition*, *cancellative superposition*, *abstraction*, and *cancellative equality factoring.* Ground versions of these rules are given below.

The following conditions are common to all the inference rules: Every literal involved in some inference must be maximal in the respective premise (except for the last but one literal in *cancellative equality factoring* inferences). A positive literal involved in a *superposition* or *abstraction* inference must be strictly maximal in the respective clause. In all *superposition* and *abstraction* inferences, the left premise is smaller than the right premise. In *standard superposition* and *abstraction* inferences, if s is a proper sum, then t (or w, respectively) occurs in a maximal atomic subterm of s.

Cancellation

$$\frac{C' \vee mu + s \mathrel{\dot{\approx}} m'u + s'}{C' \vee (m - m')u + s \mathrel{\dot{\approx}} s'}$$

if $m \geq m' \geq 1$ and $u \succ s$, $u \succ s'$.

Equality Resolution

$$\frac{C' \vee 0 \not\approx 0}{C'}$$

Standard Superposition

$$\frac{D' \vee t \approx t' \qquad C' \vee s[t] \mathrel{\dot{\approx}} s'}{D' \vee C' \vee s[t'] \mathrel{\dot{\approx}} s'}$$

if t occurs below a free function symbol in s, and $s[t] \succ s'$, $t \succ t'$.

Canc. Superposition

$$\frac{D' \vee nu + t \approx t' \qquad C' \vee mu + s \mathrel{\dot{\approx}} s'}{D' \vee C' \vee \psi s + \chi t' \mathrel{\dot{\approx}} \chi t + \psi s'}$$

if $m \geq 1$, $n \geq 1$, $\psi = n/\mathrm{gcd}(m,n)$, $\chi = m/\mathrm{gcd}(m,n)$, and $u \succ s$, $u \succ s'$, $u \succ t$, $u \succ t'$.

Abstraction

$$\frac{D' \vee nu + t \approx t' \qquad C' \vee s[w] \mathrel{\dot{\approx}} s'}{C' \vee y \not\approx w \vee s[y] \mathrel{\dot{\approx}} s'}$$

if $n \geq 1$, $w = mu + q$ occurs in s immediately below some free function symbol, $m \geq 1$, $nu + t$ is not a subterm of w, and $u \succ t$, $u \succ t'$, $s[w] \succ s'$.

[5] In [14], this inference system is denoted by *CS-Inf*$_{\mathbf{N}^{>0}}$.

Canc. Eq. Factoring $\dfrac{C' \vee nu + t \approx n'u + t' \vee mu + s \approx s'}{C' \vee \psi t + \chi s' \not\approx \chi s + \psi t' \vee nu + t \approx n'u + t'}$

if $m \geq 1$, $n > n' \geq 0$, $\nu = n - n'$, $\psi = m/\gcd(m, \nu)$, $\chi = \nu/\gcd(m, \nu)$, and $u \succ s$, $u \succ s'$, $u \succ t$, $u \succ t'$.

The inference system $\mathfrak{K}$ is sound with respect to ACUKT. In other words, for every inference with premises $C_1, \ldots, C_n$ and conclusion C_0, we have $\{C_1, \ldots, C_n\} \models_{\mathrm{ACUKT}} C_0$.

Lifting the inference rules to non-ground clauses is relatively straightforward as long as we restrict to clauses without unshielded variables. For the inference rules *equality resolution* and *standard superposition*, we proceed as in the standard superposition calculus (Bachmair and Ganzinger [2]). For the inference rules *cancellation*, *cancellative superposition*, and *cancellative equality factoring*, we have to take into account that, in a clause $C = C' \vee A$, the maximal literal A need no longer have the form $mu + s \mathrel{\dot\approx} s'$, where u is the unique maximal atomic term. Rather, a non-ground literal such as $f(x) + 2f(y) + b \not\approx c$ may contain several (distinct but ACU-unifiable) maximal atomic terms u_k with multiplicities m_k, where k ranges over some finite non-empty index set K. We obtain thus $A = \sum_{k \in K} m_k u_k + s \mathrel{\dot\approx} s'$, where $\sum_{k \in K} m_k$ corresponds to m in the ground literal above. As in the *standard superposition* rule, the substitution σ that unifies all u_k (and the corresponding terms v_l from the other premise) is applied to the conclusion. For instance, the *cancellative superposition* rule has now the following form:

Cancellative Superposition

$$\frac{D' \vee A_2 \qquad C' \vee A_1}{(D' \vee C' \vee A_0)\sigma}$$

if the following conditions are satisfied:

- $A_1 = \sum_{k \in K} m_k u_k + s \mathrel{\dot\approx} s'$.
- $A_2 = \sum_{l \in L} n_l v_l + t \approx t'$.
- $m = \sum_{k \in K} m_k \geq 1$, $n = \sum_{l \in L} n_l \geq 1$.
- $\psi = n/\gcd(m, n)$, $\chi = m/\gcd(m, n)$.
- u is one of the u_k or v_l ($k \in K$, $l \in L$).
- σ is a most general ACU-unifier of all u_k and v_l ($k \in K, l \in L$).
- $u \not\preceq s$, $u \not\preceq s'$, $u \not\preceq t$, $u \not\preceq t'$.
- $A_0 = \psi s + \chi t' \mathrel{\dot\approx} \chi t + \psi s'$.

The lifted versions of the rules *cancellation* and *cancellative equality factoring* are obtained analogously. The only inference rule for which lifting is not so straightforward is the *abstraction* rule. Here we have to take into account that the term to be abstracted out may be a sum containing variables at the top [14].

In the presence of unshielded variables, it is still possible to devise (more complicated) lifted inference rules that produce only finitely many conclusions for a given tuple of premises. We do not repeat these rules here, as the additional theory axioms DivInvNt make it possible to eliminate unshielded variables completely. The elimination of unshielded variables happens in two stages. First we show that every clause is logically equivalent to a clause without unshielded variables. Then this elimination algorithm has to be integrated into the cancellative superposition calculus. Our main tool for the second step is the concept of redundancy.

Let $C_0, C_1, \ldots, C_k$ be clauses and let θ be a substitution such that $C_i\theta$ is ground for all $i \in \{1, \ldots, k\}$. If there are inferences

$$\frac{C_k \ \ldots \ C_1}{C_0}$$

and

$$\frac{C_k\theta \ \ldots \ C_1\theta}{C_0\theta}$$

then the latter is called a ground instance of the former.

Let N be a set of clauses, let $\overline{N}$ be the set of ground instances of clauses in N. An inference is called ACUKT-redundant with respect to N if for each of its ground instances with conclusion $C_0\theta$ and maximal premise $C\theta$ we have $\{\, D \in \overline{N} \mid D \prec_{\mathrm{C}} C\theta \,\} \models_{\mathrm{ACUKT}} C_0\theta$.[6] A clause C is called ACUKT-redundant with respect to N, if for every ground instance $C\theta$, $\{\, D \in \overline{N} \mid D \prec_{\mathrm{C}} C\theta \,\} \models_{\mathrm{ACUKT}} C\theta$.

A set N of clauses is called saturated with respect to an inference system and a redundancy criterion, if every inference from clauses in N is redundant with respect to N.

THEOREM 3.1 *The inference system $\mathfrak{K}$ is refutationally complete with respect to* ACUKT, *that is, a $\mathfrak{K}$-saturated set of clauses is unsatisfiable modulo* ACUKT *if and only if it contains the empty clause (Waldmann [14]).*

One application of the redundancy concept is simplification: A prover produces a saturated set of clauses by computing inferences according to some fair strategy and adding the conclusions of non-redundant inferences to the current set of clauses. At any time of the saturation process, the prover is permitted to replace a clause by an equivalent set of new clauses, provided the new clauses make the simplified clause redundant. As we will see later, in the calculus for DTAGs, redundancy is already essential to prove the refutational completeness of the inference rules themselves.

[6]For *abstraction* inferences one has to consider all ground instances $C_0\theta\rho$ of $C_0\theta = y \not\approx w\theta \lor C_0'\theta[y]$ with $y\rho \prec w\theta$.

4 Variable Elimination: The Logical Side

It is well-known that the theory of DTAGs allows quantifier elimination: For every quantified formula over 0, +, and $\approx$ there exists an equivalent quantifier-free formula. In the presence of free function symbols, there is of course no way to eliminate all variables from a clause, but we can at least give an effective method to eliminate all *unshielded* variables.

Let x be a variable. We define a binary relation $\to_x$ over clauses by

CancelVar $\quad C' \vee mx + s \mathrel{\dot{\approx}} m'x + s' \quad \to_x \quad C' \vee (m{-}m')x + s \mathrel{\dot{\approx}} s'$
if $m \geq m' \geq 1$.

ElimNeg $\quad C' \vee mx + s \not\approx s' \quad \to_x \quad C'$
if $m \geq 1$ and x does not occur in C', s, s'.

ElimPos $\quad C' \vee m_1 x + s_1 \approx s'_1 \vee \ldots \vee m_k x + s_k \approx s'_k \quad \to_x \quad C'$
if $m_i \geq 1$ and x does not occur in C', s_i, s'_i, for $1 \leq i \leq k$.

Coalesce $\quad C' \vee mx + s \not\approx s' \vee nx + t \mathrel{\dot{\approx}} t'$
$\to_x \quad C' \vee mx + s \not\approx s' \vee \psi t + \chi s' \mathrel{\dot{\approx}} \psi t' + \chi s$
if $m \geq 1$, $n \geq 1$, $\psi = m/\gcd(m,n)$, $\chi = n/\gcd(m,n)$, and x does not occur at the top of s, s', t, t'.

The relation $\to_x$ is noetherian. Let the binary relation $\to_{\text{elim}}$ over clauses be defined in such a way that $C_0 \to_{\text{elim}} C_1$ if and only if C_0 contains an unshielded variable x and C_1 is a normal form of C_0 with respect to $\to_x$. Then $\to_{\text{elim}}$ is again noetherian. For any clause C, let $\text{elim}(C)$ denote some (arbitrary but fixed) normal form of C with respect to the relation $\to_{\text{elim}}$.

LEMMA 4.1 *For every clause C, $\text{elim}(C)$ contains no unshielded variables.*

LEMMA 4.2 *For every clause C, $\{C\} \cup \text{DivInvNt} \models_{\text{ACUKT}} \text{elim}(C)$ and $\{\text{elim}(C)\} \models_{\text{ACUKT}} C$. For every ground instance $C\theta$, $\{\text{elim}(C)\theta\} \models_{\text{ACUKT}} C\theta$.*

Using the technique sketched so far, every clause C_0 can be transformed into a clause $\text{elim}(C_0)$ that does not contain unshielded variables, follows from C_0 and the divisible torsion-free abelian group axioms, and implies C_0 modulo ACUKT. Obviously, we can perform this transformation for all initially given clauses *before* we start the saturation process. However, the set of clauses without unshielded variables is not closed under the inference system $\mathfrak{K}$, i.e., inferences from clauses without unshielded variables may produce clauses with unshielded variables. To eliminate these clauses *during* the saturation process, it is not sufficient that they follow logically from *some* other clauses: redundancy requires that they follow from *some sufficiently* *small* clauses. Unfortunately, under certain circumstances the transformed clause $\mathrm{elim}(C_0)$ may not be small enough. Hence, to integrate the variable elimination algorithm into the cancellative superposition calculus, it has to be supplemented by a case analysis technique.

5 Variable Elimination: The Operational Side

Let ι be an inference. We call the unifying substitution σ that is computed during ι and applied to the conclusion the pivotal substitution of ι. (For *abstraction* inferences and all ground inferences, the pivotal substitution is the identity mapping.) If A is the last literal of the last premise of ι, we call $A\sigma$ the pivotal literal of ι. Finally, if u_0 is the atomic term that is cancelled out in ι, or in which some subterm is replaced or abstracted out,[7] then we call $u_0\sigma$ the pivotal term of ι. Pivotal terms have two important properties: First, whenever an inference ι from clauses *without* unshielded variables produces a conclusion *with* unshielded variables, then all these unshielded variables occur in the pivotal term of ι. Second, no atomic term in the conclusion of ι can be larger than the pivotal term of ι.

A clause C is called fully abstracted, if no non-variable term occurs below a free function symbol in C. Every clause C can be transformed into an equivalent fully abstracted clause $\mathrm{abs}(C)$ by iterated rewriting

$$C[f(\dots,t,\dots)] \quad \to \quad x \not\approx t \vee C[f(\dots,x,\dots)]\,,$$

where x is a new variable and t is a non-variable term occurring immediately below the free function symbol f in C. It should be noted that the variable elimination algorithm preserves full abstraction, so that for every clause C, $\mathrm{elim}(\mathrm{abs}(C))$ is a logically equivalent clause that is fully abstracted and does not contain unshielded variables.

In the sequel we assume that every clause C in the input of the inference system is replaced by $\mathrm{elim}(\mathrm{abs}(C))$ before we start the saturation process. The inference system $\mathfrak{D}^{abs}$ that we will describe now preserves both properties: the set of all fully abstracted clauses without unshielded variables is closed under $\mathfrak{D}^{abs}$. The system $\mathfrak{D}^{abs}$ is given by two meta-inference rules:

Eliminating Inference

$$\frac{C_n \quad \dots \quad C_1}{\mathrm{elim}(C_0)}$$

if the following condition is satisfied:

[7]More precisely, u_0 is the maximal atomic subterm of s containing t (or w) in *standard superposition* or *abstraction* inferences, and the term u in all other inferences.

- $\dfrac{C_n \quad \ldots \quad C_1}{C_0}$ is a non-*abstraction* and non-*standard superposition* $\mathfrak{K}$-inference.[8]

Instantiating Inference

$$\frac{C_n \quad \ldots \quad C_1}{C_0 \tau}$$

if the following conditions are satisfied:

- $\dfrac{C_n \quad \ldots \quad C_1}{C_0}$ is a non-*abstraction* and non-*standard superposition* $\mathfrak{K}$-inference with pivotal literal A and pivotal term u.
- The multiset difference $\mathrm{elim}(C_0) \setminus C_0$ contains a literal A_1 with the same polarity as A.
- An atomic term u_1 occurs at the top of A_1.
- τ is contained in a minimal complete set of ACU-unifiers of u and u_1.

The redundancy of $\mathfrak{D}^{abs}$-inferences is defined in a slightly complicated way. Essentially, a $\mathfrak{D}^{abs}$-inference is redundant if sufficiently many ground instances of the $\mathfrak{K}$-inference on which it is based are redundant. For our purposes, it is sufficient to know that any inference is redundant with respect to a set N of clauses as soon as its conclusion (or a simplified version thereof) is present in N.

THEOREM 5.1 *If a set of fully abstracted clauses is saturated with respect to $\mathfrak{D}^{abs}$ and none of the clauses contains unshielded variables, then it is also saturated with respect to $\mathfrak{K}$, and it is unsatisfiable modulo* ACUKT $\cup$ DivInvNt *if and only if it contains the empty clause (Waldmann [14, 16]).*

If all clauses are fully abstracted, then the terms that have to be compared during the saturation have the property that they do not contain the operator $+$. In this situation, the requirement that the ordering $\succ$ has to be ACU-compatible becomes void, and we may use an arbitrary reduction ordering over terms not containing $+$ that is total on ground terms and for which 0 is minimal. As every ordering of this kind can be extended to an ordering that is ACU-compatible and has the multiset property (Waldmann [15]), the completeness proof is still justified.

[8] In the one-sorted case considered in this paper, *standard superposition* inferences from fully abstracted clauses are impossible. In the general many-sorted case, *standard superposition* inferences must not be ignored.

6 Deciding the Theory of DTAGs

A refutationally complete calculus derives a contradiction (and terminates) whenever the set of input formulae is inconsistent. To show that a refutationally complete calculus is actually a decision procedure, one has to prove that it terminates even on consistent inputs. Following this general scheme, we will now demonstrate that the calculus $\mathfrak{D}^{abs}$ is a decision procedure for the theory of divisible torsion-free abelian groups.

Let us denote by $\mathcal{D}$ the class of all closed first-order formulae with arbitrary quantifiers and logical connectives and containing not more than the function symbols $+$ (binary), 0 (constant), $-$ (unary), $\mathit{div\text{-}by}_k$ (unary) for $k \in \mathbf{N}^{>0}$, and the binary predicate symbol $\approx$. Given a formula $F \in \mathcal{D}$, our task is to decide whether F is equivalent to true or to false with respect to the theory of divisible torsion-free abelian groups. As the theory of DTAGs is complete, *every* formula in $\mathcal{D}$ is equivalent either to true or to false, hence F is equivalent to true if and only if it is satisfiable.

We can first of all eliminate the symbols $-$ and $\mathit{div\text{-}by}_k$ from F by recursively replacing any atom $s[-t] \approx s'$ by $\forall x(\neg\, x + t \approx 0 \,\vee\, s[x] \approx s')$ and any atom $s[\mathit{div\text{-}by}_k(t)] \approx s'$ by $\forall x(\neg\, kx \approx t \,\vee\, s[x] \approx s')$, where x is a new variable. The resulting formula F_1 is then converted into a formula F_2 in prenex normal form. By skolemization, F_2 can be further translated into a formula F_3 without existentially quantified variables, such that F_3 is satisfiable if and only if F is satisfiable. Skolemization replaces the existentially quantified variables of F_2 by terms $f_k(x_1, \dots, x_i)$, where the x_j are universally quantified variables and f_k is a new free function symbol. Finally, the formula F_3 can be transformed into conjunctive normal form, which we represent as a finite set of clauses. This set of clauses is a subset of the class $\mathcal{D}_{\mathrm{c}}$ defined as follows: A clause C is contained in $\mathcal{D}_{\mathrm{c}}$ if and only if there exists a finite sequence of distinct variables $x_1, \dots, x_n$ such that, for every literal $s \mathrel{\dot{\approx}} s'$ in C, both s and s' are sums $\sum n_k t_k$, and each t_k is either a variable x_i or an atomic term $f(x_1, \dots, x_i)$ for some $i \leq n$. The class of all clauses C in $\mathcal{D}_{\mathrm{c}}$ without unshielded variables is denoted by $\mathcal{D}_{\mathrm{c}}^{\mathrm{elim}}$. We claim that there is a strategy for $\mathfrak{D}^{abs}$-superposition that is guaranteed to terminate on every finite subset of $\mathcal{D}_{\mathrm{c}}^{\mathrm{elim}}$. Termination implies that with this strategy $\mathfrak{D}^{abs}$-superposition becomes a decision procedure for the satisfiability of finite subsets of $\mathcal{D}_{\mathrm{c}}^{\mathrm{elim}}$ (and hence of formulae in $\mathcal{D}$) with respect to $\mathrm{ACUKT} \cup \mathrm{DivInvNt}$.

In the rest of this paper, we assume $\succ$ to be a lexicographic path ordering based on a precedence relation that respects the arity of function symbols (greater arity implying higher precedence). Apart from satisfying this restriction, the precedence can be arbitrary (but has to be total). Without loss of generality, we assume that the function symbols occurring in the input clauses are $f_m \succ \cdots \succ f_1$. We note that $f_j(x_1, \dots, x_l) \succ f_k(x_1, \dots, x_i)$ if and only if $f_j \succ f_k$ if and only if $j > k$.

In the one-sorted case, the inference system $\mathfrak{D}^{abs}$ consists of the eliminating and the instantiating variants of the rules *cancellation*, *equality resolution*, *cancellative superposition*, and *cancellative equality factoring.* We will show that for the special class of clauses $\mathcal{D}_{\mathrm{c}}^{\mathrm{elim}}$, instantiating inferences are not needed:

LEMMA 6.1 *Every $\mathfrak{D}^{abs}$-inference from clauses in $\mathcal{D}_{\mathrm{c}}^{\mathrm{elim}}$ is an eliminating inference.*

PROOF. Assume that there is an instantiating $\mathfrak{D}^{abs}$-inference

$$\frac{C_n \quad \ldots \quad C_1}{C_0\tau}$$

with premises in $\mathcal{D}_{\mathrm{c}}^{\mathrm{elim}}$. Then

$$\frac{C_n \quad \ldots \quad C_1}{C_0}$$

is a $\mathfrak{K}$-inference with pivotal literal A, pivotal term u, and pivotal substitution σ. Furthermore, the multiset difference $\mathrm{elim}(C_0) \setminus C_0$ contains a literal A_1 with the same polarity as A, and $u\tau = u_1\tau$ for some atomic term u_1 occurring at the top of A_1. As $\mathrm{elim}(C_0) \neq C_0$, the clause C_0 must contain some unshielded variable x, and since the premises have no unshielded variables, x must occur in the pivotal term u. Now, as the premises C_i are clauses in $\mathcal{D}_{\mathrm{c}}^{\mathrm{elim}}$, there exists a fixed list of variables $x_1, x_2, \ldots$ such that all atomic terms in $C_i\sigma$, and thus in C_0 and $\mathrm{elim}(C_0)$, have the form $f_j(x_1, \ldots, x_l)$ for some j and l. Consequently, any two atomic terms in $C_i\sigma$, C_0, and $\mathrm{elim}(C_0)$ are either equal or not unifiable. By assumption, u and u_1 have the unifier τ, hence $u = u_1$. So x occurs in u_1, and thus in an atomic term in $\mathrm{elim}(C_0)$, and thus in an atomic term in C_0. Hence x is shielded in C_0, which refutes our assumption. $\square$

For a clause C, let $\mathrm{sfact}(C)$ be the clause obtained from C by syntactic factoring, that is, by replacing every repeated literal $A \vee \ldots \vee A$ by A. Let $\mathrm{scanc}(C)$ be the clause obtained from C by syntactic cancellation, that is, by replacing every literal $s + t \mathrel{\dot{\approx}} s' + t$ with non-zero t by $s \mathrel{\dot{\approx}} s'$.

Unlike syntactic factoring, syntactic cancellation may introduce unshielded variables (if the term that was cancelled out was the last term shielding some variable). During elimination of these unshielded variables, the *Coalesce* rule may again produce syntactically equal terms on both sides of a literal. Let the binary relation $\rightarrow_{\mathrm{sce}}$ over clauses be defined in such a way that $C_0 \rightarrow_{\mathrm{sce}} C_1$ if and only if $C_1 = \mathrm{elim}(\mathrm{scanc}(C))$ and $C_1 \neq C_0$. It is easy to show that $\rightarrow_{\mathrm{sce}}$ terminates. Let us denote the normal form of a clause C with respect to $\rightarrow_{\mathrm{sce}}$ by $\mathrm{scanc}^*(C)$, and let $\mathrm{simp}(C)$ be the clause $\mathrm{sfact}(\mathrm{scanc}^*(C))$.

LEMMA 6.2 *For every clause C in $\mathcal{D}_{\rm c}^{\rm elim}$, replacing C by* simp(C) *is a simplification.*[9]

In descriptions of resolution or paramodulation style inference systems, one assumes conventionally that all clauses are variable disjoint, so that overlapping terms or literals can always be unified in the inference rules. To simplify the termination proof, we will exploit the fact that the particular structure of $\mathcal{D}_{\rm c}^{\rm elim}$ allows us to use quite the opposite approach: Consider a $\mathfrak{D}^{abs}$-inference from two clauses C_2 and C_1 in $\mathcal{D}_{\rm c}^{\rm elim}$. During this inference, the maximal atomic term of C_2, say $f_k(x_1'', \dots, x_i'')$, and the maximal atomic term of C_1, say $f_k(x_1', \dots, x_i')$, are overlapped. By definition of the ordering and of the class $\mathcal{D}_{\rm c}^{\rm elim}$, the set of variables of C_1 is exactly $\{x_1', \dots, x_i'\}$, and all atomic terms in C_1 have the form $f_j(x_1', \dots, x_l')$ with $j \leq k$ and $l \leq i$ (and analogously for C_2). Therefore, essentially the same inference is also possible, if we assume that all clauses share the *same* variables $x_1, x_2, \dots$, and all non-variable terms occurring in the *clause set* have the form $f_j(x_1, \dots, x_l)$ for some j and l. The pivotal substitution can then always be assumed to be the identity mapping, and it is trivial to check that the conclusion of any $\mathfrak{D}^{abs}$-inference uses again the variables $x_1, x_2, \dots$ in the required way.

To saturate a given finite subset of the class $\mathcal{D}_{\rm c}^{\rm elim}$, we use the following strategy:

Let N be the set of all input clauses.
Let $f_m \succ \cdots \succ f_1$ be the function symbols occurring in N.
Let $N_{m+1}^* = \{\, \mathrm{sfact}(C) \mid C \in N \,\}$.
For $k = m, m-1, \dots, 1$:
 If N_{k+1}^* is defined, let N_k^0 be the set obtained from N_{k+1}^* by replacing every clause C whose maximal function symbol is f_k by simp(C).
 For $r = 0, 1, \dots$:
  If N_k^r is defined and if there are non-redundant cancellative superposition or cancellative equality factoring $\mathfrak{D}^{abs}$-inferences from clauses in N_k^r with pivotal term $f_k(x_1, \dots, x_i)$, pick one of them "don't care" non-deterministically, let C be its conclusion, and let $N_k^{r+1} = N_k^r \cup \{\mathrm{sfact}(C)\}$;
  if N_k^r is defined and if there is no such inference, let $N_k^* = N_k^r$.
If N_1^* is defined, let N^* be the union of N_1^* and the set of all conclusions of all non-redundant equality resolution $\mathfrak{D}^{abs}$-inferences from clauses in N_1^*.

LEMMA 6.3 *Let $k \in \{1, \dots, m\}$. If N_{k+1}^* is defined, then there exists an $r \in \mathbf{N}$ such that there is no non-redundant cancellative superposition or cancellative equality factoring $\mathfrak{D}^{abs}$-inference from clauses in N_k^r with pivotal term $f_k(x_1, \dots, x_i)$.*

[9]The restriction to clauses in $\mathcal{D}_{\rm c}^{\rm elim}$ is crucial for the correctness of this lemma.

Proof. Every $\mathfrak{D}^{abs}$-inference is redundant with respect to N_k^r if its conclusion C or an equivalent smaller clause, such as $\mathrm{sfact}(C)$, is contained in N_k^r. All inclusions in the sequence $N_k^0 \subseteq N_k^1 \subseteq \dots \subseteq N_k^r \subseteq \dots$ must therefore be strict. A clause can participate in an inference with pivotal term $f_k(x_1, \dots, x_i)$ only if it contains f_k and if it does not contain any f_j with $j > k$, or in other words, if $f_k(x_1, \dots, x_i)$ is its maximal atomic term. The set of all such clauses in N_k^0 is obviously finite. We will show below that the number of such clauses in $\bigcup_r N_k^r$ is finitely bounded. From these finitely many clauses only finitely many conclusions of inferences can be derived, hence $\bigcup_r N_k^r$ must be finite. As the inclusions in the sequence are strict, the sequence is finite.

It remains to be proved that the number of clauses with maximal atomic term $f_k(x_1, \dots, x_i)$ in $\bigcup_r N_k^r$ is finitely bounded. Let M be the subset of N_k^0 containing all clauses with maximal atomic term $f_k(x_1, \dots, x_i)$. Let L be the set of all literals of clauses in M, let L_1 be the set of all literals in L in which f_k occurs, and let $L_0 = L \setminus L_1$. Note that there is no literal in L_1 in which f_k occurs on both sides. Let L_0' be the set of all literals A, such that there is a cancellative superposition $\mathfrak{K}$-inference

$$\frac{A_2 \qquad A_1}{A}$$

with literals A_1 and A_2 from L_1. Let L_0'' be the set of all literals A, such that there is a cancellative equality factoring $\mathfrak{K}$-inference

$$\frac{A_2 \vee A_1}{A \vee A_2}$$

with literals A_1 and A_2 from L_1. Note that f_k does not occur in literals from $L_0' \cup L_0''$. Let M^* be the set of all clauses consisting of literals in $L_0 \cup L_0' \cup L_0'' \cup L_1$ (without duplicated literals).

Consider an arbitrary eliminating cancellative superposition or cancellative equality factoring $\mathfrak{D}^{abs}$-inference

$$\frac{C_n \quad \dots \quad C_1}{\mathrm{elim}(C_0)}$$

from premises in M^* with pivotal term $f_k(x_1, \dots, x_i)$ and conclusion $D = \mathrm{elim}(C_0)$. If $f_k(x_1, \dots, x_i)$ occurs in $\mathrm{sfact}(D)$, then it occurs also in C_0. In this case, all variables in C_0 are shielded, thus $\mathrm{elim}(C_0) = C_0$. Since

$$\frac{C_n \quad \dots \quad C_1}{C_0}$$

is a cancellative superposition or cancellative equality factoring $\mathfrak{K}$-inference, $\mathrm{sfact}(D) = \mathrm{sfact}(C_0)$ is again contained in M^*. As $M \subseteq M^*$, we can conclude that all clauses in $\bigcup_r N_k^r$ with maximal atomic term $f_k(x_1, \dots, x_i)$ are contained in M^*. Since M^* is finite, this completes the proof. $\square$

Corollary 6.4 *N_k^* and N^* are defined for every $k \in \{1, \ldots, m+1\}$.*

Corollary 6.5 *$N \vdash N_{m+1}^* \vdash N_m^0 \vdash N_m^1 \vdash \ldots \vdash N_m^* \vdash \ldots \vdash N_1^0 \vdash N_1^1 \vdash \ldots \vdash N_1^* \vdash N^*$ is a finite theorem proving derivation; N and N^* are equivalent modulo* ACUKT $\cup$ DivInvNt.

Lemma 6.6 *Let $1 \leq k \leq j \leq m$. Then all $\mathfrak{D}^{abs}$-inferences with pivotal term $f_j(x_1, \ldots, x_l)$ from clauses in N_k^* are redundant with respect to N_k^*.*

Proof. By induction, we may assume that all $\mathfrak{D}^{abs}$-inferences with pivotal term $f_p(x_1, \ldots, x_l)$, $p > k$ from clauses in N_{k+1}^* are redundant with respect to N_{k+1}^*.

The clauses in $N_k^* \setminus N_{k+1}^*$ contain only function symbols f_p with $p \leq k$. Therefore, every $\mathfrak{D}^{abs}$-inference from clauses in N_k^* with pivotal term $f_p(x_1, \ldots, x_i)$ and $p > k$ is an inference from clauses in N_{k+1}^*, hence it is redundant with respect to N_{k+1}^*. As all clauses in $N_{k+1}^* \setminus N_k^*$ are redundant with respect to N_{k+1}^*, every inference that is redundant with respect to N_{k+1}^* is also redundant with respect to N_k^*. Therefore it suffices to show that all $\mathfrak{D}^{abs}$-inferences with pivotal term $f_k(x_1, \ldots, x_i)$ from clauses in N_k^* are redundant with respect to N_k^*.

It is easy to check that literals with f_k occurring on both sides cannot occur at all in clauses in $N_k^* \setminus N_k^0$, and that they can occur in a clause C in N_k^0 only if some f_p with $p > k$ occurs in C. Hence there are no cancellation inferences with pivotal term $f_k(x_1, \ldots, x_i)$ from clauses in $N_k^* = N_k^0 \cup (N_k^* \setminus N_k^0)$. This means that all inferences from clauses in N_k^* with pivotal term $f_k(x_1, \ldots, x_i)$ are either cancellative superposition or cancellative equality factoring inferences, hence they are redundant with respect to N_k^* by construction of N_k^*. $\square$

Theorem 6.7 *All inferences from clauses in N^* are redundant with respect to N^*.*

Proof. By the previous lemma, all $\mathfrak{D}^{abs}$-inferences with pivotal terms $f_j(x_1, \ldots, x_l)$ from clauses in N_1^* are redundant with respect to N_1^* (and hence with respect to N^*). Furthermore, by construction of N^*, all equality resolution inferences from clauses in N_1^* are redundant with respect to N^*. Since equality resolution applies only to clauses with *maximal* literals $0 \not\approx 0$ and since no clause in N_1^* contains repeated literals, no inferences are possible from clauses in $N^* \setminus N_1^*$. $\square$

As N^* is saturated, it contains the empty clause if and only if it is unsatisfiable modulo ACUKT $\cup$ DivInvNt. Since N and N^* are equivalent modulo the theory axioms, the main theorem of the this paper is proved:

Theorem 6.8 *A finite set $N \subseteq \mathcal{D}_{\mathrm{c}}^{\mathrm{elim}}$ is unsatisfiable modulo* ACUKT $\cup$ DivInvNt *if and only if the saturation strategy derives the empty clause from N.*

7 Conclusions

In previous work, we have demonstrated that the cancellative superposition calculus $\mathfrak{K}$ can be augmented by a variable elimination algorithm for DTAGs. The resulting calculus $\mathfrak{D}^{abs}$ is refutationally complete with respect to the axioms of divisible torsion-free abelian groups and allows us to dispense with variable overlaps altogether. As variable overlaps are one of the most prolific types of inferences in resolution or superposition style calculi, integration of the variable elimination algorithm leads to a dramatically reduced search space compared with the usual cancellative superposition calculus or, even worse, AC or ACU superposition calculi.

Since 1976 several resolution or superposition calculi have been shown to be decision procedures for certain classes of formulae (e.g., [3, 5, 7]). If the calculi in question are known to be refutationally complete, then showing that they are actually decision procedures amounts to proving that they terminate even on consistent inputs. In the present paper we have demonstrated that the calculus $\mathfrak{D}^{abs}$ is powerful enough to solve the decision problem for divisible torsion-free abelian groups. Following the general scheme described above, the termination proof is peculiar in two respects: First, we require that the set of clauses is saturated in a stratified way. Termination follows from the two facts that the number of strata is finite and that the number of new clauses derived during each stratum is finite. Second, the particular structure of the literals and clauses makes it possible to assume that all clauses share the *same* variables and that the pivotal substitution is always the identity mapping – in some sense, variables are treated as if they were constants.

What remains open at present is the precise computational complexity of our decision procedure. The time bound that can be derived in a straightforward manner from the saturation strategy is non-elementary. Possibly significantly better bounds can be obtained for subclasses of $\mathcal{D}_{\mathrm{c}}^{\mathrm{elim}}$, but this is still a matter of further research.

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