FACTORIZATION PROPERTIES OF GENUS-TWO BOSONIC AND FERMIONIC STRING PARTITION FUNCTIONS

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The genus-two bosonic and fermionic string partition functions factorize, in the separating pinching limit, into genus-one partition functions after integrating out the supermoduli with super-Beltrami support on the puncture points in the fermionic and heterotic case.

One very important aspect of string theory is the consistency of higher loop scattering amplitudes. In contrast to the one-loop expressions [1,2], the results for higher loops [3] are much less straightforward to obtain. The difficulties originate in the complicated structure of moduli or supermoduli space. In this letter we will show that the two-loop partition functions of the bosonic and fermionic string, derived from a path integral approach [4], correctly factorize into the known one-loop expressions pinching the zero homology cycle. For the bosonic string, the contribution of the reparametrization ghosts does not simply cancel two bosonic degrees of freedom as in the one-loop case. This follows from the non-existence of a globally defined light-cone gauge for more than one loop. However, the dependence of the insertion points of the three ghost fields disappears in this pinching limit. For the fermionic string we follow the approach of ref. [4] by first integrating out the supermoduli. Then, by identifying the insertion points of the super-Beltrami differentials with the two puncture points, in the pinching limit [5,6], the one-loop partition function of the fermionic string is obtained for each spin structure. Following this procedure we will show that only the matter part of the supercurrent contributes in the pinching limit [#1].

In the path integral formulation the two-loop partition function of the bosonic string is given by

$$\Gamma_2=\int\prod_{i=1}^{3}\mathrm{d}^2m_i\prod_{\bar{i}=1}^{3}\mathrm{d}^2\bar{m}_{\bar{i}}\int\mathscr{D}X\,\mathscr{D}b\,\mathscr{D}\bar{b}\,\mathscr{D}c\,\mathscr{D}\bar{c}\,\exp(-S)\prod_{i=1}^{3}(\eta_i|b)\prod_{\bar{i}=1}^{3}(\bar{\eta}_{\bar{i}}|\bar{b})\,. \tag{1}$$

The m_i denote the three complex moduli of the genus-two Riemann surface Σ_2 and the η_i are the three Beltrami differentials dual to the one-forms $\mathrm{d}m_i$. The insertion of the $(\eta_i|b)$ is necessary for the absorption of the three b-zero modes. (Any non-vanishing correlation function requires $(N+3)$ b and $\bar{b}$, and N (c and $\bar{c}$ insertions.) Thus, the computation of this path integral involves the following correlation functions:

$$\langle 1\rangle_X=\int\mathscr{D}X\exp(-S)=\int\mathrm{d}p_1^\mu\,\mathrm{d}p_2^\nu\exp[-2\pi p_i^\mu(\mathrm{Im}\,\Omega)_{ij}p_j^\nu]\,|Z_1(\Omega)|^{-26}=\left|\frac{1}{\det\mathrm{Im}\,\Omega}\right|^{13}|Z_1(\Omega)|^{-26}\,, \tag{2}$$

$$\langle b(w_1)\,b(w_2)\,b(w_3)\rangle\sim[Z_2(w_1,w_2,w_3)/Z_1^{1/2}](\Omega)\,, \tag{3}$$

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[#1] Recently Yasuda [7] has discussed the factorization behaviour of the four-graviton superstring amplitude.

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where Ω is the genus-two period matrix. Z_1 can be computed by relating it to the correlation function of two b and one c field of conformal spin 1 and 0, respectively, and background charge $Q=1$ [8]

$$Z_1^{3/2}=\frac{1}{\det \omega_i(z_j)} Z_1(z_1, z_2, w)=\frac{1}{\det \omega_i(z_j)} \theta(z_1+z_2-w-\Delta\,|\,\Omega) \frac{E(z_1, z_2)}{E(z_1, w)\, E(z_2, w)} \frac{\sigma(z_1)\, \sigma(z_2)}{\sigma(w)} . \tag{4}$$

Similarly one obtains for Z_2 [8]

$$Z_2(w_1, w_2, w_3)=\theta(w_1+w_2+w_3-3\Delta\,|\,\Omega)\, E(w_1, w_2)\, E(w_1, w_3)\, E(w_2, w_3)\, \sigma^3(w_1)\, \sigma^3(w_2)\, \sigma^3(w_3) . \tag{5}$$

The $\theta(z|\Omega)$ are the Riemann theta functions, $E(z, w)$ the prime form, Δ the Riemann theta constant and $\sigma(z)$ a holomorphic section of a trivial line bundle of rank $g/2$ tensors (see e.g. ref. [8]).

We will study the partition function near the boundary Δ_1 of moduli space which corresponds to pinching Σ_2 along the trivial homology cycle. Near Δ_1 the period matrix Ω of Σ_2 can be expanded as

$$\Omega=\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}+\alpha\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}+\mathrm{O}(t^2) , \tag{6}$$

where τ_1 and τ_2 are the Teichmüller parameters of the tori T_1 and T_2 into which Σ_2 decomposes at Δ_1, i.e. $t=0$. We can picture Σ_2 near Δ_1 as two tori T_1 and T_2 connected by a cylinder C which becomes infinitely long as $t\to 0$. The length of the cylinder, T, is related to t by $|t|=\exp(-T)$. Alternatively, we can view the surface as two tori connected by an annulus A whose inner radius shrinks to zero as $t\to 0$. A point z on A is related to a point z' on C via $z'=(1/2\pi \mathrm{i}) \ln z$.

Let us list the pinching limits of various quantities needed below [8]:

$$E(z_1, z_2)\to t^{-1/2} E(z_1, p_1)\, E(p_2, z_2) (\mathrm{d}p_1)^{1/2} (\mathrm{d}p_2)^{1/2} ,$$

$$E(z_1, z_3)\to -t^{-1/4} E(z_1, p_1)\, z_3 (\mathrm{d}z_3)^{-1/2} (\mathrm{d}p_1)^{1/2} ,$$

$$E(z_2, z_3)\to t^{-1/4} E(z_2, p_2) (\mathrm{d}z_3)^{-1/2} (\mathrm{d}p_2)^{1/2} ,$$

$$\sigma(z_1)\to \frac{\sigma(z_1)}{\sigma(p_1)} \frac{1}{E(z_1, p_1)} , \quad \text{likewise for } \sigma(z_2) , \quad \sigma(z_3)\to \frac{1}{z_3}\, \mathrm{d}z_3 . \tag{7}$$

and

$$\theta[\delta]\left(\sum_{i=1}^{m} z_1^{(i)}+\sum_{j=1}^{n} z_2^{(j)}+\sum_{k=1}^{p} z_3^{(k)}-Q\Delta\,|\,\Omega\right)$$

$$\to \theta[\delta_1]\left(\sum_{i=1}^{m} z_1^{(i)}-m p_1-Q\Delta_1\,|\,\tau_1\right) \theta[\delta_2]\left(\sum_{j=1}^{n} z_2^{(j)}-n p_2-Q\Delta_2\,|\,\tau_2\right) . \tag{8}$$

Here z_1 (z_2) denote points on the tori T_1 (T_2) and z_3 a point on the annulus A. p_1 and p_2 are the puncture points on T_1 and T_2, respectively. For genus one the prime form is

$$E(z, w)=\frac{\theta_1(z-w\,|\,\tau)}{\theta'(0\,|\,\tau) (\mathrm{d}z\, \mathrm{d}w)^{1/2}} , \quad \frac{\sigma(z_1)}{\sigma(z_1')}=\frac{(\mathrm{d}z_1)^{1/2}}{(\mathrm{d}z_1')^{1/2}} .$$

$\boldsymbol{\delta}=(\delta_1, \delta_2)$; for $\boldsymbol{\delta}$ an even spin structure, δ_1 and δ_2 are either both even or both odd. Also, we will use the notation $\theta[\boldsymbol{\delta}=0](z|\Omega)=\theta(z|\Omega)$ and $\theta_1=\theta[{}^{1/2}_{1/2}]$. At $g=1$, $\Delta=\frac{1}{2}(1+\tau)$ and $\theta(z+2\Delta\,|\,\tau)\sim\theta(z\,|\,\tau)$, $\theta(z+Q\Delta\,|\,\tau)\sim\theta_1(z+(Q-1)\,\Delta\,|\,\tau)$, up to z-independent phases which cancel in our final results; furthermore $\theta_1(0\,|\,\tau)=0$. Note that the factorization properties of the Riemann θ-functions are exactly such that at the torus the net ghost number is zero if one imagines $n_{1(2)}$ c-ghost insertions at the puncture points $p_{1(2)}$. This has also important consequences when determining the lowest $(\text{mass})^2$ string state which is exchanged between the two tori. The Beltrami differentials become [6]

$$\eta_{\tau_1}=\frac{1}{\operatorname{Im}\tau_1}\,\mathrm{d}\bar z\otimes(\mathrm{d}z)^{-1}\,,\quad \eta_{\tau_2}=\frac{1}{\operatorname{Im}\tau_2}\,\mathrm{d}\bar z\otimes(\mathrm{d}z)^{-1}\,,$$

$$\eta_t=-\frac{1}{t}\frac{z}{\bar z}\frac{1}{\log|t|}\,\mathrm{d}\bar z\otimes(\mathrm{d}z)^{-1}=-\frac{1}{t}\frac{1}{\log|t|}\,\mathrm{d}\bar z'\otimes(\mathrm{d}z')^{-1}\,, \tag{9}$$

where η_{τ_1}, η_{τ_2} and η_t have support on T_1, T_2 and C, respectively.

Applying these formulae to eqs. (3)–(5), it is easy to verify that Z_1, $Z_2(w_1, w_2, w_3)$ factorize as follows:

$$Z_1^{1/2}\to\eta(\tau_1)\,\eta(\tau_2)\,,\quad Z_2(w_1,w_2,w_3)\to(1/tw_3^2)\,\eta^3(\tau_1)\,\eta^3(\tau_2) \tag{10,11}$$

[$\eta(\tau)$ being the Dedekind η-function]. The dependence on the ghost insertion points disappears in this pinching limit as $\theta_1(w_1-p_1)/E(w_1,p_1)\sim\theta_1'(0)$. The w_3-integral implied by $(\eta_t|b)$, together with the factor $1/w_3^2$ in eq. (11) gives 1. Combining all the various factors and performing the w_1, w_2, w_3 integrals with the Beltrami differentials as given in eq. (9), we obtain for the bosonic string partition function

$$\Gamma_2=\int \mathrm{d}^2\tau_1\,\mathrm{d}^2\tau_2\int \mathrm{d}^2t\,\frac{1}{|t|^4}\left(\frac{1}{\operatorname{Im}\tau_1}\right)^{13}|\eta(\tau_1)|^{-48}\left(\frac{1}{\operatorname{Im}\tau_2}\right)^{13}|\eta(\tau_2)|^{-48}\,. \tag{12}$$

To recognize the implications of the factorization it is convenient to transform the annular coordinates w into the coordinate w' on the cylinder. Then, the t-integral turns into $\int_0^\infty \mathrm{d}T\exp(m^2T)$ $(m^2=-2)$ describing the exchange of the tachyon between the two tori.

To get a better understanding of the tachyon exchange let us examine the operators appearing in the degenerate limit [9,10]

$$\Gamma_2\sim{}_{\mathrm{T}_1}\left\langle b\left(\sum_h|\Phi_h\rangle\langle\Phi_h|\right)(\eta_t|b)\left(\sum_h|\Phi_h\rangle\langle\Phi_h|\right)b\right\rangle_{\mathrm{T}_2}\,, \tag{13}$$

where a complete set of states has been inserted at each end of the cylinder. The states have been labelled by their conformal weights. The cylinder will contribute a factor $(1/t)\,\mathrm{d}t\,t^h$ from the state Φ_h propagating through the cylinder. To leading order in the pinching parameter t the state with the lowest conformal weight h will dominate. From the factorization properties of the θ-function we see that this state is the ghost field c with $h=-1$, which is just the tachyon vertex operator at zero momentum. This ghost state is also needed to get non-vanishing correlation functions on the two tori as well as on the cylinder, according to the Riemann–Roch theorem for Riemann surfaces with punctures.

Finally let us mention that the expression (1) has exactly the same factorization behaviour as the two-loop partition function obtained by considering modular forms of weight ten [3,11,12]

$$\Gamma_2=\int \mathrm{d}^2\Omega\,\frac{1}{(\det\operatorname{Im}\Omega)^{13}}\,|\Delta_{(2)}(\Omega)|^{-4}\,. \tag{14}$$

The cusp-form $\Delta_{(2)}$

$$\Delta_{(2)}(\Omega)=\prod_{i=1}^{10}\theta[\delta_i](0|\Omega)\to t\eta^{12}(\tau_1)\,\eta^{12}(\tau_2) \tag{15}$$

leads again to the $|t|^{-4}$ dependence in the pinching limit. Thus, we conjecture that the path integral expression is identical to eq. (14). It is clear that the b, c ghosts make a qualitatively different contribution to Γ_2 than the bosonic X-system.

For the heterotic string the two-loop partition function is

$$\Gamma_2 = \int \prod_{i,\bar{i}=1}^{3} \mathrm{d}m_i\,\mathrm{d}\bar{m}_{\bar{i}} \prod_{a=1}^{2} \mathrm{d}\zeta_a \sum_{\delta} \epsilon_{\delta} W_{\delta} \,, \tag{16}$$

where

$$W_{\delta} = \int \mathscr{D}X\,\mathscr{D}\psi_{\delta} \exp(-S[X,\psi]) \int \mathscr{D}\bar{b}\,\mathscr{D}\bar{c} \exp(-S[\bar{b},\bar{c}]) \int \mathscr{D}B_{\delta}\,\mathscr{D}C_{\delta} \exp(-S[B,C])$$
$$\times \prod_{a} \delta((\chi_a|B)) \prod_{i,\bar{i}} (\eta_i|B)(\eta_{\bar{i}}|B) \,. \tag{17}$$

Here C and B are superfields containing the right-moving conformal and superconformal ghosts (c, γ) and anti-ghosts (b, β). η_i, $\eta_{\bar{i}}$ and χ_a are the Beltrami and super-Beltrami differentials. They project out the zero-modes of B corresponding to changes in the (super)conformal structure of the genus-two Riemann surface Σ_2 and the insertions $(\eta_i|B)$, $\delta((\chi_a|B))$ restrict the B integration to modes orthogonal to the zero modes. The sum over spin structures δ reflects the fact that we can impose periodic or antiperiodic boundary conditions for ψ, β and γ along the homology cycles of Σ_2. Since the ψ^{μ} and the gravitino transform under supersymmetry with the same supertransformation parameter, the ψ^{μ}, β and γ carry the same spin structure δ. ϵ_{δ} are phases that ensure modular invariance of the partition function. For odd spin structures, W_{δ} vanishes identically due to the ψ zero-modes. From now on δ will always denote an even spin structure on Σ_2. Finally, m_i, $\bar{m}_{\bar{i}}$ and ζ_a are the moduli and supermoduli. Following the literature [6] we choose a metric on Σ_2 such that $\eta_{\bar{i}\bar{z}}{}^{z} = \eta_{iz}{}^{\bar{z}} = \eta_{a\bar{z}}{}^{z} = \eta_{az}{}^{\bar{z}} = 0$ and for the gravitini $\chi_{\bar{z}}{}^{\theta} = \sum_a \zeta_a \delta^{(2)}(z - z_a)$ where $z_{1,2}$ are as yet arbitrary points on Σ_2 (however, c.f. below). We can then perform the integration over the supermoduli to arrive at [13]

$$W_{\delta} = \left\langle \prod_{\bar{i}} (\bar{\eta}_{\bar{i}}|\bar{b}) \right\rangle \left\langle \xi(z_0)\, Y(z_1)\, Y(z_2) \prod_{i} (\eta_i|b) + \xi(z_0)\, Y(z_1)\, \partial\xi(z_2) \sum_{j=1}^{3} (-1)^{j+1} \frac{\partial z_2}{\partial m_j} \prod_{i \neq j} (\eta_i|b) \right.$$
$$\left. + \xi(z_0)\, Y(z_2)\, \partial\xi(z_1) \sum_{j=1}^{3} (-1)^{j+1} \frac{\partial z_1}{\partial m_j} \prod_{i \neq j} (\eta_i|b) \right\rangle_{\delta} . \tag{18}$$

Here we have bosonized the β, γ ghost system and dropped terms which vanish by $(b, \bar{b})$ ghost number conservation. The picture changing operator Y splits into three parts labelled by their superconformal ghost charge

$$Y = Y^{(0)} + Y^{(1)} + Y^{(2)} \tag{19}$$

with

$$Y^{(0)}(z) = c\partial\xi(z) \,, \quad Y^{(1)}(z) = \exp(\phi)\, T_{\mathrm{F}}^{\mathrm{matter}}(z) = \exp(\phi)\, \psi^{\mu}\partial X_{\mu}(z) \,,$$

$$Y^{(2)}(z) = -\tfrac{1}{4}[2(\partial\eta)\exp(2\phi)\, b(z) + \eta\partial(\exp(2\phi)\, b(z))] = -\tfrac{1}{4}(2\partial_{z'} + \partial_z)\, \eta(z') \exp[2\phi(z)] b(z) \Big|_{z'=z} .$$

We will show that near Δ_1 only the first term in W_{δ} contributes and factorizes into the product of two genus-one partition functions for each spin structure separately if we put z_1 and z_2, the support of the super-Beltramis, or equivalently, the insertion points of the picture changing operators, to the puncture points p_1 and p_2, as we are instructed to do following refs. [5,6]. We will first take the limit $t \to 0$ and then put $z_{1,2}$ to the puncture points. Let us start with the term quadratic in the picture changing operator. Conservation of conformal or superconformal ghost charge dictates that only three out of nine possible terms contribute, namely $Y^{(1)}(z_1)\, Y^{(1)}(z_2) + Y^{(0)}(z_1)\, Y^{(2)}(z_2) + Y^{(2)}(z_1)\, Y^{(0)}(z_2)$. We will denote their contributions to the partition function by $W_{\delta}^{(11)} + W_{\delta}^{(02)} + W_{\delta}^{(20)}$. Let us consider them in turn. The various correlation functions can be found in ref. [8].

$W_{\delta}^{(11)}$. For the holomorphic part we need the following correlators and their pinching limits. (We will only keep terms to lowest order in t and $\bar{t}$.)

$$\langle \partial X^\mu(z_1)(\mathrm{d}z_1)\,\partial X^\nu(z_2)(\mathrm{d}z_2)\rangle \sim |Z_1|^{-10}\frac{1}{(\det \operatorname{Im}\tau)^5}\,[-2\pi^2\omega_i(z_1)(\operatorname{Im}\tau)^{-1}_{ij}\omega_j(z_2)+\partial_1\partial_2\ln E(z_1,z_2)]\delta^{\mu\nu}$$

$$\to \bar{t}\left(\frac{1}{|Z_1|^{10}}\frac{\mathrm{d}z_1}{(\operatorname{Im}\tau_1)^6}\right)\left(\frac{1}{|Z_1|^{10}}\frac{\mathrm{d}z_2}{(\operatorname{Im}\tau_2)^6}\right)\delta^{\mu\nu}+\mathrm{O}(t)\,, \tag{20}$$

where we have dropped constant factors.

$$\langle \psi_\mu(z_1)\,\psi_\nu(z_2)\rangle_{\boldsymbol{\delta}}=\frac{1}{Z_1^{5/2}}\,\theta^4[\boldsymbol{\delta}](0|\Omega)\,\frac{\theta[\boldsymbol{\delta}](z_1-z_2|\Omega)}{E(z_1,z_2)}\,\delta_{\mu\nu}$$

$$\to t^{1/2}\left(\frac{1}{Z_1^{5/2}}\,\theta^4[\delta_1](0|\tau_1)\,\frac{\theta_1[\delta_1](z_1-p_1|\tau_1)}{E(z_1,p_1)}\right)\left(\frac{1}{Z_1^{5/2}}\,\theta^4[\delta_2](0|\tau_2)\,\frac{\theta_1[\delta_2](z_2-p_2|\tau_2)}{E(z_2,p_2)}\right)\delta_{\mu\nu}\,. \tag{21}$$

Notice that this vanishes if δ_1 or δ_2 are odd spin structures.

$$\langle \xi(x)\exp[\phi(z_1)](\mathrm{d}z_1)^{-3/2}\exp[\phi(z_2)](\mathrm{d}z_2)^{-3/2}\rangle_{\boldsymbol{\delta}}=Z_1^{1/2}\,\frac{1}{\theta[\boldsymbol{\delta}](z_1+z_2-2\Delta|\Omega)}\,\frac{1}{E(z_1,z_2)}\,\frac{1}{\sigma(z_1)^2\sigma(z_2)^2}$$

$$\to t^{1/2}\left(\frac{Z_1^{1/2}}{\theta[\delta_1](z_1-p_1|\tau_1)}\,E(z_1,p_1)\,\frac{\sigma(p_1)^2}{\sigma(z_1)^2}\right)\left(\frac{Z_1^{1/2}}{\theta[\delta_2](z_2-p_2|\tau_2)}\,E(z_2,p_2)\,\frac{\sigma(p_2)^2}{\sigma(z_2)^2}\right)\,. \tag{22}$$

The remaining correlation functions have been given before. From the antiholomorphic part we also get a contribution from the sixteen-dimensional lattice corresponding to the spin structure $\boldsymbol{\delta}$. Its contribution is given in terms of θ-functions and they factorize in the pinching limit according to eq. (7). Putting everything together we finally get

$$\Gamma_2^{\boldsymbol{\delta}}=\int\frac{\mathrm{d}^2t}{|t|^2}\left(\int\frac{\mathrm{d}^2\tau_1}{(\operatorname{Im}\tau_1)^2}\,\frac{1}{(\operatorname{Im}\tau_1)^4}\,\frac{\theta^4[\delta_1](0|\tau_1)}{\eta^{12}(\tau_1)}\,\frac{P^{\delta_1}_{\text{lattice}}(\bar{\tau}_1)}{\bar{\eta}^{24}(\bar{\tau}_1)}\right)\left(\int\frac{\mathrm{d}^2\tau_2}{(\operatorname{Im}\tau_2)^2}\,\frac{1}{(\operatorname{Im}\tau_2)^4}\,\frac{\theta^4[\delta_2](0|\tau_2)}{\eta^{12}(\tau_2)}\,\frac{P^{\delta_2}_{\text{lattice}}(\bar{\tau}_2)}{\bar{\eta}^{24}(\bar{\tau}_2)}\right)\,. \tag{23}$$

From this we see that the lightest exchanged particle is the dilaton (cf. the discussion of the superstring below).

$W^{(02)}_{\boldsymbol{\delta}}$. We will show that $W^{(02)}_{\boldsymbol{\delta}}$ vanishes in the pinching limit. We write

$$W^{(02)}_{\boldsymbol{\delta}}=\int_{w_i}\prod_{i=1}^{3}\eta_i(w_i)\,C^{(02)}_{\boldsymbol{\delta}}\,, \tag{24}$$

where

$$C^{(02)}_{\boldsymbol{\delta}}=-\tfrac{1}{4}(2\partial_{z_2'}+\partial_{z_2})\left\langle c(z_1)(\mathrm{d}z_1)^{-1}b(z_2)(\mathrm{d}z_2)^2\prod_{i=1}^{3}b(w_i)(\mathrm{d}w_i)^2\right\rangle$$

$$\times\,\langle \xi(z_0)\,\partial_{z_1}\xi(z_1)(\mathrm{d}z_1)\,\eta(z_2')(\mathrm{d}z_2')\exp[2\phi(z_2)](\mathrm{d}z_2)^{-4}\rangle_{\boldsymbol{\delta}}\Big|_{z_2=z_2'}(\mathrm{d}z_2)\,. \tag{25}$$

It is straightforward to show that in the pinching limit it has the following w_3-dependence ($w_3'=(1/2\pi\mathrm{i})\times\log w_3\in\mathrm{C}$): $C^{(02)}_{\boldsymbol{\delta}}\sim w_3^{-3}(\mathrm{d}w_3)^2$. This comes entirely from the bc-correlator, as is easy to see. Using the expression for $\eta_i(w_3)$ given in eq. (8) and integrating over w_3 we get zero. I.e. $W^{(02)}_{\boldsymbol{\delta}}$ vanishes in the pinching limit.

$W^{(20)}_{\boldsymbol{\delta}}$. It vanishes by an argument identical to the one above.

Let us now turn our attention to the terms in eq. (18) linear in the picture changing operator. Here, by ghost charge conservation only $Y^{(2)}$ contributes. Let us concentrate on the second line in eq. (18). We will show that each term in the sum vanishes separately. The third line then vanishes by symmetry under interchange of z_1 and z_2. Near Δ_1 the moduli m_i are τ_1, τ_2 and t. The term proportional to $\partial z_2/\partial\tau_2$ contains a factor

$$\langle b(z_1)(\mathrm{d}z_1)^2(\eta_{\tau_1}|b)(\eta_t|b)\rangle = \int\limits_{w_i} \eta_{\tau_1}(w_1)\,\eta_t(w_3)\langle b(z_1)(\mathrm{d}z_1)^2 b(w_1)(\mathrm{d}w_1)^2 b(w_3)(\mathrm{d}w_3)^2\rangle\,. \tag{26}$$

The b-correlation function is

$$\langle b(z_1)(\mathrm{d}z_1)^2 b(w_1)(\mathrm{d}w_1)^2 b(w_3)(\mathrm{d}w_3)^2\rangle$$
$$= Z_1^{-1/2}\theta(\boldsymbol{z}_1+\boldsymbol{w}_1+\boldsymbol{w}_3-3\Delta\,|\,\Omega)\,E(z_1,w_1)\,E(z_1,w_3)\,E(w_1,w_3)\,\sigma^3(z_1)\,\sigma^3(w_1)\,\sigma^3(w_3)\,. \tag{27}$$

In the pinching limit z_1, $w_1\in \mathrm{T}_1$, $w_3'=(1/2\pi\mathrm{i})\log w_3\in \mathrm{C}$, where the restriction on w_1 and w_3 comes from the support of η_{τ_1} and η_t respectively. Then

$$\theta(\boldsymbol{z}_1+\boldsymbol{w}_1+\boldsymbol{w}_3-3\Delta\,|\,\Omega)\to\theta(z_1+w_1-2p_1-3\Delta_1\,|\,\tau_1)\,\theta(-3\Delta_2\,|\,\tau_2)=0\,. \tag{28}$$

The reason why this term vanishes in the pinching limit is clear. There is no b-insertion on T_2.

The term proportional to $\partial z_2/\partial\tau_1$ contains a factor

$$(2\partial_{z_1'}+\partial_{z_1})\langle b(w_2)(\mathrm{d}w_2)^2 b(z_1)(\mathrm{d}z_1)^2 b(w_3)(\mathrm{d}w_3)^2\rangle\langle\xi(z_0)\,\eta(z_1')\,\partial\xi(z_2)\,\exp[2\phi(z_1)]\rangle_{\boldsymbol{\delta}}\Big|_{z_1'=z_1}\,. \tag{29}$$

In the pinching limit, $z_1\in\mathrm{T}_1$, z_2, $w_2\in\mathrm{T}_2$ and $w_3\in\mathrm{A}$. In this limit the above expression becomes

$$(2\partial_{z_1'}+\partial_{z_1})\left((2\pi)^{-1}\eta^{-3}(\tau_2)\,\frac{\theta[\delta_2](2z_2-2p_2)}{\theta^2[\delta_2](z_2-p_2)\,\theta_1^2(z_2-p_2)}\right)$$
$$\times\left((2\pi)^3\eta^9(\tau_1)\,\frac{\theta[\delta_1](2z_1-2z_1')}{\theta^2[\delta_1](2z_1-z_1'-p_1)}\,\frac{\theta_1^4(z_1-p_1)}{\theta_1^2(z_1'-p_1)}\right)\frac{1}{w_3^2}\Bigg|_{z_1'=z_1}\,, \tag{30}$$

which vanishes for even spin structure δ_1 upon differentiation and setting $z_1'=z_1$ since then $\theta'[\delta_1](0)=0$. For odd spin structure δ_1 the factor coming from the ψ integration, $\langle 1\rangle_\psi$ vanishes in the pinching limit. This is easy to understand since for δ_1 and δ_2 odd spin structures, the ψ zero-modes on the two tori lead to a vanishing result. A different argument why this term is zero is that $(1/\tau_1)\,\partial z_2$ itself vanishes as $z_2\to p_2$. The puncture point on T_2 should not depend on the Teichmüller parameter of T_1.

The last term finally vanishes since $\partial z_2/\partial t=0$ for $z_2\to p_2$. This is so because t parametrizes a family of surfaces near Δ_1 of the moduli space of Σ_2 all of which have the same pinching limit (namely $t\to 0$) and therefore the same puncture points.

The pinching limit of the type II superstring partition function is obtained by repeating the calculation of the holomorphic sector of the heterotic string for the antiholomorphic sector. The only difference is the correlation function of the bosonic matter fields $\langle\partial X^\mu(z_1)\,\partial X^\nu(z_2)\,\bar\partial X^\rho(\bar z_1)\,\bar\partial X^\sigma(\bar z_2)\rangle$ which now produces a factor (to lowest order in t) $(\mathrm{Im}\,\tau_1)^{-6}(\mathrm{Im}\,\tau_2)^{-6}$. Also, the terms proportional to $\partial z_a/\partial\bar m_{\bar i}$ and $\partial\bar z_{\bar a}/\partial m_i$ which spoil the factorization of the two-loop partition function into an holomorphic and an anti-holomorphic part, can be shown to vanish either by $(b,\bar b)$ conservation or by arguments identical to the ones discussed in the heterotic case. Multiplying the holomorphic and anti-holomorphic parts we finally obtain for a given even spin structure $\boldsymbol{\delta}=(\delta_1,\delta_2)$

$$\Gamma_2^{\boldsymbol{\delta}}=\int \mathrm{d}^2\tau_1\,\mathrm{d}^2\tau_2\int\limits_{\mathrm{A}}\frac{\mathrm{d}^2 t}{|t|^2}\left(\frac{1}{(\mathrm{Im}\,\tau_1)^6}\left|\frac{\theta^4[\delta_1](0|\tau_1)}{\eta(\tau_1)^{12}}\right|^2\right)\left(\frac{1}{(\mathrm{Im}\,\tau_2)^6}\left|\frac{\theta^4[\delta_2](0|\tau_2)}{\eta(\tau_2)^{12}}\right|^2\right). \tag{31}$$

The lightest exchanged particle, corresponding to the lowest power in t, is a massless dilaton, as indicated by the factor $1/|t|^2$ in eq. (31). This can be seen if we replace eq. (13) by

$$\Gamma_2 \sim {}_{T_1}\left\langle \partial X \psi \exp(\phi)\, b \left(\sum_h |\Phi_h\rangle\langle\Phi_h| \right) (\eta_t|b) \left(\sum_h |\Phi_h\rangle\langle\Phi_h| \right) \partial X \psi \exp(\phi)\, b \right\rangle_{T_2} . \tag{32}$$

Now the state Φ with lowest weight h inserted at the puncture points p_1 and p_2 can be deduced from the factorization properties of the θ-functions of the ψ, (b, c) and (β, γ)-correlators and the requirement of ghost number conservation. We find $\Phi = c\exp(-\phi)\,\psi$. Since this holds for both the left- and the right-moving sectors, we conclude that the lightest exchanged state is a massless dilaton. This is true for each spin structure separately and due to the integration over the supermoduli.

In conclusion, we have shown that only $W_\delta^{(11)}$ does contribute to the partition function in the pinching limit. Furthermore, the factorization behaviour into genus-one partition functions is true for each spin structure separately. Therefore, these arguments can be easily generalized for general (heterotic) string theories in an arbitrary number of dimensions.

Note added in proof. The observation of the absence of tachyon exchange in the superstring and heterotic string even before the sum over spin structures is performed, is already contained in ref. [14]. We want to thank Professor I. Iengo for bringing this to our attention.

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