A steady-state Fokker-Planck solver for wave-particle interactions in solar coronal holes

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Resonant wave-ion interactions are considered one possible mechanism at the basis of the solar corona heating and the solar wind acceleration [1, 2, 3]. Generally, wave-particle interactions can be described by the quasi-linear theory [4] which is a well-established tool in plasma physics and fusion research [4, 5]. The Fokker-Planck solver SSFPQL (steady state Fokker-Planck quasi-linear, [5]), in particular, has been widely applied to fusion plasma, and the aim of our work is to transfer those techniques into the context of the longstanding solar corona heating problem. As a necessary step, we have to demonstrate the feasibility of using a Legendre polynomial representation of the pitch-angle dependence of ion velocity distribution functions, as this representation is crucial for the application of SSFPQL. In this paper we report the first numerical implementation of the model equations in terms of Legendre polynomials and related numerical problems.

We consider the case of coronal funnels, that are magnetic field structures roughly modeled as half of a magnetic bottle having the high-field side attached to the solar surface (photosphere) with field lines going into the interplanetary space. An axisymmetric analytical solution [6] is our standard model for a funnel magnetic field. Particles move in spirals with the center of gyration attached to a magnetic field line, parameterized, say, by s; averaging over the gyration phase, a particle at the point s is fully characterized by its parallel and perpendicular (to the magnetic field) speeds, $v_{\|}$ and $v_{\perp}$, i.e., a population of ions of the species j at the position s can be described by a velocity distribution function of the form $f_j(s, v_{\|}, v_{\perp})$. As for the dynamics, in addition to the bounding force of gravity, one should consider the outwardly-oriented electric field due to the escaping electrons, as well as the magnetic mirror force due to the abrupt expansion of the magnetic field lines in the funnel. Both these forces reduce the gravitational potential well for ions, so that, when enough energy is supplied, e.g., through absorption of waves, they can flow out of the funnel at high speed ($\approx$ 700km/s) and relatively low-densities (from $10^{10}\mathrm{cm}^{-3}$ at the boundary of the solar corona $z = 0.01R_{\odot}$ down to $10^{7}\mathrm{cm}^{-3}$ in the lower corona at $z = 0.3R_{\odot}$, where z is the altitude from the photosphere and $R_{\odot}$ is the solar radius). Such low density values imply, particularly, that collisions are rare, and cannot fully restore the local thermodynamical equilibrium [2]. At last, observed distributions exhibit evidences of

interaction of ions with a spectrum of ion cyclotron waves which we describe in the framework of the quasi-linear theory. The dynamics is summarized in the following set of equations, [7],

$$\mu v\frac{\partial f_j}{\partial s}+\left(\frac{q_j}{m_j}E_\parallel-g_\parallel\right)\left[\mu\frac{\partial f_j}{\partial v}+\frac{1-\mu^2}{v}\frac{\partial f_j}{\partial \mu}\right]-\frac{B_0' v(1-\mu^2)}{2B_0}\frac{\partial f_j}{\partial \mu}=C_j(f_j)+Q_j(\mathscr{E},f_j), \quad (1)$$

$$-eN_eE_\parallel(s)=\nabla_\parallel p_e, \quad p_e=k_BN_eT_e. \quad (2)$$

Equation (1) is the steady-state Fokker-Planck equation averaged over the fast gyration of the particles and written in spherical velocity coordinates $v=(v_\parallel^2+v_\perp^2)^{1/2}$, $\mu=v_\parallel/v$; here, $E_\parallel$ and $g_\parallel$ are the parallel components of the self-consistent electric field generated by the electron pressure gradient, equation (2), and of the gravitational acceleration; the magnetic field strength $B_0(s)$ is obtained from the standard analytical solution [6]. The right-hand side of equation (1) comprises the Coulomb collision operator C_j linearized on a Maxwellian background and the quasi-linear diffusion operator [4] which depends on the wave energy spectrum $\mathscr{E}$. Equation (2) expresses the equilibrium conditions for the mass-less fluid of electrons. An equation for the self-consistent evolution of $\mathscr{E}$ should be added which, however, is not addressed here.

It is worth noting that the system (1)-(2) with $Q_j=0$ has a simple static solution of the form $f_j\propto\exp(-\varepsilon)$ where ε is the total energy of an ion normalized to k_BT_0, T_0 being a constant equilibrium temperature (the same for all species). According to this solution the ion gravitational potential energy is twice as large as the electric potential energy, so that ions are strongly gravitationally bounded and their density profiles drop down quickly with the parameter s which for a straight field line can be identified with the altitude normalized to the solar radius $s=z/R_\odot$; more specifically, the electric potential drop from the transition region to the interplanetary space is ≈ 1000V (Pannekoek-Rosseland potential), and, for a proton-electron plasma one can prove the density scaling $\propto 1/(R_\odot+z)$. In general, the Fokker-Planck equation (1) describes a two-way diffusion process [8, 9], as the orientation of the evolution parameter s changes depending of the sign of the pitch-angle μ.

In order to take advantage of the routine SSFPQL [5], we need to exploit Legendre polynomials $P_n(\mu)$, $n\geq 0$, that constitute a basis for squared-integrable functions on the interval $-1\leq\mu\leq 1$; we write the distribution functions as

$$f_j(s,v,\mu)=\frac{\bar{N}}{\bar{V}_j^3}\chi_j(s)e^{-u^2/u_j^2}\sum_{n\geq 0}F_{j,n}(s,u)P_n(\mu). \quad (3)$$

Here, $\bar{N}$ and $\bar{V}_j=(2k_B\bar{T}/m_j)^{1/2}$ are reference values for the density and the speed, the latter being defined in terms of a reference value $\bar{T}$ for the temperature. With a slight abuse of notation, the velocity coordinates $u=v/\bar{V}_j$ and μ used here are different from those in (1) as they are

referred to the velocity frame co-moving with the plasma parallel fluid motion at the speed $V_{j,\|}$. A Maxwellian pre-factor with the normalized thermal speed $u_j^2 = (2k_BT_j/m_j)\overline{V}_j^{-2} = T_j/\overline{T}$, has been introduced for technical reasons: this allows us to avoid the fine-tuned balance between terms in the collision operator C_j as well as in the transport operator at the left-hand side of the Fokker-Planck equation (1). The substitution of (3) into (1) yields a degenerate evolution equation along the magnetic field line for the coefficients $F_{j,n}(s,u)$,

$$A\frac{\partial F}{\partial s} = \frac{\partial}{\partial u}\left[\Gamma^{11}\frac{\partial F}{\partial u} + \Gamma^{10}F\right] - \Gamma^{01}\frac{\partial F}{\partial u} - \Gamma^{00}F, \tag{4}$$

where $F = \big(F_{j,n}(s,u)\big)$ is the vector of coefficients $F_{j,n}$ in (3) and A, Γ^{ab}, $a,b = 0,1$, are matrices. Equation (4) must be supplied with appropriate boundary conditions; with this aim, we note that the results available for the two-way diffusion problem [8, 9] no longer apply as the Legendre series expansion unavoidably mixes the positive-μ and the negative-μ contributions. The character of the operator $\mu\partial/\partial s$ is mapped into the matrix operator $A\partial/\partial s$ with A being a strongly ill-conditioned matrix, i.e.,: A is formally invertible, but $\det A$ is close to zero. However, it appears physically natural to set up an initial/boundary value problem for (4), with initial position $s = s_1 = z_1/R_\odot = 0.01$ set at the transition region between the chromosphere and the lower corona. The initial distributions can be local Maxwell distributions with parameters typical of the chromosphere. We also need the values of the vector $F(s,u)$ for $u = 0$ which is the boundary in velocity space. When $u = 0$ the distribution functions cannot depend on the pitch-angle μ and this condition gives $F_{j,n}(s,0) = 0$ for every $n \geq 1$; as for the zero-order coefficient we set $F_{j,0}(s,0) = F_{j,0}(s_1,0) =$ constant, the information of the physical boundary condition being retained through the factor $\chi_j(s)$ in (3). The latter is determined in terms of the density profile, hence, coupled to (4) we have fluid equations obtained by parallel velocity moments of (1), namely,

$$(N_jV_{j,\|})' = (B_0'/B_0)N_jV_{j,\|}, \tag{5a}$$

$$(N_jV_{j,\|}^2 + \tfrac{k_B}{2m_j}N_jT_{j,\|})' = \tfrac{B_0'}{B_0}\left[N_jV_{j,\|}^2 + \tfrac{k_B}{2m_j}N_j(T_{j,\|} - T_{j,\perp})\right] + (\tfrac{q_j}{m_j}E_\| - g_\|)N_j + (\tfrac{dU_{j,\|}}{ds})_{\text{coll.}}, \tag{5b}$$

$$N_j = 4\pi\chi_jI_j, \quad \frac{1}{2}k_BN_jT_{j,\|} = U_{j,\|}, \quad k_BN_jT_{j,\perp} = U_{j,\perp}, \tag{5c}$$

where $I_j = I_j(F)$ and $U_{j,\|}$, $U_{j,\perp}$ are the parallel and perpendicular internal energies, to be evaluated as velocity integrals of the coefficients F. This is a system of fully nonlinear ordinary differential/algebraic equations coupled to the system (4) via the integrals I_j and internal energies. The prime denotes differentiation with respect to s and the operator $\frac{d}{ds} - \frac{B_0'}{B_0}$ is just the divergence of a parallel vector field.

We have designed a numerical routine that can solve rather general systems of degenerate evolution equations coupled to general nonlinear ordinary differential/algebraic equations. The algorithm is based on the combination of Galerkin finite elements in velocity u with stiffly-accurate Runge-Kutta methods in position s, according to the so called Rothe method [10]. We are currently applying this tool to the system (4)-(5) and the work is still in progress.

More specifically, we have been able to run some test cases solving separately the partial differential equations (4) with given plasma profiles, as well as the fluid equations (5) with given distribution functions. The static solution is particularly useful in this testing phase: the solutions of both (4) and (5) reproduce the corresponding quantities of the static equilibrium with good accuracy. For the specific case of the numerical solution for the distribution functions, equation (4), we have successfully reproduced the deformation of the ion distribution functions due to a uniform electric field imposed artificially for the purpose of the test.

On the other hand, the full self-consistent solution, in which the internal energies and the integral $I_k(s)$ in (5) are computed from the solution F of (4), exhibits numerical problems that requires more efforts and this will be addressed in the future.

References

[1] M.J. Aschwanden, *Physics of the Solar Corona* Springer-Verlag Berlin (2005).

[2] E. Marsch, Living Rev. Solar Phys. **3**, 1 (2006).

[3] S.R. Cranmer, Living Rev. Solar Phys. **6**, 3 (2009).

[4] M. Brambilla, *Kinetic Theory of Plasma Waves*, Oxford Science Publ. (1998).

[5] M. Brambilla, Nuclear Fusion **34**, 1121 (1994).

[6] P. Hackenberg, G. Mann, and E. Marsch, Space Science Reviews **87**, 207 (1999).

[7] C. Vocks and E. Marsch, AstroPhys. J. **568**, 1030 (2002).

[8] R. Beals, J. Math. Phys. **22**, 954 (1981)

[9] F. Andersson etal., Phys. Rev. E **65**, 036502 (2002).

[10] E. Emmrich and M. Thalhammer, Mathematics of Computation **79**, 785 (2010).