Testing the Master Constraint Programme for Loop Quantum Gravity V. Interacting Field Theories

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Abstract

This is the final fifth paper in our series of five in which we test the Master Constraint Programme for solving the Hamiltonian constraint in Loop Quantum Gravity. Here we consider interacting quantum field theories, specifically we consider the non − Abelean Gauss constraints of Einstein − Yang − Mills theory and 2+1 gravity. Interestingly, while Yang − Mills theory in 4D is not yet rigorously defined as an ordinary (Wightman) quantum field theory on Minkowski space, in background independent quantum field theories such as Loop Quantum Gravity (LQG) this might become possible by working in a new, background independent representation.

While for the Gauss constraint the Master constraint can be solved explicitly, for the 2+1 theory we are only able to rigorously define the Master Constraint Operator. We show that the, by other methods known, physical Hilbert is contained in the kernel of the Master Constraint, however, to systematically derive it by only using spectral methods is as complicated as for 3+1 gravity and we therefore leave the complete analysis for 3+1 gravity.

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Contents

1 Introduction

We continuue the test of the Master Constraint Programme [1] for Loop Quantum Gravity (LQG) [6, 7, 8] which we started in the companion papers [2, 3, 4, 5]. The Master Constraint Programme is a new idea to improve on the current situation with the Hamiltonian constraint operator for LQG [9]. In short, progress on the solution of the Hamiltonian constraint has been slow because of a technical reason: the Hamiltonian constraints themselves are not spatially diffeomorphism invariant. This means that one cannot first solve the spatial diffeomorphism constraints and then the Hamiltonian constraints because the latter do not preserve the space of solutions to the spatial diffeomorphism constraint [10]. On the other hand, the space of solutions to the spatial diffeomorphism constraint [10] is relatively easy to construct starting from the spatially diffeomorphism invariant representations on which LQG is based [11] which are therefore very natural to use and, moreover, essentially unique. Therefore one would really like to keep these structures. The Master Constraint Programme removes that technical obstacle by replacing the Hamiltonian constraints by a single Master Constraint which is a spatially diffeomorphism invariant integral of squares of the individual Hamiltonian constraints which encodes all the necessary information about the constraint surface and the associated invariants. See e.g. [1, 2] for a full discussion of these issues. Notice that the idea of squaring constraints is not new, see e.g. [13], however, our concrete implementation is new and also the Direct Integral Decomposition (DID) method for solving them, see [1, 2] for all the details.

The Master Constraint for four dimensional General Relativity will appear in [15] but before we test its semiclassical limit, e.g. using the methods of [16, 17] and try to solve it by DID methods we want to test the programme in the series of papers [2, 3, 4, 5]. In the previous papers we focussed on finite dimensional systems of various degrees of complexity and free field theories. In this paper we consider interacting quantum field theories, in particular in four dimensions and 2+1 gravity which is interacting before one solves the constraint (the reduced phase space is of course finite dimensional). This might be a surprise given the fact that no interacting Wightman quantum field theories have been constructed so far in four dimensions. The resolution of the puzzle is that instead of working on a Minkowski background we couple gravity to matter and therefore arrive at a background independent quantum field theory. That these display a better ultraviolet behaviour than background dependent quantum field theories, at least for a large class of operators was already shown in [9] and we will see that the same mechanism is at work here and makes the associated Master Constraint Operators well defined. This does not mean, however, that we are done and have a consistent quantum theory of both matter and geometry because only 1. after having solved the Master Constraint for *all* constraints (here we only consider the matter Gauss constraints but not, in particular, the Hamiltonian constraints) of gravity coupled to the standard model and 2. after having shown that the solution theory is consistent with experiment (reduces to QFT on (curved) backgrounds in the semiclassical limit of low geometry fluctuations) do we have a viable candidate theory. It is precisely the purpose of the Master Constraint Programme to complete this task and the present series of papers serves the purpose to make sure that the Master Constraint Programme reproduces the established results in solvable cases.

2 Infinite Number of Non-Abelean First Class Constraints Non – Polynomial in the Momenta with Structure Constants

In a previous article [5] we discussed a background dependent quantum field theory, specifically free Maxwell theory. Even in this free theory the definition of the Master Constraint had to involve a non-trivial integral kernel C which depended on the background metric in oder that the Master Constraint Operator was densely defined. In this section we discuss two background independent theories, namely Einstein – Yang – Mills theory and pure Einstein gravity. Notice that these theories are no longer free, both the Yang – Mills and gravitational degrees of freedom are self-interacting and interacting with each other. One therefore expects severe ultraviolet divergence problems, the more as we are not allowed to use a background metric in order to regulate the Master Constraint Operator. Nevertheless it turns out that **precisely because of the background independence the ultraviolet problems can be overcome, background independent theories regulate themselves!** The way this works is identical to the mechanism discovered in [9] for the definition of various Hamiltonian constraints in Loop Quantum Gravity (LQG) (see [7, 8] for an introduction). Hence we will construct a corresponding Master Constraint Operator for the Gauss constraints of Einstein – Yang – Mills theory using LQG techniques.

In what follows we consider canonical Yang – Mills fields for a compact gauge group G with canonicaly conjugate pair $(\underline{A}_a^J, \underline{E}_J^a)$. Here $a, b, .. = 1, 2, 3$ are spatial tensor indices while $J, K, .. = 1, .., \dim(G)$ are Lie(G) indices. Likewise for the gravitational sector we consider the canonical pair (A_a^j, E_j^a) where $j, k, .. = 1, 2, 3$ are $su(2)$ indices. From the kinematical point of view the gravitational phase space is identical with that for a Yang – Mills theory with gauge group $SU(2)$. We assume both principal bundles to be trivial for simplicity. The non-vanishing Poisson brackets are

$$\begin{aligned} \{\underline{E}_J^a(x), \underline{A}_b^K(y)\} &= g^2 \delta_b^a \delta_J^K \delta(x, y) \\ \{E_j^a(x), A_b^k(y)\} &= \kappa \delta_b^a \delta_j^k \delta(x, y) \end{aligned} \tag{2.1}$$

where g^2 and $\kappa = 8\pi G_{Newton}$ denote the Yang – Mills and gravitational coupling constant respectively. We are using units in which both connections have dimension cm^{-1} while the Yang – Mills and gravitational electric fields respectively have dimension cm^{-2} and cm^0 respectively. As a result the Feinstrukturkonstante $\alpha = \hbar g^2$ is dimensionless while $\ell_p^2 = \hbar\kappa$ is the Planck area.

The kinematical phase space of the theory is subject to the (non – Abelean) Gauss constraints

$$\begin{aligned} \underline{G}_J &= \partial_a \underline{E}_J^a + f_{JK}{}^L \underline{A}_a^K \underline{E}_L^a \\ G_j &= \partial_a E_j^a + \epsilon_{jk}{}^l A_a^k E_l^a \end{aligned} \tag{2.2}$$

where in terms of a basis $\underline{\tau}_J, \tau_j$ for Lie(G) and $su(2)$ respectively the structure constants are defined by $[\underline{\tau}_J, \underline{\tau}_K] = f_{JK}{}^L \tau_L$ and $[\tau_j, \tau_k] = \epsilon_{jk}{}^l \tau_l$ respectively. We normalize the anti Hermitean generators such that $\mathrm{Tr}(\underline{\tau}_J \underline{\tau}_K) = -\delta_{JK}/2$ and $\mathrm{Tr}(\tau_j \tau_k) = -\delta_{jk}/2$. The phase space is subject to further spatial diffeomorphism and Hamiltonian constraints but these we reserve for a separate paper [15].

We now review the kinematical Hilbert space of the theory which is actually selected by requiring spatial diffeomorphism invariance [11]. It is easiest described in terms of so – called spin (or charge, flavour, colour..) network functions. We do this for a general compact group.

Definition 2.1.

i)

A spin network s for a gauge theory with compact gauge group G over a manifold σ consists of a quadruple $s = (\gamma(s), \pi(s), m(s), n(s))$ consisting of an oriented (piecewise analytic) graph $\gamma(s)$ embedded into σ as well as a labelling of each of the edges e of $\gamma(s)$ with a nontrivial irreducible representation $\pi_e(s)$ of G and corresponding matrix element labels $m_e(s), n_e(s) = 1, .., \dim(\pi_e(s))$. Here for each equivalence class of irreducible representations we have chosen once and for all an arbitrary representative.

ii)

A spin network function is simply the following complex valued function on the space $\mathcal{A}$ of smooth connections over σ

$$T_s(A) := \prod_{e \in E(\gamma(s))} [\sqrt{\dim(\pi_e(s))} \; \{[\pi_e(s)](A(e))\}_{m_e(s) n_e(s)}] \tag{2.3}$$

where

$$A(e) := \mathcal{P} \exp_e(\int_e A^J \tau_J) \tag{2.4}$$

denotes the holonomy of A along e and $E(\gamma(s))$ is the set of edges of $\gamma(s)$.

iii)

The kinematical Hilbert space $\mathcal{H}_{Kin}$ is the closure of the finite linear span of spin network functions which define a basis. Hence the kinematical inner product is given by

$$< T_s, T_{s'} >_{Kin} = \delta_{s,s'} \tag{2.5}$$

The holonomy operators $\widehat{A(e)}$ act by multiplication while the conjugate electric flux operators correspondinng to

$$E_J(S) = \int_S (*E_J) \tag{2.6}$$

act by differentiation whose details [8] we will not need in what follows. Here $*E_J$ denotes the pseudo two – form dual to the vector density E^a_J of weight one.

One can show that $\mathcal{H}_{Kin} = L_2(\overline{\mathcal{A}}, d\mu_0)$ is a space of square integrable functions over a distributional extension $\overline{\mathcal{A}}$ of $\mathcal{A}$ with respect to a probability Borel measure [12, 8] but this will not be needed in what follows. For the Yang – Mills and gravitational sector respectively we have one kinematical Hilbert space each corresponding to the gauge group under question and $SU(2)$ respectively and we will denote them as $\mathcal{H}^{YM}_{Kin}$ and $\mathcal{H}^{GR}_{Kin}$ respectively. The total kinematical Hilbert space for the Einstein – Yang – Mills theory is simply the tensor product $\mathcal{H}_{Kin} = \mathcal{H}^{YM}_{Kin} \otimes \mathcal{H}^{GR}_{Kin}$. A convenient basis is given by the states $T_c \otimes T_s$ where c, s respectively are spin networks for G, called colour networks in what follows, and for $SU(2)$ respectively. Please refer to [8] and the sixth and eigth reference of [9] for more details.

The functions (2.2) are scalar densities of weight one with respect to spatial diffeomorphisms Diff(σ) and they transform in the adjoint representation of G and $SU(2)$ respectively. In [8] we showed that their smeared form admits a well-defined quantization as essentially self-adjoint operators on $\mathcal{H}_{Kin}$. More specifically, let $\underline{\Lambda}, \Lambda$ be a $Lie(G)-$ or $su(2)-$valued functions on σ respectively (not necessarily smooth!) and let

$$\underline{G}(\underline{\Lambda}) := \int_\sigma d^3x \underline{\Lambda}^J \underline{G}_J \tag{2.7}$$

and similarly for $G(\Lambda)$. Then

$$\begin{aligned}
\widehat{\underline{G}(\underline{\Lambda})}\; T_c \otimes T_s &= \{i\alpha \sum_{v\in V(\gamma(c)} \underline{\Lambda}^J(v)[\sum_{e\in E(\gamma(c));b(e)=v} \underline{R}^e_J - \sum_{e\in E(\gamma(c));f(e)=v} \underline{L}^e_J]\; T_c\} \otimes T_s \\
&=: \{i\alpha \sum_{v\in V(\gamma(c)} \underline{\Lambda}^J(v)\underline{X}^v_J\; T_c\} \otimes T_s \\
\widehat{G(\Lambda)}\; T_c \otimes T_s &= T_c \otimes \{i\ell_p^2 \sum_{v\in V(\gamma(s))} \Lambda^j(v)[\sum_{e\in E(\gamma(s));b(e)=v} R^e_j - \sum_{e\in E(\gamma(s));f(e)=v} L^e_j]\; T_s\} \\
&=: T_c \otimes \{i\ell_p^2 \sum_{v\in V(\gamma(c))} \Lambda^j(v)X^v_j\; T_s\}
\end{aligned} \tag{2.8}$$

Here $V(\gamma)$ denotes the set of vertices of a graph γ, $b(e), f(e)$ respectively denote beginning and final point of an edge e and the operators R^e_j, L^e_j respectively act as follows (the description is similar for the Yang – Mills counterparts): We may view spin network states as so – called cylindrical functions

$$T_s(A) = f_{\gamma(s)}(\{A(e)\}_{e\in E(\gamma(s))}) \tag{2.9}$$

where $f_{\gamma(s)}$ is a complex valued function on $SU(2)^{|E(\gamma(s))|}$. Then

$$\begin{aligned}
(R^e_j T_s)(A) &:= (\frac{d}{dt})_{t=0} f_{\gamma(s)}(\{e^{t\tau_j\delta_{ee'}} A(e')\}_{e'\in E(\gamma(s))}) \\
(L^e_j T_s)(A) &:= (\frac{d}{dt})_{t=0} f_{\gamma(s)}(\{A(e') e^{t\tau_j\delta_{ee'}}\}_{e'\in E(\gamma(s))})
\end{aligned} \tag{2.10}$$

This ends our review of the kinematical LQG decription of Einstein – Yang – Mills theory. In the next two subsections we will construct and solve the associated Master Constraint Operator corresponding to both Gauss constraints.

2.1 Einstein – Yang – Mills Gauss Constraint

In [9] it was shown that background independent theories offer the possibility to define ultraviolet finite operators for classical integrals of scalar densities of weight one. The intuitive reason for this is as follows: If $F = \int d^3x f(x)$ is the classical integral to be quantized then the density n scalar $f(x)$ becomes an operator valued distribution of the form $\hat{f}(x) = \sum_\alpha (\delta(x, x_\alpha))^n \hat{f}_\alpha$ where the sum is over some index set depending on the theory and f. Here $\hat{f}_\alpha$ is an actual operator (not a distribution). Notice that the cassical density weight of f is correctly carried by the $n - th$ power of the $\delta-$distribution but unless $n = 1$ this is ill-defined and requires a point splitting regularization ("operator product expansion") with subsequent renormalization. This is actually the reason why in [5] we needed a trace class operator in order to split the points of the square $(\partial E)^2$ which is a scalar density of weight two. Splitting points requires a background metric which is not allowed in background independent theories, however, on the other hand in background independent theories the density weight comes out to be **always unity because of spatial diffeomorphism invariance!** Namely only integrals of scalar densities of weight one are spatially diffeomorphism invariant. Hence in background independent theories renormalization is not only not allowed but also not needed. The interested reader is referred to the second reference in [6] and to [8] for more details.

In keeping with this spirit we want to define a Master Constraint which is the integral of a density of weight one, quadratic in the Gauss constraint and independent of a background metric. The simplest and most natural choice is

$$\mathsf{M} := \int_\sigma d^3x \, \frac{\underline{G}_J \underline{G}_K \delta^{JK}}{\sqrt{\det(q)}} \tag{2.11}$$

Here the spatial metric q_{ab} is defined via its inverse $\det(q)q^{ab} = E^a_j E^b_k \delta^{jk}$. Notice that the overall density weight of (2.11) equals unity indeed. In fact, (2.11) is both $G-$invariant and Diff$(\sigma)-$invariant. The choice (2.11) is natural because $\sqrt{\det(q)}$ is the simplest scalar density of weight one which at least classically is nowhere vanishing. We could not have proceeded similarly with pure Maxwell theory because there the simplest scalar density is $\partial \cdot E$ itself which would have resulted in $\mathsf{M} = \int d^3x|\partial \cdot E|$. Not only would it be impossible to quantize this expression on Fock space, it is also not a classically differentiable function on the classical phase space and hence inacceptable. In non – Abelean Yang – Mills theory it is also possible to construct the "metric" Q_{ab} whose inverse is defined by $\det(Q)Q^{ab} = \underline{E}^a_J \underline{E}^b_K \delta^{JK}$, however, the corresponding scalar density $\sqrt{\det(Q)}$ which does not vanish identically if $\dim(G) \geq 3$ is classically not constrained to be non – vanishing everywhere. Hence the fact that we have coupled gravity cannot be avoided.

To quantize (2.11) we will proceed similarly as in [9]. Hence we will be brief, referring the interested reader to the literature. The idea is to make use of the identity

$$1 = \frac{[\det(e)]^2}{\det(q)} \tag{2.12}$$

where e^j_a is the co – triad defined up to $SU(2)$ transformations by $q_{ab} = \delta_{jk} e^j_a e^k_b$. Let $\mathcal{P}$ be a partition of σ into mutually disjoint regions R. Then the integral (2.11) is the limit as $\mathcal{P} \to \sigma$ of the corresponding Riemann sum

$$\begin{aligned} \mathsf{M} &= \lim_{\mathcal{P}\to\sigma} \sum_{R\in\mathcal{P}} \frac{\underline{G}_J(R)\underline{G}_K(R)\delta^{JK} e(R)^2}{V(R)^3} \\ \underline{G}_J(R) &= \int_R d^3x \underline{G}_J(x) \\ e(R) &= \int_R d^3x \det(e)(x) \\ V(R) &= \int_R d^3x \sqrt{\det(q)}(x) \end{aligned} \tag{2.13}$$

We now observe the further identity

$$e(R) = (\frac{1}{\kappa})^3 \int_R \epsilon_{jkl} \{A^j(x), V(R)\} \wedge \{A^k(x), V(R)\} \wedge \{A^l(x), V(R)\} \tag{2.14}$$

hence

$$\frac{e(R)}{V(R)^{3/2}} = (\frac{2}{\kappa})^3 \int_R \epsilon_{jkl} \{A^j(x), V(R)^{1/2}\} \wedge \{A^k(x), V(R)^{1/2}\} \wedge \{A^l(x), V(R)^{1/2}\} =: (\frac{2}{\kappa})^3 \tilde{e}(R) \tag{2.15}$$

Thus (2.13) becomes

$$\mathsf{M} = \lim_{\mathcal{P}\to\sigma} (\frac{2}{\kappa})^6 \sum_{R\in\mathcal{P}} \delta^{JK} \underline{G}_J(R)\underline{G}_K(R)\tilde{e}(R)^2 \tag{2.16}$$

The reason for writing the Master Constraint in this form is that all quantities involved in (2.16) admit a well-defined quantization at finite partition $\mathcal{P}$: First of all $\underline{G}_J(R) = \underline{G}(\underline{\Lambda})$ with $\underline{\Lambda}^K = \chi_R \delta^K_J$ where χ_R is the characteristic function of R admits a well-defined quantization according to (2.8). Next the volume $V(R)$ of R is well-defined as a positive essentially self-adjoint operator on $\mathcal{H}_{Kin}$, its explicit action on spin network functions being given by

$$\hat{V}(R)T_s = \ell_p^3 \sum_{v\in V(\gamma(s))\cap R} \sqrt{\frac{1}{48} | \sum_{e_1,e_2,e_3\in E(\gamma(s));b(e_1)=b(e_2)=b(e_3)=v} \epsilon^{jkl}\epsilon(e_1,e_2,e_3) R^{e_1}_j R^{e_2}_k R^{e_3}_l |} \; T_s \tag{2.17}$$

Here we have assumed that all edges are outgoing from a vertex (split edges into two halves to do that) and $\epsilon(e_1, e_2, e_3)$ is the sign of the determinant of the matrix defined by the column vectors $\dot{e}_1(0), \dot{e}_2(0)\dot{e}_3(0)$ in this sequence where $e_I(0) = b(e_I),\ I = 1, 2, 3$.

It remains to quantize $\tilde{e}(R)$ itself and to take the limit $\mathcal{P} \to \sigma$. To that effect, notice that the limit in (2.13) is independent of the choice of the sequence of partitions $\mathcal{P}$. We may therefore without loss of generality assume that the partition is actually a triangulation consisting of tetrahedra R. In the limit $\mathcal{P} \to \sigma$ each R can be described by a base point $v(R)$ and a right oriented triple of edges $e_I(R) \subset \partial R,\ I = 1, 2, 3$, incident at and outgoing from $v(R)$. Then it is easy to see that in the limit $\mathcal{P} \to \sigma$ the expression $\tilde{e}(R)$ can be replaced by

$$\begin{aligned} -8e'(R) &= -8\epsilon^{jkl}\epsilon^{JKL}\mathrm{Tr}(\tau_j(A(e_J(R)))^{-1}\{A(e_J(R)), V(R)^{1/2}\}) \times \\ &\quad \times\mathrm{Tr}(\tau_k(A(e_K(R)))^{-1}\{A(e_K(R)), V(R)^{1/2}\})\ \ \mathrm{Tr}(\tau_l(A(e_L(R)))^{-1}\{A(e_L(R)), V(R)^{1/2}\}) \end{aligned} \tag{2.18}$$

Expression (2.18) can now be quantized by replacing Poisson brackets by commutators divided by $i\hbar$.

Anticipating that the strong limit $\mathcal{P} \to \sigma$ exists as a symmetric operator we define

$$\widehat{\mathbf{M}} = \lim_{\mathcal{P}\to\sigma} (\frac{4}{\kappa})^6 \sum_{R\in\mathcal{P}} \delta^{JK}\widehat{\underline{G}_J(R)}\widehat{e'(R)}^2\widehat{\underline{G}_K(R)} \tag{2.19}$$

Notice that no ordering ambiguities arise because the gravitational and Yang – Mills degrees of freedom commute with each other. It will be sufficient to define (2.18) on the basis $T_c \otimes T_s$.

In performing the limit a regularization ambiguity arises: The quantum limit when applied to $T_c \otimes T_s$ depends on the sequence $\mathcal{P} \to \sigma$ while the classical limit was independent of that choice. We will therefore proceed as in [9] and choose a sequence for each $T_c \otimes T_s$ individually which is justified by the fact that classically the choice of the sequence does not make any difference. The result is the following: We notice first of all from (2.8) that there is no contribution in (2.18) from those tetrahedra R which do not contain a vertex of $\gamma(c)$. Next, $[\hat{A}(e), \hat{V}(R)^{1/2}]$ evidently vanishes for those R which do not contain a vertex of $\gamma(s)$ no matter how we choose $e \subset \partial R$. Hence, for sufficiently fine $\mathcal{P}$ we can focus attention on those R which contain a common vertex of $\gamma(s)$ and $\gamma(c)$. The idea is now to average over all triangulations. Let $n(v)$ be the gravitational valence of such a vertex then the detailed averaging performed in [9] results in the following final expression after taking the limit $\mathcal{P} \to \sigma$

$$\begin{aligned} \widehat{\mathbf{M}}\, T_c \otimes T_s &= \alpha^2(\frac{4}{\ell_p})^6 \sum_{v\in V(\gamma(c))\cap V(\gamma(s))} \delta^{JK}\underline{X}_J(v)\widehat{e'(v)}^2\underline{X}_K(v)\ \ T_c \otimes T_s \\ \widehat{e'(v)} &= \frac{1}{n(v)(n(v)-1)(n(v)-2)}\epsilon^{jkl}\epsilon^{JKL} \sum_{e_1,e_2,e_3\in E(\gamma(s));\ b(e_1)=b(e_2)=b(e_3)=v} \times \\ &\quad \times\mathrm{Tr}(\tau_j(A(s(e_I)))^{-1}[A(s(e_I)), \hat{V}(v)^{1/2}])\ \ \mathrm{Tr}(\tau_k(A(s(e_K)))^{-1}[A(s(e_K)), \hat{V}(v)^{1/2}])\ \times \\ &\quad \times\mathrm{Tr}(\tau_l(A(s(e_L)))^{-1}[A(s(e_L)), \hat{V}(v)^{1/2}]) \end{aligned} \tag{2.20}$$

We have displayed the action for vertices of gravitational valence at least three. For gravitationally bi-valent vertices the action looks similar and is non-vanishing [9]. These details will not be important for what follows.

Here $s(e)$ denotes an infinitesimal beginning segment of an edge e and $\hat{V}(v)$ denotes the volume operator for an infinitesimal region $R(v)$ containing v. By the methods of [9] it is easy to see that (2.20) is a positive, essentially self-adjoint, gauge invariant and spatially diffeomorphism invariant operator on $\mathcal{H}_{Kin}$. Moreover, the choice of the segments and regions $s(e), R(v)$ respectively is completely irrelevant as long as they contain $b(e), v$ respectively.

To solve the Master Constraint (2.20) is now surprisingly simple because the spectrum of the Master Constraint Operator (2.20) is *pure point*. Moreover, we can determine it sufficiently explicitly in order to actually derive the physical Hilbert space. To see this, notice that $\mathcal{H}^{YM}_{Kin}$ can be decomposed as

$$\mathcal{H}^{YM}_{Kin} = \overline{\oplus_\gamma \mathcal{H}^{YM}_{Kin;\gamma,\{\pi\}}} \tag{2.21}$$

where $\mathcal{H}^{YM}_{Kin;\gamma,\pi}$ is the finite linear span of charge network states c with $\gamma(c) = \gamma, \pi(c) = \pi$ and the overline denotes closure. Now each $\mathcal{H}^{YM}_{Kin;\gamma,\pi}$ can be further decomposed as

$$\mathcal{H}^{YM}_{Kin;\gamma,\pi} = \oplus_\Pi \mathcal{H}^{YM}_{Kin;\gamma,\pi,\Pi} \tag{2.22}$$

where $\Pi = \{\Pi_v\}_{v\in V(\gamma)}$ is a collection of equivalence classes of irreducible representations of G, one for each vertex of v, with the following meaning: From (2.3) one easily verifies that the colour network state $T_c,\ \gamma(c) = \gamma, \pi(c) = \pi$ transforms under local $G-$gauge transformations with support at $v \in V(\gamma)$ in the representation

$$[\otimes_{e\in E(\gamma);\ b(e)=v}\pi_e] \otimes [\otimes_{e\in E(\gamma);\ f(e)=v}\pi^c_e] \tag{2.23}$$

where π^c denotes the representation contragredient to π. Since G is compact, every representation is completely reducible and (2.23) can be decomposed into mutually orthogonal but not necessarily inequivalent irreducible representations. Let the orthogonal projector on the representation space consisting of mutually orthogonal representations equivalent to the equivalence class Π_v be denoted by

$$i^{\Pi_v}_{[\otimes_{b(e)=v}\pi_e]\otimes[\otimes_{f(e)=v}\pi^c_e]} \tag{2.24}$$

also called an intertwiner. Notice that (2.24) vanishes for all Π_v except for finitely many. We have the completeness relation

$$\sum_\Pi i^\Pi_{[\otimes_{b(e)=v}\pi_e]\otimes[\otimes_{f(e)=v}\pi^c_e]} = \mathrm{id}_{[\otimes_{b(e)=v}\pi_e]\otimes[\otimes_{f(e)=v}\pi^c_e]} \tag{2.25}$$

Using (2.25) for every $v \in V(\gamma)$ we can decompose each colour network state and arrive at (2.22). Finally we define

$$\mathcal{H}^{YM}_{Kin;\gamma,\Pi} = \overline{\oplus_\pi \mathcal{H}^{YM}_{Kin;\gamma,\Pi,\pi}} \tag{2.26}$$

and thus have the identity

$$\mathcal{H}^{YM}_{Kin} = \overline{\oplus_{\gamma,\Pi} \mathcal{H}^{YM}_{Kin;\gamma,\Pi}} \tag{2.27}$$

The point of the decomposition (2.27) is that the operator $\Delta_{\gamma,v} := \delta^{JK} X^v_J X^v_K$ which appears in (2.20) is diagonal on $\mathcal{H}^{YM}_{Kin;\gamma,\Pi}$ with eigenvalue $-\lambda_{\Pi_v} \le 0$ given by the eigenvalue of the Laplace operator on G in any representation equivalent to Π_v. This also demonstrates that the sum in (2.27) is indeed orthogonal.

We can now proceed similarly with $\mathcal{H}^{GR}_{Kin}$ and decompose it as

$$\mathcal{H}^{GR}_{Kin} = \overline{\oplus_{\gamma,\lambda} \mathcal{H}^{GR}_{Kin;\gamma,\lambda}} \tag{2.28}$$

where $-\lambda = \{-\lambda_v\}_{v\in V(\gamma)} \le 0$ are the eigenvalues of $(\widehat{e'(v)})^2$. That this operator is diagonizable in this fashion follows from a similar property for the volume operator itself. The direct integral decomposition of $\mathcal{H}_{Kin}$ with respect to the Master Constraint Operator is therefore simply

$$\mathcal{H}_{Kin} = \overline{\oplus_{(\gamma,\Pi),(\gamma',\lambda)} \mathcal{H}^{YM}_{Kin;\gamma,\Pi} \otimes \mathcal{H}^{GR}_{Kin;\gamma',\lambda}} \tag{2.29}$$

Since the spectrum of $\widehat{\mathbf{M}}$ is pure point we just need to identify the zero eigenspace in (2.29) as the physical Hilbert space which, as a subspace of $\mathcal{H}_{Kin}$, carries the kinematical inner product as the physical inner product. To that end we notice that the eigenvalue on $\mathcal{H}^{YM}_{Kin;\gamma,\Pi} \otimes \mathcal{H}^{GR}_{Kin;\gamma,\lambda}$ is given by

$$\alpha^2 (\frac{4}{\ell_p})^6 \sum_{v \in V(\gamma) \cap V(\gamma')} \lambda_{\Pi_v} \lambda_v \tag{2.30}$$

If the $\lambda_v > 0$ are not vanishing, the only way to make (2.30) vanish for all γ is to require $\lambda_{\Pi_v} = 0$ for all $v \in V(\gamma)$. Hence $\Pi_v = \text{Triv}$ must be the equivalence class of the trivial representation in that case.

If $\lambda_v = 0$ then Π_v is arbitrary, which is different from what the ordinary Gauss constraint would select. We actually do not know explicitly the space of states with $\lambda_v = 0$ but it contains at least states with at least one vertex which is not at least trivalent with respect to the gravitational spin network structure. These zero eigenvalues are related to zero volume eigenstates because $e'(B)$ is related to the volume of the region B and we know by now that there are many such states even for higher valent vertices [19]. To see that this does not pose any problem, notice that a zero volume vertex physically corresponds to a region which actually does not exist. In other words, whether the Gauss constraint holds there or not is unaccessable by any observer[1]. Likewise, the results of [19] demonstrate that the number of zero volume eigenvalues is neglible in a semiclassical sense because they correspond to spin configurations whose number compared to the number of all spin configurations at an $n-$valent vertex with maximal spin j decreases roughly as $j^{-(n-1)}$ in the semiclassical limit of large quantum numbers j. Hence, semiclassical states are linear combinations of spin network states with almost always $\lambda_v > 0$ as it should be since classical General Relativity is about non – degenerate metrics. In what follows we will simply drop that unphysical subsector of zero volume states for notational simplicity.

Then, denoting

$$\mathcal{H}^{YM}_{Inv} := \overline{\oplus_\gamma \mathcal{H}^{YM}_{Kin;\gamma,\text{Triv}}} \tag{2.31}$$

the physical Hilbert space is given by the subspace

$$\mathcal{H}^{YM}_{Phys} = \mathcal{H}^{YM}_{Inv} \otimes \mathcal{H}^{GR}_{Kin} \subset \mathcal{H}_{Kin} \tag{2.32}$$

consisting of the closed linear span of states $T_c \otimes T_s$ where T_c is a gauge invariant charge network state constructed by using the intertwiner for the trivial representation. This is precisely the same phyiscal Hilbert space as selected by the Gauss constraint (2.8) itself.

2.2 Einstein Gauss Constraint

We can proceed completely similarly with respect to the gravitational Gauss constraint for which the Master Constraint is given by

$$\mathbf{M} := \int_\sigma d^3x \, \frac{G_j G_k \delta^{jk}}{\sqrt{\det(q)}} \tag{2.33}$$

[1] One could object that the zero volume region has a closed two – boundary for which we could construct the electric flux operator. However, notice that in non – Abelean gauge theories the electric flux operator for a closed two surface 1. does not measure the charge contained in the region bounded by the surface because it cannot be obtained by the (non – Abelean) Stokes theorem from the Gauss constraint and 2. it is not a Dirac observable even with respect to the Gauss constraint, that is, it is not gauge invariant and therefore does not correspond to a physical (measurable) quantity. The area operator *is* gauge inavariant but does not measure the charge neither, in fact, the charge is associated with vertices while the area is associated with edges. For Abelean gauge theories the objection is valid but here we are interested in a gravity coupled situation and then we have to consider also the spatial diffeomorphism and Hamiltonian constraints respectively for which the flux is not a Dirac observable again.

Its quantization proceeds entirely analogous to that of the Einstein – Yang – Mills Gauss constraint, just that it acts trivially on the Yang – Mills sector so that we will drop $\mathcal{H}^{YM}_{Kin}$ for the remainder of this section. Its action on spin network functions is given by

$$\widehat{\mathbf{M}}\, T_s = \ell_p^4 (\frac{4}{\ell_p})^6 \sum_{v\in V(\gamma(s))} \delta^{jk} X_j(v) \widehat{e'(v)}^2 X_k(v)\ \ T_s \tag{2.34}$$

Now one could imagine that an ordering issue arises: In the ordering (2.34) the operator is manifestly positive and essentially self-adjoint. However, in order to conclude similarly as in the previous section we need the ordering in which $\delta^{jk}X_j(v)X_k(v)$ stands to the outmost right. In the Yang – Mills case there was no issue because the gravitational and Yang – Mills degrees of freedom commute. Fortunately also here there is no problem because $X_j(v)$ is the generator of $SU(2)$ gauge transformations at v and the operator $\widehat{e'(v)}$ is manifestly gauge invariant. Hence the operators $\delta^{jk}X_j(v)X_k(v)$ and $\widehat{e'(v)}^2$ commute and can be diagonalized simultaneously. Hence we can perform exactly the same steps as in section 2.1 and conclude that the physical Hilbert space is given by

$$\mathcal{H}_{Phys} = \mathcal{H}_{Inv} \subset \mathcal{H}_{Kin} \tag{2.35}$$

3 Infinite Number of Non-Abelean First Class Constraints Non – Polynomial in the Momenta with Structure Functions

Euclidean 2+1 gravity can be formulated in complete analogy to Lorentzian 3+1 gravity (see e.g the fifth reference in [9]). In this form the constraints are as difficult to solve as for the 3+1 theory which we reserve for future work [15]. In particular, they are non-polynomial in the momenta and only close with structure functions rather than structure constants. However, there is a classically equivalent way (when the spatial metric is not degenerate) to write the constraints linear in the momenta. Namely Euclidean 2+1 gravity can be written as an $SU(2)-$gauge theory over a two dimensional Riemann surface σ subject to a Gauss constraint and the curvature constraints

$$C^j = \frac{1}{2}\epsilon^{ab} F^j_{ab} \tag{3.1}$$

where F^j_{ab} is the curvature of A^j_a and ϵ^{ab} is the density one valued skew tensor in two dimensions. It follows that (3.1) is a scalar density of weight one. Accordingly we define the corresponding Master Constraint as

$$\mathbf{M}_E := \frac{1}{2}\int_\sigma d^2x \frac{F^j F^k \delta_{jk}}{\sqrt{\det(q)}} \tag{3.2}$$

where $\det(q) q^{ab} = E^a_j E^b_k \delta^{jk}$ defines the spatial metric. Of course the pair (A^j_a, E^a_j) is canonically conjugate up to the gravitational coupling constant κ. The significance of the label E will be explained momentarily.

To quantize (3.2) we will work directly on the space of gauge invariant states $\mathcal{H}_{Inv}$ given by the closed linear span of gauge invariant spin network functions derived in the previous section. We proceed as in the fifth reference of [9] and define the "degeneracy vector"

$$E^j = \frac{1}{2}\epsilon_{ab}\epsilon^{jkl} E^a_k E^b_l \tag{3.3}$$

called this way because $\det(q) = \delta_{jk} E^j E^k$. Using this we can establish an interesting identity:

Notice that the combinations (we use the identity $F^j_{ab} = \epsilon_{ab} F^j$)

$$\begin{aligned} C_a &:= C_j \epsilon_{ab} E^b_j = F^j_{ab} E^b_j \\ C &:= \frac{C_j E^j}{\sqrt{\det(q)}} = \frac{1}{2} \frac{C_j \epsilon_{ab} E^a_k E^b_l \epsilon^{jkl}}{\sqrt{\det(q)}} = \frac{1}{2} \frac{F^j_{ab} E^a_k E^b_l \epsilon^{jkl}}{\sqrt{\det(q)}} \end{aligned} \tag{3.4}$$

look excacly like the spatial diffeomorphism and (Euclidean) Hamiltonian constraint of the 3+1 theory. Now one immediately verifies that

$$\mathbf{M}_E = \frac{q^{ab} C_a C_b + C^2}{\sqrt{\det(q)}} \tag{3.5}$$

Hence we arrive at the following crucial observation:

The Master Constraint (3.2) for the curvature constraints (3.1) coincides with the extended Master Constraint [1] **for the spatial Diffeomorphism and Hamiltonian constraints**.
Hence 2+1 gravity not only tests the extended Master Constraint idea but also can be considered as an example with a non – Abelean constraint algebra with structure functions.

Similar to the previous section we define for a two dimensional region R the following smeared quantities

$$\begin{aligned} F_j(R) &= \int_R d^2x F_j \\ E^j(R) &= \int_R d^2x E^j \\ V(R) &= \int_R d^2x \sqrt{\det(q)} \end{aligned} \tag{3.6}$$

and write the classical Master Constraint as the Riemann sum limit for a family of partitions $\mathcal{P}$, that is

$$\mathbf{M}_E = \lim_{\mathcal{P}\to\sigma} \frac{1}{2} \sum_{R\in\mathcal{P}} \frac{(F_j(R))^2 (E^k(R))^2}{(V(R))^3} \tag{3.7}$$

Using the classical identity for $x \in R$

$$E^j(x) = \frac{1}{\kappa^2} \epsilon^{ab} \epsilon_{jkl} \{A^k_a(x), V(R)\}\{A^l_b(x), V(R)\} \tag{3.8}$$

we have

$$\frac{E^j(R)}{V(R)^{3/2}} = (\frac{4}{\kappa})^2 \epsilon_{jkl} \int_R \{A^k, V(R)^{1/4}\} \wedge \{A^l, V(R)^{1/4}\} =: (\frac{4}{\kappa})^2 E'_j(R) \tag{3.9}$$

Hence the Master Constraint becomes

$$\mathbf{M}_E = \lim_{\mathcal{P}\to\sigma} \frac{1}{2} (\frac{4}{\kappa})^4 \sum_{R\in\mathcal{P}} (F_j(R))^2 (E'_k(R))^2 \tag{3.10}$$

Again we specialize to a simplicial decomposition $\mathcal{P}$. We single out a corner $v(R)$ for each triangle R and denote by $e_I(R)$ the two edges of ∂R starting at $v(R)$. Then

$$\begin{aligned} \mathbf{M}_E &= \lim_{\mathcal{P}\to\sigma} 2(\frac{4}{\kappa})^4 \sum_{R\in\mathcal{P}} (\tilde{F}_j(R))^2 (\tilde{E}'_k(R))^2 \qquad (3.11) \\ \tilde{F}_j(R) &= \mathrm{Tr}(\tau_j A(\partial R)) \\ \tilde{E}'_j(R) &= \epsilon^{jkl} \sum_{K,L=1,2} \epsilon^{IJ} \times \\ & \times \mathrm{Tr}(\tau_k A(e_K(R))\{A(e_K(R))^{-1}, V(R)^{1/4}\}) \ \mathrm{Tr}(\tau_l A(e_L(R))\{A(e_L(R))^{-1}, V(R)^{1/4}\}) \end{aligned}$$

Now (3.11) is written in terms of holonomies and the volume operator both of which admit well-defined quantizations on $\mathcal{H}_{Kin}$ (see the fifth reference of [9] for the 2+1 volume operator).

There is a difference in formulating the Master Constraint Operator $\widehat{\mathbf{M}}$ for the Hamiltonian constraint and the extended Master Constraint Operator $\widehat{\mathbf{M}}_E$ for the combined spatial diffeomorphism and Hamiltonian constraint: As shown in [1] the former can only be defined, in its fundamental form, on the spatially diffeomorphism invariant Hilbert space $\mathcal{H}_{Diff}$ [10] of solutions to the spatial diffeomorphism constraint while the second of course must be defined on the kinematical Hilbert space $\mathcal{H}_{Kin}$ since it solves both types of constraints in one step. However, since $\widehat{\mathbf{M}}_E$ is nevertheless a spatially diffeomorphism invariant operator, it can be defined on $\mathcal{H}_{Kin}$ only if it does not change the graph of a spin network state on which it acts as was shown in detail [10]. We must keep this in mind when quantizing (3.11).

Our heuristic ansatz will be to define the following quadratic form in which we order the various operator factors judiciously (if it exists)

$$
\begin{aligned}
&Q_{\mathbf{M}_E}(T_s, T_{s'}) = \lim_{\mathcal{P}\to\sigma} 2(\frac{4}{\ell_p^2})^4 \sum_{R\in\mathcal{P}} \delta^{jk}\delta^{mn} \sum_{s_1} \times \qquad (3.12)\\
&\times < T_s, \widehat{\tilde{F}_j(R)}\widehat{\tilde{E}'_m(R)} T_{s_1} >_{Kin} \overline{< T_{s'}, \widehat{\tilde{F}_k(R)}\widehat{\tilde{E}'_n(R)} T_{s_1} >_{Kin}}\\
=\; & 2(\frac{4}{\ell_p^2})^4 \sum_{s_1} \lim_{\mathcal{P}\to\sigma} \sum_{R\in\mathcal{P}} \delta^{jk}\delta^{mn} \times\\
&\times < T_s, \widehat{\tilde{F}_j(R)}\widehat{\tilde{E}'_m(R)} T_{s_1} >_{Kin} \;\; \overline{< T_{s'}, \widehat{\tilde{F}_k(R)}\widehat{\tilde{E}'_n(R)} T_{s_1} >_{Kin}} \text{ where}\\
&\widehat{\tilde{E}'_j(R)} T_{s_1} = \epsilon^{jkl} \sum_{K,L=1,2} \epsilon^{KL} \times\\
&\times \mathrm{Tr}(\tau_k A(e_K(R))[A(e_K(R))^{-1}, \hat{V}(R)^{1/4}]) \;\; \mathrm{Tr}(\tau_l A(e_L(R))[A(e_L(R))^{-1}, \hat{V}(R)^{1/4}]) \;\; T_{s_1}
\end{aligned}
$$

and $\widehat{\tilde{F}_j(R)}$ acts by multiplication. Notice that the sum over s_1 is an uncountably infinite sum but we will see soon that for given s, s' only a finite number of terms contribute so that the interchange of the limit with the sum in the second step is indeed justified. One may wonder why we have introduced this insertion of unity at all. This is motivated by the theory in 3+1 dimensions [1, 15] where this insertion of unity is mandatory when working at the level of $\mathcal{H}_{Diff}$ and where $T_s, T_{s'}$ are replaced by diffeomorphism invariant distributions.

It remains to take the limit $\mathcal{P} \to \sigma$. First of all, as the regions R shrink to points, every term in the sum over the R at given s_1 vanishes anyway if $\gamma(s) \neq \gamma(s')$ due to the orthogonality of the spin network functions so that both graphs must be subgraphs of $\gamma(s_1) \cap \partial R$, so they must equal each other for sufficiently small R. Thus:

The $\widehat{\mathbf{M}}_E$ regularized this way in terms of a quadratic form is automatically not graph changing.

Next, no matter how the limit $\mathcal{P} \to \sigma$ is performed, given s, for sufficiently fine $\mathcal{P}$ the last line in (3.12) vanishes unless R intersects $\gamma(s)$. Now in contrast to the situation in 3+1 dimensions, if R intersects $\gamma(s)$ but does not contain a vertex of $\gamma(s)$, the last line in (3.12) is not automatically vanishing which leads to a divergence when the limit $\mathcal{P} \to \sigma$ is not carefully performed. Following the proposal in the fifth reference of [9] we adapt the limit $\mathcal{P} \to \sigma$ for the matrix element (3.12) to the states T_{s_1} in the following sense:

1.

The triangles R are chosen to saturate each vertex v of $\gamma(s_1)$, that is, for each adjacent pair of edges e, e' there is precisely one triangle R with $v(R) = v$ and $e_1(R) \subset e, e_2(R) \subset e'$.

2.

Away from the vertices, the triangles R intersect $\gamma(s)$ only in such a way that the vertices of the graph $\partial R \cap \gamma(S)$ which are not vertices of $\gamma(s)$ are co-linear, that is, the tangents of the respective intersecting edges are linearly dependent there.

This results in a different limit $\mathcal{P} \to \sigma$ for each $\gamma(s_1)$ which is justified by the fact that classically the Riemann sum converges to the same limit no matter which sequence is chosen. The second requirement now makes sure that the only contributions come from the triangles with $v(R) \in V(\gamma(s))$, hence (3.12) simplifies to

$$\begin{aligned} Q_{\mathbf{M}_E}(T_s, T_{s'}) &= \delta_{\gamma(s),\gamma(s')} \sum_{s_1} \lim_{\mathcal{P}\to\sigma} 2(\frac{4}{\ell_p^2})^4 \sum_{v\in V(\gamma(s))} \sum_{R\in\mathcal{P};\, v(R)=v} \delta^{jk}\delta^{mn} \times \\ &\times < T_s, \widehat{\tilde{F}_j(R)}\widehat{\tilde{E}'_m(R)} T_{s_1} >_{Kin} \overline{< T_{s'}, \widehat{\tilde{F}_k(R)}\widehat{\tilde{E}'_n(R)} T_{s_1} >_{Kin}} \end{aligned} \quad (3.13)$$

Now a miracle happens: The limit involved in (3.13) is already trivial because, as the triangles $R(v)$ shrink, this shrinking can be absorbed by an (analytical) diffeomorphism of σ which preserves $\gamma(s) = \gamma(s')$. However, since the diffeomorphism group is represented unitarily on $\mathcal{H}_{Kin}$ [10], the number on the right hand side of (3.13) actually no longer depends on the "size and shape" of the triangles R but only on their diffeomorphism invariant characteristics (this is easiest to see by performing the sum over s_1 thus resulting in a unit operator). This reduces the number of regularization ambiguities tremendously from uncountably infinite to countably infinite (namely the C^n class of the intersections at the two vertices of $\partial R \cap \gamma(s)$ different from $v(R)$).

Hence, we are left with making that choice. The most natural choice is the one made in [9] namely such that the the edge $\partial R - \gamma(s_1)$ intersects $\gamma(s_1)$ transversally at both end points (corresponding to a C^0 intersection). However, it is easy to see that the limit in (3.13) is not satisfactory unless $\partial R \subset \gamma(s)$. The reason is that up to the commutator between $\widehat{\tilde{F}_k(R)}, \widehat{\tilde{E}'_n(R)}$ (which is of higher order in $\hbar$) after summation over s_1 the matrix element is proportional to a sum of matrix elements between spin network states over $\gamma(s)$ of the operator

$$(\widehat{\tilde{F}_j(R)})^\dagger \widehat{\tilde{F}_j(R)} = 3 - \chi_1(\hat{A}(\partial R))$$

where χ_j is the character of the irreducible representation of $SU(2)$ with weight j. Since $\gamma(s) = \gamma(s')$ and since $\widehat{\tilde{E}'_j(R)}$ only acts on the intertwiners of a spin-network state but not the spins, it follows that the piece $\chi_1(\hat{A}(\partial R))$ drops out of (3.13) for sufficiently small ∂R. Hence the matrix element would not contain any information about the curvature F_j. This happens because we are working at the kinematical level and not at the spatially diffeomorphism invariant level: If T_s would be diffeomorphism invariant and depend only on the diffeomorphism class of s [15] then there would be a non-vanishing contribution no matter how small ∂R because we could for instance choose s to be in the class of the spin network state $\chi_1(A(\partial R))T_{s'}$.

It follows that at the level of $\mathcal{H}_{Kin}$ we must proceed differently than for $\mathcal{H}_{Diff}$ in order to obtain a satisfactory operator. The necessary idea was outlined already in [1, 16]:

Definition 3.1.
i)
Let γ be a graph, $v \in V(\gamma)$ a vertex and $e, e' \in E(\gamma)$ two different edges starting at v (reverse orientation if necessary). A loop $\alpha_{\gamma,v,e,e'}$ within γ starting at v along e and ending at v along $(e')^{-1}$ is said to be minimal if there is no other loop within γ with the same properties and less edges of γ traversed.
ii)
Given the data γ, v, e, e' we denote by $L(\gamma, v, e, e')$ the set of minimal loops compatible with those data.

Notice that the notion of a minimal loop does not refer to a background structure such as a metric. The set $L(\gamma, v, e, e')$ is never empty for the closed graphs that we are considering here and it is always finite.

The idea is now that semiclassical states must necessarily have very complex graphs in order that, e.g. the volume operator for every macroscopic region has non-zero expection values. Moreover, they will involve very high spins because high spin means large volume so that the correspondence principle is satisfied. For such complex graphs the limiting loops ∂R are of the same order of magnitude as the minimal loops associated with it. Hence on semiclassical states it makes sense to replace the limit by those minimal loops. This is justified in the sense that we are quantizing an operator corresponding to a classical quantity which we can hope to be approximated well for semiclassical states only. Since then there is no consistency check for the details at the microscopic level, our procedure is justified in this semiclassical sense. Of course, the argument is not entirely satisfactory and can at best result in an effective description $\widehat{\mathsf{M}}_{E,eff}$ of the fundamental operator $\widehat{\mathsf{M}}_E$ which, however, must then be defined on $\mathcal{H}_{Diff}$ rather than $\mathcal{H}_{Kin}$ in order to obtain a non-trivial result. Hence, at the level of $\mathcal{H}_{Kin}$ this is the best we can do. The more fundamental programme will be carried out in [15] directly for the 3+1 case.

Summarizing, the close to final matrix element is defined by

$$
\begin{aligned}
& Q_{\mathsf{M}_{E,eff}}(T_s, T_{s'}) = \delta_{\gamma(s),\gamma(s')} \times \qquad (3.14) \\
& \times \sum_{s_1} 2(\frac{4}{\ell_p^2})^4 \sum_{v\in V(\gamma(s))} \delta^{jk}\delta^{mn} \sum_{e_1,e_2\in E(\gamma(s)); v=e_1\cap e_2} \sum_{\alpha\in L(\gamma(s),v,e_1,e_2)} \frac{1}{|L(\gamma(s),v,e_1,e_2)|} \times \\
& \times < T_s, \widehat{\tilde{F}_j(R(\alpha))}\widehat{\tilde{E}'_m(v,e_1,e_2)} T_{s_1} >_{Kin} \overline{< T'_s, \widehat{\tilde{F}_k(R(\alpha))}\widehat{\tilde{E}'_n(v,e_1,e_2)} T_{s_1} >_{Kin}} \\
& \widehat{\tilde{E}'_j(v,e,e')} T_{s_1} = \epsilon^{jkl} \sum_{K,L=1,2} \epsilon^{KL} \times \\
& \times \mathrm{Tr}(\tau_k A(s(e_K))[A(s(e_K))^{-1}, \hat{V}(v)^{1/4}])\ \mathrm{Tr}(\tau_l A(s(e_L))[A(s(e_L))^{-1}, \hat{V}(v)^{1/4}])\ T_s
\end{aligned}
$$

Here we have averaged over the number of minimal loops with data $\gamma(s), v, e, e'$ in order not to overcount as compared to (3.13) and $R(\alpha)$ is the unique, interior, **contractable** region enclosed by α, i.e. $\partial R(\alpha) = \alpha$. As before $s(e)$ denotes an infinitesimal beginning segment of an edge e and $\hat{V}(v)$ the volume operator for an infinitesimal region containing v. Notice that (3.14) is a drastic modification of (3.13) unless the graph $\gamma(s)$ is very complex, hence we have put the subscript "effective".

Formula (3.14) is not yet quite what we want because we have to impose the constraint $\delta_{\gamma(s),\gamma(s')}$ explicitly. We can avoid that by inserting a projection operator $\hat{P}_\alpha$ which projects onto the closed subspace of spin network states which have spin higher than 1/2 for all edges of α. This again modifies $\widehat{\mathsf{M}}_E$for spin network states involving low spins but does not change its semiclassical behaviour for the reason mentioned above. Hence the final expression for the effective Master Constraint Operator is

$$
\begin{aligned}
& Q_{\mathsf{M}_{E,eff}}(T_s, T_{s'}) = \sum_{s_1} 2(\frac{4}{\ell_p^2})^4 \sum_{v\in V(\gamma(s))} \delta^{jk}\delta^{mn} \times \qquad (3.15) \\
& \times \sum_{e_1,e_2\in E(\gamma(s)); v=e_1\cap e_2} \sum_{\alpha\in L(\gamma(s),v,e_1,e_2)} \frac{1}{|L(\gamma(s),v,e_1,e_2)|} \times \\
& \times < T_s, \widehat{\tilde{F}_j(R(\alpha))}\widehat{\tilde{E}'_m(v,e_1,e_2)} \hat{P}_\alpha T_{s_1} >_{Kin} \overline{< T'_s, \widehat{\tilde{F}_k(R(\alpha))}\widehat{\tilde{E}'_n(v,e_1,e_2)} \hat{P}_\alpha T_{s_1} >_{Kin}}
\end{aligned}
$$

Due to the projections the state

$$\widehat{\tilde{F}_j(R(\alpha))}\widehat{\tilde{E}'_m(v,e_1,e_2)} \hat{P}_\alpha T_{s_1}$$

is a linear combination of spin network states over the graph $\gamma(s_1) \cup \alpha = \gamma(s_1)$, hence we must have $\gamma(s) = \gamma(s_1) = \gamma(s')$ in order that the matrix element for s_1, α does not vanish. Thus the condition $\gamma(s) = \gamma(s')$ is implicit.

Expression (3.15) defines a positive quadratic form. To see the positivity we take a generic linear combination of spin network functions

$$f = \sum_{p=1}^{M} \sum_{r=1}^{N_p} z_p^r T_{s_p^r}$$

where z_p^r are complex numbers and we have adopted a labelling such that $\gamma(s_p^r) = \gamma_p$ for all $r = 1, .., N_p$ and the graphs γ_p are mutually different. Then we simply compute, using the fact that the matrix elements vanish between spin network states over different graphs

$$\begin{aligned} Q_{\mathbf{M}_{E,eff}}(f,f) &= \sum_{p=1}^{N} \sum_{r,s=1}^{N_p} \bar{z}_p^r z_p^l \times \qquad (3.16) \\ &\times \sum_{s_1} 2(\frac{4}{\ell_p^2})^4 \sum_{v \in V(\gamma_p)} \delta^{jk} \delta^{mn} \sum_{e_1,e_2 \in E(\gamma_p); v = e_1 \cap e_2} \sum_{\alpha \in L(\gamma_p, v, e_1, e_2)} \frac{1}{|L(\gamma_p, v, e_1, e_2)|} \times \\ &\times < T_{s_p^r}, \tilde{F}_j\widehat{(R(\alpha))} \tilde{E}'_m\widehat{(v, e_1}, e_2) \hat{P}_\alpha T_{s_1} >_{Kin} \overline{< T_{s_p^s}, \tilde{F}_k\widehat{(R(\alpha))} \tilde{E}'_n\widehat{(v, e_1}, e_2) \hat{P}_\alpha T_{s_1} >_{Kin}} \\ = & \sum_{p=1}^{N} \sum_{s_1} 2(\frac{4}{\ell_p^2})^4 \sum_{v \in V(\gamma_p)} \sum_{j,m} \sum_{e_1,e_2 \in E(\gamma_p); v = e_1 \cap e_2} \sum_{\alpha \in L(\gamma_p, v, e_1, e_2)} \frac{1}{|L(\gamma_p, v, e_1, e_2)|} \times \\ &\times |\sum_{r=1}^{N_p} \bar{z}_p^r < T_{s_p^r}, \tilde{F}_j\widehat{(R(\alpha))} \tilde{E}'_m\widehat{(v, e_1}, e_2) \hat{P}_\alpha T_{s_1} >_{Kin} |^2 \end{aligned}$$

which is manifestly non – negative.

Not every positive quadratic form defines an operator but if it does (technically, if it is closable), then the corresponding positive, self-adjoint operator is unique (i.e. there is no choice in its self-adjoint extension).

Theorem 3.1.
The quadratic from (3.15) defines a positive self – adjoint operator.

Proof of theorem 3.1:
To see this we notice that the would be operator is given by

$$\widehat{\mathbf{M}}_{E,eff} T_s = \sum_{s'} Q_{\mathbf{M}_{E,eff}}(T_{s'}, T_s) T_{s'} \qquad (3.17)$$

This defines a (densely defined) operator if and only if the right hand side of (3.17) is normalizable, that is,

$$|| \widehat{\mathbf{M}}_{E,eff} T_s||^2_{Kin} = \sum_{s'} |Q_{\mathbf{M}_{E,eff}}(T_{s'}, T_s)|^2 \qquad (3.18)$$

converges. Notice that the sum on the right hand side is over an uncountably infinite set, so (3.17) looks dangerous. However, we know already that (3.18) reduces to the countable sum

$$|| \widehat{\mathbf{M}}_{E,eff} T_s||^2_{Kin} = \sum_{\gamma(s') = \gamma(s)} |Q_{\mathbf{M}_{E,eff}}(T_{s'}, T_s)|^2 \qquad (3.19)$$

Now the matrix elements (3.15) are finite for given s, s' since we must have $\gamma(s_1) = \gamma(s)$ and the states $\tilde{F}_j\widehat{(R(\alpha))} \tilde{E}'_m\widehat{(v, e_1}, e_2) \hat{P}_\alpha T_{s_1}$ are a finite linear combination of spin network states of which at most one can coincide with either T_s or $T_{s'}$. Since there are only $\sum_{v\in V(\gamma(s))} n(v)$ such terms involved in (3.15) where $n(v)$ denotes the valence of v, the assertion follows.

Hence, to show convergence of (3.19) it is sufficient to show that for given s there are only a finite number of s' for which $Q_{\mathbf{M}_{E,eff}}(T_{s'}, T_s) \neq 0$. Now we have seen that $Q_{\mathbf{M}_{E,eff}}(T_{s'}, T_s) \neq 0$ implies that in the sum over s_1 only a finite number of terms contribute which must satisfy $s_1 \in S(s)$ where $S(s)$ is a finite set of spin network labels. For each $s_1 \in S(s)$ consider the finite number of spin network states with labels in $S(s_1)$ contained in the span of the states

$$\tilde{F}_j\widehat{(R(\alpha))}\tilde{E}'_m\widehat{(v, e_1, e_2)}\hat{P}_\alpha T_{s_1}$$

as $v, e_1, e_2, \alpha \in L(\gamma(s), v, e_1, e_2)$ vary. Thus s' must be in the finite set $\cup_{s_1\in S(s)} S(s_1)$.

This shows that (3.17) defines a positive symmetric operator on the finite linear span of spin network states. However, positive symmetric operators have a preferred self – adjoint extension, the Friedrichs extension, which is given by the unique closure of the corresponding quadratic form $Q_{\mathbf{M}_{E,eff}}$ with the same domain of definition.
$\square$

We now can proceed to solve $\widehat{\mathbf{M}}_{E,eff}$. Let Φ_{Kin} be the dense subset of $\mathcal{H}_{Kin}$ given by the finite linear span of spin network functions. We begin by verifying that for any $f \in \Phi_{Kin}$ the distribution

$$\eta(f) := \bar{f}\delta[F], \quad \delta[F] := \int_{\mathcal{M}_{flat}} d\nu_0(A_0)\; \delta_{A_0} \tag{3.20}$$

solves the effective extended Master Constraint Operator. Here $\mathcal{M}_{flat}$ is the moduli space of flat connections on σ, δ_{A_0} is the $\delta-$distribution supported at A_0 and ν_0 is the following measure on $\mathcal{M}_{flat}$: Any function on $\mathcal{M}_{flat}$ is of the form $f(A_0) = f_n(A_0(\alpha_1), .., \alpha_n(A_0))$ where $\alpha_1, .., \alpha_n$ are any generators of the fundamental group $\pi_1(\sigma)$. Then

$$\nu_0(f) = \int_{\mathcal{M}_{flat}} d\nu_0(A_0) f(A_0) = \int_{SU(2)^n} d\mu_H(h_1)..d\mu_H(h_n)\;\; f_n(h_1, .., h_n) \tag{3.21}$$

The interested reader is referred to [9] for more details.

Now we may check whether (3.20) is a generalized solution of $\widehat{\mathbf{M}}_{E,eff} = 0$. We have for any $f' \in \Phi_{Kin}$

$$\begin{aligned}\eta(f)[\widehat{\mathbf{M}}_{E,eff}\, f'] &= \int_{\mathcal{M}_{flat}} d\nu_0(A_0)\; \overline{f(A_0)}\;\; \delta_{A_0}[\widehat{\mathbf{M}}_{E,eff}\, f'] \\ &= \int_{\mathcal{M}_{flat}} d\nu_0(A_0)\; \overline{f(A_0)}[\widehat{\mathbf{M}}_{E,eff}\, f'](A_0) \end{aligned} \tag{3.22}$$

This will be zero if and only if it is zero for any spin network state $f' = T_s$, hence we compute

$$
\begin{aligned}
& \eta(f)[\widehat{\mathbf{M}}_{E,eff}\, T_s] = \sum_{s'} < f, T_{s'} >_{Flat}\, Q_{\widehat{\mathbf{M}}_{E,eff}}(T_{s'}, T_s) \\
= \quad & 2(\frac{4}{\ell_p^2})^4 \sum_{s',s_1} < f, T_{s'} >_{Flat} \sum_{v\in V(\gamma(s))} \delta^{jk}\delta^{mn} \times \\
& \times \sum_{e_1,e_2\in E(\gamma(s));v=e_1\cap e_2} \quad \sum_{\alpha\in L(\gamma(s),v,e_1,e_2)} \frac{1}{|L(\gamma(s),v,e_1,e_2)|} \times \\
& \times < T_{s'}, \tilde{F}_j\widehat{(R(\alpha))}\tilde{E}'_m\widehat{(v,e_1},e_2)\hat{P}_\alpha T_{s_1} >_{Kin} \overline{< T_s, \tilde{F}_k\widehat{(R(\alpha))}\tilde{E}'_n\widehat{(v,e_1},e_2)\hat{P}_\alpha T_{s_1} >_{Kin}} \\
= \quad & 2(\frac{4}{\ell_p^2})^4 \sum_{s_1} \sum_{v\in V(\gamma(s))} \delta^{jk}\delta^{mn} \sum_{e_1,e_2\in E(\gamma(s));v=e_1\cap e_2} \sum_{\alpha\in L(\gamma(s),v,e_1,e_2)} \frac{1}{|L(\gamma(s),v,e_1,e_2)|} \times \\
& \times < f, \tilde{F}_j\widehat{(R(\alpha))}\tilde{E}'_m\widehat{(v,e_1},e_2)\hat{P}_\alpha T_{s_1} >_{Flat} \overline{< T_s, \tilde{F}_k\widehat{(R(\alpha))}\tilde{E}'_n\widehat{(v,e_1},e_2)\hat{P}_\alpha T_{s_1} >_{Kin}} \\
= \quad & 0
\end{aligned}
\tag{3.23}
$$

In the last step we have used the completeness relation with respect to s' and the fact that $\mathrm{Tr}(\tau_j A_0(\alpha)) = 0$ for every $A_0 \in \mathcal{M}_{flat}$. We could interchange the various summations because the non-vanishing terms reduce the sums to finite ones.

It follows that

$$\widehat{\mathbf{M}}'_{E,eff}\, \eta(f) = 0 \tag{3.24}$$

where the prime denotes the dual of the operator on the space Φ^*_{Kin} of linear functionals on Φ_{Kin} (without continuity requirement) defined in general by $[O'l](f) = l[O^\dagger f]$. Hence we see that the point $\lambda = 0$ lies at least in the continuous part of $\widehat{\mathbf{M}}_{E,eff}$ since the $\eta(f)$ are not normalizable with respect to $< .,. >_{Kin}$, provided that we can write $\eta(f)[f'] =< \tilde{f}(0), \tilde{f}'(0) >_{\mathcal{H}_0^{c\oplus}}$ where $\tilde{f},\ \tilde{f}'$ are representatives of f, f' corresponding to a direct integral decomposition of $\mathcal{H}$ subordinate to $\widehat{\mathbf{M}}_{E,eff}$. One should now complete the analysis and compute the full spectrum of $\widehat{\mathbf{M}}_{E,eff}$ to see whether that is indeed the case. Assuming that to be true, for the purposes of this paper it is sufficient to note that whatever the complete $\mathcal{H}_{Phys}$ might be, it contains the closed linear span of the $\eta(f)$ as a subspace with the induced physical inner product

$$< \eta(f), \eta(f') >_{Phys} := \nu_0(f\overline{f'}) \tag{3.25}$$

which is of course well known in the literature [20]. We leave a more complete analysis for future work and just remark that with the techniques of [17] it is not difficult to show that the operator (3.17), although we have performed rather drastic manipulations, indeed has the correct classical limit.

4 Conclusions and Outlook

What we have learnt in this paper is that the Master Constraint Programme is also able to deal with the case of interacting quantum field theories. By this we mean that we can solve, e.g. the Gauss contraint of Non – Abelean Yang Mills theory *when coupled to gravity*. Classical Non – Abelean Yang – Mills theory on a background spacetime is a self – interacting field theory and in four dimensions nobody was able to show that the corresponding interacting quantum field theory exists (in the continuum; on the lattice there are no problems). The Gauss constraint of the theory is a quadratic polynomial in the fields and its square id a fourth order polynomial. Therefore, from a QFT on curved background point of view the Master Constraint should be as UV singular as the square of the curvature that defines the Yang – Mills action. Things become even worse because on top of that we have multiplied the square of the Gauss constraint by a factor which depends non – polynomially on the the degrees of freedom of the gravitational field.

Yet, we were able to quantize the integral of the resulting expression on the Hilbert space which is used in Loop Quantum Gravity (LQG) without encountering UV problems. The technical reason for why this happened is that the Master Constraint is spatially diffeomorphism invariant integral of a scalar density. As was shown in the sixth reference of [9], for such quantities, in a very precise sense, the UV regulator gets swallowed up by spatial diffeomorphism group: The theory does not depend on a background metric, thus all "distances" are gauge equivalent.

Let us compare the situation with our previous paper [5]: There we were looking at free field theories on a Minkowski background and we could use the associated background dependent Fock representations. The square of the Maxwell Gauss constraint is too singular and cannot be employed to define the Master Constraint, thus we had to use the flexibility of the Master Constraint Programme to regularize the Master Constraint by a background dependent kernel. In the present situation the singularity of the square is expected, according to perturbation theory arguments, to be even worse. However, fundamentally the Fock representation is not a valid representation for interacting quantum field theories. A possible representation is the one that uses LQG techniques, however, that representation is only valid when we couple the gravitational field as otherwise e.g. the Yang – Mills Hamiltonian is ill – defined in that representation.

It is precisely this observation which has lead us to quantize the Master Constraint in this kind of LQG representation. It turns out that the non – polynomial, gravitational field dependent function mentioned above that enters the Master Constraint plays exactly the same role as for the Yang – Mills Hamiltonian (constraint): It serves as a background independent UV regulator. Moreover, while in the free field case the regulator could be chosen to be background dependent, here we are not allowed to do that, however, *background independent theories, through gravity, have the tendency to regulate themselves*. Thus the factor of $1/\sqrt{\det(q)}$ which typically enters the background independent Master Constraints becomes a quantum operator (rather than a $\mathbb{C}$ number valued expression) but otherwise plays the same role as the trace class operator K in [5]: It removes the UV singularities of the square of the Gauss constraints.

We stress again that this does not mean at all that we have proved the existence of, say, QCD. This is because, while we are able to avoid certain singularities, now the burden is on us to show that the theory can also successfully deal with the additional symmetries that have entered the stage by coupling matter to the gravitational field. This is the spacetime diffeomorphism symmetry which finds its way into the canonical framework in the form of the spatial diffeomorphism and the Hamiltonian constraint. It is precisely for this reason that the Master Constraint Programme was created. One now has to apply it to all symmetries of General Relativity, solve the full Master Constraint and establish that we have captured a quantum theory of General Relativity rather than a mathematically consistent but physically uninteresting quantum theory of geometry and matter. This is what has to be done in the close future and finally the mathematical techniques are available in order to make progress.

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References

[1] T. Thiemann, "The Phoenix Project: Master Constraint Programme for Loop Quantum Gravity", gr-qc/0305080

[2] B. Dittrich, T. Thiemann, "Testing the Master Constraint Programme for Loop Quantum Gravity I. General Framework", gr-qc/0411138

[3] B. Dittrich, T. Thiemann, "Testing the Master Constraint Programme for Loop Quantum Gravity II. Finite Dimensional Systems", gr-qc/0411139

[4] B. Dittrich, T. Thiemann, "Testing the Master Constraint Programme for Loop Quantum Gravity III. SL(2,R) Models", gr-qc/0411140

[5] B. Dittrich, T. Thiemann, "Testing the Master Constraint Programme for Loop Quantum Gravity IV. Free Field Theories", gr-qc/0411141

[6] C. Rovelli, "Loop Quantum Gravity", Living Rev. Rel. **1** (1998) 1, gr-qc/9710008
T. Thiemann,"Lectures on Loop Quantum Gravity", Lecture Notes in Physics, **631** (2003) 41 – 135, gr-qc/0210094
A. Ashtekar, J. Lewandowski, "Background Independent Quantum Gravity: A Status Report", Class. Quant. Grav. **21** (2004) R53; [gr-qc/0404018]
L. Smolin, "An Invitation to Loop Quantum Gravity", hep-th/0408048

[7] C. Rovelli, "Quantum Gravity", Cambridge University Press, Cambridge, 2004

[8] T. Thiemann, "Modern Canonical Quantum General Relativity", Cambridge University Press, Cambridge, 2005, gr-qc/0110034

[9] T. Thiemann, "Anomaly-free Formulation of non-perturbative, four-dimensional Lorentzian Quantum Gravity", Physics Letters **B380** (1996) 257-264, [gr-qc/9606088]
T. Thiemann, "Quantum Spin Dynamics (QSD)", Class. Quantum Grav. **15** (1998) 839-73, [gr-qc/9606089]; "II. The Kernel of the Wheeler-DeWitt Constraint Operator", Class. Quantum Grav. **15** (1998) 875-905, [gr-qc/9606090]; "III. Quantum Constraint Algebra and Physical Scalar Product in Quantum General Relativity", Class. Quantum Grav. **15** (1998) 1207-1247, [gr-qc/9705017]; "IV. 2+1 Euclidean Quantum Gravity as a model to test 3+1 Lorentzian Quantum Gravity", Class. Quantum Grav. **15** (1998) 1249-1280, [gr-qc/9705018]; "V. Quantum Gravity as the Natural Regulator of the Hamiltonian Constraint of Matter Quantum Field Theories", Class. Quantum Grav. **15** (1998) 1281-1314, [gr-qc/9705019]; "VI. Quantum Poincaré Algebra and a Quantum Positivity of Energy Theorem for Canonical Quantum Gravity", Class. Quantum Grav. **15** (1998) 1463-1485, [gr-qc/9705020]; "Kinematical Hilbert Spaces for Fermionic and Higgs Quantum Field Theories", Class. Quantum Grav. **15** (1998) 1487-1512, [gr-qc/9705021]

[10] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão, T. Thiemann, "Quantization for diffeomorphism invariant theories of connections with local degrees of freedom", Journ. Math. Phys. **36** (1995) 6456-6493, [gr-qc/9504018]

[11] H. Sahlmann, "When do Measures on the Space of Connections Support the Triad Operators of Loop Quantum Gravity?", gr-qc/0207112; "Some Comments on the Representation Theory of the Algebra Underlying Loop Quantum Gravity", gr-qc/0207111
H. Sahlmann, T. Thiemann, "On the Superselection Theory of the Weyl Algebra for Diffeomorphism Invariant Quantum Gauge Theories", gr-qc/0302090; "Irreducibility of the Ashtekar-Isham-Lewandowski Representation", gr-qc/0303074
A. Okolow, J. Lewandowski, "Diffeomorphism Covariant Representations of the Holonomy Flux Algebra", gr-qc/0302059

[12] A. Ashtekar, C.J. Isham, "Representations of the Holonomy Algebras of Gravity and Non-Abelean Gauge Theories", Class. Quantum Grav. **9** (1992) 1433, [hep-th/9202053]
A. Ashtekar, J. Lewandowski, "Representation theory of analytic Holonomy $C^\star$ algebras", in "Knots and Quantum Gravity", J. Baez (ed.), Oxford University Press, Oxford 1994

[13] J. Klauder, “Universal Procedure for Enforcing Quantum Constraints”, Nucl.Phys.B547:397-412,1999, [hep-th/9901010]; “Quantization of Constrined Systems”, Lect. Notes Phys. **572** (2001) 143-182, [hep-th/0003297]
A. Kempf, J. Klauder, “On the Implementation of Constraints through Projection Operators”, J. Phys. **A34** (2001) 1019-1036, [quant-ph/0009072]

[14] J.E. Marsden, P.R. Chernoff, “Properties of Infinite Dimensional Hamiltonian Systems”, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1974

[15] T. Thiemann, “Quantum Spin Dynamics (QSD): VIII. The Master Constraint”, in preparation

[16] H. Sahlmann, T. Thiemann, “Towards the QFT on Curved Spacetime Limit of QGR. 1. A General Scheme”, [gr-qc/0207030]; “2. A Concrete Implementation”, [gr-qc/0207031]

[17] T. Thiemann, “Quantum Spin Dynamics (QSD): VII. Symplectic Structures and Continuum Lattice Formulations of Gauge Field Theories”, Class.Quant.Grav.18:3293-3338,2001, [hep-th/0005232]; “Gauge Field Theory Coherent States (GCS): I. General Properties”, Class.Quant.Grav.18:2025-2064,2001, [hep-th/0005233]; “Complexifier Coherent States for Canonical Quantum General Relativity”, gr-qc/0206037
T. Thiemann, O. Winkler, “Gauge Field Theory Coherent States (GCS): II. Peakedness Properties”, Class.Quant.Grav.18:2561-2636,2001, [hep-th/0005237]; “III. Ehrenfest Theorems”, Class. Quantum Grav. **18** (2001) 4629-4681, [hep-th/0005234]; “IV. Infinite Tensor Product and Thermodynamic Limit”, Class. Quantum Grav. **18** (2001) 4997-5033, [hep-th/0005235]
H. Sahlmann, T. Thiemann, O. Winkler, “Coherent States for Canonical Quantum General Relativity and the Infinite Tensor Product Extension”, Nucl.Phys.B606:401-440,2001 [gr-qc/0102038]

[18] M. Reed, B. Simon, “Functional Analysis”, vol. 1, Academic Press, New York, 1980
675-693 (1994)

[19] J. Brunnemann, T. Thiemann, “Simplification of the Spectral Analysis of the Volume Operator in Loop Quantum Gravity”, gr-qc/0405060

[20] E. Witten, Nucl. Phys. **B311** (1988) 46

[21] D. Giulini, D. Marolf , “A Uniqueness Theorem for Constraint Quantization” Class. Quant. Grav. 16:2489-2505,1999, [gr-qc/9902045]; “On the Generality of Refined Algebraic Quantization”, Class. Quant. Grav. 16:2479-2488,1999, [gr-qc/9812024]

[22] I. M. Gel’fand, N. Ya. Vilenkin, “Generalized Functions”, vol. 4, Applications of Harmonic Analysis, Academic Press, New York, London, 1964