Propagation of High-Frequency Electromagnetic Waves Through a Magnetized Plasma in Curved Spaces-Time. II. Application of the Asymptotic Approximation

R. A. Breuer; J. Ehlers

Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, Vol. 374, No. 1756. (Jan. 7, 1981), pp. 65-86.

Stable URL:

http://links.jstor.org/sici?sici=0080-4630%2819810107%29374%3A1756%3C65%3APOHEWT%3E2.0.CO%3B2-8

Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences is currently published by The Royal Society.

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at http://www.jstor.org/about/terms.html. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at http://www.jstor.org/journals/rsl.html.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

The JSTOR Archive is a trusted digital repository providing for long-term preservation and access to leading academic journals and scholarly literature from around the world. The Archive is supported by libraries, scholarly societies, publishers, and foundations. It is an initiative of JSTOR, a not-for-profit organization with a mission to help the scholarly community take advantage of advances in technology. For more information regarding JSTOR, please contact support@jstor.org.

http://www.jstor.org
Fri Feb 8 08:27:11 2008

Proc. R. Soc. Lond. A **374**, 65–86 (1981)
Printed in Great Britain

Propagation of high-frequency electromagnetic waves through a magnetized plasma in curved space–time. II.† Application of the asymptotic approximation

By R. A. Breuer and J. Ehlers
Max-Planck-institut für Physik und Astrophysik, Föhringer Ring 6, D-8 München 40, F.R.G.

(*Communicated by R. Penrose, F.R.S. – Received* 12 *May* 1980)

This is the second of two papers on the propagation of high-frequency electromagnetic waves through an inhomogeneous, non-stationary plasma in curved space–time. By applying the general two-scale W.K.B. method developed in part I to the basic wave equation, derived also in that paper, we here obtain the dispersion relation, the rays, the polarization states and the transport laws for the amplitudes of these waves. In an unmagnetized plasma the transport preserves the helicity and the eccentricity of the polarization state along each ray; the axes of the polarization ellipse rotate along a ray, relative to quasiparallely displaced directions, at a rate determined by the vorticity of the electron fluid; and the norm of the amplitude changes according to a conservation law which can be interpreted as the constancy of the number of quasiphotons. In a magnetized plasma the polarization state changes differently for ordinary and extraordinary waves, according to the angle between the wave-normal and the background magnetic field, and under specified approximation conditions the direction of polarization of linearly polarized waves undergoes a generalized Faraday rotation.

1. Introduction

The observed phenomenon of X-rays emanating from matter which accretes onto neutron stars within compact binary systems has posed the problem of electromagnetic high-frequency waves propagating through the magnetosphere of the neutron star which is filled with the accreting plasma (Mészáros 1978) or a self-magnetized accretion disk around a black hole (Mészáros *et al.* 1977). Similar processes will occur in other more catastrophic events, for example those in the early universe just before decoupling time, in quasars if we assume an accretion scenario around a super-massive black hole, and also in the late stage of a collapsing star. There, in fact, the production and dispersion of gravitational waves may also play an important role.

In this second paper on the propagation of electromagnetic waves through a magnetized plasma in some arbitrary space–time we will not deal with gravitational

† Part I appeared in *Proc. R. Soc. Lond.* A **370**, 389–406 (1980).

radiation; however, we consider the results of this paper also as a preparatory calculation for the analysis of the interaction of gravitational waves with dissipative matter. Previous attempts to deal with this topic and the topic of this paper (Madore 1974; Bičák & Hadrava 1975; Anile & Pantano 1977, 1979) seem to us to be incomplete.

Our aim is to apply the general methods developed in Breuer & Ehlers (1980, henceforth referred to as paper I) to the fundamental perturbation equation derived there. We here give a brief summary of this derivation.

Let $(M, g^a{}_b)$ be a space–time populated with a cold, pressure-free, two-component plasma. The number density and four velocity of the electrons are denoted by n and u^a; J^a denotes the ion-current density. If e and m are the charge and mass of the electron and F_{ab} is the electromagnetic field, we consider the following *background system* of differential equations:

$$\nabla_{[a} F_{bc]} = 0; \tag{1.1}$$

$$\nabla_b F^{ab} = enu^a + J^a; \tag{1.2}$$

$$u^b \nabla_b u^a = (e/m)\, F^a{}_b u^b; \tag{1.3}$$

$$\nabla_a (nu^a) = 0; \tag{1.4}$$

$$u_a u^a = -1. \tag{1.5}$$

Small perturbations $\hat{F}_{ab}$, $\hat{n}$ and $\hat{u}^a$ of the background system obey the following *perturbed system* of equations:

$$\nabla_{[a} \hat{F}_{bc]} = 0; \tag{1.6}$$

$$\nabla_b \hat{F}^{ab} = e(\hat{n} u^a + n\hat{u}^a); \tag{1.7}$$

$$u^b \nabla_b \hat{u}^a + \hat{u}^b \nabla_b u^a = (e/m)\,(F^a{}_b \hat{u}^b + \hat{F}^a{}_b u^b); \tag{1.8}$$

$$\nabla_a (\hat{n} u^a + n \hat{u}^a) = 0; \tag{1.9}$$

$$u_a \hat{u}^a = 0. \tag{1.10}$$

By projecting (1.7) parallel to u^a and orthogonal to the tangent rest-space of u^a by $h^a{}_b := \delta^a{}_b + u^a u_b$, the unknowns $\hat{n}, \hat{u}^a$ can be eliminated from (1.7):

$$e\hat{n} = -u_a \nabla_b \hat{F}^{ab}, \tag{1.11}$$

$$en\hat{u}^a = h^a{}_b \nabla_c \hat{F}^{bc}. \tag{1.12}$$

Introducing the electromagnetic four-potential of the perturbation, $\hat{A}^a$, via $\hat{F}_{ab} = 2\nabla_{[a} \hat{A}_{b]}$ and imposing the Landau gauge condition $u^a \hat{A}_a = 0$, we have shown in paper I that (1.6)–(1.10) are equivalent to the *fundamental equation governing the perturbations*:

$$\begin{aligned} \mathrm{D}^{ab} \hat{A}_b := \{ & h^{ac} u^d \nabla_d (\nabla^b{}_c - \delta^b{}_c \nabla^e{}_e) + (\omega^{ac} + \omega_{\mathrm{L}}{}^{ac} + \theta^{ac} + \theta h^{ac} + \\ & + (e/m)\, E^a u^c)\,(\nabla^b{}_c - \delta^b{}_c \nabla^e{}_e) + \omega_p^2 h^{ab} u^d \nabla_d + \omega_p^2 (\theta^{ab} - \omega^{ab}) \} \hat{A}_b = 0. \end{aligned} \tag{1.13}$$

This is a constraint-free homogeneous linear system of three partial differential equations of order three for the three unknown spatial components of $\hat{A}^a$. In (1.13) we used the notation

$$\left.\begin{aligned} E^a &:= F^a{}_b u^b, & B_{ab} &:= h^c{}_a h^d{}_b F_{cd}, \\ \omega_{\mathrm{L}}{}^a{}_b &:= (-e/m) B^a{}_b, & \omega_{\mathrm{L}}{}^a &:= -\tfrac{1}{2}\eta^{abcd} u_b \omega_{\mathrm{L}cd}, \\ \omega_{\mathrm{L}} &:= (\omega_{\mathrm{L}\,a}\omega_{\mathrm{L}}{}^a)^{\frac{1}{2}}, & \omega_{\mathrm{p}} &:= (ne^2/m)^{\frac{1}{2}}, \\ \nabla_b u^a &= \omega^a{}_b + \theta^a{}_b - u_b u^c \nabla_c u^a, & \theta &:= \theta^a{}_a = \nabla_a u^a, \\ \omega^a{}_b &= -\omega_{ba}, \quad \theta_{ab} = \theta_{ba}. \end{aligned}\right\} \quad (1.14)$$

In paper I we proved existence, uniqueness and linearization stability of solutions of the Cauchy problem for the background system and the perturbed system. In addition we adapted the W.K.B. method for the construction of oscillatory asymptotic solutions to systems such as (1.13) and stated a sufficient condition for asymptotic solutions of finite order to be, in fact, approximate solutions of the respective systems. By these theorems the approximation scheme to follow has been put on a firm basis.

In this paper we shall apply the method to (1.13) and thus study the propagation of high-frequency electromagnetic waves through plasmas embedded in (possibly) strong gravitational fields, emphasizing those properties which arise from inhomogeneities of the plasma and the background electromagnetic and gravitational fields.

The plan of this paper is as follows. In § 2 the two-timing method is employed to reformulate (1.13). In § 3 we study in detail the rays associated with the ordinary and extraordinary waves, and the phase-propagation along them. In § 4 we determine the polarization states of these waves. Then, in § 5, we set up the transport equations for the lowest-order amplitudes of the waves and consider their implications for the propagation of polarization states and wave intensities. In particular, we generalize the elementary treatment of Faraday rotation to non-stationary, inhomogeneous backgrounds. We summarize our results and state some problems in the final § 6. Appendix A deals with the role of gauge conditions in asymptotic expansions, and in Appendix B we define a quasiparallel transport of vectors which is needed to interpret the transport of polarization vectors along rays in curved space–times. The notation is as in paper I and follows Misner *et al.* (1973).

2. A two-scale method

According to a theorem of paper I, the perturbation equation (1.13) has, in particular, solutions which are approximately plane and monochromatic on a scale λ much smaller than the one, say L, on which the background quantities vary; appropriate initial conditions determine such waves. By means of the method described in paper I, § 4 it should be possible to approximate such short-wave solutions by asymptotic series, provided the small scale-ratio $\epsilon := \lambda/L$ is introduced into (1.13) suitably.

To introduce the background scale L we assume that the connection coefficients $\Gamma^{(a)}{}_{(b)(c)}$ associated with the orthonormal tetrad field $E_{(a)}{}^{b}$ used in relation to (1.13), defined by

$$\nabla_{E_{(a)}} E_{(b)} = \Gamma^{(c)}{}_{(b)(a)} E_{(c)}, \tag{2.1}$$

are at most of the order of magnitude L^{-1}, and that the directional derivatives of low order† of the tetrad components of the background fields are at most of the order of magnitude L^{-1} times the field components themselves. Clearly, L is the scale on which the background fields change. In astrophysical applications L will be a 'large' length like a stellar radius. (Examples: In a Schwarzschild space–time filled with a stationary test fluid, L can be chosen to be the Schwarzschild radius, provided the domain considered excludes a neighbourhood of the absolute event horizon. For a not-too-large part of a Friedmann–Robertson–Walker space–time filled with the standard cosmological fluid, L can be chosen to be of the order of the Hubble age of that domain.)

We define the 'dimensionless covariant directional derivative' operators

$$\mathrm{D}_a := L E_{(a)}{}^{b} \nabla_b = L \nabla_{E_{(a)}}. \tag{2.2}$$

These operators do not change the physical dimensions of the fields on which they act, and if C_b is a slowly varying quantity, then (by assumption)

$$|\mathrm{D}_a C_b| \lesssim |C_b|. \tag{2.3}$$

(Here and in the sequel we no longer ornate tetrad-indices by brackets.)

Our assumptions and (1.3) and (1.14) imply that

$$\left.\begin{aligned} \omega^{ab} &= L^{-1}\,\underset{0}{\omega}{}^{ab}, & \theta &= L^{-1}\underset{0}{\theta}, \\ \theta^{ab} &= L^{-1}\underset{0}{\theta}{}^{ab}, & E^{a} &= L^{-1}\underset{0}{E}{}^{a}, \end{aligned}\right\} \tag{2.4}$$

where all symbols with an underzero denote dimensionless, slowly varying functions of (at most) order unity.

In terms of the dimensionless operators (2.2) and coefficient functions $\underset{0}{\omega}{}^{ab}$ etc. the perturbation equation (1.13) reads, if λ denotes a – at this stage still arbitrary – length, and $\epsilon := \lambda/L$:

$$\begin{aligned} \{[\epsilon^{3}(\underset{0}{h}{}^{ac}\underset{0}{u}{}^{d}\mathrm{D}_d + \underset{0}{\theta}{}^{ac} + \underset{0}{\theta}\underset{0}{h}{}^{ac} + \underset{0}{\omega}{}^{ac} + (e/m)\,\underset{0}{E}{}^{a}\underset{0}{u}{}^{c}) + \epsilon^{2}\lambda\omega_{\mathrm{L}}{}^{ac}]\,[\mathrm{D}^{b}{}_{c} - \delta^{b}{}_{c}\mathrm{D}^{e}{}_{e}] \\ + \epsilon\lambda^{2}\omega_p^{2}(\underset{0}{h}{}^{ab}\underset{0}{u}{}^{d}\mathrm{D}_d + \underset{0}{\theta}{}^{ab} - \underset{0}{\omega}{}^{ab})\}\,\hat{A}_b = 0. \end{aligned} \tag{2.5}$$

As mentioned previously, (2.5) is a special case of (4.1) of paper I for $p = 3$, $m = 3$, and $n = 4$. We have written $\underset{0}{h}{}^{ab}$, $\underset{0}{u}{}^{a}$ instead of h^{ab}, u^{a} to emphazise that all quantities with an underzero are slowly variable and at most of order one.

† Precise assumptions of this kind would be required if errors were to be estimated.

We now specialize the perturbations to locally approximately plane, monochromatic waves, choose λ to be of the order of the wavelength, and assume $\epsilon := \lambda/L \ll 1$. In accordance with this assumption we put

$$\hat{A}_a(x,\epsilon) \underset{\epsilon\to 0}{\sim} \exp\left[\frac{\mathrm{i}}{\epsilon} S(x)\right] \sum_{n=0}^{\infty} \left(\frac{\epsilon}{\mathrm{i}}\right)^n \underset{n}{\hat{A}_a}(x) \tag{2.6}$$

with

$$u^a \underset{n}{\hat{A}_a} = 0. \tag{2.7}$$

The amplitudes $\underset{n}{\hat{A}_a}$ have to be chosen so small that the linear approximation discussed in paper I is valid.

The method of paper I, § 4 can be applied to (2.5) in two distinct ways:

(*a*) One considers the background scale L as fixed and takes the limit $\lambda \to 0$. In this case it is reasonable to put

$$\lambda \omega_{\mathrm{L}}{}^{ab} = \epsilon L \omega_{\mathrm{L}}{}^{ab}, \quad \lambda \omega_{\mathrm{p}} = \epsilon L \omega_{\mathrm{p}}$$

and to treat $L\omega_{\mathrm{L}}{}^{ab}$, $L\omega_{\mathrm{p}}$ as ϵ-independent, bounded functions like the other coefficients of the operator in (2.5). All terms of (2.5) then contain the factor ϵ^3 which therefore can be dropped. In this case it is convenient to choose L as the unit of length, $L = 1$, so that $\epsilon = \lambda$.

(*b*) One chooses a fixed wavelength scale λ and takes $L \to \infty$. Then it is reasonable to consider $\lambda\omega_{\mathrm{L}}{}^{ab}$ and $\lambda\omega_{\mathrm{p}}$ as bounded, ϵ-independent coefficients in (2.5). The operator then contains different powers of ϵ. In this case it is convenient to take λ as the unit of length, $\lambda = 1$, so that $L^{-1} = \epsilon$, and to work with D_a.

Both methods are based on the smallness of the scale-ratio ϵ, and both use the concepts of geometrical optics – rays and phase-hypersurfaces (wavefronts) – as the tools by means of which approximate periodic waves are constructed. The difference between the two methods resides in their domains of validity: In version (*a*) one approximates the wave in a given domain with a fixed inhomogeneity such that the error decreases with decreasing wavelength. In version (*b*) one keeps a specified wavelength range and improves the approximation by shifting the space–time domain towards regions of smaller inhomogeneity. In accordance with this, in (*b*) the influence of matter on the wave (for example, dispersion) is taken into account already in the lowest approximation. This is not the case in version (*a*).

In reality the values of L, λ, ω^{ab} etc. are, of course, given numerically by the physics; the 'limit process' $\epsilon \to 0$ is a formal device only. In order that the formal order-of-magnitude assignment corresponds roughly to the actual numerical values, one has to have (with $\omega_{\mathrm{r}} := (\frac{1}{2}\omega_{ab}\omega^{ab})^{\frac{1}{2}}$)

$$\omega_{\mathrm{L}} \sim \omega_{\mathrm{r}} \sim L\omega_{\mathrm{p}}{}^2 \quad \text{for case } (a),$$
$$\omega_{\mathrm{L}} \sim (L/\lambda)\,\omega_{\mathrm{r}} \sim \lambda\omega_{\mathrm{p}}{}^2 \quad \text{for case } (b),$$

i.e. the numbers will determine whether method (*a*) or (*b*) is more appropriate. A rigorous comparison of the methods requires error estimates which, unfortunately, appear to be unknown for most cases of physical interest.

Since we use the model of an 'electron fluid' we have to restrict λ by

$$n^{-\frac{1}{3}} \ll \lambda. \tag{2.8}$$

For this reason and since we are interested in dispersion effects *we shall henceforth use method* (b), and accordingly simplify the notation by putting $\lambda = 1$. Then in (2.6) the phase $\epsilon^{-1}S$ varies rapidly, on the scale 1, whereas the wave-covector

$$l_a := \nabla_a \epsilon^{-1} S = \mathrm{D}_a S, \tag{2.9}$$

the frequency

$$\omega := -u^a l_a = -u^a \mathrm{D}_a S, \tag{2.10}$$

with respect to the electron background, and the amplitude $\sum_{n=0}^{\infty} (\epsilon/\mathrm{i})^n \underset{n}{A}_a$ vary slowly, on the scale L, provided $u^a \mathrm{D}_a S$ is of order one and the $\mathrm{D}_a \mathrm{D}_b S$ and $\mathrm{D}_a \underset{n}{\hat{A}}_b$ are at most of order one. These latter conditions can be satisfied partly by the choice of initial conditions, and partly they determine the domain in which the approximation is valid.

Having obtained (2.5) and decided to use alternative (b), we can now apply the method of paper I, § 2. There is one formal difference, however: in (2.5) the covariant derivative operators D_a do not commute and, if written in coordinate language, contain first-order and zero-order terms. Nevertheless, it is easy to check that the method generalizes without any substantial change to the case where the $\partial_a = \partial/\partial x^a$ are replaced by D_a as defined in (2.2). In fact, the local theory can be globalized and reformulated in invariant, geometric language (Duistermaat 1974). In particular, it does not matter whether one uses, in $T_* M$, canonical coordinates (x^a, l_a) or any other coordinates.

For the lowest-order steps, the ordering of the operators D_a, which in general is relevant because of curvature, does not matter. This is obvious from (2.5) and (2.6): in zero order no derivatives are involved, in first order only one derivative appears. Hence for the first two steps one could equally well work in flat space–time and later use the principle of 'minimal coupling', replacing partial by covariant derivatives, thus obtaining the same results. Only for the third and higher-order steps is the curved-space treatment necessary to obtain the correct forms of the functions appearing as 'source terms' in the transport equations.

We note how one obtains the electric and magnetic fields from the potential. By (3.11) of paper I and (2.2) the perturbed field $\hat{F}_{ab}$ is

$$\begin{aligned}\hat{F}_{ab} &= 2\,\mathrm{Re}\,\{\nabla_{[a} \hat{A}_{b]}\} = 2L^{-1}\,\mathrm{Re}\,\{\mathrm{D}_{[a} \hat{A}_{b]}\} \\ &= 2\epsilon\,\mathrm{Re}\,\{\mathrm{D}_{[a} \hat{A}_{b]}\}.\end{aligned} \tag{2.11}$$

The asymptotic expansion (2.6) together with (2.9) then implies

$$\hat{F}_{ab} = 2\mathrm{i} l_{[a}\,\mathrm{Re}\,\{\mathrm{e}^{(\mathrm{i}/\epsilon)S} \underset{0}{\hat{A}}_{b]}\} + O(\epsilon). \tag{2.12}$$

Hence we get for the electric field $\hat{F}_{ab} u^b$ in lowest order

$$\underset{0}{\hat{E}}_a = \mathrm{Re}\{\mathrm{i}\omega\, \mathrm{e}^{(\mathrm{i}/\epsilon)S} \underset{0}{\hat{A}}_a\}, \tag{2.13}$$

i.e. $\mathrm{i}\omega \underset{0}{\hat{A}}_a$ is the complex amplitude of the lowest-order electric field of the wave. Correspondingly, the magnetic field is given by

$$\underset{0}{\hat{B}}_a = \mathrm{Re}\,(\mathrm{i}\,\mathrm{e}^{(\mathrm{i}/\epsilon)S}\, n_{ab}{}^{cd}\, u^b l_{[c}\, \underset{0}{\hat{A}}_{d]}). \tag{2.14}$$

In Appendix A it is shown that if the field strength admits an asymptotic expansion analogous to (2.6), then because of (1.6) there exists a potential of the form (2.6) which obeys the Landau gauge condition (2.7), and for given $\hat{F}_{ab}$ and S the amplitudes $\underset{n}{\hat{A}}_a$ are then uniquely determined, in accordance with the preceding remark about $\mathrm{i}\omega \underset{0}{\hat{A}}_a$.

3. Dispersion relation, eiconal equation and rays

We now apply the two-scale method to (2.5), using version (*b*) of § 2 – following closely the exposition of the method of paper I, § 4. We insert the asymptotic expansion (2.6) and take, formally, the limit $\epsilon \to 0$, which yields

$$\underset{0}{L}^{ab} \underset{0}{A}_b =: \{(\omega h^a{}_c + \mathrm{i}\omega_{\mathrm{L}}{}^a{}_c)\,(l^2 h^{cb} - k^c k^b) + \omega\omega_{\mathrm{p}}^2 h^{ab}\}\, \underset{0}{A}_b = 0, \tag{3.1}$$

where

$$k_a = h_a{}^b\, l_b$$

is the wave three-vector in the local rest frames of the background plasma, so that

$$l^a = k^a + \omega u^a, \quad l^2 = k^2 - \omega^2. \tag{3.2}$$

In (3.1), and subsequently, we drop the caret on the potential A_a.

Equation (3.1) is a linear homogeneous *polarization condition* for the lowest-order amplitude. Henceforth, we always restrict our attention to the cases where $k \neq 0$, $\omega \neq 0$ and $\omega^2 \neq \omega_{\mathrm{p}}^2$. To simplify the polarization condition we decompose $\underset{0}{A}_a$ into longitudinal and transverse parts with respect to the wave two-surfaces, by introducing the unit normal vector n^a and the transverse projector $p^a{}_b$:

$$n^a := k^{-1} k^a, \quad p^a{}_b := h^a{}_b - n^a n_b, \tag{3.3}$$

$$\left.\begin{aligned} \underset{0}{A}_{\parallel} &:= n^a \underset{0}{A}_a, \quad \underset{0}{A}_{\perp a} := p_a{}^b \underset{0}{A}_a, \\ \underset{0}{A}_a &= \underset{0}{A}_{\perp a} + n_a \underset{0}{A}_{\parallel}. \end{aligned}\right\} \tag{3.4}$$

The 'longitudinal part' of (3.1) then gives

$$\omega k(\omega^2 - \omega_{\mathrm{p}}{}^2)\, \underset{0}{A}_{\parallel} = \mathrm{i}(\omega^2 - k^2)\, \omega_{\mathrm{L}}{}^a{}_b\, k_a \underset{0}{A}_{\perp}{}^b. \tag{3.5}$$

After elimination of $\underset{0}{A}_{\parallel}$, via (3.5), from (3.1), the 'transverse part' of (3.1), obtained by projecting with $p^a{}_b$, reads

$$\begin{aligned} \underset{0}{\overset{\perp}{L}}{}^a{}_b \underset{0}{A}_{\perp}{}^b := \{&(\omega_{\mathrm{p}}{}^2 - \omega^2)\,[\omega(l^2 + \omega_{\mathrm{p}}{}^2)\, p^a{}_b + \mathrm{i} l^2 \overset{\perp}{\omega}_{\mathrm{L}}{}^a{}_b] \\ &+ l^2 \omega\omega_{\mathrm{L}}{}^a{}_c\, \omega_{\mathrm{L}bd} n^c n^d\}\, \underset{0}{A}_{\perp}{}^b = 0, \end{aligned} \tag{3.6}$$

where

$$\overset{\perp}{\omega}_{\mathrm{L}}{}^{a}{}_{b} := p^{a}{}_{c}\,\omega_{\mathrm{L}}{}^{c}{}_{d}\,p^{d}{}_{b}. \tag{3.7}$$

Note that in contrast to the effectively 3×3, matrix $\underset{0}{L}{}^{ab}$ in (3.1), the, effectively 2×2, matrix $\underset{0}{\overset{\perp}{L}}{}^{ab}$ is Hermitian.

Dispersion relation

Except for the factor ω corresponding to a zero-frequency mode, the dispersion relation implied by (3.1) is identical to the one belonging to (3.6):

$$H(x^a, l_b) := \omega^2(\omega^2-\omega_{\mathrm{p}}{}^2)\,(\omega^2-k^2-\omega_{\mathrm{p}}{}^2)^2-(\omega^2-k^2)\,[\omega^2\omega_{\mathrm{L}}^2(\omega^2-k^2-\omega_{\mathrm{p}}{}^2) + \omega_{\mathrm{p}}{}^2(\boldsymbol{\omega}_{\mathrm{L}}\cdot\boldsymbol{k})^2] = 0, \tag{3.8}$$

where the Larmor-frequency vector $\omega_{\mathrm{L}}{}^a$ or $\boldsymbol{\omega}$ was defined in (1.14). (We sometimes write $\boldsymbol{\omega}_{\mathrm{L}}$ to emphasise that it is a three-vector in the rest frame of the electron fluid, just like $\boldsymbol{k}$.)

Equation (3.8) is the standard dispersion relation for a magnetized plasma (see, for example, Stix 1962). In accordance with a remark made above, it is unaffected by curvature or inhomogeneities of the background. (Heintzmann & Schrüfer (1977) have derived a dispersion relation for a cold plasma which differs from (3.8) by a term containing the fluid's vorticity. That term occurs since the authors assumed the vorticity to be of order λ^{-1}, rather than of order L^{-1} as we have assumed; compare our discussion in the previous section.)

If the definitions

$$\omega = -u^a \mathrm{D}_a S, \quad k_a = h_a{}^b \mathrm{D}_b S$$

are inserted into (3.8), the *eiconal equation* for S is obtained – in this case a first-order partial differential equation of degree eight. Equation (3.8) shows that if S is a solution then so is $-S$. We shall, therefore, without loss of generality, always take $\omega > 0$, as usual.

In the absence of a magnetic field and except at the plasma frequency (3.8) reduces to the well known dispersion relation

$$\omega = (\omega_{\mathrm{p}}^2 + k^2)^{\frac{1}{2}} \tag{3.9}$$

for phenomenological photons.

Restricting attention to frequencies such that $\omega^2 > \omega_{\mathrm{p}}^2 + \omega_{\mathrm{L}}^2$, and introducing the angle α between the magnetic field and the wavenormal,

$$k\omega_{\mathrm{L}}\cos\alpha = \boldsymbol{k}\cdot\boldsymbol{\omega}_{\mathrm{L}}, \tag{3.10}$$

we solve the dispersion relation (3.8) for k^2, obtaining

$$(k^2)_{\pm} = \omega^2 - \omega_{\mathrm{p}}^2 + \omega_{\mathrm{p}}^2(\omega_{\mathrm{L}}/\omega)\,F_{\pm}(\omega_{\mathrm{L}}/\omega, \alpha), \tag{3.11}$$

where

$$F_{\pm} := \frac{-\omega\omega_{\mathrm{L}}[\frac{1}{2}\omega^2 + (\frac{1}{2}\omega^2-\omega_{\mathrm{p}}^2)\cos^2\alpha] \mp \omega^2[(\omega^2-\omega_{\mathrm{p}}^2)^2\cos^2\alpha + \frac{1}{4}\omega_{\mathrm{L}}^2\omega^2\sin^4\alpha]^{\frac{1}{2}}}{\omega^2(\omega^2-\omega_{\mathrm{p}}^2-\omega_{\mathrm{L}}^2)+\omega_{\mathrm{p}}^2\omega_{\mathrm{L}}^2\cos^2\alpha}. \tag{3.12}$$

For a given, non-vanishing magnetic field $\boldsymbol{\omega}_{\mathrm{L}}$ and wavenormal $\boldsymbol{n}$, there are thus two propagating modes, which are called the *ordinary wave* (upper sign in (3.11), (3.12) and subsequent equations) and the *extraordinary wave* (lower sign), respectively. Each characteristic strip and each eiconal S belong to exactly one of these modes. Henceforth, k_+ and k_- denote the functions of ω and $\boldsymbol{n}$ defined by (3.11) and (3.12). Alternatively, one may consider ω_+ and ω_- as functions of $\boldsymbol{k}$.

From now on we shall be concerned with high-frequency waves only, i.e.

$$\omega^2 \gg \omega_{\mathrm{p}}^2 + \omega_{\mathrm{L}}^2, \tag{3.13}$$

and we shall write '$\approx$' to indicate approximations in which only the dominant terms in the small variables $\omega_{\mathrm{p}}/\omega$, $\omega_{\mathrm{L}}/\omega$ are retained. (We do not restrict the relative magnitudes of ω_{p} and ω_{L}.) Then, from (3.12),

$$F_{\pm} \approx -\tfrac{1}{2}(\omega_{\mathrm{L}}/\omega)(1+\cos^2\alpha) \mp [\cos^2\alpha + \tfrac{1}{4}(\omega_{\mathrm{L}}/\omega)^2 \sin^4\alpha]^{\frac{1}{2}}. \tag{3.14}$$

These functions depend strongly on α and obey $|F_{\pm}| \lesssim 1$.

The three terms in (3.11) satisfy, under the condition (3.13), the inequality $\omega^2 \gg \omega_{\mathrm{p}}^2 \gg \omega_{\mathrm{p}}^2(\omega_{\mathrm{L}}/\omega)F_{\pm}$. The vacuum term dominates the isotropic plasma term which, in turn, is larger than the anisotropic magnetic contribution.

It is essential that the approximation (3.14) holds uniformly for all angles α since in curved space–times or in an inhomogeneous plasma there are no plane waves, and in general the angle α will not be constant for a solution of the eiconal equation. In fact, for some astrophysical field configurations α varies over the full range $0 \leqslant \alpha \leqslant \pi$ along a single ray. (See, for example, figure 3.) A linearization of $F_{\pm}$ with respect to $\omega_{\mathrm{L}}/\omega$ is not possible uniformly in α.

From (3.11) and (3.14) we obtain the *phase velocities*

$$(v_{\mathrm{p}})_{\pm} \approx 1 + (\omega_{\mathrm{p}}^2/2\omega^2)[1-(\omega_{\mathrm{L}}/\omega)F_{\pm}] \tag{3.15}$$

and, equivalently, the *indices of refraction* $n_{\pm} = v_{\pm}^{-1}$ of the two waves. Note that $(v_{\mathrm{p}})_{+} > (v_{\mathrm{p}})_{-}$.

Rays

Next we determine the rays along which the amplitudes are transported. According to §4 of paper I the Hamiltonian $H(x^a, l_a)$, governing the characteristic strips from which the solutions of the eiconal equation can be constructed, can be taken to be the left-hand side of (3.8) with $\omega = -u^a(x)\,l_a$ and $k^2 = k^{ab}(x)\,l_a l_b$ inserted. If S is an eiconal, $H(x^a, S_{,b}) = 0$, then the associated rays are the integral curves of the transport vector field $T^a = \partial H(x^b, S_{,c})/\partial l_a$.

To calculate T^a we introduce, for given background fields u^a, ω_{L}^a, an orthonormal frame field $E_{(a)}$ with $E_{(0)}^a = u^a$ and $E_{(1)}^a = \omega_{\mathrm{L}}^{-1}\omega_{\mathrm{L}}^a$. With respect to its dual co-basis $E^{(a)}$, the wave-covector has the components $l_a = (-\omega, k_1, k_2, k_3)$. Computing $\partial H/\partial l_a$ from (3.8) and inserting the approximate solutions (3.11) and (3.14) of the dispersion relation we get

$$T_{\pm}^{a} \approx 4\omega_{\mathrm{p}}^2\omega_{\mathrm{L}}\omega^3\{\pm l_{\pm}^a[\cos^2\alpha + \tfrac{1}{4}(\omega_{\mathrm{L}}/\omega)^2\sin^4\alpha]^{\frac{1}{2}} - \tfrac{1}{2}(\omega_{\mathrm{p}}/\omega)^2\omega_{\mathrm{L}}^a\cos\alpha\}. \tag{3.16}$$

The *transport vectors* are thus almost collinear with $l^a_\pm$. The 'correction' in the direction of the magnetic field vanishes if $\alpha = \frac{1}{2}\pi$ and is always smaller than the leading term by a factor of order $\omega_L \omega_p/\omega^2$.

The spatial *ray-velocities* relative to the unperturbed plasma, which equal the *group velocities*, are given by $v^\lambda_g = T^\lambda/T^0$, i.e. according to (3.16), by

$$(\boldsymbol{v}_g)_\pm \approx \left[1-\frac{\omega_p^2}{2\omega^2}\left(1-\frac{\omega_L}{\omega}F_\pm\right)\right]\boldsymbol{n} \pm \frac{\omega_p^2}{2\omega^2}\cos\alpha\,\frac{\boldsymbol{\omega}_L}{\omega}\left[\cos^2\alpha+\tfrac{1}{4}\left(\frac{\omega_L}{\omega}\right)^2\sin^4\alpha\right]^{-\frac{1}{2}}.$$

Decomposing $\boldsymbol{\omega}_L$ into longitudinal and transverse parts, and using (3.14), one finds

$$(\boldsymbol{v}_g)_\pm \approx \left(1-\frac{\omega_p^2}{2\omega^2}\right)\boldsymbol{n} \pm \frac{\sin\alpha\cos\alpha}{[\cos^2\alpha+\frac{1}{4}(\omega_L/\omega)^2\sin^4\alpha]^{\frac{1}{2}}}\,\frac{\omega_p^2}{2\omega^2}\left(\frac{\boldsymbol{\omega}_L}{\omega}\right)_\perp, \tag{3.17}$$

where terms of order $(\omega_p/\omega)^2(\omega_L/\omega)^2$ have been neglected. The rays are, therefore, not exactly orthogonal to the wave-surfaces, except for $\alpha = 0$ and $\alpha = \frac{1}{2}\pi$; and to within the specified accuracy *the magnitudes of the two ray velocities,*

$$(v_g)_+ \approx (v_g)_- \approx 1-\omega_p^2/2\omega^2, \tag{3.18}$$

are direction-independent and equal to the ray speed in an unmagnetized plasma, in contrast to the phase speeds (3.15).

Equation (3.17) gives the ray velocities for both modes as functions of frequency and wavenormal. Inversion gives, to the same accuracy, the dependence of the two possible wavenormals on frequency and ray velocity:

$$\boldsymbol{n}_\pm \approx \left(1+\frac{\omega_p^2}{2\omega^2}\right)\boldsymbol{v}_g \mp \frac{\sin\alpha\cos\alpha}{[\cos^2\alpha+\frac{1}{4}(\omega_L/\omega)^2\sin^4\alpha]^{\frac{1}{2}}}\,\frac{\omega_p^2}{2\omega^2}\left(\frac{\boldsymbol{\omega}_L}{\omega}\right)_\perp. \tag{3.19}$$

In this equation α may be identified with the angle between the magnetic field and the ray, and $\perp$ may be taken to refer to the plane orthogonal to the ray.

The preceding results show that *for high frequencies, ordinary and extraordinary waves propagate approximately along the same rays with different phase speeds.* This fact is essential for the phenomenon of Faraday rotation to be discussed subsequently.

Let a ray $x^a(t)$ be parametrized such that $u_a\dot{x}^a = -1$. (This equation means that t coincides with proper time measured by clocks comoving with the unperturbed plasma, Einstein-synchronized along the ray.) Suppose that an ordinary and an extraordinary wave both travel along that ray. Then the rates of change of their respective phases along the ray are

$$\epsilon^{-1}\dot{S}_\pm = \dot{x}^a\epsilon^{-1}\nabla_a S_\pm$$

or, because of (2.9) and (3.2),

$$(u^a+v^a_g)\,l^\pm_a = -\omega+(\boldsymbol{k}\cdot\boldsymbol{v}_g)_\pm.$$

We define the *Faraday rotation rate* ω_F by

$$\omega_F := \tfrac{1}{2}\epsilon^{-1}(\dot{S}_- - \dot{S}_+) = \tfrac{1}{2}[(\boldsymbol{k}\cdot\boldsymbol{v}_g)_- - (\boldsymbol{k}\cdot\boldsymbol{v}_g)_+]$$

and obtain from (3.11), (3.14), and (3.17)

$$\omega_{\mathrm{F}} \approx \frac{\omega_{\mathrm{p}}^2\,\omega_{\mathrm{L}}}{2\omega^2}\left[\cos^2\alpha + \tfrac{1}{4}\left(\frac{\omega_{\mathrm{L}}}{\omega}\right)^2 \sin^4\alpha\right]^{\frac{1}{2}}, \tag{3.20}$$

which reduces in the most frequent case $\cos^2\alpha \gg \frac{1}{4}(\omega_{\mathrm{L}}/\omega)^2 \sin^1\alpha$ to the standard formula

$$\omega_{\mathrm{F}} \approx (\omega_{\mathrm{p}}^2\,\omega_{\mathrm{L}}/2\omega^2)\,|\cos\alpha|. \tag{3.21}$$

The interpretation of this angular velocity under the general circumstances considered here will be discussed in § 5.

As the preceding considerations indicate, one can calculate the rays in a magnetic plasma very precisely by neglecting the magnetic terms in the canonical equations, and afterwards apply (3.20) to compute the Faraday rotation.

We end this section with some remarks about the *rays in an unmagnetized plasma.* Then one can take the Hamiltonian

$$H = \tfrac{1}{2}(\omega_{\mathrm{p}}^{-2} g^{ab} l_a l_b + 1) \tag{3.22}$$

to calculate the characteristic strips and the rays, using the canonical equations together with the constraint $H = 0$. (It must be remembered that *any* function $H(x^a, l_b)$, such that $H = 0$ gives the dispersion relation, and $\partial H/\partial l_a$ does not vanish on $H = 0$, can be used as a Hamiltonian.) The form (3.22) of H shows that *the rays are the timelike geodesics associated with the metric* $\omega_{\mathrm{p}}^2\, g_{ab}$, inverse to $\omega_{\mathrm{p}}^{-2} g^{ab}$, conformal to the basic space–time metric; they are thus also characterized by the (parameter-independent) variational principle

$$\delta \int \omega_{\mathrm{p}}(-g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b)^{\frac{1}{2}} = 0. \tag{3.23}$$

Another possibility is to choose the Hamiltonian $H' = \frac{1}{2}(g^{ab} l_a l_b + \omega_{\mathrm{p}}^2)$ and to pass to a Langrian by Legendre transformation. One obtains

$$L(x^a, \dot{x}^b) = \tfrac{1}{2}(g_{ab}\,\dot{x}^a \dot{x}^b - \omega_{\mathrm{p}}^2),$$

to be combined with the constraint $g_{ab}\,\dot{x}^a\dot{x}^b = -\omega_{\mathrm{p}}^2$. This method covers also the vacuum case which is excluded, of course, in (3.23).

It is easy to derive, for a stationary space–time occupied by a stationary background plasma the worldlines of which are the timelike Killing orbits, a *Fermat principle* (Pham Mau Quan 1959, Synge 1964). If this principle is applied, for example, to the deflexion of radar waves by the combined influence of the Sun's corona and gravitational field, one recognizes that the total effective index of refraction is the *product* of that due to the plasma and that duc to the gravitational field. The latter is given, in lowest weak-field approximation, by

$$n_{\mathrm{grav}} = 1 - 2 \times (\text{Newtonian potential}).$$

(see, for example, Fock 1960.) More details and generalizations to magnetized plasmas will be published separately.

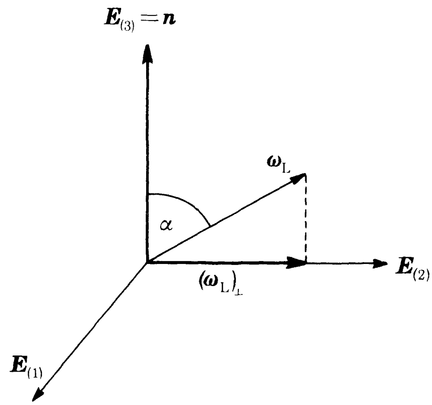

FIGURE 1. The space part of the adapted orthonormal frame.

4. Polarization states and Faraday rotation

If there is *no magnetic field* (3.9) holds, and because of (3.5) the longitudinal component of the potential vanishes in lowest order, $\underset{0}{A}_{\parallel} = 0$. Then (3.6) implies that $\underset{0}{\overset{\perp}{L}}{}^a{}_b = 0$, i.e. the polarization states at an event x^a and for a given wave vector l_a form a two-dimensional complex Hilbert space $\mathscr{H}(x,l) := \{\underset{0}{A}{}^a \mid u_a \underset{0}{A}{}^a = 0, l_a \underset{0}{A}{}^a = 0\}$ with the inner product $\langle \underset{0}{A}, \underset{0}{B} \rangle := \underset{0}{\bar{A}}{}_a \underset{0}{B}{}^a$, as in vacuum, although the rays are timelike. We shall show in the next section that any such state is preserved along a ray, i.e. the maps between these spaces $\mathscr{H}(x,l)$ which are defined by the transport equation are unitary except for a positive scale factor.

To discuss the *polarization states when there is a magnetic field* it is useful to work with an orthonormal frame adapted to the background plasma, the background magnetic field $\boldsymbol{\omega}_{\mathrm{L}}$ and a wave direction n as shown in figure 1. (If $\sin\alpha = 0$, $\boldsymbol{E}_{(1)}$ and $\boldsymbol{E}_{(2)}$ are taken to be an arbitrary orthonormal pair orthogonal to u^a and n^a, except that the tetrad field $E^b{}_{(a)}$ is assumed smooth everywhere.) With respect to this frame the spatial components of the tensors appearing in the polarization condition (3.6) are as follows:

$$\boldsymbol{k} = (0, 0, k), \boldsymbol{\omega}_{\mathrm{L}} = \omega_{\mathrm{L}}(0, \sin\alpha, \cos\alpha), \tag{4.1}$$

$$\left.\begin{aligned} &\omega_{\mathrm{L}}{}^{ij} = \omega_{\mathrm{L}} \begin{bmatrix} 0 & \cos\alpha & -\sin\alpha \\ -\cos\alpha & 0 & 0 \\ \sin\alpha & 0 & 0 \end{bmatrix}, \quad \overset{\perp}{\omega}_{\mathrm{L}}{}^{ij} = \cos\alpha\, \omega_{\mathrm{L}} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ &(\boldsymbol{\omega}_{\mathrm{L}} \wedge \boldsymbol{n})^i = -\omega_{\mathrm{L}}{}^i{}_a n^a = \omega_{\mathrm{L}}(\sin\alpha, 0, 0), \\ &\omega_{\mathrm{L}}{}^i{}_a n^a \omega_{\mathrm{L}}{}^j{}_b n^b = \sin^2\alpha\, \omega_{\mathrm{L}}^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned}\right\} \tag{4.2}$$

$$\underset{0}{\overset{\perp}{(L)}}_{\pm} = (-\omega)\begin{bmatrix} (\omega^2-\omega_{\mathrm{p}}^2)(k_{\pm}^2-\omega^2+\omega_{\mathrm{p}}^2)-(k_{\pm}^2-\omega^2)\,\omega_{\mathrm{L}}^2\sin\alpha^2, & \mathrm{i}(\omega^2-\omega_{\mathrm{p}}^2)(k_{\pm}^2-\omega^2) \\ -\mathrm{i}(\omega^2-\omega_{\mathrm{p}}^2)(k_{\pm}^2-\omega^2)(\omega_{\mathrm{L}}/\omega)\cos\alpha, & (\omega^2-\omega_{\mathrm{p}}^2)(k_{\pm}^2-\omega^2+\omega_{\mathrm{p}}^2) \end{bmatrix} \tag{4.3}$$

In (4.3) $k_{\pm}$ are the values given exactly in (3.11) and approximately in (3.14).

Since the matrices given in (4.3) have rank one for each mode, the zero-order, transverse amplitudes $\underset{0}{A}{}_{\perp}{}^{a}$ are determined, by the polarization condition $\underset{0}{\overset{\perp}{L}}{}^{a}{}_{b}\,\underset{0}{A}{}_{\perp}{}^{b}=0$, uniquely up to a scalar factor; the modes are non-degenerate. One obtains by means of (4.3) and (3.11)

$$[1-(\omega_{\mathrm{L}}/\omega)\,F_{\pm}]\cos\alpha(\underset{0}{A}_1)_{\pm} = \mathrm{i}F_{\pm}(\underset{0}{A}_2)_{\pm}, \tag{4.4}$$

and in the special case $\boldsymbol{n}\perp\boldsymbol{\omega}_{\mathrm{L}}$ for the extraordinary wave

$$(\underset{0}{A}_1)_{-} = 0. \tag{4.5}$$

The longitudinal component of $\underset{0}{A}{}^{a}$ is determined by the transverse components, via (3.5), which in adapted components gives

$$\begin{aligned}(\underset{0}{A}_{\parallel})_{\pm} &= \mathrm{i}\,\frac{1-(\omega_{\mathrm{L}}/\omega)\,F_{\pm}}{1-(\omega_{\mathrm{p}}/\omega)^2}\left(\frac{\omega_{\mathrm{p}}}{\omega}\right)^2\frac{\omega_{\mathrm{L}}}{\omega}\sin\alpha(\underset{0}{A}_1)_{\pm} \\ &\approx \mathrm{i}\left(\frac{\omega_{\mathrm{p}}}{\omega}\right)^2\frac{\omega_{\mathrm{L}}}{\omega}\sin\alpha(\underset{0}{A}_1)_{\pm}.\end{aligned} \tag{4.6}$$

Thus for high frequencies the amplitude is always very nearly transverse, as is well known.

The information contained in (4.4), (4.5) and (3.14) is summarized in figure 2.

As figure 2 and (4.4) show, the waves are almost circularly polarized except for the very narrow range of angles given by $\cos^2\alpha < (\omega_{\mathrm{L}}^2/4\omega^2)\sin^4\alpha$, i.e. $\boldsymbol{n}$ nearly orthogonal to $\boldsymbol{\omega}_{\mathrm{L}}$, in which there is elliptic and, for $\boldsymbol{n}_{\perp}\boldsymbol{\omega}_{\mathrm{L}}$, linear polarization. It is nevertheless important to have approximations valid for all angles since α may vary along a single ray over the full range $0 \leqslant \alpha \leqslant \pi$, so that the sequence of states indicated in the figure, with a change from right- to left-circular polarization, for example, is realized along one ray.

5. Transport laws for the amplitudes

Each solution of the dispersion relation (3.8) specifies a class of polarization states via (3.5) and (3.6). The data needed to determine a particular amplitude $\underset{0}{A}{}^{a}$, within this class, along a ray can be chosen arbitrarily at one point of the ray; they are then determined all along the ray by the transport equation corresponding to the dispersion branch considered.

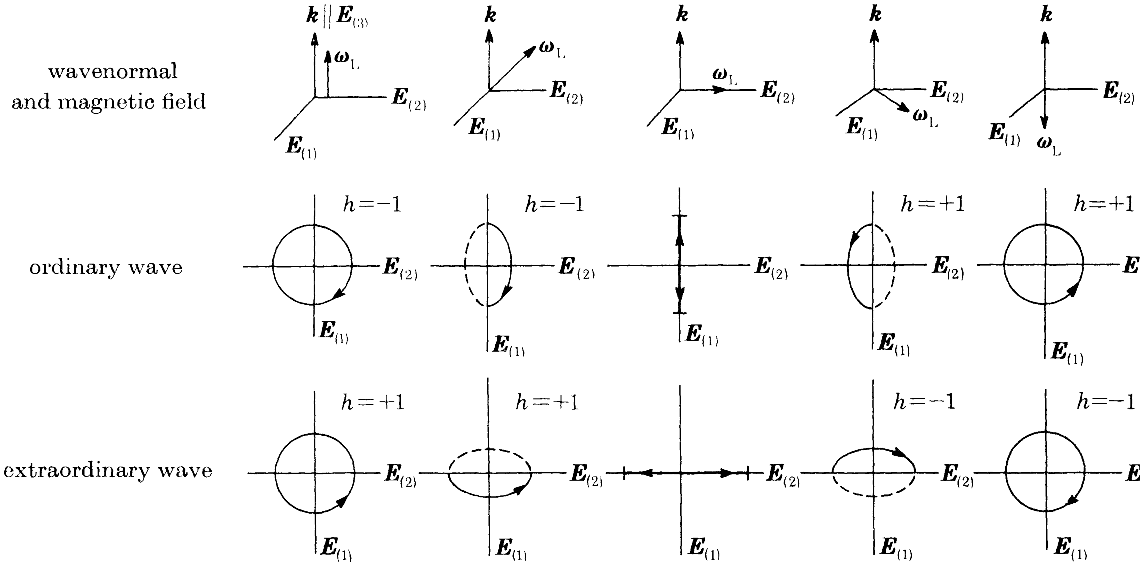

FIGURE 2. Dependence of the polarization state, i.e. the oscillation pattern of the electric field $\boldsymbol{E}$, on the angle α between the wavevector $\boldsymbol{k} \parallel \boldsymbol{n} = \vec{\boldsymbol{E}}_3$ and the background magnetic field $\boldsymbol{\omega}_{\mathrm{L}}$ for high-frequency waves ($\omega^2 \gg \omega_{\mathrm{p}}^2 + \omega_{\mathrm{L}}^2$). The ordinary (extraordinary) wave is characterized by the property that $\boldsymbol{E}$ rotates in the same (opposite) sense in which positively charged particles gyrate. The ray direction is contained in the $\boldsymbol{E}_{(2)}$, $\boldsymbol{E}_{(3)}$-plane and very close to the wavenormal $\boldsymbol{n}$. The polarization coefficient $\mathrm{i}\underset{0}{A_1}/\underset{0}{A_2}$, the modulus of which is the ratio of the axes of the polarization ellipse (see, for example, Ginzburg 1964, Breuer 1975), can be read off from (4.4). The helicity of the wave is denoted by h. The phase speed of the ordinary wave is larger than the one of the extraordinary wave.

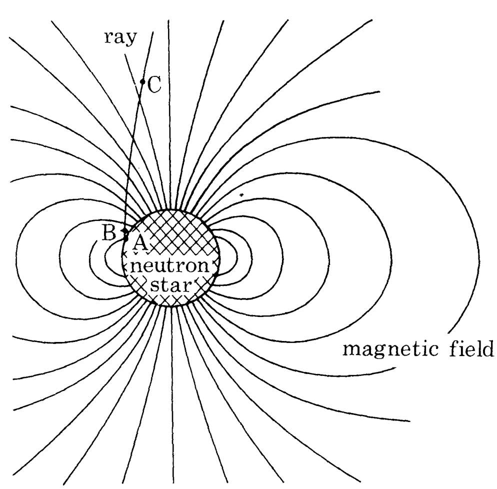

FIGURE 3. Variation of polarization state with changing relative orientation of ray and magnetic field directions. Along a ray, the state of polarization may vary from right-circular (near (A)) via linear (at B) to left-circular (near C) polarization.

The transport equations for the lowest-order amplitudes result from the terms of order ϵ of (2.5), once the expansion (2.6) has been inserted. These terms give

$$\underset{0}{L}^{ab}\underset{1}{A}_b + \underset{1}{L}^{ab}\underset{0}{A}_b = 0, \tag{5.1}$$

where $\underset{0}{L}^{ab}$ is the tensor defined in (3.1) and $\underset{1}{L}^{ab}$ is the following first-order differential operator:

$$\begin{aligned}\underset{1}{L}^{ab} = {} & 2(\omega h^{ab} + \mathrm{i}\omega_{\mathrm{L}}{}^{ab})(\mathrm{D}_l + \underset{0}{\hat{\theta}}) + (k^a k^b - [l^2 + \omega_{\mathrm{p}}^2]\, h^{ab})\,\mathrm{D}_u \\ & - (\omega h^{ac} + \mathrm{i}\omega_{\mathrm{L}}{}^{ac})(k^b \mathrm{D}_c + k_c \mathrm{D}^b) \\ & - (\omega h^{ac} + \mathrm{i}\omega_{\mathrm{L}}{}^{ac})\,\mathrm{D}_c k^b - h^{ab}\mathrm{D}_u l^2 + k^a \mathrm{D}_u k^b \\ & + \omega_{\mathrm{p}}^2(\underset{0}{\omega}^{ab} - \underset{0}{\theta}^{ab}) - l^2(\underset{0}{\omega}^{ab} + \underset{0}{\theta}^{ab} + \underset{0}{\theta} h^{ab}) \\ & + (h^{ac}\mathrm{D}_u l_c + [\underset{0}{\omega}^{ac} + \underset{0}{\theta}^{ac}]\, k_c + \underset{0}{\theta} k^a - \omega(e/m)\underset{0}{E}^a)\, k^b. \end{aligned} \tag{5.2}$$

Here we have put

$$\mathrm{D}_l := l^a \mathrm{D}_a, \quad \mathrm{D}_u := u^a \mathrm{D}_a, \quad \underset{0}{\hat{\theta}} := \tfrac{1}{2}\mathrm{D}_a l^a. \tag{5.3}$$

The other symbols in (5.2) have been defined in the introduction and in § 2. (Symbols like $\mathrm{D}_c k^b$ denote functions, not differential operators.)

We first consider an *unmagnetized plasma*, $\omega_{\mathrm{L}} = 0$. Then $\underset{0}{A}^a$ is transverse, $l^2 + \omega_{\mathrm{p}}^2 = 0$, and $\underset{0}{L}^{ab} = -\omega k^a k^b$, whence (5.1) implies $p^a{}_b \underset{1}{L}^{bc} \underset{0}{A}_{\perp c} = 0$. To obtain this equation explicitly we use (5.2) and the consequence $\mathrm{D}_u \omega_{\mathrm{p}}^2 = -\omega_{\mathrm{p}}^2 \underset{0}{\theta}$ of the electron-number conservation law. This gives the *transport law*

$$\{p^a{}_b(\nabla_l + \hat{\theta}) + (\omega_{\mathrm{p}}^2/\omega)\, \overset{\perp}{\omega}{}^a{}_b\}\, \underset{0}{A}_\perp{}^b = 0, \tag{5.4}$$

in which we have used the original expressions $\nabla_l, \omega^a{}_b$ and $\hat{\theta} := \tfrac{1}{2}\nabla_a l^a$ instead of their dimensionless counterparts $\mathrm{D}_l, \underset{0}{\omega}{}^a{}_b$ and $\underset{0}{\hat{\theta}}$. The derivative in (5.4) acts in the ray direction l^a, as expected.

Equation (5.4) defines a vector-space isomorphism from the space $\mathscr{H}_{\mathrm{p}}$ of transverse amplitudes at p onto that at q, $\mathscr{H}_q$, for any two events p, q on a ray. This isomorphism maps real vectors into real ones. The Hilbert inner product $\langle \underset{0}{A}, \underset{0}{B} \rangle$ (defined in § 4) of two solutions of (5.4) changes along a ray according to the conservation law

$$(\nabla_l + 2\hat{\theta})\langle \underset{0}{A}, \underset{0}{B} \rangle = \nabla_a(\langle \underset{0}{A}, \underset{0}{B} \rangle\, l^a) = 0, \tag{5.5}$$

and the same equation holds for the Euclidean inner product $\underset{0}{A}_{\perp a} \underset{0}{B}{}^a_\perp$. Hence, the isomorphism from $\mathscr{H}_{\mathrm{p}}$ to $\mathscr{H}_q$ is unitary or orthogonal, respectively, up to a positive factor, with respect to these inner products. These facts imply that *the transport preserves linear, circular, elliptic polarization, helicity and ellipticity.*

Equation (5.5) implies in particular

$$\nabla_a(\|\underset{0}{A}\|^2 l^a) = 0 \tag{5.6}$$

for the Hilbert norm $\|\underset{0}{A}\|$ of $\underset{0}{A}{}_{\perp}{}^a$. This equation can be interpreted as the law of *conservation of a quasiphoton number*, for according to (2.13) $\|\underset{0}{A}\|^2$ is related to the time average of the squared electric field of the wave by $\overline{\underset{0}{E}{}_a \underset{0}{E}{}^a} = \frac{1}{2}\omega^2\|\underset{0}{A}\|^2$, whence one may consider $\frac{1}{2}\omega\|\underset{0}{A}\|^2$ as the quasiphoton number density in the electron fluid's rest frame (except for the factor $\hbar^{-1}$), $\frac{1}{2}\omega_{\mathrm{p}}\|\underset{0}{A}\|^2$ as the proper number density of the quasiphoton stream, and $N^a = \frac{1}{2}\|\underset{0}{A}\|^2 l^a$ as the quasiphoton four-current density. Equation (5.6) gives a simple and useful relation if integrated over a narrow bundle of rays. It is also easy to combine (5.6) with the equation of motion for the quasi-photon four-momentum l^a derived in § 3 and thus to obtain the formal energy–momentum conservation law:

$$\bar{\nabla}_b(\omega_{\mathrm{p}}^{-6}\|\underset{0}{A}\|^2 l^a l^b) = 0, \tag{5.7}$$

where $\bar{\nabla}$ denotes the covariant derivative associated with the 'plasma metric' $\bar{g}_{ab} := \omega_{\mathrm{p}}^2 g_{ab}$ considered previously.

To interpret the transport of *directions* implied by (5.4) we introduce a real, orthonormal, quasiparallel basis of transverse vectors along a ray. (For the definition and properties of quasiparallel transport see Appendix B.) We then find that the direction of a real solution $\underset{0}{A}{}_{\perp}{}^a$ of (5.4) rotates, relative to quasiparallel axes and with respect to electron proper time, with the angular velocity $-(\omega_{\mathrm{p}}/\omega)^2\,\omega_{\mathrm{e}\parallel}\boldsymbol{n}$. Here, $\omega_{\mathrm{e}\parallel} := \frac{1}{2}\eta_{abcd}n^a u^b \omega^{cd}$ is the component of the vorticity of the electron fluid in the ray direction. (This rotation is counter-clockwise if one looks in the direction $\boldsymbol{n}$ of the wave, provided $\omega_{\mathrm{e}\parallel} > 0$.)

Once (5.4) has been solved (5.1) determines $\underset{1}{A}{}_{\parallel}$, the longitudinal component of the first-order-amplitude $\underset{1}{A}{}^a$. This, then, exhausts the information contained in the first approximation.

In the case of a *magnetized plasma* the matrix $\underset{0}{L}{}^{ab}$, restricted to one of the two high-frequency modes, has rank 2 as was shown in § 3. Multiplication of (5.1) with a left null vector N_a (which is unique up to a factor of $\underset{0}{L}{}^{ab}$) gives

$$N_a \underset{1}{L}{}^{ab} \underset{0}{A}{}_b = 0. \tag{5.8}$$

N_a can be decomposed like $\underset{0}{A}{}_a$ as $N_a = N_{\perp a} + N_{\parallel} n_a$;

one finds
$$(\omega^2 - \omega_{\mathrm{p}}^2) N_{\parallel} = \mathrm{i}\omega\omega_{\mathrm{L}}{}^a{}_b n_a N_{\perp}{}^b \tag{5.9}$$

and

$$N_{\perp a} \underset{0}{\overset{\perp}{L}}{}^{ab} = 0. \tag{5.10}$$

Let $(R_\perp{}^a)_\pm$ be arbitrary, but fixed, non-vanishing solutions of the polarization conditions $\overset{\perp}{\underset{0}{L}}{}^a{}_b R_\perp{}^b = 0$ for the two modes. Since $\overset{\perp}{\underset{0}{L}}{}^{ab}$ is Hermitian we can and shall satisfy (5.10) by taking

$$N_\perp{}^a = \overline{R_\perp{}^a} \tag{5.11}$$

(the bar denotes complex conjugate). The general amplitudes are then

$$(A^a)_\pm = v_\pm (R^a)_\pm, \tag{5.12}$$

and the complex scalars $v_\pm$ have to obey the transport equations

$$N_a \underset{1}{L}{}^{ab}(R_b v) = 0 \tag{5.13}$$

in which we have suppressed the indices $\pm$. The general theory of paper I guarantees that the derivative operators $(N_a R_b \underset{1}{L}{}^{ab})$ act in the ray directions $T^a_\pm$ considered in § 3.

The explicit, exact equations (5.13) – found by taking the $R_\perp{}^a$ from (4.4) and (4.5), $R_\parallel$ from (4.6), $N_\perp{}^a$ from (5.11) and $N_\parallel$ from (5.9) and inserting into (5.2) – are lengthy and do not seem enlightening; therefore we shall not display them. Instead, we only write down that approximate version of (5.13) in which all terms involving the small parameter $\omega_{\rm L}/\omega$ are neglected. It is equivalent to

$$\overline{\underset{0}{A}{}_\perp{}^a} \nabla_l \underset{0}{A}{}_{\perp a} + \hat{\theta} \, \|\underset{0}{A}{}_\perp\|^2 + (\omega_{\rm p}^2/\omega) \, \overset{\perp}{\omega}_{ab} \overline{\underset{0}{A}{}_\perp{}^a} \underset{0}{A}{}_\perp{}^b = 0. \tag{5.14}$$

For amplitudes corresponding to (nearly) circular polarization this equation contains, not surprisingly, the same information as (5.4). Thus if in the case $\omega_{\rm L} \neq 0$, those short and rare parts of rays where the waves are not nearly circularly polarized (i.e. where $\boldsymbol{n}$ is perpendicular to $\boldsymbol{\omega}_{\rm L}$) are excluded, the amplitudes propagate approximately according to the law (5.4). If this result is combined with the analogous statement concerning rays, the following conclusion emerges:

In a magnetized plasma and for high frequencies $(\omega^2 \gg \omega_{\rm p}^2 + \omega_{\rm L}^2)$ the ordinary and the extraordinary waves have rays which are approximately timelike geodesics of $\bar{g}_{ab} = \omega_{\rm p}^2 g_{ab}$, and their circularly polarized amplitudes are transported approximately according to (5.4). The phases of the ordinary and extraordinary waves, however, change differently along the rays, according to (3.20).

This statement permits us to generalize the theory of *Faraday rotation* to an inhomogeneous plasma in curved space–time. If a wave enters a magnetized region of a plasma linearly polarized it leaves that region again linearly polarized. However, owing to the different phase speeds of its circularly polarized components in the intervening region and possibly because of the rotation of the electron fluid, the direction of polarization will have changed, relative to quasiparallely transported axes (see Appendix B), by the angle

$$\Delta\alpha = \int \left(\frac{\omega_{\rm p}}{\omega}\right)^2 (\tfrac{1}{2}\omega_{{\rm L}\parallel} - \omega_{{\rm e}\parallel}) \, {\rm d}t, \tag{5.15}$$

where the integral is to be taken along the ray, and t denotes electron proper time (§ 3). (The first part, which usually dominates, corresponds to a clockwise rotation if one looks in the ray direction $\boldsymbol{n}$, and $\omega_{L\parallel} > 0$.) This description of Faraday rotation is valid only as long as the exact rays belonging to the ordinary and the extraordinary waves, respectively, deviate by less than the coherence length of the radiation. Only then can the left- and right-circularly polarized components superpose again when leaving the magnetized plasma. Relativistic (Doppler and gravitational) frequency changes along a ray are, of course, included in (5.15); but it must be remembered that t refers to the plasma, not to an arbitrarily moving observer. The special case of rays along which the helicity switches (see figure 3) will not be treated here, although the tools for studying this case have been provided. In principle, though hardly in practice, one could extend the approximation to higher order in ϵ.

6. Summary and discussion

In this and the preceding paper (Breuer & Ehlers 1980) we have treated the propagation of high-frequency electromagnetic waves through an inhomogeneous, in general non-stationary, moving plasma in a general, curved space–time by means of a J.W.K.B. method adapted to this purpose. The aim was to provide a formal approximation scheme applicable also to similar wave propagation problems *and* to justify it mathematically. To do this we have kept the plasma model as simple as possible and have focused our attention on the essential mathematical structure and its physical meaning. More complicated, and therefore more realistic, models could, however, be included without major difficulties. (For example, a pressure could be taken into account. In fact, some of the basic arguments become simpler in that case than in the 'degenerate' pressure-free model since the corresponding system of differential equations is then symmetric-hyperbolic in the sense of Friedrichs. The equations, however, become formally even more complicated than they are without a pressure.)

For a two-component, cold (pressureless) plasma we have shown that the Cauchy problem for the magnetohydrodynamic system of differential equations on an arbitrary space–time possesses locally a unique solution for appropriate initial data. Also, this system was shown to be linearization stable at any of its solutions.

These properties of the basic equations imply that the associated evolution equations for linear perturbations also have unique solutions for initial data satisfying the linearly perturbed constraints. Because of the linearization stability of the original system, the sum of a background solution and an 'infinitesimal' perturbation indeed approximates a solution of the full system. Moreover, the wave equation for a suitably gauged four-potential of the electromagnetic field perturbation has locally unique solutions for unconstrained initial data, although it is not strictly hyperbolic.

We can assert that the nth-order field, computed by means of the approximation algorithm applied to the perturbation equation, which is an asymptotic solution of order n in the small parameter ϵ = wavelength of the perturbation–length scale of the background, actually approximates a solution provided the inverse of a certain differential operator (given in (1.13) of paper I) is small in a certain sense. We have not established this property for the operator in question, however; this remains to be done.

Using the two-scale perturbation scheme, we obtained the dispersion relation, the rays, the corresponding polarization states and the transport equations for high-frequency electromagnetic waves, and by means of them considered the appropriately generalized Faraday rotation.

An important physical question not discussed in these papers concerns the applicability of the method to high-frequency waves in low-density plasmas when the inequality $n^{-\frac{1}{3}} \ll \lambda$ does not hold. For densities of typical accretion gas, for example, the method appears to be justified up to optical frequencies (only). It seems that certain properties of X-rays emerging from accretion processes onto compact binaries – to which the theory was aimed at initially – can still be explained by means of a formalism which treats the plasma as a continuum even at these wavelengths (see, for example, Chanan *et al.* 1979), but a justification of this procedure, to our knowledge, is lacking.

References

Anile, A. M. & Pantano, P. 1977 *Phys. Lett.* A **61**, 215.

Anile, A. M. & Pantano, P. 1979 *J. math. Phys.* **20**, 177.

Bičák, J. & Hadrava, P. 1975 *Astron. Astrophys.* **44**, 389.

Breuer, R. A. 1975 *Gravitational perturbation theory and synchrotron radiation*; Lecture notes in physics, vol. 44. Berlin, Heidelberg, New York: Springer-Verlag.

Breuer, R. A. & Ehlers, J. 1980 *Proc. R. Soc. Lond.* A **370**, 389–406. (Paper I.)

Chanan, G. A., Novick, R. & Silver, E. H. 1979 *Astrophys. J.* **228**, L71.

Courant, R. & Hilbert, D. 1962 *Methods of mathematical physics*, pp. 636–640. New York, London: Interscience Publications.

Duistermaat, J. J. 1974 *Communs pure. appl. Math.* **27**, 207.

Fock, V. 1960 *Theorie von Raum, Zeit und Gravitation*. Berlin: AK-Verl.

Ginzburg, V. L. 1964 *The propagation of electromagnetic waves in plasmas*. Oxford, New York: Pergamon Press.

Heintzmann, H. & Schrüfer, E. 1977 *Phys. Lett.* A **60**, 79.

Lassen, H. 1947 *Annls Phys.* **1**, 415.

Madore, J. 1974 *Communs math. Phys.* **38**, 103.

Mészáros, P. 1978 *Astron. Astrophys.* **63**, L19.

Mészáros, P., Meyer, F. & Pringle, J. E. 1977 *Nature, Lond.* **268**, 420.

Misner, C. W., Thorne, K. S. & Wheeler, J. A. 1973 *Gravitation*. San Francisco: Freeman.

Pham, M. Q. 1962 *Colloques int. Cent. natn. Rech. Scient.* **61**, 165–171.

Stix, T. H. 1962 *The theory of plasma waves*. New York: McGraw-Hill.

Synge, J. L. 1964 *Relativity: the general theory*, ch. XI. Amsterdam: North Holland.

Appendix A. Gauge conditions and asymptotic expansions

It will be shown here that if a one-parameter family of complex, closed bivector fields, $F(x,\epsilon)$, $\mathrm{d}F = 0$, admits an asymptotic expansion

$$F(x,\epsilon) \sim \mathrm{e}^{(\mathrm{i}/\epsilon)S} \sum_{n=-1}^{\infty} \left(\frac{\epsilon}{\mathrm{i}}\right)^n \underset{n}{F}(x), \tag{A 1}$$

then there exists a four-potential $A(x,\epsilon)$, $F = \mathrm{d}A$, which also admits an expansion

$$A(x,\epsilon) \sim \mathrm{e}^{(\mathrm{i}/\epsilon)S} \sum_{n=0}^{\infty} \left(\frac{\epsilon}{\mathrm{i}}\right)^n \underset{n}{A}(x). \tag{A 2}$$

If U is a vector field which is nowhere tangent to the hypersurfaces $S =$ constant, then A can be chosen to obey the Landau gauge condition

$$\langle A, U\rangle = 0. \tag{A 3}$$

The corresponding amplitudes $\underset{n}{A}$ which, of course, satisfy

$$\langle \underset{n}{A}, U\rangle = 0, \tag{A 4}$$

are then uniquely determined by F and S. Note that these remarks do not require a metric or connection, or any field equations for F besides the 'homogeneous-half' of Maxwell's equation, $\mathrm{d}F = 0$. On the other hand, if there is a metric, then the possibility of imposing the Lorentz gauge condition

$$\nabla_a A^a = 0 \tag{A 5}$$

on the potential (A 2) depends on additional properties of S, $\underset{-1}{F}$ and F which we shall not discuss exhaustively, since that is not necessary for our purposes. We see at any rate that the radiation gauge (A 3) *is* compatible with (A 2), and can be satisfied trivially.

To establish these claims, verify that (A 1) and $\mathrm{d}F = 0$ imply

$$\mathrm{d}S \wedge \underset{-1}{F} = 0, \quad \mathrm{d}S \wedge \underset{n+1}{F} + \mathrm{d}\underset{n}{F} = 0 \quad (n = -1, 0, 1, \ldots).$$

Therefore, there exists $\underset{0}{A}$ such that $\underset{-1}{F} = \mathrm{d}S \wedge \underset{0}{A}$. Hence $0 = \mathrm{d}S \wedge \underset{0}{F} + \mathrm{d}\underset{-1}{F} = \mathrm{d}S \wedge (\underset{0}{F} - \mathrm{d}\underset{0}{A})$, whence there is an $\underset{1}{A}$ such that $\underset{0}{F} = \mathrm{d}\underset{0}{A} + \mathrm{d}S \wedge \underset{1}{A}$, etc. With the $\underset{n}{A}$ so chosen, form (A 2). It is easily checked that $F = \mathrm{d}A$. This establishes the existence of an A obeying (A 2).

The construction of the sequence $\{\underset{n}{A}\}$ shows that the following 'gauge transformations' are permitted:

$$\underset{0}{A} \to \underset{0}{A} + \underset{0}{\Lambda}\,\mathrm{d}S, \quad \underset{n}{A} \to \underset{n}{A} + \underset{n}{\Lambda}\,\mathrm{d}S + \mathrm{d}\underset{n-1}{\Lambda} \quad (n = 1, 2, \ldots) \tag{A 6}$$

where the $\underset{n}{A}$ are arbitrary functions. This change corresponds to $A \rightarrow A + \mathrm{d}\Lambda$, where $\Lambda \sim \mathrm{e}^{(\mathrm{i}/\epsilon)S} \sum_0^\infty (\epsilon/\mathrm{i})^n \underset{n}{\Lambda}$. Obviously, if $\langle \mathrm{d}S, U \rangle \neq 0$, then (A 3) or, equivalently, (A 4) for all n can be satisfied by taking a suitable Λ, and the $\underset{n}{\Lambda}$ are thereby uniquely determined.

Let us now turn to the Lorentz gauge, (A 5). Equation (A 2) will satisfy it if and only if

$$\underset{0}{A}^a S_{,a} = 0 \quad \text{and} \quad \underset{n}{A}^a S_{,a} + \nabla_a \underset{n-1}{A}^a = 0, \quad n = 1, 2, \ldots . \tag{A 7}$$

The left-hand sides of these equations change under a gauge transformation (A 6) by the terms

$$\underset{0}{\Lambda} S_{,a} S'^a, \ \underset{n-2}{\Lambda}{}^{;a}{}_a + \underset{n-1}{\Lambda} S^{;a}{}_a + \underset{n}{\Lambda} S_{,a} S'^a,$$

($\underset{-1}{\Lambda} := 0$). Hence, if $S_{,a} S'^a \neq 0$, i.e. if the phase velocity is not 1, then one *can* satisfy (A 7) by a unique choice of the $\underset{n}{\Lambda}$. Suppose, however, that $S_{,a} S'^a = 0$. Then $\underset{0}{A}^a S_{,a}$ is gauge invariant and vanishes exactly if $\underset{-1}{F} = \mathrm{d}S \wedge \underset{0}{A}$ is a null bivector. If, in addition, $S^{;a}{}_a = 0$ – non-diverging rays – then also $A^a S_{,a} + \nabla_a \underset{0}{A}^a$ is gauge invariant. These remarks substantiate our claims.

Appendix B. Quasiparallel transport of vectors

To interpret geometrically the transport of amplitudes along rays in curved space–times it is useful to define a quasiparallel transport of vectors along curves that preserves conditions such as $p^a{}_b \underset{0}{A}^b = \underset{0}{A}^a$, which requires $\underset{0}{A}^a$ to be contained in a two-dimensional subspace of the tangent space. We shall here define such a transport generally and then specialize it to the case needed in this paper.

Let M be an n-manifold, ∇ a linear connection on M. Suppose $u \rightarrow x^a(u)$ represents a curve in M with tangent $T^a = \dot{x}^a$. Moreover, let $T_{x(\lambda)} = P_\lambda \oplus Q_\lambda$ be a direct decomposition of the tangent space at $x^a(\lambda)$ such that $p^a{}_b$ projects onto P_λ along Q_λ, and $q^a{}_b$ projects onto Q_λ along P_λ, with $p^a{}_b$, $q^a{}_b$ depending smoothly on λ. Then

$$\left.\begin{aligned} p^a{}_b p^b{}_c &= p^a{}_c, \quad q^a{}_b q^b{}_c = q^a{}_c, \\ p^a{}_b q^b{}_c &= 0, \quad p^a{}_b + q^a{}_b = \delta^a{}_b \end{aligned}\right\} \tag{B 1}$$

and

$$\nabla_T q^a{}_c = q^a{}_b \nabla_T q^b{}_c + q^b{}_c \nabla_T q^a{}_b. \tag{B 2}$$

A vector X^a is contained in P_λ if and only if one of the equivalent equations

$$p^a{}_b X^b = X^a, \quad q^a{}_b X^b = 0 \tag{B 3}$$

holds.

We shall say that $\lambda \to X^a(\lambda)$ defined on the curve $\lambda \to X^a(\lambda)$ and contained in P, is *quasiparallel* if

$$p^a{}_b \nabla_T X^b = 0. \tag{B 4}$$

Suppose this is the case. Then, by (B 1) and (B 3)

$$\nabla_T X^a = -X^b \nabla_T q^a{}_b. \tag{B 5}$$

This equation has a unique solution for any initial value $\underset{0}{X}{}^a$ at $x^a(0)$. Equation (B 5) implies, by (B 2),

$$\nabla_T (q^a{}_b X^b) = (\nabla_T q^a{}_b)(q^b{}_c X^c). \tag{B 6}$$

Hence, if the initial value is in P_0, then the solution of (B 5) is in P_λ all along the curve, because of uniqueness of solutions of (B 6). Moreover, that solution also obeys (B 4). That is, *a vector $\underset{0}{X}{}^a$ in P_0 determines a unique quasiparallel field $\lambda \to X^a(\lambda)$ on the curve.* Obviously quasiparallel transport maps P_{λ_1} *isomorphically* onto P_{λ_2} for any pair (λ_1, λ_2).

Let us now assume that ∇ is the *Levi–Civita connection* of a pseudo-Riemannian metric g_{ab} and that $p^a{}_b$ is an *orthogonal* projection:

$$p_{[ab]} = 0. \tag{B 7}$$

Then *quasiparallel transport preserves inner products*:

$$\begin{aligned}\nabla_T (g_{ab} X^a Y^b) &= g_{ab}(X^a \nabla_T Y^b + Y^a \nabla_T X^b)\\ &= p_{ab}(X^a \nabla_T Y^b + Y^a \nabla_T X^b).\end{aligned}$$

This case applies, in particular, to space–time (M, g_{ab}) and the projector $p^a{}_b$ defined in (3.3). Clearly, quasiparallel transport can be applied to complex vectors, too, and it preserves also the Hilbert inner product defined in § 4. If the transport equation (5.4) is written in components with respect to a quasiparallel, orthonormal, transverse basis and if the ray is parametrized in terms of the electron proper time defined in connection with (3.20), it reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\underset{0}{A}_1\\ \underset{0}{A}_2\end{bmatrix} = \begin{bmatrix}-\hat{\theta}/\omega & (\omega_\mathrm{p}/\omega)^2\,\omega_3\\ -(\omega_\mathrm{p}/\omega)^2\,\omega_3 & -\hat{\theta}/\omega\end{bmatrix}\begin{bmatrix}\underset{0}{A}_1\\ \underset{0}{A}_2\end{bmatrix},$$

This equation justifies the interpretation of (5.4) given in § 5.

http://www.jstor.org

LINKED CITATIONS

- Page 1 of 1 -

You have printed the following article:

Propagation of High-Frequency Electromagnetic Waves Through a Magnetized Plasma in Curved Spaces-Time. II. Application of the Asymptotic Approximation

R. A. Breuer; J. Ehlers

Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, Vol. 374, No. 1756. (Jan. 7, 1981), pp. 65-86.

Stable URL:

http://links.jstor.org/sici?sici=0080-4630%2819810107%29374%3A1756%3C65%3APOHEWT%3E2.0.CO%3B2-8

This article references the following linked citations. If you are trying to access articles from an off-campus location, you may be required to first logon via your library web site to access JSTOR. Please visit your library's website or contact a librarian to learn about options for remote access to JSTOR.

[Footnotes]

[†] **Propagation of High-Frequency Electromagnetic Waves Through a Magnetized Plasma in Curved Space-Time. I**

R. A. Breuer; J. Ehlers

Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, Vol. 370, No. 1742. (Mar. 28, 1980), pp. 389-406.

Stable URL:

http://links.jstor.org/sici?sici=0080-4630%2819800328%29370%3A1742%3C389%3APOHEWT%3E2.0.CO%3B2-E

References

Propagation of High-Frequency Electromagnetic Waves Through a Magnetized Plasma in Curved Space-Time. I

R. A. Breuer; J. Ehlers

Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, Vol. 370, No. 1742. (Mar. 28, 1980), pp. 389-406.

Stable URL:

http://links.jstor.org/sici?sici=0080-4630%2819800328%29370%3A1742%3C389%3APOHEWT%3E2.0.CO%3B2-E

NOTE: *The reference numbering from the original has been maintained in this citation list.*