

FOUR-DIMENSIONAL HETEROTIC STRINGS AND CONFORMAL FIELD THEORY

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Received 10 July 1987

The techniques of (super) conformal field theory are applied to 4-dimensional heterotic string theories. We discuss certain aspects of 4-dimensional strings in the framework of the bosonic lattice approach such as the realization of superconformal symmetry, character valued partition functions, construction of vertex operators and ghost picture changing. As an application we compute all possible 3- and 4-point tree amplitudes of the massless fields and derive from them the low energy effective action of the massless modes. Some effects for the massless spectrum due to one-loop string effects are also mentioned.

1. Introduction

The greatest challenge for string theory [1] as a unifying framework for quantum gravity and particle interactions is to make contact with experiment. Naturally, string theories which are formulated [2–10] directly in four dimensions are most closely related to this problem. In this context, all approaches known so far have in common that they describe two independent (“left” and “right”) two-dimensional (super) conformal field theories [11–16]. For the case of the phenomenologically most promising chiral heterotic string theories, the left-moving sector consists of 26 bosonic fields; the right-moving part possesses superconformal symmetry and contains 10 fermionic as well as bosonic fields*. The number of space-time dimensions d depends on how the Lorentz rotation group $SO(d)$ is generated by the

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* In refs. 17–20 it was discussed how to obtain the 10-dimensional fermionic string from the purely 26-dimensional bosonic model.

two-dimensional fields such that d left- and right-moving bosonic coordinates cannot be treated independently (like the remaining ones).

In this paper we will follow the formulation of 4-dimensional heterotic string theories in the covariant lattice approach [21–24]. Bosonizing the Neveu-Schwarz-Ramond fermions ψ^μ as well as the superconformal ghosts, modular invariant d -dimensional heterotic string theories are characterized by odd self-dual lorentzian lattices $\Gamma_{26-d; 15-d, 1}$. In this way, the correct averaging over different spin structures together with correct spin-statistics relations in the fermionic description [2, 25, 26] is automatically ensured. However, not only the proof of consistency (like modular invariance [22]) and the construction [3] of the spectrum of 4-dimensional string theories turns out to be very convenient in the covariant lattice approach; this scheme becomes especially attractive when calculating string scattering amplitudes directly in four dimensions without reference to any compactification scheme by using the techniques of (super) conformal field theory, since it is most closely related to the construction of covariant vertex operators of ref. [15]. The objective of this paper is to demonstrate this.

Because we are formulating the 4-dimensional string theory entirely in terms of bosonic self-dual lattices, all vertex operators are basically given by exponentials of free bosonic fields which permit explicit calculation of all correlation functions. Unlike the case of a “general asymmetric orbifold” [5] twist operators [27, 28] which are in general not known explicitly, are not needed in our approach.

Thus, applying the superconformal field theory, we compute all three- and four-point scattering amplitudes of the massless 4-dimensional fields. Some of these amplitudes are identical to those in 10 dimensions like graviton- and gauge boson scattering. However others which involve massless scalars and fermions were not discussed before. As a result, we derive the string tree level effective action of quantum gravity coupled to fermions, scalars and Yang-Mills gauge bosons in zeroth and first order in $\sqrt{2\alpha'}$. At one-loop string perturbation theory the amplitudes get additional contributions, and we discuss some of the modifications.

The paper is organized as follows. In sect. 2 we recall the main features of 4-dimensional heterotic string theories in the covariant lattice approach. We also discuss, in the context of character valued partition functions, in how far the 4-dimensional string theories can be regarded as compactifications of higher dimensional string theories by turning on suitable background fields in the underlying non-linear σ -model. Finally, the vertex operators for the massless states are constructed in two “ghost pictures” where, because of superconformal invariance, picture changing also involves internal degrees of freedom. In sect. 3 we apply these results to calculate various string tree level amplitudes like graviton scattering, gauge boson interactions, scalar and fermion self-interactions and Yukawa interactions. The discussion is held as model independent as possible, however, in some cases reference to a specific 4-dimensional model is also made. In sect. 4 the effective action of the massless fields is obtained as well as the fixed relations among the

various coupling constants. Corrections due to one-loop string effects are discussed in sect. 5 where also one specific diagram, namely the antisymmetric tensor field – gauge boson mixing is calculated explicitly. Sect. 6 summarizes the paper.

2. Four-dimensional heterotic strings and covariant lattices

The formulation of the d -dimensional ($d \leq 10$) heterotic string in the covariant lattice approach originates in the work of Friedan, Martinec and Shenker [15] about the construction of covariant vertex operators for the fermionic string. The key point is the covariant quantization of the fermionic string and the bosonization of the 10 world sheet fermions ψ^μ ($\mu = 0, \dots, 9$) as well as the superconformal ghosts β, γ .

Consider first only the right-moving part of the 10-dimensional fermionic string. It consists of 10 matter superfields $X^\mu(z)$,

$$X^\mu(z) = X^\mu(z) + \theta\psi^\mu(z), \quad (2.1)$$

and also of ghost superfields $B(z)$ and $C(z)$,

$$\begin{aligned} B(z) &= \beta(z) + \theta b(z), \\ C(z) &= c(z) + \theta\gamma(z). \end{aligned} \quad (2.2)$$

The bosonization of the fermions $\psi(z)$ provides five bosonic fields $H^i(z)$, ($i = 1, \dots, 5$),

$$\psi^{2i}\psi^{2i-1}(z) = \partial_z H^i(z), \quad (2.3)$$

where the H^i parametrize a five-dimensional torus and $\partial_z H^i$ are the generators of the Cartan subalgebra of $O(10)$. The superconformal ghosts β, γ with conformal dimensions $\frac{3}{2}$ and $-\frac{1}{2}$ respectively are bosonized according to [15]:

$$\begin{aligned} \gamma(z) &= e^\phi \eta(z) = e^{\phi-x}(z), \\ \beta(z) &= e^{-\phi} \partial \xi(z) = e^{-\phi+x} \partial \chi(z). \end{aligned} \quad (2.4)$$

The super-stress-energy tensor as generator of the superconformal transformations is given by:

$$T(z) = T_F(z) + \theta T_B(z). \quad (2.5)$$

$T(z)$ is a chiral superfield of dimension $\frac{3}{2}$. T_B is the bosonic stress-energy tensor

$$\begin{aligned} T_B(z) &= -\frac{1}{2} \partial X^\mu \partial X_\mu(z) - \frac{1}{2} \partial \psi^\mu \psi_\mu(z) \\ &= -\frac{1}{2} \partial X^\mu \partial X_\mu(z) - \frac{1}{2} \partial H^i \partial H_i(z), \end{aligned} \quad (2.6)$$

T_F is its superpartner,

$$T_F(z) = -\frac{1}{2} \partial X_\mu \psi^\mu(z), \tag{2.7}$$

where we neglected the ghost part of T_B and T_F . The operator products of T_F and T_B are given by:

$$T_B(z)T_B(w) \sim \frac{15/2}{(z-w)^4} + \frac{2T_B(w)}{(z-w)^2} + \frac{\partial T_B(w)}{(z-w)} + \text{finite}, \tag{2.8a}$$

$$T_F(z)T_F(w) \sim \frac{15/6}{(z-w)^3} + \frac{\frac{1}{2}T_B(w)}{(z-w)} + \text{finite}, \tag{2.8b}$$

$$T_B(z)T_F(w) \sim \frac{\frac{3}{2}T_F(w)}{(z-w)^2} + \frac{\partial T_F(w)}{(z-w)} + \text{finite}. \tag{2.8c}$$

Physical states are generated by covariant vertex operators $V(z)$:

$$V_w(z) = (\text{derivatives}) \times e^{i\lambda \cdot H} e^{q \cdot \phi}(z) c_w, \quad w = (\lambda, q). \tag{2.9}$$

λ denotes a vector of the $O(10)$ weight lattice D_5 , q the ghost charge of the state and c_w a cocycle generating factor. The mass of the state generated by V_w is given by:

$$\frac{1}{8}m^2 = \frac{1}{2}\lambda^2 - \frac{1}{2}q^2 - q + N - 1. \tag{2.10}$$

N counts the number of derivatives in (2.9) and will be neglected in the following discussion. All states in the Neveu-Schwarz (NS) [29] sector have ghost charge $q \in \mathbb{Z}$ whereas states in the Ramond (R) [30] sector are characterized by $q \in \mathbb{Z} + \frac{1}{2}$. However, as discussed in ref. [15], sectors differing by integer units of q are physically equivalent and related by the “picture changing operation”. In the canonical picture, physical states in the NS-sector have $q = -1$ where in the R-sector $q = -\frac{1}{2}$. This choice leads in eq. (2.10) to the correct normal ordering constant.

Since the superconformal ghosts β, γ have bosonic statistics, the ghost part of the covariant vertex operator (2.9) leads to a minus sign in the operator product

$$V_{w=(\lambda, q)}(z)V_{w'=(\lambda', q')}(w) \sim (z-w)^{\lambda \cdot \lambda' - qq'} V_{(\lambda+\lambda', q+q')}(z) \epsilon(w, w') + \dots \tag{2.11}$$

($\epsilon(w, w')$ is a cocycle factor). Therefore, it is very suggestive to consider $w = (\lambda, q)$ as a vector of the lorentzian lattice $\Gamma_{5,1}$ with metric $(++++-)$. Analogously to the D_n weight lattices, the vectors of $\Gamma_{5,1}$ can be assigned into conjugacy classes 0, V, S or C. In particular, 0 contains the massless vector $(\lambda = (\pm 1, 0, 0, 0, 0) +$

permutations, $q = -1$), V the scalar tachyon ($\lambda = \mathbf{0}$, $q = -1$) while S contains the massless spinor $\lambda = (\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})$ (even number of “-” signs, $q = -\frac{1}{2}$) and finally C spinors of opposite helicity $\lambda = (\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})$ (odd number of “-” signs, $q = -\frac{1}{2}$).

In general, operator products (2.11) lead to noninteger powers in $(z - w)$ so that the theory is non-local. However, the GSO projection [31] reduces $\Gamma_{5,1}$ to an odd self-dual lorentzian lattice implying the theory to become local [21]. Actually, demanding modular invariance of the partition function of the right-moving superstring forces $\Gamma_{5,1}$ to be odd self-dual [22]. Then the (one-loop) partition function is just given by the sum over physical state vectors which belong to the 0 and S conjugacy classes of $\Gamma_{5,1}$. This vanishes because of space-time supersymmetry.

Let us also briefly discuss the implications of the “right-moving” world sheet supersymmetry for the partition function. Chiral space time fermions are due to world sheet fermions with periodic boundary conditions (PP) along both noncontractable loops on the world sheet torus. In this sector, the partition function $\chi_{PP}(\tau)$,

$$\chi_{PP}(\tau) \sim \text{Tr}(-1)^{F_R} q^{H_R} \sim \frac{\theta_1^4(0|\tau)}{\eta(\tau)^4}, \quad q = e^{2\pi i\tau} \tag{2.12}$$

is the character valued index [32–34], also called elliptic genus [34], of the supercharge $Q = \int dz T_F$ which vanishes due to the zero mode of the Dirac operator on the torus. However, factorizing out the zero mode from eq. (2.12) only the massless chiral ground state contributes to $\chi'_{PP}(\tau)$ which becomes a constant [32]. This is due to the fact that the massive excitations come in chiral symmetric pairs and their contribution together with the bosonic excitations cancels due to world sheet supersymmetry.

Now, let us proceed discussing the case of the heterotic string theory in dimensions $d \leq 10$ (d even). The additional left-moving degrees of freedom are described by 26 bosons $X(\bar{z})$. In dimension d , $26 - d$ left-moving bosons $X^I(\bar{z})$ ($I = 1, \dots, 26 - d$) and $10 - d$ right-moving bosons $X^{I'}(z)$ ($I' = 1, \dots, 10 - d$) are circle valued. Thus, combining with the 6 bosonic coordinates $H^i(z)$, $\phi(z)$ from the right-moving fermionic string, the heterotic string theories are described by a lorentzian lattice $\Gamma_{26-d;15-d,1} = \Gamma_{26-d;15-\frac{3}{2}d} \otimes \Gamma_{\frac{1}{2}d,1}$. The semicolon separates left and right movers and the metric is $((-)^{26-2d}, (+)^{15-d}, (-))$. The $\Gamma_{\frac{1}{2}d,1}$ part of the lattice describes the space-time degrees of freedom of the bosonized fermions. For a sensible space-time interpretation, $\Gamma_{\frac{1}{2}d,1}$ has to contain only the four conjugacy classes S , C , V and 0 – the part $\Gamma_{\frac{1}{2}d}$ describes the Lorentz group $SO(d)$.

Modular invariance of the one loop partition function (see also subsect. 5.1) again forces $\Gamma_{26-d;15-d,1}$ to be odd self-dual where of course the left- resp. right-moving sublattices Γ_{26-d} and $\Gamma_{15-d,1}$ do not necessarily have to be self-dual separately*.

* Higher loop modular invariance [35] is ensured by the inclusion of proper spin-statistic factors [22].

Therefore, in general only after combining left- and right-moving degrees of freedom does one obtain a local theory.

How is superconformal invariance preserved in the most general d -dimensional heterotic string theory? To investigate this question let us decompose the lattice $\Gamma_{26-d;15-d,1}$ into its part which comes from the X -coordinates and into the part corresponding to the bosons H and ϕ :

$$\Gamma_{26-d;15-d,1} = (\Gamma_{26-d;10-d})_X \otimes (\Gamma_{0;5-\frac{1}{2}d} \otimes \Gamma_{0;\frac{1}{2}d,1})_{H,\phi}. \tag{2.13}$$

The case where Γ_X is even self-dual and $\Gamma_{H,\phi}$ odd self-dual corresponds to ‘‘Narain’’ compactifications [36] of the ten-dimensional supersymmetric heterotic string theory. However, all these theories are non-chiral and have $N = 4$ supersymmetry. Superconformal symmetry is preserved trivially in this case because the conjugacy classes of $\Gamma_{0;5-\frac{1}{2}d}$ and $\Gamma_{0;\frac{1}{2}d,1}$ are identical. Therefore the supercurrent T_F has still the form of eq. (2.7). However, formulating the most general conformal and modular invariant heterotic string theory there is a priori no reason to treat the right-moving bosonic coordinates $X^{I'}(z)$ and $H(z)$ on different grounds. Namely one has to consider situations where the conjugacy classes of $\Gamma_{0;5-\frac{1}{2}d}$ and the space time lattice $\Gamma_{0;\frac{1}{2}d,1}$ are no longer coupled. This corresponds to performing Lorentz rotations which mix the lattices Γ_X and $\Gamma_{H,\phi}$. Thus we are led to consider a more general lattice $\Gamma_{0;15-\frac{3}{2}d}$ where the associated bosons $\tilde{X}^{I''}(z)$, ($I'' = 1, \dots, 15 - \frac{3}{2}d$) are the combined set of $X^{I'}(z)$ and $H^i(z)$:

$$\Gamma_{26-d;15-d,1} = (\Gamma_{26-d;0}) \otimes (\Gamma_{0;15-\frac{3}{2}d}) \otimes (\Gamma_{0;\frac{1}{2}d,1}). \tag{2.14}$$

Like in [3] we assume that $\Gamma_{0;15-\frac{3}{2}d}$ can be written in terms of D_n factors and all states can be characterized by their transformation properties under these D_n 's.

On the other hand one cannot perform arbitrary Lorentz transformations on $\Gamma_{26-d;15-d,1}$, i.e. consider arbitrary lattices $\Gamma_{0;15-\frac{3}{2}d}$. The restrictions arise due to preserving the supersymmetry on the world sheet [8] or, equivalently, demanding space-time Lorentz invariance, i.e. the absence of massive chiral spinors [3]. Both considerations imply the existence of constraint vectors t which must be lattice vectors of $\Gamma_{0;15-\frac{3}{2}d}$. Specifically, in four dimensions one obtains the following 24 vectors t (they have to be combined with a V conjugacy class of $\Gamma_{0,2,1}$):

$$t = \begin{pmatrix} (\pm 1, 0, 0, \pm 1, 0, 0, \pm 1, 0, 0) \\ (0, \pm 1, 0, 0, \pm 1, 0, 0, \pm 1, 0) \\ (0, 0, \pm 1, 0, 0, \pm 1, 0, 0, \pm 1) \end{pmatrix}. \tag{2.15}$$

Now we can also consider the generalized right-moving supercurrent $T_F = T_F^{\text{ext}} + T_F^{\text{int}}$. The external part T_F^{ext} has the standard form $T_F^{\text{ext}} = \frac{1}{2} \partial X^\mu \psi_\mu$. According to the

interpretation of the lattice $\Gamma_{0,9}$ (in four dimensions), the internal part of the supercurrent T_F^{int} is constructed out of the nine bosons $\tilde{X}^{\prime\prime}(z)$ and has the general form

$$T_F^{\text{int}}(z) = \sum_{\substack{\mathbf{t} \in \Gamma_{0,9} \\ \mathbf{t}^2=3}} c(\mathbf{t}) e^{i\mathbf{t} \cdot \tilde{X}(z)}. \quad (2.16)$$

The vectors \mathbf{t} , $(\mathbf{t}^2=3)^*$ appearing in the expression for T_F^{int} have to be lattice vectors of $\Gamma_{0,9}$ since the supercurrent T_F acts as part of the picture changing operator (see end of this section) – adding or subtracting \mathbf{t} to a lattice vector leads again to a lattice vector. In fact, the vectors \mathbf{t} can be chosen to be exactly the 24 constraint vectors of eq. (2.15). Then the constants $c(\mathbf{t})$ are determined to be $c(\mathbf{t}) = \frac{1}{4}$ using eq. (2.8b). In addition, the validity of our ansatz is checked by reproducing exactly the term $\partial T_F(w)/(z-w)$ in eq. (2.8c).

One important question is whether four-dimensional string theories which are obtained from one another by a (discrete) Lorentz rotation on the lattice $\Gamma_{26-d,15-d,1}$ can be regarded as compactifications of a higher dimensional string theory. First, Lorentz rotations acting only inside the first bracket of eq. (2.13) are not restricted – all these theories are related to one another by giving expectation values to couplings of the underlying σ -model action [37]. They correspond to equivalent σ -models and are compactifications of higher dimensional models.

To answer this question for more general Lorentz rotations acting on the whole lattice $\Gamma_{26-d,15-d,1}$, consider again the elliptic genus, i.e. the character valued partition function in the PP sector of the theory:

$$\chi'_{\text{PP}}(q, \bar{q}) \sim \text{Tr} \bar{q}^{H_L} q^{H_R} (-1)^{F_R}. \quad (2.17)$$

Preserving world sheet supersymmetry implies that massive states cancel in this trace such that the elliptic genus is actually a holomorphic function of \bar{q} [32–34]:

$$\chi'_{\text{PP}}(\bar{q}) \sim \sum_{\lambda} b_{\lambda} \bar{q}^{\lambda}. \quad (2.18)$$

The coefficients b_{λ} are the indices of the generalized Dirac operator at any mass level and can be determined for a specific model. The crucial point is that the b_{λ} are topological invariants of the underlying superconformal σ -model and cannot be changed by changing background fields. On the other hand, for arbitrary Lorentz rotations with \mathbf{t} not being a lattice vector of $\Gamma_{0,9}$ the elliptic genus is not holomorphic in \bar{q} since superconformal invariance is destroyed. Then the coefficients b_{λ} are no longer invariant and are in general altered by Lorentz rotations^{**}.

* The restriction $\mathbf{t}^2=3$ is necessary to give T_F^{int} the correct conformal dimension, namely $\frac{3}{2}$.

** A similar remark was also made in ref. [5] in the context of asymmetric orbifolds.

This means that one cannot in general interpolate between different four-dimensional string theories by turning on background fields of the σ -model; they are based on topologically different σ -models and cannot be regarded as compactification of a higher dimensional heterotic string theory.

For practical purposes one can consider a different topological invariant of the σ -model namely the U(1) character valued index, resp. partition function [32–34]. It is given by

$$\chi'_{\text{pp}}(q, \bar{q}, F) \sim \text{Tr} \bar{q}^{H_L} q^{H_R} e^{i\theta K} (-1)^{F_R}. \quad (2.19)$$

F indicates the U(1) field strength and K is the U(1) generator. Again, superconformal invariance implies that (2.19) is a function of \bar{q} only and the coefficients of the power series expansion in \bar{q} are topological invariants. For theories where the U(1) gauge symmetry is not anomalous, this expression reduces just to the previously discussed elliptic genus. However there exist also four-dimensional string theories with anomalous U(1) symmetry; the anomaly will be cancelled by the Green-Schwarz mechanism [38] due to the one-loop BF diagram (see sect. 5). In this case we can extract additional topological information from the U(1)-character valued partition function.

To make contact to physical quantities, e.g. the trace of U(1)-charges of the massless fermionic states, let us rewrite $\chi'_{\text{pp}}(\bar{q}, F)$ in a slightly different way. In [32] it was shown that $\chi'_{\text{pp}}(\bar{q}, F)$ is a sum of products of ratios of θ -functions $\theta_i(\nu|\tau)$ where ν is basically one of the 2-form skew eigenvalues of F . In four dimensions it must be a modular function of weight -1 . Then $\chi'_{\text{pp}}(\bar{q}, F)$ can be expanded in terms of mixed traces of F where the coefficients of a term of order k in F is a modular function of weight $k - 1$ and can be expressed in terms of two Eisenstein functions G_4 and G_6 (see ref. [3]). Therefore, the topological invariants in their power series expansion in \bar{q} are just combinations of traces of F . In particular, the term of order \bar{q}^0 contains (besides other traces) as coefficient the trace over the U(1) charges of the massless fermions. So we are led to conjecture that two string theories which have different value of $\text{Tr} F$ belong to topologically different σ -models, and one cannot smoothly interpolate among them. Especially, this is true comparing theories with anomalous resp. non-anomalous U(1) symmetry.

Let us now come to a brief discussion of the main features of the spectrum of 4-dimensional heterotic string theories on covariant lattices. Specific models are most easily constructed from even self-dual lattices [3]; here we want to use only the odd self-dual lattices since they are most closely related to the covariant vertex operators of the conformal field theory.

Any 4-dimensional state is characterized by a lattice vector $w = (\lambda_L, \lambda_R, \lambda_R, q)$. λ_L is a vector of the 22-dimensional left-moving lattice $\Gamma_{22,0}$, λ_R a vector of $\Gamma_{0,9}$ and (λ_R, q) a vector of the space-time plus ghost part $\Gamma_{0,2,1}$. The lattice momenta plus Cartan subalgebra excitations produce states in representations of $G_L \times G_R \times$

SO(4). The masses are given by

$$\begin{aligned} \frac{1}{8}m_L^2 &= \frac{1}{2}\lambda_L^2 + N_L - 1, \\ \frac{1}{8}m_R^2 &= \frac{1}{2}(\lambda_R'^2 + \lambda_R^2) - \frac{1}{2}q^2 - q + N_R - 1, \\ m_L^2 &= m_R^2, \\ m^2 &= \frac{1}{2}(m_L^2 + m_R^2). \end{aligned} \tag{2.20}$$

Space-time spinors $S^\alpha, S^{\dot{\alpha}}$ are obtained if $\lambda_R = (\pm \frac{1}{2}, \pm \frac{1}{2})$ (even or odd number of “-” signs) of D_2 , vectors ψ^μ for $\lambda_R = (\pm 1, 0), (0, \pm 1)$ and scalars if $\lambda_R = 0$. For the spinors and scalars to be massless also a non-trivial lattice vector λ_R' is needed: $\lambda_R'^2 = \frac{3}{4}$, resp. $\lambda_R'^2 = 1$. Massless gauge particles are only obtained from the left part of the lattice, the gauge bosons of G_L correspond to root vectors $\lambda_L^2 = 2$. On the other hand, the symmetry G_R acts only as a global symmetry without massless gauge particles*.

Before we proceed, we introduce some notation which will allow us to write our results in an almost model independent way. Lower case Greek letters refer to 4-dimensional space-time, lower case Roman letters to the internal (ungauged) part of the right lattice and capital Roman letters to the left (gauged) part of the lattice. Letters from the beginning of the alphabets refer to spinor weights and letters from the middle of the alphabets to vector weights or roots. The left as well as the right internal degrees of freedom transform in general under a product of simple factors, $\otimes D_n; \sum_i n_i = 9$ for the right part and $\sum_i n_i = 22$ for the left part. The collection of the non-vanishing weight vectors of a given state in the different simple factors is denoted by $\{A\}$ for the left lattice and $\{a\}$ for the internal part of the right lattice. The corresponding vertex operators are denoted by $S^{(a)}(z) = \exp(i\lambda_R' \cdot \tilde{X}(z))$, resp. $S^{(A)}(\bar{z}) = \exp(i\lambda_L \cdot X(\bar{z}))$. Masslessness restricts the length of the left and the right vector. Further restrictions on the weight vectors are imposed by the self-duality of the lattice. We will discuss only the case where all the non-vanishing weights in the left and the right lattice are elements of a spinor conjugacy class, a vector conjugacy class or a 0 conjugacy class. All the examples we have encountered fall into this category. The generalization to other cases is straightforward.

To formulate the complete vertex operators we also have to determine the ghost charge of the state. Let us first consider the (canonical) picture with ghost charge $-\frac{1}{2}$, resp. -1 . The vertex operator for a massless spinor Ψ is given by:

$$V_{-\frac{1}{2}}^\Psi \equiv V_{-\frac{1}{2}}^{\{a\}\{A\}} = u_\alpha S^\alpha(z) e^{-\phi/2}(z) S^{(a)}(z) S^{(A)}(\bar{z}) \exp(ik_\mu X^\mu)(z, \bar{z}). \tag{2.21}$$

k^μ is the four-dimensional momentum of the state. Similarly the vertex operator for the charged gauge bosons A_μ is given by

$$V_{-1}^A \equiv V_{-1}^i = \epsilon^\mu \psi_\mu(z) e^{-\phi}(z) J^{MN}(\bar{z}) \exp(ik_\mu X^\mu)(z, \bar{z}) (T^i)^{MN}. \tag{2.22}$$

The vertex operator for the neutral (U(1)) gauge bosons is obtained replacing the

* Otherwise there would be no chiral fermions in the spectrum [3].

current $J^{MN} = \frac{1}{2}\psi^M\psi^N$ by the generators of the Cartan subalgebra, ∂X_I , of G_L . For a massless scalar particle we get:

$$V_{-1}^\Phi \equiv V_{-1}^{m\{M\}} = e^{-\phi}(z)\psi^m(z)\psi^{\{M\}}(\bar{z})\exp(ik_\mu X^\mu)(z, \bar{z}). \quad (2.23)$$

Finally, the vertex operator for a graviton $G_{\mu\nu}$, antisymmetric tensor $B_{\mu\nu}$ or dilaton D reads:

$$V_{-1}^{G, B, D} \equiv V_{-1} = \varepsilon^{\mu\nu}\psi_\mu(z)e^{-\phi}(z)\partial X_\nu(\bar{z})\exp(ik_\mu X^\mu)(z, \bar{z}), \quad (2.24)$$

with appropriate choice for the polarization tensor $\varepsilon_{\mu\nu}$. In order to preserve ghost charge calculating amplitudes, one needs in addition vertex operators \tilde{V} in the second picture [15] with ghost charge $\frac{1}{2}$, resp. 0. Picture changing is performed by acting with $2T_F e^\phi$ on V :

$$\tilde{V}_{0(\frac{1}{2})}(z) = \lim_{w \rightarrow z} [2T_F e^\phi(w)V_{-1(-\frac{1}{2})}(z)] + \text{spurious terms}. \quad (2.25)$$

For vertex operators with no $\Gamma_{0,9}$ excitation λ'_R the picture changing is analogous to the one in ten dimensions – only the external part T_F^{ext} gives a contribution to \tilde{V} . Specifically, we obtain:

$$\tilde{V}_0^A = \varepsilon^\mu [\partial X_\mu(z) + i(k \cdot \psi)\psi_\mu](z) J^{MN}(\bar{z})\exp(ik_\mu X^\mu)(z, \bar{z})(T^i)^{MN}, \quad (2.26)$$

$$\tilde{V}_0^{G, B, D} = \varepsilon^{\mu\nu} [\partial X_\mu(z) + i(k \cdot \psi)\psi_\mu(z)] \partial X_\nu(\bar{z})\exp(ik^\mu X_\mu)(z, \bar{z}). \quad (2.27)$$

On the other hand, for states which have also non-trivial transformation properties under G_R like massless fermions or scalars, the internal part T_F^{int} gives contributions to \tilde{V} . In general, $\tilde{V}_{+\frac{1}{2}}^\Psi$ and \tilde{V}_0^Φ take the following form:

$$\begin{aligned} \tilde{V}_{+\frac{1}{2}}^\Psi = & u^\alpha \left\{ \sqrt{\frac{1}{2}} [\partial X^\mu(z) + \frac{1}{4}i(k \cdot \psi)\psi^\mu(z)] e^{\phi/2}(z)(\gamma_\mu)_\alpha^\beta S_\beta^{(a)}(z) \right. \\ & \left. + \lim_{z \rightarrow w} \left(\frac{1}{2} \sum_{t \in \Gamma_{0,9}} e^{it \cdot \tilde{X}}(w) e^\phi(w) S^{(a)}(z) e^{-\phi/2}(z) \right) S_\alpha^{(a)}(z) \right\} \\ & \times S^{(A)}(\bar{z})\exp(ik_\mu X^\mu)(z, \bar{z}), \end{aligned} \quad (2.28)$$

$$\begin{aligned} \tilde{V}_0^\Phi = & \left\{ ik_\mu \psi^\mu(z)\psi^m(z) + \lim_{z \rightarrow w} \left(\frac{1}{2} \sum_{\substack{t \in \Gamma_{0,9} \\ t^2=3}} \exp(it \cdot \tilde{X})(w) e^\phi(w) \psi^m(z) e^{-\phi}(z) \right) \right\} \\ & \times \psi^{\{M\}}(\bar{z})\exp(ik_\mu X^\mu)(z, \bar{z}). \end{aligned} \quad (2.29)$$

Finally, at the end of this section we want to give as an example a 4-dimensional, non-space-time supersymmetric model with chiral fermions. Many different models with or without chiral fermions and with $N = 0, 1, 2, 4$ supersymmetry can be constructed [3]. Although we intend to be as model independent as possible, calculating string scattering amplitudes in the following sections, we will refer to this model whenever necessary.

In ref. [3] a model was constructed from the euclidean even self-dual Niemeier lattice $A_{11} \times D_7 \times E_6$. It can be mapped onto the lorentzian odd self-dual lattice $(D_5 \times D_5 \times D_5 \times D_2 \times D_2 \times D_2 \times D_1)_L \times (D_3 \times D_3 \times D_3 \times D_{2,1})_R$. Thus, the rank 22 gauge group is $[SO(10) \times SO(4)]^3 \times U(1)$ where the $U(1)$ will turn out to be anomalous. The right part gives rise to the global symmetry $[SO(6)]^3$. The 24 constraint vectors t (eq. (2.15)) are in the (V, V, V) conjugacy class of $[D_{3R}]^3$. Besides the massless gauge particles and the graviton, the model possesses chiral (left-handed) massless fermions transforming under $([SO(10)]^3 \times [SO(4)]^3 \times U(1))_{\text{local}} \times [SO(6)]^3_{\text{global}}$ as follows:

$$\begin{aligned}\Psi_L^1 &\sim (16, 1, 1, 2, 1, 1, \frac{1}{2}; 4, 1, 1) + (16, 1, 1, \bar{2}', 1, 1, \frac{1}{2}; \bar{4}, 1, 1), \\ \Psi_L^2 &\sim (1, 16, 1, 1, 2, 1, \frac{1}{2}; 1, 4, 1) + (1, 16, 1, 1, \bar{2}', 1, \frac{1}{2}; 1, \bar{4}, 1), \\ \Psi_L^3 &\sim (1, 1, 16, 1, 1, 2, \frac{1}{2}; 1, 1, 4) + (1, 1, 16, 1, 1, \bar{2}', \frac{1}{2}; 1, 1, \bar{4}).\end{aligned}\quad (2.30)$$

The $SO(10)$'s act as three grand unification groups (two of them play the role of hidden sectors!) where the $SO(4)$'s and $SO(6)$'s can be interpreted as local, resp. global horizontal symmetries. As mentioned before, the $U(1)$ is not traceless, $\text{tr } F = 768$, has anomalies with each of the $SO(10)$ and $SO(4)$ groups. λ'_R has the form $(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, (0)^6)$ (even number of “-” signs), it is the spinor weight of one of the three D_{3R} 's. For example, $V_{-\frac{1}{2}}^{\Psi_L^1}$ can be written as:

$$V_{-\frac{1}{2}}^{\Psi_L^1} = u_\alpha S^\alpha(z) e^{-\phi/2}(z) S^a(z) S^A(\bar{z}) S^{A'}(\bar{z}) S^{A''}(\bar{z}) \exp(ik_\mu X^\mu)(z, \bar{z}). \quad (2.31)$$

$S^a, S^A, S^{A'}, S^{A''}$ are the spin fields of $SO(6), SO(10), SO(4)$ and $U(1)$.

Similarly, one obtains massless scalars in the following representations:

$$\begin{aligned}\phi_1 &= (10, 1, 1, 1, 1, 1, \pm 1; 6, 1, 1), \\ \phi_2 &= (1, 10, 1, 1, 1, 1, \pm 1; 1, 6, 1), \\ \phi_3 &= (1, 1, 10, 1, 1, 1, \pm 1; 1, 1, 6), \\ \phi_4 &= (1, 1, 1, 4, 4, 1, 0; 6, 1, 1), \\ \phi_5 &= (1, 1, 1, 4, 1, 4, 0; 1, 6, 1), \\ \phi_6 &= (1, 1, 1, 1, 4, 4, 0; 1, 1, 6), \\ \phi_7 &= (10, 10, 1, 1, 1, 1, 0; 6, 1, 1), \\ \phi_8 &= (10, 1, 10, 1, 1, 1, 0; 1, 6, 1), \\ \phi_9 &= (1, 10, 10, 1, 1, 1, 0; 1, 1, 6).\end{aligned}\quad (2.32)$$

Finally, the vertex operator $\tilde{V}_0^{\phi_1}$ takes the following form:

$$\tilde{V}_0^{\phi_1} = \left\{ i(k \cdot \psi) \psi^m(z) + \frac{1}{2} \sum_{\substack{u=i+\lambda_R \\ t:\lambda_R=-1}} \exp(iu \cdot \tilde{X})(z) \right\} \psi^M(\bar{z}) \psi^{M'}(\bar{z}) \exp(ik_\mu X^\mu)(z, \bar{z}). \tag{2.33}$$

Here $\psi^m, \psi^M, \psi^{M'}$ are vectors of SO(6), SO(10) and U(1) respectively.

3. Tree level amplitudes

In this section we calculate various tree level amplitudes of massless particles. Some of the results derived below are well known, others are new; they describe processes such as Yukawa couplings and scalar self-interactions absent in ten-dimensional string theories. This section also serves to demonstrate how easy it is to do explicit calculations using the covariant lattice approach. The techniques employed here are those of conformal field theory. The calculation of amplitudes then amounts to the evaluation of correlation functions of the appropriate vertex operators. However, the vertex operator that creates a given physical on-shell state is not unique; different versions are related by the picture changing operation. To obtain a non-vanishing correlation function, two conditions have to be met: (i) the vertex operators, integrated over the world sheet have to be of conformal dimension zero, (ii) they have to be chosen such that the zero modes of the conformal and superconformal ghosts are soaked up.

On the sphere (tree level) there are three c zero modes, two γ zero modes and no b or β zero modes, which have to be soaked up, or equivalently, the functional integrations over c and γ have to be restricted to ghost fields orthogonal to the zero-modes. The ghosts c and γ have conformal weight -1 and $-\frac{1}{2}$ respectively. The only BRST invariant way to make a c -insertion on the sphere is to attach $c(z)\bar{c}(\bar{z})$ to a physical vertex operator of conformal dimension 2 to get a conformally invariant object. In other words, we replace the integral over the world sheet by $c(z)c(\bar{z})$. Since $c(z) = \delta(c(z))$ we see that the ghost field insertions restrict the functional integral over $c(z)$ to those ghost field configurations orthogonal to the zero modes. Since $c(z)$ has three zero modes on the sphere we conclude that 0, 1 and 2 point tree level amplitudes vanish identically. In particular, there are no tree level mass terms. At the one and higher loop level this conclusion is avoided by the occurrence of zero modes of b . For the zero modes of γ we first restrict our discussion to the case of only bosonic vertex operators. They can be written as superfields in the following way [39]: $V(z) = V_0(z) + \theta \tilde{V}_0(z)$, where the subscript denotes superconformal ghost charge and V_0 differs from V_{-1} only by its ghost charge. $d^2z d^2\theta V(z)$ is of conformal dimension zero. So is $\theta \delta(\gamma(z))$, and we can

insert it into two of the vertices. Integration over θ then results for a N -point function in $2 V_0 \delta(\gamma)$ and $(N-2) \tilde{V}_0$ vertex operators. Then $\delta(\gamma)$ again restricts the γ -functional integration to fields orthogonal to the γ zero modes. If we now use the bosonization prescription [40] $\delta(\gamma) = e^{-\phi}$, we get $V_0 \delta(\gamma) = V_{-1}$, i.e. the non-vanishing correlation functions always contain 2 vertex operators of the form V_{-1} and the remaining ones of the form \tilde{V}_0 .

In the case of fermions, we have to deal with spin fields and vertex operators with half-integer ghost charge. Here it is not possible to combine $\tilde{V}_{+\frac{1}{2}}$ and $V_{-\frac{1}{2}}$ into one superfield, and the reasoning above does not apply. In order to absorb the γ zero modes, we have to choose vertex operators $V_{\pm\frac{1}{2}}$ such that the total ghost charge is minus the number of γ zero modes, i.e. -2 at tree level.

Since we attached the ghost fields $c(z)c(\bar{z})$ to three vertex operators to absorb the c zero modes, there are only $N-3$ integrations over the world sheet for an amplitude with N external fields. In addition, the c -ghost correlations produce a factor $|(z_i - z_j)(z_i - z_k)(z_j - z_k)|^2$. Therefore, the insertion of the c -ghosts has exactly the effect of factorizing out the volume of the conformal group $SL(2, R)$. Eg. in three-point functions we drop all three integrations to set the positions of all three vertex operators to fixed values, and the amplitudes are independent of any particular choice because of BRST invariance. In four-point functions we fix the positions of three vertices and integrate over the fourth. Only after performing the integral do we end up with an expression independent of any particular choice. We have followed the literature [15] and set $z_1 = \infty$, $z_2 = 1$, $z_4 = 0$ and we have integrated over z_3 .

A general N -point function then has the following form:

$$A = g^{N-2} \left(\frac{1}{2\pi} \right)^{N-3} \int d^2 z_3 \prod_{i=5}^N d^2 z_i \times \left\langle c\bar{c}V_{q_1}(z_1, \bar{z}_1) c\bar{c}V_{q_2}(z_2, \bar{z}_2) V_{q_3}(z_3, \bar{z}_3) c\bar{c}V_{q_4}(z_4, \bar{z}_4) \prod_{j=5}^N V_{q_j}(z_j, \bar{z}_j) \right\rangle, \quad (3.1)$$

with $\sum_{i=1}^N q_i = -2$, where g is the string coupling constant and will be related to the coupling constants in four-dimensional field theory later.

We also have to comment on the correct normalization of the vertex operators which is needed to obtain the scattering amplitudes of the four-dimensional field theory. This implies that the vertex operators create states which are normalized according to particles in the field theory. The normalization constants are determined once for a specific process like the three gauge boson scattering and the two fermion-one gauge boson scattering and have to lead in the following to consistent results for all other processes. In this way we obtained that every V_{-1} of eqs. (2.22)–(2.24) appearing in eq. (3.1), has to be multiplied by 2 and every $V_{-\frac{1}{2}}$ by $2^{-1/4}$.

The vertex operators needed in the evaluation of the various scattering amplitudes below, have been given in sect. 2. To evaluate the correlation functions we need the following basic free field propagators:

$$\begin{aligned}
 \langle X^\mu(z) X^\nu(w) \rangle &= -g^{\mu\nu} \ln(z-w), \\
 \langle \psi^\mu(z) \psi^\nu(w) \rangle &= -g^{\mu\nu} (z-w)^{-1}, \\
 \langle c(z) c(w) \rangle &= (z-w), \\
 \langle \phi(z) \phi(w) \rangle &= -\ln(z-w).
 \end{aligned}
 \tag{3.2}$$

The correlation functions involving spinors are most easily evaluated using bosonization techniques. For details we refer to ref. [24]. The advantage of this approach lies in the fact that after bosonizing ψ and S all correlation functions are those of free fields. The price we have to pay for this is that we have to recast the correlation functions into a manifestly Lorentz covariant form. To get non-vanishing correlation functions, we have to impose momentum conservation on each factor of the lattice (except the ghost part, where the momentum has to be -2).

We are now ready to list the amplitudes. First the three-point functions:

(i) *Three gauge boson amplitude.* The amplitude is non-vanishing only if all three gauge bosons belong to the same simple factor of the gauge group; then it is given by:

$$\begin{aligned}
 A^{ijk} &= 4g \langle c\bar{c}V_{-1}^{\mu i}(1) c\bar{c}V_{-1}^{\nu j}(2) c\bar{c}V_0^{\rho k}(3) \rangle \epsilon_\mu^{(1)} \epsilon_\nu^{(2)} \epsilon_\rho^{(3)} \\
 &= gf^{ijk} \left[\epsilon^{(1)} \cdot (k_3 - k_2) (\epsilon^{(2)} \cdot \epsilon^{(3)}) + \epsilon^{(2)} \cdot (k_1 - k_3) (\epsilon^{(1)} \cdot \epsilon^{(3)}) \right. \\
 &\quad \left. + \epsilon^{(3)} \cdot (k_2 - k_1) (\epsilon^{(1)} \cdot \epsilon^{(2)}) \right].
 \end{aligned}
 \tag{3.3}$$

We have used the normalization $\text{Tr}(T^i T^j) = 2\delta^{ij}$, corresponding to root vectors of $(\text{length})^2 = 2$. f^{ijk} are the structure constants. The above result holds for the scattering of gauge bosons of any one of the simple factors of the gauge group and is identical to the heterotic string result [41, 42]. (The superstring three gauge boson amplitude has additional terms fourth order in the momenta*.)

(ii) *Two scalar – one gauge boson amplitude*

$$\begin{aligned}
 A^{i, m\{M\}, n\{N\}} &= 4g \langle c\bar{c}V_{-1}^{m\{M\}}(1) c\bar{c}V_{-1}^{n\{N\}}(2) c\bar{c}V_0^{\mu i}(3) \rangle \epsilon_\mu \\
 &= g\epsilon \cdot (k_2 - k_1) \delta^{mn} (T^i)^{\{M\}\{N\}},
 \end{aligned}
 \tag{3.4}$$

where $(T^i)^{\{M\}\{N\}} = (T^i)^{M_1 N_1} \delta^{M_2 N_2}$ if e.g. the gauge boson belongs to the first factor of the gauge group under which the scalars transform non-trivially.

* Three-point amplitudes involving on-shell vector or tensor particles actually vanish for real momenta due to the kinematic constraint that all three (real) momenta are colinear. To get non-zero amplitudes we have to allow for complex momenta (analyticity).

(iii) *Two fermion – one gauge boson amplitude*

$$\begin{aligned} A^{i, \{a\}\{A\}, \{b\}\{B\}} &= \sqrt{2} g \left\langle c\bar{c}V_{-\frac{1}{2}}^{\alpha\{a\}\{A\}}(1) c\bar{c}V_{-\frac{1}{2}}^{\beta\{b\}\{B\}}(2) c\bar{c}V_{-1}^{\mu i}(3) \right\rangle u_{\alpha}^{(1)} u_{\beta}^{(2)} \epsilon_{\mu} \\ &= \frac{1}{4} g (\Gamma^{MN})^{\{A\}\{B\}} (T^i)^{MN} C^{\{a\}\{b\}} u^{(1)} \not{u}^{(2)}. \end{aligned} \quad (3.5)$$

For the calculation of this amplitude we have used the following correlation functions:

$$\begin{aligned} \langle \psi^{\mu}(z_1) S^{\alpha}(z_2) S^{\beta}(z_3) \rangle &= \sqrt{\frac{1}{2}} (\gamma^{\mu})^{\alpha\beta} (z_{12})^{-1/2} (z_{13})^{-1/2}, \\ \langle S^{\{a\}}(z_2) S^{\{b\}}(z_3) \rangle &= C^{\{a\}\{b\}} (z_{23})^{-3/4}, \\ \langle J^i(\bar{z}_1) S^{\{A\}}(\bar{z}_2) S^{\{B\}}(\bar{z}_3) \rangle &= \frac{(T^i)^{\{A\}\{B\}}}{(\bar{z}_{12})(\bar{z}_{13})(\bar{z}_{23})}. \end{aligned} \quad (3.6)$$

Here $C^{\{a\}\{b\}}$ is a product of charge conjugation matrices with indices in the factors of the (ungauged) symmetry group associated with the right lattice under which the fermions transform non-trivially, $J^i = \frac{1}{2} \psi^M \psi^N (T^i)^{MN}$, and $(T^i)^{\{A\}\{B\}} = (T^i)^{A_1 B_1} \delta^{A_2 B_2} \dots$ as above with $(T^i)^{AB} = \frac{1}{4} (\Gamma^{MN})^{AB} (T^i)^{MN}$; $z_{ij} = z_i - z_j$.

(iv) *Two fermion – one scalar (Yukawa) amplitude*

$$\begin{aligned} A^{\{a\}\{A\}, \{b\}\{B\}, m\{M\}} &= \sqrt{2} g \left\langle c\bar{c}V_{-\frac{1}{2}}^{\alpha\{a\}\{A\}}(1) c\bar{c}V_{-\frac{1}{2}}^{\beta\{b\}\{B\}}(2) c\bar{c}V_{-1}^{m\{M\}}(3) \right\rangle u_{\alpha}^{(1)} u_{\beta}^{(2)} \\ &= \sqrt{2} g (u^{(1)} C u^{(2)}) \left(\sqrt{\frac{1}{2}} (\Gamma^m)^{\{a\}\{b\}} \right) \left(\sqrt{\frac{1}{2}} (\Gamma^{\{M\}})^{\{A\}\{B\}} \right), \end{aligned} \quad (3.7)$$

where $\sqrt{\frac{1}{2}} (\Gamma^m)^{\{a\}\{b\}}$, $\sqrt{\frac{1}{2}} (\Gamma^{\{M\}})^{\{A\}\{B\}}$ denote products of Dirac and charge conjugation matrices depending on whether both fermions and the scalar or only the fermions transform non-trivially under a particular factor of the symmetry group. For the calculation of this amplitude we have used correlation functions similar to the ones given above in eq. (3.6).

(v) *Three – graviton amplitude*

$$\begin{aligned} A &= 4g \left\langle c\bar{c}V_{-1}^{\mu\nu}(1) c\bar{c}V_{-1}^{\rho\sigma}(2) c\bar{c}V_0^{\alpha\tau}(3) \right\rangle \epsilon_{\mu\nu}^{(1)} \epsilon_{\rho\sigma}^{(2)} \epsilon_{\alpha\tau}^{(3)} \\ &= g \left\{ (k_1 \epsilon^{(2)} k_1) (\epsilon^{(1)} \epsilon^{(3)\top}) + (k_2 \epsilon^{(3)} k_2) (\epsilon^{(2)} \epsilon^{(1)\top}) + (k_3 \epsilon^{(1)} k_3) (\epsilon^{(3)} \epsilon^{(2)\top}) \right. \\ &\quad + (k_1 \epsilon^{(2)} \epsilon^{(1)\top} \epsilon^{(3)} k_2) + (k_2 \epsilon^{(3)} \epsilon^{(2)\top} \epsilon^{(1)} k_3) + (k_3 \epsilon^{(1)} \epsilon^{(3)\top} \epsilon^{(2)} k_1) \\ &\quad + (k_2 \epsilon^{(3)} \epsilon^{(1)\top} \epsilon^{(2)} k_1) + (k_3 \epsilon^{(1)} \epsilon^{(2)\top} \epsilon^{(3)} k_2) + (k_1 \epsilon^{(2)} \epsilon^{(3)\top} \epsilon^{(1)} k_3) \\ &\quad \left. - \frac{1}{4} \left[(k_1 \epsilon^{(2)} k_1) (k_2 \epsilon^{(3)} \epsilon^{(1)\top} k_3) + (k_2 \epsilon^{(3)} k_2) (k_3 \epsilon^{(1)} \epsilon^{(2)\top} k_1) \right. \right. \\ &\quad \left. \left. + (k_3 \epsilon^{(1)} k_3) (k_1 \epsilon^{(2)} \epsilon^{(3)\top} k_2) \right] \right\}, \end{aligned} \quad (3.8)$$

where we have used the notation $(\epsilon^{(i)} \epsilon^{(j)\top}) = \text{Tr} \epsilon^{(i)} \epsilon^{(j)\top}$. $\epsilon^{(i)}$ are the polarization tensors. Above amplitude is actually valid for the scattering of gravitons, anti-symmetric tensor particles and dilatons, depending on whether we choose $\epsilon^{(i)}$ to be symmetric and traceless, antisymmetric or transverse diagonal. (In the amplitudes below, “graviton” always refers to all three of these particles unless otherwise

specified.) Above amplitude agrees with the corresponding one for the heterotic string [41, 42] in 10-dimensions.

(vi) *Two gauge boson – one graviton amplitude*

$$\begin{aligned}
 A^{ij} &= 4g \langle c\bar{c}V_{-1}^{\mu i}(1) c\bar{c}V_{-1}^{\nu j}(2) c\bar{c}V_0^{\rho\sigma}(3) \rangle \epsilon_{\mu}^{(1)} \epsilon_{\nu}^{(2)} \epsilon_{\rho\sigma} \\
 &= g \delta^{ij} \{ (\epsilon^{(1)} \cdot \epsilon^{(2)}) (k_1 \epsilon k_1) - (k_3 \cdot \epsilon^{(2)}) (k_1 \epsilon \epsilon^{(1)}) - (k_2 \cdot \epsilon^{(1)}) (k_1 \epsilon \epsilon^{(2)}) \}.
 \end{aligned}
 \tag{3.9}$$

(vii) *Two fermion – one graviton amplitude*

$$\begin{aligned}
 A^{(a)\{A\}, (b)\{B\}} &= \sqrt{2} g \langle c\bar{c}V_{-\frac{1}{2}}^{\alpha(a)\{A\}}(1) c\bar{c}V_{-\frac{1}{2}}^{\beta(b)\{B\}}(2) c\bar{c}V_{-1}^{\mu\nu}(3) \rangle u_{\alpha}^{(1)} u_{\beta}^{(2)} \epsilon_{\mu\nu} \\
 &= \frac{1}{4} g C^{(a)\{b\}} C^{(A)\{B\}} u^{(1)} \gamma^{\mu} u^{(2)} (k_1 - k_2)^{\nu} \epsilon_{\mu\nu}.
 \end{aligned}
 \tag{3.10}$$

This amplitude vanishes for the case of a dilaton due to on-shell conditions.

(viii) *Two scalar – one graviton amplitude*

$$\begin{aligned}
 A^{m\{M\}, n\{N\}} &= 4g \langle c\bar{c}V_{-1}^{m\{M\}}(1) c\bar{c}V_{-1}^{n\{N\}}(2) c\bar{c}V_0^{\mu\nu}(3) \rangle \epsilon_{\mu\nu} \\
 &= g (k_1 \epsilon k_1) \delta^{mn} \delta^{(M)\{N\}}.
 \end{aligned}
 \tag{3.11}$$

This amplitude vanishes for the case of a dilaton or antisymmetric tensor.

This exhausts the list of non-vanishing 3-particle scattering amplitudes*. Next we turn to the four-particle amplitudes. We omit the explicit calculation of the 4 graviton, 2 graviton-2 gauge boson and 1 graviton-3 gauge boson amplitudes; they were already computed and discussed for the 10-dimensional heterotic string in refs. [41, 42].

(i) *Four gauge boson amplitudes.* Here we have to distinguish between two cases: all four gauge bosons belong to the same simple factor of the gauge group or they do belong pairwise to different simple factors. In the first case we get the well known [41] result:

$$\begin{aligned}
 A^{ijkl} &= \frac{2g^2}{\pi} \int dz d\bar{z} \langle c\bar{c}V_{-1}^{\mu i}(1) c\bar{c}V_{-1}^{\nu j}(2) V_0^{\rho k}(z, \bar{z}) c\bar{c}V_0^{\sigma l}(4) \rangle \epsilon_{\mu}^{(1)} \epsilon_{\nu}^{(2)} \epsilon_{\rho}^{(3)} \epsilon_{\sigma}^{(4)} \\
 &= g^2 K \frac{\Gamma(-\frac{1}{8}s) \Gamma(-\frac{1}{8}t) \Gamma(-\frac{1}{8}u)}{\Gamma(\frac{1}{8}s) \Gamma(\frac{1}{8}t) \Gamma(\frac{1}{8}u)} \\
 &\quad \times \left\{ \frac{\frac{1}{4} \text{Tr}(T^i T^k) \text{Tr}(T^j T^l)}{u(1 + \frac{1}{8}u)} + \frac{\frac{1}{4} \text{Tr}(T^i T^j) \text{Tr}(T^k T^l)}{s(1 + \frac{1}{8}s)} + \frac{\frac{1}{4} \text{Tr}(T^i T^l) \text{Tr}(T^j T^k)}{t(1 + \frac{1}{8}t)} \right. \\
 &\quad \left. - \frac{8 \text{Tr}(T^i T^j T^l T^k)}{us} - \frac{8 \text{Tr}(T^i T^j T^k T^l)}{st} - \frac{8 \text{Tr}(T^i T^k T^j T^l)}{ut} \right\},
 \end{aligned}
 \tag{3.12}$$

* There exists no three scalar amplitude because of the internal quantum numbers of the scalars.

where the kinematical factor K is given by

$$\begin{aligned}
 K = & -\frac{1}{4}(st + su + tu) \\
 & + \frac{1}{2}s [(\varepsilon^{(1)} \cdot \varepsilon^{(3)})(k_3 \cdot \varepsilon^{(2)})(k_1 \cdot \varepsilon^{(4)}) + (\varepsilon^{(1)} \cdot \varepsilon^{(4)})(k_4 \cdot \varepsilon^{(2)})(k_1 \cdot \varepsilon^{(3)}) \\
 & \quad + (\varepsilon^{(2)} \cdot \varepsilon^{(3)})(k_3 \cdot \varepsilon^{(1)})(k_2 \cdot \varepsilon^{(4)}) + (\varepsilon^{(2)} \cdot \varepsilon^{(4)})(k_4 \cdot \varepsilon^{(1)})(k_2 \cdot \varepsilon^{(3)})] \\
 & + \frac{1}{2}t [(\varepsilon^{(1)} \cdot \varepsilon^{(2)})(k_1 \cdot \varepsilon^{(3)})(k_2 \cdot \varepsilon^{(4)}) + (\varepsilon^{(1)} \cdot \varepsilon^{(3)})(k_3 \cdot \varepsilon^{(4)})(k_1 \cdot \varepsilon^{(2)}) \\
 & \quad + (\varepsilon^{(2)} \cdot \varepsilon^{(4)})(k_4 \cdot \varepsilon^{(3)})(k_2 \cdot \varepsilon^{(1)}) + (\varepsilon^{(3)} \cdot \varepsilon^{(4)})(k_3 \cdot \varepsilon^{(1)})(k_4 \cdot \varepsilon^{(2)})] \\
 & + \frac{1}{2}u [(\varepsilon^{(1)} \cdot \varepsilon^{(2)})(k_1 \cdot \varepsilon^{(4)})(k_2 \cdot \varepsilon^{(3)}) + (\varepsilon^{(1)} \cdot \varepsilon^{(4)})(k_1 \cdot \varepsilon^{(2)})(k_4 \cdot \varepsilon^{(3)}) \\
 & \quad + (\varepsilon^{(2)} \cdot \varepsilon^{(3)})(k_3 \cdot \varepsilon^{(4)})(k_2 \cdot \varepsilon^{(1)}) + (\varepsilon^{(3)} \cdot \varepsilon^{(4)})(k_3 \cdot \varepsilon^{(2)})(k_4 \cdot \varepsilon^{(1)})]. \quad (3.13)
 \end{aligned}$$

In the case when two gauge bosons are from one simple factor and two from another, the amplitude is

$$A^{ijk'l'} = \frac{g^2}{4} \frac{\Gamma(-\frac{1}{8}s)\Gamma(-\frac{1}{8}t)\Gamma(-\frac{1}{8}u)}{\Gamma(\frac{1}{8}s)\Gamma(\frac{1}{8}u)\Gamma(\frac{1}{8}t)} K \frac{\text{Tr}(T^i T^j) \text{Tr}(T^{k'} T^{l'})}{s(1 + \frac{1}{8}s)}. \quad (3.14)$$

(ii) Four scalar amplitude

$$\begin{aligned}
 A = & \frac{2g^2}{\pi} \int dz d\bar{z} \langle c\bar{c}V_0^{mMM''}(1)c\bar{c}V_0^{nNN''}(2)V_{-1}^{pPP''}(z,\bar{z})c\bar{c}V_{-1}^{qQQ''}(4)\rangle \\
 = & 2g^2 \frac{\Gamma(-\frac{1}{8}s)\Gamma(-\frac{1}{8}t)\Gamma(-\frac{1}{8}u)}{\Gamma(\frac{1}{8}s)\Gamma(\frac{1}{8}t)\Gamma(\frac{1}{8}u)} \{ ut\delta^{mn}\delta^{pq} + ts\delta^{mp}\delta^{nq} + us\delta^{mq}\delta^{np} \} \\
 & \times \left\{ \frac{1}{8s(1 + \frac{1}{8}s)} \delta^{MN}\delta^{PQ}\delta^{M''N''}\delta^{P''Q''} + \frac{1}{8u(1 + \frac{1}{8}u)} \delta^{MP}\delta^{NQ}\delta^{M''P''}\delta^{N''Q''} \right. \\
 & \quad + \frac{1}{8t(1 + \frac{1}{8}t)} \delta^{MQ}\delta^{NP}\delta^{M''Q''}\delta^{N''P''} \\
 & \quad + \frac{1}{ut} [\delta^{MP}\delta^{NQ}\delta^{M''Q''}\delta^{N''P''} + \delta^{MQ}\delta^{NP}\delta^{M''P''}\delta^{N''Q''}] \\
 & \quad + \frac{1}{st} [\delta^{MQ}\delta^{NP}\delta^{M''N''}\delta^{P''Q''} + \delta^{MN}\delta^{PQ}\delta^{M''Q''}\delta^{N''P''}] \\
 & \quad \left. + \frac{1}{us} [\delta^{MP}\delta^{NQ}\delta^{M''N''}\delta^{P''Q''} + \delta^{MN}\delta^{PQ}\delta^{M''P''}\delta^{N''Q''}] \right\}. \quad (3.15)
 \end{aligned}$$

This amplitude describes the scattering of 4 scalars ϕ_1 (cf. eq. (2.32)). The vertex operator V_0 is given by eq. (2.33) and involves the picture changing in the internal right-moving lattice Γ_9 . The corresponding correlation function is given by:

$$\frac{1}{4} \sum_{u, u'} \langle e^{u \cdot \tilde{X}(z_1)} e^{u' \cdot \tilde{X}(z_2)} \rangle \langle \psi^p(z_3) \psi^q(z_4) \rangle = \frac{\delta^{mn} \delta^{pq}}{(z_{12})^2 (z_{34})}. \quad (3.16)$$

(iii) Four fermion amplitude*

$$\begin{aligned} A &= \frac{g^2}{4\pi} \int dz d\bar{z} \langle c\bar{c}V_{-\frac{1}{2}}^{\alpha\alpha A A' A''}(1) c\bar{c}V_{-\frac{1}{2}}^{\beta\beta B B' B''}(2) V_{-\frac{1}{2}}^{\gamma\gamma C C' C''}(z, \bar{z}) c\bar{c}V_{-\frac{1}{2}}^{\delta\delta D D' D''}(4) \rangle \\ &\times u_{\alpha}^{(1)} u_{\beta}^{(2)} u_{\gamma}^{(3)} u_{\delta}^{(4)} \\ &= \frac{1}{8} g^2 \frac{\Gamma(-\frac{1}{8}s) \Gamma(-\frac{1}{8}t) \Gamma(-\frac{1}{8}u)}{\Gamma(\frac{1}{8}s) \Gamma(\frac{1}{8}t) \Gamma(\frac{1}{8}u)} (u^{(1)} C u^{(2)}) (u^{(3)} C u^{(4)}) \\ &\times (C^{A'' C''} C^{\dot{B}'' D''} + C^{A'' D''} C^{\dot{B}'' C''}) \\ &\times \left\{ C^{a\dot{c}} C^{bd} \left[C^{A\dot{C}} C^{B\dot{D}} \left(\frac{8}{u} C^{A'D'} C^{B'C'} - \frac{t}{u(1+\frac{1}{8}u)} C^{A'C'} C^{D'B'} \right) \right. \right. \\ &\quad + C^{A\dot{D}} C^{B\dot{C}} \left(-\frac{8}{u} C^{A'C'} C^{D'B'} + \frac{1}{1+\frac{1}{8}t} C^{A'D'} C^{B'C'} \right) \\ &\quad \left. \left. + \frac{1}{2} (\Gamma^M)^{AB} (\Gamma^M)^{\dot{C}\dot{D}} \left(\frac{8}{s} C^{A'B'} C^{C'D'} - \frac{8}{u} C^{A'C'} C^{D'B'} \right) \right] \right. \\ &\quad - C^{a\dot{d}} C^{b\dot{c}} \left[C^{A\dot{D}} C^{B\dot{C}} \left(\frac{8}{t} C^{A'C'} C^{D'B'} - \frac{u}{t(1+\frac{1}{8}t)} C^{A'D'} C^{B'C'} \right) \right. \\ &\quad \left. + C^{A\dot{C}} C^{B\dot{D}} \left(-\frac{8}{t} C^{A'D'} C^{B'C'} + \frac{1}{1+\frac{1}{8}u} C^{A'C'} C^{D'B'} \right) \right. \\ &\quad \left. \left. + \frac{1}{2} (\Gamma^M)^{AB} (\Gamma^M)^{\dot{C}\dot{D}} \left(\frac{8}{s} C^{A'B'} C^{C'D'} - \frac{8}{t} C^{A'D'} C^{B'C'} \right) \right] \right\}. \quad (3.17) \end{aligned}$$

This amplitude describes the scattering of two fermions which transform under the symmetry group $(SO(10) \otimes SO(4) \otimes U(1))_{\text{local}}$ as $(16, 2, \frac{1}{2})$ with two fermions which transform under this group as $(\bar{16}, 2, -\frac{1}{2})$ (cfr. eq. (2.30)). The computation

* A different four-fermion amplitude was calculated in the context of the 10-dimensional $O(16) \otimes O(16)$ model in ref. [43].

of this amplitude involves the following correlation functions containing spin fields:

$$\begin{aligned}
 \text{SO}(4)_{\text{Lorentz}}: & \langle S_{-\frac{1}{2}}^{\alpha}(z_1) S_{-\frac{1}{2}}^{\beta}(z_2) S_{-\frac{1}{2}}^{\gamma}(z_3) S_{-\frac{1}{2}}^{\delta}(z_4) \rangle \\
 & = C^{\alpha\beta} C^{\gamma\delta} (z_{12})^{-3/4} (z_{13})^{-1/4} (z_{14})^{-1/4} (z_{23})^{-1/4} (z_{24})^{-1/4} (z_{34})^{-3/4}, \\
 \text{SO}(6)_{\text{global}}: & \langle S^a(z_1) S^b(z_2) S^c(z_3) S^d(z_4) \rangle \\
 & = C^{a\dot{c}} C^{b\dot{d}} (z_{12})^{-1/4} (z_{13})^{-3/4} (z_{14})^{1/4} (z_{23})^{1/4} (z_{24})^{-3/4} (z_{34})^{-1/4} \\
 & \quad - C^{ad} C^{b\dot{c}} (z_{12})^{-1/4} (z_{13})^{1/4} (z_{14})^{-3/4} (z_{23})^{-3/4} (z_{24})^{1/4} (z_{34})^{-1/4}, \\
 \text{SO}(10)_{\text{local}}: & \langle S^A(\bar{z}_1) S^B(\bar{z}_2) S^{\dot{C}}(\bar{z}_3) S^{\dot{D}}(\bar{z}_4) \rangle \\
 & = \{ C^A \dot{C} C^B \dot{D} (\bar{z}_{13})^{-1} (\bar{z}_{24})^{-1} - C^A \dot{D} C^B \dot{C} (\bar{z}_{14})^{-1} (\bar{z}_{23})^{-1} \\
 & \quad + \frac{1}{2} (\Gamma^M)^{AB} (\Gamma^M)^{\dot{C}\dot{D}} (\bar{z}_{12})^{-1} (\bar{z}_{34})^{-1} \} \\
 & \quad \times \{ (\bar{z}_{12})^{1/4} (\bar{z}_{13})^{-1/4} (\bar{z}_{14})^{-1/4} (\bar{z}_{23})^{-1/4} (\bar{z}_{24})^{-1/4} (\bar{z}_{34})^{1/4} \}, \\
 \text{SO}(4)_{\text{local}}: & \langle S^{A'}(\bar{z}_1) S^{B'}(\bar{z}_2) S^{C'}(\bar{z}_3) S^{D'}(\bar{z}_4) \rangle \\
 & = \frac{1}{3} \prod_{i < j} (\bar{z}_{ij})^{-1/2} \{ C^{A'B'} C^{C'D'} [\bar{z}_{12} \bar{z}_{24} + \bar{z}_{14} \bar{z}_{23}] \\
 & \quad + C^{A'C'} C^{D'B'} [\bar{z}_{14} \bar{z}_{32} + \bar{z}_{34} \bar{z}_{12}] \\
 & \quad + C^{A'D'} C^{B'C'} [\bar{z}_{12} \bar{z}_{43} + \bar{z}_{13} \bar{z}_{42}] \}. \tag{3.18}
 \end{aligned}$$

(iv) Two fermion – two scalar amplitude

$$\begin{aligned}
 A & = \frac{g^2}{\sqrt{2} \pi} \int dz \bar{d}z \langle c\bar{c} V_{-\frac{1}{2}}^{\alpha a A A' \dot{A}''}(1) c\bar{c} V_{-\frac{1}{2}}^{\beta \dot{b} B B' B''}(2) V_{-1}^{m M M''}(z, \bar{z}) \\
 & \quad \times c\bar{c} V_0^{n N N''}(4) \rangle u_{\alpha}^{(1)} u_{\beta}^{(2)} \\
 & = \frac{g^2}{2} \frac{\Gamma(-\frac{1}{8}s) \Gamma(-\frac{1}{8}t) \Gamma(-\frac{1}{8}u)}{\Gamma(\frac{1}{8}s) \Gamma(\frac{1}{8}t) \Gamma(\frac{1}{8}u)} C^{A'B'} \delta^{M'N'} u^{(1)} (\not{k}_3 - \not{k}_4) u^{(2)} \\
 & \quad \times \left\{ \frac{1}{8(1 + \frac{1}{8}s)} \left[\frac{u-t}{s} C^{ab} \delta^{mn} + (\Gamma^{mn})^{ab} \right] C^{A\dot{B}} \delta^{MN} C^{\dot{A}''B''} \delta^{M''N''} \right. \\
 & \quad + \frac{1}{4t} (\Gamma^n \Gamma^m)^{ab} (\Gamma^N \Gamma^M)^{A\dot{B}} (\Gamma^{N''} \Gamma^{M''})^{\dot{A}''B''} \\
 & \quad + \frac{1}{4u} (\Gamma^m \Gamma^n)^{ab} (\Gamma^M \Gamma^N)^{A\dot{B}} (\Gamma^{M''} \Gamma^{N''})^{\dot{A}''B''} \\
 & \quad \left. + \frac{1}{s} [(\Gamma^{MN})^{A\dot{B}} C^{\dot{A}''B''} + C^{A\dot{B}} (\Gamma^{M''N''})^{\dot{A}''B''}] C^{ab} \delta^{mn} \right\}. \tag{3.19}
 \end{aligned}$$

(v) Two fermion – one graviton – one scalar amplitude

$$\begin{aligned}
 A &= \frac{g^2}{\sqrt{2}\pi} \int dz d\bar{z} \left\langle c\bar{c}V_{-\frac{1}{2}}^{\alpha\{a\}\{A\}}(1) c\bar{c}V_{-\frac{1}{2}}^{\beta\{b\}\{B\}}(2) \right. \\
 &\quad \left. \times V_0^{\mu\nu}(z, \bar{z}) c\bar{c}V_{-1}^{m\{M\}}(4) \right\rangle u_\alpha^{(1)} u_\beta^{(2)} \varepsilon_{\mu\nu} \\
 &= \sqrt{2} g^2 \frac{\Gamma(-\frac{1}{8}s)\Gamma(-\frac{1}{8}t)\Gamma(-\frac{1}{8}u)}{\Gamma(\frac{1}{8}s)\Gamma(\frac{1}{8}t)\Gamma(\frac{1}{8}u)} \frac{(\Gamma^m)^{\{a\}\{b\}}}{\sqrt{2}} \frac{(\Gamma^M)^{\{A\}\{B\}}}{\sqrt{2}} \\
 &\quad \times \left[\frac{1}{s} (k_4 \varepsilon k_4) u^{(1)} C u^{(2)} - \frac{1}{2t} k_2^\nu k_4^\rho u^{(1)} \gamma^\rho \gamma^\mu u^{(2)} \varepsilon_{\mu\nu} - \frac{1}{2u} k_1^\nu k_4^\rho u^{(1)} \gamma^\mu \gamma^\rho u^{(2)} \varepsilon_{\mu\nu} \right].
 \end{aligned} \tag{3.20}$$

(vi) Two scalar – two gauge boson amplitude

$$\begin{aligned}
 A^{ij} &= \frac{2g^2}{\pi} \int dz d\bar{z} \left\langle c\bar{c}V_{-1}^{m\{M\}}(1) c\bar{c}V_{-1}^{n\{N\}}(2) V_0^{\mu i}(z, \bar{z}) c\bar{c}V_0^{j\nu}(4) \right\rangle \varepsilon_\mu^{(1)} \varepsilon_\nu^{(2)} \\
 &= \frac{g^2}{2} \frac{\Gamma(-\frac{1}{8}s)\Gamma(-\frac{1}{8}t)\Gamma(-\frac{1}{8}u)}{\Gamma(\frac{1}{8}s)\Gamma(\frac{1}{8}t)\Gamma(\frac{1}{8}u)} \\
 &\quad \times \left[\frac{\frac{1}{4}(T^i T^j) \delta^{\{M\}\{N\}}}{s(1+\frac{1}{8}s)} + \frac{4(T^i T^j)^{\{M\}\{N\}}}{su} + \frac{4(T^j T^i)^{\{M\}\{N\}}}{ts} \right] \\
 &\quad \times \left[-u(\varepsilon^{(1)} \cdot \varepsilon^{(2)}) + 2t(\varepsilon^{(1)} \cdot k_3)(\varepsilon^{(2)} \cdot k_4) + 2u(\varepsilon^{(1)} \cdot k_4)(\varepsilon^{(2)} \cdot k_3) \right].
 \end{aligned} \tag{3.21}$$

Here we have again to distinguish between two cases: (a) the two scalars do not couple directly to the gauge bosons, i.e. transform non-trivially under a different factor of the gauge group; (b) scalar and gauge particles transform under a common gauge group. The formula eq. (3.21) is true for case (b), whereas for case (a) only the first term in eq. (3.21) survives.

(vii) Two fermion – two gauge boson amplitudes

$$\begin{aligned}
 A^{ij} &= \frac{g^2}{\sqrt{2}\pi} \int dz d\bar{z} \left\langle c\bar{c}V_{-\frac{1}{2}}^{\alpha\{a\}\{A\}}(1) c\bar{c}V_{-\frac{1}{2}}^{\beta\{b\}\{B\}}(2) \right. \\
 &\quad \left. \times V_0^{\mu i}(z, \bar{z}) c\bar{c}V_{-1}^{j\nu}(4) \right\rangle u_\alpha^{(1)} u_\beta^{(2)} \varepsilon_\mu^{(3)} \varepsilon_\nu^{(4)}.
 \end{aligned} \tag{3.22}$$

Here we have to distinguish again between two possibilities: (a) The two gauge

bosons belong to different simple factors of the gauge group, say D_2 and D_5 . In this case eq. (3.22) leads to

$$A^{ij'} = g^2 \frac{\Gamma(-\frac{1}{8}s)\Gamma(-\frac{1}{8}t)\Gamma(-\frac{1}{8}u)}{\Gamma(\frac{1}{8}s)\Gamma(\frac{1}{8}t)\Gamma(\frac{1}{8}u)} C^{(a)\{b\}} (T^i)^{A\dot{B}} (T^{j'})^{A'\dot{B}'}$$

$$\times \left\{ \frac{1}{t} u^{(1)} \not{k}^{(4)} (\not{k}_1 + \not{k}_4) \not{k}^{(3)} u^{(2)} + \frac{1}{u} u^{(1)} \not{k}^{(3)} (\not{k}_1 + \not{k}_3) \not{k}^{(4)} u^{(2)} \right\}. \quad (3.23)$$

(b) In the case where both gauge bosons belong to the same simple factor of the gauge group, eq. (3.22) gives:

$$A^{ij} = g^2 \frac{\Gamma(-\frac{1}{8}s)\Gamma(-\frac{1}{8}t)\Gamma(-\frac{1}{8}u)}{\Gamma(\frac{1}{8}s)\Gamma(\frac{1}{8}t)\Gamma(\frac{1}{8}u)} C^{(a)\{b\}}$$

$$\times \left\{ \left[\frac{1}{8} \frac{u}{s(1+\frac{1}{8}s)} \delta^{ij} C^{(B)\{A\}} + \frac{1}{t} (T^i T^j)^{(B)\{A\}} \right] u^{(1)} \not{k}^{(4)} (\not{k}_1 + \not{k}_4) \not{k}^{(3)} u^{(2)} \right.$$

$$- \left[\frac{1}{8} \frac{t}{s(1+\frac{1}{8}s)} \delta^{ij} C^{(A)\{B\}} + \frac{1}{u} (T^i T^j)^{(A)\{B\}} \right] u^{(1)} \not{k}^{(3)} (\not{k}_1 + \not{k}_3) \not{k}^{(4)} u^{(2)}$$

$$+ f^{ijk} (T^k)^{(A)\{B\}} [(k_4 \cdot \varepsilon^{(3)}) u^{(1)} \not{k}^{(4)} u^{(2)} + (\varepsilon^{(3)} \cdot \varepsilon^{(4)}) u^{(1)} \not{k}_3 u^{(2)}$$

$$\left. - (k_3 \cdot \varepsilon^{(4)}) u^{(1)} \not{k}^{(3)} u^{(2)} \right\}. \quad (3.24)$$

(viii) Two fermion – two graviton amplitude

$$A = \frac{g^2}{\sqrt{2}\pi} \left\langle c\bar{c}V_{-\frac{1}{2}}^{\alpha(a)\{A\}}(1) c\bar{c}V_{-\frac{1}{2}}^{\beta(b)\{B\}}(2) V_0^{\mu\nu}(z, \bar{z}) c\bar{c}V_{-1}^{\rho\sigma}(4) \right\rangle u_a^{(1)} u_b^{(2)} \varepsilon_{\mu\nu}^{(3)} \varepsilon_{\rho\sigma}^{(4)}$$

$$= \frac{g^2}{4} \frac{\Gamma(-\frac{1}{8}s)\Gamma(-\frac{1}{8}t)\Gamma(-\frac{1}{8}u)}{\Gamma(\frac{1}{8}s)\Gamma(\frac{1}{8}t)\Gamma(\frac{1}{8}u)} C^{(A)\{B\}} C^{(a)\{b\}}$$

$$\times \left\{ \left[-\frac{1}{t} k_1^\rho k_2^\nu - \frac{1}{2} \frac{u}{s(1+\frac{1}{8}s)} \left(\eta^{\rho\sigma} - \frac{k_4^\nu k_3^\sigma}{4} \right) \right] u^{(1)} \gamma^\rho (\not{k}_1 + \not{k}_4) \gamma^\nu u^{(2)} \right.$$

$$+ \left[-\frac{1}{u} k_1^\nu k_2^\sigma - \frac{1}{2} \frac{t}{s(1+\frac{1}{8}s)} \left(\eta^{\rho\sigma} - \frac{k_4^\nu k_3^\sigma}{4} \right) \right] u^{(1)} \gamma^\nu (\not{k}_1 + \not{k}_3) \gamma^\rho u^{(2)}$$

$$\left. + \frac{1}{s} [k_4^\mu u^{(1)} \gamma^\rho u^{(2)} + \eta^{\mu\rho} u^{(1)} \not{k}_3 u^{(2)} - k_3^\rho u^{(1)} \gamma^\mu u^{(2)}] (k_1^\nu k_2^\sigma - k_2^\nu k_1^\sigma) \right\} \varepsilon_{\mu\nu}^{(3)} \varepsilon_{\rho\sigma}^{(4)}. \quad (3.25)$$

(ix) Two scalar – one graviton – one gauge boson amplitude

$$\begin{aligned}
 A^i &= \frac{2g^2}{\pi} \int dz d\bar{z} \langle c\bar{c}V_{-1}^{m\{M\}}(1) c\bar{c}V_{-1}^{n\{N\}}(2) V_0^{\mu\nu}(z, \bar{z}) c\bar{c}V_0^{\rho i}(4) \rangle \varepsilon_{\mu\nu}^{(3)} \varepsilon_{\rho}^{(4)} \\
 &= g^2 \frac{\Gamma(-\frac{1}{8}s)\Gamma(-\frac{1}{8}t)\Gamma(-\frac{1}{8}u)}{\Gamma(\frac{1}{8}s)\Gamma(\frac{1}{8}t)\Gamma(\frac{1}{8}u)} \delta^{mn} (T^i)^{\{M\}\{N\}} \\
 &\quad \times \left\{ (\varepsilon^{(4)} \varepsilon^{(3)} (k_2 - k_1)) - \frac{2}{u} (k_1 \varepsilon^{(3)} k_1) (k_2 \cdot \varepsilon^{(4)}) - \frac{2}{t} (k_2 \varepsilon^{(3)} k_2) (k_1 \cdot \varepsilon^{(4)}) \right. \\
 &\quad \left. + \frac{1}{s} \left[\frac{t-u}{2} (\varepsilon^{(4)} \varepsilon^{(3)} k_4) + 2(k_1 \varepsilon^{(3)} k_4) (k_2 \cdot \varepsilon^{(4)}) - 2(k_2 \varepsilon^{(3)} k_4) (k_1 \cdot \varepsilon^{(4)}) \right] \right\}. \quad (3.26)
 \end{aligned}$$

(x) Two fermion – one graviton – one gauge boson amplitude

$$\begin{aligned}
 A^i &= \frac{g^2}{\sqrt{2}\pi} \int dz d\bar{z} \langle c\bar{c}V_{-\frac{1}{2}}^{\alpha\{a\}\{A\}}(1) c\bar{c}V_{-\frac{1}{2}}^{\beta\{b\}\{B\}}(2) V_0^{\mu\nu}(z, \bar{z}) c\bar{c}V_{-1}^{\rho i}(4) \rangle u_{\alpha}^{(1)} u_{\beta}^{(2)} \varepsilon_{\mu\nu}^{(3)} \varepsilon_{\rho}^{(4)} \\
 &= g^2 \frac{\Gamma(-\frac{1}{8}s)\Gamma(-\frac{1}{8}t)\Gamma(-\frac{1}{8}u)}{\Gamma(\frac{1}{8}s)\Gamma(\frac{1}{8}t)\Gamma(\frac{1}{8}u)} C^{\{a\}\{b\}}(T^i)^{\{A\}\{B\}} \\
 &\quad \times \left\{ \frac{1}{s} [k_4^{\mu} k_4^{\nu} u^{(1)} \gamma^{\rho} u^{(2)} + \eta^{\mu\rho} k_4 u^{(1)} \not{k}_3 u^{(2)} - k_3^{\rho} k_4^{\nu} u^{(1)} \gamma^{\mu} u^{(2)}] \right. \\
 &\quad \left. + \frac{1}{2u} k_1^{\nu} u^{(1)} \gamma^{\mu} (\not{k}_1 + \not{k}_3) \gamma^{\rho} u^{(2)} + \frac{1}{2t} k_2^{\nu} u^{(1)} \gamma^{\rho} (\not{k}_2 + \not{k}_3) \gamma^{\mu} u^{(2)} \right\} \varepsilon_{\mu\nu}^{(3)} \varepsilon_{\rho}^{(4)}. \quad (3.27)
 \end{aligned}$$

(xi) Two scalar – two graviton amplitude

$$\begin{aligned}
 A &= \frac{2g^2}{\pi} \int dz d\bar{z} \langle c\bar{c}V_{-1}^{m\{M\}}(1) c\bar{c}V_{-1}^{n\{N\}}(2) V_0^{\mu\nu}(z, \bar{z}) c\bar{c}V_0^{\rho\sigma}(4) \rangle \varepsilon_{\mu\nu}^{(3)} \varepsilon_{\rho\sigma}^{(4)} \\
 &= g^2 \frac{\Gamma(-\frac{1}{2}s)\Gamma(-\frac{1}{8}t)\Gamma(-\frac{1}{8}u)}{\Gamma(\frac{1}{8}s)\Gamma(\frac{1}{8}t)\Gamma(\frac{1}{8}u)} \delta^{mn} \delta^{\{M\}\{N\}} \\
 &\quad \times \left\{ \left[\frac{1}{2} \frac{u}{s(1+\frac{1}{8}s)} \left(\eta^{\nu\sigma} - \frac{k_4^{\nu} k_3^{\sigma}}{4} \right) + \frac{1}{s} (k_2^{\nu} k_1^{\sigma} - k_1^{\nu} k_2^{\sigma}) + \frac{1}{t} k_2^{\nu} k_1^{\sigma} \right] k_2^{\mu} k_1^{\rho} \right. \\
 &\quad \left. + \left[\frac{1}{2} \frac{t}{s(1+\frac{1}{8}s)} \left(\eta^{\nu\sigma} - \frac{k_4^{\nu} k_3^{\sigma}}{4} \right) + \frac{1}{s} (k_1^{\nu} k_2^{\sigma} - k_2^{\nu} k_1^{\sigma}) + \frac{1}{u} k_1^{\nu} k_2^{\sigma} \right] k_1^{\mu} k_2^{\rho} \right. \\
 &\quad \left. + \left[-\frac{1}{4} \frac{ut}{s(1+\frac{1}{8}s)} \left(\eta^{\nu\sigma} - \frac{k_4^{\nu} k_3^{\sigma}}{4} \right) + \frac{t}{s} k_1^{\nu} k_2^{\sigma} + \frac{u}{s} k_1^{\sigma} k_2^{\nu} \right] \delta^{\mu\rho} \right\} \varepsilon_{\mu\nu}^{(3)} \varepsilon_{\rho\sigma}^{(4)}. \quad (3.28)
 \end{aligned}$$

(xii) Two fermion – one gauge boson – one scalar amplitudes

$$A^i = \frac{g^2}{\sqrt{2}\pi} \int dz d\bar{z} \langle c\bar{c}V_{-\frac{1}{2}}^{\alpha\{a\}\{A\}}(1)c\bar{c}V_{-\frac{1}{2}}^{\beta\{b\}\{B\}}(2)V_0^{\mu i}(z, \bar{z})c\bar{c}V_{-1}^{m\{M\}}(4) \rangle u_{\alpha}^{(1)}u_{\beta}^{(2)}\epsilon_{\mu} \tag{3.29}$$

Here we have to distinguish two cases: (a) The scalar does not couple directly to the gauge boson. For our specific model this is the case if the gauge boson belongs to SO(4). We then get

$$A^i = \frac{4g^2}{\sqrt{2}} \frac{\Gamma(-\frac{1}{8}s)\Gamma(-\frac{1}{8}t)\Gamma(-\frac{1}{8}u)}{\Gamma(\frac{1}{8}s)\Gamma(\frac{1}{8}t)\Gamma(\frac{1}{8}u)} (T^i)^{A'B'} \left\{ \sqrt{\frac{1}{2}}(\Gamma^m)^{ab}\sqrt{\frac{1}{2}}(\Gamma^M)^{AB}\sqrt{\frac{1}{2}}(\Gamma^{M''})^{A''B''} \right\} \\ \times \left[-\frac{1}{t}u^{(1)}\not{k}_4\not{u}^{(2)} + \frac{1}{u}u^{(1)}\not{k}_4\not{u}^{(2)} \right]. \tag{3.30}$$

(b) In the second case the scalar does couple directly to the gauge boson which, for concreteness we chose to be a gauge boson of SO(10). We obtain for the amplitude

$$A^i = 2\sqrt{2}g^2 \frac{\Gamma(-\frac{1}{8}s)\Gamma(-\frac{1}{8}t)\Gamma(-\frac{1}{8}u)}{\Gamma(\frac{1}{8}s)\Gamma(\frac{1}{8}t)\Gamma(\frac{1}{8}u)} \left\{ \sqrt{\frac{1}{2}}(\Gamma^m)^{\{a\}\{b\}}\sqrt{\frac{1}{2}}(\Gamma^{M''})^{A''B''}C^{A'B'} \right\} \\ \times \left[-\frac{1}{2t}\frac{1}{\sqrt{2}}(\Gamma^M T^i)^{AB}u^{(1)}\not{k}_4\not{u}^{(2)} + \frac{1}{2u}\frac{1}{\sqrt{2}}(T^i \Gamma^M)^{AB}u^{(1)}\not{k}_4\not{u}^{(2)} \right. \\ \left. + \frac{1}{s}\frac{1}{\sqrt{2}}(T^i)^{PM}(\Gamma^P)^{AB}(\epsilon \cdot k_4)u^{(1)}C u^{(2)} \right]. \tag{3.31}$$

All other four-point amplitudes vanish by conservation of lattice momentum or are already given in ref. [42].

4. Effective action for four-dimensional heterotic strings

In this section we will derive the effective action for the massless fields of four-dimensional heterotic string theory*. The first indication that string theory is a low energy expansion about a point particle theory arose in the paper of Neveu and Scherk [46]. In the following it was shown by Yoneya [47] and Scherk, Schwarz [48] that the effective action of the massless spin two state is in the zero slope limit given by the Einstein action of gravitation. Later the effective action for the superstring [49] and the bosonic part of the heterotic string [41, 42] was derived.

* In refs. [44, 45] the low energy effective supergravity field theory of four-dimensional strings was obtained by a truncation of $N = 4$ supergravity.

In general, the (classical) effective action of string theory is obtained by calculating the scattering amplitudes of the massless particles in tree approximation*. One then constructs an effective lagrangian, invariant under all the symmetries of the theory, which reproduces these amplitudes. The terms up to cubic order, $\mathcal{L}_{3\text{pt}}$, are determined by the string 3-point amplitudes. The coefficients of some of these terms can in principle contain arbitrary parameters which represent the fact that on-shell amplitudes can be independent under field redefinitions of the massless fields. $\mathcal{L}_{3\text{pt}}$ already allows to relate various coupling constants of the effective action to the string coupling constant g^{**} and to the string tension α' if we replace the external momenta k by $k\sqrt{2\alpha'}$. In this way the expansion of the effective action in numbers of derivatives is nothing else but an expansion in powers of $\sqrt{2\alpha'}$.

At the next step one then considers the four-point amplitudes. Unitarity guarantees that the massless poles will be those generated from the tree graphs of $\mathcal{L}_{3\text{pt}}$. This allows us to check again the relations between the relevant coupling constants. The remainder is in general due to massive particle exchange and will be reproduced in the effective action either by four-point interactions already present in $\mathcal{L}_{3\text{pt}}$ or by new local vertices $\mathcal{L}_{4\text{pt}}$ which are of quartic order in the massless fields.

Our strategy for obtaining the effective action will be that we derive all terms cubic in the massless fields from the 3-point scattering amplitudes. For the vertices $\mathcal{L}_{4\text{pt}}$ coming from the four-point amplitudes, we restrict ourselves to consider only terms which contain at the most two derivatives. Therefore, we compute the complete effective action to quartic order in the massless fields to order $\sqrt{2\alpha'}$. However, our action does not contain all terms linear in $2\alpha'$.

Let us now consider the three point amplitudes of the four-dimensional heterotic string. The amplitudes, eqs. (3.3), (3.8), (3.9) which involve only the gauge bosons and the graviton, antisymmetric tensor field or dilaton are identical to those of the 10-dimensional heterotic string [41,42] and therefore also the effective action is practically the same. We use again the relation between metric $g_{\mu\nu}$ and graviton field $h_{\mu\nu}$, $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa_4 h_{\mu\nu}$, where κ_4 is the four-dimensional gravitational constant related to Newton's constant by $\kappa_4 = \sqrt{8\pi G}$. The only difference to ten dimensions arises due to the normalization of the dilaton field D . It is the transverse diagonal part of $\varepsilon_{\mu\nu}$: $\varepsilon_{\mu\nu} = \sqrt{\frac{1}{2}}(\eta_{\mu\nu} - k_\mu \bar{k}_\nu - \bar{k}_\mu k_\nu)D$ with $\bar{k} \cdot \bar{k} = 0$, $k \cdot \bar{k} = 1$. $\mathcal{L}_{3\text{pt}}$ corresponding to these fields is given by [42]:

$$\mathcal{L}_{3\text{pt}} = \sqrt{g} \left\{ \frac{1}{2\kappa_4^2} R - \frac{1}{6} \exp(-2\sqrt{2}\kappa_4 D) H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{2} (\nabla_\nu D) (\nabla^\nu D) - \frac{1}{8} \exp(-\sqrt{2}\kappa_4 D) \text{Tr}(F_{\mu\nu} F^{\mu\nu}) \right\} + \text{other terms}, \quad (4.1)$$

* Alternatively one can consider the β -functions of the corresponding σ -model [50,51].

** We will assume that the dilaton vacuum expectation value is constant such that the string coupling constant is given by g .

with

$$H_{\mu\nu\lambda} = \partial_{[\lambda} B_{\mu\nu]} - \frac{\kappa_4}{4} (\Omega_{3Y})_{\mu\nu\lambda} + \frac{\kappa_4}{4g_4^2} (\Omega_{3L})_{\mu\nu\lambda},$$

$$(\Omega_{3Y})_{\mu\nu\lambda} = \text{Tr} \left(A_{[\lambda} F_{\mu\nu]} - \frac{1}{3} g_4 A_{[\mu} A_{[\nu} A_{\lambda]}] \right),$$

$$(\Omega_{3L})_{\mu\nu\lambda} = \text{Tr} \left(\omega_{[\lambda} R_{\mu\nu]} - \frac{1}{3} \omega_{[\mu} \omega_{[\nu} \omega_{\lambda]}] \right),$$

“other terms” involve various couplings between $R_{\mu\nu\lambda\rho}$, $H_{\mu\nu\lambda}$ and D (see ref. [42], eq. (3.8)). As in ten dimensions we obtain the following relation between gravitational κ_4 , gauge g_4 and string coupling constant g in four dimensions

$$g_4 = g, \quad \kappa_4 = \frac{1}{2} g \sqrt{2\alpha'}. \quad (4.2)$$

In addition we recognize that all simple factors of the gauge group have identical coupling constants. The exponential factors containing the dilaton field in eq. (4.1) and equations below are required by (four-dimensional) conformal invariance. The general rule is to multiply each term in the lagrangian with conformal weight w by $\exp c(1+w)D$ [42]. The conformal weight is determined as follows: $g_{\mu\nu}(\gamma_\mu)$ and $g^{\mu\nu}(\gamma^\mu)$ have conformal weight $+1(+\frac{1}{2})$ and $-1(-\frac{1}{2})$ respectively and Ψ has conformal weight $-\frac{1}{4}$. All other fields have conformal weight zero. The constant c can be determined, for instance, by comparing the field theory with the one dilaton-two antisymmetric tensor string amplitude with the result (valid in d dimensions): $c = (2/\sqrt{d-2})\kappa_d$. On the other hand the terms which involve the scalar field Φ and the fermion Ψ are not discussed in the ten-dimensional heterotic string and lead to additional terms in the effective action.

The two scalar-one graviton amplitude eq. (3.11) together with the two scalar-one gauge boson amplitude eq. (3.4) is reproduced by the covariant kinetic energy for the scalars:

$$\mathcal{L}_{3\text{pt}} = \frac{1}{2} \sqrt{g} g^{\mu\nu} (D_\mu \Phi)^{m\{M\}\dagger} (D_\nu \Phi)^{m\{M\}}, \quad (4.3)$$

with

$$(D_\mu \Phi)^{m\{M\}} = \partial_\mu \Phi^{m\{M\}} - g_4 A_\mu^{\{M\}\{N\}} \Phi^{m\{N\}}.$$

Similarly, the two fermion – one graviton amplitude eq. (3.10) and the two fermion-one gauge boson amplitude eq. (3.5) corresponds to:

$$\mathcal{L}_{3\text{pt}} = \sqrt{g} \Psi^{\alpha\{a\}\{A\}} (\not{D}\Psi)_{\alpha\{a\}\{A\}}, \quad (4.4)$$

where

$$(\not{D}\Psi)_{\alpha\{a\}\{A\}} = g^{\mu\nu} \left(C_{\{A\}}^{\{B\}} \gamma_\mu \nabla_\nu - \frac{1}{4} i g_4 (\Gamma^{\{M\}\{N\}})_{\{A\}}^{\{B\}} \gamma_\mu A_\nu^{\{M\}\{N\}} \right)_\alpha \beta C_{\{a\}}^{\{b\}} \Psi_{\beta\{b\}\{B\}}$$

However, in the case of an antisymmetric tensor the amplitude eq. (3.10) induces a new term in the effective action:

$$\mathcal{L}_{3\text{pt}} = \lambda_4^{(1)} C^{(a)\{b\}} C^{(A)\{B\}} \exp(-\sqrt{2} \kappa_4 D) \Psi_{\{a\}\{A\}} \gamma^{\mu\nu\rho} \Psi_{\{b\}\{B\}} H_{\mu\nu\rho}, \quad (4.5)$$

with

$$\lambda_4^{(1)} = \frac{1}{2} \kappa_4 = \frac{1}{4} g_4 \sqrt{2\alpha'}. \quad (4.6)$$

Finally, the two fermion-one scalar amplitude eq. (3.7) leads to the following Yukawa coupling in the effective action:

$$\begin{aligned} \mathcal{L}_{3\text{pt}} = & \frac{1}{2} \sqrt{g} \exp\left(\frac{\kappa_4}{\sqrt{2}} D\right) h_4 C^{\alpha\beta} \frac{(\Gamma^m)^{(a)\{b\}}}{\sqrt{2}} \frac{(\Gamma^M)^{(A)\{B\}}}{\sqrt{2}} \\ & \times \Psi_{\alpha\{a\}\{A\}} \Psi_{\beta\{b\}\{B\}} \Phi^{m\{M\}}, \end{aligned} \quad (4.7)$$

h_4 is the four-dimensional Yukawa coupling constant and has a fixed relation to the gauge coupling constant:

$$h_4 = \sqrt{2} g_4. \quad (4.8)$$

Thus, in string theory the Yukawa coupling matrix is directly calculable and expressed in terms of known coupling constants. For example, taking the 16 fermions F_1 of eq. (2.30) ($\text{SO}(6)_{\text{global}} \times \text{SO}(4)_{\text{local}}$ acts as horizontal symmetry group) which interact with ϕ_1 of eq. (2.32), we obtain an explicit Yukawa coupling matrix. Specifically the fermions which transform as 2 and $\bar{2}$ under $\text{SO}(4)_{\text{local}}$ are not coupled such that the Yukawa coupling matrix is a block-diagonal 16×16 matrix where each block has the following form:

$$H_{ij} = \frac{h_4}{2\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.9)$$

Next let us discuss the field theory connected to the four-point amplitudes. Since we look only at terms with maximal two derivatives, we can set the Γ -function factor equal to unity.

For $k \rightarrow 0$ the four gauge boson amplitude is fully reproduced by the graviton and gauge boson poles, i.e. by the exchange of massless gravitons and gauge bosons (see

also ref. [41]). This can be proved using the three gauge boson, resp. two gauge boson-graviton vertex derived from eq. (4.1). We again find that the gauge coupling for the different simple factors are identical and that $\kappa_4 = \frac{1}{2}g_4\sqrt{2\alpha'}$.

The four scalar amplitude is constituted in the limit $k \rightarrow 0$ by the exchange of a massless graviton and gauge bosons and also by direct contact terms. The graviton poles in the s, t, u -channel (in the third and fourth line of eq. (3.15)) are reproduced in the field theory by using the one graviton-two scalar vertex obtained from eq. (4.3):

$$V_{\mu\nu} = \kappa_4(\eta_{\mu\nu}(k_1 \cdot k_2) - k_{1\mu}k_{2\nu} - k_{2\mu}k_{1\nu}). \tag{4.10}$$

Analogously the gauge boson poles are contained in the last 3 brackets of eq. (3.15). Then, after subtracting these pole terms, we obtain for $k \rightarrow 0$ the following four-scalar interactions:

$$\begin{aligned} \mathcal{L}_{4pt} = \frac{1}{4}\exp(\sqrt{2}\kappa_4 D) \{ & \lambda_4^{(2)} [\Phi^{mMM''}\Phi^{nNN''}(\Phi^{mMN''}\Phi^{nNM''} + \Phi^{mNM''}\Phi^{nMN''}) \\ & + \lambda_4^{(3)}\Phi^{mMM''}\Phi^{nNM''}\Phi^{mNN''}\Phi^{nMN''}] \}. \end{aligned} \tag{4.11}$$

From eq. (3.15) we deduce that the scalar self-couplings $\lambda_4^{(2,3)}$ can be expressed in terms of the gauge coupling as follows:

$$\lambda_4^{(2)} = 2g_4^2, \quad \lambda_4^{(3)} = -g_4^2. \tag{4.12}$$

The four-fermion coupling has only poles due to graviton exchange in the t - and u -channel. This comes from the fact that in the s -channel the two fermions cannot be pairwise neutral under the U(1)-charge. The graviton poles are reproduced using the following two fermion - one graviton vertex:

$$V_{\mu\nu} = \frac{1}{4}\kappa_4 C^{ab}C^{A\dot{B}}C^{A'B'} [(\gamma_\mu)^{\alpha\dot{\beta}}(k_1 - k_2)_\nu + (\gamma_\nu)^{\alpha\dot{\beta}}(k_1 - k_2)_\mu]. \tag{4.13}$$

Analogously gauge boson poles appear only in the t - and u -channel. In addition, the amplitude eq. (3.17) has poles due to the exchange of the U(1)-charged scalar ϕ_1 (see eq. (2.32)) in the s -channel. This can be seen by using the SO(6) Fierz-identity

$$C^{a\dot{c}}C^{b\dot{d}} - C^{a\dot{d}}C^{b\dot{c}} = -\frac{1}{2}(\Gamma^m)^{ab}(\Gamma^m)^{\dot{c}\dot{d}}. \tag{4.14}$$

Then, after subtracting all these poles, the following four-fermion interaction survives:

$$\begin{aligned} \mathcal{L}_{4pt} = & -\lambda_4^{(4)}\Psi_{\alpha a A A' \dot{A}'}\Psi_{\beta b B B' \dot{B}'}\Psi_{\gamma \dot{c} C C' \dot{C}'}\Psi_{\delta \dot{d} D D' \dot{D}'} \\ & \times C^{\alpha\beta}C^{\gamma\dot{\delta}}(\Gamma^m)^{ab}(\Gamma^m)^{\dot{c}\dot{d}}(C^{A''C''}C^{\dot{B}''D''} + C^{A''D''}C^{\dot{B}''C''}) \\ & \times (C^{A\dot{C}}C^{B\dot{D}}C^{A'C'}C^{D'B'} + C^{A\dot{D}}C^{B\dot{C}}C^{A'D'}C^{B'C'}), \end{aligned} \tag{4.15}$$

with

$$\lambda_4^{(4)} = \frac{1}{8} g_4^2 (2\alpha'). \quad (4.16)$$

Next, let us examine the two fermion-two scalar amplitude. The pole terms can be completely accounted for by the terms in $\mathcal{L}_{3\text{pt}}$: the s -poles arise due to graviton and gauge boson exchange and the t - and u -poles due to fermion exchange. The remaining term gives rise to a new contact interaction:

$$\mathcal{L}_{4\text{pt}} = \lambda_4^{(5)} \Psi_{\alpha a \{A\}} \Psi_{\beta b \{B\}} \Phi_{m \{M\}} \tilde{\delta} \Phi_{n \{N\}} C^{\{A\}\{B\}} \delta^{\{M\}\{N\}} (\gamma^\mu)^{\alpha\beta} (\Gamma^{mn})^{ab} \quad (4.17)$$

with

$$\lambda_4^{(5)} = \frac{1}{16} g^2 (2\alpha'). \quad (4.18)$$

The two fermion-one graviton-one scalar amplitude eq. (3.20) is, for the case of an external on-shell graviton, completely reproduced by the couplings contained in eqs. (4.3), (4.4), (4.7), with the s -channel pole arising due to scalar exchange and the t - and u -channel poles resulting from fermion exchange. In the case where the external particle is an antisymmetric tensor, the s -channel pole vanishes and the t - and u -channel poles are again due to fermion exchange, with the coupling of the antisymmetric tensor to fermions given by eq. (4.5). If the external particle is a dilaton, the amplitude eq. (3.20) reduces to an expression which is fully reproduced by the two fermion-one dilaton-one scalar four-point coupling contained in eq. (4.7), i.e. no new contribution to $\mathcal{L}_{4\text{pt}}$ is necessary.

The two scalar, two gauge boson amplitude eq. (3.21) produces no new interactions in $\mathcal{L}_{4\text{pt}}$. This amplitude is fully reproduced by the vertices of $\mathcal{L}_{3\text{pt}}$. Graviton and gauge boson exchange both give rise to a pole in the s -channel, whereas the poles in the u - and t -channels are solely due to gauge boson exchange. Finally, also the direct two scalar-two gauge boson coupling in eq. (4.3) contributes to this amplitude.

The two fermion, two gauge boson amplitude is obtained again entirely by the interactions of $\mathcal{L}_{3\text{pt}}$ – the exchange of a graviton and gauge bosons in the s -channel as well as the exchange of massless fermions in the t - and u -channels generate the amplitudes eqs. (3.23), (3.24) for low momenta.

In the same way, the exchange of a graviton in s -channel and of massless fermions in the t - and u -channels and, in addition, also the direct contact term in eq. (4.4) give the leading contribution to the two fermion, two graviton amplitude.

Likewise the remaining four-point amplitudes are reproduced by the terms in $\mathcal{L}_{3\text{pt}}$, partially via exchange contributions and partially by direct couplings, and no new contact terms are needed.

In summary, to lowest order in $\sqrt{2\alpha'}$ the low energy effective action consists just of gauge fields and gravity coupled to scalars and fermions plus some self-interac-

tions of scalars and fermions, with all the coupling constants related to the string coupling in a definite way.

5. One-loop string perturbation theory

As it is well known one-loop amplitudes in first quantized string theory are obtained performing the path integral over genus-one Riemann surface – the torus. This involves the calculation [52, 53] of correlation functions on the torus which are replacing the corresponding correlators eq. (3.2) on the sphere.

In addition, further interesting problems arise beyond the tree level calculation due to zero modes of ghost and fermionic fields. These issues were discussed recently in refs. [54, 40], and we want to mention some of the main aspects needed for our discussion.

For the bosonic string there are in the case of $g = 1$ one complex zero mode of the b and c ghost fields. The zero modes of b are known as moduli and correspond to deformations of the world sheet metric which change its conformal structure. The zero modes of c correspond to the translational invariance on the 2-dimensional torus. Performing the path integral one has to integrate over all conformally inequivalent surfaces. Therefore, the amplitudes involve the integration over the moduli τ and $\bar{\tau}$.

In addition, one has to soak up the zero modes of the b, c ghosts. This can be done in a BRST invariant fashion by inserting $\langle b\bar{b}|c\bar{c}\rangle = \int d^2z b\bar{b}c\bar{c}$ into the path integral. This factor leads, together with the proper normalization of the b zero mode to a factor proportional to $(\text{Im } \tau)^{-1}$ yielding exactly the correct τ integration measure. We see that in the one loop case the b, c zero modes can be already absorbed if no external vertex is attached in contrast to the sphere where 0, 1, 2-point amplitudes vanish.

Next we turn to the β, γ -superconformal ghost system. For even spin structure on the torus (AP, PA, AA) there are no zero modes of these fields. Then, the path integral contains just the product of N vertex operators such that the ghost charge adds up to zero, e.g.:

$$A = g^N \int \prod_{i=1}^N d^2z_i \int \frac{d^2\tau}{\text{Im } \tau} \int dX d\psi db d\bar{b} dc d\bar{c} d\beta d\gamma e^{-S} \prod_{i=1}^N V_{q_i}(z_i, \bar{z}_i), \quad (5.1)$$

with

$$\sum_i q_i = 0.$$

However, for odd (PP) spin structure, β and γ have zero modes like in the case of higher genus Riemann surfaces. Specifically, there is a complex β zero mode – the supermoduli – and, using the Riemann-Roch index theorem, also a complex γ -zero mode.

To deal with the supermoduli, it turns out [40] to be most convenient to perform the integration over the supermoduli explicitly. Since the supermoduli are coupled to the fermionic stress energy tensor T_F in the action, this integration brings one factor of $\langle \chi | T_F \rangle = \int d^2z \chi(z) T_F(z)$ down to the path integral, where χ are the super-Beltrami differentials dual to the supermoduli. In addition, one has to insert an operator $\delta(\langle \chi | \beta \rangle)$ to absorb the β -zero mode. The combination $\delta(\langle \chi | \beta \rangle) \langle \chi | T_F \rangle$ is BRST invariant [15, 40] and is actually nothing else than the picture changing operator. This can be seen explicitly by setting $\chi(z) = \delta(z - z_a)$ and using that $\delta(\beta(z_a)) = e^\phi(z_a)$. Thus, the path integral contains one factor of $e^\phi(z_a) \cdot T_F(z_a)$ at an arbitrary point z_a .

In order to take the γ -zero modes into account, we restrict the γ -functional integral again to fields orthogonal to the γ -zero modes, i.e. inserting one factor of $\theta\delta(\gamma(z))$ at one of the vertices. Performing the θ -integration we obtain amplitudes of the following form (we do not consider the case when spin fields are present):

$$A = g^N \int \prod_{i=1}^N d^2z_i \int \frac{d^2\tau}{\text{Im}\tau} \int dX d\psi db d\bar{b} dc d\bar{c} d\beta d\gamma e^{-S} \times \prod_{i=1}^{N-1} V_0(z_i, \bar{z}_i) V_{-1}(z_N, \bar{z}_N) (e^\phi T_F(z_a)). \tag{5.2}$$

This expression involves the correlation function of the β - γ ghost fields on the torus. However, calculating 2-point amplitudes in the next section, we will not use the explicit form of these correlation functions because we compare the amplitude with the already known expression for the partition function which contains the same ghost correlations.

Finally, let us also mention the zero modes of the world sheet fermions ψ^μ . For even spin structures there are no zero modes. On the other hand, for odd spin structures in the 4-dimensional heterotic string theory there are 4 ψ zero-modes. Thus, in the PP sectors one needs at least 3 additional fermions ψ_μ from the external vertices to absorb the ψ zero-modes to get a non-vanishing result (T_F already contains one field ψ^μ).

5.1. THE ONE-LOOP PARTITION FUNCTION

For even spin structure the vacuum amplitude Z is given by:

$$Z = \int \frac{d^2\tau}{\text{Im}\tau} \int dX d\psi db d\bar{b} dc d\bar{c} d\beta d\gamma e^{-S}. \tag{5.3}$$

However, in order to obtain the correct normalization of Z we will not consider this expression, but we construct the one-loop partition function of the 4-dimensional

heterotic string using the hamiltonian formalism. The left- and right-moving Hamilton operators take the following forms:

$$\begin{aligned}
 H_L &= \frac{1}{8}p_\mu^2 + \frac{1}{2}\lambda_L^2 + N_L - 1, \\
 H_R &= \frac{1}{8}p_\mu^2 + \frac{1}{2}(\lambda_R^2 + \lambda'_R{}^2) - \frac{1}{2}q^2 - q + N_R - 1.
 \end{aligned}
 \tag{5.4}$$

p_μ are the four-dimensional momenta and $(\lambda_L, \lambda'_R, \lambda_R, q) \in \Gamma_{22,11,1}$. The vacuum energy is given as

$$\begin{aligned}
 Z &= \int_F \frac{d^2\tau}{(\text{Im } \tau)^2} \chi(\tau, \bar{\tau}), \\
 \chi(\tau, \bar{\tau}) &= \text{Im } \tau \frac{1}{32} \pi^2 \text{Tr } \bar{q}^{H_L} q^{H_R} \Phi,
 \end{aligned}
 \tag{5.5}$$

where Φ is a phase and F the fundamental region in Teichmüller space. The factor $\frac{1}{32}\pi^2$ was obtained by comparing this expression with the partition function computed in the operator formalism ($\alpha' = \frac{1}{2}$). With eq. (5.4) we obtain for $\chi(\tau, \bar{\tau})$:

$$\begin{aligned}
 \chi(\tau, \bar{\tau}) &= \frac{1}{\text{Im } \tau} \frac{1}{128\pi^2} \frac{1}{\eta^{12}(\tau)} \frac{1}{\eta^{24}(\bar{\tau})} \frac{1}{\Theta_{1,1}(\tau)} \\
 &\times \sum_{w=(\lambda_L, \lambda'_R, \lambda_R, q) \in \Gamma_{22,11,1}} \bar{q}^{\frac{1}{2}\lambda_L^2} q^{\frac{1}{2}\lambda_R^2 + \frac{1}{2}\lambda'_R{}^2 - \frac{1}{2}q^2 - q - \frac{1}{2}} e^{2\pi i q}.
 \end{aligned}
 \tag{5.6}$$

Here, $\Theta_{1,1}(\tau)$ is the lattice partition function of an even self-dual lorentzian lattice $\Gamma_{1,1}$ [22] and reduces the above expression to the partition function of the physical light-cone states. The factor $e^{2\pi i q}$ ensures the correct spin-statistics relation. As mentioned in the introduction, $\chi(\tau, \bar{\tau})$ is modular invariant if $\Gamma_{22,11,1}$ is odd self-dual [22].

For arbitrary non-supersymmetric 4-dimensional heterotic string theories, Z is in general non-vanishing and leads to a cosmological constant in the effective action. This also results in a non-vanishing dilaton tadpole and the question of finiteness of the non-supersymmetric theory arises. However there may be still a chance that Z is zero even in a non-supersymmetric theory. In a recent, very interesting paper Moore [55] showed that, if $\chi(\tau, \bar{\tau}) \neq 0$ has ‘‘Atkin-Lehner’’ symmetry, integration over the fundamental region in moduli space forces Z to vanish. Whether, however, models with Atkin-Lehner symmetry can be found within the covariant lattices is not yet clear to us.

Let us briefly investigate the partition function in the PP sector of the theory. It is given by

$$Z = \int \frac{d^2\tau}{\text{Im } \tau} \int dX d\psi db d\bar{b} dc d\bar{c} d\beta d\gamma e^{-S} e^{-\phi(z)} (e^\phi T_F)(z_a). \tag{5.7}$$

In the hamiltonian formalism it is defined as

$$\chi(\tau, \bar{\tau}) = \text{Im } \tau \frac{1}{32} \pi^2 \text{Tr } \bar{q}^{H_L} (-1)^{F_R} q^{H_R}, \tag{5.8}$$

and can be also written in terms of lattice vectors of $\Gamma_{22,11,1}$:

$$\begin{aligned} \chi(\tau, \bar{\tau}) = & \frac{1}{\text{Im } \tau} \frac{1}{128 \pi^2} \frac{1}{\eta^{12}(\tau)} \frac{1}{\eta^{24}(\bar{\tau})} \frac{1}{\Theta_{1,1}(\tau)} \\ & \times \left(\sum_{\lambda_R \in S \text{ of SO}(4)} - \sum_{\lambda_R \in C \text{ of SO}(4)} \right) \left(\bar{q}^{\frac{1}{2} \lambda_L^2} q^{\frac{1}{2} (\lambda_R^2 + \lambda_R^2) - \frac{1}{2} q^2 - q - \frac{1}{2}} \right). \tag{5.9} \end{aligned}$$

This vanishes as explained in the first section.

5.2. TWO-POINT FUNCTIONS

Unlike the tree-level calculations, the two-point functions in general do not vanish on the torus. One interesting example is the two-point function of 2 scalar particles which leads to a radiative (mass)² term in the effective action and can cause spontaneous gauge symmetry breaking. In ref. [56] this amplitude was calculated explicitly for an arbitrary supersymmetry preserving background with the result that scalars, which are charged under an anomalous U(1)-symmetry, get radiative mass contribution. In this way a Fayet-Ilioupoulos term is created which can trigger spontaneous supersymmetry breaking. For arbitrary non-supersymmetric models m_ϕ^2 cannot be computed explicitly; it depends on the (in principle calculable) one-loop partition function and needs explicit (probably numerical) integration in moduli space. This comes from the fact that, using the vertex operators of eq. (2.32), m_ϕ^2 is non-zero only for the even spin-structure. It vanishes however for the odd spin structures since there are not enough ψ fields to absorb the zero modes in this sector.

As an example for a one-loop two point function which can be calculated model independently we consider the “mass” mixing term between the antisymmetric tensor fields B and U(1) gauge field which belongs to an “anomalous” U(1)-gauge symmetry. This $B \text{Tr } F$ term was recently calculated in the old operator formalism in ref. [57]. Here, we like to emphasize some of the aspects in conformal field theory.

In this context one means with anomalous U(1)-symmetry, a U(1) with non-vanishing trace over the charge of the massless fermions. However, the theory possesses no true anomaly; the contribution of the anomalous triangle diagram is cancelled exactly via the Green-Schwarz mechanism [38] by the one-loop diagram which involves the $B \text{Tr } F$ term. This amplitude can be calculated in a model independent way since it is related to the partition function in the PP sector which is (after factorizing out the zero modes) a holomorphic function of \bar{q} and does not

depend at all on the massive spectrum. Only the massless fermions which have non vanishing U(1) charge will contribute to this amplitude.

First, let us briefly see the vanishing of the amplitude in the even spin structure sector; it is given by:

$$A = g^2 \int d^2z_1 d^2z_2 d^2\tau \langle V_0^B(z_1, \bar{z}_1) V_0^A(z_2, \bar{z}_2) \rangle_{\text{even}}. \tag{5.10}$$

The correlation functions of the various fields on the torus are given in refs. [52, 53]. However, one can see immediately that the amplitude vanishes because of the on-shell conditions $k_1 = -k_2$, $k_1^2 = k_2^2 = \epsilon^{\mu\nu} k_{1\mu} = \epsilon^\rho k_{2\rho} = 0$.

Thus the only non vanishing contribution comes from the PP sector of the theory. Here the amplitude has the following form:

$$A = g^2 \int d^2z_1 d^2z_2 d^2\tau \langle V_0^B(z_1, \bar{z}_1) V_{-1}^A(z_2, \bar{z}_2) (e^\phi(z_a) T_F(z_a)) \rangle_{\text{PP}}. \tag{5.11}$$

First we recognize, since the vertex operators do not involve internal lattice excitations, that only the external part of T_F contributes to (5.11):

$$\begin{aligned} A &= g^2 \epsilon^{(1)\mu\nu} \epsilon^{(2)\rho} \\ &\times \int d^2z_1 d^2z_2 d^2\tau \langle (\partial X_\nu + ik_\lambda^{(1)} \psi^\lambda \psi_\nu)(z_1) \partial X_\mu(\bar{z}_1) \exp(ik_\mu^{(1)} X^\mu)(z_1, \bar{z}_1) \\ &\quad \times \psi_\rho(z_2) e^{-\phi(z_2)} \partial X^I(\bar{z}_2) \exp(ik_\mu^{(2)} X^\mu)(z_2, \bar{z}_2) \\ &\quad \times \frac{1}{2} e^\phi(z_a) \psi^\sigma(z_a) \partial X_\sigma(z_a) \rangle_{\text{PP}}. \end{aligned} \tag{5.12}$$

As we have mentioned before, we need 4 zero mode ψ^μ -fields in the correlator. Then replacing $\psi^\mu(z)$ by the constant zero mode piece ψ_0^μ eq. (5.12) becomes:

$$\begin{aligned} A &= ig^2 \epsilon^{(1)\mu\nu} \epsilon^{(2)\rho} k_\lambda^{(1)} \int d^2z_1 d^2z_2 d^2\tau \langle \psi_0^\lambda \psi_0^\nu \psi_0^\rho \psi_0^\sigma \partial X_\mu(\bar{z}_1) \exp(ik_\mu^{(1)} X^\mu)(z_1, \bar{z}_1) \\ &\quad \times e^{-\phi(z_2)} \partial X^I(\bar{z}_2) \exp(ik_\mu^{(2)} X^\mu)(z_2, \bar{z}_2) \frac{1}{2} e^\phi(z_a) \partial X_\sigma(z_a) \rangle_{\text{PP}}. \end{aligned} \tag{5.13}$$

In the PP sector of the theory we have that

$$\langle \psi_0^\lambda \psi_0^\nu \psi_0^\rho \psi_0^\sigma \rangle_{\text{PP}} = -4i \epsilon^{\lambda\nu\rho\sigma}. \tag{5.14}$$

The $\partial X_\mu(\bar{z}_1), \partial X_\sigma(z_a)$ contribute also only through their zero mode part [57], $\partial X_\mu(z) = \frac{1}{2}p_\mu$, since otherwise the correlators vanish in the on-shell limit $k^2 \rightarrow 0$. For the same reason, the correlator between $\exp(ik_\mu^{(1)}X^\mu)$ and $\exp(ik_\mu^{(2)}X^\mu)$ can be set to 1. Finally, the internal momentum $\partial X^I(z)$ is again equal to its constant zero mode piece, $\partial X^I(z) = p^I$. So we obtain the following expression:

$$A = \frac{1}{4}g^2 \epsilon_{\mu\nu}^{(1)} \epsilon_\rho^{(2)} \epsilon^{\lambda\nu\rho\mu} k_\lambda^{(1)} \times \int d^2z_1 d^2z_2 d^2\tau \langle p^2 \rangle_{\text{PP}} \langle p^I \rangle_{\text{PP}} \langle e^{-\phi}(z_2) e^\phi(z_a) \rangle_{\text{PP}}. \tag{5.15}$$

Apart from the factor in front of the integral this expression looks actually very similar to the partition function in the PP sector. The only differences are the integrations over z_1 and z_2 which yield an additional factor of $4(\text{Im } \tau)^2$, and the trace over the momenta p^2 bringing a factor of $4i/(\pi \text{Im } \tau)$. (Equivalently this factor is obtained by evaluating the correlation function of $\partial X_\mu(\bar{z}_1)$ and $\partial X_\sigma(z_a)$ where only the zero mode part contributes.) Of most important consequences is, however, the zero mode factor $\langle p^I \rangle_{\text{PP}}$. It is the appearance of this factor which makes the result non-vanishing. First we see as in the case of the PP partition function, that the right-moving massive states do not contribute such that the integrand is a function of \bar{q} only. But, unlike the case of the partition function, the massless fermions now do give a non-vanishing effect since their contribution is weighted by p^I which is nothing else than the U(1)-charge of the chiral massless fermions. The reason is that the left-handed fermions have opposite U(1)-charge compared to the right-handed antifermions. Therefore, because of the ‘‘anomaly’’ of the U(1)-symmetry, the vacuum has a non-vanishing U(1)-charge, and the expectation value $\langle p^I \rangle_{\text{PP}}$ does not vanish as one would naively expect – it is proportional to the trace over the U(1) charges of the massless fermions $\text{Tr}(p^I)$.

Now, we can compare eq. (5.15) with the expression of the partition function, and we finally obtain:

$$A = \frac{ig^2}{32\pi^3} \epsilon_{\mu\nu}^{(1)} \epsilon_\rho^{(2)} \epsilon^{\lambda\nu\rho\mu} k_\lambda^{(1)} \int_{\text{F}} \frac{d^2\tau}{(\text{Im } \tau)^2} \text{Tr}(p^I) = \frac{ig^2}{48\pi^2} \epsilon_{\mu\nu}^{(1)} \epsilon_\rho^{(2)} \epsilon^{\lambda\nu\rho\mu} k_\lambda^{(1)} \text{Tr}(p^I). \tag{5.16}$$

The dependence of the integrand on $\bar{\tau}$ disappeared, because it must be a modular function of weight zero. This result agrees with the one of ref. [57].

6. Summary

In this paper we have discussed features of four-dimensional heterotic string theories and have calculated scattering amplitudes of massless fields using the bosonic formulation of odd self-dual lattices.

Four-dimensional heterotic strings have many phenomenological advantages which makes it attractive to regard them as a promising candidate for a unified theory of gravity and matter. The spectrum of four-dimensional heterotic strings contains chiral fermions in complex representations of the gauge groups which are – in contrast to the type II theories [6–9] – large enough to contain the gauge symmetry of the standard model. Furthermore, there exist massless scalars transforming non-trivially under the gauge symmetry, the graviton field, antisymmetric tensor field and also a dilaton field. $N = 1, 2, 4$ local supersymmetry is also possible.

The four-dimensional heterotic string theories cannot, in general, be regarded as compactifications of higher dimensional string theories. Classes of models are described by topologically distinct superconformal field theories – the topological characterization is based on the recently discussed index of the supercharge, also called elliptic genus. This character valued index can be used in an especially convenient way in models which possess an $U(1)$ -symmetry whose “anomaly” is cancelled by the Green-Schwarz mechanism.

Although the bosonic lattice formulation is not the most general way to describe four-dimensional strings, it turns out to be most practical when calculating the scattering amplitudes of the massless fields using the techniques of superconformal field theory. We computed all possible tree level 3- and 4-point amplitudes. Treating the superconformal ghost zero modes properly, we were led to consider picture changing also in the “internal” dimensions which give rise to the (global) symmetries of the models. From the amplitudes we could deduce the low energy effective action up to order $\sqrt{2\alpha'}$. It contains phenomenologically favoured interactions, namely Einstein gravitation coupled to fermions, scalars and gauge bosons, Yang-Mills interactions of fermions and scalars, Yukawa interactions and also 4-particle scalar and Fermi self-interactions. However at tree level, there are no linear, quadratic and cubic terms in the scalar potential. The various coupling constants are fixed and related to the string coupling constant.

Of course, there are also many phenomenological questions, e.g. how to perform the breaking of the too large gauge symmetry. This could be due to negative (mass)² terms of the scalars induced by higher loop string interactions. To make this work also additional scalar multiplets are very likely to be needed. Furthermore, one could reduce the gauge symmetry by dividing out discrete groups along the lines of [58–60].

Finally, we also discussed one-loop string effects and calculated the mixing diagram between the antisymmetric tensor field and the $U(1)$ -gauge field which is relevant for the Green-Schwarz mechanism. On the other hand, not many model

independent conclusions can be drawn about the cosmological constant or the scalar mass terms in the general non-supersymmetric four-dimensional heterotic models. Only under very special circumstances the cosmological constant is expected to vanish at one loop. To find satisfactory answers to these as well as related questions, like finiteness of the four-dimensional string theories, one needs more intimate understanding of supermoduli space and its possible symmetries.

We would like to thank P. Forgács, J. Lauer, W. Lerche and C. Vafa for useful discussions.

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