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# Effective Lagrangian for Quantum Black Holes

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**Abstract.** We discuss the most general effective Lagrangian obtained from the assumption that the degrees of freedom to be quantized, in a black hole, are on the horizon. The effective Lagrangian depends only on the induced metric and the extrinsic curvature of the (fluctuating) horizon, and the possible operators can be arranged in an expansion in powers of $M_{\rm Pl}/M$, where $M_{\rm Pl}$ is the Planck mass and $M$ the black hole mass. We perform a semiclassical expansion of the action with a formalism which preserves general covariance explicitly. Quantum fluctuations over the classical solutions are described by a single scalar field living in the 2+1 dimensional world-volume swept by the horizon, with a given coupling to the background geometry. We discuss the resulting field theory and we compute the black hole entropy with our formalism.

# 1 Introduction

Effective Lagrangians are one of the most powerful tools of theoretical physics. They allow us to investigate physics at large distances or low energies when either the fundamental microscopic theory is unknown, as is the case in quantum gravity, or when an explicit derivation of large distance physics from the underlying fundamental theory is technically very difficult, as, for instance, in *QCD*. The only ingredients that are needed in an effective Lagrangian approach are: 1) to know the symmetries of the underlying theory, and to know how they are realized in the vacuum, and 2) to understand what are the relevant degrees of freedom at low energies. These, in a non-perturbative regime, are very different from the fields which appear in the underlying Lagrangian, and are in fact collective excitations of the fundamental variables.

An example which immediately comes to mind is the chiral Lagrangian for pions. In this case one knows that the relevant degrees of freedom at low energies are not the fundamental fields, quarks and gluons, but rather the pion fields. Then, in order to write down the most general effective Lagrangian, we do not need to know that the fundamental theory is *QCD*, but we only need to know how chiral symmetry is realized. Where we pay for our ignorance of short distance physics is in the fact that we must introduce phenomenological couplings, like $f_\pi$. Then, if the microscopic theory is known, we can in principle derive these couplings from it, although this is in general a difficult non-perturbative problem.

The purpose of this paper is to set up a similar formalism for black holes in quantum gravity. The first question to be answered is: what are the relevant degrees of freedom in terms of which we should write our effective Lagrangian? We answer this question with the following

> *Basic Postulate:* from the point of view of a static observer, in a black hole the degrees of freedom to be quantized are on the horizon. The effective Lagrangian is written in terms of the variables $\zeta^\mu(\tau, \sigma_1, \sigma_2)$ which define the position of the quantum, fluctuating, horizon.

The variables $\tau, \sigma_1, \sigma_2$ parametrize the world-volume swept by the horizon. For the Schwarzschild black hole, we will usually fix the gauge $\tau = t, \sigma_1 = \theta, \sigma_2 = \phi$, where $t$ is Schwarzschild time and $\theta, \phi$ are the polar angles.

The symmetries to be respected by the effective Lagrangian are: general covariance in the embedding spacetime and reparametrization invariance in the world-volume.

The above postulate, in our opinion, is in the spirit of ideas of 't Hooft [1] and of Susskind and coworkers [2]. In the framework of classical black holes, a 'membrane paradigm' was discussed in ref. [3]. Further investigations along similar lines have been recently presented in refs. [4-8].

In order to have a useful effective Lagrangian we need, of course, a small parameter, so that it will be possible to limit ourselves to the first few terms in the effective theory. Our small parameter, in the case of Schwarzschild black holes, is $M_{\rm Pl}/M$, where $M$ is the black hole mass and $M_{\rm Pl}$ is the Planck mass. We will show that higher dimension operators in the effective Lagrangian correspond to higher order terms in an expansion in powers of $M_{\rm Pl}/M$.

The paper is organized as follows. In sect. 2 we show how to construct the most general effective Lagrangian compatible with the basic postulate and with the symmetries of the problem and we write down explicitly the lowest dimension operators. In sect. 3, using a generally covariant background field method, we expand the action around the classical solutions and we write it in terms of a single scalar field $\phi(\tau, \sigma_1, \sigma_2)$ living in the world-volume. In sect. 4 we compute the black hole entropy with this formalism, and we find corrections to the relation $S = A/4$. Sect. 5 contains the conclusions. Some technical issues are discussed in Appendix A, while in Appendix B we repeat our considerations in the case of the 2+1 dimensional black hole.

## 2 The effective Lagrangian

The fields $\zeta^\mu(\xi)$, $\xi^i = (\tau, \sigma_1, \sigma_2)$, describe a 2+1 dimensional timelike hypersurface (the world-volume) embedded in 3+1 dimensional spacetime. If $g_{\mu\nu}$ is the spacetime metric[1] (to be specified below), the intrinsic properties of the surface are completely characterized by the induced metric $h_{ij}$,

$$h_{ij} = g_{\mu\nu}\partial_i\zeta^\mu\partial_j\zeta^\nu \,, \tag{1}$$

where $\partial_i = \partial/\partial\xi^i$, and world-volume indices take values $i = 0, 1, 2$. The extrinsic properties of the surface, instead, are completely characterized by the extrinsic curvature tensor $K_{ij}$, defined by

$$K_{ij} = n_{\mu;\nu}\partial_i\zeta^\mu\partial_j\zeta^\nu \,. \tag{2}$$

Here $n_\mu$ is the outer normal of the surface $x^\mu = \zeta^\mu(\xi)$, and the semicolon denotes as usual the covariant derivative in the embedding space. The equations defining the normal $n^\mu$ are

$$g_{\mu\nu}n^\mu\partial_i\zeta^\nu = 0 \,, \quad i = 0, 1, 2 \tag{3}$$

together with the normalization condition $g_{\mu\nu}n^\mu n^\nu = 1$. The trace of the extrinsic curvature is $K = h^{ij}K_{ij} = K^i_i$ (in the following, world-volume indices are raised or lowered with the induced metric.) We recall that $K_{ij}$ is a symmetric tensor (see e.g. ref. [10]). Since $h_{ij}$ determines completely the intrinsic properties of the surface while $K_{ij}$ determines completely how the surface is embedded in 3+1 dimensional space, the effective action can depend only on these quantities, and must respect general covariance in the embedding spacetime and reparametrization invariance in the world-volume.

Under parity transformations in the world-volume (e.g. $\tau \to \tau, \sigma_1 \to \sigma_1, \sigma_2 \to -\sigma_2$) the outer normal remains unchanged, and therefore $K$ is a real scalar under these transformations. In some special case, there is a symmetry transformation under which $K$ transform as a pseudoscalar. This happens, for instance, if we consider a planar membrane located at $z = 0$, in $\mathbf{R}^4$. In this case the reflection $z \to -z$ is a symmetry, and this symmetry forbids the presence of odd powers of $K$ in the effective action. However, in the general case, these terms will be present.

[1] We use the sign conventions of Misner, Thorne and Wheeler [9].

Let us group the possible terms that we can write down on the basis of their dimensions. For the purpose of power counting, it is convenient to assign dimensions of length to each of the $\xi^i$. Then $h_{ij}$ is dimensionless and $K_{ij}$ has dimensions of mass (if we use units $\hbar = c = 1$ and we keep $G$ explicit). The most general effective action compatible with our symmetries can be written as

$$S_{\rm eff} = -\mathcal{T} \int d^3\xi \sqrt{-h} \left[ 1 + \sum_a c_a^{(1)} O_a^{(1)} + \sum_a c_a^{(2)} O_a^{(2)} + \ldots \right] . \tag{4}$$

Here $\mathcal{T}$ is the membrane tension and has dimensions of $(\text{mass})^3$. The first term in this expansion is just the standard Dirac membrane action [11], i.e., the generalization to membranes of the Nambu-Goto action for strings and of the action $S = -m \int ds$ for pointlike particles. The operators $O_a^{(1)}$, enumerated by the index $a$, are all possible operators with dimensions of mass, the operators $O_a^{(2)}$ have dimensions of $(\text{mass})^2$, etc. Correspondingly, the phenomenological parameters $c_a^{(n)}$ have dimensions of $(\text{mass})^{-n}$.

These operators must be constructed with $K_{ij}$, $h_{ij}$ and their derivatives. Because of reparametrization invariance the derivatives of $h_{ij}$ will only enter through its Riemann tensor $R_{ijkl}$, which in the $2+1$ dimensional world-volume is fixed by the Ricci tensor $R_{ij}$, and through the covariant derivatives of the Riemann tensor. In the following $R_{ijkl}, R_{ij}, R$ will always refer to the Riemann tensor, Ricci tensor and scalar curvature of the world-volume, while script letters $\mathcal{R}_{\mu\nu\rho\sigma}, \mathcal{R}_{\mu\nu}, \mathcal{R}$ refer to the embedding space. Since $K_{ij}$ has dimensions of mass while $R_{ij}$ has dimensions of $(\text{mass})^2$, at level $n = 1$ the only possible operator is $O^{(1)} = K$. At the level $n = 2$ we have three different operators,

$$O_1^{(2)} = R, \quad O_2^{(2)} = K^2, \quad O_3^{(2)} = K_{ij}K^{ij} . \tag{5}$$

However in the vacuum, where $\mathcal{R}_{\mu\nu} = 0$, they are not independent because of the Gauss-Codazzi relation, $R = K^2 - K_{ij}K^{ij}$. At order $n = 3$ we have

$$O_1^{(3)} = R_{ij}K^{ij}, \quad O_2^{(3)} = K^3, \quad O_3^{(3)} = K^i_j K^j_l K^l_i, \quad O_4^{(3)} = K_{ij}K^{ij}K \tag{6}$$

where we have eliminated terms related by the Gauss-Codazzi relation and total derivatives. At order $n = 4$, even if we make use of the Gauss-Codazzi relation, we have many independent terms:

$$\begin{gathered} K^4, \; K^2 R, \; R^2, \; R_{ij}R^{ij}, \; R_{ij}K^i_l K^{lj}, \; R_{ij}K^{ij}K, \; K^j_i K^l_j K^m_l K^i_m, \; K K^j_i K^l_j K^i_l, \\ D_i K_{jl} D^i K^{jl}, \; D_i K_{jl} D^j K^{il}, \; D_i K D^i K, \; D^i K_{ij} D^j K, \; D^i K_{ij} D_l K^{jl} , \end{gathered} \tag{7}$$

where $D_i$ is the covariant derivative in the world-volume. Limiting ourselves to order $n = 2$, and considering for the moment a metric $g_{\mu\nu}$ generic, we therefore write

$$\begin{aligned} S_{\rm eff} &= -\mathcal{T}\int d^3\xi\sqrt{-h}\,\Big[1 + C_0 K + C_{I} R + C_{II} K^2 \\ &\quad + C_{III} K_{ij}K^{ij} + (\text{operators with}\, n\geq 3)\Big] \,. \end{aligned} \tag{8}$$

This effective action has already been found by Carter and Gregory [12] in the context of domain walls in flat space. In this case one has the 'microscopic' theory, which is the theory of a scalar field in a double well potential, and one can compute explicitly the effective action for a thin domain wall. The result of the explicit calculation turns out to be of the form (8), in agreement with the general arguments presented above.[2]

Dimensionally, we have $C_0 =$ (length), $C_I, C_{II}, C_{III} =$ (length)$^2$. As in any effective theory, the scale for these coefficients is fixed by the intrinsic cutoff of the effective Lagrangian, i.e. by the length scale over which we have performed a coarse graining. For domain walls, in fact, $C_I, C_{II}, C_{III}$ turn out to be proportional to the square of the thickness of the wall [12]. In our case, as we will discuss in sect. 3, the membrane is a coarse graining over a region of thickness $O(L_{\rm Pl}^2/R_S)$ or at most $O(L_{\rm Pl})$, where $L_{\rm Pl}$ is the Planck length and $R_S = 2M$ is the Schwarzschild radius. If superstrings are the fundamental theory we should consider the string length rather then $L_{\rm Pl}$. The two scales differ by a numerical factor which is not very relevant for us at the moment. Thus, $C_0$ is expected to be $O(L_{\rm Pl}^2/R_S)$ or at most $O(L_{\rm Pl})$, $C_I, C_{II}, C_{III}$ are expected to be $O(L_{\rm Pl}^4/R_S^2)$ or at most $O(L_{\rm Pl}^2)$, etc.

The length scale characterizing $K_{ij}, R_{ij}$ is instead the curvature radius of the embedding spacetime. Thus, for dimensional reasons, in the Schwarzschild metric

$$K_{ij} = O\left(\frac{1}{R_S}\right)\,, \qquad R_{ij} = O\left(\frac{1}{R_S^2}\right)\,. \tag{9}$$

[2] In the case of ref. [12], however, $C_0$ vanishes because the membrane divides the embedding spacetime (in this case $\mathbf{R}^4$) into two identical part. Then the reflection about the membrane is a symmetry of the problem. Under this transformation $K \to -K$ and therefore odd terms are forbidden. This is not anymore the case if we consider, for instance, a domain wall located at $z = 0$ in a finite volume, $-L_1 \leq z \leq L_2$ with $L_1 \neq L_2$. In this case a repetition of the computation of ref. [12] shows that $C_0 \neq 0$.

Therefore the expansion in higher dimension operators in the effective Lagrangian is an expansion in powers of $(L_{\rm Pl}/R_S)^n$, with $n = 2$ or at least $n = 1$. This is nothing but the thin wall approximation used for domain walls.

A final ingredient for setting up our formalism is the choice of the background metric $g_{\mu\nu}$. In the limit of infinite black hole mass any backreaction on the classical metric due to the motion of the membrane is clearly neglegible, and we can simply use the Schwarzschild metric (or the Rindler metric, depending on the case in which we are interested.) Note that, since the basic postulate makes explicit reference to a static observer outside the horizon, the Schwarzschild metric must necessarily be expressed in Schwarzschild coordinates. More in general, the metric $g_{\mu\nu}$ should also include the backreaction of the membrane, and this will be a source of finite mass corrections.

Note that on the Schwarzschild and on the Rindler metric the three operators appearing at order $n = 2$ are not independent, because of the Gauss-Codacci relation, and therefore we can limit ourselves to two of them.

# 3 The semiclassical expansion

The structure of the effective action can be greatly clarified by expanding it around the classical solution of the equations of motion. In this section and in sect. 4 we limit ourselves to the leading term, i.e. to the Dirac membrane action.

The equations of motion obtained by variation of the Dirac membrane action can be written in terms of the trace of the extrinsic curvature as $K = 0$. In Rindler space we use coordinates $x^\mu = (t, x, y, z)$ and we fix the gauge $\tau = t, \sigma_1 = x, \sigma_2 = y$. From now on we use Planck units, setting $G = 1$. The metric is $g_{\mu\nu} = \mathrm{diag}(-g^2z^2, 1, 1, 1)$. The Rindler metric is the limit of the Schwarzschild metric if we send $M \to \infty$, while remaining at a fixed, limited, distance from the horizon, if $g$ is identified with $1/(4M)$. If we look for planar solutions of the form $z = z(\tau)$, the equation of motion $K = 0$ takes the form [5]

$$z\ddot{z} - 2\dot{z}^2 + g^2z^2 = 0\,, \tag{10}$$

and has the solution

$$z_{\rm cl}(\tau) = \frac{z_0}{\cosh g\tau}\,. \tag{11}$$

For Schwarzschild black holes, we use coordinates $x^\mu = (t, r, \theta, \phi)$ and $g_{\mu\nu} = \mathrm{diag}(-\alpha, \alpha^{-1}, r^2, r^2 \sin^2\theta)$, with $\alpha = 1 - (2M)/r$. In order to facilitate the comparison with the Rindler limit, we use the notation $g = 1/(4M)$ and we define $z = \alpha^{1/2}/g$. The equation of motion of a spherical membrane, written in terms of $z(\tau)$, is

$$z\ddot{z} - 2\dot{z}^2 + g^2 z^2 (1 + 3g^2 z^2)(1 - g^2 z^2)^3 = 0\,. \tag{12}$$

This equation can be integrated exactly in terms of the elliptic function of the third kind and the explicit expression is quite complicated. However, one can observe that, if $|\tau| \to \infty$, the solution approaches the horizon and therefore $gz(\tau)$ is small. Then the equation can be solved perturbatively; the non linear terms in eq. (12) can be neglected and the solution reduces to eq. (11).

Now we can expand the action considering fluctuations around a classical solution of the equations of motion. The expansion can be performed in a way which preserves general covariance explicitly. The technique is basically the same which was used in the classical works on the background field method for the nonlinear $\sigma$-model [13], and in the case of domain walls it has been discussed in refs. [14, 15].

One starts with the observation [14] that fluctuations along the surface are 'pure gauge' and can be reabsorbed with a reparametrization. The only physical quantum fluctuations are the ones perpendicular to the surface, and can be parametrized by a single scalar field $\phi(\xi)$ living in the world-volume. Let us denote by $\bar{n}^\mu(\xi)$ the normal to the classical solution $\bar{\zeta}^\mu(\xi)$. In a generic coordinate system, for any given value of $\xi$ we consider the (spacelike) geodesic $\zeta^\mu_{\mathrm{geod}}(\phi, \bar{\zeta}, \bar{n})$ parametrized by an affine parameter $\phi$, which at $\phi = 0$ goes through the point $\bar{\zeta}^\mu$, with a tangent at $\phi = 0$ equal to $\bar{n}^\mu$. Then one writes [13]

$$\zeta^\mu(\xi) = \zeta^\mu_{\mathrm{geod}}\left(\phi(\xi), \bar{\zeta}(\xi), \bar{n}(\xi)\right)\,. \tag{13}$$

Expanding the geodesic equation in powers of $\phi$ one gets

$$\zeta^\mu = \bar{\zeta}^\mu + \phi\bar{n}^\mu - \frac{\phi^2}{2}\bar{\Gamma}^\mu_{\rho\sigma}\bar{n}^\rho\bar{n}^\sigma - \frac{\phi^3}{3!}(\overline{\partial_\tau\Gamma}^\mu_{\rho\sigma} - \bar{\Gamma}^\alpha_{\tau\rho}\bar{\Gamma}^\mu_{\alpha\sigma} - \bar{\Gamma}^\alpha_{\tau\sigma}\bar{\Gamma}^\mu_{\alpha\rho})\bar{n}^\rho\bar{n}^\sigma\bar{n}^\tau + \ldots\,, \tag{14}$$

where the overbar denotes the value at $\bar{\zeta}$. In order to expand the action in powers of $\phi$ it is extremely convenient to use Riemann normal coordinates.

In these coordinates the geodesics are straight lines, which means that in these coordinates $\bar{\Gamma}^{\mu}_{\rho\sigma} = 0$ and all the derivatives $\partial_{\tau_1} \ldots \partial_{\tau_n} \Gamma^{\mu}_{\rho\sigma}$ vanish when evaluated at $\bar{\zeta}$ and fully symmetrized in the indices $(\tau_1, \ldots, \tau_n, \rho, \sigma)$. Then eq. (14) becomes

$$\zeta^{\mu}(\xi) = \bar{\zeta}^{\mu}(\xi) + \phi(\xi)\bar{n}^{\mu}(\xi) \,. \tag{15}$$

The expansion for $\partial_i \zeta^{\mu}$ can be evaluated by taking the first derivative of eq. (14), and using the fact that in Riemann normal coordinates

$$\overline{\partial_{\nu}\Gamma^{\mu}_{\rho\sigma}} = -\frac{1}{3}(\bar{\mathcal{R}}^{\mu}{}_{\rho\sigma\nu} + \bar{\mathcal{R}}^{\mu}{}_{\sigma\rho\nu}) \,, \tag{16}$$

with similar relations for higher order derivatives [16]. Then at quadratic order one has

$$\partial_i \zeta^{\mu} = \partial_i \bar{\zeta}^{\mu} + \partial_i(\phi \bar{n}^{\mu}) + \frac{\phi^2}{3} \bar{\mathcal{R}}^{\mu}{}_{\rho\sigma\alpha} \bar{n}^{\rho} \bar{n}^{\sigma} \partial_i \bar{\zeta}^{\alpha} + O(\phi^3) \,. \tag{17}$$

At the same time, in Riemann normal coordinates the metric $g_{\mu\nu}$ has an expansion in terms of the Riemann tensor and its covariant derivatives which at quadratic order reads (see e.g. ref. [16]),

$$g_{\mu\nu}(\zeta) = \bar{g}_{\mu\nu} - \frac{\phi^2}{3} \bar{\mathcal{R}}_{\mu\rho\nu\sigma} \bar{n}^{\rho} \bar{n}^{\sigma} + O(\phi^3) \,. \tag{18}$$

Using eqs.(17,18) we can compute the expansion of the induced metric $h_{ij} = g_{\mu\nu}(\zeta)\partial_i \zeta^{\mu} \partial_j \zeta^{\nu}$ and therefore of $\sqrt{-h}$. The term linear in $\phi$ is simply $\phi \bar{K}$ and vanishes because of the equation of motion $\bar{K} = 0$, and at quadratic order one gets (after reabsorbing a factor $\mathcal{T}^{-1/2}$ into the definition of $\phi$)

$$S_{\rm eff} = S_{\rm cl} - \frac{1}{2} \int d^3\xi \sqrt{-\bar{h}} \left[ \partial_i \phi \partial^i \phi - (\bar{K}^j_i \bar{K}^i_j + \bar{\mathcal{R}}_{\mu\nu} \bar{n}^{\mu} \bar{n}^{\nu}) \phi^2 \right] \,, \tag{19}$$

where $S_{\rm cl} = -\mathcal{T} \int d^3\xi (-\bar{h})^{1/2}$, and the indices are raised and lowered with $\bar{h}_{ij}$. Eq. (19) agrees with the result found by Carter with more geometrical methods [17].

Eq. (19) is the action of a scalar field in curved space in 2+1 dimensions. For a planar membrane in Rindler space, $\bar{\zeta}^{\mu} = (\tau, x, y, z_{\rm cl}(\tau))$ with $z_{\rm cl}$ given in eq. (11). Then

$$\bar{h}_{ij} = {\rm diag}(-g^2 z_{\rm cl}^2 + \dot{z}_{\rm cl}^2, 1, 1) = {\rm diag}(-\frac{g^2 z_0^2}{\cosh^4 g\tau}, 1, 1) \,. \tag{20}$$

and $\bar{K}_i^j \bar{K}_j^i = 0, \mathcal{R}_{\mu\nu} = 0$. Thus eq. (19) describes a massless scalar field living in a 2+1 dimensional space with

$$ds^2 = -\frac{g^2 z_0^2}{\cosh^4 g\tau} d\tau^2 + dx^2 + dy^2 \,. \tag{21}$$

Introducing $\tilde{\tau} = z_0 \tanh g\tau$, it becomes a flat metric with boundaries in the time direction, since $-z_0 < \tilde{\tau} < z_0$. The propagator of the $\phi$ field in this background is discussed in Appendix A.

Let us consider now the Schwarzschild background. The membrane represents a coarse graining over the region of space where the local Hawking temperature reaches the scale where unknown physics sets in. This certainly happens at least at the Planck mass, or at $T = O(1)$ in Planck units. From the expression of the local temperature near the horizon,

$$T = \frac{1}{8\pi M \sqrt{-g_{00}}} \simeq \frac{\sqrt{2M}}{8\pi M \sqrt{r - 2M}} \,, \tag{22}$$

we see that $T \gtrsim O(1)$ when $r - 2M \lesssim O(1/M)$. Thus, this is the region over which we have to perform a coarse graining, if we do not want to be faced with physics beyond the Planck mass. This corresponds to considering membranes whose area is $A = 4\pi R_S^2 + \delta A$, with $\delta A \sim 4\pi L_{\rm Pl}^2$. Note that, instead, when $r - 2M$ is on the order of a few Planck lengths, the local Hawking temperature is $\sim M^{-1/2}$ and it is still small for black holes with large $M$.

Then, let us study the theory obtained by expanding around a spherical solution of the equations of motion with initial conditions $r(0) = 2M + \delta r$, $\dot{r}(0) = 0$. The minimum value of $\delta r$ in which we are interested is $\delta r = O(1/M)$, since at this scale we enter the super-Planckian region. The largest value of interest are instead $\delta r = O(1)$, since this is the scale where quantum effects on the horizon start to become important [18]. In terms of the variable $z = \alpha^{1/2}/g$, using the notation $z_0 = z(\tau = 0)$, this gives $z_0$ between $O(1)$ and $O(M^{1/2})$ and therefore $g z_0$ between $O(1/M)$ and $O(1/M^{1/2})$. Since $g z(\tau) \leq g z_0 \ll 1$, the equation of motion for a spherical membrane, eq. (12), can be solved perturbatively in $g z_0$ and to lowest order it reduces to the Rindler equation of motion. Then, to lowest order in $g z_0$, the interval in the $2+1$ dimensional world-volume can be written as

$$ds^2 \simeq -\frac{g^2 z_0^2}{\cosh^4 g\tau} d\tau^2 + R_S^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \,. \tag{23}$$

As $\tau \to \infty$ any deviation of a classical solution from spherical symmetry is washed out exponentially [5], so the result obtained expanding around a non-spherical background are qualitatively similar.

The main difference with the Rindler case is that now the mass term in the Lagrangian, eq. (19), is non zero. Strictly speaking, the quadratic term is not exactly a mass term, since it depends on $\tau$. However, it has a finite limit for $\tau \to \pm\infty$ (which is independent of whether we take an exactly spherical configuration or not) which, to lowest order in $\alpha_0 = (gz_0)^2$, is

$$\bar{K}_i^j \bar{K}_j^i \to \frac{6\alpha_0}{R_S^2} \tag{24}$$

On the Schwarzschild metric $\mathcal{R}_{\mu\nu} = 0$ and therefore it does not contribute to the mass term. Thus, similarly to what happens for domain walls in flat space [14], the mass term has a tachyonic sign. Let us consider the evolution of a generic perturbation $\phi(\tau, \theta, \phi)$. We expand the perturbation in spherical harmonics,

$$\phi(\tau, \theta, \phi) = \sum_{lm} \phi_l(\tau) Y_{lm}(\theta, \phi) \tag{25}$$

and we define

$$m^2 = \frac{6\alpha_0}{R_S^2}\,. \tag{26}$$

In the range of values of $r(0)$ which is relevant for us, $m^2$ varies between $O(1/M^4)$ and $O(1/M^3)$. The equation of motion of the perturbation in the background given by eq. (23) is

$$\left[\frac{\partial^2}{\partial\tilde{\tau}^2} - m^2 + \frac{l(l+1)}{R_S^2}\right] \phi_l(\tau) = 0\,, \tag{27}$$

with $\tilde{\tau} = z_0 \tanh g\tau$. In the large $M$ limit only the mode with $l = 0$ is unstable, and solving the equation of motion for the fluctuations at quadratic order, we get

$$\phi_0(\tau) = \phi_0(0) e^{\pm m\tilde{\tau}} = \phi_0(0) \exp\{\pm m z_0 \tanh g\tau\}\,. \tag{28}$$

As $\tau \to \infty$ the unstable mode grows, but only increases up to the finite value $\phi_0(0)\exp\{mz_0\}$. In the range of values of $z_0$ which is more interesting for us, $mz_0$ varies between $O(1/M^2)$ and $O(1/M)$ and therefore the fluctuations only increase up to a small value.

We have computed higher order terms in the potential part of the Lagrangian, i.e., in the part which does not involve derivatives of $\phi$. Writing

$$V(\phi) = -\frac{1}{2}m^2\phi^2 + v_3\phi^3 + v_4\phi^4 + \dots \tag{29}$$

we find, for a generic background,

$$v_3 = \frac{1}{3}\bar{n}^\rho\bar{n}^\sigma\bar{\mathcal{R}}_{\mu\rho\nu\sigma}\bar{K}^{\mu\nu} + \frac{1}{3}\bar{K}^i_j\bar{K}^j_l\bar{K}^l_i - \frac{1}{6}\bar{n}^\rho\bar{n}^\sigma\bar{n}^\tau\bar{\mathcal{R}}_{\rho\sigma;\tau}\,, \tag{30}$$

$$\begin{aligned} v_4 \;=\; & \frac{1}{12}\bar{n}^\rho\bar{n}^\sigma\bar{n}^\tau\bar{K}^\nu_\mu\bar{\mathcal{R}}^\mu{}_{\rho\nu\sigma;\tau} - \frac{1}{12}\bar{n}^\alpha\bar{n}^\beta\bar{n}^\gamma\bar{n}^\delta\bar{\mathcal{R}}^\rho{}_{\delta\tau\alpha}\bar{\mathcal{R}}^\tau{}_{\beta\rho\gamma} \\ & -\frac{1}{3}\bar{n}^\rho\bar{n}^\sigma\bar{K}^\alpha_\mu\bar{K}^\nu_\alpha\bar{\mathcal{R}}^\mu{}_{\rho\nu\sigma} - \frac{1}{4}\bar{K}^i_j\bar{K}^j_l\bar{K}^l_m\bar{K}^m_i + \frac{1}{8}(\bar{K}^i_j\bar{K}^j_i)^2 \\ & -\frac{1}{12}\bar{n}^\mu\bar{n}^\nu\bar{n}^\rho\bar{n}^\sigma\bar{\mathcal{R}}_{\mu\nu;\rho\sigma} + \frac{1}{8}(\bar{n}^\mu\bar{n}^\nu\bar{\mathcal{R}}_{\mu\nu})^2 + \frac{1}{4}\bar{n}^\mu\bar{n}^\nu\bar{\mathcal{R}}_{\mu\nu}\bar{K}^i_j\bar{K}^j_i\,, \end{aligned} \tag{31}$$

where $K_{\mu\nu} = (\delta^\alpha_\mu + n_\mu n^\alpha)n_{\nu;\alpha}$ is the extrinsic curvature in 4-dimensional notation. Evaluating these quantities on the Schwarzschild metric, taking the limit $\tau \to \pm\infty$ and limiting ourselves to the leading term in $\alpha_0$ we get

$$v_3 \;\to\; -\frac{\alpha_0^{1/2}}{R_S^3} \tag{32}$$

$$v_4 \;\to\; -\frac{1}{8R_S^4}\,. \tag{33}$$

Thus, also the cubic and quartic term give a negative contribution to the potential, which (without reabsorbing $\mathcal{T}$ into $\phi$) can be written as

$$V(\phi) \;=\; -\alpha_0^2\mathcal{T}\left[3\varphi^2 + \varphi^3 + \frac{1}{8}\varphi^4 + \dots\right]\,, \tag{34}$$

$$\varphi \;\stackrel{\text{def}}{=}\; \frac{\phi}{\alpha_0^{1/2}R_S}\,. \tag{35}$$

The fact that the quadratic term in the potential have a negative sign shows that, at the quantum level, $\phi = 0$ is not the true ground state of the theory. The problem is due to the spherical mode, $l = 0$, since for all higher modes the term $l(l+1)/R_S^2$ dominates over $m^2$, if $M$ is large.

The spherical mode, however, has been treated exactly, i.e. without performing a semiclassical expansion, in ref. [5], where it has been obtained the Schroedinger equation and therefore the wave function of a spherical membrane. The resulting distribution probability of the radial mode is peaked at a non-zero value of $r - R_S$, which depends also on the membrane tension.

The fact that in the semiclassical expansion the spherical mode is unstable therefore reflects an intrinsic limitation of the semiclassical approximation. Another, related, limitation is due to the fact that the solution $\zeta^\mu_{\rm cl}$ of the classical equation of motion approaches asymptotically the horizon. However, the variables $\zeta^\mu$ emerge only after performing a coarse graining over the appropriate scale of distances, and therefore in their definition is implicit an uncertainty on the order of this length scale. Therefore, it is not legitimate to extrapolate the solution of the equation of motion down to distances very close to the horizon. When the classical solution $\zeta^\mu_{\rm cl}$ formally reaches a value of $r = R_S + O(L^2_{\rm Pl}/R_S)$, it would be physically more sensible to use, instead of $r_{\rm cl}(\tau)$, a membrane essentially static at an average value of $r = R_S + aL^2_{\rm Pl}/R_S$, with $a$ some numerical constant. This approach was recently discussed by Lousto and one of the authors [19]. In this case one gets again a Klein-Gordon equation for the fluctuations, in terms of a variable $\tilde{\tau}$ which now is defined as $\alpha_M^{1/2}\tau$, where $\alpha_M$ is the value of $\alpha$ at the average membrane location. Therefore $\tilde{\tau}$ is just the local time, and the energy conjugate to it is the local energy.

# 4 Black hole entropy

## 4.1 The entropy of the $\phi$ field

In our approach, the horizon is described by a scalar field living in the horizon world-volume, and the microscopic degrees of freedom of the black hole (or its 'quantum hair') are the modes of this field. Their contribution to the entropy can be estimated as follows.

Let us consider first the Rindler metric. In this case the field $\phi$ is massless, and therefore we have to compute the entropy of a massless boson gas in 2+1 dimensions. Let us call $A = L^2$ the area of the horizon. For Rindler space this should be eventually sent to infinity, so what matters is the entropy per unit area. The modes of the field $\phi$ are labeled by $\mathbf{k} = (k_x, k_y)$ and the free energy $F$ of a 2+1 dimensional massless boson gas taken at the inverse temperature $\beta$ is

$$\beta F = \sum_{\mathbf{k}} \log(1 - e^{-\beta E})\,, \tag{36}$$

with $E = |\mathbf{k}|$. For large $L$ the summation over modes can be replaced with an integration, using

$$\sum_{\mathbf{k}} = \left(\frac{L}{2\pi}\right)^2 \int d^2k\,. \tag{37}$$

The integral over $k$ converges both at small and at large values of $k$. However, physically it does not make much sense to include in the integral over $d^2k$ modes of the $\phi$ field with arbitrarily high energy. Then we get for the free energy and the entropy

$$F = -\frac{c(\Lambda)}{2\pi\beta^3}\,A \tag{38}$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_A = \frac{3c(\Lambda)}{2\pi\beta^2}\,A\,, \tag{39}$$

where

$$c(\Lambda) = -\int_0^\Lambda dx\, x \log(1 - e^{-x})\,, \tag{40}$$

and $\Lambda = \beta E_{\mathrm{max}}$. If $\Lambda \to \infty$ then $c(\Lambda) \to \zeta(3) \simeq 1.2$, where $\zeta(x)$ is the Riemann zeta function. With a finite cutoff we get a smaller value. For instance, $c(1) \simeq 0.4$.

The same result is obtained for a Schwarzschild black hole in the large mass limit. In fact for Schwarzschild black holes in the large $M$ limit we still have massless bosons, since the (tachyonic) mass term is $\sim M^{-2}$. The modes are labelled by the angular momentum quantum numbers $(l, m)$, and

$$\beta F = \sum_{l=0}^{\infty} (2l+1) \log\left(1 - e^{-\beta E_l}\right)\,, \tag{41}$$

$$E_l = \frac{\sqrt{l(l+1)}}{R_S}\,. \tag{42}$$

For large $R_S$ the spacing between the levels is small and we can approximate the sum over $l$ with an integral. After an integration by parts we get

$$F = -\frac{1}{2R_S}\int_0^\infty dl\, \frac{(2l+1)\sqrt{l(l+1)}}{\exp\left(\frac{\beta\sqrt{l(l+1)}}{R_S}\right) - 1}\,. \tag{43}$$

The integral is dominated by $l \sim R_S/\beta$; in our case $\beta \sim 1$ (see below) and therefore $R_S/\beta \gg 1$. Then we can approximate $l(l+1) \simeq l^2$ and we find that the sum over modes is the same as in the Rindler case, with now $A = 4\pi R_S^2$.

The value of $\beta$ which enters in the above expressions is the value of the *local* inverse temperature, since this is the temperature felt by the membrane. By definition, the membrane is a coarse graining over the region where new physics sets in. As discussed in sect. 3, the distances where the local Hawking temperature becomes of order one are $r = R_S + O(L_{\rm Pl}^2/R_S)$. Then, let us consider a membrane which represents a coarse graining over the region

$$r \le R_S + a\frac{L_{\rm Pl}^2}{R_S}\,, \tag{44}$$

with $a$ a numerical constant to be discussed below. The corresponding local inverse temperature is $\beta \simeq 4\pi\sqrt{a}$. Then, for the entropy we get

$$S = \gamma\,\frac{A}{4}, \qquad \gamma = \left(\frac{3c(\Lambda)}{8\pi^2}\right)\frac{1}{\pi a}\,. \tag{45}$$

This result is very similar to the result obtained by 't Hooft [1] and by Frolov and Novikov [20], with approaches which, at first sight, are very different from ours. The relation with their result and with the total black hole entropy is discussed in the next subsection.

## 4.2 The total entropy

While the computation of the entropy of the $\phi$ field is technically straightforward, its interpretation and its relation to the total black hole entropy is not immediate. A number of authors, see e.g. [1, 20, 21], have presented computations in which some form of mode counting is performed. In all these computations the resulting entropy diverges unless one introduces a cutoff near the horizon, like our constant $a$ introduced in the previous subsection, and the divergence is $\sim 1/a$. One can fix the cutoff by requiring that it gives just the standard value for the proportionality constant, $S = A/4$.

From the point of view of the effective Lagrangian approach, however, a different interpretation seems more natural. In the membrane approach, the result $S = A/4$ can be obtained from a 'zeroth order' term [22], and the contribution of the $\phi$ field is more naturally interpreted as a correction to it. The reasoning presented in ref. [22] is as follows. We want to define a path integral over the quantum field $g_{\mu\nu}$ in the presence of a black hole (or, more generally, of a horizon), from the point of view of an external, static observer. In intuitive terms, we would like to limit ourselves to integration variables which live outside the horizon. However, very close to the horizon we have to face the problem that we are entering the region of super-Planckian temperatures. To cope with this difficulty we divide the space outside the black hole into the region $r < R_S + a(L_{\rm Pl}^2/R_S)$, and the region $\Omega$ defined by $r \geq R_S + a(L_{\rm Pl}^2/R_S)$. The partition function in the region $\Omega$ is simply

$$Z_{\rm grav} = \int_\Omega \mathcal{D}g_{\mu\nu}\, e^{iS_{\rm grav}}\,, \tag{46}$$

where $S_{\rm grav}$ is the standard Einstein-Hilbert action supplemented by the boundary term on $\partial\Omega$. However, at leading order in the large $M$ expansion we can neglect the quantum fluctuations of the metric $g_{\mu\nu}$ in the region $\Omega$, and we simply use the classical Schwarzschild metric. This also implies that, at this order, we do not have to worry about the measure of integration over $g_{\mu\nu}$, and we write the Euclidean partition function of the region $\Omega$ as

$$Z_{\rm grav} \simeq e^{-S_{\rm grav}}|_{g_{\mu\nu}=g_{\mu\nu}^{\rm cl}}\,. \tag{47}$$

On the classical metric $g_{\mu\nu}^{\rm cl}$, the volume term in $S_{\rm grav}$ is zero and only the boundary term survives. In the region $r < R_S + aL_{\rm Pl}^2/R_S$ we use instead our effective action. If we call $Z_{\rm eff}$ the partition function of the $\phi$ field, the total partition function is

$$Z = Z_{\rm grav} Z_{\rm eff}\,. \tag{48}$$

If now we fix the local inverse temperature $\beta$ at the standard value, we can compute the free energy $F$ from $Z = e^{-\beta F}$, and therefore the entropy $S$. As shown in ref. [22], the boundary term in the gravitational action gives just the result $S = A/4$. The computation is formally the same as the well-known computation of Gibbons and Hawking in Euclidean quantum gravity [23]. The only difference is that we are evaluating the boundary term on the surface $r = R_S + aL_{\rm Pl}^2/R_S$ while in ref. [23] it is computed on a surface at infinity. However, as shown by York [24], if one computes the derivatives of the thermodynamic potentials with respect to the local inverse temperature $\beta$, rather than with respect to the inverse of the temperature at infinity, the result for the entropy is independent of the position of the surface used. This explains why we obtain the same result as the one in ref. [23].

The contribution to the entropy which comes from $Z_{\rm eff}$ is instead what we have computed in the previous subsection. Therefore we find, for the black hole entropy $S_{\rm bh}$

$$S_{\rm bh} = \frac{1}{4}(1+\gamma)\,A\,. \tag{49}$$

It remains to estimate the order of magnitude of $a$ and therefore of $\gamma$. In our approach $a$ is not a cutoff put in by hand, and to be removed with a renormalization procedure, but it is a number fixed by physics, and it depends on the scale at which new physics sets in. If new physics sets in at the Planck scale, the constant $a$ is, parametrically, of order one. Thus $\gamma$ is not parametrically small compared to one, even if, for typical values of $a$ and $\Lambda$, it might be numerically small.

If however superstrings are the fundamental theory, new physics sets in at the string scale. If we denote by $\lambda_s$ the string length, we start to perform the coarse graining when the local temperature $T$ reaches a value $\sim 1/\lambda_s$, or $\beta \sim \lambda_s$. Since $\beta$ is related to $a$ by $\beta \simeq 4\pi\sqrt{a}$, this gives (writing explicitly also the Planck length $L_{\rm Pl}$),

$$a = O\left(\frac{\lambda_s^2}{L_{\rm Pl}^2}\right)\,. \tag{50}$$

(Note that in heterotic string theory this means $a \sim \alpha_{\rm GUT}^{-2}$, since $L_{\rm Pl}/\lambda_s = \alpha_{\rm GUT}/4$.) Thus $\gamma \sim 1/a$ is a parametrically small quantity, and we get

$$S = \frac{A}{4}\left(1 + O(\frac{L_{\rm Pl}^2}{\lambda_s^2})\right)\,. \tag{51}$$

The interpretation of the contribution of the $\phi$ field as a parametrically small correction to the entropy also allow us to see in a different light the results of refs. [1, 20], where it is computed the contribution of external matter fields to the entropy. If matter fields $\Phi$ are present, we would write for the Euclidean partition function $Z = Z_{\rm grav} Z_{\rm eff} Z_\Phi$, where $Z_\Phi$ is the partition function of the matter fields restricted to the region $\Omega$. Their contribution to the entropy has been computed by 't Hooft using a brick wall regulator, and the horizon contribution is $(A/4)(Z/(360\pi a))$, where we have written 't Hooft regulator $h$ as $h = a/R_S$, $Z$ is the number of fields, and we have used the standard value of the temperature. Note that this is identical, even in the numerical constant, to the result found by Frolov and Novikov [20] with a different method. If, instead of choosing $a = 1/(360\pi)$ in order to reproduce the black hole entropy, we rather say that $a$ is fixed by the scale where new physics sets in, and we identify this scale with $\lambda_s$, then this contribution is just another parametrically small correction to the black hole entropy.

This interpretation allows us to get rid of some inconsistencies which are inherent in the attempt to identify the contribution of matter fields with the total black hole entropy. In particular the horizon contribution to the internal energy becomes parametrically smaller than $M$, while if one choses $a = 1/(360\pi)$ one gets [1] $U = (3/8)M$, which is a sizeable fraction of the total black hole mass. The same happens of the specific heat, which with the choice $a = 1/(360\pi)$ is positive and even larger, in module, than the black hole value $-8\pi M^2$ [25]. Another hint in favor of this interpretation comes from the study of the brick wall model for a black hole with charge $Q \neq 0$. With a straightforward repetition of 't Hooft computation for the case $Q \neq 0$ we have found for the entropy of near-extremal black holes $S \sim (A/a)(1 - (Q^2/M^2))^{1/2}$. If we want to fix $h = a/R_S$ in such a way as to obtain the result $S = A/4$ we must chose $h \sim (1 - (Q^2/M^2))^{1/2}/R_S$, which is quite inconsistent, since for near-extremal black hole it is well within the region of super-Planckian temperatures. If, on the contrary, we chose $h$ as the scale at which string physics comes in, we simply obtain a correction to the $A/4$ result which is suppressed both by the small parameter $(L_{\rm Pl}/\lambda_s)^2$ and by the factor $(1 - (Q^2/M^2))^{1/2}$.

# 5 Conclusions

Following the pioneering work of 't Hooft [1], a number of authors have recently considered the possibility that, in spite of the fact that in classical general relativity the horizon merely represents a coordinate singularity, at the quantum level it is the place where the quantum degrees of freedom of black holes should be found, at least from the point of view of a static observer.

In this approach, when we are sufficiently close to the horizon we are entering a region of very high temperatures (because of the blue-shift factor in the Hawking temperature), and unknown physics sets in. Thus, to describe properly the region close to the horizon is a very difficult and essentially non-perturbative problem. On the other hand, it is a very important problem, which might be the key to understanding the statistical origin of black hole entropy and the information loss paradox.

In order to cope with our ignorance of short distance physics we have proposed an effective Lagrangian approach. In this paper we have investigated the general formalism which follows from the 'Basic Postulate' given in the Introduction. We have shown how to construct the most general effective action, and we have found that the various terms are organized as an expansion in powers of $M_{\rm Pl}/M$. We have discussed a semiclassical expansion of the action which is in a sense complementary to the treatement given in ref. [5]. In ref. [5] the field theory resulting from the membrane action was truncated to the radial mode, and the resulting quantum mechanical problem was discussed exactly, i.e., without performing a semiclassical expansion. In this paper, instead, we have retained all modes of the membrane, but we have used a semiclassical approximation.

Probably the main message of this paper is that at this effective Lagrangian level the region very close to the horizon can be studied by rather standard field theoretical methods. The fluctuations of the horizon are described by a single scalar field living in a 2+1 dimensional curved space, governed by an action which is completely fixed, apart from a number of phenomenological constants, like the membrane tension, in which it is summarized all our ignorance of short distance physics.

With this approach it is possible to compute the black hole entropy, and we find the result $S = A/4$ plus corrections which depend on the scale at which new physics sets in.

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**Appendix A. The propagator in Rindler space.**

Let us compute the propagator $D(\xi,\xi')$ of the field $\phi(\xi)$ in the background given by eq. (21). In terms of the variable $\tilde{\tau}=z_0\tanh g\tau$, eq. (21) becomes simply $ds^2=-d\tilde{\tau}^2+dx^2+dy^2$. Since $\tilde{\tau}=\pm z_0$ corresponds to $\tau=\pm\infty$, we impose the boundary conditions $D(\tilde{\tau}=\pm z_0,\tilde{\tau}';\mathbf{x}-\mathbf{x}')=D(\tilde{\tau},\tilde{\tau}'=\pm z_0;\mathbf{x}-\mathbf{x}')=0$, where we have used the notation $\mathbf{x}=(x,y)$. Thus we are dealing with a massless propagator with time boundary conditions, in flat space. We compute it following ref. [26], with some simple modifications due to the fact that in our case the boundaries are in Minkowski, rather then in Euclidean space. It is convenient to consider the Green's function $D(\tau,\tau';\mathbf{k})$ obtained by performing the Fourier transform only with respect to the spatial coordinates $\mathbf{x}$. A Green's function with the $\epsilon$ prescription appropriate for the Feynman propagator, but which does not obey the boundary conditions, is

$$D_F^{(0)}(\tilde{\tau},\tilde{\tau}';\mathbf{k})=\frac{1}{2z_0}\sum_{n=-\infty}^{\infty}\frac{e^{-i\omega_n(\tilde{\tau}-\tilde{\tau}')}}{\omega_n^2-\mathbf{k}^2-i\epsilon}\,,\qquad\qquad (\omega_n=\frac{\pi n}{z_0})\,. \tag{52}$$

The Green's function which vanishes at the time boundaries is then

$$D_F(\tilde{\tau},\tilde{\tau}')=D_F^{(0)}(\tilde{\tau},\tilde{\tau}')-\frac{D_F^{(0)}(\tilde{\tau},z_0)D_F^{(0)}(z_0,\tilde{\tau}')}{D_F^{(0)}(z_0,z_0)}\,. \tag{53}$$

Using $D_F^{(0)}(\tilde{\tau},z_0)=D_F^{(0)}(\tilde{\tau},-z_0)$ we see that $D_F(\tilde{\tau},\pm z_0)=D_F(\pm z_0,\tilde{\tau}')=0$. The sum in eq. (52) can be performed explicitly and, expressing the result in terms of $\tau,\tau'$, we get

$$\begin{aligned}&D_F(\tau,\tau';\mathbf{k})=-\frac{1}{|\mathbf{k}|\sin(2z_0|\mathbf{k}|)}\times\\&[\theta(\tau-\tau')\sin\left(z_0|\mathbf{k}|\tanh(g\tau)-z_0|\mathbf{k}|\right)\sin\left(z_0|\mathbf{k}|\tanh(g\tau')+z_0|\mathbf{k}|\right)\\&+(\tau\leftrightarrow\tau')]\;.\end{aligned} \tag{54}$$

Here $|\mathbf{k}|$ actually is $|\mathbf{k}|+i\epsilon$. This result can also be directly obtained from eq. (B.2) of ref. [26], where the boundary conditions are in Euclidean space, with the formal replacement $\mathbf{k}\to i\mathbf{k}$.

The field theory obtained by expanding around a generic, non-planar, solution of the equations of motion is qualitatively similar to that obtained expanding about a planar solution; in fact, it is easy to check from the equations of motion that, for an arbitrary solution, any 'bump' is smoothed out exponentially with $\tau$. Thus, asymptotically $z_{\rm cl} \to z_0/\cosh(g\tau)$, independently of the initial conditions, and therefore the background metric is still given, asymptotically, by eq. (21).

**Appendix B. The black hole in 2+1 dimensions**

It might be interesting to examine our formalism also for the 2+1 dimensional black hole solution discovered by Bañados, Teitelboim and Zanelli (BTZ) [27]. In spite of the many differences between three-dimensional and four-dimensional gravity, the BTZ black hole has remarkable similarities with its four-dimensional analog, and a number of investigations of its geometrical and thermodynamical properties have appeared recently, see e.g. [27, 28, 29, 6] and references therein. One considers the theory defined by the action

$$I_{\rm grav} = \frac{1}{16\pi G}\int d^3x\,\sqrt{-g}\left[\mathcal{R} + 2l^{-2}\right] + B \tag{55}$$

where $l$ is related to the cosmological constant $\Lambda$ by $\Lambda = -l^{-2} < 0$ and $B$ is the boundary term. Writing the metric in the ADM form (and setting to zero the shift functions $N^i$), $ds^2 = N^2dt^2 - g_{ij}dx^i dx^j$, the boundary term is given by

$$B = -\frac{1}{8\pi G}\int d^3x\,\partial_i\left[g^{ij}\partial_j N\,\sqrt{^{(2)}g}\right]\,, \tag{56}$$

where $^{(2)}g = \det g_{ij}$. Hereafter, following [27], we will use units $G = 1/8$. As before, the embedding spacetime curvature is denoted by $\mathcal{R}$ while $R$ will be used for the world-sheet curvature. The BTZ black hole, limiting ourselves to zero angular momentum, is given by

$$ds^2 = -\alpha dt^2 + \alpha^{-1}dr^2 + r^2 d\theta^2\,, \qquad \alpha = \frac{r^2 - r_+^2}{l^2}\,, \tag{57}$$

where $r_+ = l\sqrt{M}$. In this case the collective coordinates $\zeta^\mu$ depend on two variables $\xi^i = (\tau, \sigma)$. Instead of a membrane, the horizon is now described by a string, with induced metric $h_{ij}$ and extrinsic curvature $K_{ij}$. Embedding space indices now take values $\mu = 0, 1, 2$ and world-sheet indices take values

$i = 0, 1$. The effective action can be written using the same arguments as in sect. 2. However, now we have some simplifications; at order $n = 2$ the term proportional to the world-sheet curvature does not contributes since $(-h)^{1/2}R$ is a total derivative. At this order, therefore, the most general effective action is

$$S = -\mathcal{T} \int d^2\xi \, \sqrt{-h} \, \left[ 1 + C_0 K + C_I K^2 + C_{II} K_i^j K_j^i \right] . \tag{58}$$

On the BTZ metric the terms $K^2$ and $K_i^j K_j^i$ are not independent and we can restrict ourselves to one of them. When $C_0 = 0$ this is the action for the rigid string discussed by Polyakov [30]. The equation of motion at order $n = 0$ is again $K = 0$. Defining $y = \alpha^{1/2}/g$ and $g = r_+/l^2 = \sqrt{M}/l$, the equation of motion $K = 0$ for a circular string reads

$$y\ddot{y} - 2\dot{y}^2 + g^2 y^2 + (\frac{2g^2}{l^2})y^4 - 2\frac{y^2\dot{y}^2}{l^2 + y^2} = 0 \, . \tag{59}$$

This equation can be integrated exactly. With the initial conditions $y(0) = al, \dot{y}(0) = 0$, where $a$ is a dimensionless parameter, the exact solution is

$$g\tau = \frac{1}{a}\sqrt{\frac{1 + a^2}{1 + 2a^2}}\, \Pi \left( \psi, \frac{1 + a^2}{a^2}, \sqrt{\frac{1 + a^2}{1 + 2a^2}} \right) . \tag{60}$$

Here $\Pi(\psi, \alpha^2, k)$ is the elliptic integral of the third kind, and $\sin^2 \psi = (a^2 - (y/l)^2)/(a^2 + 1)$. However, as in the 3+1 dimensional case, it is more convenient to consider the Rindler limit rather than working with the exact solution. Let us examine the Rindler limit for the BTZ black hole. In terms of $y = \alpha^{1/2}/g$, and of $x$ defined by $x = r_+\theta$, the BTZ metric can be written as

$$ds^2 = -g^2 y^2 dt^2 + (1 + \frac{y^2}{l^2})^{-1} dy^2 + (1 + \frac{y^2}{l^2}) dx^2 \, . \tag{61}$$

If we are sufficiently close to the horizon, $y \ll l$, we get a $2 + 1$ dimensional Rindler metric, $ds^2 = -g^2 y^2 dt^2 + dx^2 + dy^2$. It is interesting to observe that, since $l = \sqrt{M}/g$, the large $l$ limit can be obtained by taking the black hole mass $M$ large, with $g$ held fixed and arbitrary, while in $3 + 1$ dimensions the Rindler limit necessarily implies $g \to 0$. In the $2 + 1$ Rindler metric the equation of motion of the string reads $y\ddot{y} - 2\dot{y}^2 + g^2 y^2 = 0$, which is the same as in the $3 + 1$ dimensional case.

The semiclassical expansion can be performed in analogy with the 3+1 dimensional case, and the effective action, at quadratic order, is formally identical to eq. (19), with $d^3\xi \to d^2\xi$. The computation of the entropy of the $\phi$ field proceeds along the lines of sect. 4 and gives a result proportional to the perimeter of the horizon. The zeroth order term, instead, has already been computed in ref. [31]: we define the local inverse temperature $\beta = \beta_\infty \alpha^{1/2}$ and we compute the boundary term on a spherical surface with a generic radius $r$. Considering $M$ as a function of $\beta$ and $r$ defined implicitly by $\beta = \beta_\infty \alpha^{1/2}$, we find $B = B(\beta, r)$, and the entropy is given by

$$S = \beta \left( \frac{\partial B}{\partial \beta} \right)_r - B \, . \tag{62}$$

A simple computation gives

$$B = -2\beta_\infty \frac{r^2}{l^2} = -4\pi r \left[ 1 + (\frac{\beta}{2\pi l})^2 \right]^{1/2} \, , \tag{63}$$

and therefore

$$S = 4\pi r_+ = 2 \times \text{perimeter} \, , \tag{64}$$

in agreement with ref. [27].

# References


[1] G. 't Hooft, Nucl. Phys. B256 (1985) 727, ibid. B335 (1990) 138; Physica Scripta T15 (1987) 143; ibid. T36 (1991) 247.

[2] L. Susskind, L. Thorlacius and J. Uglum, Phys. Rev. D48 (1993) 3743;

L. Susskind, Phys. Rev. Lett. 71 (1993) 2367; Phys. Rev. D49 (1994) 6606.

L. Susskind and L. Thorlacius, Phys. Rev. D49 (1994) 966.

[3] K.S. Thorne, R.H. Price and D.A. Macdonald, *"Black Holes: The Membrane Paradigm"*, Yale Univ. Press, 1986, and references therein.

[4] M. Maggiore, Phys. Rev. D49 (1994) 2918.

[5] M. Maggiore, Nucl. Phys. B429 (1994) 205.

[6] S. Carlip, Phys. Rev. D51 (1995) 632.

[7] A.P. Balachandran, L. Chandar, Arshad Momen, *"Edge States in Gravity and Black Hole Physics"*, preprint SU-4240-590, December 1994.

[8] C.O. Lousto, Phys. Rev. D51 (1995) 1733.

[9] C.W. Misner, K.S. Thorne and J.A. Wheeler, *"Gravitation"*, Freeman, San Francisco, CA, 1973.

[10] S.W. Hawking and G.F.R. Ellis, *"The large scale structure of space-time"*, Cambridge University Press, Cambridge 1973, pag. 46.

[11] P.A.M. Dirac, Proc. Roy. Soc. London A268 (1962) 57.

[12] B. Carter and R. Gregory, *"Curvature corrections to dynamics of domain walls"*, preprint DAMTP/R-94/41.

[13] J. Honerkamp, Nucl. Phys. B36 (1972) 130;

L. Alvarez-Gaumé, D.Z. Freedman and S. Mukhi, Ann. Phys. 134 (1981) 85.

[14] J. Garriga and A. Vilenkin, Phys. Rev. D44 (1991) 1007; ibid. D45 (1992) 3469.

[15] J. Guven, Phys. Rev. D48 (1993) 4604.

[16] A.Z. Petrov, *"Einstein Spaces"*, Pergamon Press, Oxford (1969).

[17] B. Carter, J. Geom. Phys. 8 (1992) 53; Phys. Rev. D48 (1993) 4835; Class. Quantum. Grav. 11 (1994) 2677.

[18] M. Maggiore, Phys. Lett. B304 (1993) 65.

[19] C.O. Lousto and M. Maggiore, *"On the energy spectrum of quantum black holes"*, preprint IFUP-TH 53/94, (unpublished).

[20] V. Frolov and I. Novikov, Phys. Rev. D48 (1993) 4545.

[21] J.W. York, Jr., Phys. Rev. D28 (1983) 2929.

[22] M. Maggiore, Phys. Lett. B333 (1994) 39; Erratum, B341 (1995) 450.

[23] G.W. Gibbons and S.W. Hawking, Phys. Rev. D15 (1977) 2752.

[24] J.W. York, Jr., Phys. Rev. D33 (1986) 2092.

[25] F. Belgiorno and S. Liberati, *"Divergences problem in the black hole brick-wall model"*, preprint Roma-1093-1995, March 1995, gr-qc 9503022.

[26] G. Curci and P. Menotti, Zeit. Phys. C21 (1984) 281.

[27] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69 (1992) 1849.

[28] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D48 (1993) 1506.

[29] F. Englert and B. Reznik, Phys. Rev. D50 (1994) 2692.

[30] A. Polyakov, Nucl. Phys. B268 (1986) 406.

[31] M. Maggiore, *"Horizon quantization and thermodynamics of the 2+1 black hole"*, preprint IFUP-TH 43/94, July 1994, gr-qc/9407016 (unpublished).