Elements of Vasiliev theory

V.E. DIDENKO* AND E.D. SKVORTSOV*†

*Lebedev Institute of Physics, Moscow, Russia

† Albert Einstein Institute, Potsdam, Germany

Abstract

We propose a self-contained description of Vasiliev higher-spin theories with the emphasis on nonlinear equations. The main sections are supplemented with some additional material, including introduction to gravity as a gauge theory; the review of the Fronsdal formulation of free higher-spin fields; Young diagrams and tensors as well as sections with advanced topics. The general discussion is dimension independent, while the essence of the Vasiliev formulation is discussed on the base of the four-dimensional higher-spin theory. Three-dimensional and d-dimensional higher-spin theories follow the same logic.

*didenko@lpi.ru
†skvortsov@lpi.ru

1

Contents

1 Introduction

One of the goals of quantum field theory is to explore the landscape of consistent theories. Normally one starts with some set of free, noninteracting fields and then tries to make them interact. Given the fact that the free fields are characterized by spin and mass we can ask the following question: which sets of fields specified by their spins and masses admit consistent interaction? Massless fields of spins $s = 1, \frac{3}{2}, 2, ...$ being gauge fields are of particular interest because their interactions are severely constrained by gauge symmetry. The well known examples include: Yang-Mills theory as a theory of massless spin-one fields; the gravity as a theory of a massless spin-two; supergravities as theories of a number of spin-$\frac{3}{2}$ fields, graviton and possibly some other fields required for consistency; string theory which spectrum contains infinitely many fields of all spins, mostly massive being highly degenerate by spin.

There is a threshold value of spin, which is 2. Once a theory contains fields of spins not greater than two its spectrum can be finite. If there is at least one field of spin greater than two, a higher-spin field, the spectrum is necessarily infinite, containing fields of all spins. String theory is an example of such theory. The Vasiliev higher-spin theory is the missing link in the evolution from the field theories of lower spins, $s \leq 2$, to string theories. The Vasiliev theory is the minimal theory whose spectrum contains higher-spin fields. Its spectrum consists of massless fields of all spins $s = 0, 1, 2, 3, ...$, each appearing once in the minimal theory, and in this respect it is much simpler than string theory. The important difference of Vasiliev theory in comparison with string theory is that while the latter has dimensionful parameter α', the former does not being the theory of gauge fields based on the maximal space-time symmetry. This higher-spin symmetry is an ultimate symmetry in a sense that it can not result from spontaneous symmetry breaking. The Vasiliev theory therefore has no energy scale and can be thought of as a toy model of the fundamental theory beyond Planck scale.

In this note we would like to give a self-contained review on some aspects of higher-spin theory. The subject dates back to the work of Fronsdal, [1], who has first found the equations of motion and the action principle for free massless fields of arbitrary spin. His equations naturally generalize those of Maxwell for $s = 1$ and the linearized Einstein equations for $s = 2$. At the same time, as spin gets larger than three (in bosonic case) the Fronsdal fields reveal certain new features. It is a subject of many no-go theorems stating that interacting higher-spin fields cannot propagate in Minkowski space. Once a yes-go result is available thanks to Fradkin and Vasiliev, [2–4] we are not going to discus these theorems in any detail referring to excellent review [5]. The crucial idea that allowed one to overcome all no-go theorems was to replace flat Minkowski background space with the anti-de Sitter one, i.e. to turn on the cosmological constant. That higher-spin theory seems to be ill-defined on the Minkowski background is not surprising from the point of view of absence of the dimensionful parameter. In other words the interaction vertices that carry space-time derivatives would be of different dimension in that case. The *AdS* background simply introduces such a dimensionful parameter, the cosmological constant.

A canonical way of constructing interacting theories is order by order. In the realm of perturbative interaction scheme one begins with a sum of quadratic Lagrangians for a given spectrum of spins and masses (these are zero since we consider gauge fields) and

tries to deform them by some cubic terms while maintaining the gauge invariance, then by some quartic terms, etc. The gauge transformations perturbatively get deformed as well. If a nontrivial solution to cubic deformations is found we are said to have cubic interaction vertices. A lot is known about cubic interactions both in the metric-like approach of Fronsdal [6–23] and in the frame-like approach of Vasiliev and Fradkin-Vasiliev [2–4, 24–30].

The cubic level, however, is insensitive to the spectrum of fields, i.e. given a set of consistent cubic couplings among various fields one can simply sum up all cubic deformations into a single Lagrangian, which is again consistent up to the cubic level. For the simplest case of a number of spin-one fields one finds that the cubic consistency relies on some structure constants f_{abc} being antisymmetric. The Jacobi identity telling us that there is a Lie algebra behind the Yang-Mills theory arises at the quartic level only.

The technical difficulty of a theory with at least one higher-spin field is that it is not even possible to consistently consider an interaction of finite amount of fields, [6,17,31,32]. Altogether, this makes Fronsdal program (construction of interacting theory for higher-spin gauge fields) extremely difficult to implement. At present, the traditional methods of metric-like approach has led to little progress in this direction, basically their efficiency stops at the level of cubic interaction, see however [31, 33]. This state of affairs made it clear that some other tools were really relevant to push the problem forward.

A very fruitful direction initiated by Fradkin and Vasiliev and then largely extended by Vasiliev that eventually foster the appearance of complete nonlinear system for higher-spin fields, [34–39], is the so called unfolded approach, [40, 41], to dynamical systems. It rests on the frame-like concept rather than the metric-like and the differential form language which makes the whole formalism explicitly diffeomorphism invariant. This is the branch of higher-spin theory that we want to discuss in this lectures. As we will see the concept of gauge symmetry intrinsically resides in the unfolded approach and, therefore, it suits perfectly for analysis of gauge systems.

The other advantage is that being applied to a free system it reveals all its symmetries and the spectrum of (auxiliary) fields these symmetries act linearly on. Particularly, the higher-spin algebra is something that one can discover from free field theory analysis using the unfolded machinery. This is one of the corner stones towards the nonlinear higher-spin system. Nevertheless, the unfolded approach is just a tool that controls gauge symmetries, degrees of freedom and coordinate independence while containing no extra physical input. It turned out to be extremely efficient for higher-spin problem providing us with explicit nonlinear equations, still it gives no clue for their physical origin.

The Vasiliev equations, [34, 36, 39, 42, 43], are background independent. The AdS vacuum relevant for propagation of higher-spin gauge fields arises as some particular exact vacuum solution, while propagation around other vacuums have received no physical interpretation so far, neither the geometry of space-time is known. Another issue about the unfolded formalism is its close relation to integrability. In its final form the space-time equations acquire zero-curvature condition which states that space-time dependence gets reconstructed from a given point in a pure gauge manner. Although in principle any dynamical system can be put into the unfolded form, in practice it is rarely possible to do so explicitly. Surprisingly, the unfolded form of unconstrained Yang-Mills ($s = 1$) and gravity ($s = 2$) in four dimensions is still not known unlike the complete theory of all

massless spins.

We consummate these lectures with the equations of motion for nonlinear higher-spin (HS) bosonic fields which are known in any space-time dimension. A substantial progress in these theories has been achieved in lower dimensions, $d = 3$ and $d = 4$. These are the dimensions where the two-component spinor formalism is available. Not only it simplifies formulation of the equations, it as well reduces technical difficulties in their analysis drastically. We restrict ourselves to the simplest case of four dimensional bosonic system in these notes. The main reason why this simplification really takes place is due to unconstrained spinorial realization of HS algebras accessible in lower dimensions. To give an idea how it works, we briefly touch on HS algebras. The effect of spinorial realization is to a large extent similar to the one in General Relativity in four dimensions when using Newman-Penrose formalism.

There are plenty of issues of higher-spin theory left untouched in this review. Particularly, we do not discuss questions on the structure of higher-spin cubic vertices, string inspired, [44–46], and BRST-type formulations [47]. We totally left aside issues related to alternative parent type formalisms, [48–50], developing side by side with the unfolded approach as being still much less purposeful for interacting theories. And even within the Vasiliev theory there are a lot of interesting problems that we consciously evade. Among those are problems of AdS/CFT correspondence, [51–54], and related subjects of exact solutions, [55–57]. This field has recently received great deal of attention especially in $d = 3$, [58–62], and develops rapidly. We believe it deserves a separate review, the beginning of the story is in [63].

Our goal was to make the reader familiar with the apparatus of unfolding approach which sometimes gives an impression as being bordered on tautology with unexpected power for unconstrained spinorial systems like those in three and four dimensions. Finally and most importantly we wanted to introduce the Vasiliev nonlinear equations and the technique to operate with them. In our own perception of the field the absence of underlying physical grounds and strictly speaking the absence of derivation of the equations themselves have always been a great source of confusion. It made us wonder of any smooth way of presenting the subject. In writing this self-contained and non-technical review we tried to emphasize the issues we had problems with ourselves while studying higher-spin theory. The review looks quite lengthy, but it is the price we pay for being elementary and we believe that some parts can be dropped depending on the reader's background.

There is a number of reviews devoted to different aspects of higher-spin theory, [38, 64–67]. These notes are based on the lectures on Vasiliev higher-spin theories given by the authors at the Galileo Galilei Institute, Florence. We appreciate any comments, suggestions on the structure of the lectures as well as pointing out missing references. Please feel free to contact us in case you have any questions or found some places difficult to understand.

The outline of these notes is as follows. We begin with two sections that are not directly related to the Vasiliev higher-spin theory — the review of the Fronsdal formulation and introduction to the frame-like formulation of gravity. The logic of the rest of the sections is first to convert the Fronsdal theory into the unfolded form, Section 4 for the gravity case and Section 5 for arbitrary spin fields. Once a number of unfolded examples is collected

we turn to a more abstract description of what the unfolded approach is, Section 6, where we emphasize its relation to Lie algebras and representation theory. The reason why lower dimensions are more tractable is thanks to exceptional isomorphisms, of which we need $so(3,1) \sim sl(2,\mathbb{C})$. The dictionary for tensors of $so(3,1)$ and spin-tensors of $sl(2,\mathbb{C})$ is explained in Section 8. In Section 9 using the vector-spinor dictionary we reformulate the unfolded equations found for any d in Section 5, which allows one to switch the cosmological constant on easily. The AdS_4 unfolded equations for all spins already contain certain remnants of higher-spin algebra, which is discussed in Section 10. With all ingredients being available we proceed to Vasiliev equations in Section 11.

There are also a number of extra sections, e.g. the one devoted to the MacDowell-Mansouri-Stelle-West formulation of gravity, which are not necessarily needed to proceed to Vasiliev equations. Other extra sections are devoted to more advanced topics. There is also a number of appendices containing our index conventions and an introduction to Young diagrams and tensors which is of some importance since the higher-spin theory is first of all a theory of arbitrary rank tensors.

2 Metric-like formulation for free HS fields

In this section we collect some useful facts about metric-like description of spin-s fields, [1, 68–70]. Following traditional field theory the subject is pretty standard and have been reviewed many times, see e.g. [64, 71, 72]. For the sake of simplicity we deal with bosonic fields only. Description of fermions is in many respects qualitatively similar. Although interacting massless higher spin fields are believed not to exist in flat space-time, which is a subject of many no-go theorems, see e.g. [5] for a review, the free fields do – making it useful firstly to discuss the case of spin-s fields in Minkowski space and then proceed with (anti)-de Sitter.

2.1 Massless fields on Minkowski background

Standard way of thinking of *massless fields* is as those that admit local gauge invariance responsible for a reduced number of physical degrees of freedom as compared to nongauge, massive, fields. This is generally true except for matter fields, $s = 0, 1/2$ which being nongauge still can be massless. In Minkowski space-time a massless spin-zero field $\phi(x)$ is the one that obeys $\Box \phi(x) = 0$, i.e. have zero mass-like term. Spin-one is a gauge field[1] that is described by a gauge potential $A_m(x)$ obeying Maxwell equation $\Box A_m - \partial_m \partial^n A_n = 0$ which remains invariant under local gauge transformation $A_m \sim A_m + \partial_m \xi$. These are the well known examples of lower spin massless fields $s = 0, 1$ which, as we will see, naturally fit the general spin-s free field description developed first by Fronsdal, [1, 68]. Still, these lower spin examples are to some extent degenerate exhibiting no special features characteristic for $s \geq 3$.

The less degenerate case of spin-two field can be described by a symmetric tensor $\phi_{mn} = \phi_{nm}$ with gauge symmetry $\delta\phi_{mn} = \partial_m \xi_n + \partial_m \xi_n$. This can be easily achieved

---

[1]Lowercase Latin letters $a, b, c, ...$ are for the indices in the flat space, which are raised and lowered with $\eta_{mn} = diag(-, +, ..., +)$.

from the fully nonlinear classical theory of Einstein gravity through its linearization. To do so, one identifies ϕ_{mn} with the fluctuations $g_{mn} = \eta_{mn} + \kappa\phi_{mn}$ of the metric field g_{mn} over the Minkowski background η_{mn}. Here κ is a formal expansion parameter. The gauge symmetry $\delta\phi_{mn} = \partial_m\xi_n + \partial_m\xi_n$ comes about as linearized diffeomorphism $\delta g_{mn} = \partial_m\xi^c g_{cn} + \partial_n\xi^c g_{cm} + \xi^c\partial_c g_{mn}$. Indeed, to the lowest order we have

$$\delta g_{mn} = \delta\phi_{mn} = \partial_m\xi^c\eta_{cn} + \partial_n\xi^c\eta_{cm} = \partial_m\xi_n + \partial_m\xi_n\,. \tag{2.1}$$

The equations for ϕ_{mn} can be obtained via linearization of the Einstein equations $R_{mn} - \frac{1}{2}g_{mn}R = 0$ and coincide with the Fronsdal equations for $s = 2$, see below. Of course, the free spin-two field can be defined without any reference to Einstein-Hilbert action, differential geometry and diffeomorphisms.

As it was shown by Fronsdal, [1,68], a massless spin-s field can be described by a totally symmetric rank-s tensor field[2] $\phi^{a(s)} \equiv \phi^{a_1\ldots a_s}$ which obeys an unusual trace constraint

$$\phi^{a(s-4)bcde}\eta_{bc}\eta_{de} \equiv \phi^{a(s-4)mn}{}_{mn} \equiv 0\,. \tag{2.2}$$

Clearly, the trace constraint becomes effective starting from $s = 4$ being irrelevant for $s = 0, 1, 2, 3$. It tells us that the Fronsdal field consists of two irreducible (symmetric and traceless) Lorentz tensors of ranks s and $s - 2$. Indeed, having an arbitrary symmetric tensor $\phi_{m_1\ldots m_k}$ one can always decompose it into a sum

$$\phi_{m(k)} = \phi'_{m(k)} + \eta_{mm}\phi''_{m(k-2)} + \eta_{mm}\eta_{mm}\phi'''_{m(k-4)} + \ldots\,, \tag{2.3}$$

where all 'primed' fields are traceless. Eq. (2.2) states then that Fronsdal field $\phi_{m(s)}$ is only allowed to have $\phi'_{m(s)}$ and $\phi''_{m(s-2)}$ to be non-zero. It is already this odd constraint that puzzles and complicates things a lot at interacting level as it would be more natural to work with fully unconstrained tensors or totally traceless ones. We will see in Section 5 that (2.2) arises naturally from the extension of the frame-like formulation of gravity to fields of any spin.

The dynamical input is given by Fronsdal equations

$$F^{a(s)} = \Box\phi^{a(s)} - \partial^a\partial_m\phi^{ma(s-1)} + \partial^a\partial^a\phi^{a(s-2)m}{}_m = 0\,, \tag{2.4}$$

which are invariant under the gauge transformations[3]

$$\delta\phi^{a(s)} = \partial^a\xi^{a(s-1)}\,, \qquad\qquad \xi^{a(s-3)m}{}_m \equiv 0\,, \tag{2.5}$$

where the gauge parameter is traceless (this becomes effective for $s \geq 3$).

This generalizes the cases of spin-$0, 1, 2$ to any s. The Fronsdal equations are valid for $s = 2$ as well, coinciding in this case with the linearized Einstein equations. They are valid for $s = 1$ and $s = 0$ if we notice that the third and the second terms are absent for $s = 0, 1$ and $s = 0$, respectively.

---

[2]Since the number of indices that a tensor can carry is now arbitrary we need a condensed notation. All indices in which a tensor is symmetric or needs to be symmetrized are denoted by the same letter and a group of s symmetric indices $a_1\ldots a_s$ is abbreviated to $a(s)$. The operator of symmetrization sums over all *necessary* permutations only, e.g. $V^aU^a \equiv V^{a_1}U^{a_2} + V^{a_2}U^{a_1}$. More info is in Appendix B.

[3]The verification of this fact is in Appendix B.

The tracelessness of $\xi^{a(s-1)}$ goes hand in hand with the double-tracelessness of the Fronsdal field. Indeed, from (2.5) it follows that $\phi^{a(s)}$ should be double traceless once $\xi^{a(s-1)}$ is traceless. The second trace of $\phi^{a(s)}$ is not affected by the gauge symmetry. In playing with trace constraints for field/gauge parameter one finds that the Fronsdal theory is essentially unique[4]. To this effect it is instructive to look at the variation of the Fronsdal operator under (2.5) with $\xi^{a(s-1)}$ not satisfying any trace constraints

$$\delta F^{a(s)} = \left(3\partial^a\partial^a\partial^a - 2\eta^{aa}\partial^a\right)\xi^{a(s-3)m}{}_m \, . \tag{2.6}$$

Going on-shell.   In order to see that the solutions to the Fronsdal equation do carry a spin-s representation of the Poincare group it is useful to define de-Donder tensor

$$D^{a(s-1)} = \partial_m\phi^{ma(s-1)} - \frac{1}{2}\partial^a\phi^{a(s-2)m}{}_m \, , \tag{2.7}$$

$$\delta D^{a(s-1)} = \Box\xi^{a(s-1)} \, , \tag{2.8}$$

$$F^{a(s)} = \Box\phi^{a(s)} - \partial^a D^{a(s-1)} \, , \tag{2.9}$$

which transforms in a simple way under the gauge transformation and constitutes the non-$\Box\phi$ part of the Fronsdal operator. Since the de-Donder tensor carries as many components as the gauge parameter (2.8) it is possible to gauge it away $D^{a(s-1)} = 0$. One is left with the gauge transformations $\xi^{a(s-1)}$ obeying $\Box\xi^{a(s-1)} = 0$. Since $\xi^{a(s-1)}$ can be nonzero one can further impose one more condition, $\phi^{a(s-2)m}{}_m = 0$. Indeed, $\phi^{a(s)}$ is now on-shell, i.e. $\Box\phi^{a(s)} = 0$, and $\delta\phi^{a(s-2)m}{}_m = 2\partial_m\xi^{ma(s-2)}$. The gauge-fixed equations and constraints read

$$\Box\phi^{a(s)} = 0 \, , \qquad\qquad\qquad \Box\xi^{a(s-1)} = 0 \, , \tag{2.10}$$

$$\partial_m\phi^{ma(s-1)} = 0 \, , \qquad\qquad\qquad \partial_m\xi^{ma(s-2)} = 0 \, , \tag{2.11}$$

$$\phi^{a(s-2)m}{}_m = 0 \, , \qquad\qquad\qquad \xi^{a(s-3)m}{}_m = 0 \, , \tag{2.12}$$

$$\delta\phi^{a(s)} = \partial^a\xi^{a(s-1)} \, . \tag{2.13}$$

These are all the Lorentz covariant conditions one can impose without trivializing the solution space. There is still a leftover on-shell gauge symmetry, which manifests inde-composable structure of the representation carried by $\phi^{a(s)}$.

---

[4]In principal one can relax the tracelessness of the gauge parameter preserving double-tracelessness of the field. However, as is seen from (2.5), this would result in differential constraint on a gauge parameter which is not a safe thing for it typically affects the number of physical degrees of freedom and having a differential constraints on fields/gauge parameters complicates the study of interactions a lot. If the field were irreducible, i.e. it is symmetric and traceless, one would derive $\partial_m\xi^{a(s-2)m} = 0$ by taking trace of (2.5), which is again a differential constraint. Imposing such a constraint makes sense at the fully nonlinear level in the case of $s = 2$, which corresponds to volume-preserving diffeomorphisms, see e.g. [73]. The generalization to the spin-s case is also possible, [72, 74]. Alternatively, one could try to project $\partial^a\xi^{a(s-1)}$ onto the traceless component, in which case one would find no gauge invariant equations, so this option is unacceptable.

Counting degrees of freedom. To count degrees of freedom[5] one solves (2.10) by performing the Fourier transform

$$\phi^{a(s)}(x) = \int d^d p \, \delta(p^2) \, \varphi^{a(s)}(p) e^{ipx} \tag{2.14}$$

and analogously for $\xi^{a(s-1)}$.

Since the equations are Lorentz covariant all points in momentum space are equivalent and we can look at any p_m, $p_m p^m = 0$ to count the number of independent functions. A convenient choice is given by light-cone coordinates

$$x_{\pm} = \frac{1}{\sqrt{2}}(x_1 \pm x_0), \qquad x_i = \{x_2, ..., x_{d-1}\}. \tag{2.15}$$

Indices range $a = \{+, -, i\}$, $i = 1...(d-2)$ in these coordinates with the metric $\eta^{+-} = \eta^{-+} = 1$, $\eta^{ij} = \delta^{ij}$ and all other components being zero. Let us take $p_m = e\delta_m^+$, where e is some constant and δ_m^+ is the Kronecker delta. The components of $\varphi^{a(s)}$ can be split into

$$\varphi^{\overbrace{+...+}^{N}\overbrace{-...-}^{M}i(s-M-N)}. \tag{2.16}$$

Eq. (2.11) tells that all components with at least one $+$ direction vanish, $\varphi^{+a(s-1)} = 0$ (analogously for $\xi^{a(s-1)}$). Now one can use gauge symmetry (2.13) $\delta\varphi^{a(s)} = e\eta^{+a}\xi^{a(s-1)}$, i.e. $\delta\varphi^{-a(s-1)} = e\eta^{+-}\xi^{a(s-1)}$ to set $\varphi^{-...-i...i} = 0$. Finally, we are left with $\varphi^{i(s)}(p)$ that is symmetric and $so(d-2)$-traceless, i.e. traceless with respect to δ^{ij}. Indeed, $\varphi^{a(s)}$ is $so(d-1,1)$-traceless, which can be rewritten as

$$\varphi^{a(s-2)m}{}_m \equiv \varphi^{a(s-2)ij}\delta_{ij} + 2\varphi^{a(s-2)+-}\eta_{+-} \equiv 0. \tag{2.17}$$

Then, we note that the last term carries at least one index along '+'-direction, so it is zero by (2.11). To conclude, the degrees of freedom are those of an irreducible rank-s $so(d-2)$-tensor times the dependence on p_m that lives on $(d-1)$-dimensional cone. That $\varphi^{i(s)}(p)$ is an irreducible rank-s tensor at each p, i.e. symmetric and traceless, implies that it describes a single spin-s particle.

Lagrangian. The Fronsdal Lagrangian reads

$$S = -\frac{1}{2}\int_{M^d}\left(\partial_m\phi^{a(s)}\partial^m\phi_{a(s)} - \frac{s(s-1)}{2}\partial_m\phi^n{}_n{}^{a(s-2)}\partial^m\phi_k{}^k{}_{a(s-2)} + \right.$$
$$+ s(s-1)\partial^m\phi^n{}_n{}^{a(s-2)}\partial^k\phi_{kma(s-2)} - s\partial_m\phi^{ma(s-1)}\partial^n\phi_{na(s-1)}+ \tag{2.18}$$
$$\left. -\frac{s(s-1)(s-2)}{4}\partial_m\phi^n{}_n{}^{ma(s-3)}\partial^r\phi_r{}^k{}_{ka(s-3)}\right).$$

It is fixed up to an overall factor and total derivatives by the gauge symmetry, (2.5), [76]. It can be put into a more compact form by integrating by parts

$$S = \frac{1}{2} \int_{M^d} \phi_{a(s)} \, G^{a(s)} \,, \qquad\qquad G^{a(s)} = F^{a(s)} - \frac{1}{2} \eta^{aa} F^{a(s-2)m}{}_m \,, \qquad\qquad (2.19)$$

where the trace of the Fronsdal operator is

$$F[\phi]^{a(s-2)m}{}_m = 2\Box\phi^{a(s-2)m}{}_m - 2\partial_n\partial_m\phi^{mna(s-2)} + \partial^a\partial_m\phi^{a(s-3)mn}{}_n \,. \qquad\qquad (2.20)$$

The gauge invariance of the action implies the Bianchi identities

$$0 = \delta S = s \int_{M^d} \partial_b\xi_{a(s-1)} \, G^{a(s-1)b} = -s \int_{M^d} \xi_{a(s-1)} \, \partial_m G^{a(s-1)m} = -s \int_{M^d} \xi_{a(s-1)} B^{a(s-1)} \,,$$

i.e. the following linear operator annihilates the equations of motion[6]

$$B[F]^{a(s-1)} = \partial_m F^{a(s-1)m} - \frac{1}{2}\partial^a F^{a(s-2)m}{}_m \,, \qquad\qquad B[F[\phi]] \equiv 0 \,. \qquad\qquad (2.21)$$

Again, it is instructive to see how the Bianchi identities get violated if ϕ does not obey the double-trace constraint,

$$B[F[\phi]]^{a(s-1)} = \left(-\frac{3}{2}\partial^a\partial^a\partial^a + \eta^{aa}\partial^a \right) \phi^{a(s-4)mn}{}_{mn} \,. \qquad\qquad (2.22)$$

Let us note that the equations that come from Lagrangian, (2.19) or (2.18), is $G^{a(s)} = 0$ and is a little bit different from (2.4). They are equivalent in fact. Indeed, taking the trace

$$G^{a(s-2)m}{}_m = -\frac{d + 2s - 6}{2} F^{a(s-2)m}{}_m \,, \qquad\qquad (2.23)$$

we see that $G^{a(s)} = 0$ implies $G^{a(s-2)m}{}_m = 0$ and hence $F^{a(s-2)m}{}_m = 0$. On substituting this back to $G^{a(s)} = 0$ one finds $F^{a(s)} = 0$. It was important that both $G^{a(s)}$ and $F^{a(s)}$ are double traceless as a consequence of $\phi^{a(s-4)mn}{}_{mn} \equiv 0$.

The long and winding road from representations to Lagrangians. It is worth stressing that a systematic approach to Lagrangians can be quite difficult requiring to answer a priori four different questions.

(i) to classify all unitary irreducible representations of the space-time symmetry group, Poincare in our case. Postulates of Quantum Mechanics combined with the Special Relativity (i.e. the idea that the physical laws are covariant under the Poincare algebra) results in the statement that all systems, e.g. particles, must carry a unitary representation of the Poincare algebra, [77]. This is where the notion of spin and mass comes out as parameters specifying a representation. The representation theory of Poincare algebra has little to do with the space-time directly.

---

[6]Let us note that $\partial_m G^{a(s-1)m}$ has one more term, $-\frac{1}{2}\eta^{aa}\partial_m F^{a(s-3)mn}{}_n$, which is projected out thanks to traceless $\xi^{a(s-1)}$, which again shows the importance of $\xi^{a(s-3)m}{}_m \equiv 0$.

(ii) to realize these representations on the solutions of certain P.D.E's imposed on certain tensor fields over the Minkowski space. We have seen that a spin-s representation is realized on $\varphi^{i(s)}(p)$ where $p^2 = 0$ and has again a little to do with the space-time. An on-shell description is given by (2.10)-(2.13) in terms of $\phi^{a(s)}(x)$ that is defined up to a gauge transformation. At this stage we see that $\varphi^{i(s)}$ comes as projection/factor of $\varphi^{a(s)}$ and in principle one can imagine embedding $\varphi^{i(s)}$ into an $so(d-1,1)$-tensor with more indices such that the equations/gauge symmetries project out redundant components. The number of indices that a field may carry is not directly related to the spin as a parameter of an irreducible representation, one has to take equations/gauge symmetries into account. There are generally infinitely many descriptions of one and the same representation by different combinations of field/P.D.E./gauge-symmetry. The simplest example is a spin-one particle, photon, which can be equally well described by gauge potential A_m, $\Box A_m - \partial_m \partial^n A_n = 0$, $\delta A_m = \partial_m \xi$ or by nongauge field strength $F_{mn} = -F_{nm}$, $\partial^n F_{mn} = 0$, $\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0$. There is a generalization of this example to fields of all spins higher than one, which is in Section 5.5.

(iii) to find an off-shell description, i.e. to extend fields/gauge parameters in such a way that no differential constraints like (2.10)-(2.12) remain. An off-shell description as we have seen requires adding a traceless rank-$(s-2)$ tensor to the traceless $\phi^{a(s)}$ to be combined together into a double traceless Fronsdal field.

(iv) to get these P.D.E.'s (or equivalent to them) as variational equations for certain Lagrangian. Coincidentally, in the case of massless spin-s fields in Minkowski or anti-de Sitter space the same field content as we used for an off-shell description is sufficient to write down a Lagrangian, which is not true for the massive spin-s field, [78, 79].

2.2   Massless fields on (anti)-de Sitter background

The Fronsdal theory can be easily extended to the constant curvature background, [69, 70], which are the maximally symmetric solutions of Einstein equations with cosmological constant Λ. These are known as de Sitter, $\Lambda > 0$, and anti-de Sitter, $\Lambda < 0$ spaces. The algebraic double trace constraint (2.2) remains unchanged but η_{mn} gets replaced with[7] the (anti)-de Sitter metric[8] $g_{\underline{mn}}(x)$

$$\phi_{\underline{m(s-4)nnrr}}\, g^{\underline{nn}} g^{\underline{rr}} \equiv 0\,. \tag{2.25}$$

All derivatives should be covariantized and we use the normalization

$$[D_{\underline{m}}, D_{\underline{n}}]V^{\underline{a}} = \Lambda \delta^{\underline{a}}_{\underline{m}} g_{\underline{nb}} V^{\underline{b}} - \Lambda \delta^{\underline{a}}_{\underline{n}} g_{\underline{mb}} V^{\underline{b}} \tag{2.26}$$

---

[7]From now on we reserve lowercase Latin letters $a, b, c, ...$ for the indices in the flat space, which are raised and lowered with η_{mn}. Underlined lowercase Latin letters $\underline{a}, \underline{m}, \underline{n}, \underline{r}, \underline{k}, ...$ are the 'world' indices being raised and lowered with some generally nonconstant metric $g_{\underline{mn}}$.

[8]Equivalently we can transfer all indices to the fiber with the help of the tetrad/frame/vielbein field $h^a_{\underline{m}}$, which will be introduced systematically in the next Section. In fact, it has to be introduced once we would like to include fermions. Then the algebraic constraints do not change at all

$$\phi^{a(s-4)bbcc}\eta_{bb}\eta_{cc} \equiv 0\,, \qquad\qquad \phi^{a(s)} = \phi_{\underline{m(s)}}h^{\underline{m}a}...h^{\underline{m}a}\,. \tag{2.24}$$

In taking derivatives we can use $[D_{\underline{m}}, D_{\underline{n}}]V^{a(s)} = \Lambda h^a_{\underline{m}} h_{\underline{n}b} V^{ba(s-1)} - \Lambda h^a_{\underline{n}} h_{\underline{m}b} V^{ba(s-1)}$.

and Λ is the cosmological constant. The gauge transformation law now reads

$$\delta\phi_{\underline{a}(s)} = \nabla_{\underline{a}}\xi_{\underline{a}(s-1)} \,. \tag{2.27}$$

The Fronsdal operator is promoted to

$$F^{\underline{a}(s)}[\phi] = \Box\phi^{\underline{a}(s)} - \nabla^{\underline{a}}\nabla_{\underline{m}}\phi^{\underline{m}\underline{a}(s-1)} + \frac{1}{2}\nabla^{\underline{a}}\nabla^{\underline{a}}\phi^{\underline{a}(s-2)\underline{m}}{}_{\underline{m}} - m_\phi^2\phi^{\underline{a}(s)} + 2\Lambda g^{\underline{a}\underline{a}}\phi^{\underline{a}(s-2)\underline{m}}{}_{\underline{m}} \,,$$
$$m_\phi^2 = -\Lambda((s-2)(d+s-3) - s) \,, \tag{2.28}$$

where we note the appearance of the mass-like terms. Mind that $\partial^a\partial^a$ in the third term has changed to $\frac{1}{2}\nabla^{\underline{a}}\nabla^{\underline{a}}$ since covariant derivatives do not commute and $s(s-1)$ terms are now needed to symmetrize $\underline{a}(s)$ as contrast to $s(s-1)/2$ terms in flat space. In checking the gauge invariance of the equations we find that the leading terms vanish thanks to the gauge invariance of the Fronsdal operator in flat space. However, in order to cancel the gauge variation we have to commute some of the derivatives. The commutators produce certain new terms that can fortunately be canceled by adding mass-like terms. The strange value of m_ϕ^2 can be derived using the representation theory of $so(d-1,2)$ or $so(d,1)$, [80,81] which are the symmetry algebras of anti-de Sitter and de Sitter spaces, respectively. It is related to the Casimir operator in the corresponding representation. We will have more to say about (anti)-de Sitter later on. The lesson is that masslessness that imply gauge invariance does not necessarily imply the absence of mass-like terms. Protection of gauge invariance is more important than the presense/absence of mass-like terms as it guarantees that the number of physical degrees of freedom does not change when switching on the cosmological constant. The constant curvature of (anti)-de Sitter space acts effectively as a harmonic potential that renormalizes the value of the mass term. Let us also note, that the precise value of the mass-like term depends also on the way the covariant derivatives in the second term of (2.28) are organized.

The action has formally the same form

$$S = \frac{1}{2}\int \phi_{\underline{a}(s)}\, G^{\underline{a}(s)} \,, \qquad\qquad G^{\underline{a}(s)} = F^{\underline{a}(s)} - \frac{1}{2}g^{\underline{a}\underline{a}}F^{\underline{a}(s-2)\underline{m}}{}_{\underline{m}} \,, \tag{2.29}$$

where the trace of the Fronsdal operator reads

$$F[\phi]^{\underline{a}(s-2)\underline{m}}{}_{\underline{m}} = 2\Box\phi^{\underline{a}(s-2)\underline{m}}{}_{\underline{m}} - 2\nabla_{\underline{n}}\nabla_{\underline{m}}\phi^{\underline{m}\underline{n}\underline{a}(s-2)} + \nabla^{\underline{a}}\nabla_{\underline{m}}\phi^{\underline{a}(s-3)\underline{m}\underline{n}}{}_{\underline{n}} - m_1^2\phi^{\underline{a}(s-2)\underline{m}}{}_{\underline{m}} \,,$$
$$m_1^2 = -2\Lambda(s-1)(d+s-3) \,. \tag{2.30}$$

Finally let us note that (2.6) and (2.22) are still valid upon replacing $\partial_a \to \nabla_{\underline{a}}$.

The process of imposing gauges is analogous to the case of the Minkowski background. Given this, without going into details, we can conclude that equation (2.30) describes the same number of physical degrees of freedom since it preserves the same amount of gauge symmetry and the order of equations and Bianchi identities remains unchanged.

It is worth stressing that by going from Minkowski to more general backgrounds one can lose certain amount of physical interpretation. For example, in anti-de Sitter space the space-time translations, P_a, do not commute so one cannot diagonalize all P_a simultaneously. In particular, P_aP^a is no longer a Casimir operator as it is the case in the

13

Minkowski space. Nevertheless, in some sense anti-de Sitter algebra $so(d-1, 2)$ is better than Poincare one in being one of the classical Lie algebras, while Poincare algebra is not semi-simple, which leads to certain peculiarities in constructing representations of the latter. The Poincare algebra can be viewed as a contraction of $so(d-1, 2)$. The particles in the case of anti-de Sitter algebra $so(d-1, 2)$ should be defined as Verma modules with spin and mass being related to the weights of $so(d-1, 2)$, which is somewhat technical and we refer to [80, 82, 83].

As for the field description the best one can do on a general background is to ensure that the number and order of gauge symmetries/equations/Bianchi identities remains unchanged (or get changed in a coherent way) so as to preserve the number of degrees of freedom, [84].

Once the gravity is dynamical or the background is different from (anti)-de Sitter or Minkowski, the Fronsdal operator is no longer gauge invariant. Indeed, in verifying the gauge invariance we have to commute covariant derivatives ∇'s. For the case of the Minkowski space ∇'s just commute. For the case of (anti)-de Sitter (constant curvature) space the commutator is proportional to the background metric, so the commutators produce mass-like terms. In generic background we are left with

$$\delta F = R_{...} \nabla \xi^{...} \neq 0 \tag{2.31}$$

where $R_{...}$ is the Riemann tensor. Therefore the Fronsdal operator becomes inconsistent on more general configurations of metric.

In particular when the metric g_{mn} becomes a dynamical field we face the problem of how to make higher-spin fields interact with gravity. This was the starting point for the no-go [85] by Deser and Aragone and then yes-go results by Fradkin and Vasiliev [3, 4] (see review [5] on various no-go theorems related to higher-spins). Some comments on the Fronsdal theory on general Riemannian manifolds can be found in extra Section 12.1.

We see that there is something special about higher-spin fields, the threshold being $s = 2$, since all lower-spin fields, $s = 0, \frac{1}{2}, 1$ can propagate on any background, g_{mn}, and the graviton is self-consistent on any background of its own.

Summary. There is a well-defined theory of free fields of any spin-s on the specific backgrounds, which are Minkowski and (anti)-de Sitter — maximally symmetric solutions of Einstein equations with cosmological constant. The fields and gauge parameters have to obey certain trace constraints, (2.2), (2.5.b).

3   Gravity as gauge theory

Among theories of fundamental interactions there are Yang-Mills gauge theories based on (non)abelian Lie algebras and General Relativity (GR) that stands far aside and is typically viewed as essentially different from gauge theories. Particularly, the way it was formulated by Einstein, GR does not rest on any gauge group. On the other hand, gravity clearly has a gauge symmetry represented by arbitrary coordinate transformations and diffeomorphisms. From that perspective it seems quite natural to address a question of a gauge form of GR.

This section is aimed to demonstrate that gravity can in many respects be thought of as a gauge theory. The relevant variables to see this are the so called vielbein $e^a_{\underline{m}}$ and spin-connection $\omega^{a,b}_{\underline{m}}$, which can to some extent be treated as components of a Yang-Mills connection of Poincare, $iso(d-1,1)$, de Sitter, $so(d,1)$, or anti-de Sitter, $so(d-1,2)$, algebras. The reader familiar with Cartan formulation of gravity can skip the entire section. We begin with a very short and elementary introduction to the Cartan geometry, then proceed to various ways of thinking of gravity as a gauge theory. The MacDowell-Mansouri-Stelle-West formulation of gravity is left to the extra Section 12.2. The relevant references include [86–89] and [90] for the references on the original papers by Cartan, Weyl, Sciama, Kibble.

3.1   Tetrad, Vielbein, Frame, Vierbein,....

In differential geometry one deals with manifolds – something that can be built up from several copies of the Euclidian space. The point is that not every hyper-surface we can imagine is homeomorphic to a Euclidian space and hence can be covered by some global coordinates. Therefore we have to cut a generic manifold into smaller overlapping pieces each of which can be thought of as a copy of Euclidian space. We need transition functions that allow one to identify the regions of two copies of Euclidian space whose images overlap on the manifold. A manifold itself then comes as a number of copies of Euclidian space (patches) together with the transition functions that are defined for certain pairs of copies and obey certain consistency relations.

In differential geometry framework the objects, tensors, one uses transform properly under the change of the coordinates so that the scalar (physical) quantities we compute do not depend on the choice of coordinates. Despite the fact that differential geometry is designed in a democratic way with respect to different coordinates, this is not fully so for tensors. Indeed, given a tensor T its components $T^{\underline{m}\cdots}{}_{\underline{n}\cdots}$ are given with respect to the basis in the tangent space that is induced from the coordinates in the current chart. The basis vectors at a given point are vectors that are tangent to the coordinate lines, see the figure. We will refer to such 'bare' tensors as to world tensors and to the indices they carry, $\underline{m}, \underline{n}, ...$, as to world indices. To disentangle the basis in the chart and in the tangent space we may introduce an auxiliary nondegenerate matrix $e^a_{\underline{m}}(x)$ that transfers tensor indices from the basis induced by the particular coordinates to some other basis in the tangent space we might prefer more. With the help of $e^a_{\underline{m}}(x)$ each world tensor acquires an avatar

$$T^{\underline{m}\cdots}{}_{\underline{n}\cdots} \longrightarrow T^{a\cdots}{}_{b\cdots} = e^a_{\underline{m}}... T^{\underline{m}\cdots}{}_{\underline{n}\cdots} \, (e^{-1})^{\underline{n}}_b... \tag{3.1}$$

and we refer to the tensor in the new basis as to the fiber (tangent) tensor and to the indices it carries, $a, b, ...$ as to fiber (tangent) indices. In principle, tensors of mixed type, i.e. those that carry both world and fiber indices simultaneously are possible and such tensors do appear in our study. But the rule of course is that only indices of the same type can be contracted with either δ^a_b or $\delta^{\underline{m}}_{\underline{n}}$.

If no derivatives are around it is obvious that one can use either of the bases for algebraic computations, e.g. taking tensors products or contracting indices, i.e. the

Figure 1: *The basis vectors in the tangent plane TM_O at some point O are by definition the vectors that are tangent to the coordinate lines. Let $p(x_1, ..., x_n)$ be the point on the manifold parameterized by Cartesian coordinates $(x_1, ..., x_n)$ in some chart, we can think of it as a point in a bigger Euclidian space where the given manifold is embedded. Then, $\vec{e}_i = \frac{d}{dt} p(x_1, ..., x_i + t, ..., x_n)|_{t=0}$*
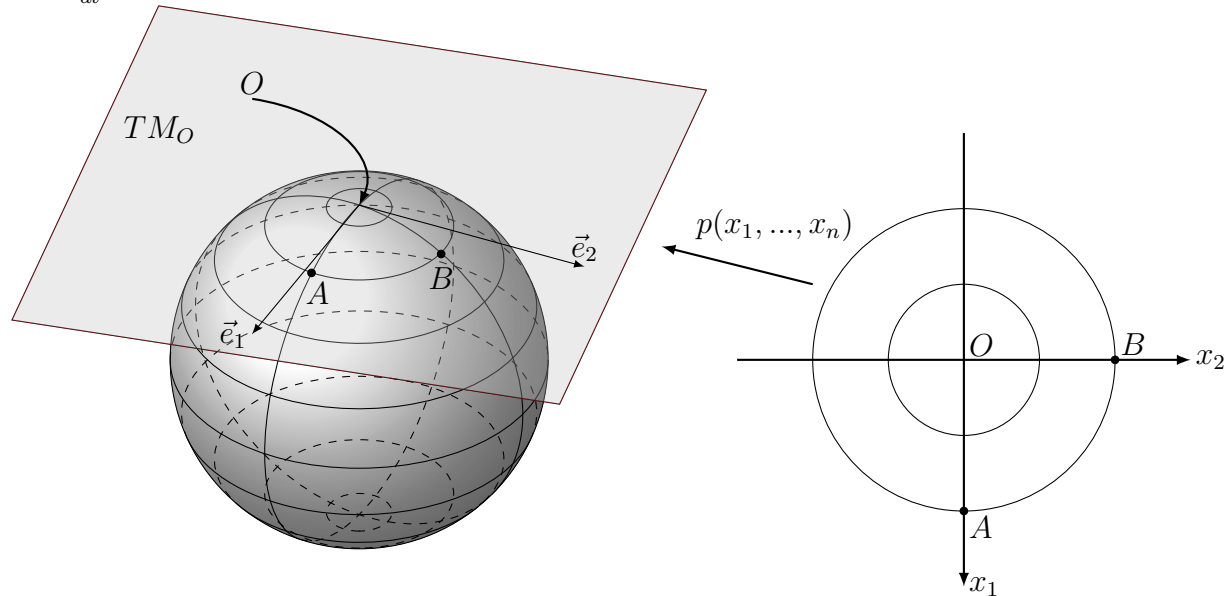
following diagram commutes

$$
\begin{array}{ccc}
\text{set of world tensors } T^{\underline{m}\cdots}{}_{\underline{n}\cdots}, ... & \xrightarrow{\;e,e^{-1}\;} & \text{set of fiber tensors } T^{a\cdots}{}_{b\cdots}, ... \\
\Big\downarrow \text{operations} & & \Big\downarrow \text{operations} \qquad (3.2) \\
\text{derived set of world tensors } T^{\underline{n}\cdots}{}_{\underline{n}\cdots}, ... & \xrightarrow{\;e,e^{-1}\;} & \text{derived set of fiber tensors } T^{a\cdots}{}_{a\cdots}
\end{array}
$$

In other words, having a set of world tensors first, we can either do some algebraic computations like taking tensor products or contracting the indices and then transfer all indices left free into the fiber ones with the help of $e^a_{\underline{m}}$, $(e^{-1})^{\underline{m}}_a$ or we can first transfer all the indices to fiber ones and then perform identical computations but in the fiber.

There is a strong motivation from physics to introduce $e^a_{\underline{m}}$ — the equivalence principle. In the famous Einstein's thought experiment an experimentalist, when put into a free falling elevator without windows, cannot tell whether she is falling freely in the gravitational field or is left abandoned in the open space far away from any sources of gravitational field[9]. This has led Einstein to the equivalence principle (EP). EP implies that locally one can always eliminate the gravitational field by taking a freely falling elevator. This statement lies at the core of all problems in defining stress-tensor of gravity. As we are going to consider gravity, there is a preferred set of bases given by Einstein's elevators — elevators that are freely falling in the local gravitational field — the physics in this elevator is locally as in the Special Relativity (SR). The latter is true for physically small elevators, i.e. up to tidal forces, etc.

[9]Equivalently, gravitational field is locally indistinguishable from the accelerating frame.

The EP tells us that the metric in the new basis, which is associated with the elevator, is constant, for example, $\eta^{ab} = \text{diag}(-++...+)$, i.e. we have

$$\eta^{ab} = e^a_{\underline{m}}(x)\, g^{\underline{mn}}(x)\, e^b_{\underline{n}}(x)\,, \qquad\qquad \eta = e^T g\, e\,, \qquad\qquad (3.3)$$

or, equivalently, one can always recover the original metric[10] $g_{\underline{mn}}$

$$g_{\underline{mn}} = e^a_{\underline{m}}(x)\, \eta_{ab}\, e^b_{\underline{n}}(x)\,. \qquad\qquad\qquad (3.4)$$

The object $e^a_{\underline{m}}(x)$, i.e. the Einstein's elevator, is called *tetrad* or *vierbein* in the case of four dimensional space-time; *vielbein, soldering form* or *frame* in arbitrary d; *zweibein, dreibein, etc* in case of two, three, etc. dimensions.

It is worth noting that the metric as a function of the vielbein is defined in such a way that different ways of raising and lowering indices lead to the same result. For example, the inverse vielbein $e^{\underline{m}}_a$ is just the matrix inverse of $e^a_{\underline{m}}$, but it can also be viewed as $e^a_{\underline{m}}$ whose indices were raised/lowered with $g^{\underline{mn}}$ and η_{ab},

$$e^{\underline{m}}_a = (e^{-1})^{\underline{m}}_a = g^{\underline{mn}}\, e^b_{\underline{n}}\, \eta_{ba}\,. \qquad\qquad\qquad (3.5)$$

Obviously, $e^a_{\underline{m}}(x)$, being a $d \times d$ matrix that depends on x, has enough components to guarantee (3.3). A change of coordinates $x^{\underline{m}} \to y^{\underline{m}}$ amounts to defining d functions $y^{\underline{m}} = f^{\underline{m}}(x^{\underline{n}})$, i.e. it has less 'degrees of freedom' as compared to the vielbein. Indeed, in order for $e^a_{\underline{m}}(x)$ to be equivalent to a change of coordinates it must be $e^a_{\underline{m}} = \partial_{\underline{m}} f^a(x)$. The integrability of this condition, i.e. $0 \equiv (\partial_{\underline{n}}\partial_{\underline{m}} - \partial_{\underline{m}}\partial_{\underline{n}}) f^a(x)$, implies $\partial_{\underline{m}} e^a_{\underline{n}} - \partial_{\underline{n}} e^a_{\underline{m}} = 0$, which is generically not true. It is obvious that one cannot remove gravitational field everywhere just by a coordinate transformation since there are tensor quantities like Riemann tensor $R_{\underline{mn},\underline{rk}}$ and $R_{\underline{mn},\underline{rk}} = 0$ is a coordinate independent statement.

The equivalence principle leads to an idea of General Relativity (GR) as being a localization (gauging) of Special Relativity (SR) and this is the idea we would like to follow and to generalize to fields of all spins. SR can be thought of as the theory of the global Poincare invariance, i.e. a theory of $ISO(d-1,1)$ as a rigid symmetry. Which amount of this symmetry gets localized in GR? Apparently the elevators form an equivalence class since given an elevator $e^a_{\underline{m}}(x)$ one can rotate it and boost it at any velocity v. These transformations belong to the Lorentz group $SO(d-1,1)$. At each point (or physically speaking at small neighborhood of each point) we have a different elevator and hence the Lorentz transformations can depend on x. To put it formally, $e^a_{\underline{m}}(x)$ and

$$e'^a_{\underline{m}}(x) = A^a{}_b(x) e^b_{\underline{m}}(x)\,, \qquad\qquad\qquad (3.6)$$

where $A(x) \in SO(d-1,1)$, i.e. $A^T \eta A = \eta$, produce the same $g_{\underline{mn}}$ and does not change η^{ab}. If the transformation is small, i.e. $A^a{}_b$ is close to the unit matrix, we can write $A^a{}_b = \delta^a{}_b - \epsilon^a{}_{,b}$ and $\epsilon^{a,b} \equiv \epsilon^a{}_c \eta^{cb}$ is antisymmetric, $\epsilon^{a,b} = -\epsilon^{b,a}$. Then a small change in $e^a_{\underline{m}}$ results in

$$\delta e^a_{\underline{m}}(x) = -\epsilon^a{}_{,b}(x) e^b_{\underline{m}}(x)\,, \qquad\qquad\qquad (3.7)$$

which is a localized version of (3.6).

---

[10]If we forget about the x dependence, $e^a_{\underline{m}}$ is a matrix that diagonalizes the given quadratic form $g_{\underline{mn}}$.

However, we lost translations of $ISO(d-1,1)$ as the local symmetry. Translation brings an elevator to another point where the gravitational field may differ. As we will see local translations are not genuine symmetries.

Now the metric g_{mn} can be viewed as a derived object and not as a fundamental. Every statement in the language of g_{mn} can be always rewritten in the language of $e^a_{\underline{m}}$ and not vice verse because $e^a_{\underline{m}}$ is defined up to an x-dependent Lorentz rotation in accordance with the fact the Einstein's elevator is not unique. We can also see this by counting independent components, g_{mn} has $d(d+1)/2$ components, while $e^a_{\underline{m}}$ has d^2 components. The Lorentz transformations form a $d(d-1)/2$-dimensional group, so

$$\#\text{vielbein} - \#\text{Lorentz} = \#\text{metric}, \qquad (3.8)$$

which means that we did not lose or gain any new 'degrees of freedom'.

There is one more fundamental reason to introduce the vielbein — matter fields, e.g. electrons, protons, neutrons, which are fermions and thus are represented by spinors. They still do experience gravitational interaction and we have to deal with this experimental fact. Let us emphasize that the very definition of spinors relies on the representation theory of the Lorentz algebra $so(d-1,1)$, which in the Minkowski space of Special Relativity is a subalgebra of the full Poincare symmetry algebra $iso(d-1,1)$. The notion of spin and mass rests on the representation theory of $iso(d-1,1)$ too. These are the parameters that define unitary irreducible representations of $iso(d-1,1)$. The existence of spinors, which is due to the first homotopy group of $SO(d-1,1)$ being nontrivial, makes it possible to consider the action of the group up to a phase which distinguishes between contractible and non-contractible paths on the group. This leads to a bizarre consequences, e.g. the electron wave function changes its sign upon 2π-rotation.

A theory formulated in terms of some tensor representations of the Lorentz algebra, which are then used to define tensor fields over the Minkowski space, can be straightfor-wardly extended to a theory that has $gl(d)$ as a symmetry algebra and then to a diffeo-morphism invariant theory. Clearly, having an $so(d-1,1)$-tensor $T^{abc\cdots}$ in some theory we can replace it with a tensor of $gl(d)$ of the same type. Then, having a tensor of $gl(d)$ we can turn it into a field $T^{abc\cdots}(x)$ and make it transform under diffeomorphisms, see the table below for some examples. However, there is no straightforward lift of spin-tensor representations of $so(d-1,1)$ to $gl(d)$. Apparently[11] we do not know of what replaces spin-tensors in the case of $gl(d)$. The vielbein solves this problem as we can put ourselves into the reference frame where the symmetry algebra is $so(d-1,1)$, the difference is that it is a local statement. In order to construct Lagrangians and field equations we need to extend the covariant derivative to tensors with fiber indices.

Covariant derivative. It is necessary to define the covariant derivative in the fiber, then we can make it act in any representation of the Lorentz algebra, $so(d-1,1)$, in

---

[11]Formally the fundamental group of $SL(d)$, $GL(d) = SL(d) \times GL(1)$, is the same as for $SO(d)$, because it is determined by the maximal compact subgroup. However, the double-valued representations of $SL(d)$ are infinite-dimensional. A possible way out is to take infinite-dimensional spinorial representations of $SL(d)$ seriously, [91]. Such representations, when restricted to $SO(d)$, decompose into an infinite sum of spin-tensor representations of all spins and hence contain higher-spin fields. As we will learn the consistency of higher-spin theory requires infinite number of higher-spin fields, so at the end of the day $SL(d)$-spinors may not be so far away, [92].

	SR	GR
general Lagrangian	$\int d^d x\, \mathcal{L}(\phi, \partial_{\underline{m}}\phi, \eta^{ab})$	$\int \sqrt{\lvert g \rvert}\, d^d x\, \mathcal{L}(\phi, \nabla_{\underline{m}}\phi, g^{\underline{mn}})$
spin-zero	$\frac{1}{2}\int d^d x\, \partial_a\phi\, \partial_b\phi\, \eta^{ab}$	$\frac{1}{2}\int \sqrt{\lvert g \rvert}\, d^d x\, \partial_{\underline{m}}\phi\, \partial_{\underline{n}}\phi\, g^{\underline{mn}}$
spin-one	$\frac{1}{4}\int d^d x\, F_{ab}\, F^{ab}$	$\frac{1}{4}\int \sqrt{\lvert g \rvert}\, d^d x\, F_{\underline{mn}}\, F_{\underline{rk}}\, g^{\underline{mr}} g^{\underline{nk}}$
spin-half	$\int d^d x\, \bar{\psi}\gamma^a\partial_a\psi$	???, wait for (3.27)

particular on spin-tensors and hence be able to write down the Dirac Lagrangian in the gravitational field. The covariant derivative needs to be defined in a way that the following diagram commutes, otherwise there will be too many problems in comparing the results of differentiation in the two bases (we still think that the simple recipe to replace ∂ with $D = \partial + \Gamma$ works well for world tensors so we do not want to abandon this knowledge),

$$
\begin{array}{ccc}
V_{\underline{m}} & \xrightarrow{\;e_a^{\underline{m}}\;} & V_a \\[4pt]
\Big\downarrow{\scriptstyle D_{\underline{n}}} & & \Big\downarrow{\scriptstyle D_{\underline{n}}} \\[4pt]
D_{\underline{n}}V_{\underline{m}} & \xrightarrow{\;e_a^{\underline{m}}\;} & D_{\underline{n}}V_a
\end{array}
\tag{3.9}
$$

The diagram implies that we can first differentiate a tensor, then transfer it to another basis, or first transfer it to another basis and then differentiate. The results must coincide. Since in the world basis a vector in two coordinate frames can be related by any $GL(d)$ matrix the Christoffel symbol $\Gamma^{\underline{m}}_{\underline{nr}}$ is a generic matrix in $\underline{m}, \underline{r}$. In the fiber basis any change of coordinates must be a Lorentz transformation. For the same reason that we used to introduce Γ we introduce the spin-connection $\omega_{\underline{n}}{}^{a,}{}_b$. It has two types of indices, the world index is due to $D_{\underline{n}}$ and the two fiber indices makes it a matrix in the fiber. Inside the covariant derivative each fiber index acts upon by the spin connection and each world index by the Christoffel symbol, e.g.

$$
D_{\underline{n}}V_{\underline{m}} = \partial_{\underline{n}}V_{\underline{m}} + \Gamma^{\underline{r}}_{\underline{nm}}V_{\underline{r}}, \qquad\qquad D_{\underline{n}}V^{\underline{m}} = \partial_{\underline{n}}V^{\underline{m}} - \Gamma^{\underline{m}}_{\underline{nr}}V^{\underline{r}}, \tag{3.10}
$$
$$
D_{\underline{n}}V^a = \partial_{\underline{n}}V^a + \omega_{\underline{n}}{}^{a,}{}_b\, V^b, \tag{3.11}
$$

$$
D_{\underline{n}}T^{abc...}_{\underline{m}...} = \partial_{\underline{n}}T^{abc...}_{\underline{m}...} + \omega_{\underline{n}}{}^{a,}{}_u\, T^{ubc...}_{\underline{m}...} + \omega_{\underline{n}}{}^{b,}{}_u\, T^{auc...}_{\underline{m}...} + \Gamma^{\underline{r}}_{\underline{nm}}T^{abc}_{\underline{r}} + ... \tag{3.12}
$$

Since only Lorentz rotations are allowed in the fiber we must have $\omega_{\underline{m}}{}^{a,b} = -\omega_{\underline{m}}{}^{b,a}$, where we have used the right to raise and lower fiber indices with the help of η^{ab}. Equivalently we can impose $D_{\underline{m}}\eta^{ab} = 0$ to find $\omega_{\underline{m}}{}^{a,b}$ antisymmetric. The consistency condition, the condition for the diagram to commute, leads to

$$
e_{\underline{m}}^a D_{\underline{n}}V_a = D_{\underline{n}}V_{\underline{m}}, \qquad\qquad V_{\underline{m}} = e_{\underline{m}}^a V_a. \tag{3.13}
$$

Since this must hold for any $V_{\underline{m}}$ we get

$$
D_{\underline{n}}e_{\underline{m}}^a = \partial_{\underline{n}}e_{\underline{m}}^a + \Gamma^{\underline{r}}_{\underline{nm}}e_{\underline{r}}^a + \omega_{\underline{n}}{}^{a,}{}_b\, e_{\underline{m}}^b = 0. \tag{3.14}
$$

This is called the vielbein postulate. Several comments can be made about the postulate

- The vielbein postulate is analogous to $D_{\underline{m}} g_{\underline{nr}} = 0$ postulate in that it is designed to disentangle algebraic manipulations with the help of e (or g) and covariant derivatives, i.e. it ensures that contractions of indices commute with covariant derivatives. Note that (3.14) induces $D_{\underline{m}} g_{\underline{nr}} = 0$.

- (3.14) can be solved both for Γ and ω as functions of e and its first derivatives, see Appendix C. This is supported by comparing the number of equations d^3 with the total number of components of $\#\Gamma = d \times d(d+1)/2$ and $\#\omega = d \times d(d-1)/2$, $\#\Gamma + \#\omega = \#eqs$.

- In the solution $\Gamma(e)$ the vielbein comes all the way in combinations that can be recognized as g and ∂g. One recovers the usual Christoffel symbols.

- On the contrary, the solution $\omega(e)$ cannot be rewritten in terms of the metric $g_{\underline{mn}}$, which supports the vielbein being a fundamental field.

If we anti-symmetrize in (3.14) over $\underline{mn}$ and use that $\Gamma^{\underline{r}}_{\underline{mn}}$ is symmetric, we find

$$T^a_{\underline{nm}} = \partial_{\underline{n}} e^a_{\underline{m}} - \partial_{\underline{m}} e^a_{\underline{n}} + \omega_{\underline{n}}{}^{a}{}_{,b}\, e^b_{\underline{m}} - \omega_{\underline{m}}{}^{a}{}_{,b}\, e^b_{\underline{n}} = 0\,, \tag{3.15}$$

i.e. Γ disappears and the system of equations turns out to have a triangular form. We can first solve for ω and then for Γ. Explicit solution for ω is given in Appendix C. In case there is no need for Γ we can use (3.15). It can be more compactly rewritten if we hide the world indices by saying that $e^a_{\underline{m}}$ and $\omega_{\underline{m}}{}^{a}{}_{,b}$ are differential forms. A short introduction to the language of differential forms can be found in Appendix D.

Thinking of $e^a_{\underline{n}}$ and $\omega_{\underline{n}}{}^{a}{}_{,b}$ as degree-one differential forms, $e^a = dx^{\underline{n}} e^a_{\underline{n}}$, $\omega^{a}{}_{,b} = dx^{\underline{n}} \omega_{\underline{n}}{}^{a}{}_{,b}$ one can rewrite (3.15) as

$$T^a = de^a + \omega^{a}{}_{,b} \wedge e^b = De^a = 0\,. \tag{3.16}$$

Two-form $T^a \equiv \frac{1}{2} T^a_{\underline{mn}} dx^{\underline{m}} \wedge dx^{\underline{n}}$ is called the torsion. We can check the integrability of (3.16) applying d to (3.16) and using that $d^2 \equiv 0$ and then using (3.16) again to express de^a. We have nothing to say on how $d\omega^{a,b}$ looks like so we keep it as it is. The result[12] is

$$F^{a}{}_{,b} \wedge e^b = 0\,, \qquad\qquad F^{a}{}_{,b} = d\omega^{a}{}_{,b} + \omega^{a}{}_{,c} \wedge \omega^{c}{}_{,b}\,. \tag{3.17}$$

The two-form $F^{a}{}_{,b}$ has four indices in total and is in fact related to the Riemann tensor

$$R_{\underline{mn},\underline{ru}} = F_{\underline{mn}}{}^{a}{}_{,b}\, e_{a\underline{r}}\, e^b_{\underline{u}}\,. \tag{3.18}$$

It is a painful computation to solve $T^a = 0$ for $\omega^{a,b}$ as a function of e^a and then compute $F^{a,b}$ to see that e^a appears in combinations that can be rewritten in terms of the metric. Fortunately, there is a back-door. Let us compute the commutator of two covariant derivatives on some vector $V^{\underline{m}}$ and the same for $V^a = V^{\underline{m}} e^a_{\underline{m}}$, i.e. $[D_{\underline{m}}, D_{\underline{n}}] V^{\underline{r}}$ and $[D_{\underline{m}}, D_{\underline{n}}] V^a$. The two results must match after transferring all the indices to fiber ones or

---

[12] In its simplest form this is just the Frobenius integrability condition. Given a set of PDE's $\partial_\mu \phi(x) = f_\mu(x)$ the commutativity of partial derivatives imply $0 \equiv (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \phi(x) = \partial_\mu f_\nu - \partial_\mu f_\nu = 0$. The last equality does not hold for a generic vector-function f_μ, which means that the system can be inconsistent.

to world ones. We already know that $[D_{\underline{m}}, D_{\underline{n}}]V^{\underline{r}}$ is expressed in terms of the Riemann tensor. Analogously, $[D_{\underline{m}}, D_{\underline{n}}]V^a$ can be expressed in terms of $F_{\underline{mn}}{}^{a}{}_{,b}$, which gives

$$R_{\underline{mn}}{}^{\underline{r}}{}_{\underline{u}} V^{\underline{u}} = \left(F_{\underline{mn}}{}^{a}{}_{b} V^b \right) e_a^{\underline{r}} = \left(F_{\underline{mn}}{}^{a}{}_{b} e_{\underline{u}}^b V^{\underline{u}} \right) e_a^{\underline{r}} = \left(F_{\underline{mn}}{}^{a}{}_{b} e_{\underline{u}}^b e_a^{\underline{r}} \right) V^{\underline{u}}. \qquad (3.19)$$

The identity (3.17) can be then recognized as the first Bianchi identity for the Riemann tensor, it being a three-form anti-symmetrizes over the three indices in square brackets,

$$R_{[\underline{mn},\underline{r}]\underline{u}} \equiv R_{\underline{mn},\underline{ru}} + R_{\underline{nr},\underline{mu}} + R_{\underline{rm},\underline{nu}} \equiv 0. \qquad (3.20)$$

Since everything in the metric-like formulation can be derived from the frame-like one, it is not surprising that the Einstein-Hilbert action[13]

$$S_{EH} = \int \sqrt{\det g}\, R \qquad (3.21)$$

can be rewritten in Cartan-Weyl form[14]

$$S_{CW} = \int F^{a,b}(\omega(e)) \wedge e^c \wedge ... \wedge e^u\, \epsilon_{abc...u}. \qquad (3.22)$$

The integrand is a top-form, i.e. the form of maximal degree, which is the space-time dimension, and can be integrated. Let us note that ω used in the action is assumed to be expressed in term of e via the vielbein postulate, (3.14), or the torsion constraint, (3.16), which obscures the interpretation of ω as a gauge field of the Lorentz algebra. This is what we would like to improve on.

3.2 Gravity as a gauge theory

Short summary on Yang-Mills. The deeper we go into the gravity the more similarities with the Yang-Mills theory we find with some important differences though. From this perspective let us collect basic formulas of Yang-Mills theory. The main object in Yang-Mills theory is the gauge potential $A_{\underline{m}}$ that takes values in some Lie algebra, say $\mathfrak{g}$. We will treat it as a degree-one form $A = A_{\underline{m}} dx^{\underline{m}}$ with values in the adjoint representation of $\mathfrak{g}$. The index of the Lie algebra is implicit but we can always recover it $A = A^{\mathcal{I}} t_{\mathcal{I}}$ with $t_{\mathcal{I}}$ being the generators of $\mathfrak{g}$, i.e. there is a Lie bracket $[t_{\mathcal{I}}, t_{\mathcal{J}}] = f_{\mathcal{I}\mathcal{J}}{}^{\mathcal{K}} t_{\mathcal{K}}$.

There can also be matter fields, i.e. fields taking values in arbitrary representation of $\mathfrak{g}$. For example, let $\phi(x) = \phi^{\mathsf{a}}(x)$ be a vector in some vector space V that carries a representation ρ of $\mathfrak{g}$, i.e. $\rho : \mathfrak{g} \rightarrow End(V)$, which means that we have matrices $\rho(t_{\mathcal{I}})^{\mathsf{a}}{}_{\mathsf{b}}$ associated with each of the generators $t_{\mathcal{I}}$ such that $[\rho(t_{\mathcal{I}}), \rho(t_{\mathcal{J}})] = \rho([t_{\mathcal{I}}, t_{\mathcal{J}}]) = f_{\mathcal{I}\mathcal{J}}{}^{\mathcal{K}} \rho(t_{\mathcal{K}})$, i.e. the matrix commutator is expressed via the Lie bracket and hence in terms of the structure constants.

In the table below we collect some formulae that we will use many times in what follows

---

[13]We omit the gravitational constant everywhere from our formulae. Our excuse is that we are not going to compute the precession of the perihelion of Mercury or anything like that in these notes.

[14]The integration of differential forms is explained at the end of Appendix D.

description	formula
gauge transformation (ϵ is a zero-form with values in $\mathfrak{g}$, $\epsilon = \epsilon^{\mathcal{I}} t_{\mathcal{I}}$)	$\delta A = D\epsilon \equiv d\epsilon + [A, \epsilon]$ $\delta\phi = -\rho(\epsilon)\phi$
curvature or field strength	$F(A) = dA + \frac{1}{2}[A, A]$
covariant derivative	$D\phi = d\phi + \rho(A)\phi$
generic variation δA of F	$\delta F = D\delta A \equiv d\delta A + [A, \delta A]$
gauge variation of F	$\delta F = [F, \epsilon]$
gauge variation of $D\phi$	$\delta D\phi = -\rho(\epsilon)D\phi$
Bianchi identity	$DF \equiv dF + [A, F] \equiv 0$
Jacobi identity	$[A, [A, A]] \equiv 0$
the commutator of two $D's$ ($\bullet$ is a placeholder)	$D^2\bullet = F\bullet,\ D^2\phi = \rho(F)\phi$

For example, the Jacobi identity acquires a simpler form $[A, [A, A]] \equiv 0$ because A is a one-form and hence, $[A_{\underline{m}}, [A_{\underline{n}}, A_{\underline{k}}]] \, dx^{\underline{m}} \wedge dx^{\underline{n}} \wedge dx^{\underline{k}}$ implicitly imposes anti-symmetrization over the three slots, which is the Jacobi identity. Analogously, D^2 computes the commutator of two D's, $DD = D_{\underline{m}} D_{\underline{n}} \, dx^{\underline{m}} \wedge dx^{\underline{n}} \equiv \frac{1}{2}[D_{\underline{m}}, D_{\underline{n}}] \, dx^{\underline{m}} \wedge dx^{\underline{n}}$, which is the field-strength.

Back to gravity. The theory of gravity in terms of vielbein/spin-connection variables must be invariant under the local Lorentz transformations. Now we can simply say that $\omega^{a,b}$ is a gauge field (Yang-Mills connection) of the Lorentz algebra, $so(d-1, 1)$. Denoting the generators as $L_{ab} = -L_{ba}$ we have the following commutation relations

$$[L_{ab}, L_{cd}] = L_{ad}\eta_{bc} - L_{bd}\eta_{ac} - L_{ac}\eta_{bd} + L_{bc}\eta_{ad}\,. \tag{3.23}$$

The Yang-Mills connection is then $\omega = \frac{1}{2}\omega^{a,b}L_{ab}$, which already looks like spin-connection. For a moment we will treat e^a as a vector matter, i.e. with ρ given by $\rho(L_{ab})^c{}_d = -\eta_{ad}\delta^c_b + \eta_{bd}\delta^c_a$. As a connection, ω possesses its own gauge parameter $\epsilon = \frac{1}{2}\epsilon^{a,b}L_{ab}$.

Specializing the formulas from the table above we find the gauge transformations

$$\delta\omega^{a,b} = d\epsilon^{a,b} + \omega^{a,}{}_c\,\epsilon^{c,b} + \omega^{b,}{}_c\,\epsilon^{a,c} \equiv D\epsilon^{a,b}\,, \tag{3.24}$$

$$\delta e^a = -\epsilon^{a,}{}_b\,e^b\,, \tag{3.25}$$

which correspond to infinitesimal Lorentz rotations. The last line is exactly (3.7). The transformation law for the spin-connection can be derived without making any reference to the Yang-Mills rules — one can apply the same reasonings as for the Christoffel symbols, i.e. use (3.25) and the fact that $D_{\underline{m}}V^a$ must be a tensor quantity (Lorentz vector in the index a).

The Yang-Mills field-strength $F(\omega)$ is exactly $F^{a,b}(\omega)$ found above, (3.17). The torsion constraint $T^a = 0$, (3.16), is just the condition for the covariant derivative De^a of e^a to

vanish. We also find that $DF^{a,b} \equiv 0$ as a Bianchi identity. Taking into account the relation between $F^{a,b}$ and the Riemann tensor we recover the second Bianchi identity $D_{[m}R_{nr],ku} \equiv 0$.

In particular we can now solve the problem of extending Dirac Lagrangian to curved manifolds since the covariant derivative can act in any representation of the Lorentz algebra. To be precise, we define fiber spinor field $\psi^{\mathsf{a}}(x)$, the fiber γ-matrices $\gamma_a = \gamma_a{}^{\mathsf{a}}{}_{\mathsf{b}}$, $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$. Then the generators of the Lorentz algebra in the spinor representation are given by $\rho(L_{ab}) = \frac{1}{4}[\gamma_a, \gamma_b]$ and the covariant derivative acts as

$$D\psi^{\mathsf{a}} = \partial_{\underline{m}}\psi^{\mathsf{a}} + \frac{1}{2}\omega^{a,b}\rho(L_{ab})^{\mathsf{a}}{}_{\mathsf{b}}\,\psi^{\mathsf{b}}\,. \tag{3.26}$$

Finally, the Dirac action on a curved background reads,

$$S_D[\psi, e, \omega] = \int \det e\,(i\bar{\psi}\gamma^a e^{\underline{n}}_a \overrightarrow{D}_{\underline{n}}\psi - i\bar{\psi}\gamma^a e^{\underline{n}}_a \overleftarrow{D}_{\underline{n}}\psi - m\bar{\psi}\psi)\,. \tag{3.27}$$

What are the symmetries of the frame-like action (3.22)? All the fiber indices are contracted with the invariant tensor $\epsilon_{ab\ldots u}$ of the Lorentz algebra and e^a as well as $F^{a,b}(\omega)$ transform homogeneously under local Lorentz rotations, i.e. like a vector and a rank-two antisymmetric tensor. This implies that the action has local $so(d-1,1)$-symmetry. It is also diffeomorphism invariant since it is an integral of a top-form.

There are still some subtleties that prevent one from simply stating that gravity is the Yang-Mills theory. Namely, $\omega^{a,b}$ is a function of vectorial matter e^a via the torsion constraint, (3.16); e^a is a one-form rather than pure vector matter; $e^a_{\underline{m}}$ must be invertible since $\det g \neq 0$; the action does not have the Yang-Mills form. Nevertheless, by going to the first-order formulation of gravity one can further improve the interpretation of gravity as a gauge theory.

Note on first-order actions. We are not aiming at rigorous definitions here. The field equations are usually second-order P.D.E.s for bosonic fields. We call the actions that immediately lead to second-order equations the second-order actions. For example, classical action for a free particle $\int \frac{1}{2}\dot{q}_i\dot{q}^i$, the Fronsdal action, (2.19), or the Einstein-Hilbert action are second-order actions because the variational equations are of the second-order.

Let us begin with the free particle. The Hamiltonian is $H = \frac{1}{2}p_ip^i$, $p_i = \dot{q}_i$. We can express the Lagrangian back using $L = p\dot{q} - H$, where we would like to treat p^i as an independent variable for a moment, so we have

$$S(q, p) = \int (\dot{q}^i - \frac{1}{2}p^i)p_i\,. \tag{3.28}$$

Now there are two variational equations

$$\frac{\delta S}{\delta p^i} = \dot{q}^i - p^i = 0\,, \qquad\qquad \frac{\delta S}{\delta p^i} = \dot{p}^i = 0\,. \tag{3.29}$$

The first equation is algebraic with respect to momenta p^i and is solved as $p^i = \dot{q}^i$. Then the second equation reduces to $\dot{p}^i = \ddot{q}^i = 0$ as desired.

To make notation coherent we can use dq^i, where $d = dt\frac{\partial}{\partial t}$ instead of $\dot{q}^i dt$, so that we treat q^i as a vector valued zero-form over one-dimensional manifold, which is the world-line of the particle parameterized by t. We can also introduce a one-dimensional einbein $e = dt$ to write

$$S(q,p) = \int (dq^i - \frac{1}{2}ep^i)p_i \, . \tag{3.30}$$

This is how a typical first-order action looks like. The ideology is that one introduces additional fields, the analogs of momenta p, such that the new, first-order, action depends on the original fields and momenta. The action now contains first-order derivatives only. The equations for momenta are algebraic and express momenta as first-order derivatives of the original fields. On substituting the solutions for the momenta into the action one gets back to the original action. In many cases the advantage of the first-order approach is that the action is simpler, less nonlinear and the new fields, momenta, as independent fields may have certain interpretation (this is what happens to $\omega^{a,b}$).

As an example, it is well-known that in the case of gravity one can treat $\Gamma^k_{\underline{mn}}$ as an independent variable in the action (Palatini formulation), writing

$$S_P(g,\Gamma) = \int \sqrt{\det g}\, g^{\underline{mn}} R_{\underline{mn}}(\Gamma) \, . \tag{3.31}$$

The equations of motion for Γ are equivalent to $\nabla_{\underline{m}} g^{nk} = 0$ and imply that $\Gamma^k_{\underline{mn}}$ is the Levi-Civita connection. On substituting this to the action one gets back to the pure Einstein-Hilbert.

One can do something more by replacing $g_{\underline{mn}}$ as independent variable with[15] $\mathbf{g}^{\underline{mn}} = \sqrt{\det g}\, g^{\underline{mn}}$. Then the action for gravity is schematically $\mathbf{g}(\partial\Gamma + \Gamma^2)$, i.e. at most cubic. All non-polynomial nonlinearities of gravity are removed by using the first-order approach and new appropriate variable $\mathbf{g}$. There are two sources of nonlinearities. The first one, is in $\sqrt{\det g}\, g^{\underline{mn}}$. The second one arises when solving for Γ as one has to invert the metric.

Back to gravity again. Let us take the route of first-order actions and see if we can treat $\omega^{a,b}$ as an independent variable (the analog of Palatini formulation in terms of vielbein and spin-connection). Generally, one cannot just isolate a bunch of derivatives and fields (the expression for ω in terms of e^a) and call it a new field to get the first-order formulation. Fortunately, this is not the case with ω. The variation of the action

$$S_f(e,\omega) = \int F^{a,b}(\omega) \wedge e^c \wedge ... \wedge e^u \, \epsilon_{abc...u} \, , \tag{3.32}$$

where ω is now an independent field, reads[16]

$$\delta S_f(e,\omega) = (d-2)\int \left(\delta\omega^{a,b} \wedge T^c \wedge ... \wedge e^u + F^{a,b}(\omega) \wedge \delta e^c \wedge ... \wedge e^u\right)\epsilon_{abc...u} \, , \tag{3.33}$$

---

[15] This is a purely algebraic change of variable, nonlinear though. It is invertible in $d > 2$.

[16] It is very useful not to split covariant derivative D into two pieces $D = d + \omega$. This can be supported by the Stokes theorem $0 = \int d(A_{\boldsymbol{p}} \wedge B_{\boldsymbol{q}}) = \int dA_{\boldsymbol{p}} \wedge B_{\boldsymbol{q}} + (-)^p A_{\boldsymbol{p}} \wedge dB_{\boldsymbol{q}}$, where $p + q = d$, which holds true for covariant derivatives as well, provided the integrand is a scalar. First, one checks that if I is a scalar, i.e. in a trivial representation of the Lie algebra we are considering, then $dI \equiv DI$. Then if I is a composite, e.g., $I = A_{\boldsymbol{p}} \wedge B_{\boldsymbol{q}}$, then D satisfies the chain rule, which gets modified by a sign factor in front of the second term since all the objects are differential forms, i.e. $d(A_{\boldsymbol{p}} \wedge B_{\boldsymbol{q}}) = D(A_{\boldsymbol{p}} \wedge B_{\boldsymbol{q}}) = DA_{\boldsymbol{p}} \wedge B_{\boldsymbol{q}} + (-)^p A_{\boldsymbol{p}} \wedge DB_{\boldsymbol{q}}$.

where we used that $\delta F^{a,b} = D\delta\omega^{a,b}$ and integrated by parts to find $T^a = De^a$. Assuming the frame field be invertible we find the following equations

$$\delta\omega_{\underline{m}}^{a,b} : \qquad\qquad T^a = De^a = de^a + \omega^{a}{}_b \wedge e^b = 0 \,, \qquad\qquad (3.34)$$

$$\delta e_{\underline{m}}^{a} : \qquad\qquad F_{\underline{mn}}^{a,b}\, e_b^{\underline{n}} = 0 \,. \qquad\qquad (3.35)$$

The first equation allows us to solve for the spin-connection. Under the condition that $T^a = 0$ the second equation, when rewritten in the metric-like language, gives $R_{\underline{mn}} = 0$, which is the vacuum Einstein equation. Up to the moment when we have to solve the torsion constraint, (3.16), $\omega^{a,b}$ can be treated as a Yang-Mills connection of the Lorentz algebra.

Let us note that the first-order action is polynomial as compared to the second order action where the nonlinearities come from $\omega(e)$ that involves inverse of the vielbein.

It is worth stressing that second-order and first-order approaches may lead to different results under certain conditions. For example, if we add matter, such as spin-$\frac{1}{2}$ fields, (3.27), to the gravity action, i.e. $S = S_f + S_D$ then the torsion constraint gets modified inasmuch as $\omega^{a,b}$ contributes to the matter action, $S_D = S_D[\psi, e, \omega]$. In the second order approach $\omega = \omega(e)$. In the first order approach instead of $T^a = 0$ we find $T \sim \bar\psi\psi$, i.e. the torsion is fixed in terms of the matter fields. One can still solve for $\omega = \omega(e, \psi)$. Restoring the gravitational constant one finds the difference between the first-order and second order gravity to be quadratic in the gravitational constant which has never been tested experimentally. Within the second-order approach one can always reproduce the corrections due to $\omega(e) - \omega(e, \psi) \neq 0$ by adding them to the action by hand. In supergravity, thinking of ω as an independent field leads to more compact expressions. Namely one can start from the second-order approach and then find that certain terms have to be added to the action to make it invariant under super-transformations. These terms can be reproduced automatically by considering ω as an independent field, i.e. within the first-order approach.

The cosmological term $\Lambda\sqrt{\det g}$ can be represented in the frame-like form

$$S_\Lambda = \Lambda \int e^a \wedge e^b \wedge ... \wedge e^u \, \epsilon_{abc...u} \,. \qquad\qquad (3.36)$$

In some applications the gravity Lagrangians are allowed to have a more general form, suggested first by Lovelock, [93]. One adds scalar polynomials in the Riemann tensor such that the equations of motion are still second-order. The Lovelock terms have the following simple form within the frame-like approach

$$S_{L,n} = \int \underbrace{F^{a,b} \wedge ... \wedge F^{c,d}}_{n} \wedge e^f \wedge e^g \wedge ... \wedge e^u \, \epsilon_{ab...cdfg...u} \,. \qquad\qquad (3.37)$$

In particular it is easy to see that there are $[d/2]$ Lovelock terms in dimension d with $S_{L,1}$ corresponding to the pure Einstein-Hilbert. The equations of motion involve at most two time derivatives. Indeed, each F, $F = d\omega + ...$, is of the second order with respect to e, $F_{0\underline{i}} = \dot\omega_{\underline{i}} + ...$, $\underline{i} = 1...d-1$. Since the $\wedge$-product anti-symmetrizes over the indices, $F_{0\underline{i}}$ can appear only once in $F \wedge ... \wedge F$.

In even dimension $d = 2n$ the top Lovelock term is topological

$$S_{L,n} = \int \underbrace{F^{a,b} \wedge ... \wedge F^{c,d}}_{n} \epsilon_{ab...cd} \,, \tag{3.38}$$

its variation vanishes up to boundary term because of $\delta F = D\delta\omega$ and $DF \equiv 0$.

Other possibilities include Yang-Mills-type action

$$S_{L,n} = \int F_{\underline{mn}}^{a,b} F_{\underline{rk}}^{c,d} \eta_{ac}\eta_{bd} g^{\underline{mr}} g^{\underline{nk}} = \int tr(F_{\underline{mn}} F_{\underline{rk}}) g^{\underline{mr}} g^{\underline{nk}} \,, \tag{3.39}$$

which turns out to be higher-derivative (it is not topological or of Lovelock type) and does not lead to the Einstein-Hilbert action.

More on gravity as a gauge theory. That spin-connection $\omega^{a,b}$ and vielbein e^a are both one-forms makes one expect they should have a similar interpretation. Later we find that it will be possible to consider all one-forms as taking values in some Lie algebra. We look for a unifying connection $A = \frac{1}{2}\omega^{a,b}L_{ab} + e^a P_a$, where P_a are generators associated with gauge field e^a. We already know the commutator $[L, L]$ and also know that e^a behaves as a vector of $so(d - 1, 1)$, which fixes the commutator $[L, P]$. There are not so many things one can write for $[P, P]$ and we are left with a one-parameter family

$$\begin{aligned}
[L_{ab}, L_{cd}] &= L_{ad}\eta_{bc} - L_{bd}\eta_{ac} - L_{ac}\eta_{bd} + L_{bc}\eta_{ad} \,, \\
[L_{ab}, P_c] &= P_a\eta_{bc} - P_b\eta_{ac} \,, \\
[P_a, P_b] &= -\Lambda L_{ab} \,,
\end{aligned} \tag{3.40}$$

where Λ is some constant and the Jacobi identities are satisfied for any Λ. Freedom in rescaling the generators leaves us with three distinct cases: $\Lambda > 0$, $\Lambda < 0$ and $\Lambda = 0$. These three cases can be easily identified with de-Sitter algebra $so(d, 1)$, anti-de Sitter algebra $so(d - 1, 2)$ and Poincare algebra $iso(d - 1, 1)$, respectively.

That $\Lambda = 0$ corresponds to $iso(d - 1, 1)$ is obvious. Let $T_{\mathsf{AB}} = -T_{\mathsf{BA}}$, where $\mathsf{A}, \mathsf{B}, ...$ range over a and one additional direction, denoted by 5, i.e. $\mathsf{A} = \{a, 5\}$, be the generators of $so(d, 1)$ or $so(d - 1, 2)$. They obey

$$[T_{\mathsf{AB}}, T_{\mathsf{CD}}] = T_{\mathsf{AD}}\eta_{\mathsf{BC}} - T_{\mathsf{BD}}\eta_{\mathsf{AC}} - T_{\mathsf{AC}}\eta_{\mathsf{BD}} + T_{\mathsf{BC}}\eta_{\mathsf{AD}} \,. \tag{3.41}$$

Defining $L_{ab} = T_{ab}$, $\sqrt{|\Lambda|}P_a = T_{a5}$ we find (3.40) with the last relation being $[P_a, P_b] = -\eta_{55}|\Lambda|L_{ab}$, which explains the minus.

The Yang-Mills curvature[17] $F = \frac{1}{2}R^{a,b}L_{ab} + T^a P_a$ and gauge transformations $\delta A = D\epsilon$, where $\epsilon = \frac{1}{2}\epsilon^{a,b}L_{ab} + \epsilon^a P_a$ read

$$R^{a,b} = d\omega^{a,b} + \omega^{a,}{}_c \wedge \omega^{c,b} - \Lambda e^a e^b \,, \qquad T^a = De^a \,, \tag{3.42}$$

$$\delta\omega^{a,b} = D\epsilon^{a,b} - \Lambda e^a \epsilon^b + \Lambda e^b \epsilon^a \,, \qquad \delta e^a = D\epsilon^a - \epsilon^{a,}{}_b e^b \,, \tag{3.43}$$

---

[17]We reserve $F^{a,b}$ for the Riemann two-form, while $R^{a,b}$ contains the cosmological term.

where D is the Lorentz covariant derivative $d + \omega$. The torsion is now one of the components of the Yang-Mills field-strength! According to the general Yang-Mills formulae, the curvatures transform as

$$\delta R^{a,b} = -\epsilon^{a,}{}_c R^{cb} - \epsilon^{b,}{}_c R^{a,c} - \Lambda T^a \epsilon^b + \Lambda T^b \epsilon^a \,, \tag{3.44}$$

$$\delta T^a = -\epsilon^{a,}{}_b T^b + R^{a,}{}_b \epsilon^b \,. \tag{3.45}$$

The Bianchi identity for the Yang-Mills field-strength, $DF = 0$, when written in components is

$$DT^a - e_m \wedge R^{a,m} \equiv 0 \,, \tag{3.46}$$

$$DR^{a,b} - \Lambda T^a \wedge e^b + \Lambda T^b \wedge e^a \equiv 0 \tag{3.47}$$

If the torsion constraint (3.16) is imposed the first equation simplifies[18] to $e_m \wedge F^{a,m} \equiv 0$, which otherwise results from differentiation of torsion (3.17). The second one simplifies to $DF^{a,b} \equiv 0$, which is the second Bianchi identity for the Riemann tensor, $\nabla_{[u} R_{mn],kr} \equiv 0$. Consequently, all useful relations automatically arise when both e and ω are combined into a single connection.

The cosmological term (3.36) is included into the action (3.32) if F is taken to be the field-strength of the (anti)-de Sitter algebra, $R^{a,b}$, rather than Poincare one, i.e.

$$S = \int R^{a,b}(\omega) \wedge e^c \wedge ... \wedge e^u \, \epsilon_{abc...u} = S_f + S_\Lambda \,. \tag{3.48}$$

There appeared a new gauge symmetry with parameter ϵ^a, the local translations, which we did not observe before. However, the action (3.32) is invariant under $so(d-1,1)$-part of gauge transformations, i.e. $\epsilon^{a,b}$, and it is not invariant under local translations with ϵ^a

$$\delta S_f = -(d-2)(d-3) \int R^{a,b}(\omega) \wedge T^c \wedge \epsilon^c \wedge e^f \wedge ... \wedge e^u \, \epsilon_{abcf...u} + \tag{3.49}$$

$$+2\Lambda(d-3) \int T^a \wedge \epsilon^b \wedge e^c \wedge ... \wedge e^u \, \epsilon_{abc...u} \,. \tag{3.50}$$

It is not a new symmetry. Local translations become a symmetry of the action when torsion is zero[19], $T^a = 0$. We stress that $T^a = 0$ is not a dynamical equation, it is a constraint that allows one to solve for $\omega^{a,b}$ as a function of e^a.

When torsion is zero the local translations can be identified with diffeomorphisms. Indeed, there is a general identity[20]

$$\mathcal{L}_\xi A = D(i_\xi A) + i_\xi F(A) \,, \tag{3.51}$$

i.e. the Lie derivative of any Yang-Mills connection $A = A^{\mathcal{I}} t_{\mathcal{I}}$ can be represented as a sum of gauge transformation with $\epsilon = i_\xi A$, i.e. $\epsilon^{\mathcal{I}} = \xi^{\underline{m}} A^{\mathcal{I}}_{\underline{m}}$, and a curvature term. Specializing to our case we derive

$$\mathcal{L}_\xi e^a = \delta_\xi e^a + i_\xi T^a \,, \qquad\qquad \mathcal{L}_\xi \omega^{a,b} = \delta_\xi \omega^{a,b} + i_\xi R^{a,b} \,, \tag{3.52}$$

---

[18]Mind that $e_b \wedge R^{a,b}$ can be replaced by $e_b \wedge F^{a,b}$ since $e_a \wedge e^a \equiv 0$.

[19]Let us note that there is something special about $d = 3$. For example, for $\Lambda = 0$ we have $\delta \int F^{a,b} \wedge e^c \, \epsilon_{abc} = \int F^{a,b} \wedge D\epsilon^c \, \epsilon_{abc}$ which vanishes upon integrating by parts and using $DF^{a,b} \equiv 0$.

[20]Remember that $\mathcal{L}_\xi = di_\xi + i_\xi d$, where $\mathcal{L}_\xi$ and i_ξ are Lie and inner derivatives, respectively, see Appendix D. Then, one completes $i_\xi dA$ to $F(A)$ which completes $d(i_\xi A)$ to a gauge transformation.

where δ_ξ means the gauge variation with $\epsilon^a = \xi^{\underline{m}} e^a_{\underline{m}}$ and $\epsilon^{a,b} = \xi^{\underline{m}} \omega^{a,b}_{\underline{m}}$. When torsion is zero we have $\mathcal{L}_\xi e^a = \delta_\xi e^a$, i.e. diffeomorphisms acting on e^a can be represented as particular gauge transformations. This is in accordance with the invariance of the action under local translations for vanishing torsion. Diffeomorphisms acting on $\omega^{a,b}$ are not equivalent to gauge transformations because $R^{a,b}$, which is related to the Riemann tensor, is generally non-zero. This does not cause a problem since the dynamical variable is the vielbein. A diffeomorphism performed on e induces a diffeomorphism for $g_{\underline{mn}} = e^a_{\underline{m}} \eta_{ab} e^b_{\underline{n}}$.

That there are three ways, (3.40), to unify $\omega^{a,b}$ and e^a within one Lie algebra is directly related to the fact there are three most symmetric solutions to Einstein equations with cosmological constant Λ. These are de Sitter space, $\Lambda > 0$, anti-de Sitter space, $\Lambda < 0$, and Minkowski space, $\Lambda = 0$.

Despite the unification of $\omega^{a,b}$ and e^a into a single Yang-Mills connection there is an important difference between the two. We use $\omega^{a,b}$ to construct Lorentz-covariant derivative D and couple matter to gravity, e.g. as in (3.27), but we do not use e^a inside D. The frame field is always outside and is used to built a volume form and contract indices. Let us mention that within the higher-spin theory the difference between $\omega^{a,b}$ and e^a to some extent vanishes as we will see that e^a does contribute to the covariant derivative!

Most symmetric background is equivalent to $dA + \frac{1}{2}[A, A] = 0$. The important observation is that de Sitter, anti-de Sitter and Minkowski space-times are solutions of $F(A) = 0$ with $A = \frac{1}{2}\omega^{a,b}L_{ab} + e^a P_a$ being the gauge field of the corresponding symmetry algebra where the commutation relations are given in (3.40) and Λ distinguishes between the three options. In terms of the Riemann tensor these space-times are defined by the following constraint

$$R_{\underline{mn,rk}} = \Lambda(g_{\underline{mr}}g_{\underline{nk}} - g_{\underline{nr}}g_{\underline{mk}}) \,. \tag{3.53}$$

Within the frame-like approach this corresponds to

$$F(A) = 0 \qquad \Longleftrightarrow \qquad \begin{cases} T^a = 0 \\ F^{a,b}(\omega(e)) = \Lambda e^a \wedge e^b \end{cases}, \tag{3.54}$$

where ω is expressed in terms of e via the torsion constraint and the second equation imposes (3.53) once we remember the relation between $F^{a,b}$ and $R_{\underline{mn,rk}}$. This is equivalent to $F(A) = 0$. As always it is implied that $\det e^a_{\underline{m}} \neq 0$.

It is not hard to write down some explicit solutions. If $\Lambda = 0$, i.e. the space-time is Minkowski, a useful choice is given by Cartesian coordinates

$$e^a_{\underline{m}} \, dx^{\underline{m}} = \delta^a_{\underline{m}} \, dx^{\underline{m}} = dx^a \,, \qquad\qquad \omega^{a,b}_{\underline{m}} \, dx^{\underline{m}} = 0 \,, \tag{3.55}$$

i.e., the vielbein is just a unit matrix and there is no difference between world and fiber. The spin-connection is identically zero. This choice leads to $g_{\underline{mn}} = \eta_{\underline{mn}}$ and $\Gamma^k_{\underline{mn}} = 0$.

If $\Lambda \neq 0$ a useful choice is given by the Poincare coordinates $x^{\underline{m}} = (z, x^i)$ where

$$g_{\underline{mn}} = \frac{1}{|\Lambda|}\frac{1}{z^2}(dz^2 + dx^i dx^j \eta_{ij}) = \frac{1}{|\Lambda|}\frac{1}{z^2}(dx^{\underline{m}} dx^{\underline{n}} \eta_{\underline{mn}}) \,, \tag{3.56}$$

where z is an analog of radial direction and x^i are the coordinates on the leaves of constant z. Then we can use, for example,

$$e_{\underline{m}}^a \, dx^{\underline{m}} = \frac{1}{\sqrt{|\Lambda|}} \frac{1}{z} \delta_{\underline{m}}^a \, dx^{\underline{m}} \qquad \omega_{\underline{m}}^{a,b} \, dx^{\underline{m}} = -\frac{1}{z} \left(\delta_{\underline{m}}^a \eta^{bz} - \delta_{\underline{m}}^b \eta^{az} \right) dx^{\underline{m}} . \qquad (3.57)$$

Note that $g_{\underline{mn}} = e_{\underline{m}}^a e_{\underline{n}}^b \eta_{ab}$ applies, of course.

As a side remark let us take the most general action composed of Lovelock-type terms, (3.37), which reads

$$S = \sum_n \frac{c_n}{d - 2n} S_{L,n} . \qquad (3.58)$$

It is interesting that it can have (anti)-de Sitter spaces with different cosmological constants as solutions. Indeed, assuming that the torsion is zero and taking the variation with respect to e^a we get

$$\delta S = \sum c_n \int F^{a,b} \wedge ... \wedge F^{c,d} \wedge \delta e^f \wedge e^g \wedge ... \wedge e^u \epsilon_{abcdfg...u} . \qquad (3.59)$$

In looking for the (anti)-de Sitter type solutions we replace $F^{a,b}$ with $\Lambda e^a \wedge e^b$ to find

$$0 = \sum c_n \Lambda^n = (\Lambda - \Lambda_1)...(\Lambda - \Lambda_n) . \qquad (3.60)$$

There are further improvements possible when cosmological constant is non-zero, the MacDowell-Mansouri-Stelle-West approach, which is reviewed in extra Section 12.2.

Summary. We have shown that the tetrad e^a and spin-connection $\omega^{a,b}$ can be unified as gauge fields of Poincare or (anti)-de Sitter algebra, $A = \frac{1}{2}\omega^{a,b} L_{ab} + e^a P_a$. The Yang-Mills curvature then delivers constituents of various actions and contains Riemann two-form and torsion.

There are at least three ways to treat Yang-Mills connections:

- As in the genuine Yang-Mills theory, i.e. $\int tr(F_{\underline{mn}} F^{\underline{mn}}) + \text{matter}$.

- As in Chern-Simons theory. We are in $3d$ with the action $\int tr(FA - \frac{1}{3}A^3)$.

- As in gravity. Here we found several options how to treat vielbein and spin-connection. The least we can do is to say that $\omega^{a,b}$ is an $so(d-1,1)$-connection and e^a is a vector-valued one-form. Another option is to unify $\omega^{a,b}$ and e^a as gauge fields of one of the most symmetric Einstein's vacua, i.e. (anti)-de Sitter or Minkowski. The case of (anti)-de Sitter is less degenerate since the symmetry algebras are semi-simple. Any theory can be re-expanded over one of its vacuum and the expansion is covariant with respect to the symmetries of the vacuum. In the case of Einstein theory, depending on the cosmological constant, there are three maximally symmetric vacua.

We found several reasons to replace metric with the vielbein or frame and spin-connection.

29

1. To have freedom in introducing general basis in the tangent space.

2. To make transition from a generic curved coordinates to the ones of the Einstein's elevators, where the local physics is as in SR. This leads us to the idea that GR is a localized (gauged) version of SR. At any rate we expect that we should gauge the Lorentz algebra $so(d-1,1)$. This makes us feel that GR should be close to the Yang-Mills theory. There are also important differences with the Yang-Mills theory, which we discuss below.

3. To make half-integer spin fields, in particular matter fields, interact with gravity. This is the strongest motivation, of course. Since it is the frame-like approach that allows matter fields to interact with gravity it is promising to stick to this approach and look for its generalization to fields of all spins.

The similarities and distinctions between gravity and Yang-Mills theory include

+ Spin-connection is a gauge field of the Lorentz algebra.

− On-shell it is not an independent propagating field, rather it is expressed in terms of the vielbein field via the torsion constraint $T^a = 0$.

+ The matter fields interact with ω through the covariant derivative, i.e. minimally, e.g. $D\psi$, like in Yang-Mills theory.

+ The action is cooked up from the Yang-Mills curvatures.

− In any case the action does not have the Yang-Mills form.

− There is a condition $\det e_{\underline{m}}^a \neq 0$, i.e. $\det g_{\underline{mn}} \neq 0$, that is hard to interpret within the Yang-Mills theory.

+ One can unify both vielbein and spin-connection as gauge fields of some Lie algebra.

− There are several options to achieve that (Poincare, de Sitter or anti-de Sitter).

− While $\omega^{a,b}$ appears in covariant derivative D only, vielbein e^a is always around when building a volume form or contracting indices, so the unification of e and ω within one gauge field is not perfect — their appearance is different, both in the gravity and in the matter Lagrangians.

− The local translation symmetry associated with frame field e^a becomes a symmetry of the action only when the torsion constraint is imposed, then it can be identified with diffeomorphisms.

− The full group structure for a diffeomorphism invariant theory with some internal local symmetry (Yang-Mills or gravity in terms of e^a and $\omega^{a,b}$) is that of a semidirect product of diffeomorphisms and local symmetry group, see extra Section 12.3.

4 Unfolding gravity

Let us abandon the action principle and concentrate on the equations of motion. The appropriate variables we need are vielbein e^a and $so(d-1,1)$ gauge field $\omega^{a,b}$, spin-connection. As we have already seen, they can be viewed as the gauge fields associated with either Poincare or (anti)-de Sitter algebra, the gravity then shares many features of Yang-Mills theory. We aim to write the Einstein equations by making use of the language of differential forms. Particularly, it means that field equations are necessarily of first order. It may seem to be too restrictive since many known dynamical equations of interest contain higher derivatives. However, pretty much as any system of differential equations can be reduced to a first order form by means of extra variables, so is any classical field theory can be rewritten in differential form language by virtue of auxiliary fields. In practice one typically needs infinitely many of those. Such equations are called the *unfolded*, [40, 41], and it is in this form that the Vasiliev theory is given in. The formulation of gravity obtained in this section admits a natural extension to all higher-spin fields.

Our starting point is the torsion constraint, (3.16), and the definition of the $so(d-1,1)$-curvature, (3.17),

$$T^a = De^a = de^a + \omega^{a,}{}_b \wedge e^b = 0 \,, \tag{4.1}$$

$$F^{ab} = d\omega^{a,b} + \omega^{a,}{}_c \wedge \omega^{c,b} \,. \tag{4.2}$$

The Einstein equations without matter and the cosmological constant

$$R_{\underline{mn}} - \frac{1}{2}g_{\underline{mn}}R = 0 \qquad\qquad \Longleftrightarrow \qquad\qquad R_{\underline{mn}} = 0 \tag{4.3}$$

are equivalent in $d > 2$ to $R_{\underline{mn}} = 0$ and hence

$$R_{\underline{mn}} = 0 \qquad\qquad \Longleftrightarrow \qquad\qquad F^{a,b}_{\underline{mn}}(e^{-1})^{\underline{n}}_b = 0 \,. \tag{4.4}$$

There are two clumsy properties of the latter expression: (i) we had to undress the differential form indices of the curvature two-form; (ii) we needed the inverse of $e^a_{\underline{m}}$ to contract indices, i.e. from the Yang-Mills point of view we had to take the 'inverse of the Yang-Mills field'. One may ask a naive question: whether is it possible to formulate the gravity entirely in the language of differential forms and connections? Indeed, this is possible and it is the starting point for the higher-spin generalizations. In Section 6 we explain that the equations formulated solely in the language of differential forms, *unfolded* equations, have a deep algebraic meaning. For a moment let us just explore this path blindly.

First of all, the Riemann tensor $R_{\underline{mn,kr}}$ is traceful and has the following decomposition

$$R_{\underline{mn,kr}} = W_{\underline{mn,kr}} + \alpha(g_{\underline{mk}}R_{\underline{nr}} - g_{\underline{nk}}R_{\underline{mr}} - g_{\underline{mr}}R_{\underline{nk}} + g_{\underline{nr}}R_{\underline{mk}}) + \beta(g_{\underline{mk}}g_{\underline{nr}} - g_{\underline{nk}}g_{\underline{mr}})R \,,$$

$$\alpha = \frac{2}{d-2} \,, \qquad\qquad \beta = -\frac{1}{(d-2)(d-1)} \,, \tag{4.5}$$

$$R_{\underline{mn,kr}}g^{\underline{nr}} = R_{\underline{mk}} \,, \qquad\qquad R_{\underline{mk}}g^{\underline{mk}} = R \,, \tag{4.6}$$

where $R_{\underline{mk}}$ is the Ricci tensor, R is the scalar curvature and $W_{\underline{mn,kr}}$ is the traceless part of the Riemann tensor, called Weyl tensor. The coefficients are fixed by the normalization in the last line. Weyl tensor has the same symmetry properties as the Riemann one, i.e.

$$W_{\underline{mn,kr}} = -W_{\underline{nm,kr}} = -W_{\underline{mn,rk}}, \qquad W_{[\underline{mn,k}]\underline{r}} \equiv 0, \qquad (4.7)$$

where the second property in (4.7) is the algebraic Bianchi identity. The Weyl tensor is by definition traceless

$$W_{\underline{mn,kr}}g^{\underline{nr}} \equiv 0. \qquad (4.8)$$

In the Young diagram language[21] the Weyl tensor is depicted as the 'window'-like diagram

$$W_{\underline{mn,kr}} : \; \boxed{\begin{array}{cc} & \\ & \end{array}} \qquad\qquad R_{\underline{nr}} : \; \boxed{\begin{array}{cc} & \end{array}} \oplus \bullet \qquad\qquad R : \bullet \qquad (4.9)$$

The free of matter Einstein equations imply $R_{\underline{mn}} = 0$, but the whole Riemann tensor, of course, may not be zero. Vanishing Riemann tensor, $R_{\underline{mn,kr}} = 0$, describes empty Minkowski space. While $R_{\underline{mn}} = 0$ has a rich set of solutions corresponding to various configurations of the gravitational field, e.g. gravitational waves, black holes etc. The difference between very strong $R_{\underline{mn,kr}} = 0$ and $R_{\underline{mn}} = 0$ is exactly the Weyl tensor. One can say that it is the Weyl tensor that is responsible for the richness of gravity. The crucial step is the equivalence of the two equations

$$R_{\underline{mn}} = 0 \qquad\qquad \Longleftrightarrow \qquad\qquad R_{\underline{mn,kr}} = \mathcal{W}_{\underline{mn,kr}}, \qquad (4.10)$$

where $\mathcal{W}_{\underline{mn,kr}}$ has algebraic properties of the Weyl tensor otherwise left unspecified at a point. While in the first form the equation directly imposes $R_{\underline{mn}} = 0$, in the second form it tells us that the only non-zero components of the Riemann tensor are allowed to be along the Weyl tensor direction, i.e. the Ricci, $R_{\underline{mn}}$ must vanish. Formally, taking trace of the second equation one finds $R_{\underline{mn,kr}}g^{\underline{nr}} = R_{\underline{mk}} = \mathcal{W}_{\underline{mn,kr}}g^{\underline{nr}} = 0$. It is also important that the second Bianchi identity, $\nabla_{[\underline{u}}R_{\underline{mn}],\underline{kr}} \equiv 0$, implies

$$\nabla_{[\underline{u}}\mathcal{W}_{\underline{mn}],\underline{kr}} \equiv 0, \qquad (4.11)$$

i.e. $\mathcal{W}_{\underline{mn,kr}}$ is arbitrary at a point but its first-derivatives are constrained. Formally, (4.11) can be viewed as one more consequence of (4.10).

The general idea behind the 'unfolding' of gravity is that *instead of specifying which components of the Riemann tensor and its derivatives have to vanish we can parameterize those that do not vanish by new fields*. This is applicable to any set of fields subject to some differential equations. Instead of imposing equations directly we can specify which derivatives of the fields may not vanish on-shell.

Let us transfer the above consideration to the frame-like approach. Instead of $R_{\underline{mn,kr}}$ we have two-form $F^{ab}(\omega)$. Converting the differential form indices of the $F^{ab} \equiv F^{ab}_{\underline{mn}} dx^{\underline{m}} \wedge dx^{\underline{n}}$ to the fiber we get a four-index object

$$F^{ab|cd} = -F^{ba|cd} = -F^{ab|dc} = F^{ab}_{\underline{mn}} e^{\underline{mc}} e^{\underline{nd}}. \qquad (4.12)$$

---

[21]An introductory course on the Young diagrams language is in the Appendix E. It is really necessary to have some understanding of what the possible symmetry types of tensors are in order to proceed to the higher-spin case.

When no torsion constraint is imposed, $F^{ab|cd}$ has more components than the Riemann tensor. It is antisymmetric in each pair and there is no algebraic Bianchi identity implied. We find the following decomposition into irreducible components

$$F^{ab|cd} \sim \Box \otimes \Box = \begin{array}{c}\Box\\\Box\end{array} \oplus \left(\begin{array}{c}\Box\Box\\\Box\end{array}^{[a} \oplus \begin{array}{c}\Box\Box\\\Box\end{array} \right) \oplus \left(\begin{array}{c}\Box\Box\\\Box\Box\end{array} \oplus \Box\Box\Box \oplus \bullet \right) \tag{4.13}$$

When torsion constraint is imposed one derives the following consequence, the algebraic Bianchi identity, (3.17), (3.20), which we will refer to as the integrability constraint

$$0 \equiv dde^a = -d(\omega^a{}_b \wedge e^b) \qquad \Longrightarrow \qquad F^{a,b} \wedge e_b \equiv 0. \tag{4.14}$$

The components with the symmetry of the first three diagrams do not pass the integrability test as they are too antisymmetric. For example, if $F^{ab} = e_m \wedge e_n C^{a,b,m,n}$, where $C^{a,b,m,n}$ is antisymmetric then $F^{ab} \wedge e_b = e_m \wedge e_n \wedge e_b \, C^{a,b,m,n}$, which does not vanish since $e^a_{\underline{m}}$ is invertible. Analogously[22], if $F^{ab} = e_m \wedge e_n C^{am,n,b}$, where $C^{aa,b,c}$ has the symmetry of the second component and contains the trace, which is the third component, we find $F^{ab} \wedge e_b = e_m \wedge e_n \wedge e_b \, C^{am,n,b}$, which does not vanish identically. Since F^{ab} is related to the Riemann tensor when the torsion constraint is imposed, it comes as no surprise that the last three components form a decomposition of the Riemann tensor into irreducible tensors. From now on the torsion constraint is implied. Analogously to (4.5), the (fiber) Riemann tensor $F^{ab|cd}$ can be decomposed as

$$F^{ab|cd} = C^{ab,cd} + \alpha(\eta^{ac}\mathcal{R}^{bd} - \eta^{bc}\mathcal{R}^{ad} - \eta^{ad}\mathcal{R}^{bc} + \eta^{bd}\mathcal{R}^{ac}) + 2\beta(\eta^{ac}\eta^{bd} - \eta^{bc}\eta^{ad})\mathcal{R}, \tag{4.15}$$

$$F^{am|b}{}_m = \mathcal{R}^{ab}, \qquad \mathcal{R}^n{}_n = \mathcal{R}, \tag{4.16}$$

$$C^{ab,cd} : \begin{array}{c}\Box\Box\\\Box\Box\end{array}, \qquad \mathcal{R}^{ab} : \Box\Box \oplus \bullet, \qquad \mathcal{R} : \bullet \tag{4.17}$$

We treat $C^{ab,cd}$, $\mathcal{R}^{ab}$, $\mathcal{R}$ as zero-forms. $C^{ab,cd}$, which is the fiber Weyl tensor, has the symmetry of the Riemann tensor, i.e. of the window Young diagram, but it is traceless, $C^{am,b}{}_m \equiv 0$. The trace of the fiber Riemann tensor is the fiber Ricci tensor $\mathcal{R}^{ab}$, whose trace is the scalar curvature $\mathcal{R}$, the latter coincides with R because it is a scalar.

It is easy to see that (4.15) can be rewritten solely in terms of differential forms as

$$F^{ab} = e_m \wedge e_n C^{ab,mn} + \alpha e^{[a} \wedge e_n \mathcal{R}^{b],n} + 2\beta e^a \wedge e^b \mathcal{R} \tag{4.18}$$

and it does not restrict F^{ab}, just telling which components of F^{ab} may in principle be non-zero when the torsion constraint is imposed.

---

[22]It is obvious that $e^a_{\underline{m}} V_a = 0$ implies $V_a = 0$ since the vielbein is invertible, e.g. we can always choose $e^a_{\underline{m}} = \delta^a_{\underline{m}}$ at a point. Similarly, operators of the form $e_m \wedge ... \wedge e_n C^{...m...n}$ just antisymmetrize over the indices $m, ..., n$ that are contracted with vielbein one-forms. Decomposing $C^{...m...n}$ into irreducible symmetry types one checks using the Young properties which of the components are annihilated. Say, for $C^{a|b}$ equation $e_m \wedge e_n C^{m|n} = 0$ implies that the antisymmetric component of $C^{a|b}$ must vanish while the symmetric one is arbitrary.

The Einstein equations are equivalent to $\mathcal{R}^{ab} = 0$. The same reasoning as in (4.10), compared to (4.4) forces us to require that the only non-zero components of F^{ab} should be given by the Weyl tensor. Namely, we just remove $\mathcal{R}^{ab}$, $\mathcal{R}$ from the *r.h.s.* of (4.18)

$$F^{ab}_{\underline{mn}} = e_{\underline{m}m}e_{\underline{n}n}C^{ab,mn} \qquad \Longleftrightarrow \qquad F^{ab} = e_m \wedge e_n C^{ab,mn} , \qquad (4.19)$$

which is equivalent to (4.4) in the same way as it was the case for (4.10).

Using the ambiguity in presentation of mixed-symmetry tensors[23] instead of $C^{ab,cd}$, which is manifestly antisymmetric in pairs of indices we can switch to $C'^{ac,bd}$, which is manifestly symmetric in pairs,

$$C^{ab,cd} = -C^{ba,cd} = -C^{ab,dc} \qquad \boxed{\begin{array}{cc} a & c \\ b & d \end{array}} \qquad C'^{ac,bd} = C'^{ca,bd} = C'^{ac,db} . \qquad (4.20)$$

In what follows we will use the symmetric basis for tensors, which is more convenient for higher-spin fields, and we write $C^{aa,bb}$ instead of $C'^{aa,bb}$. In the symmetric basis[24] (4.19) reads

$$F^{ab} = e_m \wedge e_n C^{am,bn} . \qquad (4.21)$$

The *r.h.s.* of (4.21) manifestly satisfies the algebraic Bianchi identity (4.14), but it is not so for the differential Bianchi identity, an analog of (4.11). The second Bianchi identity is a consequence of $DF^{ab} \equiv 0$, which implies (recall that $De^a = 0$)

$$0 \equiv DF^{ab} = D(e_m \wedge e_n C^{am,bn}) = e_m \wedge e_n \wedge DC^{am,bn} . \qquad (4.22)$$

We can either stop here and supplement (4.21) with (4.22) or try to analyze this integrability condition. (4.22) is a restriction on the derivatives of the Weyl tensor. Were $C^{ab,cd}$ arbitrary we would find several tensor types to appear in $D_{\underline{m}}C^{ab,cd}$. This is equivalent to analyzing the second term of the Taylor expansion of $C^{aa,\overline{bb}}(x)$ and decomposing Taylor coefficients into irreducible $so(d-1,1)$ tensors. In doing so it is convenient to transfer the world index of $D_{\underline{m}}$ to the fiber and define

$$B^{aa,bb|c} = e^{\underline{m}c}D_{\underline{m}}C^{aa,bb} . \qquad (4.23)$$

We use the slash notation within a group of tensor indices (e.g., $B^{aa,bb|c}$) to split the indices (tensor product) into the groups of indices in which the tensor is irreducible. Since we dropped the second Bianchi identity for a moment there are no relations between ab, cd and m, i.e. as the tensor of the Lorentz algebra it has the following decomposition into irreducibles

$$B^{aa,bb|c} \sim \;\young{\hfil\hfil\cr\hfil\hfil\cr} \otimes \;\square\; = \;\young{\hfil\hfil\hfil\cr\hfil\hfil\cr} \oplus \left(\;\young{\hfil\hfil\cr\hfil\hfil\cr\hfil\cr} \oplus \;\young{\hfil\hfil\cr\hfil\cr} \right) \qquad (4.24)$$

---

[23]This is a special feature about mixed-symmetry tensors that allows us to use dual Young tableaux presentation. More detail on mixed-symmetry tensors can be found in Appendix E.

[24]A typical exercise on Young symmetry is to show that the *r.h.s.* is antisymmetric in ab, which is not manifest. The expression is antisymmetric if its symmetrization vanishes. Symmetrizing over ab we get $C^{am,bn} + C^{bm,an}$ which is almost the Young condition $C^{am,bn} + C^{bm,an} + C^{ab,mn} \equiv 0$. Hence we get $0 \equiv F^{ab} + F^{ba} = -e_m \wedge e_n C^{ab,mn}$. Remembering that $e_m \wedge e_n$ is antisymmetric in mn while $C^{ab,mn}$ is symmetric we get the desired $e_m \wedge e_n C^{ab,mn} \equiv 0$.

To parameterize all three components we can introduce three zero-forms $C^{aaa,bb}$, $B^{aa,bb,c}$ and $B^{aa,b}$

$$DC^{aa,bb} = e_m \left(C^{aam,bb} + \frac{1}{2} C^{aab,bm} \right) + e_m B^{aa,bb,m} +$$

$$+ e^b C^{aa,b} + e^a C^{bb,a} - \frac{2}{d-2} e_m \left(\eta^{aa} C^{bb,m} + \eta^{bb} C^{aa,m} - \frac{1}{2} \eta^{ab} C^{ab,m} \right) . \qquad (4.25)$$

The first two terms project[25] onto ⊞, so do the first two terms in the second line as well. The resulting tensor is not traceless, the trace projector is imposed with the help of the last group of terms. The projector for $e_m B^{aa,bb,m}$ is trivial. We see that while it is easy to say what Young diagrams of $so(d-1,1)$-irreducible tensors that $B^{aa,bb|c}$ contain, this amounts to computing the tensor product. It is much more complicated to handle this decomposition in the language of tensors due to Young- and trace-projectors that are generally there. Luckily, many statements can be proven in terms of Young diagrams without appealing to the tensor language.

Back to (4.24), it is easy to see that the presence of the last two components is not consistent with (4.22), which is equivalent to $e_m \wedge e_n \wedge e_c B^{am,bn,c} \equiv 0$. So we have to keep the first term only in order to write solution to the differential Bianchi identity

$$DC^{aa,bb} = e_m \left(C^{aam,bb} + \frac{1}{2} C^{aab,bm} \right) , \qquad C^{aaa,bb} : \boxed{} \qquad (4.26)$$

$C^{aaa,bb}$ is the first 'descendant' of the Weyl tensor that allows us to solve the differential constraint in a constructive way. The second term together with $\frac{1}{2}$ in front of it ensures that the r.h.s. has the right Young symmetry. Again, we can either stop here or check if there are differential constraints for $C^{aaa,bb}$. So far all the equations were exact in the sense that we had not neglected any terms. In continuing the process we find a technical complication. Indeed, $D^2 = F \sim eeC$, (4.21), so checking the integrability of the equation last obtained we get stuck with

$$DDC^{aa,bb} = e_m \wedge e_n C^{am,n}{}_c C^{ac,bb} + e_m \wedge e_n C^{bm,n}{}_c C^{aa,cb} = -e_m \left(DC^{aam,bb} + \frac{1}{2} DC^{aab,bm} \right)$$

That $D^2 \sim eeC$ makes the *l.h.s.* nonlinear, so we have to solve $eeCC = eDC^1$ where C^1 denotes $C^{aaa,bb}$. Hence we have to introduce quadratic terms on the *r.h.s.* of $DC^{aaa,bb}$. This is what should be expected since gravity is a nonlinear theory.

---

[25]One more feature of mixed-symmetry tensors is that the operation of adding one index (taking tensor product) or removing (contracting with an external object) one index does not preserve the symmetry properties. Indeed, the tensor product contains in general several irreducible components and we are free to keep them all or to project onto one of them. For example, given $T^{aa,b}$ of ⊟ symmetry type, we can contract it with vielbein to get $e_m T^{am,b}$. The resulting tensor is neither symmetric nor antisymmetric in ab and contains both ⊡ and ⊟ symmetry types. It can be projected onto ⊡ simply as $e_m T^{am,b} + e_m T^{bm,a} = -e_m T^{ab,m}$ and onto ⊟ as $e_m T^{am,b} - e_m T^{bm,a}$. The operation of adding one index and projecting onto irreducible component is more complicated. For example, $e^b T^{aa,b}$ neither has any definite symmetry type nor is it traceless. Typically a number of terms is needed to project a contraction/tensor product of two irreducible tensor objects onto an irreducible component. If we need an irreducible tensor of orthogonal algebra, rather than $gl(d)$, we have in addition to project out its traces.

We will use the example of gravity in the frame-like formalism as a starting point for higher-spin generalization. The extension to higher-spin fields is to be done at the free level first and then we review the Vasiliev solution to the nonlinear problem. At the interaction level fields of all spins interact with each other and no truncations of the full system to a finite subset of fields is possible. Given that we will continue the gravity part of the story at the linearized level only, i.e. we will neglect D^2, assuming that $D^2 \sim 0$.

Repeating the derivation of the constraint on the first derivatives of $C^{aaa,bb}$ from (4.26) in the approximation $D^2 = 0$ we now find

$$DDC^{aa,bb} = 0 = -e_m \left(DC^{aam,bb} + \frac{1}{2} DC^{aab,bm} \right). \tag{4.27}$$

It can be solved analogously to the way we did before by considering all possible r.h.s. of $DC^{aaa,bb}$ resulting in

$$DC^{aaa,bb} = e_m \left(C^{aaam,bb} + \frac{1}{3} C^{aaab,bm} \right) + O(eC^2) \qquad aaaa, bb = \begin{array}{|c|c|c|c|}\hline & & & \\\hline & & \multicolumn{2}{c}{}\\\cline{1-2}\end{array} \tag{4.28}$$

Continuing this process we can derive one by one the following set of equations/constraints that describes Einstein's gravity without the cosmological constant and matter with higher-order corrections due to $D^2 \neq 0$ neglected

$$\begin{cases} T^a = De^a = 0 \\ \delta e^a = D\epsilon^a - \epsilon^{a,}{}_b \, e^b \end{cases} \tag{4.29}$$

$$\begin{cases} F^{a,b} = d\omega^{a,b} + \omega^{a,}{}_c \wedge \omega^{c,b} = e_m \wedge e_n C^{am,bn} \\ \delta \omega^{a,b} = D\epsilon^{a,b} + (\epsilon_m e_n - \epsilon_n e_m) C^{am,bn} \end{cases} \tag{4.30}$$

$$DC^{aa,bb} = e_m \left(C^{aam,bb} + \frac{1}{2} C^{aab,bm} \right) \qquad\qquad aa, bb = \begin{array}{|c|c|}\hline & \\\hline & \\\hline\end{array} \tag{4.31}$$

$$DC^{a(k+2),bb} = e_m \left(C^{a(k+2)m,bb} + \frac{1}{k+2} C^{a(k+2)b,bm} \right) + O(eC^2) \quad a(k+2), bb = \begin{array}{|c|c|c|c|c|}\hline & & & & k \\\hline & & \multicolumn{3}{c}{}\\\cline{1-2}\end{array}$$

$$\delta C^{a(k+2),bb} = -\epsilon^{a,}{}_m C^{a(k+1)m,bb} - \epsilon^{b,}{}_m C^{a(k+2),bm}$$

$$\qquad + \epsilon_m \left(C^{a(k+2)m,bb} + \frac{1}{k+2} C^{a(k+2)b,bm} \right) + O(\epsilon C^2) \tag{4.32}$$

where we marked those equations that are not exact with $+O(eC^2)$ label. Notice that all $C^{a(k+2),bb}$ transform under local Lorentz transformations.

It is worth stressing that zero-forms, like the Weyl tensor or matter fields of Yang-Mills, do not have their own gauge parameters. Nevertheless, they take advantage of gauge fields' parameters, but the gauge transformations do not contain derivatives.

Notice the local translation symmetry $D\epsilon^a$ in (4.29), (4.30), (4.32). It gets restored since the torsion constraint is imposed and we can interpret local translations as diffeomorphisms. Hence, we automatically gauge the Poincare algebra when considering equations of motion, while at the level of the action principle we had certain problems in interpreting it as resulting from gauging of the Poincare algebra.

Equivalently, we can consider the first order expansion of gravity over the Minkowski background. The background Minkowski space can be defined by vielbein $h_{\underline{m}}^a$ and by spin-connection $\varpi_{\underline{m}}^{a,b} = -\varpi_{\underline{m}}^{b,a}$ obeying the torsion constraint and zero-curvature

$$T^a = dh^a + \varpi^{a,}{}_b \wedge h^b = 0\,, \qquad\qquad d\varpi^a{}_b + \varpi^a{}_c \wedge \varpi^c{}_b = 0\,, \qquad (4.33)$$

the latter implies that the whole Riemann tensor vanishes. The convenient choice in the case of Minkowski is given by Cartesian coordinates, where

$$g_{\underline{mn}} = \eta_{\underline{mn}}\,, \qquad \Gamma^{\underline{r}}_{\underline{mn}} = 0\,, \qquad h^a_{\underline{m}} = \delta^a_{\underline{m}}\,, \qquad \varpi^{a,b}_{\underline{m}} = 0\,. \qquad (4.34)$$

It is useful to define the background Lorentz-covariant derivative $D = d + \varpi$, which we denote by the same letter as the full Lorentz-covariant derivative above. That (4.33) implies $D^2 = 0$ supports this notation as we were going to neglect D^2 anyway.

The linearization of (4.29)-(4.32) over Minkowski background reads

$$\begin{cases} T^a = De^a - h_m \wedge \omega^{a,m} = 0 \\ \delta e^a = D\epsilon^a - \epsilon^{a,}{}_b\, e^b \end{cases} \qquad\qquad (4.35)$$

$$\begin{cases} F^{a,b} = D\omega^{a,b} = h_m \wedge h_n C^{am,bn} \\ \delta\omega^{a,b} = D\epsilon^{a,b} \end{cases} \qquad aa, bb = \begin{array}{c}\square\square\\\square\square\end{array} \qquad (4.36)$$

$$DC^{aa,bb} = h_m \left(C^{aam,bb} + \frac{1}{2} C^{aab,bm} \right) \qquad aaa, bb = \begin{array}{c}\square\square\square\\\square\square\end{array}$$

$$DC^{aaa,bb} = h_m \left(C^{aaam,bb} + \frac{1}{3} C^{aaab,bm} \right) \qquad aaaa, bb = \begin{array}{c}\square\square\square\square\\\square\square\end{array}$$

$$DC^{a(k+2),bb} = h_m \left(C^{a(k+2)m,bb} + \frac{1}{k+2} C^{a(k+2)b,bm} \right) \qquad a(k+2), bb = \begin{array}{c}\square\square\boxed{k}\\\square\square\end{array}$$

$$\delta C^{a(k+2),bb} = 0 \qquad\qquad (4.37)$$

Let us make few comments on this system. D is defined with respect to the background, $d + \varpi$. The linearized torsion constraint has a slightly different form because $\omega^{a,}{}_m \wedge e^m$ yields, when linearized, two types of terms, ϖe and ωh, the first being hidden inside the background Lorentz derivative, D. Analogously, when linearized, F^{ab} has lost $\omega\omega$ piece and is simply $D\omega^{a,b}$.

Minkowski background implies $D^2 = 0$, so the linearized curvature $D\omega^{a,b}$ is gauge invariant. Therefore, $C^{aa,bb}$ on the *r.h.s.* of (4.36) should not transform under $\epsilon^{a,b}$ anymore, the same being true for all $C^{a(k+2),bb}$. This is due to the fact that D has lost dynamical spin-connection $\omega^{a,b}$, of which $\epsilon^{a,b}$ is a gauge parameter. No mixing of the form ωC implies there is no need to rotate $C^{a(k+2),bb}$ anymore and similarly is for the ϵ^a-symmetry. This all follows from linearization, of course.

The linearized Einstein equations, i.e. the Fronsdal equations for $s = 2$, are imposed by (4.35)-(4.36), which we will show for the spin-s generalization later. The rest of equations (and vanishing of the torsion) are constraints in a sense that they do not impose any differential equations merely expressing one field in terms of derivatives of the other.

Going to nonlinear level we find $O(eC^2)$ and higher order corrections on the *r.h.s.* of equations due to $D^2 \sim eeC$. Analogous type of corrections we find in the equations describing higher-spin fields.

5 Unfolding, spin by spin

In this section we begin to move towards the linearized Vasiliev equations that describe an infinite multiplet of free HS fields in AdS. Some preliminary comments are below.

When fields of all spins are combined together into the multiplet of a higher-spin algebra the equations they satisfy turn out to be much transparent and revealing than the equations for individual fields. We will not follow this idea in this section rather consider those fields individually, spin by spin. Moreover the technique that happened to be extremely efficient for HS fields may look superfluous for some simple cases like a free scalar field, which we consider at the end.

The idea we blindly follow in this section is to look for the frame-like (tetrad-like, vielbein-like, ...) formulation for fields of all spins. The simple guiding principle is to express everything in the language of differential forms. All fields in question are differential forms that may take values in some linear spaces that are viewed as the fiber over the space-time manifold. The equations of motion are required to have the following schematic form

$$d(\text{field}) = \text{exterior products of the fields themselves} \tag{5.1}$$

This is what we have already achieved for the case of gravity. Equations of this form are called the *unfolded equations*, [40, 41]. It is this simple idea that could have been used to discover the frame-like formulation of gravity and yet it also leads to the nonlinear frame-like formulation of higher-spin fields. The detailed and abstract discussion is left to Section 6, where we show that such equations are intimately connected with the theory of Lie algebras.

The structure on the base manifold that we will need is quite poor — only differential forms are allowed to be used along with the operations preserving the class of differential forms, i.e. the exterior derivative d and the exterior product, $\wedge$.

The spin-two case corresponds to the gravity itself. At the nonlinear level we find a perfect democracy among fields of all spins. This not quite so at the free level because all fields propagate over Minkowski (this section) or (anti)-de Sitter space (Section 9) which is the vacuum value of the spin-two. We have the background non-propagating gravitational fields defined by h^a, $\varpi^{a,b}$, which obey (3.54) with $\Lambda = 0$, i.e. the Yang-Mills field strength of the Poincare algebra is zero. These fields are considered to be the zero order vacuum ones within the perturbation theory, while propagating fields of all spins, including the spin-two itself, are of order one, so that the equations are linear in perturbations. If we had a master field, say W, whose components correspond to fields of all spins and the full theory in terms of W, then we could say that we expanded it as $W_0 + gW_1 + \text{higher orders}$, where g is a small coupling constant and

$$W_0 = \{\underbrace{0}_{s=0}, \underbrace{0}_{s=1}, \underbrace{h^a, \varpi^{a,b}, C^{aa,bb} = 0, ..., C^{a(k+2),bb} = 0, ...}_{s=2}, \underbrace{0}_{s=3}, \underbrace{0}_{s=4}, ...\}, \tag{5.2}$$

where the sector of spin-two is non-degenerate and contains h^a, $\varpi^{a,b}$ that obey $Dh^a = 0$ and $D^2 = 0$, $D = d + \varpi$, which is equivalent to having Minkowski space.

We would like to read off the part of the theory that is linear in W_1 and determine W_1 itself. Recall that in the case of linearized gravity W_1 was found to contain one-forms e^a,

$\omega^{a,b}$ and infinitely many zero-forms $C^{a(k+2),bb}$ that start from the Weyl tensor $C^{aa,bb}$ for $k = 0$. For the Minkowski vacuum all $C^{a(k+2),bb}$ vanish. If we wish to expand the theory over the space with say a black hole inside, then the Weyl tensor would be nonvanishing.

Each of the considered below cases consists of two parts: a quasi-derivation explaining why the solution has its particular form and a part with the results, where the system of equations is written down. The relevant original references include [39, 88, 94, 95].

5.1 $s = 2$ retrospectively

By the example of gravity we would like to show the main steps of how one could have discovered the frame-like gravity from the Fronsdal theory for $s = 2$ using the idea that the theory should be formulated in terms of differential forms and bearing in mind the Yang-Mills theory. The starting point is a symmetric traceful field ϕ^{aa}, the Fronsdal field at $s = 2$, that has a gauge symmetry, (2.5),

$$\delta\phi^{aa} = \partial^a \epsilon^a \tag{5.3}$$

and the Fronsdal equations, (2.4), specialized to $s = 2$. We would like to replace ϕ^{aa} with a yet unknown differential form e^* of a certain degree q taking values in some tensor representation of the Lorentz algebra, denoted by placeholder $*$. The gauge transformations are then $\delta e^* = d\epsilon^*$, where ϵ^* is a differential form of degree $q - 1$ that takes values in the same Lorentz representation $*$. Comparing

$$\delta\phi^{aa} = \partial^a \epsilon^a \qquad \Longleftrightarrow \qquad \delta e^* = d\epsilon^* = dx^{\underline{m}}\partial_{\underline{m}}\epsilon^* \tag{5.4}$$

we see that one index a carried by ∂^a should turn into $\partial_{\underline{m}}$, then the leftover index should belong to ϵ^*, i.e. $* = a = \square$. Since e^* must carry the same indices as its gauge parameter we have $e^a_{\underline{m}}$ and hence $\delta e^a_{\underline{m}} = \partial_{\underline{m}}\epsilon^a$. That the gauge parameter is a Lorentz vector immediately tells us that the frame field has a vector index too. Since the world indices of differential forms are disentangled from fiber indices, to write $\partial_{\underline{m}}\epsilon^a + \partial^a \epsilon_{\underline{m}}$ is meaningless. We also see that the frame field must be a one-form because the gauge parameter is naturally a zero-form. Consequently, we found

$$\delta e^a = d\epsilon^a \,. \tag{5.5}$$

That world and fiber indices in $e^a_{\underline{m}}$ are disentangled implies that there are no symmetry/trace conditions between a and $\underline{m}$ in $e^a_{\underline{m}}$. In particular $e^a_{\underline{m}}$ contains more components as compared to original ϕ^{aa}, which is symmetric. In Minkowski space in Cartesian coordinates there is no difference between world and fiber indices, but formally we can use the background $h^{\underline{m}a}$ to convert the world indices. With the help of $h^{\underline{m}a}$ we see that in addition to the totally symmetric component to be identified with ϕ^{aa} the frame field, $e^{a|b} = e^a_{\underline{m}}h^{\underline{m}b}$, contains an antisymmetric component too, i.e.

$$e^{a|b} \sim \square \otimes \square = \begin{array}{c}\square\\\square\end{array} \oplus (\square\square \oplus \bullet) \,. \tag{5.6}$$

The symmetric part of the frame field transforms as the Fronsdal field. Indeed,

$$\delta e^{a|b} = \partial^b \epsilon^a \qquad \Longrightarrow \qquad \delta(e^{a|b} + e^{b|a}) = \partial^b \epsilon^a + \partial^a \epsilon^b \,. \tag{5.7}$$

39

The antisymmetric component can be a propagating field unless we manage to get rid of it. The simplest solution is to introduce a new gauge symmetry, which acts algebraically and whose purpose is to gauge away the antisymmetric component completely,

$$\delta e^a_{\underline{m}} = \partial_{\underline{m}} \epsilon^a - \epsilon^{a,}{}_{\underline{m}} \, , \tag{5.8}$$

where $\epsilon^{a,b} = -\epsilon^{b,a}$ and we do not care about the difference between world and fiber indices since we can always set $h^a_{\underline{m}} = \delta^a_{\underline{m}}$ at a point. To make the last expression meaningful we can cure it as

$$dx^{\underline{m}} \delta e^a_{\underline{m}} = dx^{\underline{m}} \partial_{\underline{m}} \epsilon^a - dx^{\underline{m}} h^b_{\underline{m}} \epsilon^{a,}{}_b \qquad \Longleftrightarrow \qquad \delta e^a = d\epsilon^a - h_b \wedge \epsilon^{a,b} \, . \tag{5.9}$$

Now we can gauge away the antisymmetric part of the frame field

$$\delta(e^{a|b} - e^{b|a}) = \partial^b \epsilon^a - \partial^a \epsilon^b - 2\epsilon^{a,b} \, . \tag{5.10}$$

Indeed, the gauge symmetry with $\epsilon^{a,b}$ is algebraic and obviously $\epsilon^{a,b}$ has the same number of components. Therefore, we can always impose $(e^{a|b} - e^{b|a}) = 0$ and the condition for the left-over gauge symmetry

$$0 = \delta(e^{a|b} - e^{b|a}) = \partial^b \epsilon^a - \partial^a \epsilon^b - 2\epsilon^{a,b} \tag{5.11}$$

expresses $2\epsilon^{a,b} = \partial^b \epsilon^a - \partial^a \epsilon^b$ and does not restrict ϵ^a. So, when $\square = 0$ gauge is imposed the whole content of the frame field is given by the Fronsdal field with its correct gauge transformation law.

In practice we do not need to impose $\square = 0$ gauge or to go to the component form and convert indices with the inverse background frame field $h^{\underline{m}b}$. What we need is a guarantee that the theory can be effectively reduced to the Fronsdal one (at least at the linearized level) and that there are no extra propagating degrees of freedom.

We used the Fronsdal theory as a starting point for the frame-like generalization, but all the statements, e.g. that the equations to be derived below do describe a spin-s representation of the Poincare algebra, can be made without any reference to the Fronsdal theory. Since the frame field is needed anyway, e.g., to make fermions interact with gravity, there is no reason to go back to the Fronsdal theory once the frame-like higher-spin generalization is worked out.

Since the background may not be given in Cartesian coordinates the form of the gauge transformations valid in any coordinate system reads

$$\delta e^a = D\epsilon^a - h_b \wedge \epsilon^{a,b} \, , \tag{5.12}$$

where $D = d + \varpi$ is the background Lorentz derivative and we recall that $D^2 = 0$.

Now we can recognize (5.12) as the combination of local translations and local Lorentz transformations. The local translation symmetry has its roots in the Fronsdal's symmetry, while the purpose for local Lorentz transformations is to compensate the redundant components resided inside the frame field.

We found a reason for gauge parameter $\epsilon^{a,b}$. In the realm of the Yang-Mills theory, there are no homeless gauge parameters. Hence, there must be a gauge field $\omega^{a,b}$, which is a one-form and its gauge transformation law starts as $\delta\omega^{a,b} = d\epsilon^{a,b} + ...$ or equivalently

$$\delta\omega^{a,b} = D\epsilon^{a,b} + ... \quad . \tag{5.13}$$

The existence of the spin-connection goes hand in hand with the existence of $\epsilon^{a,b}$.

Given gauge transformations (5.12)-(5.13) it is easy to guess the gauge invariant field-strengths to be

$$T^a = De^a - h_b \wedge \omega^{a,b}\,, \qquad\qquad F^{a,b} = D\omega^{a,b} + ... \quad, \qquad (5.14)$$

which is the torsion and the linearized Riemann two-form. The question of whether we should impose $T^a = 0$ and what we should write instead of ... in $F^{a,b}$ is dynamical. One can see that setting T^a to zero expresses ω in terms of e and imposes no differential equations on the latter. At this point one can repeat the analysis of the previous section to find that $F^{a,b} = h_m \wedge h_n C^{am,bn}$ imposes the Fronsdal equations, etc.

The case of $s = 2$ is deceptive since h_m^a is already a vielbein. The illustration with spin-two is not self contained because the background space must have been already defined in terms of h^a and $\varpi^{a,b}$. We can imagine that we know how to define the background geometry but we are unaware of how to put propagating fields on top of this geometry. Later we will find out that propagating field e^a can be naturally combined with the background h^a into the full vielbein.

5.2 $s \geq 2$

We would like to systematically look for an analog of the frame-like formalism for the fields of any spin $s \geq 2$. The starting point is the gauge transformation law and the algebraic constraints on the Fronsdal field and its gauge parameter

$$\delta\phi^{a(s)} = \partial^a \epsilon^{a(s-1)}\,, \qquad \phi^{a(s-4)mn}{}_{nm} \equiv 0\,, \qquad \epsilon^{a(s-3)n}{}_n \equiv 0\,. \qquad (5.15)$$

The Fronsdal field has to be embedded into a certain generalized frame field $e_q^{a\cdots}$ with the form degree and the range of fiber indices to be yet determined. We understand already that writing $de_q^{a\cdots} = ...$ implies gauge transformation of the form $\delta e_q^{a\cdots} = d\xi_{q-1}^{a\cdots}$ with $\xi_{q-1}^{a\cdots}$ being the form of degree $q-1$ valued in the same representation of the fiber Lorentz algebra. This gauge symmetry is in general reducible $\delta\xi_{q-1}^{a\cdots} = d\chi_{q-2}^{a\cdots}$ unless $q = 1$. We know that there are no reducible gauge symmetries in the case of totally-symmetric fields. Therefore, $q = 1$ and gauge parameter is a zero-form, $\xi^{a\cdots}$, i.e. it has no differential form indices. In order to match

$$dx^{\underline{m}} \partial_{\underline{m}} \xi^{a\cdots} \qquad \text{with} \qquad \partial^a \epsilon^{a(s-1)} \qquad\qquad (5.16)$$

the gauge parameter must be $\xi^{a(s-1)}$, i.e. symmetric and traceless in the fiber indices. So, the frame field must be one-form $e_1^{a(s-1)}$ while the gauge transformation law now reads

$$\delta e_1^{a(s-1)} = d\xi^{a(s-1)} + ...\,, \qquad\qquad e^{a(s-3)m}{}_m = \xi^{a(s-3)m}{}_m \equiv 0\,. \qquad (5.17)$$

Let us convert the world indices in the last formula to the fiber

$$\delta e^{a(s-1)|b} = \partial^b \xi^{a(s-1)}\,, \qquad\qquad e^{a(s-1)|b} = e_{\underline{m}}^{a(s-1)} h^{\underline{m}b}\,. \qquad (5.18)$$

As in the spin-two case the frame field contains more components than the original metric-like, Fronsdal, field. The irreducible content of $e^{a(s-1)|b}$ is given by[26]

$$\boxed{s-1} \otimes \square = (\boxed{s} \oplus \boxed{s-2}) \oplus \boxed{\begin{smallmatrix} s-1 \\ \end{smallmatrix}}, \tag{5.19}$$

where the first two components, when put together, are exactly the content of the Fronsdal tensor since a double-traceless rank-s tensor $\phi^{a(s)}$ is equivalent to two traceless tensors $\psi^{a(s)}$ and $\psi^{a(s-2)}$ of ranks s and $s-2$

$$\phi^{a(s)} = \psi^{a(s)} + \frac{1}{d+2s-4}\eta^{aa}\psi^{a(s-2)}, \quad \phi^{a(s-2)m}{}_m = \psi^{a(s-2)}, \quad \phi^{a(s-4)mn}{}_{nm} \equiv 0, \tag{5.20}$$

where the coefficient is fixed by the relation in the middle. In terms of field components the decomposition of $e^{a(s-1)|b}$ into a Fronsdal-like field and the leftover traceless tensor $\psi^{a(s-1),b}$ with the symmetry of $\boxed{\begin{smallmatrix} s-1 \\ \end{smallmatrix}}$ reads

$$e^{a(s-1)|b} = \frac{1}{s}\phi^{a(s-1)b} + \psi^{a(s-1),b} +$$
$$+ \frac{1}{s(d+s-4)}\left(\frac{(s-2)}{2}\eta^{ab}\phi^{a(s-2)m}{}_m - \eta^{aa}\phi^{ba(s-3)m}{}_m\right). \tag{5.21}$$

The overall normalization is fixed in such a way that

$$e^{a(s-1)|a} = \phi^{a(s)}. \tag{5.22}$$

(remember that $\psi^{a(s-1),a} \equiv 0$ and it drops out). Note the group of terms in the second line of (5.21) which are absent in the spin-two case. The coefficients are fixed from (5.22) and from the tracelessness of $e^{a(s-1)|b}$ in a indices. In the language of differential forms with the Fronsdal field $\phi^{a(s)}$ and $\psi^{a(s-1),b}$ treated as zero-forms the embedding of the Fronsdal field reads

$$e_{\mathbf{1}}^{a(s-1)} = \frac{1}{s}h_m\,\phi^{a(s-1)m} + h_b\psi^{a(s-1),b} + \tag{5.23}$$
$$+ \frac{1}{s(d+s-4)}\left(\frac{(s-2)}{2}h^a\phi^{a(s-2)c}{}_c - \eta^{aa}h_m\phi^{a(s-3)mc}{}_c\right). \tag{5.24}$$

We see that such an unnatural within the metric-like approach condition as vanishing of the second trace

$$g^{\underline{mm}}g^{\underline{nn}}\phi_{\underline{mmnn}r(s-4)} \equiv 0 \tag{5.25}$$

comes from a very natural condition within the frame-like approach — the frame field is an irreducible fiber tensor. The condition of the type (5.25) is a source of problems in the interacting theory as it has to be either preserved or deformed when $g_{\underline{mn}}$ is not Minkowski or (anti)-de Sitter but a dynamical field. In contrast, $\eta_{bb}e^{bba(s-3)} \equiv 0$, (5.17), causes no problem since η_{aa} is a non-dynamical object, the dynamical gravity is described by the frame field $e_{\mathbf{1}}^a$. The identification of the Fronsdal tensor as the totally symmetric component of the frame field we discussed is essentially linear.

---

[26]We refer to the Appendix E.2, where the tensor product rules are discussed.

Combining (5.18) with (5.22) we recover the Fronsdal gauge transformation law for the totally symmetric component of the frame field.

As in the case of spin-two, the frame field contains an additional component $\psi^{a(s-1),b}$. To prevent it from becoming a propagating field we can introduce an algebraic gauge symmetry with a zero-form parameter $\xi^{a(s-1),b}$, so as to make it possible to gauge away $\psi^{a(s-1),b}$. As a tensor $\xi^{a(s-1),b}$ is irreducible, i.e. Young and traceless, since the unwanted component is like that. This correction can be written as follows

$$\delta e_{\mathbf{1}}^{a(s-1)} = -h_m \xi^{a(s-1),m} \tag{5.26}$$

and coincides for $s = 2$ with the action of local Lorentz rotations. Once we had to introduce $\xi^{a(s-1),b}$ there must be an associated gauge field $\omega_{\mathbf{1}}^{a(s-1),b}$, the generalization of the spin-connection. This allows us to write the first equation immediately as[27]

$$de_{\mathbf{1}}^{a(s-1)} = h_m \wedge \omega_{\mathbf{1}}^{a(s-1),m} \tag{5.27}$$

together with the gauge transformations

$$\delta e_{\mathbf{1}}^{a(s-1)} = d\xi^{a(s-1)} - h_m \xi^{a(s-1),b} , \qquad \delta\omega_{\mathbf{1}}^{a(s-1),b} = d\xi^{a(s-1),b} . \tag{5.28}$$

Once we have (5.27) we can check its integrability to read off the restrictions on $d\omega_{\mathbf{1}}^{a(s-1),b}$

$$0 \equiv dde_{\mathbf{1}}^{a(s-1)} = d\left(h_m \wedge \omega_{\mathbf{1}}^{a(s-1),m}\right) = -h_m \wedge d\omega_{\mathbf{1}}^{a(s-1),m} . \tag{5.29}$$

In principle one should write down all the tensors that can appear on the *r.h.s.* of $d\omega_{\mathbf{1}}^{a(s-1),m}$ to see which of them passes trough the integrability condition. We are not going to present this analysis and just claim that the solution has a nice form of

$$d\omega_{\mathbf{1}}^{a(s-1),b} = h_m \wedge \omega_{\mathbf{1}}^{a(s-1),bm} , \qquad a(s-1), bb = \boxed{\begin{array}{|c|c|c} \multicolumn{2}{c}{s-1} \\ \hline & & \\ \hline \\ \cline{1-1} \end{array}} , \tag{5.30}$$

where the new field is a one-form $\omega_{\mathbf{1}}^{a(s-1),bb}$ with values in the irreducible representation (Young and traceless) of the Lorentz algebra specified above. The integrability holds thanks[28] to $h_m \wedge h_n \wedge \omega_{\mathbf{1}}^{a(s-1),mn} \equiv 0$. A new one-form field comes with the associated gauge parameter, so the last equation ensures the invariance under

$$\delta\omega_{\mathbf{1}}^{a(s-1),b} = d\xi^{a(s-1),b} - h_m \xi^{a(s-1),bm} , \qquad \delta\omega_{\mathbf{1}}^{a(s-1),bb} = d\xi^{a(s-1),bb} . \tag{5.32}$$

Let us emphasize that we found a new field that was absent in the case of gravity! To have a vielbein and a spin-connection was enough for a spin-two field. $\omega_{\mathbf{1}}^{a(s-1),bb}$ is the first

---

[27]Already this equations as well as all others below, points towards some algebra, whose connection A contains the gravity sector in terms of vielbein/spin-connection and higher-spin connections, of which $e^{a(s-1)}$ is a particular component. What we see is the linearization of $F = dA + \frac{1}{2}[A, A]$ under $A_0 + gA_1$, which gives $F = dA + [A_0, A]$, $A_0 = \{h, \varpi\}$.

[28]The following trivial identities are used here and below

$$h^a \wedge h^b + h^b \wedge h^a \equiv 0 , \qquad h_m \wedge h^m \equiv 0 . \tag{5.31}$$

of the *extra fields* that appear in the frame-like formulation of higher-spin fields, [3, 95]. The necessity for the *extra fields* can be seen already in (5.27). From the pure gravity point of view this is a torsion constraint, which allows one to express spin-connection as a derivative of the vielbein. This is not so for $s > 2$. One can either find that spin-connection $\omega_{\mathbf{1}}^{a(s-1),b}$ cannot be fully solved from (5.27) or observe that (5.27) is invariant under $\delta\omega_{\mathbf{1}}^{a(s-1),b} = h_m\xi^{a(s-1),bm}$. These are equivalent statements as the component of the spin-connection affected by the extra gauge symmetry cannot be solved for. The appearance of an additional symmetry tells us there is an associated gauge field, the first extra field. The extra gauge symmetry can also be seen in the quadratic action built from $e_{\mathbf{1}}^{a(s-1)}$ and $\omega_{\mathbf{1}}^{a(s-1),b}$, [94].

At this stage it is necessary to see if the Fronsdal equations appear inside (5.30). It is a right time for them to emerge since $\omega_{\mathbf{1}}^{a(s-1),b}$ is expressed as $\partial e^{a(s-1)} \sim \partial\phi^{a(s)}$ via (5.27) and (5.30) contains $\partial\partial\phi^{a(s)}$. The Fronsdal equations are imposed by the same trick as in gravity — the *r.h.s.* of (5.30) parameterizes those derivatives that can be non-zero on-shell. We postpone this check till the summary section. Once we have found the second equation we can check its integrability. This process continues smoothly giving

$$d\omega_{\mathbf{1}}^{a(s-1),b(k)} = h_m \wedge \omega_{\mathbf{1}}^{a(s-1),b(k)m} \,, \qquad a(s-1), b(k) = \boxed{\begin{array}{c} s-1 \\ \hline k \end{array}} \,, \qquad (5.33)$$

until

$$d\omega_{\mathbf{1}}^{a(s-1),b(s-2)} = h_m \wedge \omega_{\mathbf{1}}^{a(s-1),b(s-2)m} \,, \qquad a(s-1), b(s-1) = \boxed{\begin{array}{c} s-1 \\ \hline \end{array}} \,, \qquad (5.34)$$

the integrability of which implies $h_k \wedge d\omega_{\mathbf{1}}^{a(s-1),b(s-2)k} \equiv 0$, the unique solution being

$$d\omega_{\mathbf{1}}^{a(s-1),b(s-1)} = h_m \wedge h_n C^{a(s-1)m,b(s-1)n} \,, \qquad a(s), b(s) = \boxed{\begin{array}{c} s \\ \hline \end{array}} \,, \qquad (5.35)$$

where $C^{a(s),b(s)}$ is a zero-form. It is easy to see that it is a solution. Indeed, $h_k \wedge h_m \wedge h_n \wedge C^{a(s-1)m,b(s-2)kn} \equiv 0$ since two anticommuting vielbein one-forms are contracted with two symmetric indices, (5.31).

On-shell $C^{a(s),b(s)}$ is expressed as order-s derivative of the Fronsdal field. It is called the (generalized) Weyl tensor for a spin-s field and it coincides with the (linearized) Weyl tensor if we set $s = 2$. Apart from the Fronsdal operator the Weyl tensor is also gauge invariant. One can prove that there are two basic gauge invariants, the Fronsdal operator and the Weyl tensor. The rest of invariants are derivatives of these two.

Again we have the integrability condition for the Weyl tensor,

$$h_m \wedge h_n \wedge dC^{a(s-1)m,b(s-1)n} \equiv 0 \,. \qquad (5.36)$$

This we can easily solve. First, we decompose $\partial^c C^{a(s),b(s)}$ into irreducible tensors of the Lorentz algebra

$$\boxed{\begin{array}{c} s \\ \hline \end{array}} \otimes \square = \boxed{\begin{array}{c} s \\ \hline \end{array}} \oplus \boxed{\begin{array}{c} s \\ \hline s-1 \end{array}} \oplus \boxed{\begin{array}{c} s \\ \hline \\ \hline \end{array}} \qquad (5.37)$$

All of these can in principle appear on the *r.h.s.* of $dC^{a(s),b(s)} = h....$ The third component, call it $C^{a(s),b(s),c}$ does not pass the integrability test, giving $h_m \wedge h_n \wedge h_k C^{a(s-1)m,b(s-1)n,k} \neq 0$.

The first component $C^{a(s+1),b(s)}$ passes the test $h_m \wedge h_n \wedge h_k C^{a(s-1)km,b(s-1)n} \equiv 0$. The reason is simple, three vielbeins form a rank-three antisymmetric tensor and there is not enough room in the tensor having the symmetry of a two-row diagram to be contracted with it[29]. It may seem that the second component, $C^{a(s),b(s-1)}$ passes the test too, this is not true, however. The reason is that it appears as the trace. Had we been not interested in traces, the decomposition would have contained two components only, given by the gl_d tensor product rule,

$$ \tag{5.38}$$

The point is that both the components in the decomposition above have their traces of the type $C^{a(s),b(s-1)}$ and these are the same $\partial_m C^{a(s-1),b(s-2)m}$. From the $so(d-1,1)$ decomposition, we know that there is only one trace. The traces are irrelevant for the integrability condition, so if the second component above does not go through the integrability test, so do all its traces, i.e. $C^{a(s),b(s-1)}$.

Eventually, one is left with $C^{a(s+1),b(s)}$ and the equation now reads

$$ dC^{a(s),b(s)} = h_m \left(C^{a(s)m,b(s)} + \frac{1}{2} C^{a(s)b,b(s-1)m} \right) . \tag{5.39}$$

Again, $C^{a(s)m,b(s)}$ alone does not have the symmetry of the *l.h.s*, hence we have to project it appropriately by hand, which is done with the help of the second term in the brackets. Proceeding this way we arrive at

$$ dC^{a(s+k),b(s)} = h_m \left(C^{a(s+k)m,b(s)} + \frac{1}{k+2} C^{a(s+k)b,b(s-1)m} \right) , \quad a(s+k), b(s) = \; \boxed{ s | k } \; . $$

Unfolded equations for any s. Summarizing, we get the following diverse equations

$$ \begin{cases} D\omega_1^{a(s-1),b(k)} = h_c \wedge \omega_1^{a(s-1),b(k)c} , \\ \delta\omega_1^{a(s-1),b(k)} = D\xi^{a(s-1),b(k)} - h_c \, \xi^{a(s-1),b(k)c} , \end{cases} \qquad 0 \le k < s-1 , \qquad \tag{5.40}$$

$$ \begin{cases} D\omega_1^{a(s-1),b(s-1)} = h_c \wedge h_d \, C^{a(s-1)c,b(s-1)d} , \\ \delta\omega_1^{a(s-1),b(s-1)} = D\xi^{a(s-1),b(s-1)} , \end{cases} \qquad k = s-1 , \qquad \tag{5.41}$$

$$ \begin{cases} DC^{a(s+i),b(s)} = h_c \left(C^{a(s+i)c,b(s)} + \frac{1}{i+2} C^{a(s+i)b,b(s-1)c} \right) , \\ \delta C^{a(s+i),b(s)} = 0 , \end{cases} \qquad i \in [0, \infty) , \qquad \tag{5.42}$$

where we have replaced d with the background Lorentz derivative $D = d + \varpi$, which makes the system valid in any coordinate system in Minkowski space. Note that $D^2 = 0$. Let us enlist once again the spectrum of fields (differential form degree and the Young

---

[29]It is the property of tensors having the symmetry of one- and two-row Young diagram that anti-symmetrization over any three indices vanishes.

shape of the fiber indices)

grade :	0	1	...	$s-1$	s	$s+1$	...
Young shape :	$\boxed{s-1}$	$\boxed{s-1}_\square$	...	$\boxed{s-1}_\square$	$\boxed{s}$	$\boxed{s}_\square$	...
degree :	1	1	...	1	0	0	...,

We recover below the Fronsdal equations from the unfolded ones. What is left aside is that the most of the fields are not dynamical. Except for those components of the frame field which are in one-to-one correspondence with the Fronsdal field all other components are either auxiliary or Stueckelberg. Auxiliary fields are those that can be expressed as derivatives of the Fronsdal field by virtue of the equations of motion, while Stueckelberg ones can be gauged away with the help of the algebraic gauge symmetry (similarly to how the extra components of the frame field can be gauged away by local Lorentz transformations). Rigorous proof of these facts requires an advanced technology, called $\boldsymbol{\sigma}_-$-cohomology, [95].

Fronsdal equations from unfolded ones. We already know the way the Fronsdal field is embedded into the frame-like field. Let us now show where the Fronsdal equations reside. We need the first two unfolded equations with the form indices revealed

$$\partial_{\underline{m}}e_{\underline{n}}^{a(s-1)} - \partial_{\underline{n}}e_{\underline{m}}^{a(s-1)} = h_{c\underline{m}}\omega_{\underline{n}}^{a(s-1),c} - h_{c\underline{n}}\omega_{\underline{m}}^{a(s-1),c}, \tag{5.43}$$

$$\partial_{\underline{m}}\omega_{\underline{n}}^{a(s-1),b} - \partial_{\underline{n}}\omega_{\underline{m}}^{a(s-1),b} = h_{c\underline{m}}\omega_{\underline{n}}^{a(s-1),bc} - h_{c\underline{n}}\omega_{\underline{m}}^{a(s-1),bc}. \tag{5.44}$$

Converting all indices to the fiber

$$e^{a(s-1)|b} \equiv e_{\underline{m}}^{a(s-1)}h^{b\underline{m}}, \quad \omega^{a(s-1),b|c} \equiv \omega_{\underline{m}}^{a(s-1),b}h^{c\underline{m}}, \quad \omega^{a(s-1),bb|c} \equiv \omega_{\underline{m}}^{a(s-1),bb}h^{c\underline{m}} \tag{5.45}$$

we get

$$\partial^c e^{a(s-1)|d} - \partial^d e^{a(s-1)|c} = \omega^{a(s-1),c|d} - \omega^{a(s-1),d|c}, \tag{5.46}$$

$$\partial^c \omega^{a(s-1),b|d} - \partial^d \omega^{a(s-1),b|c} = \omega^{a(s-1),bc|d} - \omega^{a(s-1),bd|c}. \tag{5.47}$$

By virtue of the first equation the spin-connection can be expressed in terms of the first derivatives of the frame field which contains the Fronsdal tensor together with a pure gauge components. Then, the second equation imposes the Fronsdal equation and expresses the second spin-connection. In order to project onto the Fronsdal equations we contract (5.47) with η_{bd} and symmetrize over c and $a(s-1)$, which gives

$$F^{a(s)} = \partial_c \omega^{a(s-1),c|a} - \partial^a \omega^{a(s-1),c|}{}_c = 0. \tag{5.48}$$

Note that the extra field disappeared because of the specific projection made. Now we symmetrize $a(s-1)$ and d in (5.46)

$$\omega^{a(s-1),c|a} = \partial^c e^{a(s-1)|a} - \partial^a e^{a(s-1)|c}, \tag{5.49}$$

46

so that we can express the first term in (5.48). To express the second of (5.48) we contract (5.46) with η_{da}, which gives

$$\omega^{a(s-1),c|}{}_c = \left(\partial_c e^{a(s-2)c|a} - \partial^a e^{a(s-2)c|}{}_c \right) . \tag{5.50}$$

Plugging the last two equations into (5.48) we find

$$\Box e^{a(s-1)|a} - \partial^a \partial_c \left(e^{a(s-2)c|a} + e^{a(s-1)|c} \right) + 2\partial^a \partial^a e^{a(s-2)c|}{}_c = 0 . \tag{5.51}$$

Now we have to remember how the Fronsdal field is embedded into the frame field

$$e^{a(s-1)|a} = \phi^{a(s)} , \qquad e^{a(s-2)c|}{}_c = \frac{1}{2} \phi^{a(s-2)c}{}_c , \qquad e^{a(s-2)c|a} + e^{a(s-1)|c} = \phi^{a(s-1)c} . \tag{5.52}$$

Magically, all the terms come exactly in the combinations above, which leads to the Fronsdal equations, (2.4),

$$\Box \phi^{a(s)} - \partial^a \partial_c \phi^{a(s-1)c} + \partial^a \partial^a \phi^{a(s-2)c}{}_c = 0. \tag{5.53}$$

We have seen already that the gauge transformations for the totally symmetric component of the frame field $e^{a(s-1)|a}$ are those of the Fronsdal field while the extra component can be gauged away.

Lower-spins. It is obvious that $s = 2$ is a particular case of the above construction. One simply sets $s = 2$ to get the unfolded equation describing free graviton of Section 4 and 5.1. While one always needs infinitely-many zero-forms, the number of one-forms equals the spin of the field, s. We will see below that lower-spin cases, $s = 1$ and $s = 0$ are just a degenerate and a very degenerate cases of the spin-s unfolded equations. In particular, there are no one-forms in the unfolded formulation of $s = 0$ in accordance with the fact that scalar field is not a gauge field. It is worth elaborating on the $s = 0, 1$ cases from scratch.

5.3 $s = 0$

A free massless scalar field in Minkowski space seems to be the case where the unfolded approach gives little advantage. At some point the formulation looks tautological. The only reason to consider the scalar case is because it is in this form the scalar field turns out to be embedded into the full higher-spin theory.

Given a scalar field $C(x)$, it is easy to put the Klein-Gordon equation $\Box C(x) = 0$ in the first order form

$$\partial_{\underline{m}} C(x) = C_{\underline{m}}(x) , \qquad\qquad \eta^{\underline{nm}} \partial_{\underline{n}} C_{\underline{m}}(x) = 0 = \eta^{\underline{nm}} \partial_{\underline{n}} \partial_{\underline{m}} C(x) , \tag{5.54}$$

which is not yet what we need. We can replace an auxiliary field $C_{\underline{m}}(x)$ with its fiber representative $C^a(x)$, $C^a = h^{\underline{m}a} C_{\underline{m}}$ and rewrite the first equation as

$$dC = h_a C^a \qquad\qquad \Longleftrightarrow \qquad\qquad dx^{\underline{m}} \left(\partial_{\underline{m}} C = \eta_{\underline{m}a} C^a \right) . \tag{5.55}$$

47

The difficulty is with the second equation, it must be of the form

$$dC^a = ... \tag{5.56}$$

where ... means some one-form. We can enumerate all the fields that can appear on the r.h.s. Indeed, l.h.s. reads $dx^{\underline{m}} \partial_{\underline{m}} C^a$. Raising all indices with the inverse vielbein we get $\partial^a C^b$, i.e. just a rank-two tensor. As such it can be decomposed into three irreducible components: traceless symmetric – $\boxed{}\boxed{}$, trace – $\bullet$ and antisymmetric – $\begin{array}{c}\boxed{}\\\boxed{}\end{array}$

$$\boxed{}\boxed{} = \partial^a C^a - \frac{1}{d}\eta^{aa}\partial_m C^m, \qquad \bullet = \partial_m C^m, \qquad \begin{array}{c}\boxed{}\\\boxed{}\end{array} = \partial^a C^b - \partial^b C^b. \tag{5.57}$$

Therefore, introducing C^{aa}, C' and $C^{a,b}$ in accordance with the pattern above, the most general r.h.s. reads

$$dC^a = h_m C^{am} + h_m C^{a,m} + \frac{1}{d}h^a C'. \tag{5.58}$$

However, C^a is not an arbitrary fiber vector, it came as a derivative of the scalar and hence $\partial^a C^b - \partial^b C^a \equiv 0$, i.e. $C^{a,b} \equiv 0$. By virtue of (5.58) $\Box C = C'$. Once C' is present on the r.h.s. there is no Klein-Gordon equation imposed. Therefore, we have to set C' to zero too[30]. Finally, C^{aa} parameterizes those second order derivatives that are generally non-zero on-shell and the second equation reads

$$dC^a = h_m C^{am}. \tag{5.59}$$

Actually, the absence of $C^{a,b}$ can be seen in checking the integrability of (5.55). Nilpotency of the exterior derivative $d^2 = 0$ implies

$$0 \equiv ddC = d(h_a C^a) = -h_a \wedge (h_m C^{am} + h_m C^{a,m} + \frac{1}{d}h^a C') = -h_a \wedge h_m C^{a,m}, \tag{5.60}$$

where we used (5.31). Since the vielbein is invertible we get $C^{a,m} \equiv 0$.

Can we stop at C^{aa}? By applying d to (5.59) we find that C^{aa} is not unconstrained

$$0 \equiv ddC^a = d(h_m C^{am}) = -h_m \wedge dC^{am}. \tag{5.61}$$

So we are led to consider possible r.h.s. in $dx^{\underline{m}}\partial_{\underline{m}} C^{aa}$. Decomposing the tensor product $\boxed{}\boxed{} \otimes \Box$ with $\boxed{}\boxed{}$ and $\Box$ representing C^{aa} and ∂^m, respectively, one finds

$$\boxed{}\boxed{} \otimes \Box = \begin{array}{c}\boxed{}\boxed{}\\\boxed{}\end{array} \oplus \boxed{}\boxed{}\boxed{} \oplus \Box, \tag{5.62}$$

i.e. the most general equation for C^{aa} reads

$$dC^{aa} = h_m C^{aam} + h_m C^{aa,m} + \left(h^a \tilde{C}^a - \frac{2}{d}\eta^{aa} h_m \tilde{C}^m\right). \tag{5.63}$$

---

[30]Another option is to express C' as a function of C. For example, $C' = m^2 C$, gives Klein-Gordon equation with a mass term. One can also introduce an arbitrary potential $C' = V(C)$.

Note that $h^a \tilde{C}^a$ is not traceless and has to be supplemented with the second term to agree with vanishing trace of C^{aa}. Both the hook ⊟ component $C^{aa,b}$ and the trace $\tilde{C}^a$ are inconsistent with (5.61) since

$$h_n \wedge dC^{an} = h_n \wedge h_m C^{an,m} - \frac{d-2}{2} h^a \wedge h_m \tilde{C}^m \neq 0 \,. \tag{5.64}$$

Therefore we have to exclude $C^{aa,b}$ and $\tilde{C}^a$ from the *r.h.s.*. The only term left parameterizes third-order derivatives of the scalar field on-shell

$$dC^{aa} = h_m C^{aam} \,. \tag{5.65}$$

Continuing this way one one arrives at the final answer. A typical situation in obtaining the unfolded equations is that one makes use of dynamical equations at first few levels which define the pattern of first auxiliary fields and then at each next level there are only Bianchi identities that further constraint the final form of equations.

Unfolded equations for $s = 0$. The full system of equations that describe a free massless scalar field reads

$$DC^{a(k)} = h_m C^{a(k)m} \,, \qquad\qquad C^{a(k-2)mn} \eta_{mn} \equiv 0 \,. \tag{5.66}$$

where the set of fields consists of totally-symmetric traceless zero-forms $C^{a(k)}$.

A field $C^{a(k)}$ with $k > 0$ is expressed as order-k derivative $\partial^a...\partial^a C$ of the lowest field $C(x)$ which can be identified with the original scalar field we started with. All derivatives $\partial^a...\partial^a C$ are traceless since $\eta_{mn} \partial^{mna(k-2)} C = \partial^{a(k-2)} \Box C = 0$, which explains why $C^{a(k)}$ has to be traceless. The Klein-Gordon equation appears thanks to the vanishing trace of C^{aa}. Simply, the second equation, with C^a already solved as $\partial^a C$, implies $\partial^a \partial^a C = C^{aa}$. Since the *r.h.s.* has vanishing trace, by contracting with η_{aa} we derive $\Box C = 0$.

In the table below we list the spectrum of fields needed. The form degree is always zero.

grade :	0	1	2	3	...
Young-shape :	●	□	⊡⊡	⊡⊡⊡	...
degree :	0	0	0	0	...

This is what one gets upon setting $s = 0$ in (5.40)-(5.42) and dropping tensors with $(s - 1) = -1$ indices.

Scalar is almost tautological. In this simple case of a scalar field there is a way to make the unfolded system tautological. Let us contract all the fiber indices with an auxiliary variable y_a to build $C(y|x)$

$$C(y|x) = \sum_k \frac{1}{k!} C^{a(k)}(x)\, y_a...y_a \tag{5.67}$$

49

then (5.66) is rewritten as

$$\left(D - h^m \frac{\partial}{\partial y^m}\right) C(y|x) = 0\,, \qquad\qquad \frac{\partial}{\partial y^m}\frac{\partial}{\partial y_m} C(y|x) \equiv 0\,, \qquad (5.68)$$

where the last condition makes $C^{a(k)}$ traceless, see Appendix E.3. Going to Cartesian coordinates where there is no difference between the world and fiber indices we find

$$\left(\frac{\partial}{\partial x^m} - \frac{\partial}{\partial y^m}\right) C(y|x) = 0\,, \qquad\qquad \Box^y C(y|x) \equiv 0\,. \qquad (5.69)$$

The first equation tells us that the dependence on x is exactly the dependence on y, the solution being $C(y|x) = C(x - y)$, while the second condition imposes Klein-Gordon equation in the fiber. Therefore, the fiber Klein-Gordon equation is mapped to the x-space Klein-Gordon equation. If we switch to a massive scalar or put a scalar on AdS instead of Minkowski or proceed to fields with spin the correspondence gets far from being tautological!

It is certainly true that one can write a tautological system that imposes the original equations in the fiber and then translates the dynamics to the base in a trivial way, like above $\partial_x = \partial_y$. The point is that such system will not have an unfolded form apart from a simple example above. The reason as we will see is that the unfolded system contains more fields, especially connections, which brings in the geometry.

5.4 $s = 1$

The starting point is the gauge transformation law

$$\delta A_{\underline{m}} = \partial_{\underline{m}} \xi\,, \qquad\qquad (5.70)$$

which suggests to treat $A_{\underline{m}}$ as a one-form $A_\mathbf{1} = A_{\underline{m}} dx^{\underline{m}}$, which is what usually done. The gauge parameter is then a zero-form and we can simply write $\delta A_\mathbf{1} = d\xi$. The equations have to start with $dA = ...$ and there are three options for what ... might be.

$$dA_\mathbf{1} = R_\mathbf{2}\,, \qquad\qquad dA = h_m \wedge \omega_\mathbf{1}^m\,, \qquad\qquad dA = h_m \wedge h_n C^{m,n}\,, \qquad (5.71)$$

where $R_\mathbf{2}$ is a two-form; $\omega_\mathbf{1}^m$ is some vector-valued one-form and $C^{m,n}$ is a zero-form that is antisymmetric in its fiber indices, thus belonging to ⊟. Note that these three options partially overlap as all of the fields on the *r.h.s.* contain ⊟, whenever all the world indices are converted to the fiber with the inverse vielbein. No matter how natural the first option is, it fails. Indeed, the first equation is invariant under $\delta A_\mathbf{1} = d\xi + \xi_\mathbf{1}$, $\delta R_\mathbf{2} = d\xi_\mathbf{1}$ where $\xi_\mathbf{1}$ is some scalar one-form, the gauge parameter for $R_\mathbf{2}$. The latter means that one can gauge away $A_\mathbf{1}$ completely with the help of $\xi_\mathbf{1}$, which is not what we need. The second option fails in a similar way once we observe that there is an additional gauge symmetry $\delta A_\mathbf{1} = -h_m\xi^m$, $\delta\omega_\mathbf{1}^a = d\xi^a$ with a vector-valued zero-form ξ^a. Therefore, we have to use the third option

$$dA_\mathbf{1} = h_m \wedge h_n C^{m,n}\,, \qquad\qquad (5.72)$$

which is just a way to parameterize the Maxwell tensor, $\partial_{\underline{m}} A_{\underline{n}} - \partial_{\underline{n}} A_{\underline{m}} = C^{a,b} h_{a\underline{m}} h_{b\underline{n}}$. The integrability of this equation implies

$$0 \equiv ddA = d(h_m \wedge h_n C^{m,n}) = h_m \wedge h_n \wedge dC^{m,n} \tag{5.73}$$

Again, as in the scalar case, $dC^{m,n}$ is restricted. The most general *r.h.s.* of $dC^{m,n}$ is given by evaluating the tensor product $\square \otimes \begin{smallmatrix}\square\\\square\end{smallmatrix}$, where $\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ correspond to ∂^a and $C^{a,b}$

$$\square \otimes \begin{smallmatrix}\square\\\square\end{smallmatrix} = \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \oplus \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} \oplus \square \,, \tag{5.74}$$

$$\begin{smallmatrix}\square\square\\\square\end{smallmatrix} = \partial^c C^{a,b} - \partial^{[c} C^{a,b]} + \frac{2}{d-1} \eta^{c[a} \partial_d C^{b,]d} \,, \tag{5.75}$$

$$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} = \partial^{[c} C^{a,b]} \,, \tag{5.76}$$

$$\square = \partial_m C^{a,m} = \square A^a - \partial^a \partial_m A^m \,, \tag{5.77}$$

so introducing $C^{aa,b}$, $C^{a,b,c}$ and C^a in accordance with the pattern above $dC^{a,b}$ reads

$$dC^{a,b} = h_m \left(C^{am,b} + \frac{1}{2} C^{ab,m} \right) + h_m C^{a,b,m} + \left(h^a C^b - h^b C^a \right) . \tag{5.78}$$

The terms in brackets make up the rank-two antisymmetric fiber tensor[31]. We see that the presence of the antisymmetric component on the *r.h.s.* violates integrability as $h_m \wedge h_n \wedge h_k C^{m,n,k} \neq 0$. The same time we know that $\partial_{[\underline{m}} C_{\underline{n},\underline{r}]} = \partial_{[\underline{m}} \partial_{\underline{n}} A_{\underline{r}]} \equiv 0$, which is the Bianchi identity, and hence there is no such component in the jet of $A_{\underline{m}}$.

   We would like to impose the Maxwell equations, but we see that keeping C^a on the *r.h.s.* we rather get $\square A^a - \partial^a \partial_m A^m = C^a$. Therefore, C^a has to be set to zero. The leftover field $C^{aa,b}$ with the symmetry of $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ parameterize the second order derivatives of $A_{\underline{m}}$ that do not vanish on-shell. Now the second equation in the hierarchy reads

$$dC^{a,b} = h_m \left(C^{am,b} + \frac{1}{2} C^{ab,m} \right) . \tag{5.79}$$

The integrability condition for this equation implies that $C^{aa,b}$ cannot be arbitrary. A straightforward analysis shows that we need to introduce fields $C^{a(k),b}$ that have the symmetry of $\begin{smallmatrix}\square\overset{k}{\square}\square\\\square\end{smallmatrix}$ and are traceless.

Unfolded equations for $s = 1$. The full system reads

$$DA_{\mathbf{1}} = h_m \wedge h_n C^{m,n} \,, \qquad\qquad \delta A_{\mathbf{1}} = D\xi \,, \tag{5.80}$$

$$DC^{a(k),b} = h_m \left(C^{a(k)m,b} + \frac{1}{k+1} C^{a(k)b,m} \right) , \tag{5.81}$$

---

[31] Let us note that $C^{am,b}$ is not antisymmetric in a, b! The relative coefficient is fixed in checking that the symmetrization over a, b does vanish and using the Young property to derive $C^{am,a} = -C^{aa,m}$.

where let us stress again that all fields are $so(d-1,1)$-irreducible as fiber tensors. It is useful to list the fields in order of their appearance by assigning certain grading. As compared to the scalar case we have more diverse structure of fiber tensors and there is one field which is not a zero-form

grade :	0	1	2	3	...
Young shape :	$\bullet$	$\square$	$\square\square$	$\square\square\square$	...
degree :	1	0	0	0	...

Again, this is the specialization of (5.40)-(5.42) to $s = 1$.

5.5 Zero-forms

The first observation is that the equations for zero-forms $C^{a(s+k),b(s)}$, (5.42), are self-consistent and do not require the presence of any one-forms. What do they describe? The answer can be read off from the spin-one case. The zero-forms begin with the Maxwell tensor $C^{a,b}$. Since we do not supplement it with gauge potential A_1 and the corresponding equation (5.72), it is the fundamental field now. Then (5.79) implies the following two equations

$$\partial^a C^{b,c} + \partial^a C^{b,c} + \partial^a C^{b,c} = 0\,, \qquad\qquad \partial_m C^{a,m} = 0\,. \qquad (5.82)$$

Indeed, the most general *r.h.s.* of (5.79) could involve three components according to (5.74), but it contains only one, (5.79). Therefore, the two combinations of first derivatives of $C^{a,b}$ are set to zero by (5.79), giving rise to the equations above. These are the standard equations that impose $dF = 0$ and $\partial_m F^{n,m} = 0$ for two-form $F_{m,n}dx^m \wedge dx^n$, $F_{m,n} = C_{m,n}$. The first equation is the Bianchi identity which implies $F = dA$ for some A, so the gauge potential is implicitly there. The second one is the Maxwell equations without any sources.

The same story for the spin-two case gives the Weyl tensor $C^{aa,bb}$ as the fundamental variable and imposes

$$\partial^{[a} C^{bc],de} = 0\,, \qquad\qquad \partial_m C^{ab,cm} = 0\,, \qquad (5.83)$$

which can be recognized as the two components of the differential Bianchi identities for the Weyl tensor.

In the case of a spin-s field we find the following equations

$$\partial^{[u} C^{a(s-1)u,b(s-1)u]} = 0\,, \qquad\qquad \partial_m C^{a(s),b(s-1)m} = 0\,, \qquad (5.84)$$

where the anti-symmetrization over three u's is implied. The rest of equations merely express the fields as derivatives of the generalized Weyl tensor. The simplest way to solve equations is to perform Fourier transform to find

$$C^{a(s),b(s)}(p) = \varphi^{a(s)}(p)\, p^b...p^b + \text{permutations}\,, \qquad (5.85)$$

where $\varphi^{a(s)}(p)$ is a symmetric tensor that depends on momentum p^a and the following additional conditions are true

$$p_m p^m = 0\,, \qquad \varphi^{a(s-2)m}{}_m = 0\,, \qquad p_m \varphi^{a(s-1)m} = 0\,. \tag{5.86}$$

We almost reproduced the on-shell conditions (2.10)-(2.12) for the Fronsdal field. Where is the gauge symmetry? The Young properties imply that $\varphi^{a(s)}$ of special form $p^a \xi^{a(s-1)}$ yields a vanishing Weyl tensor. This is the on-shell Fronsdal gauge transformations in momentum space. It is not possible to build an antisymmetric tensor with $p^a p^b$, which is automatically symmetric. Likewise, it is not possible to build $C^{a(s),b(s)}$ out of $p^b...p^b p^a \xi^{a(s-1)}$. The problem is that two p's show up in the same column of the Young diagram, $p^a p^b$. By going into antisymmetric presentation, where $C^{a(s),b(s)}$ has s pairs of antisymmetric indices, $C^{ab,cd,\cdots}$, we see that the gauge transformation of $\phi^{a(s)}$ does not affect the Weyl tensor. Therefore, $\phi^{a(s)}$ is defined up to a gauge transformation and we recover the on-shell Fronsdal theory.

The upshot is that the equations for zero-forms also describe a spin-s massless field, but in a non-gauge way. The examples of $s = 1, 2$ tells us that it is the gauge potentials A and g_{mn} (which needs to be replaced by e^a, $\omega^{a,b}$) that mediate interactions. Therefore, while the non-gauge description of massless fields is perfectly fine at the free level it is crucial to supplement it with gauge potentials in order to proceed to interactions. The full unfolded equations tells us that the gauge potentials are determined up to a gauge transformation by the gauge invariant Weyl tensors (field-strengths).

5.6 All spins together

One can write a relatively simple system that describes fields of all spins $s = 0, 1, 2, 3, ...$ together. This is a Minkowski prototype of the unfolded equations in AdS_4 of Sections 9 and 10. First we sum up all the fields that appeared above into generating functions of two auxiliary variables, y^a, p^a,

$$\omega_{\mathbf{1}}(y, p|x) = \sum_{s>0} \sum_{k=0}^{k=s-1} \frac{1}{(s-1)!k!} \, \omega_{\mathbf{1}}^{a(s-1),b(k)} y_a...y_a \, p_b...p_b \,, \tag{5.87}$$

$$C(y, p|x) = \sum_{s=0} \sum_{k=0} \frac{1}{(s+k)!s!} \, C^{a(s+k),b(s)} y_a...y_a \, p_b...p_b \,. \tag{5.88}$$

Analogous decomposition is used for the collective gauge parameter $\xi(y, p|x)$. One of the crucial observations is that one-forms $\omega_{\mathbf{1}}^{a(s-1),b(k)}$ when summed over all spins, cover all irreducible Lorentz representations with the symmetry of all two-row Young diagrams, each appearing in one copy

$$\{\omega_{\mathbf{1}}^{a(s-1),b(k)}\} \qquad a(s-1), b(k) : \begin{array}{|c|}\hline s-1 \\\hline k \\\hline\end{array} \tag{5.89}$$

The zero-forms cover the same variety of Young diagrams (all two-row)

$$\{C^{a(s+k),b(s)}\} \qquad a(s+k), b(s) : \begin{array}{|c|}\hline s+k \\\hline s \\\hline\end{array} \tag{5.90}$$

Therefore, the fiber spaces of one-forms and zero-forms are isomorphic!

We need the Taylor coefficients of (5.87)-(5.88) to obey Young symmetry and trace constraint, which implies[32]

$$\left\{ y^n \frac{\partial}{\partial p^n}, \frac{\partial^2}{\partial y_m \partial y^m}, \frac{\partial^2}{\partial y_m \partial p^m}, \frac{\partial^2}{\partial p_m \partial p^m} \right\} (\omega_{\mathbf{1}}(y,p|x), C(y,p|x), \xi(y,p|x)) \equiv 0 \qquad (5.91)$$

Then the unfolded system of equations together with the gauge transformations can be rewritten as

$$\mathcal{D}\omega_{\mathbf{1}}(y,p|x) = \delta_{N_y,N_p} h^m \wedge h^n \frac{\partial^2}{\partial y^m \partial p^n} C(y,p|x), \qquad (5.92)$$

$$\delta\omega_{\mathbf{1}}(y,p|x) = \mathcal{D}\xi(y,p|x), \qquad (5.93)$$

$$\widetilde{\mathcal{D}} C(y,p|x) = 0, \qquad (5.94)$$

$$\mathcal{D} = D - h^m \frac{\partial}{\partial p^m}, \qquad (5.95)$$

$$\widetilde{\mathcal{D}} = D - h^m \left(\frac{\partial}{\partial y^m} + \frac{1}{N_y - N_p + 2} p^n \frac{\partial}{\partial y^n} \frac{\partial}{\partial p^m} \right), \qquad (5.96)$$

$$D = d + \varpi^{a,b} \left(y_a \frac{\partial}{\partial y^b} + p_a \frac{\partial}{\partial p^b} \right), \qquad (5.97)$$

where $N_y = y^m \partial/\partial y^m$, $N_p = p^m \partial/\partial p^m$ are Euler operators that just count the number of tensor indices, i.e. $[N_y, y_a] = +y_a$, $[N_y, \partial/\partial y^a] = -\partial/\partial y^a$, etc. The Lorentz-covariant derivative, D, must hit every index contracted with y_a or p_b. This is achieved with the help of the differential operator in the last line. In fact, it makes use of the standard representation of the orthogonal algebra on functions

$$\varpi^{a,b} \left(y_a \frac{\partial}{\partial y^b} \right) = \frac{1}{2} \varpi^{a,b} \left(y_a \frac{\partial}{\partial y^b} - y_b \frac{\partial}{\partial y^a} \right). \qquad (5.98)$$

Taylor expanding this system we find all the unfolded equations that we derived above. The Kronecker symbol δ_{N_y,N_p} ensures that only the Weyl tensors of C contribute to the r.h.s. of the equations for ω. One of the effective realization of δ_{N_y,N_p} could be

$$C(yt, pt^{-1})\big|_{t=0} \qquad (5.99)$$

The system is gauge invariant and consistent thanks to $\mathcal{D}^2 = 0$, $\widetilde{\mathcal{D}}^2 = 0$. It is important that the term that sources one-forms ω by zero-forms C does not spoil the consistency. The system can be projected onto particular spin-s subsector with the help of

$$N_y \omega_{\mathbf{1}}(y,p|x) = (s-1)\omega_{\mathbf{1}}(y,p|x), \qquad N_p C(y,p|x) = s C(y,p|x). \qquad (5.100)$$

The system describes free fields of all spins $s = 0, 1, 2, 3, ...$ over the Minkowski space. It is a very nontrivial problem to find a nonlinear theory that leads to this equations in the

---

[32]See Appendix E.3 for the implementation of Young-symmetry/trace constraints in terms of generating functions. It is these constraints that will be shown to be automatically solved with the help of spinorial auxiliary variables later on, which makes $d = 3, 4$ HS theories simpler.

linearization. For this case it was proved, [96], under certain assumptions, that it is not possible to find a nonlinear completion for the system above. But it becomes possible in anti-de Sitter space.

There is even a simpler set of equations, [96], that does the same with the help of additional auxiliary fields.

6 Unfolding

In this section we would like to give a systematic overview of the unfolded approach, [40, 41], which underlies the Vasiliev HS theory. The unfolded approach is a universal method of formulating differential equations on manifolds. Any set of differential equations can be put into this form in principle. Once it is done one can learn a lot about the symmetries of the original system. The symmetries that were not at all obvious in the original formulation become manifest when the unfolded form of equations is available. This turned out to be very useful for the HS problem analysis since the HS theory is so much constrained by the symmetry.

The unfolded approach is one of the general methods and as such it is not a cure-all. It can provide some substantial progress once there is a rich hidden symmetry behind the system, but it may not give any advantage as compared to other approaches when dealing with systems that have a poor symmetry. Besides, the unfolding of some nonlinear dynamical equations can be of a great technical challenge let alone there are only few examples of nonlinear equations in the literature that acquire explicit unfolded form.

Few of the nice features of the unfolded approach include: its manifest diffeomorphism invariance; it is specifically suited to account for gauge symmetries, conserved currents and charges. Generally when looking for the interactions starting from a linear gauge theory, one is faced with several problems at once: find a deformation of Lagrangian or equations of motion and a deformation of the gauge symmetry in such a way as to leave Lagrangian or equations of motion invariant. These deformations are governed by the same set of structure constants within the unfolded approach.

The variety of unfolded equations is at least as rich as all Lie algebras together with all representations thereof and cohomologies. As we will see all of the structure constants that are at the bottom of any unfolded equations have certain interpretation within the Lie algebra theory.

Though at present it is far from being clear, the unfolded approach have a potential to be the generalization of the notion of integrability in higher dimensions. Certain aspects of the unfolded approach showed up in the context of supergravity [97–99] and in topology [100] under the name of Free Differential Algebras.

6.1 Basics

Let $W^{\mathcal{A}}$ be a set of differential forms over a certain manifold $\mathcal{M}_d$. The index $\mathcal{A}$ is a formal one that we let run over some set of fields assuming the Einstein summation convention.

In practice it runs over a set of linear spaces[33]. The differential form degree of $W^{\mathcal{A}}$ is denoted by $|\mathcal{A}|$. Equations of the following form are referred to as the unfolded equations

$$dW^{\mathcal{A}} = F^{\mathcal{A}}(W), \tag{6.1}$$

where d is the exterior (de-Rham) differential and the function $F^{\mathcal{A}}(W)$ has degree $|\mathcal{A}|+1$ and is assumed to be expandable in terms of the exterior products of the fields themselves

$$F^{\mathcal{A}}(W) = \sum_n \sum_{|\mathcal{B}_1|+...+|\mathcal{B}_n|=|\mathcal{A}|+1} f^{\mathcal{A}}{}_{\mathcal{B}_1...\mathcal{B}_n} W^{\mathcal{B}_1} \wedge ... \wedge W^{\mathcal{B}_n}, \tag{6.2}$$

The structure constants $f^{\mathcal{A}}{}_{\mathcal{B}_1...\mathcal{B}_n}$ are certain elements of $\mathrm{Hom}(\mathcal{B}_1 \otimes ... \otimes \mathcal{B}_n, \mathcal{A})$, i.e. maps from a tensor product $\mathcal{B}_1 \otimes ... \otimes \mathcal{B}_n$ to $\mathcal{A}$. They satisfy the symmetry condition induced by the form degree of the fields

$$f^{\mathcal{A}}{}_{\mathcal{B}_1...\mathcal{B}_i\mathcal{B}_{i+1}...\mathcal{B}_n} = (-)^{|\mathcal{B}_i||\mathcal{B}_{i+1}|} f^{\mathcal{A}}{}_{\mathcal{B}_1...\mathcal{B}_{i+1}\mathcal{B}_i...\mathcal{B}_n}. \tag{6.3}$$

The crucial point is to impose the integrability condition

$$F^{\mathcal{B}} \wedge \frac{\delta F^{\mathcal{A}}}{\delta W^{\mathcal{B}}} \equiv 0, \tag{6.4}$$

that arises when applying d to both sides of (6.7), using $d^2 = 0$ and the equations of motion again on the *r.h.s.* (we will use $\partial_{\mathcal{B}}$ instead of $\frac{\delta}{\delta W^{\mathcal{B}}}$)[34]

$$0 \equiv d^2 W^{\mathcal{A}} = dF^{\mathcal{A}}(W) = dW^{\mathcal{B}} \wedge \partial_{\mathcal{B}} F^{\mathcal{A}}(W) = F^{\mathcal{B}} \wedge \partial_{\mathcal{B}} F^{\mathcal{A}}(W). \tag{6.5}$$

The integrability condition implies that the equations are formally consistent and contain all their differential consequences[35]. In deriving (6.4) we also assume the absence of manifest space-time dependence in $F^{\mathcal{B}}$.

When written in terms of the structure constants the integrability condition reads

$$\sum f^{\mathcal{C}}{}_{\mathcal{D}_1...\mathcal{D}_m} f^{\mathcal{A}}{}_{\mathcal{C}\mathcal{B}_2...\mathcal{B}_n} W^{\mathcal{D}_1} \wedge .... \wedge W^{\mathcal{B}_n} = 0, \tag{6.6}$$

and reminds the Jacobi identity. Let us note that some of these equations may hold automatically if the total differential form degree exceeds the dimension of the manifold $\mathcal{M}_d$. The useful concept introduced by Vasiliev is that of the *universal* unfolded system, [65,96]. The unfolded equations are called universal iff the system is integrable irrespectively of the dimension, i.e. formally for any d. In other words, the integrability constraint in some fixed dimension $d = d_0$ may admit nontrivial solution due to the fact that any differential form of the degree $d_0 + 1$ is identically zero. The *universal* unfolded systems are supposed to be consistent for any d.

The bonus of the universal system, which has been already shown to be useful in applications, is that it remains meaningful on a manifold of any dimension. In particular

---

[33]For example, $\mathcal{A}$ runs over Lorentz tensors with the symmetry of $\boxed{\frac{s-1}{k}}$, $k = 0, ..., s-1$ and $\boxed{\frac{s+n}{s}}$, $n = 0, 1, ...$ in the case of a spin-s field as we have seen in Section 5.

[34]The derivatives carry the same grading as the fields, i.e. $\partial_{\mathcal{A}}\partial_{\mathcal{B}} = (-)^{|\mathcal{A}||\mathcal{B}|}\partial_{\mathcal{B}}\partial_{\mathcal{A}}$.

[35]Recall footnote 12 on page 20.

one can consider the same unfolded equations over different space-times (say, AdS_{d+1} and its conformal boundary M_d, which could be a way to make AdS/CFT tautological [101]).

As the unfolded equations are formulated entirely in the language of differential forms, there is no need for extra data, like metric or Christoffel symbols, that $\mathcal{M}_d$ is typically equipped with.

It is convenient to rewrite the unfolded equations as the zero-curvature condition

$$R^{\mathcal{A}} = 0, \qquad R^{\mathcal{A}} \equiv dW^{\mathcal{A}} - F^{\mathcal{A}}(W), \tag{6.7}$$

where $R^{\mathcal{A}}$ will be referred to as the curvature.

The unfolded equations turn out to implement the concept of gauge symmetry automatically. Indeed, let us define the following gauge variations

$$\delta_\epsilon W^{\mathcal{A}} = d\epsilon^{\mathcal{A}} + \epsilon^{\mathcal{B}} \partial_{\mathcal{B}} F^{\mathcal{A}}, \qquad \text{if } |\mathcal{A}| > 0, \tag{6.8}$$

$$\delta_\epsilon W^{\mathcal{A}} = +\epsilon^{\mathcal{B}'} \partial_{\mathcal{B}'} F^{\mathcal{A}}, \qquad \mathcal{B}' : |\mathcal{B}'| = 1, \qquad \text{if } |\mathcal{A}| = 0, \tag{6.9}$$

where we have to distinguish between forms of degree greater than zero and zero-forms. Each $W^{\mathcal{A}}$ with $|\mathcal{A}| > 0$ has an associated gauge symmetry with parameter $\epsilon^{\mathcal{A}}$ that is a degree $|\mathcal{A}| - 1$ form taking values in the same space as $W^{\mathcal{A}}$ does. There are no genuine gauge symmetries associated with zero-forms. They transform without the $d\epsilon^{\mathcal{A}}$-piece similar to matter fields and with the parameters of the degree-one fields among $W^{\mathcal{A}}$ they are coupled to.

One can check that the variation of the curvature $R^{\mathcal{A}}$ vanishes on-shell, i.e. when $R^{\mathcal{A}} = 0$, since

$$\delta R^{\mathcal{A}} = (-)^{\mathcal{C}+1} \epsilon^{\mathcal{C}} R^{\mathcal{B}} \partial_{\mathcal{B}} \partial_{\mathcal{C}} F^{\mathcal{A}}. \tag{6.10}$$

This is a generalization of $\delta F = [F, \epsilon]$ in the Yang-Mills theory with $\partial_{\mathcal{B}} \partial_{\mathcal{C}} F^{\mathcal{A}}$ playing the role of the structure constants of the Lie algebra, which in general are not constants and do depend on the fields.

The integrability condition can be rewritten as a Bianchi identity for the curvature

$$dR^{\mathcal{A}} + R^{\mathcal{C}} \partial_{\mathcal{C}} F^{\mathcal{A}} \equiv 0, \tag{6.11}$$

which is a generalization of $DF \equiv 0$ in the Yang-Mills.

There is also a generalization of the fact that within a gauge theory a diffeomorphism can be represented as a gauge transformation plus a correction due to nonvanishing curvature

$$\mathcal{L}_\xi A = D(i_\xi A) + i_\xi F(A), \qquad F(A) = dA + \frac{1}{2}[A, A]. \tag{6.12}$$

The analog within the unfolded approach reads

$$\mathcal{L}_\xi W^{\mathcal{A}} = \delta_{i_\xi \cdot W} W^{\mathcal{A}} + i_\xi \cdot R^{\mathcal{A}} \tag{6.13}$$

the difference is that within the unfolded approach any diffeomorphism is always a particular gauge transformation on-shell, since $R^{\mathcal{A}} = 0$.

The notion of reducible gauge symmetries is also easily implemented. Indeed, the condition for $W^{\mathcal{A}}$ to be invariant under the gauge transformation

$$\delta_\epsilon W^{\mathcal{A}} = 0, \qquad d\epsilon^{\mathcal{A}} - \epsilon^{\mathcal{B}} \partial_{\mathcal{B}} F^{\mathcal{A}} = 0, \tag{6.14}$$

can be interpreted as the unfolded-like system itself with the field content extended by the gauge parameters $\epsilon^{\mathcal{A}}$. As such it is gauge invariant under the second-level gauge transformations

$$\delta\epsilon^{\mathcal{A}} = d\xi^{\mathcal{A}} + \xi^{\mathcal{B}}\partial_{\mathcal{B}}F^{\mathcal{A}}, \tag{6.15}$$

where we neglect terms bilinear in $\epsilon^{\mathcal{A}}$ by the definition of what gauge transformations are. Again, $\delta\epsilon^{\mathcal{A}} = 0$ is an unfolded equation and so on until the zero-forms are reached. The level-two gauge parameter here $\xi^{\mathcal{A}}$ is a degree-$(|\mathcal{A}| - 2)$ form. Therefore, a degree-q field has q levels of gauge (and gauge for gauge) transformations in total.

6.2   Structure constants

In this section we unveil the algebraic meaning of the structure constants. As we will see soon, the sector of one-forms is at the core of any unfolded-system as they necessarily belong to some Lie algebra, say $\mathfrak{g}$. The rest of the fields can be interpreted as taking values in various $\mathfrak{g}$-modules. These modules can be in general glued (coupled) together via Chevalley-Eilenberg cocycles of $\mathfrak{g}$.

Lie algebras and flatness/zero-curvature.   Suppose there are one-forms only or a closed subsector thereof. Let $\Omega^I \equiv \Omega^I_\mu dx^\mu$ denote the components of the one-forms in some basis. Then the only unfolded-type equations one can write read

$$d\Omega^I + \frac{1}{2}f^I_{JK}\Omega^J \wedge \Omega^K = 0, \tag{6.16}$$

where f^I_{JK} are the structure constants that are antisymmetric in J, K since one-forms anti-commute. The integrability condition (6.4) implies the Jacobi identity

$$f^I_{JK}f^J_{LM}\Omega^K \wedge \Omega^L \wedge \Omega^M \equiv 0 \qquad \longleftrightarrow \qquad f^I_{J[K}f^J_{LM]} \equiv 0\,. \tag{6.17}$$

Therefore, f^I_{JK} define certain Lie algebra, call it $\mathfrak{g}$. Then (6.16) is the flatness or zero-curvature condition for a connection of $\mathfrak{g}$. Equivalently, (6.16) means that the Yang-Mills field-strength $F = d\Omega + \frac{1}{2}[\Omega, \Omega]$ vanishes. The gauge transformations associated with the zero-curvature equations are the standard Yang-Mills transformations, $\delta\Omega^I = D_\Omega\epsilon^I$. Such equations describe background geometry, e.g. Minkowski or AdS, (3.54).

Modules and covariant constancy.   Consider now the extension of the unfolded system with one-forms by a sector $W^{\mathcal{A}}_q$ of q-forms. The most general equations that are linear in $W^{\mathcal{A}}_q$ read

$$dW^{\mathcal{A}}_q + f_I{}^{\mathcal{A}}{}_{\mathcal{B}}\,\Omega^I \wedge W^{\mathcal{B}}_q = 0\,, \tag{6.18}$$

with Ω^I obeying (6.16) and $f_I{}^{\mathcal{A}}{}_{\mathcal{B}}$ being some new structure constants. The integrability (6.4) then implies

$$\Omega^J \wedge \Omega^K \left(-\frac{1}{2}f^I_{JK}f_I{}^{\mathcal{A}}{}_{\mathcal{B}} + f_J{}^{\mathcal{A}}{}_{\mathcal{C}}f_K{}^{\mathcal{C}}{}_{\mathcal{B}}\right) W^{\mathcal{B}}_q = 0\,, \tag{6.19}$$

where there is an implicit anti-symmetrization over J, K in the last term due to anti-commuting nature of Ω^I. Thinking of $f_I{}^{\mathcal{A}}{}_{\mathcal{B}}$ as endomorphisms $f_i \in \mathrm{Hom}(V, V)$, just matrices on the space V where the q-forms take value, we recognize the definition of a $\mathfrak{g}$-module

$$f_J \circ f_K - f_K \circ f_J = [f_J, f_K] = f_{JK}^I f_I \tag{6.20}$$

where $\circ$ denotes the usual matrix product over implicit indices $\mathcal{A}$.

Therefore, writing down (6.18)-type equations is equivalent to specifying certain representation V of $\mathfrak{g}$ and the corresponding equation is the covariant constancy condition for a field with values in V,

$$D_\Omega W_q^{\mathcal{A}} \equiv dW_q^{\mathcal{A}} + f_I{}^{\mathcal{A}}{}_{\mathcal{B}} \Omega^I \wedge W_q^{\mathcal{B}} = 0\,, \qquad \delta W_q^{\mathcal{A}} = D_\Omega \xi_{q-1}^{\mathcal{A}}\,, \tag{6.21}$$

where we added gauge transformations — the covariant derivative in the module defined by $f_I{}^{\mathcal{A}}{}_{\mathcal{B}}$. The gauge invariance is thanks to $D_\Omega D_\Omega = 0$, (6.16).

Contractible cycles and empty equations. Consider now the linear equations of the form

$$dW_q^{\mathcal{A}} = f^{\mathcal{A}}{}_{\mathsf{B}} W_{q+1}^{\mathsf{B}}. \tag{6.22}$$

By linear transformations they can always be split into two types of subsystems $f^{\mathcal{A}}{}_{\mathsf{B}}$

$$I: \qquad dW_q^{\mathsf{A}} + W_{q+1}^{\mathsf{A}} = 0, \qquad dW_{q+1}^{\mathsf{B}} = 0, \tag{6.23}$$

$$II: \qquad dW_q^{\mathsf{A}} = 0, \tag{6.24}$$

where the last equation of (6.23) is the integrability condition for the first one. (6.23) is called a contractible cycle, [100]. With the help of gauge transformations $\delta W_q^{\mathsf{A}} = d\xi_{q-1}^{\mathsf{A}} - \chi_q^{\mathsf{A}}$ we can always gauge away $W_q^{\mathcal{A}}$ since χ_q^{A} has the same number of components and enters algebraically. Then we are left with $W_{q+1}^{\mathsf{A}} = 0$. Consequently we see that the equations of type (6.23) are dynamically empty. The type-II equations can be solved in a pure gauge form $W_q^{\mathsf{A}} = d\xi^{\mathsf{A}}$ by virtue of the Poincare lemma unless $q = 0$, see extra Section 12.7. If $q = 0$ the solution is just a constant, c^{A}, which cannot be gauged away. Therefore, the equations of type (6.24) are dynamically empty unless $q = 0$. In the latter case the dimension of the solution space is that of W_0^{A} at a point, i.e. c^{A}.

In what follows we will never meet contractible cycles as parts of the unfolded equations. Clearly their presence is redundant or at least unnecessary. Being found in some unfolded system, such contractible subsystems can always be removed. By contrast, type-II for $q = 0$ and with d extended to the covariant derivative in some module will be shown to be very important since they carry all degrees of freedom.

Cocyles and couplings. Here we consider a more and the most general types of unfolded equations that can appear, while somewhat technical discussion is left to extra Section 12.4. At the free level all equations are linear in matter fields described by some $W_p^{\mathcal{A}}$, W_q^{A}, etc, but they can be nonlinear in the background fields, i.e connection Ω,

$$D_\Omega W_p^{\mathcal{A}} \equiv dW_p^{\mathcal{A}} + f_I{}^{\mathcal{A}}{}_{\mathcal{B}} \Omega^I \wedge W_p^{\mathcal{B}} = f_{I_1 \ldots I_k}{}^{\mathcal{A}}{}_{\mathsf{B}} \Omega^{I_1} \wedge \ldots \Omega^{I_k} \wedge W_q^{\mathsf{B}}\,, \tag{6.25}$$

$$D_\Omega W_q^{\mathsf{A}} = \ldots \tag{6.26}$$

As it is explained in extra Section 12.4, $f_{I_1...I_k}{}^{\mathcal{A}}{}_{\mathsf{B}}$ is a Chevalley-Eilenberg cocycle of the underlying Lie algebra, $\mathfrak{g}$, where Ω takes values in. The cocycle takes values $\mathcal{R}_1 \otimes \mathcal{R}_2^*$, where $\mathfrak{g}$-modules $\mathcal{R}_1$ and $\mathcal{R}_2$ are associated with $W_{\boldsymbol{p}}^{\mathcal{A}}$ and $W_{\boldsymbol{q}}^{\mathsf{A}}$, respectively.

Unfolded equations that have something to do with interactions have to be nonlinear. Assuming that there is a well-defined linearization, i.e. nonlinearities can be neglected in some limit, we should get the structures of the form (6.25) at the free level. This equips us with a bunch if $\mathfrak{g}$-modules, $\mathcal{R}_1$, $\mathcal{R}_2$, ... and Chevalley-Eilenberg cocyles. Nonlinear deformations of (6.25) can have the following form

$$D_\Omega W_{\boldsymbol{p}}^{\mathcal{A}} \equiv dW_{\boldsymbol{p}}^{\mathcal{A}} + f_I{}^{\mathcal{A}}{}_{\mathcal{B}} \, \Omega^I \wedge W_{\boldsymbol{p}}^{\mathcal{B}} = f_{I_1...I_k}{}^{\mathcal{A}}{}_{\mathsf{B}_1...\mathsf{B}_m} \, \Omega^{I_1} \wedge ... \Omega^{I_k} \wedge W_{\boldsymbol{q_1}}^{\mathsf{B}_1} \wedge ... \wedge W_{\boldsymbol{q_m}}^{\mathsf{B}_m} \,. \quad (6.27)$$

Again, $f_{I_1...I_k}{}^{\mathcal{A}}{}_{\mathsf{B}_1...\mathsf{B}_m}$ is a Chevalley-Eilenberg cocycle. The gauge transformations can also be written down.

Zero-forms and degrees of freedom. The sector of zero-forms is a distinguished one. First of all, the most general linear equations in the sector of zero-forms, say C^{A}, have the form of the covariant constancy condition

$$dC^{\mathsf{A}} + f_{I\mathsf{B}}^{\mathsf{A}} \, \Omega^I \wedge C^{\mathsf{B}} = 0 \qquad (6.28)$$

and no sources on the *r.h.s.* are allowed by the form-degree argument. Indeed, the sources must be at most and at least linear in Ω, which means that the sources are zero-forms, so we can move them on the *l.h.s.* Therefore, zero-forms are initial objects in a sense that they can source forms of higher degrees, but nothing can source zero-forms.

The most important property of zero-forms is that they are moduli of local solutions. Locally, by Poincare Lemma one can always solve $d\omega = 0$ as $\omega = d\xi$ unless ω is a zero-form. For zero-forms $dC^{\mathsf{A}} = 0$ has the solution space, $C^{\mathsf{A}}(x) = c^{\mathsf{A}} = const$, that is isomorphic to the space where $C^{\mathsf{A}}(x)$ takes values in, say $\mathcal{R}$. The analogous statement on the size of the solution space can be made about the covariant analog $D_\Omega C^{\mathsf{A}} = 0$ of $dC^{\mathsf{A}} = 0$. The covariant generalization of Poincare Lemma implies that all fields that are forms of non-zero degree can be represented locally as pure gauge unless they are sourced by zero-forms, in the later case they consist of two pieces, pure gauge and another one determined by zero-forms. Zero-forms are not pure gauge. There are no gauge parameters at all associated with zero-forms. Therefore, locally a solution of any unfolded system is determined by the values of all zero-forms at a point, i.e. $\mathcal{R}$, up to a gauge transformation.

That is why we find an infinitely many zero-forms when unfolding fields of different spins. Fields carry infinitely many 'degrees of freedom' in a sense that the solutions of field equations are parameterized by functions on Cauchy surface (usually it is the number of functions referred to as the number of physical degrees of freedom) and these functions are equivalent to infinitely many constants parameterized by the zero-forms at a point, c^{A}.

Combining together the fact that locally solution is reconstructed from values of zero-forms at a point and the general property of unfolded equations that zero-forms take values in certain representation, say $\mathcal{R}$, of the underlying Lie algebra, we can conclude that $\mathcal{R}$ is related to the space of one-particle states, [102, 103]. Remember that the solution space of field equations (one particle states) must carry a unitary irreducible representation, say

60

$\mathcal{R}'$ of the space-time symmetry algebra. Since the solution is parameterized by zero-forms, it is actually parameterized by $\mathcal{R}$. So $\mathcal{R}$ and $\mathcal{R}'$ must be tightly related. In fact, they are equivalent when complexified.

Let us finish with the following illustration of the unfolded approach. Up to the issues related to gauge symmetries and the fact that unfolded equations are diffeomorphism invariant and could be nonlinear the idea is to identify those derivatives of original dynamical fields that remain non-zero on-shell. Imposing equations of motions, $Eq(\phi) = 0$ sets some of the derivatives $\partial^k \phi$ to zero, the rest is then parameterized by zero-forms. Roughly speaking, at every order $\partial^k \phi$ there remains just one component, but k runs from zero to infinity.

Figure 2: *The idea of the unfolded approach.*

$$Eq(\phi) = 0,\ \delta\phi = \partial\xi$$

degrees of freedom, i.e. zero-forms

derivatives

$\partial^k \phi$

ϕ
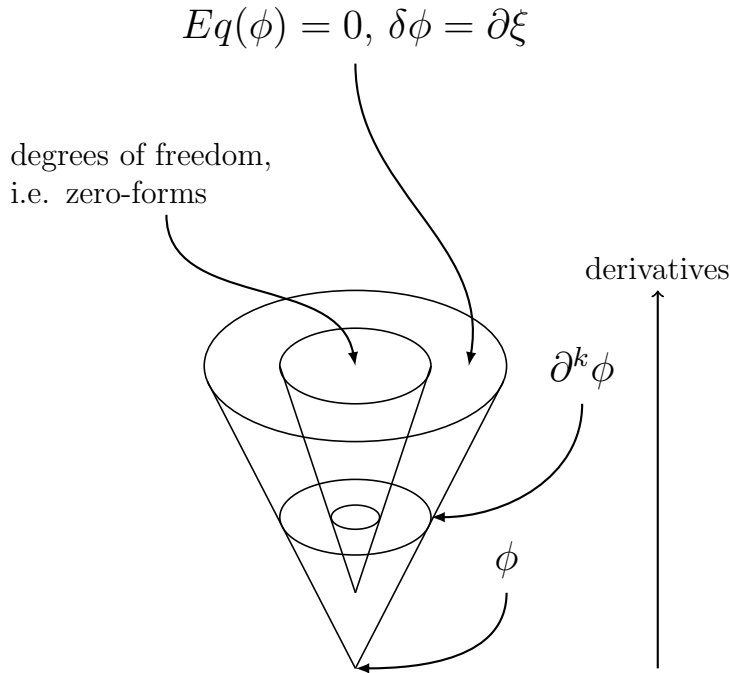
Unfolded systems of HS type. In Section 5.6 we found that the linearized unfolded equations that describe fields of all spins read schematically as

$$d\Omega^I + f^I_{JK}\Omega^J \wedge \Omega^K = 0\,, \tag{6.29}$$

$$d\omega^{\mathcal{A}} + f^{\mathcal{A}}_{J\mathcal{B}}\,\Omega^J \wedge \omega^{\mathcal{B}} = f^{\mathcal{A}}_{JK|\mathcal{B}}\,\Omega^J \wedge \Omega^K \wedge C^{\mathcal{B}}\,, \tag{6.30}$$

$$dC^{\mathcal{A}} + \bar{f}^{\mathcal{A}}_{I\mathcal{B}}\,\Omega^I \wedge C^{\mathcal{B}} = 0\,, \tag{6.31}$$

where $\Omega^I = \{h^a, \varpi^{a,b}\}$ is the Poincare connection. Recall, see Section 5.6, that the space of one-forms $\omega^{\mathcal{A}}$, when summed over all spins, is isomorphic to that of zero-forms. They cover irreducible Lorentz representations with the symmetry of all two-row Young diagrams, each appearing once. That is why we used the same index for $\omega^{\mathcal{A}}$ and $C^{\mathcal{A}}$. The cocycle on the *r.h.s.* of (6.30) was found to have the form

$$h_m \wedge h_n C^{a(s-1)m,b(s-1)n}\,. \tag{6.32}$$

The problem of interactions can be reduced to seeking for the nonlinear terms such that the integrability is preserved. The form degree argument shows that nonlinearities can be of two types: $f^{\mathcal{A}}_{\mathcal{BC}} \, \omega^{\mathcal{B}} \wedge \omega^{\mathcal{C}}$ on the *l.h.s.* of (6.30) and zero-forms can source *r.h.s.* of all equations by the terms of the form $\omega \wedge \omega CC...C$, (6.30), and $\omega CC...C$, (6.31).

Already at this point we can see that there must be an infinite-dimensional Lie algebra, $\mathfrak{g}$, of which $\omega^{\mathcal{A}}$ are gauge fields. Recall that Ω^I is a subset of $\omega^{\mathcal{A}}$ associated with the gravity sector. At the linearized level we cannot fully see this algebra. We found a part of structure constants. Namely, f^I_{JK}, which define the symmetry algebra, $\mathfrak{h}$, of the background space and $f^{\mathcal{A}}_{J\mathcal{C}}$, which show $\mathfrak{g}$ as a module under its subalgebra $\mathfrak{h}$, but not all $f^{\mathcal{A}}_{\mathcal{BC}}$. Therefore, the first question is the existence of such an algebra and secondly whether nonlinear deformations of equations are possible or not. To solve the problem within the perturbation theory one has to switch on the cosmological constant, which replaces $\mathfrak{h} = iso(d-1,1)$ with $so(d-1,2)$ or $so(d,1)$. Then all questions above can be answered in affirmative and the solution is given by the Vasiliev equations. It is important that the spectrum of the fields we found does not change when the cosmological constant is turned on. Like in the Fronsdal theory, there are some corrections proportional to Λ. Schematically the solution reads

$$d\omega^{\mathcal{A}} + f^{\mathcal{A}}_{\mathcal{BC}} \, \omega^{\mathcal{B}} \wedge \omega^{\mathcal{C}} = f^{\mathcal{A}}_{\mathcal{BC}|\mathcal{D}} \, \omega^{\mathcal{B}} \wedge \omega^{\mathcal{C}} \wedge C^{\mathcal{D}} + f^{\mathcal{A}}_{\mathcal{BC}|\mathcal{DE}} \, \omega^{\mathcal{B}} \wedge \omega^{\mathcal{C}} \wedge C^{\mathcal{D}} \wedge C^{\mathcal{E}} + ... \, , \qquad (6.33)$$

$$dC^{\mathcal{A}} + \bar{f}^{\mathcal{A}}_{\mathcal{BC}} \, \omega^{\mathcal{B}} \wedge C^{\mathcal{C}} = \bar{f}^{\mathcal{A}}_{\mathcal{B}|\mathcal{CD}} \, \omega^{\mathcal{B}} \wedge C^{\mathcal{C}} \wedge C^{\mathcal{D}} + ... \, , \qquad (6.34)$$

where we stress that zero-forms and one-forms take values in the same linear space, whose basis is given by irreducible Lorentz tensors with the symmetries of all two-row Young diagrams, each in one copy.

As was noticed in [41], the unfolded equations for higher-spin fields have the following structure

$$d\omega^{\mathcal{A}} = F^{\mathcal{A}}(\omega, C) \, , \qquad (6.35)$$

$$dC^{\mathcal{A}} = C^{\mathcal{B}} \frac{\partial}{\partial \omega^{\mathcal{B}}} F^{\mathcal{A}}(\omega, C) \, , \qquad (6.36)$$

i.e. the equations for zero-forms are built from the same structure function $F^{\mathcal{A}}(\omega, C)$ as the equations for one-forms. Formally, we applied $C^{\mathcal{B}} \frac{\partial}{\partial \omega^{\mathcal{B}}}$ to the first equation in order to get the second one. For example, this operation transforms $d\omega + \frac{1}{2}[\omega, \omega]$ into[36] $dC + [\omega, C]$. So, there is no doubling of structure constants, (6.34) is fully determined by (6.33). Let us note that (6.36) is nevertheless not an integrability consequence of (6.35).

Summary. We found that the structure constants of any unfolded system of equations have certain interpretation in terms of Lie theory. There is some underlying Lie algebra, which is related to the subsector of one-forms. The rest of the fields take values in certain modules of this Lie algebra. Couplings between different sectors of forms are given by certain representatives of the Chevalley-Eilenberg cohomology groups. Locally, solution to any unfolded system is reconstructed up to a gauge transformation from knowing the values of all zero-forms at some point.

---

[36]There is a technical detail related to fact that C takes values in the twisted-adjoint representation, which will become clear around (10.47) and it does not violate the statement that the equations for C are determined by those for ω.

7 Higher-spin theory in four dimensions

In the remaining we will consider the application of unfolded approach to four dimensional HS theory. Remarkable progress has been achieved in lower dimensions $d = 3$ and $d = 4$ due to effective twistor-like language available in these dimensions. It turns out to be perfectly suited for dealing with HS algebras and eventually brings the one to nonlinear HS equations. We restrict ourselves to the case of four dimensions, for $d = 4$ unlike $d = 3$ case admits propagation of HS fields. To proceed in this direction we need to reformulate our settings in terms of spinor language.

8 Vector-spinor dictionary

In dealing with irreducible tensors we have to preserve Young symmetry properties and the tracelessness. The advantage of the spinorial language as compared to the tensor one is that the trace and Young symmetry constraints get automatically resolved. Unfortunately, it is available in specific dimensions only. It is the condition for traces to vanish that leads to complicated trace projectors, the Young symmetry is not for free though too. We have not faced any trace projectors yet since we developed a theory in Minkowski space. The unfolded equations for higher-spin fields in AdS get modified by more complicated terms which are not seen when $\Lambda = 0$, e.g.

$$
\begin{aligned}
DC^{a(k),b(l)} = ... + \Lambda \Bigg(& h^a C^{a(k),b(l)} - \frac{k}{(d + 2k - 2)} h_m C^{a(k-1)m,b(l)} \eta^{aa} - \\
& - \frac{l}{d + k + l - 3} h_m C^{a(k),mb(l-1)} \eta^{ab} + \\
& + \frac{k}{l(d + 2k - 2)(d + k + l - 3)} h_m C^{a(k-1)b,b(l-1)m} \eta^{aa} \Bigg),
\end{aligned}
$$

(8.1)

getting more and more involved at each level of algebraic manipulations. The unfolded equations for the complete multiplet of higher-spin fields have a simpler form, but still the d-dimensional higher-spin theory, [39], is way more complicated than the $4d$ one.

The only condition that a spin-tensor has to obey is the symmetry condition — it must be symmetric in (each sort of) indices it has. To have all indices symmetrized is much simpler than to preserve a more general Young symmetry types. But the major benefit of using spin-tensors rather than tensors is that the corresponding tensors, which can be recovered by contracting all spinor indices with γ-matrices, are automatically traceless. The latter property is of great use for practical calculations. Being available for free in spinorial language it is not granted and causes a real headache in the language of tensors which is on top of having the definite Young symmetry type. There are deeper reasons of course as to why spinors are so effective, we just mention those from practical standpoint.

In the next section we establish the dictionary, which is a little bit boring, with the summary table put at the end.

8.1 $so(3,1) \sim sl(2,\mathbb{C})$

The idea behind two-component representation for $so(3,1)$ Lorentz fields is the identity $4 = 2 \times 2$. Particularly, any Lorentz vector v^a, $a = 0, ..., 3$ can be encoded by two-by-two matrix

$$v^a \leftrightarrow \begin{pmatrix} m & n \\ p & q \end{pmatrix}, \tag{8.2}$$

The convenient basis for 2×2-matrices is given by $\sigma^a_{\alpha\dot\beta} = (I, \sigma^i)$, where I - is the unit-matrix and $\sigma^i_{\alpha\dot\beta}$ — Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{8.3}$$

of these we know, that

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}I, \qquad \mathrm{Tr}\,\sigma_i = 0. \tag{8.4}$$

The corresponding v^a is then a hermitian matrix, $v = v^\dagger$,

$$v^a \leftrightarrow \begin{pmatrix} v^0 + v^3 & v^1 - iv^2 \\ v^1 + iv^2 & v^0 - v^3 \end{pmatrix}. \tag{8.5}$$

Let us choose Minkowski metric in the form $\eta_{ab} = (- + ++)$ and introduce dual set to σ-matrices $\bar\sigma^{a\dot\alpha\beta} = (I, -\sigma^i)$, then for any v^a we can define

$$v_{\alpha\dot\beta} = v^m \sigma_{m\alpha\dot\beta}, \tag{8.6}$$

such that

$$v^m = -\frac{1}{2} v_{\alpha\dot\beta} \bar\sigma^{m\dot\beta\alpha}. \tag{8.7}$$

Indeed, $-\frac{1}{2} v_{\alpha\dot\beta} \bar\sigma^{a\dot\beta\alpha} = -\frac{1}{2} v^b \sigma_{b\alpha\dot\beta} \bar\sigma^{a\dot\beta\alpha} = -\frac{1}{2} v^b \mathrm{Tr}\left(\sigma_b \bar\sigma^a\right) = -\frac{1}{4} v^b \mathrm{Tr}\{\sigma_b, \bar\sigma^a\} = v^a$. It is convenient then for each type of indices dotted and undotted to introduce antisymmetric matrices

$$\epsilon^{\alpha\beta} = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon^{\dot\alpha\dot\beta}, \tag{8.8}$$

with its inverse $-\epsilon_{\alpha\beta} = (i\sigma^2)^{-1} = -i\sigma^2$, $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$, $\epsilon_{\alpha\gamma}\epsilon^{\beta\gamma} = \delta_\alpha{}^\beta$ and define symplectic $sp(2)$-form for raising and lowering indices[37]

$$A_\alpha = A^\gamma \epsilon_{\gamma\alpha}, \qquad A^\alpha = \epsilon^{\alpha\beta} A_\beta. \tag{8.9}$$

note the position of indices, which, unlike the case of a symmetric metric, can be changed at the price of an additional sign factor[38].

---

[37]The appearance of the symplectic structure is not accidental since $sl(2) \sim sp(2)$ thanks to the following identity valid for 2×2 matrices, $A^T \epsilon A = \det A\,\epsilon$, which implies that an $SL(2)$ matrix A, $\det A = 1$, is at the same a symplectic matrix $A^T \epsilon A = \epsilon$.

[38]There are different conventions on how to work with antisymmetric metric tensor. Our convention may seem unusual at first sight, but at the end it is the simplest one. For example, raising and then lowering an index one gets back to the original expression. A convention different from ours produces a sign factor, which to us is unnatural and requires much care in computations. We are free to change the position of uncontracted indices using our convention. More detail on symplectic calculus can be found in Appendix F.

Analogously for $\epsilon_{\dot{\alpha}\dot{\beta}}$. Note, that $A_\alpha B^\alpha = -A^\alpha B_\alpha$. Using the above definitions one finds, that

$$\sigma^m_{\alpha\dot{\alpha}} = \epsilon_{\beta\alpha}\epsilon_{\dot{\beta}\dot{\alpha}}\bar{\sigma}^{m\dot{\beta}\beta}\,, \qquad \bar{\sigma}^{\dot{\alpha}\alpha}_m = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma_{m\beta\dot{\beta}}\,, \tag{8.10}$$

allowing us to identify

$$\sigma^{\alpha\dot{\alpha}}_m \equiv \bar{\sigma}^{\dot{\alpha}\alpha}_m\,. \tag{8.11}$$

Matrices $\sigma_{m\alpha\dot{\alpha}}$ with indices being raised and lowered by $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$ are called Van der Waerden symbols. Two-component spinors A_α and $A^{\dot{\alpha}}$ are called chiral and anti-chiral, correspondingly. Let us note that one should be careful with numerical factors in identifying scalar combinations, e.g.

$$v_{\alpha\dot{\alpha}}v^{\alpha\dot{\alpha}} = -2v_a v^a \tag{8.12}$$

To proceed we need a simple lemma. Any bi-spinor $A_{\alpha\beta}$ can be decomposed as

$$A_{\alpha\beta} = A_{(\alpha\beta)} + \frac{1}{2}\epsilon_{\alpha\beta}A_\gamma{}^\gamma\,. \tag{8.13}$$

The proof is straightforward, decomposing $A_{\alpha\beta} = A_{(\alpha\beta)} + A_{[\alpha\beta]}$ and noting, that any antisymmetric 2×2-matrix $A_{[\alpha\beta]} \sim \epsilon_{\alpha\beta}$ one gets (8.13). Using it we can further prove the following relation for σ-matrices

$$\sigma^{m\dot{\alpha}}_\alpha\sigma^n_{\beta\dot{\alpha}} = \eta^{mn}\epsilon_{\alpha\beta} + (\sigma^{mn})_{\alpha\beta}\,, \qquad (\sigma^{mn})_{\alpha\beta} = -(\sigma^{nm})_{\alpha\beta} = (\sigma^{mn})_{\beta\alpha} = \frac{1}{2}[\sigma^m, \sigma^n]_{\alpha\beta}\,. \tag{8.14}$$

Indeed, consider first symmetric part (mn) of $l.h.s.$

$$\sigma^{m\dot{\alpha}}_\alpha\sigma^m_{\beta\dot{\alpha}} = -\sigma^{m\dot{\alpha}}_\beta\sigma^m_{\alpha\dot{\alpha}} = \frac{1}{2}\epsilon_{\alpha\beta}\sigma^{m\dot{\alpha}}_\gamma\sigma^{m\gamma}{}_{\dot{\alpha}} = \eta^{mm}\epsilon_{\alpha\beta}\,. \tag{8.15}$$

For antisymmetric part $[ab]$ one obtains the definition of the r.h.s. The great advantage of the $4d$ spinor formalism is that it allows one to handle easily traceless Young diagrams. Particularly, let us state an important fact which will be very useful in what follows. Given a traceless Lorentz tensor $C_{a_1b_1,a_2b_2,\ldots,a_sb_s}$ corresponding to a rectangular two-row Young diagram, i.e. it is antisymmetric with respect to each pair of indices $C_{..,ab,..} = -C_{..,ba,..}$, it is traceless with respect to any indices and it satisfies Young condition — antisymmetrization of any three gives zero $C_{..,[ab,c],..} = 0$, then its spinor counterpart reads

$$C_{a_1b_1,a_2b_2,\ldots,a_sb_s} \quad \rightarrow \quad C_{\alpha(2s)}, \bar{C}_{\dot{\alpha}(2s)}\,, \tag{8.16}$$

where the repeated indices $\alpha(2s) = \alpha_1 \ldots \alpha_{2s}$ mean total symmetrization as usual.

We will demonstrate this statement with the simplest examples. Let us start with $s = 1$, i.e. $C_{ab} = -C_{ba}$. Its spinor image by definition is $C_{\alpha\dot{\alpha},\beta\dot{\beta}} = C^{ab}\sigma_{a\alpha\dot{\alpha}}\sigma_{b\beta\dot{\beta}}$. Using (8.13), one can decompose

$$C_{\alpha\dot{\alpha},\beta\dot{\beta}} = C^{ab}\left(\frac{1}{2}\sigma_{a\alpha\dot{\alpha}}\sigma_{b\beta\dot{\beta}} + \frac{1}{2}\sigma_{a\beta\dot{\alpha}}\sigma_{b\alpha\dot{\beta}} + \frac{1}{2}\epsilon_{\alpha\beta}\sigma_{a\gamma\dot{\alpha}}\sigma_b{}^\gamma{}_{\dot{\beta}}\right)\,. \tag{8.17}$$

Using now ab-antisymmetry we finally obtain

$$C_{\alpha\dot{\alpha},\beta\dot{\beta}} = \epsilon_{\alpha\beta}\bar{C}_{\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}}C_{\alpha\beta}\,, \tag{8.18}$$

65

where

$$C_{\alpha\beta} = C_{\beta\alpha} = \frac{1}{2}\sigma_{a\alpha\dot\gamma}\sigma_{b\beta}{}^{\dot\gamma}C^{ab}, \qquad \bar{C}_{\dot\alpha\dot\beta} = \bar{C}_{\dot\beta\dot\alpha} = \frac{1}{2}\sigma_{a\gamma\dot\alpha}\sigma_{b}{}^{\gamma}{}_{\dot\beta}C^{ab}. \qquad (8.19)$$

It is crucial that $C_{\alpha\beta}$ is a complex conjugate of $\bar{C}_{\dot\alpha\dot\beta}$. A symmetric rank-two spin-tensor has 3 complex components, i.e. 6 real, which is exactly the number of components of the Maxwell tensor $F_{mn} = -F_{nm}$. If $C_{\alpha\beta}$ and $\bar{C}_{\dot\alpha\dot\beta}$ were not conjugated to each other, the corresponding F_{mn} would be complex. For the same reason, a tensor is often equivalent to a pair of spin-tensors, which are complex conjugated with the only exception for $C_{\alpha(s),\dot\alpha(s)}$ which is self-conjugated. In particular, the spinorial version of x^a, which is $x^{\alpha\dot\alpha}$, is a hermitian matrix, (8.5).

The next less trivial example is $C_{ab,cd}$ whose Young symmetries are those of gravity Weyl tensor, taken again in the antisymmetric basis. Its spinor counterpart can be shown to be

$$C_{\alpha\dot\alpha\beta\dot\beta,\gamma\dot\gamma\delta\dot\delta} = C^{ab,cd}\sigma_{a\alpha\dot\alpha}\sigma_{b\beta\dot\beta}\sigma_{c\gamma\dot\gamma}\sigma_{d\delta\dot\delta} = C_{\alpha\beta\gamma\delta}\epsilon_{\dot\alpha\dot\beta}\epsilon_{\dot\gamma\dot\delta} + \bar{C}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}, \qquad (8.20)$$

where

$$C_{\alpha(4)} = \frac{1}{4}C^{ab,cd}\sigma_{a\alpha}{}^{\dot\gamma}\sigma_{b\alpha\dot\gamma}\sigma_{c\alpha}{}^{\dot\delta}\sigma_{d\alpha\dot\delta} \qquad (8.21)$$

is a totally symmetric spin-tensor. Note, that in deriving this result not only the trace-lessness and antisymmetry of ab and cd have been used, but also the Young condition $[ab,c] = 0$. On top of that one repeatedly exploits (8.13). Analogously, one shows that

$$C_{\alpha_1\dot\alpha_1\beta_1\dot\beta_1,\dots,\alpha_s\dot\alpha_s\beta_s\dot\beta_s} = C_{\alpha(s)\beta(s)}\epsilon_{\dot\alpha_1\dot\beta_1}\dots\epsilon_{\dot\alpha_s\dot\beta_s} + c.c., \qquad (8.22)$$

$$C_{\alpha(2s)} = \frac{1}{2^s}C^{a_1b_1,a_2b_2,\dots,a_sb_s}\sigma_{a_1\alpha}{}^{\dot\gamma_1}\sigma_{b_1\alpha\dot\gamma_1}\dots\sigma_{a_s\alpha}{}^{\dot\gamma_s}\sigma_{b_s\alpha\dot\gamma_s}. \qquad (8.23)$$

Finally, let us conclude this section with spinor representation for Minkowski metric η_{ab}

$$\eta_{\alpha\dot\alpha,\beta\dot\beta} = \eta_{ab}\sigma^a_{\alpha\dot\alpha}\sigma^b_{\beta\dot\beta} = 2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} \qquad (8.24)$$

8.2 Dictionary

The dictionary between tensors of the $4d$ Lorentz algebra $so(3,1)$ and spin-tensors of $sl(2,\mathbb{C})$ is summarized in the table below.

If we sum over all spins $s = 0,1,2,3,\dots$ then we see that the set of one-forms is isomorphic to the set of zero-forms in a sense that they cover the same space of tensors in the fiber space — all two-row Young diagrams, each appearing once, see Section 5.6. The main statement of this section is that the same space is isomorphic to the space of all spin-tensors of even ranks, each appearing once, i.e.

$$\sum_{s>k}\omega^{a(s-1),b(k)} \sim \sum_{s,i}C^{a(s+i),b(s)} \sim \sum_{k+m=\text{even}}\omega^{\alpha(k),\dot\alpha(m)} \sim \sum_{k+m=\text{even}}C^{\alpha(k),\dot\alpha(m)}. \qquad (8.25)$$

Allowing for spin-tensors of odd ranks brings fermions. The same space can be described as the space of all functions of two auxiliary two-component variables, y_α, $\bar{y}_{\dot\alpha}$, since the Taylor coefficients cover the same range of spin-tensors,

$$f(y,\bar{y}) = \sum_{k,m}\frac{1}{k!m!}f^{\alpha(k),\dot\alpha(m)}y_\alpha\dots y_\alpha\,\bar{y}_{\dot\alpha}\dots\bar{y}_{\dot\alpha}. \qquad (8.26)$$

$so(3,1)$ tensor	$sl(2,\mathbb{C})$ tensor	dimension
•	•	1
Dirac spinor	$T^\alpha \oplus T^{\dot\alpha}$	4
$T^a \sim \square$	$T^{\alpha\dot\alpha}$	4
$T^{a,b} \sim \begin{smallmatrix}\square\\\square\end{smallmatrix}$	$T^{\alpha\alpha} \oplus \bar{T}^{\dot\alpha\dot\alpha}$	6
$T^{aa} \sim \square\square$	$T^{\alpha\alpha,\dot\alpha\dot\alpha}$	10
$T^{a(k)} \sim \boxed{k}$	$T^{\alpha(k),\dot\alpha(k)}$	$(k+1)^2$
$T^{a(k),b(k)} \sim \boxed{k}$	$T^{\alpha(2k)} \oplus \bar{T}^{\dot\alpha(2k)}$	$2(2k+1)$
$T^{a(k),b(m)} \sim \boxed{\begin{smallmatrix}k\\m\end{smallmatrix}}$	$T^{\alpha(k+m),\dot\alpha(k-m)} \oplus \bar{T}^{\alpha(k-m),\dot\alpha(k+m)}$	$2(k+m+1)(k-m+1)$

The main advantage of using spin-tensor generating function is that $f(y,\bar{y})$ is an unconstrained function (with fermions) or obeys a simple constraint $f(-y,-\bar{y}) = f(y,\bar{y})$ (bosons only), while to describe the same space in terms of vector-like variables we need a number of differential constraints, see Section 5.6 and Appendix E.3, which are difficult to deal with, especially in interactions.

There is an alternative way to describe the same space. One can take just one auxiliary variable Y_A with the index A running over 4 values. We can think of it as a composite index $A = \{\alpha, \dot\alpha\}$. Then

$$f(Y) = \sum_s \frac{1}{s!} f^{A(s)} Y_A ... Y_A = \sum_{k,m} \frac{1}{k!m!} f^{\alpha(k),\dot\alpha(m)} y_\alpha ... y_\alpha \, \bar{y}_{\dot\alpha} ... \bar{y}_{\dot\alpha} \qquad (8.27)$$

This fact has again a group-theoretical meaning, $so(3,2) \sim sp(4,\mathbb{R})$, which is useful to remember of. It is reviewed in Appendix G together with the branching rules for $so(3,2) \downarrow so(3,1)$. The bosonic projection $f(-y,-\bar{y}) = f(y,\bar{y})$ means $f(Y) = f(-Y)$.

9 Free HS fields in AdS_4

There is no conceptual problem to extend the unfolded equations for a field of any spin to AdS_d. However, the unfolded equations for the individual fields are more complicated than for all spins gathered into an infinite multiplet of the higher-spin algebra. Thanks to the effective spinorial language the unfolded equations in AdS_d with $d = 3, 4, 5$ are simpler. We will focus on the higher-spin theory in AdS_4. There are no propagating higher-spin fields in $d < 4$, so the case of AdS_4 is the first nontrivial one and historically the first example of higher-spin theory.

A good warm up starting point on the way is to consider Klein-Gordon equation in Minkowski space-time, i.e. the spinorial version of (5.66), and then generalize the elaborated machinery to AdS_4 and to any spin. A content of this section should be

compared with Section 5 as we simply convert our d-dimensional findings into spinorial form for $d = 4$ and then extend the equations to AdS_4, which is a minor modification in the spinorial notation but a challenge in the vector one. Fortunately, one does not need to apply σ-matrices to every equation in Section 5 reaching an equivalent result entirely in the language of spin-tensors.

9.1   Massless scalar and HS on Minkowski

We are interested in obtaining first order differential equations of motion for a free massless scalar that will be equivalent to

$$\Box C = 0, \qquad \Box = \partial_m \partial^m \tag{9.1}$$

The equations that we are looking for should be formulated in terms of differential forms. The spinorial avatar of coordinate x^m is a bispinor $x^{\alpha\dot\alpha}$. So, let us introduce differential $d = dx^m \frac{\partial}{\partial x^m} = dx^{\alpha\dot\alpha} \frac{\partial}{\partial x^{\alpha\dot\alpha}}$. The most general equation that one can write down for dC has the form

$$dC = C_{\alpha\dot\alpha} h^{\alpha\dot\alpha} \qquad \leftrightarrow \qquad \partial_{\alpha\dot\alpha} C = C_{\alpha\dot\alpha}, \tag{9.2}$$

where $h_{\alpha\dot\alpha} = dx_{\alpha\dot\alpha}$ is the vielbein in Cartesian coordinates in Minkowski space-time. Formally, $h_m^{\alpha\dot\alpha} = \sigma_m^{\alpha\dot\alpha}$ or $h_{\beta\dot\beta}^{\alpha\dot\alpha} = \epsilon_\alpha{}^\beta \epsilon_{\dot\alpha}{}^{\dot\beta}$. Field $C_{\alpha\dot\alpha}$ is some arbitrary field, the spinor version of C^a. Proceeding further, one writes down the most general equation for $dC_{\alpha\dot\alpha}$, c.f. (5.58),

$$dC_{\alpha\dot\alpha} = C_{\alpha\dot\alpha|\beta\dot\beta} h^{\beta\dot\beta} \qquad \leftrightarrow \qquad \partial_{\beta\dot\beta} C_{\alpha\dot\alpha} = C_{\alpha\dot\alpha|\beta\dot\beta}, \tag{9.3}$$

where we once again have introduced a new field $C_{\alpha\dot\alpha|\beta\dot\beta}$, which can be decomposed into traceless and traceful parts, c.f. (5.57),

$$C_{\alpha\dot\alpha|\beta\dot\beta} = C_{\alpha\beta,\dot\alpha\dot\beta} + T_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} + \bar{T}_{\dot\alpha\dot\beta}\epsilon_{\alpha\beta} + T\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}. \tag{9.4}$$

On the other hand, from (9.2), (11.147) one has $C_{\alpha\dot\alpha|\beta\dot\beta} = \partial_{\beta\dot\beta}\partial_{\alpha\dot\alpha} C$. Using then (9.1) one finds that $T_{\alpha\beta} = T_{\dot\alpha\dot\beta} = T = 0$ and so, c.f. (5.59),

$$dC_{\alpha\dot\alpha} = C_{\alpha\beta,\dot\alpha\dot\beta} h^{\beta\dot\beta} \tag{9.5}$$

Keep going in similar fashion we have the decomposition

$$dC_{\alpha(2),\dot\alpha(2)} = C_{\alpha(2),\dot\alpha(2)|\beta\dot\beta} h^{\beta\dot\beta} = \left(C_{\alpha(2),\dot\alpha(2)\dot\beta} + \epsilon_{\alpha\beta} T_{\alpha,\dot\alpha(2)\dot\beta} + \epsilon_{\dot\alpha\dot\beta} T_{\alpha(2)\beta,\dot\beta} + \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} T_{\alpha,\dot\alpha} \right) h^{\beta\dot\beta}.$$

Observing that $\partial_{\beta\dot\beta} C_{\alpha(2),\dot\alpha(2)} = \partial_{\beta\dot\beta}\partial_{\alpha\dot\alpha} C_{\alpha\dot\alpha}$ and using $[\partial_{\alpha\dot\alpha}, \partial_{\beta\dot\beta}] = 0$ we immediately find that $T_{\alpha,\dot\alpha(3)} = T_{\alpha(3),\dot\alpha} = T_{\alpha,\dot\alpha} = 0$ and so,

$$dC_{\alpha(2),\dot\alpha(2)} = C_{\alpha(3)\beta,\dot\alpha(2)\dot\beta} h^{\beta\dot\beta}. \tag{9.6}$$

The chain of the obtained equations is infinite and at each level n reads

$$dC_{\alpha(n),\dot\alpha(n)} = C_{\alpha(n)\beta,\dot\alpha(n)\dot\beta} h^{\beta\dot\beta}, \qquad n \geq 0. \tag{9.7}$$

68

Its dynamical content is equivalent to that of Klein-Gordon equation for the lowest component $n = 0$, $C(x)$ expressing the higher level fields via derivatives of the physical field C, c.f. (5.66),

$$\Box C(x) = 0\,, \qquad C_{\alpha(n),\dot\alpha(n)} = \underbrace{\partial_{\alpha\dot\alpha}\dots\partial_{\alpha\dot\alpha}}_{n} C(x) \tag{9.8}$$

It is said, that fields $C_{\alpha(n),\dot\alpha(n)}$ for $n \geq 0$ constitute a scalar module. It can be shown that the obtained set of fields that describe a scalar field is the same for any space-time background. The dynamical equations for general background of course will be different, yet their form is known only for $(A)dS_d$ and Minkowski, [104].

It is convenient to pack the obtained scalar module into a generating function

$$C(y,\bar y|x) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} C_{\alpha(n),\dot\alpha(n)}(y^\alpha)^n (\bar y^{\dot\alpha})^n\,, \tag{9.9}$$

this leads to the following equation[39], c.f. (5.68),

$$dC(y,\bar y|x) - h^{\alpha\dot\alpha}\frac{\partial^2}{\partial y^\alpha \partial \bar y^{\dot\alpha}}C(y,\bar y|x) = 0\,, \tag{9.10}$$

which by construction is consistent with integrability condition $d^2 = 0$. Indeed, we need to check that

$$0 \equiv ddC(y,\bar y|x) = h^{\alpha\dot\alpha}\frac{\partial^2}{\partial y^\alpha \partial \bar y^{\dot\alpha}} \wedge h^{\beta\dot\beta}\frac{\partial^2}{\partial y^\beta \partial \bar y^{\dot\beta}}C(y,\bar y|x) \tag{9.11}$$

The latter is obvious with the help of the following identity

$$h^{\alpha\dot\alpha} \wedge h^{\beta\dot\beta} \equiv \frac{1}{2}H^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta} + \frac{1}{2}H^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta} \tag{9.12}$$

where $H^{\alpha\beta} = H^{\beta\alpha} = h^\alpha{}_{\dot\gamma} \wedge h^{\beta\dot\gamma}$, idem. for $H^{\dot\alpha\dot\beta}$, and the fact that $\xi_\alpha\xi_\beta\epsilon^{\alpha\beta} \equiv 0$ for any two-component commuting object ξ_α, even ∂_α.

The consistency does not require specific for a scalar module grading of generating function

$$\left(y^\alpha\frac{\partial}{\partial y^\alpha} - \bar y^{\dot\alpha}\frac{\partial}{\partial y^{\dot\alpha}}\right)C(y,\bar y|x) = 0\,, \tag{9.13}$$

i.e. it does not rely on the number dotted indices being equal to that of undotted. This allows us to consider generating function of the form

$$C(y,\bar y|x) = \sum_{n,m=0}^{\infty} \frac{1}{m!n!} C_{\alpha(m),\dot\alpha(n)}(y^\alpha)^m (\bar y^{\dot\alpha})^n \tag{9.14}$$

and impose the condition (9.10) to see if it reproduces anything reasonable. It is convenient to present $C(y,\bar y)$ as $C = \sum_{2s=0}^{\infty} C_s(y,\bar y|x)$, where

$$C_s = \sum_{n=0}^{\infty} \frac{1}{(2s+n)!n!}(C_{\alpha(n+2s),\dot\alpha(n)}(y^\alpha)^{n+2s}(\bar y^{\dot\alpha})^n + \bar C_{\alpha(n),\dot\alpha(n+2s)}(y^\alpha)^n(\bar y^{\dot\alpha})^{n+2s})\,, \tag{9.15}$$

---

[39]Peculiarities of differential calculus when the metric is symplectic, i.e. antisymmetric, which is our case of $\epsilon_{\alpha\beta}$, are discussed in Appendix F.

such that

$$\left(y^\alpha \frac{\partial}{\partial y^\alpha} - \bar{y}^{\dot\alpha} \frac{\partial}{\partial \bar{y}^{\dot\alpha}}\right) C_s(y,\bar{y}|x) = \pm 2s\, C_s\,, \tag{9.16}$$

Note that the grading operator above commutes with the equations of motion, so (9.10) decomposes into independent subsystems for each s. Using the spinor-tensor dictionary derived in Section 8 and the field content used in Section 5.2 it is easy to see that (9.10) is the spinor version of the unfolded equations for the zero-forms $C^{a(s+k),b(s)}$, (5.94).

The module $C_{1/2}$ describes left and right massless $s=1/2$ fields $C_\alpha, \bar{C}_{\dot\alpha}$ according to Dirac equations

$$\partial_{\beta\dot\alpha} C^\beta = 0\,, \qquad \partial_{\alpha\dot\beta}\bar{C}^{\dot\beta} = 0\,, \tag{9.17}$$

with all of the rest fields in the module $C_{1/2}$ expressed via derivatives of the fermion matter fields C_α and $\bar{C}_{\dot\alpha}$. The $s=1$ module is equivalent to Maxwell equations and Bianchi identities for (anti)self-dual parts of the Maxwell tensor $C_{\alpha\beta} \oplus \bar{C}_{\dot\alpha\dot\beta}$

$$\partial_{\gamma\dot\beta} C^\gamma{}_\alpha = 0\,, \qquad \partial_{\beta\dot\gamma}\bar{C}_{\dot\alpha}{}^{\dot\gamma} = 0\,, \tag{9.18}$$

which are spinorial versions of (5.82) and look more symmetric than (5.82). For $s=2$ we arrive at Bianchi identities for gravity Weyl tensor

$$\partial_{\gamma\dot\beta} C^\gamma{}_{\alpha(3)} = 0\,, \qquad \partial_{\beta\dot\gamma}\bar{C}_{\dot\alpha(3)}{}^{\dot\gamma} = 0\,, \tag{9.19}$$

which are spinorial versions of (5.83). One reproduces equations for generalized HS curvatures $C_{\alpha(2s)}, \bar{C}_{\dot\alpha(2s)}$ in this fashion, c.f. (5.84),

$$\partial_{\gamma\dot\beta} C^\gamma{}_{\alpha(2s-1)} = 0\,, \qquad \partial_{\beta\dot\gamma}\bar{C}_{\dot\alpha(2s-1)}{}^{\dot\gamma} = 0\,. \tag{9.20}$$

Similar analysis can be implemented for AdS_4 space-time. We will see that it does not lead to much complication.

9.2 Spinor version of AdS_4 background

To write down spinor form of equation (3.54) we need to find spinor counterparts for $so(3,2)$ generators and connection fields. Recall that Lorentz vector x^a is represented by bi-spinor $x_{\alpha\dot\alpha}$, while antisymmetric Lorentz tensor $T_{ab} = -T_{ba}$ by a pair of symmetric $T_{\alpha\beta}$ and $\bar{T}_{\dot\alpha\dot\beta}$. Therefore, the spinorial version of AdS_4 generators L_{ab}, P_a, (3.40), are the translations $P_{\alpha\dot\beta}$ and Lorentz generators $L_{\alpha\beta}$, $\bar{L}_{\dot\alpha\dot\beta}$. The representation in terms of spinor generators gives rise explicitly to the well known isomorphism[40] $so(3,2) \sim sp(4,\mathbb{R})$. Indeed, gathering $L_{\alpha\beta}, \bar{L}_{\dot\alpha\dot\beta}, P_{\alpha\dot\beta}$ together into symmetric matrix, $A, B, ... = 1, ..., 4$,

$$T_{AB} = T_{BA} = \begin{pmatrix} L_{\alpha\beta} & P_{\alpha\dot\beta} \\ P_{\beta\dot\alpha} & \bar{L}_{\dot\alpha\dot\beta} \end{pmatrix} \tag{9.21}$$

---

[40]Understanding of $sp(4,\mathbb{R}) \sim so(3,2)$ is not required in this section as all formulae will be written down using the Lorentz covariant basis of $sl(2,\mathbb{C}) \sim so(3,1)$. The discussion of $sp(4,\mathbb{R}) \sim so(3,2)$ can be found in Appendix G.2.

and defining the invariant $sp(4, \mathbb{R})$-form as

$$\epsilon_{AB} = -\epsilon_{BA} = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot\alpha\dot\beta} \end{pmatrix}. \tag{9.22}$$

the $sp(4)$ commutation relations

$$[T_{AB}, T_{CD}] = \epsilon_{BC} T_{AD} + \epsilon_{AC} T_{BD} + \epsilon_{BD} T_{AC} + \epsilon_{AD} T_{BC} \tag{9.23}$$

when rewritten in terms of $sl(2, \mathbb{C})$ with $A = \{\alpha, \dot\alpha\}$ give[41]

$$[L_{\alpha\alpha}, L_{\beta\beta}] = \epsilon_{\alpha\beta} L_{\alpha\beta}, \qquad [\bar{L}_{\dot\alpha\dot\alpha}, \bar{L}_{\dot\beta\dot\beta}] = \epsilon_{\dot\alpha\dot\beta} L_{\dot\alpha\dot\beta}, \tag{9.24}$$

$$[L_{\alpha\alpha}, P_{\beta\dot\beta}] = \epsilon_{\alpha\beta} P_{\alpha\dot\beta}, \qquad [\bar{L}_{\dot\alpha\dot\alpha}, P_{\beta\dot\beta}] = \epsilon_{\dot\alpha\dot\beta} P_{\beta\dot\alpha}, \tag{9.25}$$

$$[P_{\alpha\dot\alpha}, P_{\beta\dot\beta}] = \lambda^2 (\epsilon_{\alpha\beta} \bar{L}_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta} L_{\alpha\beta}), \tag{9.26}$$

where for historical reason the cosmological constant Λ was replaced by λ^2. Its appearance requires rescaling of the translation generators $T_{\alpha\dot\alpha} \to \lambda P_{\alpha\dot\alpha}$. For connection fields we have $\varpi^{a,b} \to \varpi^{\alpha\beta}, \bar\varpi^{\dot\alpha\dot\beta}, h^a \to h^{\alpha\dot\alpha}$. The zero-curvature condition (3.54), specialized to $so(3,2)$ in terms of spinor connections

$$\Omega = \frac{1}{2} \omega^{\alpha\beta} L_{\alpha\beta} + h^{\alpha\dot\alpha} P_{\alpha\dot\alpha} + \frac{1}{2} \bar\omega^{\dot\alpha\dot\beta} L_{\dot\alpha\dot\beta} \tag{9.27}$$

acquire the following form

$$dh^{\alpha\dot\beta} + \varpi^\alpha{}_\gamma \wedge h^{\gamma\dot\beta} + \bar\varpi^{\dot\alpha}{}_{\dot\gamma} \wedge h^{\alpha\dot\gamma} = 0, \tag{9.28}$$

$$d\varpi^{\alpha\beta} + \varpi^\alpha{}_\gamma \wedge \varpi^{\gamma\beta} = -\lambda^2 h^\alpha{}_{\dot\gamma} \wedge h^{\beta\dot\gamma}, \tag{9.29}$$

$$d\bar\varpi^{\dot\alpha\dot\beta} + \bar\varpi^{\dot\alpha}{}_{\dot\gamma} \wedge \bar\varpi^{\dot\gamma\dot\beta} = -\lambda^2 h_\gamma{}^{\dot\alpha} \wedge h^{\gamma\dot\beta}. \tag{9.30}$$

Let us define Lorentz-covariant derivative $D = d + \varpi$ as follows

$$DA^{\alpha\dot\alpha} = dA^{\alpha\dot\alpha} + \varpi^\alpha{}_\beta \wedge A^{\beta\dot\alpha} + \bar\varpi^{\dot\alpha}{}_{\dot\beta} \wedge A^{\alpha\dot\beta}, \tag{9.31}$$

where $A^{\alpha\dot\alpha}$ is any q-form. The upshot is that each index of a spin-tensor is acted upon by the appropriate half of the spin-connection, undotted indices are rotated with $\varpi^{\alpha\beta}$, while dotted indices with $\bar\varpi^{\dot\alpha\dot\beta}$. From (9.28)-(9.30) then we have

$$D^2 A^{\alpha\dot\alpha} = -\lambda^2 H^\alpha{}_\beta \wedge A^{\beta\dot\alpha} - \lambda^2 \bar{H}^{\dot\alpha}{}_{\dot\beta} \wedge A^{\alpha\dot\beta}, \tag{9.32}$$

where we recall that $H^{\alpha\beta} = h^\alpha{}_{\dot\gamma} \wedge h^{\beta\dot\gamma}$.

Altogether we have four equivalent ways of presenting the background $4d$ anti-de Sitter geometry in terms of flat connection Ω, which has 10 components, in accordance

---

[41]Four terms, 2×2, in the first line are needed to symmetrize over the indices denoted by the same letter. Two terms are in the second line with no additional terms in the third line. Note that α is totally different from $\dot\alpha$, so that no symmetrization over $\alpha, \dot\alpha$ is possible without breaking the Lorentz symmetry.

with various choices of the base

$$so(3,1): \qquad \Omega = \frac{1}{2}\omega^{a,b}L_{ab} + h^a P_a \qquad\qquad 6+4 \qquad (9.33)$$

$$sl(2,\mathbb{C}): \qquad \Omega = \frac{1}{2}\omega^{\alpha\beta}L_{\alpha\beta} + h^{\alpha\dot\alpha}P_{\alpha\dot\alpha} + \frac{1}{2}\bar\omega^{\dot\alpha\dot\beta}L_{\dot\alpha\dot\beta} \qquad 3+4+\bar 3 \qquad (9.34)$$

$$so(3,2): \qquad \Omega = \frac{1}{2}\Omega^{\mathsf{A,B}}T_{\mathsf{AB}} \qquad\qquad 5\times 4/2 \qquad (9.35)$$

$$sp(4,\mathbb{R}): \qquad \Omega = \frac{1}{2}\Omega^{AB}T_{AB} \qquad\qquad 4\times 5/2 \qquad (9.36)$$

with the commutation relations given in (3.40), (9.24), (3.41) and (9.23) in terms of $so(3,1)$, $sl(2,\mathbb{C})$, $so(3,2)$ and $sp(4,\mathbb{R})$ bases, respectively.

Poincare coordinates are useful in applications, the spinorial counterpart of (3.57), is

$$\varpi^{\alpha\alpha} = \frac{i}{2z}d\mathrm{x}^{\alpha\alpha}, \qquad \varpi^{\dot\alpha\dot\alpha} = -\frac{i}{2z}d\mathrm{x}^{\dot\alpha\dot\alpha}, \qquad h^{\alpha\dot\alpha} = \frac{1}{2z\lambda}(-d\mathrm{x}^{\alpha\dot\alpha} + i\epsilon^{\alpha\dot\alpha}dz), \qquad (9.37)$$

where the coordinates split into the radial coordinate z and three boundary coordinates x^i packed into a symmetric real bispinor $\mathrm{x}^{\alpha\beta} = \mathrm{x}^{\beta\alpha}$. All together z and $\mathrm{x}^{\alpha\beta}$ combine into hermitian $x^{\alpha\dot\alpha} = \mathrm{x}^{\alpha\beta}\delta_\beta{}^{\dot\alpha} + i\epsilon^{\alpha\dot\alpha}z$, where $i\epsilon^{\alpha\dot\alpha}$ is one of the Pauli matrices, $-\sigma_2$. The choice of coordinates breaks manifest symmetry down to the boundary Lorentz symmetry $so(2,1)$ that rotates $\mathrm{x}^{\alpha\beta}$ and dilatations, $\mathrm{x}, z \to \gamma\mathrm{x}, \gamma z$. The vierbein $h^{\alpha\dot\alpha}$ gives the expected metric tensor

$$ds^2 = \frac{1}{z^2\lambda^2}(dz^2 + dx_i dx^i). \qquad (9.38)$$

9.3 Extension to AdS_4

First of all let us extend equations (9.10) written in Cartesian coordinates to any coordinate system, where spin-connection may not be trivial. This is done similarly to the way we did in Sections 5, 5.3. One can replace d with the Lorentz-covariant derivative $D = d + \varpi$ to obtain the following equation in Minkowski space-time

$$DC(y,\bar y|x) - h^{\alpha\dot\alpha}\frac{\partial^2}{\partial y^\alpha \partial \bar y^{\dot\alpha}}C(y,\bar y|x) = 0, \qquad (9.39)$$

where the fact that D acts on every index, (9.31), can be simulated by, c.f. (5.98),

$$D = d + \varpi^{\alpha\beta}y_\alpha\partial_\beta + \varpi^{\dot\alpha\dot\beta}y_{\dot\alpha}\partial_{\dot\beta}, \qquad (9.40)$$

where $\varpi, \bar\varpi, h$ obey (9.28)-(9.30) with $\lambda = 0$, which entails $D^2 = 0$. Our next goal is to lift (9.39) to AdS_4 where $\lambda \neq 0$ and the system has to be modified.

Technical difference in deriving equations analogous to (9.7) for a scalar in AdS_4 is that covariant derivatives no longer commute in this case $D^2 \neq 0$, (9.32). Consider the case of massive scalar on AdS_4 in some detail, i.e. we aim to rewrite in the unfolded form the Klein-Gordon equation[42]

$$D_{\alpha\dot\alpha}D^{\alpha\dot\alpha}C = -m^2 C. \qquad (9.41)$$

---

[42] The sign in front of m^2 is in accordance with the signature $(+ - --)$.

The analysis is analogous to Minkowski case, we have

$$DC = C_{\alpha\dot\alpha}h^{\alpha\dot\alpha}, \tag{9.42}$$

where $C_{\alpha\dot\alpha}$ is yet unconstrained. We can write then

$$DC_{\alpha\dot\alpha} = C_{\alpha\dot\alpha|\beta\dot\beta}h^{\beta\dot\beta} = \left(C_{\alpha\beta,\dot\alpha\dot\beta} + T_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} + \bar{T}_{\dot\alpha\dot\beta}\epsilon_{\alpha\beta} + T\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\right)h^{\beta\dot\beta}.$$

From $C_{\alpha\dot\alpha} = D_{\alpha\dot\alpha}C$ and (9.41) one finds that $T_{\alpha\beta} = \bar{T}_{\dot\alpha\dot\beta} = 0$, while $T = \frac{1}{4}m^2C$. Proceeding further

$$DC_{\alpha(2),\dot\alpha(2)} = C_{\alpha(2),\dot\alpha(2)|\beta\dot\beta}h^{\beta\dot\beta} = \left(C_{\alpha(2),\dot\alpha(2)\beta} + \epsilon_{\alpha\beta}T_{\alpha,\dot\alpha(2)\dot\beta} + \epsilon_{\dot\alpha\dot\beta}T_{\alpha(2)\beta,\dot\beta} + \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}T_{\alpha,\dot\alpha}\right)h^{\beta\dot\beta}.$$

Using Bianchi identities (9.32) one finds, that $T_{\alpha(3),\dot\alpha} = T_{\alpha,\dot\alpha(3)} = 0$, and $T_{\alpha\dot\alpha} \sim C_{\alpha\dot\alpha}$. This pattern persists at higher levels and allows one to write general expression

$$DC_{\alpha(n),\dot\alpha(n)} = C_{\alpha(n)\beta,\dot\alpha(n)\dot\beta}h^{\beta\dot\beta} + f_n C_{\alpha(n-1),\dot\alpha(n-1)}h_{\alpha\dot\alpha}, \tag{9.43}$$

where coefficients f_n to be determined from Bianchi identities (9.32). Note that the field content does not change when we switch on λ, the f_n term compensates for $DD \neq 0$. On one hand, we have

$$D^2 C_{\alpha(n),\dot\alpha(n)} = -\lambda^2 H_\alpha{}^\beta C_{\beta\alpha(n-1),\dot\alpha(n)} - \lambda^2 H_{\dot\alpha}{}^{\dot\beta} C_{\alpha(n),\dot\beta\dot\alpha(n-1)} \tag{9.44}$$

On the other, using

$$DC_{\alpha(n)\beta,\dot\alpha(n)\dot\beta} = C_{\alpha(n)\beta\gamma,\dot\alpha(n)\dot\beta\dot\gamma}h^{\gamma\dot\gamma} + f_{n+1}\Big(C_{\alpha(n-1)\beta,\dot\alpha(n-1)\dot\beta}h_{\alpha\dot\alpha} + C_{\alpha(n),\dot\alpha(n)}h_{\beta\dot\beta} + \tag{9.45}$$

$$+ C_{\beta\alpha(n-1),\dot\alpha(n)}h_{\alpha\dot\beta} + C_{\alpha(n),\dot\beta\dot\alpha(n-1)}h_{\beta\dot\alpha}\Big) \tag{9.46}$$

and

$$DC_{\alpha(n-1),\dot\alpha(n-1)} = C_{\alpha(n-1)\beta,\dot\alpha(n-1)\dot\beta}h^{\beta\dot\beta} + f_{n-1}C_{\alpha(n-2),\dot\alpha(n-2)}h_{\alpha\dot\alpha} \tag{9.47}$$

one arrives at the following recurrent condition

$$-\lambda^2 = -\frac{1}{2}f_{n+1}(n+2) + \frac{1}{2}nf_n, \tag{9.48}$$

which can be easily solved by

$$f_n = +\lambda^2 + \frac{A}{(n+1)n}, \tag{9.49}$$

where A is related to the mass term in (9.41) $A = \frac{1}{2}(m^2 - 4\lambda^2)$. The pure massless case, when a scalar in question is conformal, corresponds to $A = 0$ and yields the simplest f_n

$$m^2 = -4\lambda^2. \tag{9.50}$$

In this case we have the following chain of equations

$$DC^{\alpha(n),\dot\alpha(n)} = h_{\beta\dot\beta}C^{\alpha(n)\beta,\dot\alpha(n)\dot\beta} + \lambda^2 h^{\alpha\dot\alpha}C^{\alpha(n-1),\dot\alpha(n-1)}, \tag{9.51}$$

which in terms of generating function (9.9) reduces to

$$\widetilde{\mathcal{D}} C(y, \bar{y}|x) = 0 \,, \tag{9.52}$$

$$\widetilde{\mathcal{D}} = D - h^{\alpha\dot\alpha} \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\dot\alpha}} - \lambda^2 h^{\alpha\dot\alpha} y_\alpha \bar{y}_{\dot\alpha} \,. \tag{9.53}$$

Just as in Minkowski space-time, (9.52) is consistent, i.e. $d^2 = 0$, for any $C(y, \bar{y}|x)$ no matter if it has or has not the form of a scalar module (9.9). In generic case eq. (9.52) describes all fields with $s \geq 0$ propagating in AdS_4 along the lines of Section 9.1. For $s \geq 1$ these read, c.f. (9.20),

$$D_{\beta\dot\alpha} C^\beta{}_{\alpha(2s-1)} = 0 \,, \qquad D_{\alpha\dot\beta} \bar{C}^{\dot\beta}{}_{\dot\alpha(2s-1)} = 0 \,. \tag{9.54}$$

The obtained equation has a very clear algebraic meaning which we reveal in what follows. Obviously, there is a well-defined flat limit, $\lambda \to 0$ that results in (9.39).

9.4 HS gauge potentials

So far massless higher-spin fields have been described in terms of generalized curvatures analogous to Maxwell $s = 1$ tensor and $s = 2$ Weyl tensor, which is parallel to Section 5.5. These along with $s = 0$ and $s = 1/2$ matter fields reside in zero-form Weyl module $C(y, \bar{y}|x)$. In this section we add gauge potentials aiming at the spinorial, extended to AdS_4, version of (5.92)-(5.97).

The simplest case one can start with is a massless $s = 1$ field. Its gauge potential $A = A_\mu dx^\mu$ defines Maxwell tensor which is $C_{\alpha\beta}, \bar{C}_{\dot\alpha\dot\beta}$ components of the Weyl module according to

$$dA = h^\alpha{}_{\dot\gamma} \wedge h^{\beta\dot\gamma} C_{\alpha\beta} + h_\gamma{}^{\dot\alpha} \wedge h^{\gamma\dot\beta} \bar{C}_{\dot\alpha\dot\beta} \,. \tag{9.55}$$

Gauge potential A possesses a standard gauge invariance

$$\delta A = d\xi \,. \tag{9.56}$$

Consider now massless $s = 2$ case. To do so one needs to impose Einstein equations and linearize them around AdS_4 background. Imposing Einstein equations is equivalent to state that Riemann tensor differs from that of AdS_4 by Weyl tensor. In other words, we can write down the following equations for $s = 2$ Lorentz connection $\omega^{\alpha\beta}, \omega^{\dot\alpha\dot\beta}$ and vierbein field $e^{\alpha\dot\alpha}$, which are (4.29)-(4.30) in the language of spinors,

$$De^{\alpha\dot\alpha} = de^{\alpha\dot\alpha} + \omega^\alpha{}_\gamma \wedge e^{\gamma\dot\alpha} + \bar{\omega}^{\dot\alpha}{}_{\dot\gamma} \wedge e^{\alpha\dot\gamma} = 0 \,, \tag{9.57}$$

$$d\omega^{\alpha\beta} + \omega^\alpha{}_\gamma \wedge \omega^{\gamma\beta} + \lambda^2 e^\alpha{}_{\dot\gamma} \wedge e^{\beta\dot\gamma} = e_{\gamma\dot\delta} \wedge e_\delta{}^{\dot\delta} C^{\alpha\beta\gamma\delta} \,, \tag{9.58}$$

$$d\bar{\omega}^{\dot\alpha\dot\beta} + \bar{\omega}^{\dot\alpha}{}_{\dot\gamma} \wedge \bar{\omega}^{\dot\beta\dot\gamma} + \lambda^2 e_\gamma{}^{\dot\alpha} \wedge e^{\gamma\dot\beta} = e_{\delta\dot\gamma} \wedge e^\delta{}_{\dot\delta} \bar{C}^{\dot\alpha\dot\beta\dot\gamma\dot\delta} \,. \tag{9.59}$$

This system is exact, like (4.29)-(4.30) is, in the sense that it is valid in the full gravity provided that Weyl tensor obeys the Bianchi identities. Then we linearize it around AdS_4 by replacing $\omega^{\alpha\beta} \to \varpi^{\alpha\beta} + \omega^{\alpha\beta}$, $e^{\alpha\dot\alpha} \to h^{\alpha\dot\alpha} + e^{\alpha\dot\alpha}$, where frame fields ϖ and h are of vacuum AdS_4. The Weyl tensor for the empty AdS_4 vanishes, so $C^{\alpha(4)} \oplus \bar{C}^{\dot\alpha(4)}$ should

be taken of the first order. The zeroth order yields the $so(3,2)$ zero-curvature equations, (9.28)-(9.30). The resulting linearized equations (linear in ω and e) reduce to

$$De^{\alpha\dot\alpha} + \omega^\alpha{}_\gamma \wedge h^{\gamma\dot\alpha} + \bar\omega^{\dot\alpha}{}_{\dot\gamma} \wedge h^{\alpha\dot\gamma} = 0\,, \tag{9.60}$$

$$D\omega^{\alpha\beta} + \lambda^2(h^\alpha{}_{\dot\gamma} \wedge e^{\beta\dot\gamma} + e^\alpha{}_{\dot\gamma} \wedge h^{\beta\dot\gamma}) = h_{\gamma\dot\delta} \wedge h_\delta{}^{\dot\delta} C^{\alpha\beta\gamma\delta}\,, \tag{9.61}$$

$$D\bar\omega^{\dot\alpha\dot\beta} + \lambda^2(h_\gamma{}^{\dot\alpha} \wedge e^{\gamma\dot\beta} + e_\gamma{}^{\dot\alpha} \wedge h^{\gamma\dot\beta}) = h_{\delta\dot\gamma} \wedge h^\delta{}_{\dot\delta} \bar C^{\dot\alpha\dot\beta\dot\gamma\dot\delta}\,. \tag{9.62}$$

Eq. (9.58)-(9.59), which are analogous to (4.35)-(4.36), are consistent provided the Bianchi identities for the Weyl tensor hold, therefore, (9.60)-(9.62) are consistent under the linearized Bianchi identities for Weyl tensor (9.54). To rewrite (9.60)-(9.62) in the generating form, we pack fields $\omega_{\alpha\beta}$, $\bar\omega_{\dot\alpha\dot\beta}$ and $e_{\alpha\dot\beta}$ into quadratic in $y,\bar y$ polynomial

$$\omega_{s=2} = \frac{1}{2}\omega^{\alpha\beta} y_\alpha y_\beta + \frac{1}{2}\bar\omega^{\dot\alpha\dot\beta} \bar y_{\dot\alpha} \bar y_{\dot\beta} + e^{\alpha\dot\alpha} y_\alpha \bar y_{\dot\alpha}\,, \tag{9.63}$$

then system (9.60)-(9.62) arranges into

$$\mathcal{D}\omega_{s=2} = h^{\gamma\dot\alpha} \wedge h_\gamma{}^{\dot\beta} \frac{\partial^2}{\partial\bar y^{\dot\alpha}\partial\bar y^{\dot\beta}} C_{s=2}(0,\bar y|x) + h^{\alpha\dot\gamma} \wedge h^\beta{}_{\dot\gamma} \frac{\partial^2}{\partial y^\alpha\partial y^\beta} C_{s=2}(y,0|x)\,, \tag{9.64}$$

where (since we are in AdS_4 till the very end, the cosmological constant is set to 1, $\lambda = 1$)

$$\mathcal{D} = D + h^{\alpha\dot\alpha}(y_\alpha\partial_{\dot\alpha} + y_{\dot\alpha}\partial_\alpha)\,. \tag{9.65}$$

(9.64) needs to be supplemented with (9.52) in the sector of spin-two. Setting y or $\bar y$ to zero on the *r.h.s.* of (9.64) is a way to project onto the Weyl tensor.

Again, equation (9.64) is consistent without any reference to the structure of $s = 2$ module (9.63) and so can be generalized for one-form generating function

$$\omega(y,\bar y|x) = \sum_{n,m=0}^\infty \frac{1}{m!n!}\omega_{\alpha(m),\dot\alpha(n)}(y^\alpha)^m(\bar y^{\dot\alpha})^n\,, \tag{9.66}$$

$$\mathcal{D}\omega = h^{\gamma\dot\alpha} \wedge h_\gamma{}^{\dot\beta} \frac{\partial^2}{\partial\bar y^{\dot\alpha}\partial\bar y^{\dot\beta}} C(0,\bar y|x) + h^{\alpha\dot\gamma} \wedge h^\beta{}_{\dot\gamma} \frac{\partial^2}{\partial y^\alpha\partial y^\beta} C(y,0|x)\,. \tag{9.67}$$

Eq. (9.67) is called the central on-mass-shell theorem, [41, 88]. It is consistent since $\mathcal{D}\mathcal{D} = 0$. Since the system is linear it decomposes into an infinite set of subsystems for fields of spin $s = 1, 2, 3, \ldots$. The following number operator commutes with $\mathcal{D}$ and allows one to single out a subsystem for a particular spin

$$\left(y^\alpha\frac{\partial}{\partial y^\alpha} + \bar y^{\dot\alpha}\frac{\partial}{\partial y^{\dot\alpha}}\right)\omega_s(y,\bar y|x) = \pm 2(s-1)\,\omega_s\,, \tag{9.68}$$

In particular, $\omega^{\alpha(s-1),\dot\alpha(s-1)}$ component is the spinorial avatar of the spin-s vielbein $e^{a(s-1)}$. The set of fields $\omega^{\alpha(s-1),\dot\alpha(s-1)}$, such that $n + m = 2(s-1)$, are avatars of $\omega^{a(s-1),b(k)}$. The *l.h.s.* of (9.67) reads

$$D\omega^{\alpha(n),\dot\alpha(m)} + h^\alpha{}_{\dot\gamma} \wedge \omega^{\alpha(n-1),\dot\alpha(n)\dot\gamma} + h_\gamma{}^{\dot\alpha} \wedge \omega^{\alpha(n)\gamma,\dot\alpha(m-1)}\,. \tag{9.69}$$

It is set to zero everywhere except for purely holomorphic and antiholomorphic components driven by $C(y,0)$ and $C(0,\bar y)$.

Summary. Free higher-spin fields including the scalar can be described uniformly by the following simple system of equations,

$$\mathcal{D}\omega(y,\bar{y}|x) = h^{\gamma\dot{\alpha}} \wedge h_\gamma{}^{\dot{\beta}} \frac{\partial^2}{\partial\bar{y}^{\dot{\alpha}}\partial\bar{y}^{\dot{\beta}}} C(0,\bar{y}|x) + h^{\alpha\dot{\gamma}} \wedge h^\beta{}_{\dot{\gamma}} \frac{\partial^2}{\partial y^\alpha \partial y^\beta} C(y,0|x) \tag{9.70}$$

$$\delta\omega(y,\bar{y}|x) = \mathcal{D}\epsilon(y,\bar{y}|x) \tag{9.71}$$

$$\widetilde{\mathcal{D}}C(y,\bar{y}|x) = 0 \,, \tag{9.72}$$

where we added the gauge transformation law for ω and

$$\mathcal{D} = D + h^{\alpha\dot{\alpha}}(y_\alpha \partial_{\dot{\alpha}} + y_{\dot{\alpha}}\partial_\alpha) \tag{9.73}$$

$$\widetilde{\mathcal{D}} = D - h^{\alpha\dot{\alpha}}\partial_\alpha \partial_{\dot{\alpha}} + h^{\alpha\dot{\alpha}}y_\alpha \bar{y}_{\dot{\alpha}} \tag{9.74}$$

$$D = d + \varpi^{\alpha\beta}y_\alpha \partial_\beta + \varpi^{\dot{\alpha}\dot{\beta}}y_{\dot{\alpha}}\partial_{\dot{\beta}} \tag{9.75}$$

These equations are the spinorial version of (5.92)-(5.97) extended to AdS_4.

It is a beneficial exercise to check that the system above is consistent. First one observes that $\mathcal{D}^2 = 0$ and $\widetilde{\mathcal{D}}^2 = 0$[43], so the system is gauge invariant and is consistent up to the terms on the *r.h.s.* of (9.70). The last thing to show is that these terms do not spoil the consistency.

The system describes fields of spins $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, ...$, each in one copy. Truncation to the bosonic sector can be done by requiring ω and C be even functions of $y, \bar{y}$,

$$\omega(y,\bar{y}) = \omega(-y,-\bar{y}) \,, \qquad\qquad C(y,\bar{y}) = C(-y,-\bar{y}) \,. \tag{9.76}$$

After having defined the higher-spin algebra in the next section we will show that these peculiar operators, $\mathcal{D}, \widetilde{\mathcal{D}}$ are automatically generated by representation theory.

Let us conclude with a picture that shows the links between the fields of the unfolded system. (9.70)-(9.72).
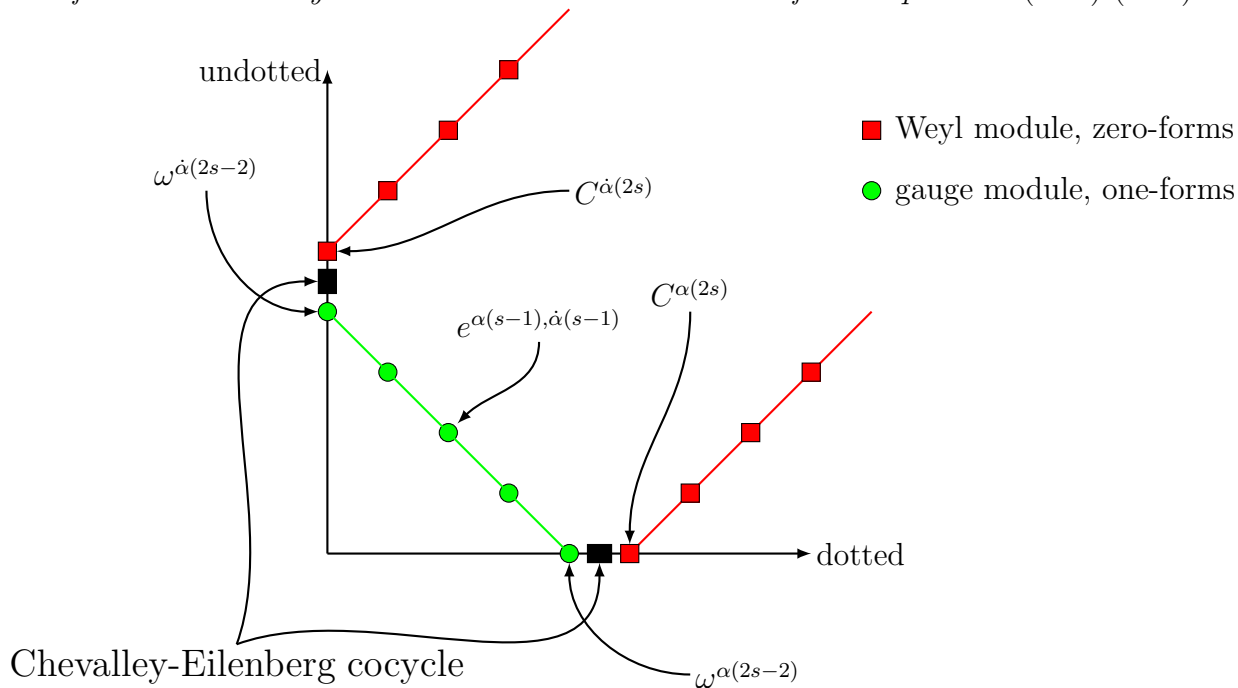
10 Higher-spin algebras

So far we have obtained linear in fluctuations unfolded equations that describe free fields of all spins, (9.70)-(9.72). At the linearized level fields of different spins are independent and the system simply decomposes into a set of decoupled equations each describing a free field of certain spin. Likewise, the $su(N)$ Yang-Mills theory at zero coupling is just a sum of $N^2 - 1$ Maxwell actions for noninteracting photons. In fact, there is a Lie algebra that unifies fields of all spins, [2, 105–107].

As we already know from the general discussion of Section 6 on the unfolded approach one-forms should take values in some Lie algebra. So, the $\Omega\omega$-type terms[44], which one sees in (5.92), (9.70), should arise from the linearization of $\frac{1}{2}[W, W]$, where W takes values

---

[43]This formula is a bit of abuse of notation as formally the operator in (9.74) does not even satisfy the chain rule. It is a differentiating operator nonetheless when treated within the corresponding twisted representation.

[44]Ω is a background, (9.33)-(9.36), i.e. h, ϖ, while ω collectively denotes free fields of all spins.

Figure 3: *A picture showing the position of various fields and their mixing via unfolded equations. The coordinates are the number of undotted/dotted indices carried by a field. The fields connected by links talk to each other via the unfolded equations (9.70)-(9.72).*

in some possibly infinite-dimensional Lie algebra, which we call the higher-spin algebra, $\mathfrak{g}$. To recover $\Omega\omega$ one replaces W with $\Omega + g\omega$, where we introduced a formal expansion parameter g. The HS algebra $\mathfrak{g}$ contains the anti-de Sitter algebra $so(3,2) \sim sp(4,\mathbb{R})$ as a subalgebra since the graviton must belong to the spectrum and it is described by an $so(3,2)$-connection. Flat $sp(4,\mathbb{R})$-connection Ω describes the vacuum over which the perturbation theory is defined. Let us expand the Yang-Mills field strength for W in terms of $\Omega + g\omega$

$$R = dW + \frac{1}{2}[W,W] = \underbrace{d\Omega + \frac{1}{2}[\Omega,\Omega]}_{\substack{\text{vanishes} \\ (9.28) - (9.30)}} + \underbrace{g(d\omega + [\Omega,\omega])}_{\substack{\text{first order correction} \\ (9.70)}} + \underbrace{g^2\frac{1}{2}[\omega,\omega]}_{\text{second order}} \qquad (10.1)$$

Since $sp(4,\mathbb{R}) \sim so(3,2)$ is a subalgebra of $\mathfrak{g}$, we can decompose $\mathfrak{g}$ into irreducible $sp(4,\mathbb{R})$-modules[45]. Since $sl(2,\mathbb{C}) \sim so(3,1)$ is a subalgebra of $so(3,2)$, we can also consider a more fine-grained decomposition of $\mathfrak{g}$ into $sl(2,\mathbb{C})$ irreducible representation. Depending on the assumptions about the spectrum of fields contained in W one may find the follow-

---

[45]It is a general property of any Lie algebra. Any Lie algebra is first of all a representation of the algebra itself, the adjoint one. For a simple algebra, the adjoint is irreducible. For any given subalgebra, the algebra as a linear space is a representation again, which is reducible in general. It is this decomposition of the algebra taken as a linear space into a direct sum of irreducible representations of its subalgebra that we consider.

ing components of ω (we remind of (8.26) vs. (8.27))[46]

spin	$sl(2,\mathbb{C})$-content	$sp(4,\mathbb{R})$-content
1	ω	$\omega \sim \bullet$
2	$\omega^{\alpha\alpha}, \omega^{\alpha\dot\alpha}, \omega^{\dot\alpha\dot\alpha}$	$\omega^{AA} \sim \boxed{}\boxed{}$
3	$\omega^{\alpha(4)}, \omega^{\alpha(3),\dot\alpha}, \omega^{\alpha(2),\dot\alpha(2)}, \omega^{\alpha(1)\dot\alpha(3)}, \omega^{\dot\alpha(4)}$	$\omega^{A(4)} \sim \boxed{}\boxed{}\boxed{}\boxed{}$
s	$\omega^{\alpha(k),\dot\alpha(m)},\ k+m=2(s-1)$	$\omega^{A(2s-2)} \sim \boxed{2s-2}$

It is a valid question to address if there is a Lie algebra, $\mathfrak{g}$, that contains $sp(4,\mathbb{R})$ as a subalgebra and has this spectrum of generators under $sp(4,\mathbb{R})$ or $sl(2,\mathbb{C})$. It does exist! The HS algebra, is the fundamental object in the HS theory. There are many different ways to define it. We present the most useful for practical computations below. That the linearized equations (9.70)-(9.72) take the simplest form when a field of every spin appears once naturally suggest this to be the property of the algebra we are looking for[47].

The existence of the algebra is important in the Yang-Mills type of theories. Had it not existed we would have proven a no-go result that these particular set of fields does not admit any interactions at all. This could have happened for a different filed multiplet, e.g., for a finite number of fields with at least one of them having spin greater than two.

The information that the free theory tells us is not enough to recover the algebra immediately. What we know is the decomposition of $\mathfrak{g}$ as a linear space under its subalgebra $\mathfrak{h} = sp(4,\mathbb{R})$, equivalently we know how W decomposes,

$$\mathfrak{g}|_{\mathfrak{h}} = \mathfrak{h} \oplus \bigoplus_s V_s, \qquad\qquad V_s = \boxed{2s-2}, \qquad (10.2)$$

$$W = \frac{1}{2}(\Omega^{AB} + \omega^{AB})T_{AB} + \sum_{s\neq 2} \omega^{A(2s-2)}T_{A(2s-2)}, \qquad\qquad (10.3)$$

where we singled out the adjoint representation of $\mathfrak{h}$ itself. By definition, we know the Lie bracket on $\mathfrak{h}$, i.e. $[\mathfrak{h},\mathfrak{h}] = \mathfrak{h}$, (9.23) or (9.24)-(9.26), and we also know $[\mathfrak{h}, V_s] = V_s$ which manifests V_i being an irreducible representation of $\mathfrak{h}$. The missing piece of information is $[V_i, V_j] = ?$. To summarize our knowledge we give the known commutation relations for the generators

$$[T_{AB}, T_{CD}] = \epsilon_{BC}T_{AD} + \epsilon_{AC}T_{BD} + \epsilon_{BD}T_{AC} + \epsilon_{AD}T_{BC}, \qquad\qquad (10.4)$$

$$[T_{AB}, T_{C(k)}] = \epsilon_{BC}T_{AC(k-1)} + \epsilon_{AC}T_{BC(k-1)}, \qquad k = 2s-2. \qquad (10.5)$$

Splitting $A = \{\alpha, \dot\alpha\}$ we can get the same relations in $sl(2,\mathbb{C})$ basis.

The most general ansatz for the missing commutators that preserves $sp(4)$ decomposition would be

$$[T_{A(k)}, T_{B(m)}] = \sum_i \alpha^i_{k,m} \underbrace{\epsilon_{AB}...\epsilon_{AB}}_{i} T_{A(k-i)B(m-i)} \qquad\qquad (10.6)$$

and we need to look for solutions to the Jacobi identity, which should give options for otherwise free coefficients $\alpha^i_{k,m}$. Here we assumed that generator of every spin can appear

---

[46]$A, B, ... = 1, ..., 4$ are $sp(4)$ indices. They can be split into a pair of $sl(2,\mathbb{C})$ indices, $A = \{\alpha, \dot\alpha\}$.

[47]This assumption turns out to be valid for bosonic algebra only. If one wishes to add fermions a single copy of every spin would be not enough.

at most once or do not appear at all, so there are no additional 'color' indices carried by $T_{A(k)}$. One can solve the Jacobi identity and find a unique higher-spin algebra, [2], but we proceed to the effective realization of this algebra in terms of oscillators.

The last comment is that any element f of the higher-spin algebra $\mathfrak{g}$ can be expanded as $f^{\mathcal{I}} e_{\mathcal{I}}$ with the basis vectors being $T_{A(k)} = e_{\mathcal{I}}$

$$f \in \mathfrak{g}, \qquad\qquad f = \sum_k f^{A(k)} T_{A(k)}, \qquad\qquad (10.7)$$

where $f^{A(k)}$ are totally symmetric tensors, which are the 'coordinates' in the linear space spanned by $e_{\mathcal{I}}$. The commutation relations $[e_{\mathcal{I}}, e_{\mathcal{J}}] = f^{\mathcal{K}}_{\mathcal{I}\mathcal{J}} e_{\mathcal{K}}$, (10.4)-(10.6), can be rewritten in terms of the coordinates $f^{\mathcal{I}} = f^{A(k)}$ as well,

$$(10.4): \qquad [f^{AB} T_{AB}, g^{CD} T_{CD}] = (4 f^A{}_C\, g^{CD}) T_{AD}, \qquad\qquad (10.8)$$

$$(10.5): \qquad [f^{AB} T_{AB}, g^{C(k)} T_{C(k)}] = (2k f^A{}_B\, g^{BC(k-1)}) T_{AC(k-1)}, \qquad\qquad (10.9)$$

which is a shorter way to encode the commutation relations.

$\star$-product realization.

Let us consider functions of an auxiliary variable Y^A that is an $sp(2n)$ vector. There is nothing special about $sp(4)$ in this section and we can extend it to $sp(2n)$. The indices $A, B, ...$ range over $2n$ values. The reason we need $sp(4)$ in the end is because $sp(4)$ is the symmetry algebra of AdS_4, which becomes relevant in the next section only. The Taylor coefficients are symmetric $sp(2n)$-tensors now

$$f(Y) = \sum_k \frac{1}{k!} f_{A(k)} Y^A ... Y^A, \qquad\qquad (10.10)$$

and, as we know, upon splitting $A = \{\alpha, \dot\alpha\}$ we get the required spin-tensor content in the case of $sp(4)$. Truncation to bosons is equivalent to $f(Y) = f(-Y)$, c.f. (9.76). So, at least there is a simple way to pack all the coordinates (10.7) into a generating function.

We want to find some operation on the space of functions $f(Y)$ in Y that induces an operation with required properties (10.4)-(10.6) or (10.8)-(10.9) and solves Jacobi. Standard product, i.e. $f(Y)g(Y)$ is a commutative operation, $[f(Y), g(Y)] \equiv 0$ and does not even lead to (10.8). We need something less trivial and less local, involving the derivatives in Y.

One more idea, that was realized, [105], after the solution was found in [2], is that the Jacobi is solved automatically if the Lie bracket $[f, g]$ comes as a commutator $[f, g] = f \star g - g \star f$ from some associative product

$$f(Y) \star (g(Y) \star h(Y)) = (f(Y) \star g(Y)) \star h(Y), \qquad\qquad (10.11)$$

and this is the case for higher-spin algebras. We have seen that the standard product is too simple. There are two ways to represent $f \star g$, which is an operation that is linear in its two arguments, f and g, and sends them to some other function of Y,

$$f(Y) \star g(Y) = \int f(U)g(V)K(U, V; Y)d^{2n}U d^{2n}V, \qquad\qquad (10.12)$$

$$f(Y) \star g(Y) = K\left(\frac{\partial}{\partial U}, Y, \frac{\partial}{\partial V}\right) f(U)g(V)\Big|_{U=V=Y}. \qquad\qquad (10.13)$$

The differential form is perfect when the arguments are polynomials. Note that the differential form can be rewritten as

$$f(Y) \star g(Y) = f(Y) K \left(\frac{\overleftarrow{\partial}}{\partial Y}, Y, \frac{\overrightarrow{\partial}}{\partial Y} \right) g(Y). \tag{10.14}$$

The integral form is suitable when arguments are more general analytic functions. The associativity of the product imposes severe restrictions on the form of the kernels K. Still, there are many solutions, which are equivalent up to a change of the basis $e_{\mathcal{J}} \to A_{\mathcal{I}}{}^{\mathcal{J}} e_{\mathcal{I}}$. We present the most convenient one. The solution we choose, which is called the Moyal $\star$-product, reads

$$f(Y) \star g(Y) = \int f(U) g(V) \exp \left(i(U_A - Y_A)(V^A - Y^A) \right) d^{2n} U d^{2n} V, \tag{10.15}$$

$$f(Y) \star g(Y) = f(Y) \exp i \left(\overleftarrow{\partial}_A \epsilon^{AB} \overrightarrow{\partial}_B \right) g(Y), \tag{10.16}$$

and there is an equivalent form of the first formula

$$f(Y) \star g(Y) = \int f(Y + U) g(Y + V) e^{(iU_A V^A)} d^{2n} U d^{2n} V. \tag{10.17}$$

The second formula is understood as

$$f(Y) \star g(Y) = f(Y) \left(1 + i \overleftarrow{\partial}_A \epsilon^{AB} \overrightarrow{\partial}_B - \frac{1}{2} (\overleftarrow{\partial}_A \epsilon^{AB} \overrightarrow{\partial}_B)^2 + ... \right) g(Y). \tag{10.18}$$

Let us prove that $\star$-product yields an example of a higher-spin algebra with all required conditions satisfied. Firstly, with the help of the differential form one sees that

$$Y_A \star Y_B = Y_A Y_B + i\epsilon_{AB} \tag{10.19}$$

and hence the algebra does not give a trivial Lie bracket

$$[Y_A, Y_B]_\star = Y_A * Y_B - Y_B * Y_A = 2i\epsilon_{AB}. \tag{10.20}$$

Moreover, one finds that the commutation relations for T_{AB}, (10.4), (10.8), holds with

$$T_{AB} = -\frac{i}{2} Y_A Y_B \tag{10.21}$$

Indeed, expanding the exp-formula up to the third term one finds

$$f^{AA} T_{AA} \star g^{BB} T_{BB} = -\frac{1}{4} f^{AA} g^{BB} Y_A Y_A Y_B Y_B + 2 f^A{}_C g^{CB} \left(\frac{i}{2} Y_A Y_B \right) - 2 f^{AB} g_{AB} \tag{10.22}$$

and then gets

$$[f^{AA} T_{AA}, g^{BB} T_{BB}]_\star = 4 f^A{}_C g^{CB} T_{AB}, \tag{10.23}$$

which is (10.8) and is equivalent to (10.4). Therefore, $sp(2n)$ is a subalgebra[48] and corresponds to quadratic polynomials in Y. The rest of generators are given by $e_{\mathcal{I}} = T_{A(k)} = Y_A...Y_A$ up to an unessential numerical prefactor.

Using exp formula one also proves (10.9), (10.5), where the following formulae are useful

$$Y_A \star g(Y) = \left(Y_A + i\overrightarrow{\partial}_A\right) g(Y), \qquad f(Y) \star Y_A = \left(Y_A - i\overrightarrow{\partial}_A\right) f(Y). \qquad (10.24)$$

Finally, we have a Lie algebra, constructed from an associative one, which obeys (10.4), (10.5) and we can look at (10.6),

$$f_{A(k)}Y^A...Y^A \star g_{B(m)}Y^B...Y^B = \qquad (10.25)$$

$$\sum_n \frac{i^n k! m!}{n!(k-n)!(m-n)!} \left(f_{A(k-n)}{}^{C(n)} g_{C(n)B(m-n)}\right) Y^A...Y^A Y^B...Y^B. \qquad (10.26)$$

Note, that in the commutator terms with even n survive only. With the same exp formula one can check that the quadratic Casimir operator of $sp(2n)$ is a fixed number $-\frac{1}{2}T_{AB} \star T^{AB} = -\frac{1}{4}n(2n+1)$.

Let us show how the integral realization of star-product works, since it is this form that will be very handy in less trivial calculations. Let us also stress, that the differential realization of the star-product strictly speaking is well defined only for polynomials, whereas its integral form acts on much broader space of functions thus being relevant for HS analysis with infinite set of fields.

It is assumed that $\int$ sign includes the numerical factor in order to ensure,

$$1 \star f(Y) = f(Y) \star 1 = f(Y), \qquad\qquad 1 \star 1 = 1, \qquad (10.27)$$

this is achieved by assuming

$$1 \star 1 = \int \exp\left(iU_A V^A\right) d^{2n}U d^{2n}V = \int \delta^{2n}(U) d^{2n}U = 1. \qquad (10.28)$$

Let us calculate $Y_A \star f(Y)$ ($f(Y) \star Y_A$ is analogous)

$$Y_A \star f(Y) = \left.\frac{\partial}{\partial p^A} e^{p^B Y_B}\right|_{p=0} \star f(Y) \qquad (10.29)$$

$$\int e^{p^B(Y_B+U_B)} f(Y+V) e^{iU_B V^B} = \int e^{iU_B(V^B - ip^B) + p^B Y_B} f(Y+V) = \qquad (10.30)$$

$$= \int \delta(V - ip) e^{p^B Y_B} f(Y+V) = f(Y+ip) e^{p^B Y_B}, \qquad (10.31)$$

from which one gets (10.24). With the help of either realization one finds the following useful formulae

$$[Y_A, f(Y)]_\star = 2i\partial_A f(Y), \qquad\qquad \{Y_A, f(Y)\}_\star = 2Y_A f(Y),$$

$$[T_{AB}, f(Y)]_\star = -\frac{i}{2}[Y_A Y_B, f(Y)]_\star = (Y_A \partial_B + Y_B \partial_A) f(Y), \qquad (10.32)$$

$$\{T_{AB}, f(Y)\}_\star = -\frac{i}{2}\{Y_A Y_B, f(Y)\}_\star = -i(Y_A Y_B - \partial_B \partial_A) f(Y).$$

---

[48]We have an associative algebra with some embedding of $sp(2n)$ into it. By definition, such an algebra has to be related to the universal enveloping algebra of $sp(2n)$, see extra Section 12.5.

Oscillator realization. Let us mention another realization of the same algebra, which is useful for Fock-type analysis in practice. One starts with good old quantum mechanical pairs of operators of coordinate/momentum or creation/annihilation (we ignore $\hbar$ and sign conventions),

$$[\hat{p}_k, \hat{q}^j] = i\delta_k^j \,. \tag{10.33}$$

The pair can be packed into one operator $\hat{Y}_A = \{\hat{q}^j, \hat{p}_k\}$. We endow $\hat{Y}_A$ since it is an operator. Then, the commutation relations

$$[\hat{Y}_A, \hat{Y}_B] = 2i\epsilon_{AB}\,, \qquad \epsilon_{AB} = \frac{1}{2}\begin{pmatrix} 0 & \delta_k^j \\ -\delta_k^j & 0 \end{pmatrix} \tag{10.34}$$

are identical to (10.20) up to a similarity transformation for ϵ_{AB}. The difference is that $\hat{Y}_A$ are operators and the product is not expected to be commutative from the very beginning, while Y_A are usual commuting variables endowed in addition to the dot-product with noncommutative $\star$-product. The two constructions are isomorphic.

The higher-spin algebra is just the algebra of all quantum-mechanical operators one can construct with $\hat{q}$ and $\hat{p}$, i.e. it is the algebra of all functions $f(\hat{Y}) = f(\hat{q}, \hat{p})$, while $\star$-product realization is just an effective way to compute the product of operators $f(\hat{q}, \hat{p})g(\hat{q}, \hat{p})$. Keeping in mind that $\hat{p}$ can be represented in the space of functions of q as $p_k = i\frac{\partial}{\partial q^k}$ we come to the conclusion that the higher-spin algebra is the algebra of all differential operators $f(\hat{Y}) = f(q, \partial_q)$, which is called the Weyl algebra.

Lorentz covariant base. Consider the $sp(4, \mathbb{R})$ case, i.e. $A, B, \dots = 1, \dots, 4$, and $A = \{\alpha, \dot{\alpha}\}$, then we find T_{AB} split as, c.f. (9.21), (9.24)-(9.26),

$$T_{AB} \longleftrightarrow L_{\alpha\beta} = T_{\alpha\beta}, P_{\alpha\dot{\alpha}} = T_{\alpha\dot{\alpha}}, \bar{L}_{\dot{\alpha}\dot{\beta}} = \bar{T}_{\dot{\alpha}\dot{\beta}} \tag{10.35}$$

with the help of (10.32) one finds the realization of Lorentz and translation generators by differential operators in $y_\alpha, \bar{y}_{\dot{\alpha}}$

$$[L_{\alpha\beta}, f(Y)]_\star = -\frac{i}{2}[y_\alpha y_\beta, f(Y)]_\star = (y_\alpha \partial_\beta + y_\beta \partial_\alpha)f(Y)\,, \tag{10.36}$$

$$[P_{\alpha\dot{\alpha}}, f(Y)]_\star = -\frac{i}{2}[y_\alpha \bar{y}_{\dot{\alpha}}, f(Y)]_\star = (y_\alpha \partial_{\dot{\alpha}} + \bar{y}_{\dot{\alpha}} \partial_\alpha)f(Y)\,. \tag{10.37}$$

We will need in what follows one more peculiar operator, given by an anticommutator,

$$\{P_{\alpha\dot{\alpha}}, f(Y)\}_\star = -\frac{i}{2}\{y_\alpha \bar{y}_{\dot{\alpha}}, f(Y)\}_\star = -i(y_\alpha \bar{y}_{\dot{\alpha}} - \partial_\alpha \partial_{\dot{\alpha}})f(Y)\,. \tag{10.38}$$

Back to the free equations. Now we can see some manifestation of the algebra in the free equations (9.70)-(9.72), where the operators can be identified as follows

$$\mathcal{D}\omega = D\omega + h^{\alpha\dot{\alpha}}(y_\alpha \partial_{\dot{\alpha}} + \bar{y}_{\dot{\alpha}} \partial_\alpha)\omega = D\omega + h^{\alpha\dot{\alpha}}[P_{\alpha\dot{\alpha}}, \omega]_\star\,, \tag{10.39}$$

$$\widetilde{\mathcal{D}}C = DC - ih^{\alpha\dot{\alpha}}\frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\dot{\alpha}}}C + ih^{\alpha\dot{\alpha}}y_\alpha \bar{y}_{\dot{\alpha}}C = DC + h^{\alpha\dot{\alpha}}\{P_{\alpha\dot{\alpha}}, C\}_\star\,, \tag{10.40}$$

$$D = d + \varpi^{\alpha\beta}y_\alpha \partial_\beta + \bar{\varpi}^{\dot{\alpha}\dot{\beta}}\bar{y}_{\dot{\alpha}}\partial_{\dot{\beta}} = d + \frac{1}{2}\omega^{\alpha\beta}[L_{\alpha\beta}, \bullet]_\star + \frac{1}{2}\bar{\omega}^{\dot{\alpha}\dot{\beta}}[\bar{L}_{\dot{\alpha}\dot{\beta}}, \bullet]_\star\,, \tag{10.41}$$

where $\bullet$ is a placeholder to be replaced with the actual expression the operator acts on. The appearance of the anti-commutator in $\widetilde{\mathcal{D}}$ is crucial and will soon be explained. The appearance of some i-factors in $\widetilde{\mathcal{D}}$ as compared to (9.74) can be compensated by rescaling $C^{\alpha(k),\dot{\alpha}(m)} \to g_{n,m} C^{\alpha(k),\dot{\alpha}(m)}$, which we did not use in Section 9. This freedom to rescale was fixed somewhat arbitrary by normalizing the first term on the *r.h.s.* of (9.43) not to have any prefactors. The HS algebra tells us that the i-factors are more natural.

Star-product allows us to define *AdS* frame fields as one-form components to various bilinears of y and $\bar{y}$ supplemented with star-product zero-curvature condition. Indeed, using (10.32) one can easily convince oneself, that the following $sp(4)$-connection

$$\Omega = \frac{1}{2}\Omega^{AB}T_{AB} = \frac{1}{2}\omega^{\alpha\beta}L_{\alpha\beta} + h^{\alpha\dot{\alpha}}L_{\alpha\dot{\alpha}} + \frac{1}{2}\bar{\omega}^{\dot{\alpha}\dot{\beta}}\bar{L}_{\dot{\alpha}\dot{\beta}} \tag{10.42}$$

$$= -\frac{i}{4}(\omega_{\alpha\beta}y^{\alpha}y^{\beta} + \bar{\omega}_{\dot{\alpha}\dot{\beta}}\bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\beta}} + 2h_{\alpha\dot{\alpha}}y^{\alpha}\bar{y}^{\dot{\alpha}}) \tag{10.43}$$

substituted into

$$d\Omega + \Omega \star \Omega = 0 \tag{10.44}$$

results in (9.28)-(9.30). Then, (10.39)-(10.40) read now[49].

$$\mathcal{D}\omega = d\omega + \Omega \star \omega + \omega \star \Omega = d\omega + \frac{1}{2}\Omega^{AB}[T_{AB}, \omega]_{\star} = d\omega + \{\Omega, \omega\}_{\star}, \tag{10.46}$$

$$\widetilde{\mathcal{D}}C = dC + \Omega \star C - C \star \pi(\Omega), \tag{10.47}$$

where π flips the sign of translations,

$$\pi(P_{\alpha\dot{\alpha}}) = -P_{\alpha\dot{\alpha}}, \qquad \pi(L_{\alpha\beta}) = L_{\alpha\beta}, \qquad \pi(L_{\dot{\alpha}\dot{\beta}}) = L_{\dot{\alpha}\dot{\beta}}, \tag{10.48}$$

eventually turning the commutator into anti-commutator inside $\widetilde{\mathcal{D}}$ in the sector of $P_{\alpha\dot{\alpha}}$.

Now the free equations (9.70)-(9.72), which to be viewed as the linearization of a yet unknown theory, acquire plain algebraic meaning in accordance with general statements made about the unfolded equations, see Section 6.

There are two master-fields, the gauge field ω that takes values in the higher-spin algebra is a usual Yang-Mills connection of that algebra with somewhat different usage. $\mathcal{D}\omega$ is a linearization of $dW + W \star W$ with $W = \Omega + g\omega$, where Ω is already flat, which guarantees $\mathcal{D}\mathcal{D} = 0$. Another master field is C which also takes values in the higher-spin algebra, but now the action of the algebra on itself is twisted by π.

---

[49]Note that $\{\Omega, \omega\}$ is a mock anti-commutator, it is a commutator since one-forms anti-commute. Indeed,

$$\{\Omega, \omega\}_{\star} = \{\Omega_{\underline{m}}dx^{\underline{m}}, \omega_{\underline{n}}dx^{\underline{n}}\}_{\star} = [\Omega_{\underline{m}}, \omega_{\underline{n}}]_{\star} \, dx^{\underline{m}} \wedge dx^{\underline{n}}. \tag{10.45}$$

This is a difference between the genuine Lie algebra bracket, where, like in Yang-Mills, we have $[\Omega, \omega]$, and the bracket constructed as a commutator in associative algebra, where $[a, b] = a \star b - b \star a$. In the latter case we have to take into account every additional grading, like the differential form degree, which elements can have, which sometimes turns commutator into anti-commutator. It is easy to translate the Yang-Mills formulae to the case when the Lie algebra is constructed from an associative one. For example, the Yang-Mills curvature is $dA + A \star A$ instead of $dA + \frac{1}{2}[A, A]$ and the commutator is implicit because one-forms anti-commute. Then $\delta A = d\epsilon + A \star \epsilon - \epsilon \star A$, etc.

Let us comment more on the twisted action. Suppose we have an algebra, say $\mathfrak{g}$, and it acts on some linear space, say V, by operators $\rho : \mathfrak{g} \rightarrow End(V)$, i.e. V is a representation of $\mathfrak{g}$. Given any automorphism π of $\mathfrak{g}$, i.e. $\pi : \mathfrak{g} \rightarrow \mathfrak{g}$ and $\pi([a, b]) = [\pi(a), \pi(b)]$, we can define another action on the same space V, called twisted action, $\rho_\pi(a) = \rho(\pi(a))$. One can check that it is a representation of $\mathfrak{g}$, i.e. $\rho_\pi([a, b]) = [\rho_\pi(a), \rho_\pi(b)]$.

Now, π as defined in (10.48) is an automorphism of the anti-de Sitter algebra, $so(3, 2)$. It acts on P only, so the nontrivial relations to check are (we drop the indices)

$$[\pi(P), \pi(P)] = [-P, -P] = [P, P] = L = \pi(L), \tag{10.49}$$

$$[\pi(L), \pi(P)] = [L, -P] = -[L, P] = -P = \pi(P). \tag{10.50}$$

Formally, π can be realized either as $y_\alpha \rightarrow -y_\alpha$ or $\bar{y}_{\dot\alpha} \rightarrow -\bar{y}_{\dot\alpha}$, since $L \sim yy$, $\bar{L} \sim \bar{y}\bar{y}$ and only $P \sim y\bar{y}$ is affected by such an action,

$$\pi(y, \bar{y}) = (-y, \bar{y}), \quad \pi \circ \pi = 1, \quad \text{or} \quad \bar{\pi}(y, \bar{y}) = (y, -\bar{y}), \quad \bar{\pi} \circ \bar{\pi} = 1. \tag{10.51}$$

With this definition π can be extended to the full higher-spin algebra, $\mathfrak{g}$, as an associative algebra, i.e. $\pi(f \star g) = \pi(f) \star \pi(g)$. Indeed, π just checks if the function is even or odd in y or $\bar{y}$ and even⋆even=even, odd⋆odd=even, even⋆odd=odd. Hence, it extends to $\mathfrak{g}$ as a Lie algebra under the commutator.

Finally, the linearized equations for massless fields of all spins combined into a single multiplet read as in (9.70)-(9.72) and $\mathcal{D}$, $\widetilde{\mathcal{D}}$ have the meaning of adjoint and twisted-adjoint covariant derivatives for the master fields ω, C taking values in the higher-spin algebra[50]. The nontrivial gluing term on the r.h.s. of (9.70) is the Chevalley-Eilenberg cocycle, see around (6.25) and extra Section 12.4 for more detail. Hence, all terms in the equations have clear representation theory meaning.

'Pure gauge' solutions. Equation (10.44) is the zero-curvature condition. Hence, any solution of (10.47) admits pure gauge form

$$\Omega = g^{-1}(Y|x) \star dg(Y|x). \tag{10.52}$$

Equation (10.47) is the covariant constancy condition in the twisted-adjoint representation of the HS algebra. The general solution of (10.47) is

$$C(Y|x) = g^{-1} \star C_0(Y) \star \pi(g), \tag{10.53}$$

where $C_0(Y)$ is an arbitrary x-independent function. Pure gauge form of (10.52) and (10.53) may look misleading for it seemingly suggests that one can gauge away any solution for dynamical fields. This is not the case. There are two restrictions that constraint gauge functions $g(Y|x)$ in (10.52). First, $g(Y|x)$ should be such that corresponding connection Ω be of the form (10.43) i.e. bilinear in y's and second, it should not provide one with degenerate vierbein $h_{\alpha\dot\alpha}$. In practical calculations the gauge function that reproduces

---

[50]That gauge fields ω and zero-forms C take values in the same space is a kind of operator-state correspondence. π is related to the Chevalley involution, on the CFT side, it is realized as an inversion.

vacuum frame fields is some exponent of bilinears in y's. The fact that higher-spin equations for zero-from $C(Y|x)$ acquire pure gauge representation is a remarkable property of the on-shell integrability of this system uncovered with the aid of the unfolding approach.

Let us note, that twisted-adjoint equation (10.47) is fixed by corresponding zero-curvature condition in the HS algebra. It means, in particular, that scalar mass term (9.50) is completely fixed by the HS algebra as well. Unfolded form of dynamical equations makes their symmetries manifest. For example, AdS_4 global symmetries are governed by the symmetry parameter $\xi(Y|x)$, collective Killing, that leaves the vacuum, Ω, invariant

$$0 = \delta\Omega = D_\Omega\xi\,, \qquad D_\Omega = d + [\Omega, \bullet]_\star\,, \tag{10.54}$$

which can be explicitly found once $g(Y|x)$ is known

$$\xi(Y|x) = g^{-1} \star \xi_0(Y) \star g\,. \tag{10.55}$$

This is a covariant generalization of the Poincare lemma, which shows that $D_\Omega\xi = 0$ and $d\xi = 0$ have isomorphic solution spaces. It is also clear why the matter and HS curvatures module $C(Y|x)$ is described by differential zero-form, rather than some p-form. It makes it gauge invariant, otherwise there would be gauge invariance $\delta C^{(p)} = \widetilde{\mathcal{D}}\xi^{(p-1)}$, where $\widetilde{\mathcal{D}}$ is the twisted-adjoint covariant derivative given in (10.40).

One may ask if it is possible to solve for ω in a 'pure gauge' form. It cannot be just $g^{-1} \star \omega_0(Y) \star g$ since the latter is a solution to the homogeneous equation $\mathcal{D}\omega = 0$. On the other hand, all gauge-invariant information is in the zero-forms, which contain Weyl tensors and derivatives thereof, so it must be possible to reconstruct ω from C up to pure gauge solutions $\mathcal{D}\xi$. The formula does exist and can be found in [108].

Klein operator. Let us have a look at the automorphism π, which is one of the key ingredients of the higher-spin theory. Formally, π can be realized as

$$\kappa = (-)^{N_y}\,, \qquad\qquad \kappa y \kappa = -y\,, \qquad\qquad \kappa \bar{y} \kappa = \bar{y}\,, \tag{10.56}$$

where N_y is the number operator $y^\gamma \partial_\gamma$ that counts y.

Let us find out whether automorphism (10.51) is the inner one or not. In other words we would like to see if it can be realized in terms of star-product (10.15)-(10.16). Assuming that such element $\varkappa$ does exist in the star-product algebra such that

$$\varkappa * f(y, \bar{y}) * \varkappa = f(-y, \bar{y}) \tag{10.57}$$

and $\varkappa \star \varkappa = 1$ since $\pi\pi = 1$ one has

$$\varkappa * y_\alpha = -y_\alpha * \varkappa\,, \qquad \varkappa * \bar{y}_{\dot\alpha} = \bar{y}_{\dot\alpha} * \varkappa\,. \tag{10.58}$$

Using (10.32) the second condition in (10.58) tells us that $\varkappa$ is $\bar{y}$-independent, while the first is equivalent to

$$\varkappa y_\alpha = 0 \quad \rightarrow \quad \varkappa \sim \delta^2(y)\,. \tag{10.59}$$

From (10.58) it follows also that $[\varkappa*\varkappa, y_\alpha]_* = 0$ and therefore $\varkappa*\varkappa \sim$ const. The constant can be chosen to be 1 which leads to

$$\varkappa = 2\pi\delta^2(y)\,. \tag{10.60}$$

Indeed,

$$\varkappa \star \varkappa = (2\pi)^2 \int du\, dv\, \delta(y+u)\delta(y+v)e^{iu_\alpha v^\alpha} = 1\,. \tag{10.61}$$

Note, that in checking (10.61) one really has to use the integral form of the star-product, for $\star$-product of δ-functions is out of reach for differential star-product. The operator $\varkappa$ which satisfies the condition

$$\{\varkappa, y_\alpha\}_\star = 0\,, \qquad \varkappa \star \varkappa = 1\,, \tag{10.62}$$

is called holomorphic Klein operator. Analogous operator can be defined in the anti-holomorphic sector. As one can see, the Klein operator, being a delta function, strictly speaking does not belong to the $\star$-product algebra and hence the automorphism (10.51) is not the inner one. Nevertheless, representation (10.60) is very useful in practice. For example the action of Klein operator on a function is just the Fourier transform[51] with respect to holomorphic oscillator y

$$\varkappa \star f(y, \bar{y}) = \int dv f(v, \bar{y})e^{iv_\alpha y^\alpha}\,. \tag{10.63}$$

Automorphism (10.51) in terms of Klein operator action is given simply by

$$\pi(f(y, \bar{y})) = \varkappa \star f \star \varkappa\,. \tag{10.64}$$

Using (10.64) one can derive the Penrose transform that maps solution of field equations (10.47) to AdS_4 HS global symmetries. Indeed, having any HS global symmetry parameter $\xi(y, \bar{y}|x)$ that satisfies (10.54) one immediately generates solution to twisted-adjoint equation (10.47)

$$C(y, \bar{y}|x) = \xi(y, \bar{y}|x) \star \varkappa = \int \xi(u, \bar{y})e^{-iu_\alpha y^\alpha}\,. \tag{10.65}$$

Let us note, that (10.65) can be applied to AdS_4 global symmetry parameter which being bilinear in Y's will generate some ill-defined solution via Fourier transform of quadratic polynomial. Generic HS global symmetry parameter $\xi_0(Y)$ in (10.55), however, can be an arbitrary star-product function which may lead to a well defined Fourier transform.

Let us stress that with the help of $\varkappa$ one can map twisted-adjoint fields in adjoint ones and vice verse. For example,

$$dC + \Omega \star C - C \star \underbrace{\varkappa \star \Omega \star \varkappa}_{\pi(\Omega)} = 0 \quad \Longleftrightarrow \quad d(C \star \varkappa) + \Omega \star (C \star \varkappa) - (C \star \varkappa) \star \Omega = 0$$

where the second equation is the first one $\star$-multiplied by $\varkappa$ since $d\varkappa = 0$ and $\varkappa \star \varkappa = 1$.

The automorphism π may look like an isomorphism between the adjoint and twisted-adjoint representations. However one has to be careful with this point of view as the field-theoretical interpretation is totally different. Particularly, the physical domains of twisted-adjoint and adjoint representations do not overlap much since polynomials are mapped by $\star\varkappa$ into derivatives of δ functions and vice versa. More detail on how to work with $\star$-products can be found in extra Section 12.6.

---

[51]The appearance of Fourier transform F should not be surprising since $(F \circ F)[f(y)] = f(-y)$.

11 Vasiliev equations

In this section we consider nonlinear interactions for bosonic HS fields governed to all orders by the Vasiliev equations. Firstly, we are going to discuss what one should expect on the way of constructing the interactions within the unfolded approach. Secondly, starting from certain mild assumptions we attempt to derive the Vasiliev equations. It is not that easy to find reasonings that lead to these assumptions, but apart from these gaps the derivation is quite solid. Thirdly, we look at the AdS_4 vacuum solution to Vasiliev equations and show that the linear equations derived in Section 9 do emerge in the first order perturbative expansion. Then, we discuss certain nontrivial properties of the equations like the local Lorentz symmetry being part of the physical requirements that should fix the form of the equations. Lastly, we make some comments on higher orders and analyze the Vasiliev system explicitly to the second order.

11.1 Generalities

Up to this point we have shown that the specific multiplet of free HS fields, where a field of every spin appears in one copy, can be described in terms of two master-fields $\omega(Y|x)$ and $C(Y|x)$, both taking values in the higher-spin algebra, with the empty anti-de Sitter space given by the flat $so(3,2) \sim sp(4,\mathbb{R})$-connection Ω, (9.28)-(9.30),

$$d\Omega = -\Omega \star \Omega\,, \tag{11.1}$$

$$d\omega = -[\Omega, \omega] + \mathcal{V}(\Omega, \Omega, C)\,, \tag{11.2}$$

$$dC = -\Omega \star C + C \star \pi(\Omega)\,, \tag{11.3}$$

where there are two crucial ingredients: automorphism π, (10.48), (10.57), and cocyle $\mathcal{V}(\Omega, \Omega, C)$, (9.70). If we believe in the existence of the consistent nonlinear theory then these equations must be the linearization of

$$d\omega = \mathcal{V}(\omega, \omega) + \mathcal{V}(\omega, \omega, C) + \mathcal{V}(\omega, \omega, C, C) + ... = F^\omega(\omega, C)\,, \tag{11.4}$$

$$dC = \mathcal{V}(\omega, C) + \mathcal{V}(\omega, C, C) + \mathcal{V}(\omega, C, C, C) + ... = F^C(\omega, C)\,, \tag{11.5}$$

where the vertices $\mathcal{V}(\bullet, ..., \bullet)$ are linear in each slot and correspond to certain couplings. The first nontrivial couplings $\mathcal{V}(\omega, \omega)$ and $\mathcal{V}(\omega, C)$ are governed by the higher-spin algebra with the help of π,

$$-\mathcal{V}(\omega, \omega) = \omega \star \omega \qquad\qquad -\mathcal{V}(\omega, C) = \omega \star C - C \star \pi(\omega) \tag{11.6}$$

The cocyle $\mathcal{V}(\omega, \omega, C)$ is not known in full, only its value in the AdS vacuum $\mathcal{V}(\Omega, \Omega, C)$ is available. Linearization is carried out by replacing $\omega \to \Omega + g\omega$ and picking up the terms of the zeroth and first order in g.

To proceed to nonlinear level one can in principle search order by order in perturbation theory for such a deformation of (11.2) and (11.3) within the set of fields $w(y, \bar{y}|x)$ and $C(y, \bar{y}|x)$ that are consistent with the integrability condition $d^2 = 0$ (hence possessing gauge symmetry) and reproduce correct free field dynamics. The perturbative analysis (11.4)-(11.5) is a challenging technical problem and gets already cumbersome at the second

order, [40, 41]. This analysis was carried out up to $\mathcal{V}(\omega, \omega, C, C)$, [109], followed by the effective method of summing all orders, [34, 36, 42, 43].

By looking at the pure gravity case we have to accept the fact that the interaction series does not truncate to a finite number of terms within the unfolded approach, i.e. all powers of C will appear. Unfortunately, no closed form for F^ω and F^C is known. Functions F^ω and F^C are analogous to the expansion of the Riemann tensor in terms of the metric field. In gravity the series goes to infinity being though ideologically simple as the source of infinite tail is the expansion of inverse metric $g_{\mu\nu}^{-1}$.

The Vasiliev equations are the generating equations written in an extended space that sum up the infinite series into the equations that are no more than quadratic in fields. Namely, the equations consist of zero-curvature equation plus certain constraints that determine the embedding of interaction vertices into the fields obeying the zero-curvature condition. The interaction vertices $\mathcal{V}(\bullet, ..., \bullet)$ can be obtained by solving the equations order by order. Vasiliev equations do not give (11.4)-(11.5) immediately! They are formulated in certain extended space with more coordinates and (11.4)-(11.5) appear as solutions in the perturbative expansion around the AdS_4 vacuum. In some sense Vasiliev equations are analogous to simple differential equations encoding complicated special functions, but in the Vasiliev case the equations encode complicated equations as solutions to additional variables. The last remark is that Vasiliev equations share many properties with integrable equations. Integrability usually means that a theory can be put into the form of some covariant constancy or zero-curvature equation, i.e. equations that are no more than quadratic in fields.

As an example we will derive second order corrections in the formal coupling constant g, $\omega \to \Omega + g\omega^1 + g^2\omega^2$ from Vasiliev system

$$\mathcal{D}\omega^2 = \mathcal{V}(\omega^1, \omega^1) + \mathcal{V}(\Omega, \Omega, C^2) + \mathcal{V}(\Omega, \omega^1, C^1) + \mathcal{V}(\Omega, \Omega, C^1, C^1) \qquad (11.7)$$

$$\widetilde{\mathcal{D}}C^2 = \mathcal{V}(\omega^1, C^1) + \mathcal{V}(\Omega, C^1, C^1) \qquad (11.8)$$

and we will show some of these vertices explicitly. And at the order N one can write

$$\mathcal{D}\omega^N = \sum_{n+m=N}{}' \mathcal{V}(\omega^n, \omega^m) + \sum_{n+m+k=N}{}' \mathcal{V}(\omega^n, \omega^m, C^k) + \mathcal{V}(\Omega, \Omega, C^N) \qquad (11.9)$$

$$\widetilde{\mathcal{D}}C^N = \sum_{n+m=N}{}' \mathcal{V}(\omega^n, C^m) \qquad (11.10)$$

where the primed sum $\sum'$ designates that the background covariant derivative has already been extracted. For future use let us note that at order N the fields of order N appear the same way as ω^1 and C^1 appear in the linearized equations. So the equations for ω^N and C^N look like the linear equations plus sources built of the lower order fields.

The perturbation theory in certain auxiliary variables for Vasiliev equations yields the perturbation theory (11.9)-(11.10) for (11.4)-(11.5), from which the vertices $\mathcal{V}$ can be recovered. It is, of course, makes more sense to look for exact solutions and try to give certain physical meaning to the extended space.

88

11.2 Quasi derivation of Vasiliev equations

It is a sort of conventional wisdom that Vasiliev equations cannot be derived rather one is welcome to check that they do satisfy all the required assumptions. Below we attempt to put things in a perspective of 'derivation' starting from the main assumption that they should have 'almost' zero-curvature/covariant constancy form as in integrable theories[52], but the field content and the space where the equations are formulated can possibly be extended.

Let $\mathcal{W}$ be the full master-field on possibly extended space, of which $\omega(Y|x)$ is a subspace. There must be $\mathcal{W} \star \mathcal{W}$-term on the r.h.s. of (11.4) where $\star$ as well acts on possibly extended space and reduces to the usual HS algebra $\star$ for $\omega \in \mathcal{W}$. Denoting all other corrections on the r.h.s. of (11.4) as Φ, which must be a two-form, and using $d^2 = 0$ one finds

$$d\mathcal{W} + \mathcal{W} \star \mathcal{W} = \Phi \,, \tag{11.11}$$

$$d\Phi + [\mathcal{W}, \Phi]_\star = 0 \,, \tag{11.12}$$

where the second equations comes from consistency requirement. Unfolding tells us that this system has local gauge invariance[53]

$$\delta\mathcal{W} = d\epsilon + [\mathcal{W}, \epsilon]_\star + \xi_{\mathbf{1}} \,, \qquad \delta\Phi = d\xi_{\mathbf{1}} + \{\mathcal{W}, \xi_{\mathbf{1}}\}_\star - [\epsilon, \Phi]_\star \,. \tag{11.13}$$

Suppose Φ is a fundamental field, i.e. is not expressed in terms of some other degree-zero or degree-one fields. Then its has its own one-form gauge parameter $\xi_{\mathbf{1}}$, which is able to kill $\mathcal{W}$. Therefore, Φ cannot be fundamental.

From free level analysis we have found an appropriate set of fields for the description of gauge field dynamics. These are the master fields — a one-form $\omega(Y|x)$ and a zero-form $C(Y|x)$. Equation (11.2) hints to identify somehow Φ with C (for a moment we forget about issues with the twisted-adjoint representation). The problem is that Φ is a two-form and hence one needs some extra two-form to identify $\Phi \sim C$. The only field independent two-form that one has is $dx^a \wedge dx^b$. This, however, carrying indices should be contracted with some derivatives in Y's of C, thus bringing us to annoying perturbative analysis.

A way out proposed by Vasiliev is to introduce some auxiliary direction to space-time that will allow one to determine the two-form carrying no indices. This can be done one way or another, but this additional direction can be anticipated to be auxiliary for Y as well. So, in spinorial formalism we are working with, let us choose some auxiliary[54] $Z_A = (z_\alpha, \bar{z}_{\dot\alpha})$ and corresponding differential dZ_A, which anticommutes with dx_m and provides us with the differential two-forms. If the auxiliary space is two-dimensional the two-form is a top-form and is unique, in particular there are no free indices it can carry,

---

[52] For example, the famous Korteweg-de Vries equation, which contains higher-derivative and is non-linear — oversimplified model of higher-spin theory, can be formulated as a zero-curvature equation in certain extended space. Upon solving for the extra functions certain first-order equations coming as the components of the zero-curvature condition one gets the KdV equation. Vasiliev equations function analogously.

[53] Mind that $[\bullet, \bullet]$ sometimes changes to $\{\bullet, \bullet\}$ due to differential form degree.

[54] We used the same notation, α, for the auxiliary indices to anticipate that z are in close interrelation with y.

$dz^\alpha \wedge dz_\alpha$ and $d\bar{z}^{\dot\alpha} \wedge d\bar{z}_{\dot\alpha}$. That we need two auxiliary two-forms is due to the two terms on the *r.h.s.* of (9.70), i.e. due to the fact that the spin-s Weyl tensor $C^{a(s),b(s)}$ splits into two complex-conjugate $C^{\alpha(2s)}$, $C^{\dot\alpha(2s)}$ that yield two sources for one-forms $\omega(Y|x)$. This allows us to write down system (11.11) in the extended space by introducing the extended connection

$$\mathcal{W} = W_m dx^m + A_\alpha dz^\alpha + \bar{A}_{\dot\alpha} d\bar{z}^{\dot\alpha} \,, \tag{11.14}$$

where the fields now depend on a new variable Z_A as well

$$\omega(Y|x) \to W(Y,Z|x)\,, \qquad C(Y|x) \to B(Y,Z|x)\,, \qquad A_A = A_A(Y,Z|x)\,. \tag{11.15}$$

The exterior differential gets enhanced too and we label it with the hat

$$d \to \hat{d} = d_x \oplus d_z \oplus d_{\bar{z}} \,. \tag{11.16}$$

As a result, one may propose the following system

$$\hat{d}\mathcal{W} + \mathcal{W} \star \mathcal{W} = \Phi + \bar\Phi \,, \tag{11.17}$$

$$\hat{d}\Phi + [\mathcal{W}, \Phi]_\star = 0 \,, \tag{11.18}$$

$$\hat{d}\bar\Phi + [\mathcal{W}, \bar\Phi]_\star = 0 \,, \tag{11.19}$$

where we have implied the following identification

$$\Phi = \frac{i}{4} dz_\alpha \wedge dz^\alpha B \,, \qquad \bar\Phi = \frac{i}{4} d\bar{z}_{\dot\alpha} \wedge d\bar{z}^{\dot\alpha} \bar{B} \,, \tag{11.20}$$

The system is formally consistent and admits local gauge invariance

$$\delta\mathcal{W} = \hat{D}\epsilon \equiv \hat{d}\epsilon + [\mathcal{W}, \epsilon]_\star \,, \qquad \delta\Phi = [\Phi, \epsilon]_\star \,. \tag{11.21}$$

Let us stress once again that eqs. (11.18) and (11.19) arise from (11.17) as consistency conditions $d^2 = 0$ and are not independent. So far nothing is said about extra spinor variables z_α and $\bar{z}_{\dot\alpha}$. Let us choose these to be dual to Y_A i.e.,

$$[Z_A, Z_B]_\star = -2i\epsilon_{AB} \,, \qquad [Z_A, Y_B]_\star = 0 \,. \tag{11.22}$$

It requires the corresponding extension of the star-product, which we will define a bit later. Recall now, that HS master field zero-form $C(Y|x)$ is subject to twisted-adjoint flatness condition rather than the adjoint one at free level. Therefore, at full level we need to redefine $B(Y,Z|x)$ using the appropriate Klein operator. Analogously to the free theory consideration, the explicit form of the Klein operator $\varkappa$ depends on the form of the automorphism through its star-product realization. At nonlinear level, the twisted-adjoint automorphism is defined as

$$\pi(y_\alpha, \bar{y}_{\dot\alpha}, z_\alpha, \bar{z}_{\dot\alpha}) = (-y_\alpha, \bar{y}_{\dot\alpha}, -z_\alpha, \bar{z}_{\dot\alpha}) \,, \tag{11.23}$$

$$\bar\pi(y_\alpha, \bar{y}_{\dot\alpha}, z_\alpha, \bar{z}_{\dot\alpha}) = (y_\alpha, -\bar{y}_{\dot\alpha}, z_\alpha, -\bar{z}_{\dot\alpha}) \,. \tag{11.24}$$

The corresponding to be found Klein operators $\varkappa$ and $\bar\varkappa$ are determined by the conditions

$$\varkappa \star F(Y,Z) = F(\pi(Y,Z)) \star \varkappa \,, \qquad \bar\varkappa \star F(Y,Z) = F(\bar\pi(Y,Z)) \star \bar\varkappa \,, \tag{11.25}$$

where $F(Y, Z)$ is an arbitrary function. Just as in the free case, the twisted-adjoint HS-curvature zero-form $B(Y, Z|x)$ is reproduced from the adjoint one as

$$B \to B \star \varkappa, \qquad \bar{B} \to B \star \bar{\varkappa}. \tag{11.26}$$

Let us note that the fields B and $\bar{B}$ are not independent as might seem from (11.20) which is in accordance with the linearized description where we had a single Weyl module $C(y, \bar{y}|x)$. This fact imposes severe restriction on the form of higher-spin interactions. Indeed if B and $\bar{B}$ were independent, equations (11.17) would be just a definition of the curvature two-form in $dz_\alpha \wedge dz^\alpha$ and $d\bar{z}_{\dot{\alpha}} \wedge d\bar{z}^{\dot{\alpha}}$ sectors on its right hand side. The explicit form of Klein operators can be derived not until the extended (Y, Z) star-product is defined, so let us proceed with its definition.

11.3 Extended star-product, Klein operators, HS equations

Commutation relations (11.22) can be reached via the following star-product

$$f(Y, Z) \star g(Y, Z) = \int dU dV f(Y + U, Z + U) g(Y + V, Z - V) e^{i U_A V^A}. \tag{11.27}$$

Mind the minus sign in the second argument of g-function which guarantees that $[Y, Z]_\star = 0$. Star-product is again can be shown to be associative. It reduces exactly to the Moyal star-product (10.17) once functions f and g are Z-independent. On the space of Y-independent functions we find a formula similar to (10.17) with an extra minus sign in the exponent that ensures (11.22.a). Commutation relations (11.22) then can be easily reproduced from definition (11.27). The following simple formulas will be useful for star-product calculations

$$e^{p^A Y_A} \star f(Y, Z) = e^{p^A Y_A} f(Y + ip, Z - ip), \quad f(Y, Z) \star e^{p^A Y_A} = e^{p^A Y_A} f(Y - ip, Z - ip),$$

$$e^{p^A Z_A} \star f(Y, Z) = e^{p^A Z_A} f(Y + ip, Z - ip), \quad f(Y, Z) \star e^{p^A Z_A} = e^{p^A Z_A} f(Y + ip, Z + ip).$$

Particularly, from these relations it follows, c.f. (10.24),

$$Y_A \star f = \left(Y_A + i\frac{\partial}{\partial Y^A} - i\frac{\partial}{\partial Z^A}\right) f, \quad f \star Y_A = \left(Y_A - i\frac{\partial}{\partial Y^A} - i\frac{\partial}{\partial Z^A}\right) f, \tag{11.28}$$

$$Z_A \star f = \left(Z_A + i\frac{\partial}{\partial Y^A} - i\frac{\partial}{\partial Z^A}\right) f, \quad f \star Z_A = \left(Z_A + i\frac{\partial}{\partial Y^A} + i\frac{\partial}{\partial Z^A}\right) f. \tag{11.29}$$

It is easy to check that thus defined star-product is associative and provides normal ordering for $a_A^\pm = Y_A \pm Z_A$, indeed, as follows from (11.28), (11.29)

$$f \star a^+ = f a^+, \quad a^- \star f = a^- f. \tag{11.30}$$

The following simple formulas we find useful below, c.f. (10.32),

$$[Y_A, f]_\star = 2i\frac{\partial}{\partial Y^A} f, \qquad\qquad [Z_A, f]_\star = -2i\frac{\partial}{\partial Z^A} f, \tag{11.31}$$

$$\{Y_A, f\}_\star = 2\left(Y_A - i\frac{\partial}{\partial Z^A}\right) f, \qquad \{Z_A, f\}_\star = 2\left(Z_A + i\frac{\partial}{\partial Y^A}\right) f, \tag{11.32}$$

91

$$[-\frac{i}{2}Y_A Y_B, f]_\star = \left(Y_B - i\frac{\partial}{\partial Z^B}\right)\frac{\partial}{\partial Y^A}f + \left(Y_A - i\frac{\partial}{\partial Z^A}\right)\frac{\partial}{\partial Y^B}f\,, \tag{11.33}$$

$$\{-\frac{i}{2}Y_A Y_B, f\}_\star = -i\left(Y_A - i\frac{\partial}{\partial Z^A}\right)\left(Y_B - i\frac{\partial}{\partial Z^B}\right)f + i\frac{\partial}{\partial Y^A}\frac{\partial}{\partial Y^B}f\,. \tag{11.34}$$

Klein operators can now be easily derived from their definition (11.25). We already know from (10.62) that in case of Z-independence the Klein operator is a delta-function of (anti)holomorphic oscillator. Similar consideration for Y-independent functions leads to delta-functions $\varkappa_z = 2\pi\delta^2(z)$ and $\bar{\varkappa}_{\bar{z}} = 2\pi\delta^2(\bar{z})$. As a result, the complete Klein operators that satisfy (11.25) are given by

$$\varkappa = \varkappa_y \star \varkappa_z = (2\pi)^2\delta^2(y) \star \delta^2(z) = e^{iz_\alpha y^\alpha}\,, \tag{11.35}$$

$$\bar{\varkappa} = \bar{\varkappa}_y \star \bar{\varkappa}_z = (2\pi)^2\delta^2(\bar{y}) \star \delta^2(\bar{z}) = e^{i\bar{z}_{\dot\alpha}\bar{y}^{\dot\alpha}}\,. \tag{11.36}$$

Note, that the Klein operators turned out to be regular functions in the extended star-product algebra and therefore automorphism (11.23), (11.24) is the inner one. One can check the following straightforward properties

$$\varkappa \star \varkappa = 1\,, \qquad\qquad \varkappa \star f(y, z) = f(z, y)e^{iz_\alpha y^\alpha}\,, \tag{11.37}$$

$$\varkappa \star f(y, z) = f(-y, -z) \star \varkappa\,. \tag{11.38}$$

analogously for antiholomorphic Klein $\bar{\varkappa}$. Note the interchange of variables y and z within the arguments of $f(y, z)$. The equations that account properly for the twist automorphism can be now obtained via substitution (11.26) and (11.35), (11.36) into (11.17)-(11.19)

$$\hat{d}\mathcal{W} + \mathcal{W} \star \mathcal{W} = \Phi \star \varkappa + \bar{\Phi} \star \bar{\varkappa}\,, \tag{11.39}$$

$$\hat{d}\Phi + \mathcal{W} \star \Phi - \Phi \star \pi(\mathcal{W}) = 0\,, \quad \Phi = \frac{i}{4}dz_\alpha \wedge dz^\alpha B\,, \tag{11.40}$$

$$\hat{d}\bar{\Phi} + \mathcal{W} \star \bar{\Phi} - \bar{\Phi} \star \bar{\pi}(\mathcal{W}) = 0\,, \quad \bar{\Phi} = \frac{i}{4}d\bar{z}_{\dot\alpha} \wedge d\bar{z}^{\dot\alpha} B\,. \tag{11.41}$$

The first equation tells us that curvature two-from (11.39) is only allowed to be nonzero in auxiliary $dZ \wedge dZ$ sector being pure gauge in space-time. This is a generic feature of the unfolded equations that tend to get rid of space-time dependence and reformulate the dynamics in the auxiliary "twistor" space. Using definition (11.14) let us rewrite the above equations in the component form.

Before doing this we make one comment. In obtaining (11.39)-(11.41) we carried out field redefinition $\Phi \to \Phi \star \varkappa$, $\bar{\Phi} \to \bar{\Phi} \star \bar{\varkappa}$ to meet the twisted-adjoint requirement. While it is fine to do so in (11.17), the substitution into (11.18) and (11.19) seemingly produces terms of the form $\partial_z(B \star \varkappa)$ and $\partial_{\bar{z}}(B \star \bar{\varkappa})$ which do not allow one to drag the Kleinians through the Z–derivative because of their Z–dependence. It does not happen though as these terms simply do not show up. Indeed the three-forms appearing on l.h.s. of (11.40) and (11.41) are identically zero for $dz \wedge dz \wedge dz \equiv 0$ and $d\bar{z} \wedge d\bar{z} \wedge d\bar{z} \equiv 0$. Another type of potentially dangerous terms $\partial_{\bar{z}}(B \star \varkappa)$ and $\partial_z(B \star \bar{\varkappa})$ are harmless since $\partial_{\bar{z}}(\varkappa) = 0$ and $\partial_z(\bar{\varkappa}) = 0$.

The fact that there are no integrability conditions in zzz and $\bar{z}\bar{z}\bar{z}$ sectors for dimensional reason (spinorial indices take two values) suggests that it might be possible to

impose some extra constraints consistent with equations (11.39)-(11.41). It turns out this is what one actually should do to describe irreducible nonlinear equations for HS bosonic fields for $d = 3$ [110] and arbitrary d [39] systems. In other words, systems (11.17)-(11.19) typically have some spurios solutions due to lack of constraint in extra twistor space. Indeed the way it is written in (11.17)-(11.19) the system just tells us in which sector the HS curvature is allowed to be non-zero, (11.17), without specifying the curvature itself — since the rest conditions (11.18) and (11.19) are simply the integrability consequences. The four dimensional case is to some extent peculiar for it was already pointed out that fields Φ and $\bar{\Phi}$ are not independent rather related to each other through the Klein operators. This fact stipulates some restriction on the possible form of higher-spin curvature that enters r.h.s. of (11.17) and takes place only in the case of four dimensional spinorial system where the two types of spinor fields — holomorphic and antiholomorphic are available. Eventually, the $4d$ extra constraint will turn out to be equivalent to kinematic condition for the system (11.39)-(11.41) to be bosonic. The required kinematic condition is not yet fully there, but it is not going to be a problem to identify it. Impressive is the fact that the desired $4d$ kinematic condition will provide us with some nontrivial algebraic constraint which is typical for all available nonlinear HS systems.

What is more, that two-form Φ is expressed in terms of zero-form B makes Φ a composite field, which solves the problem of the extra gauge symmetry, $\xi_{\mathbf{1}}$, (11.13) that would be associated with Φ if it were a fundamental field. But for now let us proceed with component form of (11.39)-(11.41) which reads

$$dW + W \star W = 0 \,, \tag{11.42}$$

$$dB + W \star B - B \star \pi(W) = 0 \,, \qquad dB + W \star B - B \star \bar{\pi}(W) = 0 \,, \tag{11.43}$$

$$\frac{\partial \bar{A}_{\dot\alpha}}{\partial z^\alpha} - \frac{\partial A_\alpha}{\partial \bar{z}^{\dot\alpha}} + [A_\alpha, \bar{A}_{\dot\alpha}]_\star = 0 \,, \tag{11.44}$$

$$dA_\alpha + [W, A_\alpha]_\star - \frac{\partial W}{\partial z^\alpha} = 0 \,, \qquad d\bar{A}_{\dot\alpha} + [W, \bar{A}_{\dot\alpha}]_\star - \frac{\partial W}{\partial \bar{z}^{\dot\alpha}} = 0 \,, \tag{11.45}$$

$$\frac{\partial A^\alpha}{\partial z^\alpha} + A_\alpha \star A^\alpha = \frac{i}{2} B \star \varkappa \,, \qquad \frac{\partial \bar{A}^{\dot\alpha}}{\partial \bar{z}^{\dot\alpha}} + \bar{A}_{\dot\alpha} \star \bar{A}^{\dot\alpha} = \frac{i}{2} B \star \bar{\varkappa} \,, \tag{11.46}$$

$$\frac{\partial B}{\partial z^\alpha} + A_\alpha \star B - B \star \bar{\pi}(A_\alpha) = 0 \,, \qquad \frac{\partial B}{\partial \bar{z}^{\dot\alpha}} + \bar{A}_{\dot\alpha} \star B - B \star \pi(\bar{A}_{\dot\alpha}) = 0 \,, \tag{11.47}$$

which results from (11.42)-(11.44) as coefficients of $dx \wedge dx$, (11.42); $dx \wedge dz \wedge dz$, (11.43.a), $dx \wedge d\bar{z} \wedge d\bar{z}$, (11.43.b); $dz \wedge d\bar{z}$, (11.44); $dx \wedge dz$, (11.45.a), $dx \wedge d\bar{z}$, (11.45.b); $dz \wedge dz$, (11.46.a), $d\bar{z} \wedge d\bar{z}$, (11.46.b); $dz \wedge d\bar{z} \wedge \bar{z}$, (11.47.a), $d\bar{z} \wedge dz \wedge z$, (11.47.b). While system (11.11)-(11.12) is manifestly integrable, it is not so obvious for the component form (11.42)-(11.47).

One may argue that the form of equations contradicts the perturbative scheme laid out in (11.4)-(11.5) as (11.42) seemingly contains no higher-spin corrections that appear on the r.h.s. of (11.4) already at free level, (11.2). Recall, however, that master fields W and B having got an extra dependence on extra Z-variable would provide the desired HS corrections for dynamical fields $w(Y|x)$ and $C(Y|x)$ through star-product (11.27). Let us now enlist some obvious properties of the obtained equations

- The equations are to be bosonic. Note, that there is π-automorphism that enters (11.43). There is however similar equation arising from (11.41) with π replaced by

93

$\bar{\pi}$. Therefore the system makes sense only for $\pi(W) = \bar{\pi}(W)$, or equivalently

$$W(-Y, -Z|x) = W(Y, Z|x) \qquad \Longleftrightarrow \qquad \varkappa \star \bar{\varkappa} \star W = W \star \varkappa \star \bar{\varkappa} \,. \qquad (11.48)$$

The parity property for master field $W(Y, Z|x)$ is a manifestation of the bosonic nature of the system[55]. It would imply $\omega(Y|x) = \omega(-Y|x)$ for the physical field.

Correspondingly, the gauge symmetry parameter ϵ must be an even function too $\epsilon(Y, Z|x) = \epsilon(-Y, -Z|x)$. It has an immediate consequence

$$A_A(Y, Z|x) = -A_A(-Y, -Z|x) \qquad \Longleftrightarrow \qquad \varkappa \star \bar{\varkappa} \star A_A = -A_A \star \varkappa \star \bar{\varkappa} \,, \qquad (11.49)$$

since W and $A_A = (A_\alpha, A_{\dot\alpha})$ are parts of the same connection. Fields and gauge parameters take values in the same space, now it is the space of even functions $\epsilon(Y, Z)$. That the gauge transformation for A, $\delta A_A = \frac{\partial}{\partial Z^A}\epsilon + ...$, contains a derivative along Z direction changes the parity of A_A as compared to W for which the gauge transformation $\delta W = d\epsilon + ...$ does not affect the parity in Y, Z space.

The bosonic projection (11.49) immediately implies from (11.45) the corresponding projection for the zero-forms[56]

$$B(Y, Z|x) = B(-Y, -Z|x) \qquad \Longleftrightarrow \qquad \varkappa \star \bar{\varkappa} \star B = B \star \varkappa \star \bar{\varkappa} \,. \qquad (11.50)$$

Let us stress that unlike (11.48), condition (11.49) and its consequence (11.50) do *not* follow from the equations. It is this missing kinematic condition (11.49) that will be equivalent to some extra algebraic constraint consistent with (11.42)-(11.47).

- Gauge symmetry. Component form of gauge transformations results from (11.21) upon redefinition (11.26)

$$\delta W = d\epsilon + [W, \epsilon] \,, \qquad (11.51)$$

$$\delta A_A = \frac{\partial \epsilon}{\partial Z^A} + [A_A, \epsilon] \,, \qquad (11.52)$$

$$\delta B = B \star \pi(\epsilon) - \epsilon \star B \qquad (11.53)$$

- Purely gauge space-time dependence. From (11.42), (11.43) and (11.45) one can always determine space-time dependence of the master fields in a pure gauge fashion

$$W = g^{-1} \star dg \,, \qquad (11.54)$$

$$B = g^{-1} \star B_0(Y, Z) \star \pi(g) \,, \qquad (11.55)$$

$$A_A = g^{-1} \star \frac{\partial g}{\partial Z^A} + g^{-1} \star A_A^0(Y, Z) \star g \,, \qquad (11.56)$$

where $g = g(Y, Z|x)$ is an arbitrary even function. This is a general statement that any covariant constancy/zero curvature equations can be at least formally solved in

---

[55]It is possible to include fermions into nonlinear system by introducing some extra Klein operators [38]. The resulting equations are supersymmetric and contain two copies of each spin.

[56]When no bosonic constraints imposed one has a strange theory containing bosons described at the free level by gauge fields and fermions described by gauge-invariant generalized Weyl tensors.

the pure gauge form. Note that the curvature for $A_\alpha, \bar{A}_{\dot\alpha}$ is not entirely zero, which explains the extra term in the last line.

These formulas suggest, for example, that one can gauge away W–field that is supposed to encode HS gauge potentials. While formally it looks as really the case, do not forget that such "gauging" when applied washes away AdS space-time itself making its frame field and Lorentz connection equal to zero, see discussion after (10.52). It raises an important question of admissible gauge transformations, which draw a line between small gauge transformations, which are true gauge transformations, and large gauge transformations that relate physically distinguishable solutions. Space-time independent functions B_0 and A_0 play a role of initial data imposed at a given space-time point x_0 where $g(Y, Z|x_0) = 1$.

- It is now obvious that $F^\omega(\omega, C)$ and $F^C(\omega, C)$ in (11.4)-(11.5) are more constrained than just by $d^2 = 0$, see discussion around (6.35)-(6.36). Indeed $dW + W \star W = 0$ yields $dB + W \star B - B \star \pi(W) = 0$ upon applying $B \star \varkappa \frac{\delta}{\delta W}$ to the former. This proves (6.35)-(6.36) to all orders.

As it was mentioned already, the system we are dealing with requires the bosonic kinematic constraint (11.49) which is not a consequence of the equations. Our goal is to rewrite (11.49) in some algebraic way and add it to the system so as to make the bosonic nature of the equations intrinsic and manifest. To proceed in this direction let us first perform some harmless field redefinition with A-field

$$A_\alpha = \frac{i}{2}(S_\alpha - z_\alpha), \qquad \bar{A}_{\dot\alpha} = \frac{i}{2}(\bar{S}_{\dot\alpha} - \bar{z}_{\dot\alpha}), \qquad (11.57)$$

where S is some new field. The shift of vacuum value of A-field is designed to eliminate partial derivatives with respect to z-variable in (11.44)-(11.47), the coefficient $i/2$ is chosen appropriately to account for interplay between $[z, f]_\star = -2i\partial_z f$ and $\partial_z f$ terms. Shift (11.57) yields

$$[S_\alpha, \bar{S}_{\dot\alpha}]_\star = 0, \qquad (11.58)$$

$$[S_\alpha, S_\beta]_\star = -2i\epsilon_{\alpha\beta}(1 + B \star \varkappa), \qquad [\bar{S}_{\dot\alpha}, \bar{S}_{\dot\beta}]_\star = -2i\epsilon_{\dot\alpha\dot\beta}(1 + B \star \bar{\varkappa}), \qquad (11.59)$$

$$[S_\alpha, B \star \bar{\varkappa}]_\star = 0, \qquad [\bar{S}_{\dot\alpha}, B \star \varkappa]_\star = 0, \qquad (11.60)$$

Note, that (11.60) is a consequence of (11.58) and (11.59). Clearly, should one had $B \star \varkappa = B \star \bar{\varkappa} = 0$ the above commutation relations would simply correspond to two copies of Weyl algebra generated by S_α and $\bar{S}_{\dot\alpha}$. If B is not equal to zero we see, that $B \star \varkappa$ is a central element for $\bar{S}_{\dot\alpha}$ and $B \star \bar{\varkappa}$ — for S_α. From (11.60) and (11.49) it immediately follows

$$\{S_\alpha, B \star \varkappa\}_\star = 0, \qquad \{\bar{S}_{\dot\alpha}, B \star \bar{\varkappa}\}_\star = 0. \qquad (11.61)$$

Algebraic condition (11.61) eventually brings us to explicitly bosonic and complete non-linear HS equations. That (11.61) respects the Jacobi identities deserves special attention. Set of equations (11.58)-(11.60) supplemented with (11.61) represents two copies of what

95

is known as deformed oscillators. Consider one copy that can be defined as follows. Let the generating elements $\hat{y}_\alpha$ and K satisfy the relations

$$[\hat{y}_\alpha, \hat{y}_\beta] = -2i\epsilon_{\alpha\beta}(1 + K), \qquad \{\hat{y}_\alpha, K\} = 0. \tag{11.62}$$

The deformed oscillators (11.62) were originally discovered by Wigner in [111] and happened to be related and Calogero model [112]. It is interesting that if one looks at the Jacobi $[[\hat{y}_\alpha, \hat{y}_\beta], \hat{y}_\gamma] + cycle = 0$ it is not going to hold in general unless indices of $\hat{y}_\alpha$ take two values so that antisymmetrization of any three gives identically zero. This is the case for two copies of deformed oscillators generated by HS master fields (11.58)-(11.60) and (11.61). Another very important property is that the deformation of Heisenberg algebra (11.62) respects $sp(2)$ symmetry. Indeed it is easy to see that generators $T_{\alpha\beta} = \frac{i}{4}\{\hat{y}_\alpha, \hat{y}_\beta\}$ form $sp(2)$ algebra

$$[T_{\alpha\beta}, T_{\gamma\delta}] = \epsilon_{\alpha\gamma}T_{\beta\delta} + (\alpha \leftrightarrow \beta) + (\gamma \leftrightarrow \delta) \tag{11.63}$$

which yet rotates $\hat{y}_\alpha$ as a vector

$$[T_{\alpha\beta}, \hat{y}_\gamma] = \epsilon_{\alpha\gamma}\hat{y}_\beta + \epsilon_{\beta\gamma}\hat{y}_\alpha. \tag{11.64}$$

Deformed oscillator properties (11.63) and (11.64) will be of utter importance in identifying local Lorentz symmetry for HS system which guarantees tensor interpretation of the dynamical fields.

Remarkably, it is the deformed oscillator constraint (11.61) that one can additionally impose to (11.39)-(11.41) leads to the nonlinear system for bosonic massless fields which we can finally write down in the form

$$dW + W \star W = 0, \tag{11.65}$$

$$dB + W \star B - B \star \pi(W) = 0, \tag{11.66}$$

$$dS_\alpha + [W, S_\alpha]_\star = 0, \qquad\qquad d\bar{S}_{\dot\alpha} + [W, \bar{S}_{\dot\alpha}]_\star = 0, \tag{11.67}$$

$$[S_\alpha, S_\beta]_\star = -2i\epsilon_{\alpha\beta}(1 + B \star \varkappa), \qquad [\bar{S}_{\dot\alpha}, \bar{S}_{\dot\beta}]_\star = -2i\epsilon_{\dot\alpha\dot\beta}(1 + B \star \bar\varkappa), \tag{11.68}$$

$$\{S_\alpha, B \star \varkappa\}_\star = 0, \qquad\qquad \{\bar{S}_{\dot\alpha}, B \star \bar\varkappa\}_\star = 0, \tag{11.69}$$

$$[S_\alpha, \bar{S}_{\dot\alpha}]_\star = 0. \tag{11.70}$$

The system of equations (11.65)-(11.70) is known as Vasiliev nonlinear equations for HS bosonic fields in four dimensions. It has a form of (11.42)-(11.47) upon field redefinition (11.57) with an extra constraint (11.69) that makes this system explicitly bosonic. Written this way it contains some flatness conditions in space-time (11.65)-(11.67) and a set of algebraic constraints (11.68)-(11.70) which are nothing but a direct sum of two deformed oscillators given by S_α and $\bar{S}_{\dot\alpha}$. The system correspondingly inherits local gauge invariance

$$\delta W = d\xi + [W, \xi]_\star, \tag{11.71}$$

$$\delta B = B \star \pi(\xi) - \xi \star B \qquad\Longleftrightarrow\qquad \delta(B \star \varkappa) = [B \star \varkappa, \xi]_\star, \tag{11.72}$$

$$\delta S_\alpha = [S_\alpha, \xi]_\star, \tag{11.73}$$

$$\delta\bar{S}_{\dot\alpha} = [\bar{S}_{\dot\alpha}, \xi]_\star. \tag{11.74}$$

As it was already mentioned, introducing the deformed oscillator anticommutator condition (11.69) one imposes extra kinematic constraints (11.49), (11.50) and vice versa.

Eq. (11.49) can be easily obtained using e.g., $[S_\alpha, B \star \bar{\varkappa}]_\star = 0$ and $\{S_\alpha, B \star \varkappa\}_\star = 0$. Once it is proved, eq. (11.50) follows immediately from the fact that $S \star S$ is an even function. Together (11.48) and (11.49), (11.50) consistently imply that the system under consideration is bosonic.

One thing that was missed so far is the reality conditions for master fields. These are dictated by the star-product properties and free level analysis. Without going into details we give the final result

$$y_\alpha^\dagger = \bar{y}_{\dot\alpha}, \qquad z_\alpha^\dagger = -\bar{z}_{\dot\alpha}, \tag{11.75}$$

$$W^\dagger = -W, \tag{11.76}$$

$$S_\alpha^\dagger = -\bar{S}_{\dot\alpha}, \tag{11.77}$$

$$B^\dagger = \pi(B). \tag{11.78}$$

An important difference between the free and interacting equations is the doubling of oscillators $Y \to (Y, Z)$ and appearance of extra S-field. A question is whether this procedure preserves physical degrees of freedom which as we have seen are encoded in a single function $C(Y|x)$ and whether the linearized approximation results in (9.70)-(9.72). We will see, that while from (11.68) one expresses B in terms of $S \star S$, perturbatively it is the S-field that appears to be totally auxiliary and is expressed on-shell in terms of B-field modulo gauge ambiguity. As for an extra Z-dependence, it turns out to be fixed again up to a gauge ambiguity by the extra condition (11.69).

Generalizations and reductions. We can ask ourselves to what extent the form of the equations (11.65)-(11.70) unique and what the possible generalizations or ambiguities in higher-spin interactions are. One assumption that was used in writing down system (11.39)-(11.41) that there are no mixing terms $dz_\alpha \wedge d\bar{z}_{\dot\alpha}$ in the auxiliary space curvature sector. These terms when present violate local Lorentz symmetry since to convert spinor indices one has to introduce some field $H_{\alpha\dot\alpha}(B)$ that breaks down the symmetry explicitly. So, this possibility is forbidden by the equivalence principle. Another possible modification which does not ruin formal integrability of the Vasiliev equations is to change $B \star \varkappa$ and $B \star \bar{\varkappa}$ on the r.h.s. of (11.68) to $f_\star(B \star \varkappa)$ and $\bar{f}_\star(B \star \bar{\varkappa})$, correspondingly. While it is possible to make such a modification one can partially eliminate its effect by field redefinition $B \to F(B)$ which leaves nontrivial only the phase in complex function f. In other words the ambiguity one cannot get rid of by field redefinition is given by arbitrary real function $\phi(B)$ that enter (11.68) as $\exp_\star(i\phi_\star(B \star \varkappa)) \star B \star \varkappa$ in holomorphic part and $\exp_\star(-i\phi_\star(B \star \bar{\varkappa})) \star B \star \bar{\varkappa}$ in the antiholomorphic. The cases of $\phi = 0, \pi/2$ corresponds to A and B models correspondingly. Finally, there is a way to introduce fermions in the system by doubling the set of fields and add some nondynamical moduli that play the role of different parameters of interaction, see [38]. It is also possible to truncate the bosonic system to the fields with even spins only, $s = 0, 2, 4, 6, ...$

11.4   Perturbation theory

In perturbation theory one starts with appropriate exact vacuum solution. For a field theory, a classical vacuum is some background having no fields propagating on it. In

case of HS gauge theory, the proper vacuum is the AdS space-time as we already know, therefore the vacuum W_0 one-form should be taken from (10.43) such that (10.44) is satisfied and as long as no dynamical fields are around we take $B = 0$ and $A_A = 0$. One has then the following exact solution of (11.65)-(11.70)

$$W_0 = \Omega = -\frac{i}{4}(\omega_{\alpha\beta}y^\alpha y^\beta + \bar{\omega}_{\dot{\alpha}\dot{\beta}}\bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\beta}} + 2h_{\alpha\dot{\alpha}}y^\alpha\bar{y}^{\dot{\alpha}}) = \frac{1}{2}T^{AB}\Omega_{AB}, \tag{11.79}$$

$$B_0 = 0, \tag{11.80}$$

$$S_{0\alpha} = z_\alpha, \qquad \bar{S}_{0\dot{\alpha}} = \bar{z}_{\dot{\alpha}}, \tag{11.81}$$

where Ω is a good old AdS_4 flat connection, (9.33)-(9.36). The vacuum for S_A is designed to undo shift (11.57) and to get back to (11.42)-(11.47), for which it is obvious that any flat Ω, $B = 0$ and $A_\alpha = A_{\dot{\alpha}} = 0$ is an exact solution.

Having fixed the vacuum we can look at perturbative expansion

$$W = W_0 + W_1 + \cdots = \Omega + W_1 + \dots, \tag{11.82}$$

$$B = B_0 + B_1 + \cdots = 0 + B_1 + \dots, \tag{11.83}$$

$$S_A = S_{0A} + S_{1A} + \cdots = Z_A + S_{1A} + \dots. \tag{11.84}$$

A general scheme for looking at perturbative series for Vasiliev equations is to first solve for the algebraic constraints (11.68)-(11.70) and then substitute the solution into space-time part of equations (11.65)-(11.67). At first order from (11.67) we have

$$z_\alpha \star B_1 \star \varkappa + B_1 \star \varkappa \star z_\alpha = 0, \qquad \bar{z}_{\dot{\alpha}} \star B_1 \star \bar{\varkappa} + B_1 \star \bar{\varkappa} \star \bar{z}_{\dot{\alpha}} = 0, \tag{11.85}$$

that using (11.25) gives

$$[z_\alpha, B_1]_\star = [\bar{z}_{\dot{\alpha}}, B_1]_\star = 0 \qquad \Longleftrightarrow \qquad \frac{\partial B_1}{\partial Z^A} = 0. \tag{11.86}$$

Its generic solution is

$$B_1 = C(y, \bar{y}|x), \tag{11.87}$$

where C is an arbitrary z-independent function. We see that at first order B-field appeared to be z-independent by (11.69) — being a manifestation of generic property stating that z-dependence of B-field is always perturbatively fixed by (11.69). Now the space-time evolution of $C(Y|x)$ is governed by (11.66) with $W = \Omega$ resulting in twisted-adjoint flatness condition (9.72). This way we find that at free level master field B indeed describes HS curvatures via (9.72), i.e. $\tilde{D}_\Omega C = 0$. The next step is to reproduce the on-mass-shell theorem (9.70) from the linearized equations. In order to do so, we first evaluate S_{1A} from (11.68) which in our case gives

$$[S_{0\alpha}, S_1^\alpha]_\star = -2iC \star \varkappa \tag{11.88}$$

and similarly for $\bar{S}_{\dot{\alpha}}$. Substituting $S_{0\alpha}$ from (11.81) and using (11.37), we get

$$\frac{\partial S_1^\alpha}{\partial z^\alpha} = C \star \varkappa = C(-z, \bar{y})e^{iz_\beta y^\beta}. \tag{11.89}$$

98

Before we proceed let us note that there are two types of equations that steadily appear in perturbation analysis, see Appendix 12.7. These are

$$\frac{\partial f^\alpha}{\partial z^\alpha} = g(z) \,, \tag{11.90}$$

$$\frac{\partial f}{\partial z^\alpha} = g_\alpha \,. \tag{11.91}$$

There is a consistency constraint for the left hand side of (11.91) which requires $\frac{\partial g^\alpha}{\partial z^\alpha} = 0$. Generic solutions to (11.90) and (11.91) can be written down in the form of homotopy integrals

$$f_\alpha = \frac{\partial \eta}{\partial z^\alpha} + z_\alpha \int_0^1 dt\, t\, g(tz) \,, \tag{11.92}$$

$$f = c + z^\alpha \int_0^1 dt\, g_\alpha(tz) \,, \tag{11.93}$$

where c is z-independent and η is an arbitrary function. The generic solution to (11.89) can be explicitly written now

$$\begin{cases} S_{1\alpha} &= \partial_\alpha \xi_1 + z_\alpha \int_0^1 dt\, t\, C(-tz, \bar{y}) e^{itz_\beta y^\beta} \,, \\ \bar{S}_{1\dot\alpha} &= \partial_{\dot\alpha} \xi_1 + \bar{z}_{\dot\alpha} \int_0^1 dt\, t\, C(y, -t\bar{z}) e^{it\bar{z}_{\dot\beta} \bar{y}^{\dot\beta}} \,, \end{cases} \tag{11.94}$$

where $\xi_1(Y, Z|x)$ is arbitrary function that plays a role of gauge ambiguity and can be set to zero for convenience. We prefer, however, to keep them nonzero so as to make sure that they will not affect on-mass-shell theorem (9.70). One can see at this level, that S indeed is auxiliary being a functional of B. The next step is to substitute (11.94) into (11.67) that gives the following equation

$$D_\Omega S_{1A} + [W_1, S_{0A}] = 0 \,, \tag{11.95}$$

where $D_\Omega = d + [\Omega, \bullet]_\star$, which is nilpotent $D_\Omega{}^2 = 0$ on account of (11.79) and its explicit action can be easily found from (11.28) to be

$$D_\Omega f = df + \Omega^{AB}\left(Y_A - i\frac{\partial}{\partial Z^A}\right)\frac{\partial}{\partial Y^B} f \,. \tag{11.96}$$

Substituting (11.81) into (11.95) we have

$$\partial_\alpha W_1 = \frac{i}{2} D_\Omega S_{1\alpha} \,, \qquad \partial_{\dot\alpha} W_1 = \frac{i}{2} D_0 \bar{S}_{1\dot\alpha} \,. \tag{11.97}$$

These are the equations of type (11.91) and we can solve them as

$$W_1 = \frac{i}{2} D_\Omega \xi_1 + \frac{i}{2} z^\alpha \int_0^1 dt\, D_\Omega \hat{S}_{1\alpha}|_{z\to tz} + c.c. + \omega(y, \bar{y}|x) \,, \tag{11.98}$$

where $\omega(y, \bar{y}|x)$ plays a role of a constant with respect to z in (11.93), while $\hat{S}_1$ means that we explicitly extracted pure gauge dependence from (11.94) which is given by the first

term in (11.98). Note that $\hat{S}_{1\alpha}$ is proportional to z_α and, since $z^\alpha z_\alpha \equiv 0$, the only terms that survive are those for which D_Ω hits z_α inside $\hat{S}_{1\alpha}$. There are several such terms,

$$D_\Omega \ni -i\omega^{\alpha\beta}\frac{\partial^2}{\partial y^\alpha \partial z^\beta} - i\bar{\omega}^{\dot{\alpha}\dot{\beta}}\frac{\partial^2}{\partial \bar{y}^{\dot{\alpha}} \partial \bar{z}^{\dot{\beta}}} - ih^{\alpha\dot{\alpha}}\frac{\partial^2}{\partial \bar{y}^{\dot{\alpha}} \partial z^\alpha} - ih^{\alpha\dot{\alpha}}\frac{\partial^2}{\partial \bar{z}^{\dot{\alpha}} \partial y^\alpha} \,, \tag{11.99}$$

which altogether yield

$$W_1 = \frac{i}{2}D_\Omega\xi_1 + \omega(y,\bar{y}|x) +$$
$$+ \frac{i}{2}\int_0^1 (1-t)\, dt \left(\left(z_\alpha \omega^{\alpha\alpha} z_\alpha t + i z_\alpha h^{\alpha\dot{\alpha}}\frac{\partial}{\partial \bar{y}^{\dot{\alpha}}} \right) C(-zt,\bar{y}) \right) e^{itz_\beta y^\beta} + c.c. \tag{11.100}$$

In accordance with our expectation that physical fields are the initial conditions at $Z=0$ we find that $W_1(Y,0)$ gives gauge connections $\omega(Y)$ up to a gauge transformation

$$W_1\Big|_{Z=0} = \frac{i}{2}D_\Omega\xi_1\Big|_{Z=0} + \omega(y,\bar{y}|x)\,. \tag{11.101}$$

Final step is to substitute (11.100) into (11.65) to determine space-time evolution of $\omega(y,\bar{y}|x)$. At first order this substitution gives for $D_\Omega W_1 = 0$

$$D_\Omega \omega(y,\bar{y}|x) = -D_\Omega \left(\frac{i}{2}\int_0^1 dt\, D_\Omega \hat{S}_{1\alpha}|_{z\to tz} + c.c. \right). \tag{11.102}$$

Note, that gauge ξ-terms do not contribute to (11.102) as they enter the right hand side via $D_\Omega{}^2\xi \equiv 0$. So, indeed, we see that arbitrary function that arises in S field as an integration constant corresponds to a gauge freedom. Now, the left hand side of (11.102) is explicitly z-independent and so should be its right hand side. In other words, the r.h.s. z-dependence is fictitious as becomes obvious after integration by parts. Equivalently,

$$\frac{\partial}{\partial z^\alpha}D_\Omega W_1 = D_\Omega \frac{\partial}{\partial z^\alpha}W_1 = D_\Omega\left(\frac{i}{2}D_\Omega S_{1\alpha} \right) = \frac{i}{2}D_\Omega D_\Omega S_{1\alpha} = 0\,. \tag{11.103}$$

It makes it convenient to calculate (11.102) at $z=0$ and be sure that the result is correct. For that reason we only need 2nd derivative terms in (11.96), i.e. (11.99), because otherwise there will be the terms on the r.h.s. of (11.102) proportional to z or zz prior to homotopy integration that do not contribute at $z=0$ anyway. Up to irrelevant after substitution in (11.102) terms we find

$$W_1 = \omega(Y) - \frac{1}{2}h^{\alpha\dot{\alpha}}z_\alpha \frac{\partial}{\partial \bar{y}^{\dot{\alpha}}}\int_0^1 dt\,(1-t)\,C(-tz,\bar{y})h^{itz_\beta y^\beta} + c.c. + O(z^2) \tag{11.104}$$

Now we need to apply D_Ω to (11.104) and set $z=0$. Again, the nontrivial part is reached only for the second derivative term in (11.96)

$$(11.102) = -h^{\beta\dot{\beta}}\wedge h^{\alpha\dot{\alpha}}\frac{i}{2}\frac{\partial}{\partial z^\beta}z_\alpha \frac{\partial}{\partial \bar{y}^{\dot{\beta}}\partial \bar{y}^{\dot{\alpha}}}\int_0^1 dt\,(1-t)\,C(0,\bar{y}) + c.c. = \tag{11.105}$$
$$= -\frac{i}{4}h_\alpha{}^{\dot{\alpha}}\wedge h^{\alpha\dot{\beta}}\frac{\partial^2 C(0,\bar{y})}{\partial \bar{y}^{\dot{\alpha}}\partial \bar{y}^{\dot{\beta}}} + c.c.\,.$$

100

So one arrives at the central on-mass-shell theorem (9.70)

$$D_\Omega \omega(y, \bar{y}|x) = -\frac{i}{4} h_\alpha{}^{\dot\alpha} \wedge h^{\alpha\dot\beta} \frac{\partial^2}{\partial \bar{y}^{\dot\alpha} \partial \bar{y}^{\dot\beta}} C(0, \bar{y}) + c.c. . \tag{11.106}$$

This completes the free level analysis. It has shown that at linearized approximation the equations do describe bosonic Fronsdal fields along with spin-zero free scalar being a part of HS multiplet in accordance with Sections 9, 10. Moreover, from perturbation theory it is clear that all degrees of freedom are encoded in a single function $C(Y|x)$. Those as many as of free fields governed by the Weyl module. It guarantees that at nonlinear level one has perturbatively the same amount of degrees of freedom, yet the unfolded form of equations (11.65)-(11.70) prevents any field redefinitions that could possibly reduce nonlinear equations to the linear ones.

The important question is whether the components of master fields W, B and S can be eventually treated as space-time tensors in accordance with the equivalence principle or not. Recall that within the unfolded approach the dictionary between fiber fields and world tensors is achieved by the local Lorentz symmetry. This raises the question whether the Lorentz symmetry acts on equations (11.65)-(11.70).

At free level the equation that we got (11.106), contained background Lorentz connections $\omega_{\alpha\beta}$ and $\bar{\omega}_{\dot\alpha\dot\beta}$ via the Lorentz covariant derivative D as a part of the AdS_4 covariant derivative D_Ω and therefore in a covariant fashion. That makes it possible to convert components of $\omega(y, \bar{y}|x)$ into space time tensors with the aid of frame field. The fact that equation (11.106) would be Lorentz covariant was not at all obvious from the point of view of initial equations (11.65)-(11.70) in the first place. Indeed, when fields depend on both of star-product variables Y and Z, (11.96) no longer acts covariantly. As an example, take $f = f_{\alpha\beta}(x) y^\alpha z^\beta$, then $D_\Omega f = Df - i\omega^{\alpha\beta} f_{\alpha\beta}$ which contains Lorentz connection in a noncovariant way. The same problem would be with (11.100) unless noncovariant terms disappeared from (11.106).

The fact that at free level noncovariant terms for one-forms in W dropped away turned out to be totally coincidental and is not going to take place any longer already at second order. This brings us to the problem of local Lorentz symmetry and laborious search for the field redefinition that would respect it. Luckily, the Lorentz symmetry happens to be intrinsically resided in the Vasiliev system and this fact is crucially related to the property of deformed oscillators (11.63). Moreover, the very existence of Lorentz symmetry to much extent fixes the equations of motion at the end of the day and could have been a guideline for their derivation.

11.5 Manifest Lorentz symmetry

Let us recall the notion of the Lorentz connection. One-form $\omega^{\alpha\beta}$ is said to be a spin-connection if it transforms under the local Lorentz transformations with parameter $\Lambda^{\alpha\beta}$ as

$$\delta\omega^{\alpha\beta} = d\Lambda^{\alpha\beta} + \omega^\alpha{}_\gamma \Lambda^{\gamma\beta} + \omega^\beta{}_\gamma \Lambda^{\alpha\gamma} \tag{11.107}$$

Having some generators $t_{\alpha\beta}$ of $sp(2)$ we can introduce $\omega = \frac{1}{2}\omega^{\alpha\beta}t_{\alpha\beta}$, $\Lambda = \frac{1}{2}\Lambda^{\alpha\beta}t_{\alpha\beta}$ and rewrite it as

$$\delta\omega = d\Lambda + [\omega, \Lambda] \tag{11.108}$$

An object $w^{\alpha(k)}$ is called a rank-k spin-tensor if it transforms as

$$\delta w^{\alpha(k)} = \Lambda^{\alpha}{}_{\gamma} w^{\gamma\alpha(k-1)}. \tag{11.109}$$

The derivative of $w^{\alpha(k)}$ must be always accompanied by a term with the spin-connection to give Lorentz-covariant derivative

$$dw^{\alpha(k)} + \omega^{\alpha}{}_{\gamma} w^{\gamma\alpha(k-1)}. \tag{11.110}$$

We are being somewhat clingy in defining Lorentz connection and local spin-tensors on purpose as we will face a problem that not every object with indices can be called a spin-tensor. Sometimes it can transform in a wrong way under local Lorentz transformations. For example, instead of (11.110) one may have something like, c.f. (11.100),

$$\omega_{\beta\gamma}w^{\beta\gamma\alpha(k-2)} \qquad \text{or} \qquad \omega^{\alpha\alpha}w^{\alpha(k)} \tag{11.111}$$

as a contribution to one of the nonlinear equations. These terms can in principle appear. And they do appear. Having such terms would lead to problems in interpretation as they violate the equivalence principle discussed at the end of the previous section. The crucial statement about the Vasiliev equations is that one can find a field redefinition that removes (11.111)-like terms and the spin-connection can then be found to appear in the form of Lorentz-covariant derivative only.

To proceed, let us search for the Lorentz generators that rotate master fields properly. At free level we know that these are given by

$$L^{y}_{\alpha\beta} = -\frac{i}{2}y_{\alpha}y_{\beta}, \qquad \bar{L}^{y}_{\dot\alpha\dot\beta} = -\frac{i}{2}\bar{y}_{\dot\alpha}\bar{y}_{\dot\beta}. \tag{11.112}$$

Their action on the free master fields is indeed of Lorentz transformation

$$\Lambda = \frac{1}{2}\Lambda^{\alpha\beta}L^{y}_{\alpha\beta} + c.c \tag{11.113}$$

$$\delta_{\Lambda}C(y,\bar{y}|x) = [\Lambda, C]_{\star} = \left(\Lambda^{\alpha\beta}y_{\alpha}\frac{\partial}{\partial y^{\beta}} + \bar{\Lambda}^{\dot\alpha\dot\beta}\bar{y}_{\dot\alpha}\frac{\partial}{\partial\bar{y}^{\dot\beta}}\right)C(y,\bar{y}|x), \tag{11.114}$$

which when rewritten in component form gives precisely (11.109). The free HS connections encoded in $\omega(y,\bar{y}|x)$ transform analogously. Indeed, from

$$\delta_{\Lambda}\omega(y,\bar{y}|x) = d\Lambda + [\omega, \Lambda] \tag{11.115}$$

it follows, that if one decomposes

$$\omega = \hat{\omega} + \omega^{L}, \qquad \omega^{L} = \frac{1}{2}\omega^{L}_{\alpha\beta}L^{\alpha\beta}_{y} + c.c. \tag{11.116}$$

in other words if one separates the bilinear in oscillators part of ω from the rest, then one arrives at the Lorentz connection and Lorentz tensor fields transformation

$$\delta_\Lambda \omega^L = d\Lambda + [\omega^L, \Lambda], \qquad \delta_\Lambda \hat{\omega} = [\hat{\omega}, \Lambda]. \qquad (11.117)$$

Note also that, while C transforms in the twisted-adjoint under HS transformations (11.72), the Lorentz subalgebra action reduces to the adjoint one, since π does not affect Lorentz generators. The transformations (11.112) clearly do not extend to nonlinear level as they do not act on Z-variable and hence fields that carry indices contracted with Z^A will not be affected by (11.112). This problem can be easily fixed by appending L^y with similar generators L^z that rotate z

$$L^y_{\alpha\beta} \rightarrow L^0_{\alpha\beta} = L^y_{\alpha\beta} + L^z_{\alpha\beta} = -\frac{i}{2}(y_\alpha y_\beta - z_\alpha z_\beta), \qquad (11.118)$$

$$[L^0_{\alpha\alpha}, f]_\star = (y_\alpha \frac{\partial}{\partial y^\alpha} + z_\alpha \frac{\partial}{\partial z^\alpha})f, \qquad (11.119)$$

analogously for $\bar{L}$. Note, that the sign in (11.118) is important as it guarantees cancelation of second derivative terms. Thus defined generators form Lorentz algebra which properly rotates spinor oscillators

$$[L^0_{\alpha\beta}, L^0_{\gamma\delta}]_\star = \epsilon_{\beta\gamma} L^0_{\alpha\delta} + \dots, \qquad (11.120)$$

$$[L^0_{\alpha\beta}, y_\gamma]_\star = \epsilon_{\beta\gamma} y_\alpha + \epsilon_{\alpha\gamma} y_\beta, \qquad (11.121)$$

$$[L^0_{\alpha\beta}, z_\gamma]_\star = \epsilon_{\beta\gamma} z_\alpha + \epsilon_{\alpha\gamma} z_\beta \qquad (11.122)$$

and implies that field $B(Y, Z|x) \star \varkappa$ does transform in Lorentz covariant fashion. The nonlinear system (11.65)-(11.70) contains field S apart from master fields B and W that on-shell perturbatively expressed via B-field. We know that in solving for Z-dependence S_α is reconstructed in terms of $B \star \varkappa$ up to a gauge freedom. The subtle point is that the gauges chosen in perturbative expansion of S_α should be such that they do not introduce any external objects. This is what makes field $S = S[B]$ purely auxiliary. In other words, at each stage of perturbative expansion with S_α defined up to a $\partial_\alpha \epsilon(Y, Z)$-term the gauge parameter ϵ is supposed to contain no extra fields in its z-dependence[57]. This is obviously true if we impose $\partial^z_\alpha \epsilon^N = 0$ at each stage, so that the exact forms $\partial^z_\alpha \epsilon^N(Y, Z)$ do not contribute to S^N_A at any order N. If that is the case S-field is purely auxiliary and should be properly rotated under Lorentz transformation

$$\delta_\Lambda S_\alpha = \frac{\delta S_\alpha}{\delta B} \delta_\Lambda B. \qquad (11.123)$$

This is not the case with (11.118). On the contrary one has

$$[L^0_{\alpha\beta}, S_\gamma] = S_\alpha \epsilon_{\beta\gamma} + S_\beta \epsilon_{\alpha\gamma} + \frac{\delta S_\gamma}{\delta B}[L^0_{\alpha\beta}, B]_\star, \qquad (11.124)$$

---

[57]There is a gauge invariance for physical fields which is given by Y-dependent functions only. That invariance is of course unconstrained. The restriction in question concerns the extra gauge freedom in auxiliary Z-sector.

where the first two terms on the r.h.s. of (11.124) arise from acting with L^0 on the spinor index of S_α, which can be carried by oscillators and their derivatives. In particular, (11.124) is obvious for $S_{1\alpha}$, (11.94), provided that $\partial_z \xi_1 = 0$. This is the point where the requirement of no extra fields gets crucial. Therefore, even though (11.118) properly rotates master field B, it still does not provide one with Lorentz generators for it acts on S_α inconsistently with the requirement $S_\alpha = S_\alpha[B]$. As a result we face the problem of proper deformation $L^0 \to L^{int}$ in such a way as to compensate the extra terms in (11.124) and yet preserve canonical Lorentz action on physical fields. A priori it is not guaranteed, that such a deformation exists and in that case one loses physical interpretation based on equivalence principle. Luckily, for HS system Lorentz symmetry does exist thanks to the property of the deformed oscillators (11.63) and (11.64) constructed from S-field. Indeed to compensate unwanted terms we subtract

$$L^S_{\alpha\beta} = \frac{i}{4}\{S_\alpha, S_\beta\}_\star \tag{11.125}$$

from $L^0_{\alpha\beta}$ and define

$$L_{\alpha\beta} = L^0_{\alpha\beta} - L^S_{\alpha\beta} \tag{11.126}$$

Note that in the vacuum $S_\alpha = z_\alpha$ we have $L^S = L^z$ and hence we are back to $L = L^y$. Using (11.68)-(11.70) one can check the following commutation relations

$$[L^S_{\alpha\beta}, S_\gamma] = S_\alpha \epsilon_{\beta\gamma} + S_\beta \epsilon_{\alpha\gamma}, \tag{11.127}$$

$$[L^S_{\alpha\beta}, L^S_{\gamma\delta}] = L^S_{\alpha\delta}\epsilon_{\beta\gamma} + ... \qquad\qquad [L^S_{\alpha\beta}, B \star \varkappa] = 0 \tag{11.128}$$

that give the desired

$$[L_{\alpha\beta}, S_\gamma]_\star = \frac{\delta S_\gamma}{\delta B}[L^0_{\alpha\beta}, B]_\star. \tag{11.129}$$

It is easy to check that L acts on $B \star \varkappa$ in a right way too due to (11.128)

$$\delta_\Lambda(B \star \varkappa) \equiv \left[\frac{1}{2}\Lambda^{\alpha\beta}L_{\alpha\beta}, B \star \varkappa\right]_\star = \left[\frac{1}{2}\Lambda^{\alpha\beta}L^0_{\alpha\beta}, B \star \varkappa\right]_\star. \tag{11.130}$$

Let us make an important comment. The generators (11.126) themselves formally do not form Lorentz algebra in a sense, that $[L, L]_\star \neq L$. Indeed, it is straightforward to check

$$[L_{\alpha\beta}, L_{\gamma\delta}]_\star = (\epsilon_{\beta\gamma}L_{\alpha\delta} + \dots) - \frac{\delta L^S_{\gamma\delta}}{\delta B}[L^0_{\alpha\beta}, B]_\star + \frac{\delta L^S_{\alpha\beta}}{\delta B}[L^0_{\gamma\delta}, B]_\star, \tag{11.131}$$

where the two last terms break down Lorentz symmetry. This apparent contradiction is in fact false. The reason is the generators (11.126) are field dependent and therefore when successively applied get differ by the corresponding change of the B-field. In other words the action of Lorentz generators (11.126) must account for the change of generators themselves[58]. Let us show that being properly treated the successive application of (11.126)

---

[58]We are very grateful to M.A. Vasiliev for explaining this point to us.

on B-field is equivalent to successive Lorentz transformations. Apply (11.126) to $B \star \varkappa$ one time

$$\delta_{\Lambda_1}(B \star \varkappa) = [L_{\Lambda_1}[B \star \varkappa], B \star \varkappa]_\star = [L^0_{\Lambda_1}, B \star \varkappa]_\star, \tag{11.132}$$

which as was shown is a proper Lorentz rotation. Now, applying the second time

$$\delta_{\Lambda_2}\delta_{\Lambda_1}(B \star \varkappa) = [L_{\Lambda_2}[B \star \varkappa], \delta_{\Lambda_1}(B \star \varkappa)]_\star + [\delta_{\Lambda_1}L^S_{\Lambda_2}[B \star \varkappa], B \star \varkappa]_\star, \tag{11.133}$$

where the second term has been added to account for the change of field dependent generator. Now, using that $[L^S, B \star \varkappa]_\star = 0$ and therefore $[\delta L^S, B \star \varkappa]_\star + [L^S, \delta(B \star \varkappa)]_\star = 0$ one finds

$$(\delta_{\Lambda_2}\delta_{\Lambda_1} - \delta_{\Lambda_1}\delta_{\Lambda_2})(B \star \varkappa) = \delta^0_{[\Lambda_2,\Lambda_1]}(B \star \varkappa). \tag{11.134}$$

So, one concludes that local Lorentz symmetry is restored. The situation with one-forms $W(Y, Z|x)$ is a bit trickier. With $\Lambda = \frac{1}{2}\Lambda^{\alpha\beta}L_{\alpha\beta} + c.c.$ looking at

$$\delta_\Lambda W \equiv d\Lambda + [W, \Lambda]_\star + c.c. = \frac{1}{2}d\Lambda^{\alpha\beta}L_{\alpha\beta} + \left[W, \frac{1}{2}\Lambda^{\alpha\beta}L^0_{\alpha\beta}\right]_\star + c.c., \tag{11.135}$$

where we made use of (11.67), we see that to extract Lorentz tensors and Lorentz connection from W we need to decompose it as follows

$$W = W^{tensor} + W^L, \qquad W^L = \frac{1}{2}\omega^{L\,\alpha\beta}L_{\alpha\beta} + \frac{1}{2}\bar\omega^{L\,\dot\alpha\dot\beta}\bar L_{\dot\alpha\dot\beta}, \tag{11.136}$$

such that transformation (11.135) reduces to

$$\delta_\Lambda W^L = d\Lambda + [W^L, \Lambda]_\star, + c.c., \tag{11.137}$$

$$\delta_\Lambda W^{tensor} = \left[W^{tensor}, \frac{1}{2}\Lambda^{\alpha\beta}L^0_{\alpha\beta}\right]_\star + c.c., \tag{11.138}$$

where again we used that in making field dependent Lorentz transformation one has to compensate the change of generators which effectively cancels the otherwise appearing extra term in (11.137)

$$\omega^{L\,\gamma\delta}\Lambda^{\alpha\beta}\frac{\delta L^S_{\gamma\delta}}{\delta(B \star \varkappa)}[L^0_{\alpha\beta}, B \star \varkappa]_\star. \tag{11.139}$$

Decomposition (11.136) may look like some arbitrary separation of W into two terms which particularly does not constraint the form of connection fields $\omega^L_{\alpha\beta}$ and $\bar\omega^L_{\dot\alpha\dot\beta}$. These are nonetheless get totally fixed by the requirement that spin-two contribution to be absent in $W^{tensor}(Y, Z|x)$ master field being encoded in the Lorentz connection term $W^L(Y, Z|x)$.

In analyzing the problem of Lorentz symmetry we had to impose a requirement of no external fields that carry indices to appear in reconstruction of Z-dependence. Does it mean that otherwise the system possesses no local Lorentz symmetry? No it does not, because this extra fields that break down explicit Lorentz symmetry appear in a pure gauge fashion. We have proven the existence of local Lorentz symmetry within certain setup which guarantees (11.124) and allows one to explicitly find Lorentz symmetry. In case of extra fields there is a gauge transformation that brings solution to the form that possesses explicit Lorentz symmetry. It is just that a concrete form of that transformation depends on the details of particular solution. Or other way around, the form of Lorentz generators in that case contains these extra fields in question. One can also have a look at [38, 110, 113, 114] where the issue of Lorentz symmetry is discussed.

11.6 Higher orders

Let us briefly consider how to operate with Vasiliev equations at higher orders. To the second order the system acquires the following form

$$D_\Omega W^2 + W^1 \wedge \star W^1 = 0, \tag{11.140}$$

$$\tilde{D}_\Omega B^2 + W^1 \star B^1 - B^1 \star \tilde{W}^1 = 0, \tag{11.141}$$

$$D_\Omega S_\alpha^2 + [W^1, S_\alpha^1] = -2i\partial_\alpha W^2, \qquad D_\Omega S_{\dot\alpha}^2 + [W^1, S_{\dot\alpha}^1] = -2i\partial_{\dot\alpha} W^2 \tag{11.142}$$

$$2i\partial_\alpha B^2 = S_\alpha^1 \star B^1 + B^1 \star \tilde{S}_\alpha^1, \qquad 2i\partial_{\dot\alpha} B^2 = S_{\dot\alpha}^1 \star B^1 + B^1 \star \tilde{\tilde{S}}_{\dot\alpha}^1, \tag{11.143}$$

$$\partial_\alpha S_\beta^2 \, \epsilon^{\alpha\beta} + \frac{i}{2} S_\alpha^1 \star S_\beta^1 \, \epsilon^{\alpha\beta} = B^2 \star \varkappa, \qquad \partial_{\dot\alpha} S_{\dot\beta}^2 \, \epsilon^{\dot\alpha\dot\beta} + \frac{i}{2} S_{\dot\alpha}^1 \star S_{\dot\beta}^1 \, \epsilon^{\dot\alpha\dot\beta} = B^2 \star \bar\varkappa, \tag{11.144}$$

$$\partial_\alpha S_{\dot\alpha}^2 - \partial_{\dot\alpha} S_\alpha^2 + \frac{i}{2} [S_\alpha^1, S_{\dot\alpha}^1]_\star = 0, \tag{11.145}$$

where $\partial_\alpha = \frac{\partial}{\partial z^\alpha}$ and tilde means the twisting, e.g., $\tilde{S}_\alpha = \pi(S_\alpha)$, $\tilde{\tilde{S}}_{\dot\alpha} = \bar\pi(S_{\dot\alpha})$. Using (11.143) and (11.93) we can first solve for Z-dependence of B field

$$2iB^2 = z^\alpha \int_0^1 dt (S_\alpha^1 \star C(Y|x) + C(Y|x) \star \tilde{S}_\alpha^1)_{z \to tz} + c.c. + C(Y|x), \tag{11.146}$$

where S_α^1 is given by (11.94). Substituting (11.146) and (11.98) into (11.141) and after some algebra and integration by parts we get

$$D^L C - ih^{\alpha\dot\alpha} \left(y_\alpha \bar y_{\dot\alpha} - \frac{\partial^2}{\partial y^\alpha \partial \bar y^{\dot\alpha}} \right) C + \omega \star C - C \star \tilde\omega +$$

$$- \frac{1}{2i} h^{\alpha\dot\alpha} \left\{ \bar y_{\dot\alpha} \int_0^1 dt\, t \left(z_\alpha C(-tz, \bar y) e^{itz_\beta y^\beta} \star C - C \star z_\alpha C(tz, \bar y) e^{itz_\beta y^\beta} \right) + c.c. \right\}_{Z=0} +$$

$$+ \frac{1}{2} h^{\alpha\dot\alpha} \int_0^1 dt\, (1-t) \left(z_\alpha \frac{\partial}{\partial \bar y^{\dot\alpha}} C(-tt'z, \bar y) e^{itz_\beta y^\beta} \star C + C \star z_\alpha \frac{\partial}{\partial \bar y^{\dot\alpha}} C(tz, \bar y) e^{itz_\beta y^\beta} + c.c. \right)_{Z=0} = 0, \tag{11.147}$$

where we also made use of the following formula

$$\int\limits_{[0,1]\times[0,1]} dt\, dt'\, t\, f(tt') = \int\limits_{[0,1]\times[0,1]} f(s)\theta(\tau - s) d\tau ds = \int_0^1 (1-s) f(s) ds. \tag{11.148}$$

Note, that at given order noncovariant terms with respect to Lorentz connection $\omega_{\alpha\beta}$ that in principle could have appeared cancel. To proceed to W-sector one solves (11.144), (11.145) for S_A^2. To do so, it is convenient to rewrite these equations as

$$\partial_A S_B^2 - \partial_B S_A^2 = R_{AB}(B^2, S^1), \qquad S_A^2 = Z^B \int_0^1 dt\, t\, R_{AB}(tZ), \tag{11.149}$$

which gives

$$S_\alpha^2 = z^\alpha \int_0^1 dt\, t\, (B^2 \star \varkappa - \frac{i}{4} [S_\alpha^1, S_\beta^1])_{z \to tz} + \partial_\alpha \epsilon^2(Y, Z|X) - \frac{i}{4} z^{\dot\alpha} \int_0^1 dt\, t\, [S_{\dot\alpha}^1, S_\beta^1]_{\bar z \to t\bar z}$$

and allows one to determine Z-dependence of W field from (11.142)

$$-2iW^2 = z^\alpha \int_0^1 dt \, (D_\Omega S^2_\alpha + [W^1, S^1_\alpha])_{z \to tz} + c.c. + \omega^2(Y|X) \,. \tag{11.150}$$

The latter when substituted into (11.140) results in second order space-time equation for $\omega^2(Y|x)$ that we omit for brevity. Higher-order analysis can be carried out along the same lines. Rewriting the equations to the N-th order one has

$$D_\Omega W^N + \sum_{n+m=N}{}' W^n \wedge \star W^m = 0 \,, \tag{11.151}$$

$$\tilde{D}_\Omega B^N + \sum_{n+m=N}{}' \left(W^n \star B^m - B^m \star \tilde{W}^n \right) = 0 \,, \tag{11.152}$$

$$D_\Omega S^N_\alpha + \sum_{n+m=N}{}' [W^n, S^m_\alpha] = -2i\partial_\alpha W^N \,, \tag{11.153}$$

$$2i\partial_\alpha B^N = \sum_{n+m=N}{}' \left(S^n_\alpha \star B^m + B^m \star \tilde{S}^n_\alpha \right) \,, \tag{11.154}$$

$$\partial_\alpha S^N_\beta \, \epsilon^{\alpha\beta} + \frac{i}{2} \sum_{n+m=N}{}' S^n_\alpha \star S^m_\beta \, \epsilon^{\alpha\beta} = B^N \star \varkappa \,, \tag{11.155}$$

$$\partial_\alpha S^N_{\dot\alpha} - \partial_{\dot\alpha} S^N_\alpha + \frac{i}{2} \sum_{n+m=N}{}' [S^n_\alpha, S^m_{\dot\alpha}]_\star = 0 \,, \tag{11.156}$$

where some of the equations need to be supplemented with their $c.c.$-versions and the prime $\sum'$ means that the contribution from the vacuum solution is extracted and written separately. Taking into account our experience in the perturbation theory up to the second order, the general recipe consists of circling order by order in the following progression

$$\cdots \longrightarrow (11.154)_N \longrightarrow (11.155)_N \longrightarrow (11.153)_N \longrightarrow \begin{cases} (11.152)_N \\ (11.151)_N \end{cases} \longrightarrow (11.154)_{N+1} \cdots$$
$$\tag{11.157}$$

which leads to

$$2iB^N = z^\alpha \int_0^1 \Big(\sum_{n+m=N}{}' S^n_\alpha \star B^m + B^m \star \tilde{S}^n_\alpha \Big)_{z \to tz} + c.c. + C^N(Y|X) \,, \tag{11.158}$$

$$S^N_\alpha = z^\alpha \int_0^1 dt t (B^N \star \varkappa - \frac{i}{4} \sum_{n+m=N}{}' [S^n_\alpha, S^m_\beta])_{z \to tz} + \partial_\alpha \epsilon^N(Y, Z|X) -$$

$$- \frac{i}{4} z^{\dot\alpha} \int_0^1 dt t \sum_{n+m=N}{}' [S^n_{\dot\alpha}, S^m_\beta]_{\bar z \to t\bar z} \,, \tag{11.159}$$

$$W^N = \frac{i}{2} z^\alpha \int_0^1 dt (D_\Omega S^N_\alpha + \sum_{n+m=N}{}' [W^n, S^m_\alpha])_{z \to tz} + c.c. + \omega^N(Y|X) \,. \tag{11.160}$$

The solutions to B^N and W^N are to be used in (11.152)-(11.151) to recover the perturbation theory in x-space. The integration constants C^N and ω^N will play the role of the order N dynamical fields while other terms will give the interaction vertices.

Let us note that B^N and hence C^N appears in (11.159) and hence in (11.160) the same way as in the first order perturbation theory. This results in the same gluing term as in the on-mass-shell theorem but with C^N. This is in agreement with the last term in (11.9).

Aknowledgements

12   Extras

12.1   Fronsdal operator on Riemannian manifolds

Let us take the kinetic part of the Fronsdal operator and put it on a general Riemannian manifold

$$F[\phi]^{\underline{a}(s)} = \Box\phi^{\underline{a}(s)} - \nabla^{\underline{a}}\nabla_{\underline{m}}\phi^{\underline{m}\underline{a}(s-1)} + \frac{1}{2}\nabla^{\underline{a}}\nabla^{\underline{a}}\phi^{\underline{a}(s-2)\underline{m}}{}_{\underline{m}}\,,$$

where $\nabla_{\underline{m}}$ is a covariant derivative with respect to some metric $g_{\underline{m}\underline{n}}$. First, we can check that the Fronsdal operator still satisfies the double trace constraint,

$$F[\phi]^{\underline{a}(s-4)\underline{m}\underline{m}\underline{n}\underline{n}}g_{\underline{m}\underline{m}}g_{\underline{n}\underline{n}} = 0\,, \tag{12.1}$$

which might not have been the case. At least we have the same number of equations as the number of fields. The Fronsdal operator is not gauge invariant in general, which is a sign of a serious problem. Once the gauge symmetry is lost, or weakened, we gained new degrees of freedom. New degrees of freedom come usually in the form of ghosts. Let us also note that the Fronsdal operator is not what one gets from the covariantization of the Fronsdal action since on has to commute the derivatives in taking the variation.

The source of non-invariance is

$$\nabla^{\underline{a}}[\nabla^{\underline{a}}, \nabla_{\underline{m}}]\xi^{\underline{m}\underline{a}(s-2)} - [\nabla^{\underline{a}}, \Box]\xi^{\underline{a}(s-1)}\,. \tag{12.2}$$

When the Riemann tensor has only scalar curvature nonvanishing, i.e. we are in (anti)-de Sitter, which is displayed by (2.26), the non-invariance is of special form, the same terms originate from mass-like terms, as we have already seen. Let us now see what happens if the only non-zero part of the Riemann tensor is the Weyl tensor (which is introduced systematically in Section 4). Then (12.2) reduces to

$$2C^{\underline{aaa},}{}_{\underline{mm}}\,\xi^{\underline{a(s-3)mm}} + C^{\underline{aa},}{}_{\underline{mm}}\,\nabla^{\underline{a}}\xi^{\underline{a(s-3)mm}} - 2C^{\underline{aa},}{}_{\underline{mm}}\,\nabla^{\underline{m}}\xi^{\underline{a(s-2)m}}, \qquad (12.3)$$

where $C^{\underline{aa},\underline{bb}}$ is the Weyl tensor and $C^{\underline{aaa},\underline{bb}}$ is its first derivative, which is constrained by the differential Bianchi identity to have ⊞-symmetry. We may try to cancel this terms by adding to the Fronsdal operator

$$C^{\underline{aa},}{}_{\underline{mm}}\,\phi^{\underline{a(s-2)mm}} + g^{\underline{aa}}C^{\underline{aa},}{}_{\underline{mm}}\,\phi^{\underline{a(s-4)mmn}}{}_{\underline{n}} \qquad (12.4)$$

but find this impossible for $s > 2$, because of the first term (12.3), which does not have derivative on the gauge parameter. The last two terms of (12.3) are problematic too because the relative coefficient does not allow one to cancel them by (12.4). So we see that it is not a piece of cake to make higher-spin fields live on a manifold, which is different from Minkowski or (anti)-de Sitter. In particular generic Ricci flat or Einstein backgrounds are not accessible by the Fronsdal theory. The solution, which is a part of the Vasiliev theory, is non-minimal in the sense that it cannot be achieved via simple modifications of the Fronsdal operator.

12.2 MacDowell-Mansouri-Stelle-West

That (anti)-de Sitter algebra $(so(d-1,2))$ $so(d,1)$ is semi-simple as compared to the Poincare one allows us to improve on the interpretation of gravity as a gauge theory even further. Recall that the curvature corresponding to L_{ab} generators of the (anti)-de Sitter algebra, (3.40), is $R^{a,b}$ and $R^{a,b} = F^{a,b} - \Lambda e^a \wedge e^b$, where $F^{a,b}$ is the L_{ab}-part of the curvature for the Poincare algebra, it is related to the Riemann tensor. It was the observation by MacDowell and Mansouri, [86] that the following $4d$-action

$$S_{MM} = -\frac{1}{2\Lambda}\int R^{a,b} \wedge R^{c,d}\epsilon_{abcd} \qquad (12.5)$$

is equivalent to the Einstein-Hilbert with cosmological term. Indeed, expanding $R = F - \Lambda ee$ we find

$$S_{MM} = \int\left(-\frac{1}{2\Lambda}F^{a,b}\wedge F^{c,d} + F^{a,b}\wedge e^c \wedge e^d - \frac{\Lambda}{2}e^a \wedge e^b \wedge e^c \wedge e^d\right)\epsilon_{abcd}. \qquad (12.6)$$

The first term is of Lovelock type, (3.37), and in fact topological in $4d$ and hence does not contribute to the equations of motion. The last two terms sum up to (3.48) with a slightly different cosmological constant, $\Lambda \to \Lambda/2$. Note that $\Lambda \to 0$ limit is singular in the action but not in the equations of motion since the singularity multiplies the topological term.

The MacDowell-Mansouri action looks similar to the Yang-Mills one, now it is quadratic in the field-strength, but it is not exactly of the Yang-Mills form, (3.39). It can be formally

rewritten as

$$S_{MM} = \int R^{a,b} \wedge R^{c,d} \epsilon_{abcd5} \,, \tag{12.7}$$

where we introduced the 5d-epsilon symbol and $\epsilon_{abcd} = \epsilon_{abcd5}$. One may wish to do that in order to keep the symmetries of the most symmetric solution, $so(3,2)$ $(so(4,1))$, which is the (anti)-de Sitter space, so it has something to do with rotations in 5d. It does not look satisfactory yet as one would like to make all symmetries manifest and the presence of 5 in ϵ_{abcd5} breaks down the symmetry. This can be fixed with a little more work.

First of all, since (anti)-de Sitter algebra $(so(d-1,2))$ $so(d,1)$ is the algebra of infinitesimal rotations, it is convenient not to split generators into L_{ab} and P_a, which already breaks the anti-de Sitter symmetry down to the Lorentz one. To accomplish this, analogously to the Lorentz generators L_{ab} themselves, we define $T_{\mathsf{AB}} = -T_{\mathsf{BA}}$, where capital Latin indices $\mathsf{A}, \mathsf{B}, ...$ run over $d+1$ values, $\mathsf{A} = \{a, 5\}$, where 5 refers to the extra $(d+1)$-th direction as compared to the Lorentz algebra. The generators T_{AB} obey

$$[T_{\mathsf{AB}}, T_{\mathsf{CD}}] = T_{\mathsf{AD}}\eta_{\mathsf{BC}} - T_{\mathsf{BD}}\eta_{\mathsf{AC}} - T_{\mathsf{AC}}\eta_{\mathsf{BD}} + T_{\mathsf{BC}}\eta_{\mathsf{AD}} \,, \tag{12.8}$$

where η_{AB} is the $(so(d-1,2))$ $so(d,1)$ invariant metric. Lorentz-covariant formulas (3.40) can be recovered upon identifying L_{ab} and P_a with T_{ab} and T_{a5}. Not surprisingly, if we define the Yang-Mills connection $\Omega = \frac{1}{2}\Omega^{\mathsf{A},\mathsf{B}}T_{\mathsf{AB}}$ of the (anti)-de Sitter algebra then the curvature

$$R^{\mathsf{A},\mathsf{B}} = d\Omega^{\mathsf{A},\mathsf{B}} + \Omega^{\mathsf{A}},_{\mathsf{C}} \wedge \Omega^{\mathsf{C},\mathsf{B}} \tag{12.9}$$

reduces to (3.42) for $R^{a,b}$ and $R^{a,5}$ as well as $\delta\Omega^{\mathsf{A},\mathsf{B}} = D\epsilon^{\mathsf{A},\mathsf{B}}$ reduces to (3.43). In particular the zero-curvature equation

$$d\Omega^{\mathsf{A},\mathsf{B}} + \Omega^{\mathsf{A}},_{\mathsf{C}} \wedge \Omega^{\mathsf{C},\mathsf{B}} = 0 \tag{12.10}$$

describes (anti)-de Sitter space analogously to (3.53)-(3.54) provided that we specified the way of splitting $\Omega^{\mathsf{A},\mathsf{B}}$ into vielbein e^a and spin-connection $\omega^{a,b}$, e.g. as $\Omega^{a,5}$ and $\Omega^{a,b}$, and required the vielbein to be nondegenerate. Let us note that (12.10) being a certain flatness condition generates its solution space locally in a pure gauge form. Indeed, one can easily check that any function $g = g(T_{\mathsf{AB}})$ gives rise to a solution[59]

$$\Omega = g^{-1}(T)dg(T) \,. \tag{12.11}$$

This fact might suggest an erroneous interpretation, namely that all such solutions are gauge equivalent to $\Omega = 0$ and, particularly, one can "gauge away" the AdS space-time itself. While formally it seems to be the case, this argument suffers from a flaw that makes the vielbein vanish. Recall that only nondegenerate frame fields admit physical

---

[59]The expression is somewhat formal. For example, $g(T)$ can be taken to be the usual exp-map from a Lie algebra to certain neighborhood of the unit element of the group. With the help of $g^{-1}g = 1$ one checks that $d\Omega + \Omega\Omega = 0$, where the product $\Omega\Omega$ is the usual group product since Ω are given in terms of group element g.

interpretation. This is another illustration that gauge formalism applied to pure gravity should be used with great care.

We would like to extend (12.7) to any d and rewrite it in manifestly (anti)-de Sitter invariant form. A natural extension seems to be of the form, [88],

$$S = \int R^{a,b} \wedge R^{c,d} \wedge e^f \wedge ...e^u \epsilon_{abcdf...u5} \tag{12.12}$$

where 5 again means $d + 1$. However, the Lovelock term $\int FFe...e$ is not topological in $d > 4$, which brings in nonlinear corrections to the Einstein equations. It is hard to tell if these corrections, however beautiful, are phenomenologically acceptable since we live, at least effectively, in $4d$.

We rewrite (12.12) by using as many uppercase indices as we can

$$S = \int R^{\mathsf{A,B}} \wedge R^{\mathsf{C,D}} \wedge e^f \wedge ...e^u \epsilon_{\mathsf{ABCD}f...u5} \,. \tag{12.13}$$

Since the value 5 is already occupied $\mathsf{A, B, C, D}$ are constrained to the Lorentz index range simply because two 5 indices cannot appear simultaneously in the ϵ-symbol. Formally we can extend e^a to the extra direction too, e.g. to define E^{A} such that $E^a = e^a$ and $E^5 = 0$ since E^5 does not contribute to the action anyway. Then we get

$$S = \int R^{\mathsf{A,B}} \wedge R^{\mathsf{C,D}} \wedge E^{\mathsf{F}} \wedge ...E^{\mathsf{U}} \epsilon_{\mathsf{ABCDF...U5}} \,. \tag{12.14}$$

The anti-de Sitter symmetry is still explicitly broken by 5 in the ϵ-symbol and by embedding of e^a into E^{A}. A natural way out is to think of 5 as of the vacuum expectation of some compensator vector field, [87], say V^{A}, and we have $V^{\mathsf{A}} = \delta_5^{\mathsf{A}}$. Now it is better,

$$S_{MMSW} = \int R^{\mathsf{A,B}} \wedge R^{\mathsf{C,D}} \wedge E^{\mathsf{F}} \wedge ...E^{\mathsf{U}} V^{\mathsf{W}} \epsilon_{\mathsf{ABCDF...UW}} \,. \tag{12.15}$$

We can make all definitions 5-independent by using V^{A} that has to have some vacuum value, e.g. δ_5^{A}. We can force V^{A} to be non-zero by imposing the following constraint

$$V^{\mathsf{A}} V^{\mathsf{B}} \eta_{\mathsf{AB}} = \sigma \,, \tag{12.16}$$

where $\sigma = 1$ for de Sitter and $\sigma = -1$ for anti-de Sitter. We refer to the choice $V^{\mathsf{A}} = \delta_5^{\mathsf{A}}$ as the *standard gauge*.

Introducing a new object V^{A} may seem to be too naive — for some reason we have to restrict ourselves to the range a rather than the full range A, this can be always achieved by introducing a (number of) vectors V^{A} whose purpose is to span the extra directions of A, so we just have a split $\mathbb{R}^{n+m}$ as $\mathbb{R}^n \oplus \mathbb{R}^m$ where (a number of) V^{A}'s span the basis of $\mathbb{R}^m$ (in our case $n = d$, $m = 1$). Such restoration of 'broken symmetry' can always be achieved. In particular instead of extending $so(d-1,1)$ to (anti)-de Sitter algebra we could choose $so(d+m)$ (the signature is immaterial now) with $m > 1$, which require m linearly independent V^{A}'s to be introduced. The reason not to go beyond the (anti)-de Sitter algebra is our desire to study field theory over (anti)-de Sitter space and we have

no evidence so far that a larger symmetry that contains $so(d-1,2)$ or $so(d,1)$ is present in the theory. From this perspective it would be great to have (anti)-de Sitter symmetry manifest. Actually, we find in Section 10 that a larger symmetry does exists and acts on the infinite multiplet of fields of all spins, but not a spin-two alone. But there are no signs of higher symmetry when dealing with pure gravity[60]. Indeed, (anti)-de Sitter and Minkowski are known to be maximally-symmetric backgrounds. Therefore, just single V^{A} should be enough.

We need some $E^{\mathsf{A}} = E^{\mathsf{A}}_{\underline{m}} dx^{\underline{m}}$ such that $E^5 = 0$, $E^a = e^a$ and $e^a = \Omega^{a,5}$ in the standard gauge. We can state this as $E^{\mathsf{A}} = \Omega^{\mathsf{A,B}} V_{\mathsf{B}}$ but it is better to use

$$E^{\mathsf{A}} = D_\Omega V^{\mathsf{A}} = dV^{\mathsf{A}} + \Omega^{\mathsf{A,}}{}_{\mathsf{B}} V^{\mathsf{B}} \,, \tag{12.17}$$

because it transforms covariantly. Now all constituents of (12.15) are well-defined. Action (12.15) is manifestly invariant under local (anti)-de Sitter transformations, i.e. Lorentz plus translations, and diffeomorphisms if we assume that the compensator transforms as a fiber vector

$$\delta\Omega^{\mathsf{A,B}} = D_\Omega \epsilon^{\mathsf{A,B}} \,, \qquad \delta V^{\mathsf{A}} = -\epsilon^{\mathsf{A,}}{}_{\mathsf{B}} V^{\mathsf{B}} \,, \qquad \delta E^{\mathsf{A}} = -\epsilon^{\mathsf{A,}}{}_{\mathsf{B}} E^{\mathsf{B}} \,. \tag{12.18}$$

The local (anti)-de Sitter invariance follows from the fact that all the (anti)-de Sitter indices in (12.15) are contracted using one η_{AB} or another invariant tensor $\epsilon_{\mathsf{AB...U}}$ of the (anti)-de Sitter algebra and all the constituents transform covariantly. The diffeomorphism invariance is explicit thanks to differential forms.

Our previous experience shows that having local (anti)-de Sitter transformations and diffeomorphisms simultaneously is too much. Recall that (3.32) was not invariant under local translations unless torsion is zero and at vanishing torsion the local translations are identical to diffeomorphisms when acting on the frame field e^a. Hence we expect $d + d(d-1)/2 = d(d+1)/2$ local gauge parameters in total.

The extension of the local symmetry algebra arises due to the presence of the compensator field. As we already mentioned, given an action/equations of motion that are invariant under local transformations belonging to some algebra $\mathfrak{h}$ and are not invariant under a bigger algebra $\mathfrak{g} \supset \mathfrak{h}$, it is always possible to restore $\mathfrak{g}$-symmetry by introducing new fields that transform in such a way as to compensate for $\mathfrak{g}/\mathfrak{h}$-noninvariance. Of course, this does not mean that the theory is invariant under $\mathfrak{g}$. A simple example was given above where we could try to extend the local Lorentz symmetry to any $so(d+m)$ with $m > 1$ by introducing m compensator fields. The genuine symmetry is the stability algebra of the compensator field. The condition for the compensator to remain invariant reads

$$(\mathcal{L}_\xi + \delta_\epsilon)V^{\mathsf{A}} = 0 \,, \tag{12.19}$$

where we employed both the diffeomorphisms and the local (anti)-de Sitter symmetry. In components it reads

$$\xi^{\underline{m}}\partial_{\underline{m}}V^{\mathsf{A}} - \epsilon^{\mathsf{A,}}{}_{\mathsf{B}} V^{\mathsf{B}} = 0 \,. \tag{12.20}$$

---

[60]In the vicinity of cosmological singularity the Einstein equations were shown to exhibit a higher symmetry, which is a certain infinite-dimensional affine algebra of hyperbolic type, see, e.g, [115]. The question of whether this symmetry acts in the gravity in higher orders and far from the singularity remains open.

It is instructive to look at this condition in the *standard gauge* $V^{\mathsf{A}} = \delta_5^{\mathsf{A}}$,

$$\epsilon^{\mathsf{A}'}{}_{\mathsf{B}} V^{\mathsf{B}} = \epsilon^{\mathsf{A}'}{}_5 = 0 \,. \tag{12.21}$$

The stability condition for V^{A} kills $\epsilon^{a,5}$ components, reducing the local symmetry group to diffeomorphisms and Lorentz rotations with $\epsilon^{a,b}$. We are left with the expected amount of symmetry, i.e. diffeomorphisms and local Lorentz transformations. Hence, in the *standard gauge* we roll back to (12.12).

One can choose a nonconstant compensator as well. In the latter case (12.19) starts to mix $\xi^{\underline{m}}$ and local translations $\epsilon^{\mathsf{A}'}{}_{\mathsf{B}} V^{\mathsf{B}}$. Our physical world (in the tangent space) is to be identified with the subspace orthogonal to the compensator. In particular we have to ensure that all the quantities, e.g. the vielbein and spin-connection, do not feel the presence of a compensator. Firstly, note that the generalized vielbein is still effectively a $d \times d$ matrix thanks to

$$E^{\mathsf{A}} V_{\mathsf{A}} \equiv 0 \,, \tag{12.22}$$

where we used $V^{\mathsf{A}} dV_{\mathsf{A}} = 0$ and the fact that $\Omega^{\mathsf{A},\mathsf{B}}$ is antisymmetric. What is the generalized spin-connection? It can be defined as follows, [88],

$$\Omega_L{}^{\mathsf{A},\mathsf{B}} = \Omega^{\mathsf{A},\mathsf{B}} - \frac{1}{V^2}(E^{\mathsf{A}} V^{\mathsf{B}} - E^{\mathsf{B}} V^{\mathsf{A}}) \,, \qquad\qquad D = d + \Omega_L \,. \tag{12.23}$$

It has some nice properties, which fix its form completely,

$$DV^{\mathsf{A}} \equiv 0 \,, \qquad\qquad\qquad DE^{\mathsf{A}} \equiv 0 \,, \tag{12.24}$$

i.e. the compensator V^{A} is insensible with respect to the physical covariant derivative, which is D. The generalized vielbein is covariantly constant, which is an analog of $De^a = 0$. In the *standard gauge* one recovers $\Omega_L^{a,b} = \Omega^{a,b}$.

Let us note that pure diffeomorphisms do not preserve $E^{\mathsf{A}} V_{\mathsf{A}} = 0$, but we can check that the combination of diffeomorphisms and gauge symmetries restricted by (12.19) do,

$$\left(\mathcal{L}_\xi E^{\mathsf{A}} - \epsilon^{\mathsf{A}'}{}_{\mathsf{B}} E^{\mathsf{B}} \right)\Big|_{(12.19)} = i_\xi R^{\mathsf{A}'}{}_{\mathsf{B}} V^{\mathsf{B}} + D(i_\xi \Omega^{\mathsf{A}'}{}_{\mathsf{B}} + \epsilon^{\mathsf{A}'}{}_{\mathsf{B}}) V^{\mathsf{B}} \,. \tag{12.25}$$

The meaning of the compensator field can be understood from the following picture. The (anti)-de Sitter space is replaced with the sphere for simplicity reason. Thinking of AdS or sphere as d-dimensional hypersurface embedded into $(d+1)$-dimensional (ambient) space, there is another natural gauge for the compensator, which is to make it lie along the radial direction. The stability algebra of the compensator is then the algebra of rotations in the plane orthogonal to V^{A} and tangent to the sphere — it is an equivalent of the Lorentz algebra.

12.3 Interplay between diffeomorphisms and gauge symmetries

A generic feature of gauge formulation is that diffeomorphisms are entangled with gauge transformations and hence we cannot treat them separately. When considering gauge fields on manifolds we have to deal with the group of diffeomorphisms[61] and group of

---

[61] Diffeomorphisms deliver a very difficult infinite-dimensional group to work with.

Figure 4: *Sphere in ambient space and the compensator*
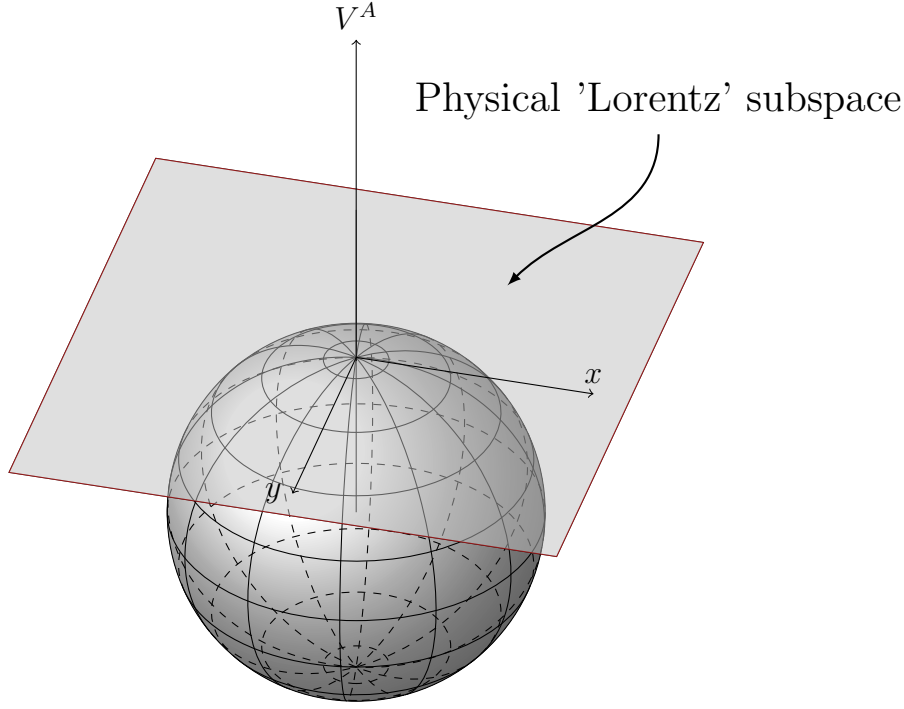
local gauge transformations. It is easier to talk about the corresponding Lie algebras. The diffeomorphisms $diff(M)$ as a Lie algebra are given by vector fields $\xi^{\underline{m}} \in diff(M)$ with the Lie commutator defined to be the Lie bracket of vector fields $[\xi,\eta]^{\underline{m}} = \xi^{\underline{n}}\partial_{\underline{n}}\eta^{\underline{m}} - \eta^{\underline{n}}\partial_{\underline{n}}\xi^{\underline{m}}$. Gauge parameters, say $\epsilon^{\mathcal{I}}(x)$, are the maps from the given manifold to some Lie algebra. Call this linear space of maps $\mathfrak{h}$. The bracket is given by the Lie algebra commutator, i.e. $[\epsilon,\theta]^{\mathcal{I}}(x) = f^{\mathcal{I}}_{\mathcal{J}\mathcal{K}}\epsilon^{\mathcal{J}}(x)\theta^{\mathcal{K}}(x)$.

Gauge parameters are not left intact by $diff(M)$, rather they transform as a number of scalars under diffeomorphisms, which affect x but do not see the Lie algebra index $\mathcal{I}$. Therefore, the elements of the gauge algebra are affected by diffeomorphisms. Indeed, the Lie derivative obeys the Leibnitz law, i.e. $\mathcal{L}_{\xi}(f(x)g(x)) = (\mathcal{L}_{\xi}f(x))g(x) + f(x)\mathcal{L}_{\xi}g(x)$, where $f(x)$, $g(x)$ are two functions and $\mathcal{L}_{\xi}f(x) = \xi^{\underline{m}}\partial_{\underline{m}}f(x)$. Then, it is obvious that

$$\mathcal{L}_{\xi}([\epsilon,\theta]^{\mathcal{I}}(x)) = [\mathcal{L}_{\xi}(\epsilon),\theta]^{\mathcal{I}}(x) + [\epsilon,\mathcal{L}_{\xi}(\theta)]^{\mathcal{I}}(x), \qquad (12.26)$$

which means that $\mathcal{L}_{\xi}$ acts as a derivation on $\mathfrak{h}$. The action of vector fields on gauge parameters, i.e. the infinitesimal change of coordinates on M, can be understood as the homomorphism $diff(M) \to Der(\mathfrak{h})$. Hence pairs $(\xi^{\underline{m}}, \epsilon^{\mathcal{I}})$ belong to the semidirect product $diff(M) \ltimes \mathfrak{h}$ and the full algebra of symmetries is the semidirect product of diffeomorphisms and gauge transformations[62].

---

[62]There are simpler finite-dimensional examples, the affine group and the Poincare group. The affine group $Aff(d)$ consists of pairs (A, a) with A belonging to $GL(d)$ and a being a d-dimensional vector. The group law $(A, a) \circ (B, b) = (AB, Ab + a)$ can be read off the action on vectors $(A, a)(\vec{x}) = A\vec{x} + a$. It resembles the semidirect origin of the affine group $Aff(d) = GL(d) \ltimes T_d$, where T_d is the group of translations in d dimensions, $a \circ b = a + b$, $a, b \in \mathbb{R}^d$.

Given the semidirect structure of the full gauge symmetry algebra it is easy to see that the two types of symmetries do not commute with the interesting part of the commutator residing in the gauge algebra sector,

$$[(\xi_1, \epsilon_1), (\xi_2, \epsilon_2)] = ([\xi_1, \xi_2], [\epsilon_1, \epsilon_2] + \mathcal{L}_{\xi_1}\epsilon_2 - \mathcal{L}_{\xi_2}\epsilon_1) \,. \tag{12.27}$$

As we already know, when the Yang-Mills curvature vanishes, every diffeomorphism can be represented as a gauge transformation. The gauge algebra at a point (just the algebra we gauge itself) can be larger or smaller than d, which is the dimension of a vector field, $\xi^{\underline{m}}(x)$. When the gauge algebra is smaller we cannot identify diffeomorphisms with gauge transformations without having to trivialize the dynamics of gauge fields due to the necessity of $F(A) = 0$. For example, this is true for $u(1)$ gauge field, $A_{\underline{m}}$. When the gauge algebra is larger one may set only part of the curvature to zero. For example, vanishing of the torsion allows us to treat diffeomorphisms as the subalgebra of local gauge transformations. Note however, that this is true only for the part of the gauge field for which the curvature vanishes, i.e. the vielbein, while the action of a diffeomorphism on the spin-connection is a sum of a gauge transformation and a curvature piece.

Let us consider an example[63]. The standard gauge transformation law of a spin-one gauge field $A \equiv A_{\underline{m}}dx^{\underline{m}}$ under gauge symmetries with $\epsilon(x)$ and diffeomorphisms with $\xi^{\underline{m}}(x)$ reads

$$\delta_{\xi,\epsilon}A = d\epsilon + \mathcal{L}_\xi A \tag{12.28}$$

The commutator of two transformations is a gauge transformation of the same type, which leads to the following algebra,

$$[\delta_{\xi',\epsilon'}, \delta_{\xi,\epsilon}] = \delta_{\mathcal{L}_{[\xi',\xi]}, \mathcal{L}_{\xi'}\epsilon - \mathcal{L}_\xi \epsilon'} \tag{12.29}$$

This form of commutator is typical of the semidirect product.

Within field theory we allow for redefinitions of fields and gauge parameters,

$$\xi \to \xi + g(\xi, \partial\xi, ...; \phi, \partial\phi, ...)\,, \tag{12.30}$$
$$\phi \to \xi + f(\phi, \partial\phi, ...)\,, \tag{12.31}$$

where the redefinition of the gauge parameters can be field-dependent and involve derivatives of the gauge parameters and fields. Such redefinitions can kill the nice Lie algebra structure found above, but they are allowed in the realm of field theory, [6]. There is a redefinition that makes the commutator look almost like a direct product. Indeed, inserting $\epsilon - i_\xi A$ instead of ϵ one finds

$$\hat\delta_{\xi,\epsilon}A = d\epsilon + i_\xi F(A)\,, \tag{12.32}$$

where $F(A) = dA$. The commutator now reads

$$[\hat\delta_{\xi',\epsilon'}, \hat\delta_{\xi,\epsilon}] = \hat\delta_{i_{[\xi',\xi]}, i_\xi i_{\xi'} F(A)} \,. \tag{12.33}$$

Going to the Lie algebras of affine or Poincare group we find $[(A, a), (B, b)] = ([A, B], Ab - Ba)$. Note that A, B are now any $d \times d$ matrices, not necessary invertible. The translations play a role analogous to that of 'gauge symmetries' in the sense that they are affected by coordinate transformations associated with A. Clearly the vector of translation has to be transformed in accordance with the change of the basis induced by A.

[63]We are grateful to T.Nutma and M.Taronna for discussions.

The diffeomorphism algebra can be seen in the first argument through the Lie commutator. Miraculously, the gauge parameters ϵ', ϵ disappeared on the right-hand side. So the commutator of diffeomorphisms and gauge transformations vanishes now. Unfortunately, we are not able to apply the Lie algebra terminology anymore. Firstly, the 'structure constants' became field-dependent via $i_\xi i_{\xi'} F(A)$. Secondly, diffeomorphisms do still contribute to the sector of gauge transformations via the same $i_\xi i_{\xi'} F(A)$.

Therefore, even ignoring the field-dependence of the structure constants we cannot say that the algebra is a direct product. Though we can say that the deformation of the gauge transformations induced by coupling of A to gravity is abelian in a sense that there are no $\epsilon - \xi$-mixing terms in the commutator (but we cannot say that it it a direct product in any sense), which is not however the case in (12.29).

Consequently, within the Lie theory the coupling is non-abelian while within the more general framework the coupling is abelian. It is a general situation that the 'structure constants' may not be constant and can be field-dependent, so there is no way to interpret it in the language of Lie algebras directly. It is the case for the higher-spin theory. It turns out that with the help of the unfolded approach, one can assign representation theory meaning to such cases as well, see Section 6. The unfolded approach disentangles nontrivial algebraic structure of field theory and diffeomorphisms. A good example of the theory where 'structure constants' are field dependent is provided by an attempt to rewrite $3d$ higher-spin theory in terms of Fronsdal fields, [116].

12.4 Chevalley-Eilenberg cohomology and interactions

Note on cohomologies of Lie algebra. Before going deep into the structure constants we would like to remind the basic definitions from the theory of Lie algebras. Given a Lie algebra $\mathfrak{g}$ together with some its representation V we can construct a cochain complex as follows. k-chains are elements of $V \otimes \mathfrak{g}^* \wedge ... \wedge \mathfrak{g}^*$, where $\mathfrak{g}^*$ is a dual of $\mathfrak{g}$ as a vector space and $\wedge$ denotes the exterior power of $\mathfrak{g}^*$ as a vector space. So k-chain C^k is a skew-symmetric functional of k elements from $\mathfrak{g}$ with values in V

$$C^k : \mathfrak{g} \wedge ... \wedge \mathfrak{g} \longrightarrow V \,, \tag{12.34}$$

$$c \in C^k, \qquad c(a_1, ..., a_k) \in V, \qquad a_i \in \mathfrak{g} \,, \tag{12.35}$$

and $c(a_1, ..., a_k)$ is antisymmetric in its arguments. The differential d_k takes C^k to C^{k+1}

$$(d_k c)(a_1, ..., a_{k+1}) = \sum_i (-)^i a_i c(a_1, ..., \hat{a}_i, ..., a_{k+1}) + \tag{12.36}$$

$$+ \sum_{i<j} (-)^{i+j+1} c([a_i, a_j], a_1, ..., \hat{a}_i, ..., \hat{a}_j, ..., a_{k+1}) \,,$$

where $a_i c(...)$ means that a_i acts on the value of $c(...)$ since it belongs to V, which is a $\mathfrak{g}$-module. Therefore, given a skew-symmetric functional in k variables from $\mathfrak{g}$ we can construct a skew-symmetric functional in $k+1$ variables. One can check that $d_q \circ d_{q-1} = 0$ so cohomology groups are defined in a standard way as

$$H^q = Ker\, d_q / Im\, d_{q-1} \,. \tag{12.37}$$

116

To compute H^q we have to find a general solution to $d_q c(a_1, ..., a_q) = 0$ for $c \in C^q$ and identify two solutions if they differ by $(d_{q-1} b)(a_1, ..., a_q)$ for some $b \in C^{q-1}$. Classes H^q are called $\mathfrak{g}$-cohomology with values in V.

Cocyles and couplings. Suppose we have an unfolded system with underlying Lie algebra $\mathfrak{g}$, which is defined via equations (6.16) for Ω^I. We filter the rest of the fields by their form degree, i.e. we have sectors of p forms, q forms etc., and without loss of generality we assume that p-forms $W_{\boldsymbol{p}}^{\mathsf{A}}$ take values in some, perhaps trivial, $\mathfrak{g}$-module defined by the structure constants of one-forms via (6.21), *idem* for q-forms etc. Consider the most general unfolded equations that are still linear in $W_{\boldsymbol{p}}^{\mathsf{A}}$, but can be nonlinear in Ω^I, i.e. we think of Ω^I as the vacuum and consider linear perturbation over it. So the unfolded equations for $W_{\boldsymbol{p}}^{\mathsf{A}}$ have the form of a covariant constancy condition (6.21) whose *r.h.s.* can be sourced by other fields.

Suppose $W_{\boldsymbol{p}}^{\mathcal{A}}$ take values in $\mathfrak{g}$-module $\mathcal{R}_1$. The simplest option, which we never find in practice, is to write ($p + 1 = k$)

$$D_\Omega W_{\boldsymbol{p}}^{\mathcal{A}} \equiv dW_{\boldsymbol{p}}^{\mathcal{A}} + f_I{}^{\mathcal{A}}{}_{\mathcal{B}}\, \Omega^I \wedge W_{\boldsymbol{p}}^{\mathcal{B}} = f_{I_1...I_k}{}^{\mathcal{A}}\, \Omega^{I_1} \wedge ... \wedge \Omega^{I_k}. \tag{12.38}$$

The *r.h.s.* can be thought of as a skew-symmetric functional $f^{\mathcal{A}}(\Omega, ..., \Omega)$ with values in $\mathcal{R}_1$. The integrability (6.4) implies that

$$f_I{}^{\mathcal{A}}{}_{\mathcal{B}}\, \Omega^I f^{\mathcal{A}}(\Omega, ..., \Omega) = k f^{\mathcal{A}}(f_{JK}^I \Omega^J \wedge \Omega^K, ..., \Omega), \tag{12.39}$$

i.e. $f^{\mathcal{A}}(\Omega, ..., \Omega)$ is the cocycle of $\mathfrak{g}$ with values in $\mathcal{R}_1$. It is a cocycle but not necessary nontrivial, i.e. not of the form db_{k-1} in the notation as above. The trivial cocyles can be removed by field redefinitions, [92]. Therefore, we see that nontrivial *r.h.s.* are in one-to-one correspondence with degree-k cohomology of $\mathfrak{g}$ with values in $\mathcal{R}_1$, i.e. are given by H^k.

More realistically, suppose we have two sectors $W_{\boldsymbol{p}}^{\mathcal{A}}$ and $W_{\boldsymbol{q}}^{\mathsf{A}}$ of p- and q-forms, each taking values in some $\mathfrak{g}$-modules, say $\mathcal{R}_1$ and $\mathcal{R}_2$. Assuming $p+1 = q+k$ the most general equations that are linear in $W_{\boldsymbol{p}}^{\mathcal{A}}$ and $W_{\boldsymbol{q}}^{\mathsf{A}}$ read

$$D_\Omega W_{\boldsymbol{p}}^{\mathcal{A}} \equiv dW_{\boldsymbol{p}}^{\mathcal{A}} + f_I{}^{\mathcal{A}}{}_{\mathcal{B}}\, \Omega^I \wedge W_{\boldsymbol{p}}^{\mathcal{B}} = f_{I_1...I_k}{}^{\mathcal{A}}{}_{\mathsf{B}}\, \Omega^{I_1} \wedge ... \Omega^{I_k} \wedge W_{\boldsymbol{q}}^{\mathsf{B}}, \tag{12.40}$$

$$D_\Omega W_{\boldsymbol{q}}^{\mathsf{A}} \equiv dW_{\boldsymbol{q}}^{\mathsf{A}} + f_I{}^{\mathsf{A}}{}_{\mathsf{B}}\, \Omega^I \wedge W_{\boldsymbol{q}}^{\mathsf{B}} = ..., \tag{12.41}$$

and we ignore the possible *r.h.s.* in the second equation that will have analogous interpretation. The integrability (6.4) implies that $f_I{}^{\mathcal{A}}{}_{\mathcal{B}}(\Omega, ..., \Omega)$ is a $\mathfrak{g}$-cocycle with values in $\mathcal{R}_1 \otimes \mathcal{R}_2^*$, where $\mathcal{R}_2^*$ is the dual of $\mathcal{R}_2$.

More generally, nonlinear deformations can look like

$$D_\Omega W_{\boldsymbol{p}}^{\mathcal{A}} \equiv dW_{\boldsymbol{p}}^{\mathcal{A}} + f_I{}^{\mathcal{A}}{}_{\mathcal{B}}\, \Omega^I \wedge W_{\boldsymbol{p}}^{\mathcal{B}} = f_{I_1...I_k}{}^{\mathcal{A}}{}_{\mathsf{B}_1...\mathsf{B}_m}\, \Omega^{I_1} \wedge ... \Omega^{I_k} \wedge W_{\boldsymbol{q_1}}^{\mathsf{B}_1} \wedge ... \wedge W_{\boldsymbol{q_m}}^{\mathsf{B}_m},$$

where p-form $W_{\boldsymbol{p}}^{\mathcal{A}}$ with values in some $\mathcal{R}$ is sourced by a number of other forms $W_{\boldsymbol{q_1}}^{\mathsf{B}_1}$ with values in $\mathcal{R}_1, ..., \mathcal{R}_m$. The integrability condition implies that $f_{I_1...I_k}{}^{\mathcal{A}}{}_{\mathsf{B}_1...\mathsf{B}_m}$ is a cocycle with values in $\mathcal{R} \otimes \mathcal{R}_1^* \otimes ... \otimes \mathcal{R}_m^*$.

12.5 Universal enveloping realization of HS algebra

Since everything below applies equally well to $sp(2n)$ for any $n > 1$, while we need $sp(4)$, we shall consider $sp(2n)$ with invariant metric ϵ_{AB}. The starting point is the $sp(2n)$ commutation relations ($\star$ will denote the product in $U(sp(2n))$)

$$[T_{AB}, T_{CD}]_\star = T_{AD}\epsilon_{BC} + T_{BD}\epsilon_{AC} + T_{AC}\epsilon_{BD} + T_{BC}\epsilon_{AD}. \tag{12.42}$$

Consider the universal enveloping algebra $U(sp(2n))$ of $sp(2n)$, i.e. the algebra of all polynomials $P(T_{AB})$ in the generators T_{AB} modulo the commutation relations (12.42) of $sp(2n)$. It is easy to work out first several levels by decomposing the Taylor coefficients

$$P(T_{AB}) = P_0 + P_1^{AB} T_{AB} + P_2^{AB|CD} T_{AB} \star T_{CD} + ... \tag{12.43}$$

into $sp(2n)$ irreducible tensors. One finds

$$U(sp(2n)) = \underbrace{\bullet}_{0} \oplus \underbrace{(\boxed{\ \ })}_{1} \oplus \underbrace{\left(\boxed{\begin{matrix}\ \\ \ \end{matrix}} \oplus \boxed{\ \ \ \ } \oplus \boxed{\begin{matrix}\ \\ \ \end{matrix}} \oplus \bullet\right)}_{2} \oplus ... \tag{12.44}$$

The lowest component is the unit $\mathbf{1}$ of $U(sp(2n))$, which is an $sp(2n)$ singlet. At the first level we find just T_{AB}. At the second level there is a number of components: the singlet $\bullet$ is the Casimir operator $C_2 = -\frac{1}{2}T_{AB} \star T^{AB}$, it corresponds to $P_2^{AB|CD} = \epsilon^{AC}\epsilon^{BD} + \epsilon^{BC}\epsilon^{AD}$; $\boxed{\ \ \ \ \ }$ is a totally symmetric $T_{AA} \star T_{AA}$ and automatically traceless since ϵ^{AB} is antisymmetric; the antisymmetric component is

$$\boxed{\begin{matrix}\ \\ \ \end{matrix}} = \frac{1}{2}\{T_{AB}, T_C{}^B\}_\star + \frac{1}{n}\epsilon_{AC}C_2. \tag{12.45}$$

There is also a window component, $\boxed{\begin{matrix}\ & \ \\ \ & \ \end{matrix}}$.

In the case of the HS theory in AdS_4 we need various totally symmetric $sp(4, \mathbb{R})$-tensor of even ranks, which are equivalent to $so(3, 2)$-tensors with the symmetry of two-row rectangular Young diagram,

$$so(5): \boxed{\begin{matrix} s-1 \\ \end{matrix}} \qquad \Longleftrightarrow \qquad sp(4): \boxed{\quad 2s \quad} \tag{12.46}$$

In the HS algebra we are looking for every symmetric $sp(4)$-tensor of even rank (generators for bosonic fields) must appear once. Already at the second level there are unwanted tensor types. The window component and the rank-two antisymmetric simply do not have the type we need. The singlet component will lead to proliferation of fields. Indeed, given some generator T and a singlet C, for any k the monomial $T \star C^k$ is a generator again.

To get rid of the unwanted diagrams one can define a two sided ideal as follows

$$I_\lambda = U(sp(2n)) \otimes \left(\boxed{\begin{matrix}\ \\ \ \end{matrix}} \oplus \boxed{\begin{matrix}\ & \ \\ \ & \ \end{matrix}} \oplus (C_2 - \lambda\mathbf{1})\right) \otimes U(sp(2n)) \tag{12.47}$$

and then try to define the HS algebra $\mathfrak{g}$ as the quotient

$$\mathfrak{g} = U(sp(2n))/I_\lambda, \tag{12.48}$$

118

with the hope that $\mathfrak{g}$ is nontrivial and free from unwanted diagrams. Note that while we have to get rid of certain nonsinglet components the singlet generator, which is the Casimir, does not necessary have to be put to zero, rather it can be made equivalent to the unit element. As we see soon, λ turns out to be fixed by the self-consistency of the procedure.

In practice it means that we try to quotient out or to put to zero certain generators within the universal enveloping algebra. However, $U(sp(2n))$ is not a free algebra, so at some point we can come to a contradiction that requiring certain generators to be zero entail via the commutation relations that $T_{AB} \sim 0$ and hence the quotient algebra is empty.

Indeed, let us look at the relations among the low lying generators. First, we combine the window component and $\boxminus$ into a single $I_{UU,VV}$

$$I_{UU,VV} = T_{UV} \star T_{UV} + \gamma C_2 \left(\epsilon_{UU}\epsilon_{VV} + \frac{1}{4}\epsilon_{UV}\epsilon_{UV} \right), \tag{12.49}$$

where the anti-symmetrization over UU and VV is implied wherever necessary. $I_{UU,VV}$ has the symmetry of the window Young diagram but it is not completely traceless, containing $\boxminus$ as a trace. The coefficient $\gamma = 4/(n(2n+1))$ is fixed by requiring $I_{UU,VV}$ not to contain C_2, i.e. $\epsilon^{UU}\epsilon^{VV}I_{UU,VV} = 0$.

Using the commutation relations we find the following identities

$$[T_{AB}, T_C{}^B]_\star = 2(n+1)T_{AC}, \tag{12.50}$$

$$T_{AB} \star T_C{}^B = \alpha\, T_{AC} + \beta\, \epsilon_{AC}C_2 + \boxminus, \qquad \alpha = n+1, \qquad \beta = -\frac{1}{n}, \tag{12.51}$$

$$[T_{AB}, T_{CD}]_\star \star T^{AB} = 4(n+1)T_{CD}. \tag{12.52}$$

Now we would like to study the self-consistency of the procedure. Since $I_{UU,VV}$ belongs to the ideal, any element of $U(sp(2n)) \otimes I_{UU,VV}$ must belong too. In particular,

$$0 \approx T^{UU} \star I_{UU,VV} = T_{UV} \star \left(C_2 \left(\frac{3}{2}\gamma - 2\beta - 4 \right) - 2\alpha(\alpha - 2) \right). \tag{12.53}$$

Either we have to fix the Casimir to be $C_2 - \lambda\mathbf{1} \sim 0$ with $\lambda = -\frac{1}{4}n(2n+1)$ or T_{AB} itself has to be quotient out. The first option leads to a nontrivial solution and fixes λ. If we ignore that $\lambda = -\frac{1}{4}n(2n+1)$ then the resulting two-sided ideal takes away everything, so the quotient is trivial.

It is far from obvious that we will not meet any inconsistencies by considering other identities in $U(sp(2n))/I_\lambda$, but it starts looking as there exist an algebra with exactly the spectrum we need

$$\mathfrak{g} = U(sp(2n))/I_\lambda = \bullet \oplus \boxbox \oplus \boxboxboxbox \oplus \dots \tag{12.54}$$

More complicated Young shapes have been taken away by $\boxbox$, $\boxminus$ and their descendants obtained by sandwiching them with $U(sp(2n))$.

We discussed the invariant definition of the HS algebra on the language of the universal enveloping algebra. In particular it is now obvious that HS algebra is an associative one since it was obtained as a quotient of the associative algebra. However, this realization is difficult to deal with in practice. There are two sources of complexity. First of all, a universal enveloping algebra is a complicated object by itself, taking into account it is not free. Secondly, we quotiented it by a two-sided ideal, which entails many more relations between the generators[64].

The advantage of the $\star$-product realization discussed in Section 10 is that the ideal is automatically resolved. We already noticed that only symmetric tensors appear as Taylor coefficients in Y_A, which implies more complicated Young shapes, including the window and the rank-two antisymmetric, be projected out.

12.6 Advanced $\star$-products: Cayley transform

There are useful tricks that sometimes make star-product calculations quite simple, [57, 121]. Particularly, the repetitive gaussian integration drastically simplifies with the aid of the so called Cayley transform. Suppose one wants to compute a bunch of gaussian integrals of the form

$$\Phi = \Phi(f_1, \xi_1, q_1) \star \cdots \star \Phi(f_n, \xi_n, q_n), \tag{12.55}$$

where

$$\Phi(f, \xi, q) = \exp i\Big(\frac{1}{2}f_{AB}Y^A Y^B + \xi^A Y_A + q\Big), \tag{12.56}$$

Star-product of two such Φ's is already complicated and given explicitly by

$$\Phi(f_1, \xi_1, 0) \star \Phi(f_2, \xi_2, 0) = \frac{1}{\sqrt{\det|1 + f_1 f_2|}} \Phi(f_{1,2}, \xi_{1,2}, q_{1,2}), \tag{12.57}$$

where

$$f_{1,2} = \frac{1}{1 + f_2 f_1}(f_2 + 1) + \frac{1}{1 + f_1 f_2}(f_1 - 1), \tag{12.58}$$

$$\xi_{1,2}^A = \xi_1^B \Big(\frac{1}{1 + f_2 f_1}(f_2 + 1)\Big)_B^A + \xi_2^B \Big(\frac{1}{1 + f_1 f_2}(1 - f_1)\Big)_B^A, \tag{12.59}$$

$$q_{1,2} = \frac{1}{2}\Big(\frac{1}{1 + f_2 f_1}f_2\Big)_{AB} \xi_1^A \xi_1^B + \frac{1}{2}\Big(\frac{1}{1 + f_1 f_2}f_1\Big)_{AB} \xi_2^A \xi_2^B - \Big(\frac{1}{1 + f_2 f_1}\Big)_{AB} \xi_1^A \xi_2^B. \tag{12.60}$$

Further multiplication with some other Φ gives cumbersome result. A systematic way to proceed is to use the map of elements (12.55) into $SpH(2n)$ group which is the semidirect product of $Sp(2n)$ and Heisenberg group, [122, 123]. Its elements consist of triplets $\mathcal{G} = (U_A{}^B, x_A, c)$, where $U_A{}^B \in Sp(2n)$ with the following product

$$\mathcal{G}_1 \circ \mathcal{G}_2 = \Big((U_1 U_2)_A{}^B, x_{1A} + U_{1A}{}^B x_{2B}, c_1 + c_2 + x_1^A U_{1A}{}^B x_{2B}\Big). \tag{12.61}$$

---

[64]Other works along the same lines include [30, 117–120].

The embedding into the star-product algebra which respects the $SpH(2n)$ group law

$$r(\mathcal{G}_1)\,\Phi\Big(f(\mathcal{G}_1),\,\xi(\mathcal{G}_1),\,q(\mathcal{G}_1)\Big)\star r(\mathcal{G}_2)\,\Phi\Big(f(\mathcal{G}_2),\,\xi(\mathcal{G}_2),\,q(\mathcal{G}_2)\Big) = \tag{12.62}$$

$$= r(\mathcal{G}_1\mathcal{G}_2)\,\Phi\Big(f(\mathcal{G}_1\mathcal{G}_2),\,\xi(\mathcal{G}_1\mathcal{G}_2),\,q(\mathcal{G}_1\mathcal{G}_2)\Big) \tag{12.63}$$

can be shown to be of the following simple form

$$f_{AB}(\mathcal{G}) = \Big(\frac{U-1}{U+1}\Big)_{AB}, \tag{12.64}$$

$$r(\mathcal{G}) = \frac{2^{n/2}}{\sqrt{\det|1+U|}}, \tag{12.65}$$

$$\xi_A(\mathcal{G}) = \pm 2\Big(\frac{1}{1+U}\Big)_A{}^B x_B, \tag{12.66}$$

$$q(\mathcal{G}) = c + \frac{1}{2}\Big(\frac{U-1}{U+1}\Big)_{AB} x^A x^B. \tag{12.67}$$

This Cayley transform[65] allows one to express (12.55) as

$$\Phi = \frac{r(G_1...G_n)}{r(G_1)...r(G_n)}\Phi\Big(f(G_1...G_n),\xi(G_1...G_n),q(G_1...G_n)\Big). \tag{12.68}$$

Let us note, that Cayley transform is not always well defined. Particularly, it is not defined if for some f_{AB}, $f_A{}^C f_C{}^B = \delta_A{}^B$. In that case the corresponding $Sp(2n)$ group element U does not exist (formally it is at the infinity). Nevertheless, the star-product with such elements is perfectly well defined as can be seen from (12.58)-(12.60). Such elements with $f^2 = 1$ turns out to be star-product projectors

$$2^n e^{\frac{i}{2}f_{AB}Y^A Y^B} \star 2^n e^{\frac{i}{2}f_{AB}Y^A Y^B} = 2^n e^{\frac{i}{2}f_{AB}Y^A Y^B}, \qquad f^2 = 1. \tag{12.69}$$

These play an important role in construction of boundary-to-bulk propagators for HS fields and in HS black hole solutions, [55–57].

12.7 Poincare Lemma, Homotopy integrals

There is a standard problem how to solve equations of the form

$$df_{\boldsymbol{k}} = g_{\boldsymbol{k+1}} \qquad\qquad dg_{\boldsymbol{k+1}} = 0 \tag{12.70}$$

where $f_{\boldsymbol{k}} \equiv f_{a[k]}(x)\,dx^a \wedge ... \wedge dx^a$ is a degree-k differential form and $g_{\boldsymbol{k+1}}$ has degree $(k+1)$. This is the simplest system of unfolded type, a contractible pair unless $g_{\boldsymbol{k+1}} \equiv 0$. The second equation is the integrability condition for the first one, (12.40). If it is not true the system is inconsistent.

---

[65]There are two well-known equivariant maps from a Lie algebra to the group, the exp-map and the Cayley one. The exp-map is in general neither injective nor surjective. In many cases the Cayley map is more useful since it is a rational map.

The general solution reads

$$f_{a[k]} = \int_0^1 t^k \, dt \, x^c g_{ca[k]}(tx) + \begin{cases} c, & k = 0, \\ dc_{\boldsymbol{k-1}}, & k > 0, \end{cases} \tag{12.71}$$

In checking that this is indeed a solution we used that the number operator $x^c \frac{\partial}{\partial x^c}$ gives the same result as $t\frac{\partial}{\partial t}$ on functions $g(xt)$, then one gets a total derivative in t. Also, the integrability condition needs to be used to transform $\partial_a g_{ca[k]}$ (anti-symmetrization over all a's is implied) into $\partial_c g_{a[k]}$.

Viewing the system of equations as an unfolded one there is a gauge symmetry, $\delta f = \xi_{\boldsymbol{k}}$, $\delta g_{\boldsymbol{k+1}} = d\xi_{\boldsymbol{k}}$ and we see that all solutions are pure gauge from the unfolded perspective[66].

Two-dimensional case. In Section 11 we will face a particular case of the above equations, which are the equations along the auxiliary z-direction. That z^α is two-dimensional leaves us with two types of equations, for $k = 0$ and $k = 1$ ($k = 2$ does not have any $g_{\boldsymbol{k+1}}$ on the *r.h.s.*, so the solution is pure gauge $d\xi$). The first one, $k = 0$ reads as

$$\partial_\alpha f(z, y) = g_\alpha(z, y), \tag{12.72}$$

where $\partial_\alpha = \frac{\partial}{\partial z^\alpha}$ and y denotes collectively all other variables functions can depend on. It is implied that $g_\alpha(z, y)$ must obey the integrability condition $\partial^\alpha g_\alpha(z, y) = 0$. The general solution can be represented as

$$f(z, y) = z^\alpha \int_0^1 dt \, g_\alpha(zt, y) + c(y) \tag{12.73}$$

The second one we meet, $k = 1$ reads as

$$\partial_\alpha f_\beta(z, y)\epsilon^{\alpha\beta} = g(z, y), \tag{12.74}$$

where $g(z, y)$ is a two form, $\frac{1}{2}dz_\alpha \wedge dz^\alpha \, g(z, y)$ and the integrability imposes no restrictions on $g(z, y)$. The general solution reads

$$f_\alpha(z, y) = z_\alpha \int_0^1 t \, dt \, g(zt, y) + \partial_\alpha c(z, y) \tag{12.75}$$

---

[66]It is a choice as to whether think of the equations as having gauge symmetries or not. For example, one can reconstruct Maxwell gauge potential from the field strength by solving $dA = F$. Despite the fact that F is treated as a two-form, it does not have its own one-form gauge parameter that is capable of gauging A away completely. So this system is not of unfolded type, see Section 5.4 for the unfolding of a spin-one field. Unfolded equations always assume the richest gauge symmetry possible.

A   Indices

indices	affiliation
$\underline{m}, \underline{n}, \underline{r}$	world indices of the base manifold, are mostly implicit thanks to the differential form language
$a, b, c, ...$	fiber vector indices of $so(d-1, 1)$
$\alpha, \beta, ..., \dot{\alpha}, \dot{\beta}, ...$	two-component spinor indices
$A, B, ...$	four-component indices of $sp(4)$-vectors or more generally $2n$-component indices of $sp(2n)$
A, B, ...	(anti)-de Sitter algebra $(so(d-1, 2))$ $so(d, 1)$ indices, range over $d+1$ values

B   Multi-indices and symmetrization

Before going to higher-spin fields we need to introduce certain condensed notation for indices. The higher the spin the more indices needed, so it can become a waste of letters. In many cases tensor expressions are symmetric in all the indices, e.g. $T^{abc...u} = T^{bac...u} = T^{acb...u} = ...$, so it is natural to write $T^{a_1 a_2 ... a_s}$ instead of $T^{abc...}$ assuming that the tensor is totally symmetric in all indices 'a'. Moreover, since all indices are denoted by the same letter now, there is no need to keep them all, to indicate the number thereof is sufficient. So $a(s)$ denotes a group of s indices $a_1 a_2 ... a_s$ such that an object (tensor) is totally symmetric with respect to all permutations of $a_1 a_2 ... a_s$.

We still need to improve notation a little bit. Sometimes a tensor we find is not totally symmetric, e.g. $\partial^m \xi^{a(s-1)}$ is only partly symmetric, and we need to make it symmetric by summing over all permutations. Ideologically the right way to achieve this is to apply the symmetrizator $\boldsymbol{P} = \dfrac{1}{s!} \displaystyle\sum_{\text{all permutations}}$, which has a nice property of being a projector $\boldsymbol{P}\boldsymbol{P} = \boldsymbol{P}$. However, in practice it is more useful to adopt a convention where the sum is taken over *all necessary* permutations without dividing by $s!$. To simplify notation, all the indices of the group of indices to be symmetrized are denoted by the same letter, so a string of indices $a(k)...a..a$ means that either the tensor is already symmetric in all of them or needs to be symmetrized. For example, (the hatted indices are omitted)

$$\partial^a \xi^{a(s-1)} = \sum_{i=1}^{i=s} \partial^{a_i} \xi^{a_1 ... \hat{a}_i ... a_s} , \tag{B.1}$$

$$\partial^a \partial^a \phi^{a(s-2)m}{}_m = \sum_{i<j} \partial^{a_i} \partial^{a_j} \phi^{a_1 ... \hat{a}_i ... \hat{a}_j ... a_s m}{}_m . \tag{B.2}$$

One should be careful with using this convention, still it leads to simpler formulas and most of the normalization coefficients simply do not appear.

For example, let us check the gauge invariance of the Fronsdal operator, (2.4),

$$\delta \Box \phi^{a(s)} = \Box \partial^a \xi^{a(s-1)}, \tag{B.3}$$

$$\partial_m \delta \phi^{ma(s-1)} = \Box \xi^{a(s-1)} + \partial^a \partial_m \xi^{a(s-2)m}, \tag{B.4}$$

$$\delta \phi^{a(s-2)m}{}_m = 2\partial_m \xi^{a(s-1)m}, \tag{B.5}$$

$$\partial^a (\partial^a \partial_m \xi^{a(s-1)m}) = 2\partial^a \partial^a \partial_m \xi^{a(s-1)m}. \tag{B.6}$$

Combining all terms together we find they cancel each other. Notice the bracket removal in the last line brings a factor of two, this is an example of a nested symmetrization. The subtle point is that in $\partial^a (\partial^a \partial_m \xi^{a(s-1)m})$ one should think of the inner expression as a generic rank-$(s-1)$ symmetric tensor, (despite the fact that it was obtained by summing over $s - 1$ permutations). It came from $\partial^a (\partial_m \phi^{a(s-1)m})$ for which s permutations are needed. Hence, to symmetrize it with ∂^a one needs s permutations. In total it gives $s(s-1)$ permutations. However, on the *r.h.s.* we symmetrize $\partial^a \partial^a$, which is already symmetric, with $\partial_m \xi^{a(s-1)m}$, so $s(s-1)/2$ permutations are needed. This brings a factor of two for the measure of all permutations that have been idle on the *l.h.s.* because we were not aware of the internal structure of $(\partial^a \partial_m \xi^{a(s-1)m})$ and the fact that $\partial^a \partial^a$ is already symmetric. Nested symmetrizations are the sources of some simple factors that are important for the final cancelation of terms.

For antisymmetric or to be anti-symmetrized indices we adopt the same rules but with indices enclosed in square brackets, e.g. $u[q]$.

C Solving for spin-connection

Christoffel symbols $\Gamma^r_{\underline{mn}}$ can be solved for as usual. Let us solve for $\omega^{a,b}$. Contracting (3.15) with $e^{\underline{mb}} e^{\underline{nc}}$ and defining $\Upsilon^{a|bc} = (\partial_{\underline{n}} e^a_{\underline{m}} - \partial_{\underline{n}} e^a_{\underline{m}}) e^{\underline{mb}} e^{\underline{nc}}$ and $\omega^{a,b|c} = \omega^{a,b}_{\underline{m}} e^{\underline{mc}}$ we find

$$\Upsilon^{a|bc} + \omega^{a,c|b} - \omega^{a,b|c} = 0. \tag{C.7}$$

As in the case of $\Gamma^r_{\underline{nm}}$ we add two more equations that differ by cyclic permutation of abc. Then it is easy to see that

$$\omega^{a,b|c} = \frac{1}{2} \left(\Upsilon^{c|ab} + \Upsilon^{b|ca} - \Upsilon^{a|cb} \right), \tag{C.8}$$

and finally

$$\omega^{a,b}_{\underline{m}} = \frac{1}{2} \left(\Upsilon^{c|ab} + \Upsilon^{b|ca} - \Upsilon^{a|cb} \right) e_{c\underline{m}}. \tag{C.9}$$

D Differential forms

Among all tensors there is a subclass of covariant totally-antisymmetric tensors that are special in many respects. Given such a tensor, say $T_{\underline{m_1} \dots \underline{m_q}}$, $T_{\underline{m_1} \dots \underline{m_i} \underline{m_{i+1}} \dots \underline{m_q}} = -T_{\underline{m_1} \dots \underline{m_{i+1}} \underline{m_i} \dots \underline{m_q}}$ for all i, it is called a differential form and its rank, q, is referred to as the form degree. Since the index structure is fixed by anti-symmetry it is useful just to indicate the form degree as a subscript, e.g. $T_{\boldsymbol{q}}$, without having to write down the indices.

A useful way to ensure that the tensor is antisymmetric is to use the Grassmann algebra. This is an associative algebra with a unit generated by $\theta^{\underline{m}}$ obeying $\theta^{\underline{m}}\theta^{\underline{n}} = -\theta^{\underline{n}}\theta^{\underline{m}}$. Consider polynomials in the Grassmann algebra

$$P(\theta) = \phi + A_{\underline{m}}\theta^{\underline{m}} + \frac{1}{2}F_{\underline{mn}}\theta^{\underline{m}}\theta^{\underline{n}} + ... + \frac{1}{d!}\omega_{\underline{m}_1...\underline{m}_d}\theta^{\underline{m}_1}...\theta^{\underline{m}_d} . \qquad (D.10)$$

The expansion coefficients are forced to be antisymmetric tensors and hence the expansion stops at the form of the highest degree possible, which is d. There are three important operations on the class of differential forms.

(i) exterior product, which is denoted usually by $\wedge$. This is just the product in the Grassmann algebra. In terms of Taylor coefficients it corresponds to first taking the tensor product and then anti-symmetrizing all the indices. It takes degree-q form $T_{\boldsymbol{q}}$ and degree-p form $R_{\boldsymbol{p}}$ to a degree-$(p+q)$ $(T_{\boldsymbol{q}} \wedge R_{\boldsymbol{p}})$ form (note that the order matters) $\frac{1}{(p+q)!}\sum_{\sigma\in S_n}(-)^{|\sigma|}\sigma \circ T_{\underline{m}_1...\underline{m}_q}R_{\underline{m}_{q+1}...\underline{m}_{p+q}}$. In the main text we sometimes omit $\wedge$ symbol when one of the factors is a zero-form since zero-forms do not carry any differential form indices and the anti-symmetrization is trivial. Zero-forms serve as purely numerical factors that can be commuted without producing a sign, $T_{\boldsymbol{q}} \wedge R_{\boldsymbol{p}} = (-)^{pq}R_{\boldsymbol{p}} \wedge T_{\boldsymbol{q}}$.

(ii) exterior derivative, d. Operator d is defined as $d = \theta^{\underline{m}}\partial_{\underline{m}}$. It is nilpotent $dd \equiv 0$ and applying $\theta^{\underline{m}}\nabla_{\underline{m}}$ produces the same result as $\theta^{\underline{m}}\partial_{\underline{m}}$, i.e. Christoffel symbols drop out from final expressions because of anti-symmetrization and are irrelevant in definition of differentiation for differential forms.

(iii) inner derivative. Given a vector field $\xi^{\underline{m}}$ one defines the inner derivative i_ξ as $\xi^{\underline{m}}\frac{\partial}{\partial\theta^{\underline{m}}}$. The Lie derivative $\mathcal{L}_\xi$ is then $\mathcal{L}_\xi = di_\xi + i_\xi d$.

In the literature instead of Grassmann algebra, symbols $dx^{\underline{m}}$ are mostly used which are required to anti-commute $dx^{\underline{m}}\wedge dx^{\underline{n}} = -dx^{\underline{n}}\wedge dx^{\underline{m}}$ with respect to exterior product $\wedge$. The inner derivative is defined as $\xi^\mu\frac{\partial}{\partial dx^\mu}$, correspondingly. We will also use $dx^{\underline{m}}$ having in mind the Grassmann algebra interpretation.

Integration and Einstein-Hilbert action. The naive way to rewrite $\int \sqrt{\det g}\, R$ is to note that[67] $\sqrt{\det g} = \det e$, and $R = F_{\underline{mn}}{}^{a,b}e^{\underline{m}}_a e^{\underline{n}}_b$. There is a fancier way by taking the advantage of the language of differential forms. Let us recall how to integrate over manifolds. The measure $d^d x$ is not invariant under a change of coordinates, $d^d x' = J d^d x$, where $J = \det|\partial x'/\partial x|$. One has to cure this non-invariance by multiplying it by something that compensates J. One can take any covariant rank-two tensor, not necessarily metric and not necessarily symmetric, because its determinant transforms as J^{-2}, so $\sqrt{\det}$ does the job. Another way is to take a rank-d totally-antisymmetric covariant tensor, say $\omega_{\underline{m}_1...\underline{m}_d}$, i.e. a differential form of top degree. As a tensor it has only one independent component and can be represented as $\omega_{\underline{m}_1...\underline{m}_d} = C(x)\epsilon_{\mu_1...\mu_d}$, where the totally antisymmetric tensor $\epsilon_{\mu_1...\mu_d}$ is normalized as $\epsilon_{12...d} = 1$. Under change of coordinates it transforms as $C' = \det|\partial x/\partial x'|C = J^{-1}C$. Therefore, given a top form, called the volume form sometimes, we can integrate over the manifold. To be precise one may use $\omega_{12...d}\, d^d x$ as an invariant measure.

---

[67]To be honest, $\det g = \det^2 e \det \eta$.

E   Young diagrams and tensors

We enter the world of Young diagrams through the $GL(d)$-tensors' door and then discuss what needs to be added to cover the case of $SO(d)$ tensors. We do not aim at comprehensive review and present mostly the facts useful for this particular course.

E.1   Generalities

$\boldsymbol{GL(d)}$. As is well-known, a rank-two tensor $T^{a|b}$ (for simplicity we deal only with either contravariant or covariant tensors) can be decomposed into its symmetric and antisymmetric components

$$T^{a|b} = T_S^{ab} + T_A^{a,b},\tag{E.11}$$

$$T_S^{ab} = T_S^{ba} = \frac{1}{2}(T^{a|b} + T^{b|a}),\qquad T_A^{a,b} = -T_A^{b,a} = \frac{1}{2}(T^{a|b} - T^{b|a}).\tag{E.12}$$

If we deal with $GL(d)$, one can easily see that under any $GL(d)$ transformation

$$T^{ab...c} \longrightarrow S^a_{\ a'}\, S^b_{\ b'}\, ...S^c_{\ c'}\, T^{a'b'...c'},\qquad S^a_{\ m} \in GL(d),\tag{E.13}$$

(i) the two parts do not mix and (ii) they preserve their symmetry type, i.e. remain (anti)symmetric. It is also obvious that there are no more $GL(d)$-invariant conditions one can impose to decompose T_S^{ab}, $T_A^{a,b}$ even further.

It is convenient to employ Young diagrams as a tool to encode graphically the symmetry type of tensors. A Young diagram is a picture made of boxes, one box per each tensor index. The rank-two (anti)-symmetric tensors are pictured by the following Young diagrams

$$T_S^{ab} \sim \boxed{a\,b},\qquad\qquad T_A^{a,b} \sim \boxed{\begin{smallmatrix}a\\b\end{smallmatrix}}.\tag{E.14}$$

A vector and scalar are denoted by $\square$ and an empty diagram $\bullet$, respectively.

As for rank-three tensors one finds something new in addition to totally (anti)-symmetric parts. Since the decomposition of rank-two tensors into irreducibles is known, we can take a rank-three tensor which is already irreducible in some two indices, say $T^{ab|c} = T^{ba|c}$. Then one finds,

$$T^{ab|c} = T_S^{abc} + H_S^{ab,c},$$

$$T_S^{abc} = \frac{1}{3}(T^{ab|c} + T^{bc|a} + T^{ca|b}),\qquad T_S^{abc} = T_S^{bac} = T_S^{acb},\tag{E.15}$$

$$H_S^{ab,c} = \frac{1}{3}(2T^{ab|c} - T^{bc|a} - T^{ca|b}),\quad H_S^{ab,c} = H_S^{ba,c},\qquad H_S^{ab,c} + H_S^{bc,a} + H_S^{ca,b} = 0.$$

The appearance of the totally symmetric component, T_S^{abc}, was expected. There is one more, $H_S^{ab,c}$, that is neither totally symmetric nor antisymmetric. It is symmetric in the first two indices, ab and the condition for it not to contain the totally symmetric component, which is already covered by T_S^{abc}, implies that the symmetrization over all three indices vanishes. With our convention on symmetrization we can rewrite it as

$$H_S^{aa,b},\qquad\qquad H_S^{aa,a} = 0.\tag{E.16}$$

Again one can check that under any $GL(d)$ transformations both T_S^{abc} and $H_S^{ab,c}$ preserve the symmetry conditions they obey and do not mix. In the language of Young diagrams these are pictured as follows

$$T_S^{abc} \sim \boxed{\text{a}\,\text{b}\,\text{c}} \qquad\qquad H_S^{ab,c} \sim \begin{array}{l}\boxed{\text{a}\,\text{b}}\\[-2pt]\boxed{\text{c}}\end{array} \qquad\qquad (\text{E.17})$$

Because of the shape, $H_S^{ab,c}$ is sometimes called 'hook'-diagram. We see tensors tend to be symmetric in the indices associated with rows of Young diagrams. Not surprisingly, a rank-s totally symmetric tensors is denoted by

$$T^{a(s)} \sim \underbrace{\boxed{\text{a}\,\text{a}}\cdots\boxed{\text{a}\,\text{a}}}_{s \text{ boxes}} \qquad\qquad (\text{E.18})$$

If instead we take a rank-three tensor that is already antisymmetric in the first two indices, $T^{ab|c} = -T^{ba|c}$, we will find a similar decomposition into $GL(d)$ irreducible components

$$T^{ab|c} = T_A^{abc} + H_A^{ab,c} \,,$$
$$T_A^{abc} = \frac{1}{3}(T^{ab|c} + T^{bc|a} + T^{ca|b})\,, \qquad T_A^{abc} = -T_A^{bac} = -T_A^{acb}\,, \qquad\qquad (\text{E.19})$$
$$H_A^{ab,c} = \frac{1}{3}(2T^{ab|c} - T^{bc|a} - T^{ca|b})\,, \quad H_A^{ab,c} = -H_A^{ba,c}\,, \qquad\qquad H_A^{ab,c} + H_A^{bc,a} + H_A^{ca,b} = 0 \,.$$

In addition to the totally antisymmetric component T_A^{abc} one finds $H_A^{ab,c}$ that is antisymmetric in ab and does not contain the totally antisymmetric component, which is the last condition[68]. In the language of Young diagrams we have

$$T_A^{abc} \sim \begin{array}{l}\boxed{\text{a}}\\[-2pt]\boxed{\text{b}}\\[-2pt]\boxed{\text{c}}\end{array} \qquad\qquad H_A^{ab,c} \sim \begin{array}{l}\boxed{\text{a}\,\text{c}}\\[-2pt]\boxed{\text{b}}\end{array} \qquad\qquad (\text{E.20})$$

Tensors tend to be antisymmetric in the indices associated with the columns. A rank-q totally antisymmetric tensor is denoted by

$$T^{u[q]} \sim \left.\begin{array}{l}\boxed{\text{u}}\\[-2pt]\vdots\\[-2pt]\boxed{\text{u}}\\[-2pt]\boxed{\text{u}}\end{array}\right\} q \text{ boxes} \qquad\qquad (\text{E.21})$$

Tensors that are neither symmetric nor antisymmetric are referred to as having mixed-symmetry. $H_S^{ab,c}$ and $H_A^{ab,c}$ have mixed-symmetry. There is a strange thing one might have noticed that the second diagram in (E.20) is identical to that of (E.17). How come that $H_S^{ab,c}$ and $H_A^{ab,c}$ have the same symmetry type but obey different symmetry conditions?

---

[68]Coincidentally, for rank-three tensors (E.15) and (E.19) look identical, the symmetry conditions they obey are different though. This is due to the fact that the cyclic permutation taking place in the definition of $H_S^{ab,c}$ and $H_A^{ab,c}$ (anti)-symmetrizes over the three indices depending on whether the original tensor is symmetric or antisymmetric in the first two indices. For tensors of higher rank one finds more difference between symmetric and antisymmetric presentations.

This is an essential feature of genuine mixed-symmetry tensors, i.e. the tensors with the symmetry of a Young diagram that is different from one-row or one-column diagram. The symmetries of (one-column) one-row Young diagrams are always presented by (anti)-symmetric tensors. There are in general many ways to present a tensor with the symmetry of more complicated Young diagrams. The simplest mixed-symmetry tensor has the symmetry of ⊞. In components we have either some $H_S^{aa,b}$, which is symmetric in the first two indices and obeys the Young condition $H_S^{aa,a} = 0$, or some $H_A^{ab,c}$, which is antisymmetric in the first two indices and obeys $H_A^{[ab,c]} = 0$.

Given $H_S^{ab,c}$ one can always construct an object of type $H_A^{ab,c}$ as

$$H_A^{ab,c} = H_S^{ac,b} - H_S^{bc,a}.$$ (E.22)
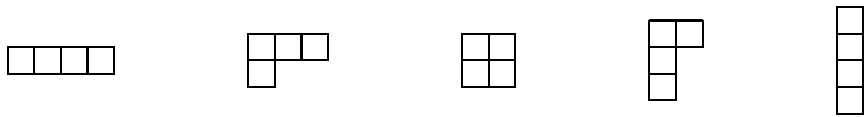
The map is invertible. Indeed, one can go back

$$H_S^{ab,c} = \alpha(H_A^{ac,b} + H_A^{bc,a}),$$ (E.23)

where α turns out to be $1/3$. Since the map is an isomorphism, it just defines two bases. We can say that there are several ways to implement symmetries of some Young diagram into a tensor. In practice it is sometimes useful to switch from one base to another one to simplify computations. It is not possible to have the symmetry properties both of $H_S^{ab,c}$ and of $H_A^{ab,c}$ realized simultaneously.

There are two bases for mixed-symmetry tensors that are most natural, symmetric and antisymmetric. In the (anti)-symmetric base tensors are explicitly (anti)-symmetric in the indices corresponding to the (columns) rows of Young diagrams, with some additional relations imposed. For example, $H_S^{ab,c}$ was given in the symmetric base, while $H_A^{ab,c}$ in the antisymmetric one.

Bearing in mind the option of having several ways to represent a tensor, we usually do not fill the boxes in Young diagrams with indices. For example, there are five different symmetry types possible for rank-four tensors

 (E.24)

The rows in a Young diagram are left aligned. The length of the rows in a proper Young diagram cannot increase downwards (the upper rows are not shorter than the lower ones).

The first and the last diagrams correspond to totally (anti)-symmetric tensors, for which there is no ambiguity in the choice of a base. There are two different presentations for the three diagrams in the middle. For example, the Riemann and Weyl tensors have the symmetry of the diagram in the exact center. Usually the Riemann tensor is defined in the antisymmetric base,

$$R_{ab,cd}^A = -R_{ba,cd}^A = -R_{ab,dc}^A = R_{cd,ab}^A, \qquad\qquad R_{[ab,c]d}^A = 0.$$ (E.25)

Seldom used is the symmetric base

$$R_{ab,cd}^S = R_{ba,cd}^S = R_{ab,dc}^S = R_{cd,ab}^S, \qquad\qquad R_{(ab,c)d}^S = 0.$$ (E.26)

Note that $R_{ab,cd}^{A(S)} = R_{cd,ab}^{A(S)}$ is not an independent relation and follows from the rest. That the two ways are equivalent is shown by writing the linear transformation explicitly

$$R_{ac,bd}^{S} = R_{ab,cd}^{A} + R_{cb,ad}^{A}\,, \qquad\qquad R_{ac,bd}^{A} = \beta(R_{ab,cd}^{S} - R_{cb,ad}^{S})\,, \qquad (\text{E.27})$$

where the first formula is treated as a definition of R^S, then the coefficient in the second formula is found to be $\beta = 1/3$. Note that in this particular case it is sufficient to (anti)-symmetrize over two indices, the (anti)-symmetry in the other two indices then follows from the properties of the original tensors.

A tensor having the symmetry of the second diagram from (E.24) in the symmetric base one has $T^{aaa,b}$ that obeys $T^{aaa,a} = 0$.

Let us consider one more example of a tensor having $\boxed{k}$ symmetry type. We begin with $T^{a(k-1)|b}$, which is symmetric in $a(k)$ and there are no symmetry conditions among b and $a(k)$. On subtracting the totally symmetric component one finds a remnant,

$$T^{a(k)|b} = T^{a(k)b} + T^{a(k),b}\,, \qquad (\text{E.28})$$

$$T^{a(k)b} = \frac{1}{k+1}\left(T^{a(k)|b} + T^{a(k-1)b|a}\right)\,, \qquad (\text{E.29})$$

$$T^{a(k),b} = \frac{1}{k+1}\left(kT^{a(k)|b} - T^{a(k-1)b|a}\right)\,, \qquad T^{a(k),a} = 0\,. \qquad (\text{E.30})$$

It is not that difficult to get the complete classification of symmetry types. For the purpose of totally symmetric higher spin fields it is sufficient to restrict ourselves to the class of two-row Young diagrams with the tensors presented in the symmetric base,

$$T^{a(k),b(m)} \sim \overbrace{\underbrace{\boxed{k}}_{m}}^{} = \overbrace{\underbrace{\boxed{a\,a\,a}\cdots\boxed{a\,a\,a}}_{}}^{k}_{\boxed{b\,b}\cdots\boxed{b}}\underbrace{}_{m} \qquad (\text{E.31})$$

The tensor $T^{a(k),b(m)}$ has two groups of indices $a(k)$ and $b(m)$; it is symmetric in k indices a and m indices b; it is irreducible under $GL(d)$ iff the symmetrization of all indices a with at least one index from the second group vanishes, i.e.

$$T^{a(k),ab(m-1)} = 0\,. \qquad (\text{E.32})$$

Note that one cannot impose more symmetry conditions in general as it would be equivalent to requiring a tensor to be symmetric and antisymmetric in some indices at the same time, which implies the tensor is identically zero. If $k < m$ the tensor vanishes identically, which explains the condition for the length of rows not to increase downwards.

For $k = 2$, $m = 0$ we have $T^{ac} = T^{ac}$, i.e. symmetric. For $k = m = 1$ we find $T^{a,b} + T^{b,a} = 0$, i.e. $T^{a,b}$ is antisymmetric. For $k = 2$, $m = 1$ we find (E.15). For $k = 2$, $m = 2$ we find the symmetry conditions for the Weyl/Riemann tensor, (E.26). There is a slightly degenerate case of rectangular Young diagrams. In the latter case the two groups of indices in the tensor, $T^{a(k),b(k)}$, are equivalent. In particular, the following relation is true

$$T^{a(k),b(k)} = (-)^k T^{b(k),a(k)}\,, \qquad (\text{E.33})$$

which is obvious for $k = 1$ and for $k = 2$, the Riemann tensor, the condition is also known to be true. The proof is easy in the antisymmetric presentation of tensors, where one has k pairs of antisymmetric indices and swapping the indices inside all pairs yields $(-)^k$.

At the end of the $GL(d)$ section let us give several identities that are frequently used to rearrange indices. Suppose we are given $T^{a(k),b(m)}$ and there is a symmetrization imposed over k indices with one of them taken from the second group of indices, i.e.

$$T^{a(k-1)c,ab(m-1)} = \sum_i T^{a_1...\hat{a}_i...a_k c, a_i b_2...b_m} \tag{E.34}$$

This is what is effectively imposed when $T^{a(k),b(m)}$ is contracted with another tensor $V_{a(k)}$ that is symmetric

$$T^{a(k-1)c,ub(m-1)} V_{a(k-1)u} . \tag{E.35}$$

The defining relation (E.32) then tells that

$$T^{a(k-1)c,ab(m-1)} = -T^{a(k),b(m-1)c} \tag{E.36}$$

but

$$T^{a(k-1)c,ub(m-1)} V_{a(k-1)u} = -\frac{1}{k} T^{a(k),b(m-1)c} V_{a(k)} , \tag{E.37}$$

the difference in $\frac{1}{k}$ is because (E.34) contains k terms explicitly, while (E.35) does not,

$$T^{a(k-1)c,ub(m-1)} V_{a(k-1)u} = \frac{1}{k} T^{a(k-1)c,ab(m-1)} V_{a(k)} \tag{E.38}$$

All k terms in the last expression are identical thanks to the symmetry of $V_{a(k)}$, hence they cancel $\frac{1}{k}$. A more general relation holds true

$$T^{a(k-n)c(n),u(n)b(m-n)} V_{a(k-n)u(n)} = \frac{(-)^n n!}{k!(k-n)!} T^{a(k),c(n)b(m-n)} V_{a(k)} \tag{E.39}$$

and allows one to always put all symmetrized indices into the first group of indices. Note that one can roll symmetrized indices to the group of indices corresponding to a longer row of Young diagram, but not in the opposite direction. Also note that (E.33) is a particular case of the identity above for $k = n = m$.

The property of being a Young diagram tells us that it is the first row/column that is the longest one. Obviously, the height of the columns in a Young diagram cannot exceed d, for we can choose the antisymmetric base to present a tensor with such a symmetry and it then will carry more than d antisymmetric indices, i.e. it has to vanish.

To draw a line, $GL(d)$ irreducibility requirements impose certain symmetry conditions on the indices carried by a tensor. Irreducibility conditions are nicely encoded by Young diagrams. There are in general several ways to present an irreducible $GL(d)$ tensor, one can transfer between different bases by (anti)-symmetrizing indices.

$SO(d)$. In case $T^{a|b}$ is an $so(d)$ tensor (signature is irrelevant) we have an invariant tensor, which is the metric η_{ab}. With the help of the metric one can do more and extract the trace

$$T^{a|b} = T_S^{ab} + T_A^{a,b} + \frac{1}{d}\eta^{ab}T \, , \tag{E.40}$$

$$T_S^{ab} = \frac{1}{2}(T^{a|b} + T^{b|a} - \frac{2}{d}\eta^{ab}T^{c|d}\eta_{cd}) \, , \qquad T_S^{ab} = T_S^{ba} \, , \qquad T_S^{ab}\eta_{ab} = 0 \, ,$$

$$T_A^{a,b} = \frac{1}{2}(T^{a|b} - T^{b|a}) \, , \qquad\qquad T_A^{a,b} = -T_A^{b,a} \, ,$$

$$T = T^{c|d}\eta_{cd} \, .$$

Again, one can check that the decomposition is stable under any $SO(d)$ transformations.

It is obvious that $SO(d)$-irreducibility requires $GL(d)$-irreducibility, i.e. Young symmetry, and one needs to supplement Young symmetry conditions with the trace constraints. The full set of constraints for tensors with the symmetry of two-row Young diagrams includes

$$T^{a(k),b(m)} \sim \boxed{\begin{array}{c} k \\ \hline m \end{array}} \, , \qquad T^{a(k),ab(m-1)} = 0 \, , \tag{E.41}$$

$$T^{a(k-2)c}{}_c{}^{,b(m)} = 0 \, , \qquad T^{a(k-1)c,\,b(m-1)}{}_c = 0 \, , \qquad T^{a(k),b(m-2)c}{}_c = 0 \, . \tag{E.42}$$

There are three types of traces one can take, depending on how the two contracted indices are distributed over the two groups of indices. Not all of these traces are independent. Indeed, assuming that $T^{a(k-2)c}{}_c{}^{,b(m)} = 0$ we can symmetrize all $a(k-2)$ with one of the b's to see, applying (E.36),

$$T^{a(k-2)c}{}_c{}^{,ab(m-1)} = -T^{a(k-1)c,\,b(m-1)}{}_c \, . \tag{E.43}$$

Symmetrizing now $a(k-1)$ with one of the b's once again we find

$$T^{a(k-1)c,\,ab(m-2)}{}_c = -T^{a(k),c\,b(m-2)}{}_c \, . \tag{E.44}$$

We see that the first trace condition implies the second and the second then implies the third, but not in the opposite direction — it is possible to have the third trace conditions satisfied without enforcing the first and the second.

Young symmetry plus trace constraints furnish the complete set of irreducibility conditions in most cases. However, when a tensor has $d/2$ antisymmetric indices (it is better to refer to the number of rows in the Young diagram), in particular we are in even dimension, one can impose (anti)-selfduality conditions,

$$T^{u[q]} = \pm(i)\epsilon^{u[q]}{}_{v[q]} T^{v[q]} \, , \qquad\qquad d = 2q \, , \tag{E.45}$$

where $\epsilon_{u[d]}$ is the totally antisymmetric tensor, which is also an invariant tensor of $so(d)$. Whether one can impose the (anti)-self duality condition with ± 1 or $\pm i$ depends on the dimension, d, modulo 4. In particular one can impose $(\pm i)$ (anti)-selfduality for tensors with the symmetry of two-row rectangular Young diagrams in the case of $so(3,1)$. This explains why a single higher-spin connection $\omega^{a(s-1),b(k)}$ splits into two complex conjugate connections $\omega^{\alpha(s+k-1),\dot\alpha(s-k-1)}$, $\omega^{\alpha(s-k-1),\dot\alpha(s+k-1)}$ — two-row Young diagrams do not

correspond to irreducible tensors in $4d$ once we allow for a pair of complex conjugated tensors.

To deal with $so(d)$ is even more restrictive. It is useful to prove the following result (we do not use this anywhere in the lectures): a traceless tensor with the symmetry of a Young diagram for which the sum of heights of the first two column exceeds d must vanish identically. Note that nothing prevents the first column to exceed $[d/2]$ at the price of all other columns being shorter than $[d/2]$, for example totally antisymmetric tensors of any rank $q = 0, ..., d$ do exist.

The existence of $\epsilon_{u[d]}$ imposes more restrictions on $so(d)$-Young diagrams. Any tensor having a symmetry of a Young diagram whose first column, say of height q, exceeds $[d/2]$ is equivalent to a tensor whose first column does not exceeds $[d/2]$, its height is $d - q$. For example, given a totally antisymmetric tensor $T^{u[q]}$, i.e. its symmetry is given by a Young diagram made of a single column, we can dualize it to a rank-$(d - q)$ tensor $T'^{u[d-q]}$

$$T'^{u[d-q]} = \epsilon^{u[d-q]}{}_{v[q]} \, T^{v[q]} \,. \tag{E.46}$$

This implies that any Young diagram of $so(d)$ should have no more than $[d/2]$ rows. As was just mentioned, this does not mean that all other Young diagrams correspond to identically vanishing tensors, but those that correspond to nontrivial tensors are equivalent via $\epsilon_{u[d]}$-dualization to Young diagrams with no more than $[d/2]$ rows.

Why Young diagrams? The rationale behind Young diagrams lies in a close connection of representation theory of $GL(d)$ and the symmetric group. The symmetric group in n letters, S_n, acts naturally on the n-th tensor power, $T^n V$ of a vector space, V, by permuting the factors $v_1 \otimes ... \otimes v_n$. Hence $T^n V$ is a representation of S_n, a reducible one. One can project onto various S_n-irreducible subspaces, which simultaneously projects onto $GL(d)$ irreducible subspaces. As is well-known the irreducible representations of S_n are in one-to-one correspondence with conjugacy classes. They can be enumerated by partitions of n into nonnegative integers, say $n = s_1 + ... + s_n$, which can be ordered $s_1 \geq s_2 \geq ... \geq s_n \geq 0$. Each partition can be encoded by a Young diagram that has rows of lengths $s_1, ..., s_n$. If we go one further we face the representation theory of Lie algebra and find out that Verma modules are parameterized by a number of constants, the weights, with the number of weights equal to the rank of a given Lie algebra. In particular the rank of $so(d)$ is $[d/2]$, so the number of parameters is in accordance with the number of rows in $so(d)$ Young diagrams. The general theory is discussed in many textbooks, see e.g. for the summary [124]. The Young diagrams are not just nice pictures and appear naturally in many different topics, [125].

E.2   Tensor products

We more or less understand now what are the conditions for a tensor to be irreducible. Let us discuss the inverse problem of how to decompose a reducible tensor into its irreducible components. This amount to computing tensor products. The main properties of tensor products are associativity, commutativity and distributivity. Actually, this is what we have already done for the tensors of simplest types. In the case of $GL(d)$ we manually

found that

$$(\text{E.12}): \qquad T^{a|b} = T_S^{ab} + T_A^{a,b} \qquad\qquad \square\otimes\square = \square\square \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix}\,, \tag{E.47}$$

$$(\text{E.15}): \qquad T^{(ab)|c} = T_S^{abc} + H_S^{ab,c} \qquad \square\square\otimes\square = \square\square\square \oplus \begin{smallmatrix}\square\square\\\square\end{smallmatrix}\,, \tag{E.48}$$

$$(\text{E.19}): \qquad T^{[ab]|c} = T_A^{abc} + H_A^{ab,c} \qquad \begin{smallmatrix}\square\\\square\end{smallmatrix}\otimes\square = \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} \oplus \begin{smallmatrix}\square\square\\\square\end{smallmatrix}\,. \tag{E.49}$$

Had we started with the most general rank-three tensor $T^{a|b|c}$ without any symmetry conditions imposed we would have to compute, $V \otimes V \otimes V$,

$$\square\otimes\square\otimes\square \tag{E.50}$$

We first find that it decomposes according to (E.47) in any pair of indices, say in a and b (tensor product is an associative, commutative operation, so we can insert brackets wherever we like as well as to permute the factors),

$$(\square\otimes\square)\otimes\square = \left(\square\square \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix}\right)\otimes\square \tag{E.51}$$

Then, with the help of distributivity and (E.48), (E.49) we get

$$(\square\square\otimes\square) \oplus \left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\otimes\square\right) = \square\square\square \oplus 2\,\begin{smallmatrix}\square\square\\\square\end{smallmatrix} \oplus \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} \tag{E.52}$$

Using (E.12), (E.15), (E.19) we could obtain a more detailed structure [69]. Note that (E.52) does not contain any information about the particular choice of indices that we made and just states that there is a totally symmetric and a totally antisymmetric component and two independent components with the symmetry of $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$.

An exercise analogous to (E.15) but for $so(d)$ reads

$$T^{aa|c} = T_S^{aac} + H_S^{aa,c} - \frac{2}{(d+2)(d-1)}\eta^{aa}T^b + \frac{d}{(d+2)(d-1)}\eta^{ab}T^a\,, \tag{E.53}$$

$$T_S^{aaa} = \frac{1}{3}\left(T^{aa|a} - \frac{2}{d+2}\eta^{aa}T^a\right)\,, \tag{E.54}$$

$$T_S^{abc} = T_S^{bac} = T_S^{acb}\,, \qquad\qquad T_S{}^{ab}{}_b = 0\,, \tag{E.55}$$

$$H_S^{aa,c} = \frac{1}{3}\left(2T^{aa|c} - T^{ac|a} + \frac{1}{d-1}(2\eta^{aa}T^c - \eta^{ac}T^a)\right)\,, \tag{E.56}$$

$$H_S^{ab,c} = H_S^{ba,c}\,, \qquad H_S^{ab,c} + H_S^{bc,a} + H_S^{ca,b} = 0\,, \qquad H_S{}^c{}_{c,}{}^a = H_S{}^{ac,}{}_c = 0\,.$$

---

[69] There is an analogy with the canonical QM textbook exercise on the $su(2)$ representation theory, namely the multiplication of quantum angular momentum. The decomposition in terms of Young diagrams contain the same information as the statement

$$j_1 \otimes j_2 = |j_1 - j_2| \oplus ... \oplus |j_1 + j_2|\,. \tag{1}$$

But representation denoted loosely by j_1 is a $(2j_1 + 1)$-dimensional vector space, analogously for j_2 and any of the spins on the r.h.s of (1). In principle we can write the decomposition in more detail, showing exactly how each of the base vectors belonging to one of $|j_1 - j_2 + 2i|$ on the *r.h.s.* decomposes into a sum of $|m_1, j_1\rangle \otimes |m_2, j_2\rangle$. This is analogous to what we did in (E.12), (E.15), (E.19). This more detailed information is not captured by (1). Luckily, in many cases knowing (1) or the decomposition in terms of Young diagrams is sufficient.

Note the appearance of an additional component, the trace T^a. We can summarize (E.40), (E.53), and the undone exercise for $T^{[ab]|c}$ with two antisymmetric indices as follows

$$(\text{E.40}): \qquad T^{a|b} \qquad \square \otimes \square = \square\square \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus \bullet \,, \qquad (\text{E.57})$$

$$(\text{E.53}): \qquad T^{(ab)|c} \qquad \square\square \otimes \square = \square\square\square \oplus \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \oplus \square \,, \qquad (\text{E.58})$$

$$\text{left as an exercise}: \qquad T^{[ab]|c} \qquad \begin{smallmatrix}\square\\\square\end{smallmatrix} \otimes \square = \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} \oplus \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \oplus \square \,. \qquad (\text{E.59})$$

The manipulations with Young diagrams are simpler than writing down the decomposition into irreducibles in the language of tensors explicitly. For example, for the most general rank-three tensor $T^{a|b|c}$ we find for $V \otimes V \otimes V$

$$\square \otimes \square \otimes \square = \left(\square\square \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus \bullet\right) \otimes \square = \square\square\square \oplus 2\,\begin{smallmatrix}\square\square\\\square\end{smallmatrix} \oplus 3\,\square \oplus \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} \qquad (\text{E.60})$$

As the rank of tensors grows the computation of tensor products become more and more involved. We will need[70]

$$gl(d): \qquad \boxed{k} \otimes \square = \boxed{k}\!\!\begin{smallmatrix}\\\square\end{smallmatrix} \oplus \boxed{k+1} \qquad (\text{E.61})$$

$$so(d): \qquad \boxed{k} \otimes \square = \boxed{k}\!\!\begin{smallmatrix}\\\square\end{smallmatrix} \oplus \boxed{k+1} \oplus \boxed{\,k-1\,} \qquad (\text{E.62})$$

and just for fun

$$gl(d): \qquad \boxed{\begin{smallmatrix}k\\m\end{smallmatrix}} \otimes \square = \boxed{\begin{smallmatrix}k\\m\end{smallmatrix}}\!\!\begin{smallmatrix}\\\\\square\end{smallmatrix} \oplus \boxed{\begin{smallmatrix}k+1\\m\end{smallmatrix}} \oplus \boxed{\begin{smallmatrix}k\\m+1\end{smallmatrix}} \qquad (\text{E.63})$$

$$so(d): \qquad \boxed{\begin{smallmatrix}k\\m\end{smallmatrix}} \otimes \square = \boxed{\begin{smallmatrix}k\\m\end{smallmatrix}}\!\!\begin{smallmatrix}\\\\\square\end{smallmatrix} \oplus \boxed{\begin{smallmatrix}k+1\\m\end{smallmatrix}} \oplus \boxed{\begin{smallmatrix}k\\m+1\end{smallmatrix}} \qquad (\text{E.64})$$

$$\oplus \boxed{\begin{smallmatrix}k-1\\m\end{smallmatrix}} \oplus \boxed{\begin{smallmatrix}k\\m-1\end{smallmatrix}} \qquad (\text{E.65})$$

The general rule for multiplying by $\square$ is quite simple. For the $gl(d)$ case one tries to add one cell to the diagram in all possible ways such that the result is again a Young diagram. The $so(d)$-rule is a combination of the $gl(d)$-rule with an additional cycle where we try to remove (take the trace) one cell in all admissible ways. One has to remember that if the height of some of the Young diagrams on the r.h.s. exceeds $[d/2]$ for $so(d)$ (or d for $gl(d)$) then one has to use the properties mentioned at the end of $gl(d)$ and $so(d)$ sections. In particular some diagrams may just vanish or should be dualized to fit in $[d/2]$ height restrictions in the case of $so(d)$. The general rules, when both factors are generic Young diagrams, are quite complicated and can be found, for example, in [126].

E.3 Generating functions

Since in the higher-spin theory we have to work with infinite collections of tensors we find it convenient, if not necessary, to contract all tensor/spinor indices with some auxiliary variables. We would like to discuss how various Young/trace constraints can be implemented on appropriate functional space.

---

[70]The tensor product rules do not depend on the signature of the metric, η_{ab}.

For example, suppose we need all symmetric tensors, say $C^{a(k)}$, with tensor of each rank appearing once, i.e. the space of tensors is multiplicity free. Then all these tensors can be collected into just one function of auxiliary variables y^a

$$C^{a(k)}, \qquad k = 0, 1, \dots \qquad \Longleftrightarrow \qquad C(y) = \sum_k \frac{1}{k!} C^{a(k)} y_a \dots y_a \,. \qquad (\text{E.66})$$

The Taylor coefficients are our original tensors. If our tensors are all traceless then the appropriate functional space is the space of harmonic functions in y^a

$$C^{a(k-2)m}{}_m = 0 \qquad \Longleftrightarrow \qquad \Box C(y) = 0 \,. \qquad (\text{E.67})$$

Indeed, on computing $\Box$ termwise and equating each Taylor coefficient to zero we get the desired

$$\Box C(y) = \frac{\partial^2}{\partial y_m \partial y^m} \sum_k \frac{1}{k!} C^{a(k)} y_a \dots y_a = \sum_k \frac{1}{(k-2)!} C^{a(k-2)m}{}_m \, y_a \dots y_a = 0 \,. \qquad (\text{E.68})$$

Suppose we need a space of tensors with the symmetry of all two-row Young diagrams, again each symmetry type appearing once, i.e.

$$C^{a(k),b(m)} \qquad C^{a(k),ab(m-1)} = 0 \qquad k = 0, 1, \dots \qquad m = 0, \dots, k \qquad (\text{E.69})$$

Two auxiliary vector-like variables are required now, say y^a and p^a, with the help of which we can build a generating function

$$C(y, p) = \sum_{k,m} \frac{1}{k! m!} C^{a(k),b(m)} y_a \dots y_a \, p_b \dots p_b \,. \qquad (\text{E.70})$$

The Taylor coefficients of a generic function of y and p do not obey any Young symmetry conditions, these are tensors $C^{a(k)|b(m)}$ symmetric in $a(k)$ and $b(m)$ with no conditions that entangle a's and b's. With a little thought the right additional restrictions on the functional space are found to be

$$y^c \frac{\partial}{\partial p^c} C(y, p) = 0 \,. \qquad (\text{E.71})$$

Indeed,

$$y^c \frac{\partial}{\partial p^c} \sum_{k,m} \frac{1}{k! m!} C^{a(k),b(m)} y_a \dots y_a \, p_b \dots p_b = \sum_{k,m} \frac{1}{k!(m-1)!} C^{a(k),cb(m-1)} y_a \dots y_a y_c \, p_b \dots p_b \quad (\text{E.72})$$

all indices contracted with the same commuting variable y^a appear automatically symmetrized, i.e.

$$C^{a(k),cb(m-1)} y_a \dots y_a y_c = \frac{1}{k+1} C^{(a(k),c)b(m-1)} y_a \dots y_a y_c \,. \qquad (\text{E.73})$$

The original sum was over all values of k and m but the Young symmetry condition $C^{a(k),ab(m-1)} = 0$ defines an identically zero tensor for $k < m$ in accordance with the restriction on Young diagrams not to have shorter rows on top of longer ones.

If tensors need to be traceless we can impose trace constraints as harmonicity with respect to

$$\frac{\partial^2}{\partial y_c \partial y^c} \,, \qquad\qquad \frac{\partial^2}{\partial y_c \partial p^c} \,, \qquad\qquad \frac{\partial^2}{\partial p_c \partial p^c} \,. \qquad (\text{E.74})$$

F   Symplectic differential calculus

Mastering symplectic calculus requires some time and a handful of examples to compare with. There are formulas that work for any dimension, i.e. with the only assumptions about the symplectic metric being

$$\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} \,, \qquad\qquad \epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha} \,, \qquad\qquad \epsilon_{\alpha\beta}\epsilon^{\gamma\beta} = \delta_{\alpha}^{\gamma} \,, \qquad (\text{F.75})$$

where δ_{α}^{γ} us the usual identity matrix. There are also formulas that work in $2d$ only, i.e. when $\alpha, \beta, ...$ runs over two values, we shall stress this below.

The main rules are on how to raise and lower indices

$$y^{\alpha} = \epsilon^{\alpha\beta} y_{\beta} \,, \qquad\qquad\qquad y_{\gamma} = y^{\alpha}\epsilon_{\alpha\gamma} \,, \qquad (\text{F.76})$$

with this one checks $y^{\alpha} = \epsilon^{\alpha\beta} y_{\beta} = \epsilon^{\alpha\beta}(y^{\gamma}\,\epsilon_{\gamma\beta}) = y^{\alpha}$, where we used $\epsilon^{\alpha\beta}\epsilon_{\gamma\beta} = \delta_{\gamma}^{\alpha}$.

If we apply raising/lowering rules to $\epsilon_{\alpha\beta}$ itself we find first of all that $\epsilon^{\alpha\beta}$ is identical to $\epsilon_{\alpha\beta}$ with all indices raised. Moreover,

$$\epsilon_{\alpha}{}^{\beta} = \delta_{\alpha}^{\beta} \,, \qquad\qquad\qquad \epsilon^{\alpha}{}_{\beta} = -\epsilon_{\alpha}{}^{\beta} = -\delta_{\alpha}^{\beta} \,. \qquad (\text{F.77})$$

If we remember these rules we can forget about δ_{β}^{α} as an independent object. The minus sign in the last expression manifests the antisymmetry of scalar products, e.g. for two vectors v_{α}, u_{β}, we have

$$v_{\alpha} u_{\beta} \epsilon^{\alpha\beta} = v_{\alpha} u^{\alpha} = -v_{\alpha} u_{\beta} \epsilon^{\beta\alpha} = -v^{\beta} u_{\beta} \,. \qquad (\text{F.78})$$

In particular, for any vector $v_{\alpha}v^{\alpha} \equiv 0$.

Partial derivative $\partial_{\alpha} = \frac{\partial}{\partial y^{\alpha}}$ is defined by the following rules

$$\partial_{\alpha} y_{\beta} = \epsilon_{\alpha\beta} \,, \qquad\qquad \partial_{\alpha} y^{\beta} = \epsilon_{\alpha}{}^{\beta} \equiv \delta_{\beta}^{\alpha} \,, \qquad (\text{F.79})$$

$$\partial^{\alpha} y^{\beta} = \epsilon^{\alpha\beta} \,, \qquad\qquad \partial^{\alpha} y_{\beta} = \epsilon^{\alpha}{}_{\beta} = -\delta_{\beta}^{\alpha} \,, \qquad (\text{F.80})$$

so we can think of anyone as the definition, the rest being consequences of raising/lowering rules. The second one is the most natural. We would like to warn everybody against using blindly the chain rule to relate ∂_{α} to ∂^{α}. The feature of symplectic calculus is that ∂^{α} is defined as ∂_{α} with the index raised according to the rules above and it does not coincide with $\frac{\partial}{\partial y_{\alpha}}$! This is because raising index of ∂_{α} follows a different rule than lowering index of y^{α} inside $\partial/\partial y^{\alpha}$. It is much more convenient to adopt the same raising/lowering rules for all objects than mess up when the rules for variables and derivatives are different. In particular, ∂_{α} is required to behave in the same way as any other vector. Another way to overcome the difficulty is to always use ∂_{α} and never try to raise the index. In practice the chain rule is not needed, the formulae above are sufficient.

The Euler or number operator is useful sometimes

$$N = y^{\gamma}\partial_{\gamma} \,, \qquad\qquad [N, y_{\alpha}] = y_{\alpha} \,, \qquad\qquad [N, \partial_{\alpha}] = -\partial_{\alpha} \,, \qquad (\text{F.81})$$

notice the position of indices since $N = -y_{\gamma}\partial^{\gamma}$.

In particular, in $2d$ we have $y^1 = y_2$, $y^2 = -y_1$ with the canonical choice $\epsilon_{12} = 1$. The main property of $2d$ symplectic world is that any tensor that is antisymmetric in two indices is proportional to $\epsilon_{\alpha\beta}$

$$T_{\alpha\beta} = -T_{\beta\alpha} \qquad \Longrightarrow \qquad T_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta}T_{\gamma\delta}\epsilon^{\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta}T_\gamma{}^\gamma \tag{F.82}$$

Therefore, every two indices can be decomposed as follows

$$F_{\alpha\beta} = \frac{1}{2}(F_{\alpha\beta} + F_{\beta\alpha}) + \frac{1}{2}(F_{\alpha\beta} - F_{\beta\alpha}) = S_{\alpha\beta} + \frac{1}{2}\epsilon_{\alpha\beta}F_\gamma{}^\gamma \ , \tag{F.83}$$

$$S_{\alpha\beta} = \frac{1}{2}(F_{\alpha\beta} + F_{\beta\alpha}) \ . \tag{F.84}$$

The big consequence is that in $2d$ all nontrivial tensors are symmetric. Whatever anti-symmetric part we find can be expressed in terms of a number of ϵ factors and a totally symmetric tensor, which is the symplectic trace of the original one.

The last thing is that in higher-spin theory one finds dotted and undotted symplectic indices running over two values, e.g. y_α, $y_{\dot\alpha}$. These are considered as totally independent objects/indices and they will never mix together (there is no such object as $\epsilon_{\alpha\dot\alpha}$) unless one considers particular solutions to the theory, where the choice of coordinates can break Lorentz symmetry, e.g. (9.37). In fact α and $\dot\alpha$ are different components of the $sp(4)$ index A, $A = \{\alpha, \dot\alpha\}$. Sometimes we use $x^{\alpha\dot\alpha}$ as a spinorial avatar for x^μ, they are related by $\sigma_m^{\alpha\dot\alpha}$. The rules for $\alpha, \dot\alpha, ...$ are the same, in particular

$$\partial_{\alpha\dot\alpha}x^{\beta\dot\beta} = \epsilon_\alpha{}^\beta\,\epsilon_{\dot\alpha}{}^{\dot\beta} \tag{F.85}$$

hence, $\partial_{\alpha\dot\alpha}x^{\alpha\dot\alpha} = 4$ in accordance with $\partial_m x^m = 4$ in $4d$ where it applies.

G   More on $so(3,2)$

Since the anti-de Sitter algebra $so(3,2)$ is at the core of the $4d$ higher-spin theory, in particular the higher-spin algebra can be described in terms of the universal enveloping algebra $U(so(3,2))$, and there is a special isomorphism $so(3,2) \sim sp(4,\mathbb{R})$ that simplifies things a lot and make the $4d$ theory special, we would like to give more info on $so(3,2)$. This section could be too much as at the end we will have representation theory of $so(3,1) \sim sl(2,\mathbb{C})$ and $so(3,2) \sim sp(4,\mathbb{R})$ expressed in two different forms each.

G.1   Restriction of $so(3,2)$ to $so(3,1)$

First of all, general arguments from the unfolded approach, see (6.18) and after, tells us that whenever we find a closed subset of fields of the same form degree, it must form certain representation of the background space symmetry algebra.

For example this should apply to the set of one-forms $\omega^{a(s-1),b(k)}$, $k = 0,...,s-1$. We considered the d-dimensional Minkowski space, but the set does not change when we switch on the cosmological constant, [95]. So, this set of fields must belong to certain

finite-dimensional representations of the anti-de Sitter algebra $so(d-1,2)$, which are known to be tensors or spin-tensors[71].

The simplest example of this kind is the pure gravity, where MacDowell-Mansouri-Stelle-West results, see Section 12.2, show that vielbein e^a and spin-connection $\omega^{a,b}$ can be viewed as different components of a single $so(d,2)$-connection $\Omega^{\mathsf{A},\mathsf{B}}$. In this case, the statement is that a $(d+1) \times (d+1)$ antisymmetric matrix can be viewed as $d \times d$ one and a d-dimensional vector. In the language of group theory we say that

$$\left.\yng(1,1)\right|_{so(d+1)\downarrow so(d)} \sim \yng(1,1) \oplus \yng(1) \tag{G.86}$$

which is called the branching rules or restriction rules. It is also easy to see that

$$\left.\yng(2)\right|_{so(d+1)\downarrow so(d)} \sim \yng(2) \oplus \yng(1) \oplus \bullet \tag{G.87}$$

Indeed, given a symmetric traceless $so(d+1)$ tensor T^{AB} we can decompose[72] it as the traceless symmetric tensor $T^{ab} - \frac{1}{d}\eta^{ab}T^m{}_m$, vector T^{a5} and scalar $T^m{}_m$. Note that $so(d+1)$ tracelessness implies $0 \equiv T^{\mathsf{A}}{}_{\mathsf{A}} = T^a{}_a + \eta_{55}T^{55}$, i.e. $T^a{}_a$ and T^{55} are the same up to a sign. A less trivial example, which is related to the spin-three case, is

$$\left.\yng(2,1)\right|_{so(d+1)\downarrow so(d)} \sim \yng(2,1) \oplus \yng(1,1) \oplus \yng(2) \tag{G.88}$$

On the r.h.s we see exactly the Young symmetries that are needed for the frame-like formulation of a spin-three field.

The full dictionary up to two-row Young diagrams is as follows

$so(d+1)$ tensor	$so(d)$ tensor content
$\bullet$	$\bullet$
$T^{\mathsf{A}} \sim \Box$	$\Box \oplus \bullet$
$T^{\mathsf{AA}} \sim \Box\Box$	$\Box\Box \oplus \Box \oplus \bullet$
$T^{\mathsf{A}(k)} \sim \boxed{\ k\ }$	$\boxed{\ k\ } \oplus \boxed{k-1} \oplus \ldots \oplus \Box \oplus \bullet$
$T^{\mathsf{A},\mathsf{B}} \sim \yng(1,1)$	$\yng(1,1) \oplus \Box$
$T^{\mathsf{A}(k),\mathsf{B}(k)} \sim \boxed{\ \ k\ \ }\!\!\!\boxed{\ \ k\ \ }$	$\boxed{k}\!\!\boxed{k} \oplus \boxed{k}\!\!\boxed{k} \oplus \ldots \oplus \boxed{k}\!\!\boxed{k} \oplus \boxed{\ k\ }$
$T^{\mathsf{A}(m+k),\mathsf{B}(m)} \sim \begin{array}{c}\boxed{m+k}\\\boxed{\ m\ }\end{array}$	$\displaystyle\bigoplus_{i=0}^{i=k}\bigoplus_{j=0}^{j=m}\begin{array}{c}\boxed{m+i}\\\boxed{\ j\ }\end{array}$

The most important line is the last but one, which shows that all higher-spin connections $\omega^{a(s-1),b(k)}$ needed for a spin-s field can be packed into just one connection of $so(d+1)$

$$W^{\mathsf{A}(s-1),\mathsf{B}(s-1)} \tag{G.89}$$

---

[71]Recall that Poincare algebra is not semi-simple and talking about Poincare tensors sounds unnatural.

[72]Again, 5 denotes the extra value of A index that is complementary to the d-dimensional a index, $\mathsf{A} = \{a, 5\}$.

which has the symmetry of the two-row rectangular Young diagram of length-$(s-1)$, [88].

Again let us note that the branching rules expressed in terms of Young diagrams are much simpler than the equivalent statements in the language of tensors.

G.2 $so(3,2) \sim sp(4,\mathbb{R})$

The second miraculous isomorphism is between the anti-de Sitter algebra $so(3,2)$ and $sp(4,\mathbb{R})$. The signature is irrelevant in the section, so one can think of complex Lie algebras, but we do not change the notation. Let us first note that according to the general relation between structure constants of unfolded equations and representation theory the set of one-forms needed for a spin-s field, i.e.

$$\omega^{\alpha(m),\dot{\alpha}(n)} \qquad\qquad m + n = 2(s-1)\,, \tag{G.90}$$

must belong to some finite-dimensional representation of $so(3,2)$, see previous section. The same time, as we noticed in Section 8, the same field content can be packed as

$$\omega^{\Omega(2s-2)}\, Y_\Omega...Y_\Omega\,, \tag{G.91}$$

where Ω runs[73] over four values, which cover $\{\alpha, \dot\alpha\}$. The reason is that $so(3,2) \sim sp(4,\mathbb{R})$ and Y_Ω is a vector of $sp(4,\mathbb{R})$, so both Lorentz algebra $so(3,1)$ and anti-de Sitter algebra $so(3,2)$ are special.

To prove the isomorphism and build the dictionary one introduces $so(3,2)$ Dirac γ-matrices $\gamma_{\mathsf{A}} \equiv \gamma_{\mathsf{A}}{}^{\Lambda}{}_{\Omega}$, $\Lambda, \Omega, ... = 1, ..., 4$, $\mathsf{A}, \mathsf{B}, ... = 0, ..., 4$.

$$(\gamma_{\mathsf{A}}\gamma_{\mathsf{B}})^{\Lambda}{}_{\Omega} + (\gamma_{\mathsf{B}}\gamma_{\mathsf{A}})^{\Lambda}{}_{\Omega} = 2\delta^{\Lambda}{}_{\Omega}\,\eta_{\mathsf{AB}} \tag{G.92}$$

The generators of $so(3,2)$ in the spinorial representation

$$T_{\mathsf{AB}} = -T_{\mathsf{BA}} = \frac{1}{4}[\gamma_{\mathsf{A}}, \gamma_{\mathsf{B}}] \tag{G.93}$$

can be observed to have a special structure,

$$T_{\mathsf{AB}}{}^{\Lambda\Omega} = T_{\mathsf{AB}}{}^{\Omega\Lambda}\,, \tag{G.94}$$

where we raise and lower $\Lambda, \Omega, ...$-indices with the charge-conjugation matrix $C_{\Lambda\Omega} = -C_{\Omega\Lambda}$ using the standard symplectic rules. The charge-conjugation matrix is going to be the invariant tensor of $sp(4,\mathbb{R})$.

In addition γ-matrices can be shown to be all antisymmetric and $C_{\Lambda\Omega}$-traceless

$$\gamma_{\mathsf{A}}{}^{\Lambda\Omega} = -\gamma_{\mathsf{A}}{}^{\Omega\Lambda}\,, \qquad\qquad \gamma_{\mathsf{A}}{}^{\Lambda\Omega}C_{\Lambda\Omega} = 0\,. \tag{G.95}$$

The latter property implies that one can use γ-matrices to map an $so(3,2)$-vector, say V_{A}, into antisymmetric rank-two tensor $V^{\Lambda\Omega} = -V^{\Omega\Lambda}$, $V^{\Lambda\Omega} = \gamma_{\mathsf{A}}{}^{\Lambda\Omega}V^{\mathsf{A}}$. This is an isomorphism, which can be proven by observing that $V^{\Lambda\Omega}$ has the same number of components,

---

[73]In the main text we use $A, B, ...$ as $sp(4,\mathbb{R})$ indices and $\mathsf{A}, \mathsf{B}, ...$ as $so(3,2)$ indices, but this could cause a confusion when $so(3,2)$ and $sp(4,\mathbb{R})$ are confronted. Therefore, $A, B, ...$ are changed to $\Lambda, \Omega, ...$.

$4 \cdot 3/2 - 1 = 5$, as V^{A} and one can find a backward transform using the properties of γ-matrices.

Analogously, a rank-two antisymmetric tensor of $so(3,2)$, say $C^{\mathsf{A},\mathsf{B}} = -C^{\mathsf{B},\mathsf{A}}$ can be mapped into rank-two symmetric tensor $C^{\Lambda\Omega} = C^{\Omega\Lambda}$, $C^{\Lambda\Omega} = T^{\Lambda\Omega}{}_{\mathsf{AB}} C^{\mathsf{A},\mathsf{B}}$. This is an isomorphism again.

Continuing along the same lines, one can derive the following $so(3,2) - sp(4,\mathbb{R})$ dictionary

$so(3,2)$ tensor	$sp(4,\mathbb{R})$ tensor	dim
$\bullet$	$\bullet$	1
Dirac spinor	$T^{\Lambda} \sim \square$	4
$T^{\mathsf{A}} \sim \square$	$T^{\Lambda,\Omega} \sim \begin{smallmatrix}\square\\\square\end{smallmatrix}$	5
$T^{\mathsf{A},\mathsf{B}} \sim \begin{smallmatrix}\square\\\square\end{smallmatrix}$	$T^{\Lambda\Lambda} \sim \square\square$	10
$T^{\mathsf{AA}} \sim \square\square$	$T^{\Lambda\Lambda,\Omega\Omega} \sim \boxplus$	15
$T^{\mathsf{A}(k)} \sim \boxed{\ \ k\ \ }$	$T^{\Lambda(k),\Omega(k)} \sim \boxed{\ \ k\ \ }$	
$T^{\mathsf{A}(k),\mathsf{B}(k)} \sim \boxed{\ \ k\ \ }$	$T^{\Lambda(2k)} \sim \boxed{\ \ 2k\ \ }$	
$T^{\mathsf{A}(k),\mathsf{B}(m)} \sim \boxed{\begin{smallmatrix}k\\m\end{smallmatrix}}$	$T^{\Lambda(k+m),\Omega(k-m)} \sim \boxed{\begin{smallmatrix}k+m\\k-m\end{smallmatrix}}$	

Irreducible $so(3,2)$ tensors are traceless with respect to symmetric η_{AB} and irreducible $sp(4,\mathbb{R})$ tensors are traceless with respect to antisymmetric $C_{\Lambda\Omega}$.

To summarize, we have the following equivalent ways to describe the space of higher-spin connections of a spin-s field in $4d$

$$so(3,1): \qquad \bigoplus_{k=0}^{s-1} \omega^{a(s-1),b(k)} \tag{G.96}$$

$$sl(2,\mathbb{C}): \qquad \bigoplus_{i+j=2(s-1)} \omega^{\alpha(i),\dot\alpha(j)} \tag{G.97}$$

$$so(3,2): \qquad \omega^{A(s-1),B(s-1)} \tag{G.98}$$

$$sp(4,\mathbb{R}): \qquad \omega^{\Lambda(2s-2)} \tag{G.99}$$

(G.96) and (G.98) are valid in any dimension $d \geq 4$.

References

[1] C. Fronsdal, *Massless fields with integer spin, Phys. Rev.* **D18** (1978) 3624.

[2] E. S. Fradkin and M. A. Vasiliev, *Candidate to the role of higher spin symmetry, Ann. Phys.* **177** (1987) 63.

[3] E. S. Fradkin and M. A. Vasiliev, *Cubic Interaction in Extended Theories of Massless Higher Spin Fields*, Nucl. Phys. **B291** (1987) 141.

[4] E. S. Fradkin and M. A. Vasiliev, *On the Gravitational Interaction of Massless Higher Spin Fields*, Phys. Lett. **B189** (1987) 89–95.

[5] X. Bekaert, N. Boulanger and P. Sundell, *How higher-spin gravity surpasses the spin two barrier: no-go theorems versus yes-go examples*, Rev.Mod.Phys. **84** (2012) 987–1009 [`1007.0435`].

[6] F. A. Berends, G. J. H. Burgers and H. van Dam, *On the theoretical problems in constructing interactions involving higher spin massless particles*, Nucl. Phys. **B260** (1985) 295.

[7] F. A. Berends, G. J. H. Burgers and H. Van Dam, *On spin three selfinteractions*, Z. Phys. **C24** (1984) 247–254.

[8] R. R. Metsaev, *Cubic interaction vertices of totally symmetric and mixed symmetry massless representations of the Poincare group in D = 6 space-time*, Phys. Lett. **B309** (1993) 39–44.

[9] R. R. Metsaev, *Generating function for cubic interaction vertices of higher spin fields in any dimension*, Mod. Phys. Lett. **A8** (1993) 2413–2426.

[10] R. R. Metsaev, *Cubic interaction vertices for massive and massless higher spin fields*, Nucl. Phys. **B759** (2006) 147–201 [`hep-th/0512342`].

[11] R. R. Metsaev, *Cubic interaction vertices for fermionic and bosonic arbitrary spin fields*, `arXiv:0712.3526 [hep-th]`.

[12] R. Metsaev, *BRST-BV approach to cubic interaction vertices for massive and massless higher-spin fields*, Phys.Lett. **B720** (2013) 237–243 [`1205.3131`].

[13] R. Manvelyan, K. Mkrtchyan and W. Ruehl, *A Generating function for the cubic interactions of higher spin fields*, Phys.Lett. **B696** (2011) 410–415 [`1009.1054`].

[14] R. Manvelyan, K. Mkrtchyan and W. Ruhl, *General trilinear interaction for arbitrary even higher spin gauge fields*, Nucl.Phys. **B836** (2010) 204–221 [`1003.2877`].

[15] N. Boulanger and S. Leclercq, *Consistent couplings between spin-2 and spin-3 massless fields*, JHEP **11** (2006) 034 [`hep-th/0609221`].

[16] N. Boulanger, S. Leclercq and P. Sundell, *On The Uniqueness of Minimal Coupling in Higher-Spin Gauge Theory*, JHEP **08** (2008) 056 [`0805.2764`].

[17] X. Bekaert, N. Boulanger and S. Leclercq, *Strong obstruction of the Berends-Burgers-van Dam spin-3 vertex*, J.Phys. **A43** (2010) 185401 [`1002.0289`].

[18] M. Taronna, *Higher Spins and String Interactions*, `1005.3061`.

[19] E. Joung and M. Taronna, *Cubic interactions of massless higher spins in (A)dS: metric-like approach*, *Nucl.Phys.* **B861** (2012) 145–174 [`1110.5918`].

[20] E. Joung, L. Lopez and M. Taronna, *Generating functions of (partially-)massless higher-spin cubic interactions*, *JHEP* **1301** (2013) 168 [`1211.5912`].

[21] M. Henneaux, G. Lucena Gomez and R. Rahman, *Higher-Spin Fermionic Gauge Fields and Their Electromagnetic Coupling*, *JHEP* **1208** (2012) 093 [`1206.1048`].

[22] Y. M. Zinoviev, *On spin 3 interacting with gravity*, *Class. Quant. Grav.* **26** (2009) 035022 [`0805.2226`].

[23] I. Buchbinder, T. Snegirev and Y. Zinoviev, *Cubic interaction vertex of higher-spin fields with external electromagnetic field*, *Nucl.Phys.* **B864** (2012) 694–721 [`1204.2341`].

[24] M. Vasiliev, *Cubic Vertices for Symmetric Higher-Spin Gauge Fields in $(A)dS_d$*, *Nucl.Phys.* **B862** (2012) 341–408 [`1108.5921`].

[25] K. B. Alkalaev and M. A. Vasiliev, *N = 1 supersymmetric theory of higher spin gauge fields in ads(5) at the cubic level*, *Nucl. Phys.* **B655** (2003) 57–92 [`hep-th/0206068`].

[26] Y. Zinoviev, *Spin 3 cubic vertices in a frame-like formalism*, *JHEP* **1008** (2010) 084 [`1007.0158`].

[27] K. Alkalaev, *FV-type action for AdS_5 mixed-symmetry fields*, *JHEP* **1103** (2011) 031 [`1011.6109`].

[28] N. Boulanger, E. Skvortsov and Y. Zinoviev, *Gravitational cubic interactions for a simple mixed-symmetry gauge field in AdS and flat backgrounds*, *J.Phys.* **A44** (2011) 415403 [`1107.1872`].

[29] N. Boulanger, D. Ponomarev and E. Skvortsov, *Non-abelian cubic vertices for higher-spin fields in anti-de Sitter space*, *JHEP* **1305** (2013) 008 [`1211.6979`].

[30] N. Boulanger and E. Skvortsov, *Higher-spin algebras and cubic interactions for simple mixed-symmetry fields in AdS spacetime*, *JHEP* **1109** (2011) 063 [`1107.5028`].

[31] R. R. Metsaev, *Poincare invariant dynamics of massless higher spins: Fourth order analysis on mass shell*, *Mod. Phys. Lett.* **A6** (1991) 359–367.

[32] N. Boulanger, D. Ponomarev, E. Skvortsov and M. Taronna, *On the uniqueness of higher-spin symmetries in AdS and CFT*, `1305.5180`.

[33] M. Taronna, *Higher-Spin Interactions: four-point functions and beyond*, *JHEP* **1204** (2012) 029 [`1107.5843`].

[34] M. A. Vasiliev, *Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions*, Phys. Lett. **B243** (1990) 378–382.

[35] M. A. Vasiliev, *Properties of equations of motion of interacting gauge fields of all spins in (3+1)-dimensions*, Class. Quant. Grav. **8** (1991) 1387–1417.

[36] M. A. Vasiliev, *More on equations of motion for interacting massless fields of all spins in (3+1)-dimensions*, Phys. Lett. **B285** (1992) 225–234.

[37] M. A. Vasiliev, *Higher-spin gauge theories in four, three and two dimensions*, Int. J. Mod. Phys. **D5** (1996) 763–797 [`hep-th/9611024`].

[38] M. A. Vasiliev, *Higher spin gauge theories: Star-product and ads space*, `hep-th/9910096`.

[39] M. A. Vasiliev, *Nonlinear equations for symmetric massless higher spin fields in (a)ds(d)*, Phys. Lett. **B567** (2003) 139–151 [`hep-th/0304049`].

[40] M. A. Vasiliev, *Equations of motion of interacting massless fields of all spins as a free differential algebra*, Phys. Lett. **B209** (1988) 491–497.

[41] M. A. Vasiliev, *Consistent equations for interacting massless fields of all spins in the first order in curvatures*, Annals Phys. **190** (1989) 59–106.

[42] M. A. Vasiliev, *Closed equations for interacting gauge fields of all spins*, JETP Lett. **51** (1990) 503–507.

[43] M. A. Vasiliev, *Algebraic aspects of the higher spin problem*, Phys. Lett. **B257** (1991) 111–118.

[44] D. Francia and A. Sagnotti, *Higher-spin geometry and string theory*, J.Phys.Conf.Ser. **33** (2006) 57 [`hep-th/0601199`].

[45] D. Francia, *String theory triplets and higher-spin curvatures*, Phys.Lett. **B690** (2010) 90–95 [`1001.5003`].

[46] A. Sagnotti and M. Taronna, *String Lessons for Higher-Spin Interactions*, Nucl.Phys. **B842** (2011) 299–361 [`1006.5242`].

[47] I. Buchbinder, V. Krykhtin and A. Reshetnyak, *BRST approach to Lagrangian construction for fermionic higher spin fields in (A)dS space*, Nucl.Phys. **B787** (2007) 211–240 [`hep-th/0703049`].

[48] G. Barnich, M. Grigoriev, A. Semikhatov and I. Tipunin, *Parent field theory and unfolding in BRST first-quantized terms*, Commun.Math.Phys. **260** (2005) 147–181 [`hep-th/0406192`].

[49] K. Alkalaev, M. Grigoriev and I. Y. Tipunin, *Massless Poincare modules and gauge invariant equations*, Nucl.Phys. **B823** (2009) 509–545 [`0811.3999`].

[50] G. Barnich and M. Grigoriev, *First order parent formulation for generic gauge field theories*, *JHEP* **1101** (2011) 122 [1009.0190].

[51] E. Sezgin and P. Sundell, *Massless higher spins and holography*, *Nucl.Phys.* **B644** (2002) 303–370 [hep-th/0205131].

[52] I. R. Klebanov and A. M. Polyakov, *AdS dual of the critical O(N) vector model*, *Phys. Lett.* **B550** (2002) 213–219 [hep-th/0210114].

[53] S. Giombi and X. Yin, *Higher Spin Gauge Theory and Holography: The Three-Point Functions*, *JHEP* **1009** (2010) 115 [0912.3462].

[54] S. Giombi and X. Yin, *Higher Spins in AdS and Twistorial Holography*, *JHEP* **1104** (2011) 086 [1004.3736].

[55] V. Didenko and M. Vasiliev, *Static BPS black hole in 4d higher-spin gauge theory*, *Phys.Lett.* **B682** (2009) 305–315 [0906.3898].

[56] C. Iazeolla and P. Sundell, *Families of exact solutions to Vasiliev's 4D equations with spherical, cylindrical and biaxial symmetry*, *JHEP* **1112** (2011) 084 [1107.1217].

[57] V. Didenko and E. Skvortsov, *Exact higher-spin symmetry in CFT: all correlators in unbroken Vasiliev theory*, *JHEP* **1304** (2013) 158 [1210.7963].

[58] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields*, *JHEP* **1011** (2010) 007 [1008.4744].

[59] M. Henneaux and S.-J. Rey, *Nonlinear $W_{infinity}$ as Asymptotic Symmetry of Three-Dimensional Higher Spin Anti-de Sitter Gravity*, *JHEP* **1012** (2010) 007 [1008.4579].

[60] M. Ammon, M. Gutperle, P. Kraus and E. Perlmutter, *Spacetime Geometry in Higher Spin Gravity*, *JHEP* **1110** (2011) 053 [1106.4788].

[61] E. Hijano, P. Kraus and E. Perlmutter, *Matching four-point functions in higher spin AdS_3/CFT_2*, *JHEP* **1305** (2013) 163 [1302.6113].

[62] A. Campoleoni, S. Fredenhagen and S. Pfenninger, *Asymptotic W-symmetries in three-dimensional higher-spin gauge theories*, *JHEP* **1109** (2011) 113 [1107.0290].

[63] G. Lucena Gomez, *Higher-Spin Theory - Part II: enter dimension three*, *PoS* **ModaveVIII** (2012) 003 [1307.3200].

[64] D. Sorokin, *Introduction to the classical theory of higher spins*, *AIP Conf. Proc.* **767** (2005) 172–202 [hep-th/0405069].

[65] X. Bekaert, S. Cnockaert, C. Iazeolla and M. Vasiliev, *Nonlinear higher spin theories in various dimensions*, hep-th/0503128.

[66] S. Giombi and X. Yin, *The Higher Spin/Vector Model Duality*, *J.Phys.* **A46** (2013) 214003 [`1208.4036`].

[67] E. Sezgin and P. Sundell, *Supersymmetric Higher Spin Theories*, *J.Phys.* **A46** (2013) 214022 [`1208.6019`].

[68] J. Fang and C. Fronsdal, *Massless Fields with Half Integral Spin*, *Phys. Rev.* **D18** (1978) 3630.

[69] C. Fronsdal, *Singletons and Massless, Integral Spin Fields on de Sitter Space (Elementary Particles in a Curved Space. 7.*, *Phys.Rev.* **D20** (1979) 848–856.

[70] J. Fang and C. Fronsdal, *Massless, half integer spin fields in de sitter space*, *Phys.Rev.* **D22** (1980) 1361.

[71] D. Francia, *Low-spin models for higher-spin Lagrangians*, *Prog.Theor.Phys.Suppl.* **188** (2011) 94–105 [`1103.0683`].

[72] A. Campoleoni and D. Francia, *Maxwell-like Lagrangians for higher spins*, *JHEP* **1303** (2013) 168 [`1206.5877`].

[73] D. Blas, *Gauge symmetry and consistent spin-two theories*, *J. Phys.* **A40** (2007) 6965–6972 [`hep-th/0701049`].

[74] E. D. Skvortsov and M. A. Vasiliev, *Transverse invariant higher spin fields*, *Phys. Lett.* **B664** (2008) 301–306 [`hep-th/0701278`].

[75] X. Bekaert and N. Boulanger, *The unitary representations of the poincare group in any spacetime dimension*, `hep-th/0611263`.

[76] T. Curtright, *Massless field supermultiplets with arbitrary spin*, *Phys. Lett.* **B85** (1979) 219.

[77] E. P. Wigner, *On unitary representations of the inhomogeneous lorentz group*, *Annals Math.* **40** (1939) 149–204.

[78] L. P. S. Singh and C. R. Hagen, *Lagrangian formulation for arbitrary spin. 2. the fermion case*, *Phys. Rev.* **D9** (1974) 910–920.

[79] L. P. S. Singh and C. R. Hagen, *Lagrangian formulation for arbitrary spin. 1. the boson case*, *Phys. Rev.* **D9** (1974) 898–909.

[80] R. Metsaev, *Massless mixed symmetry bosonic free fields in d-dimensional anti-de Sitter space-time*, *Phys.Lett.* **B354** (1995) 78–84.

[81] R. Metsaev, *Arbitrary spin massless bosonic fields in d-dimensional anti-de Sitter space*, *Lect.Notes Phys.* **524** (1997) 331–340 [`hep-th/9810231`].

[82] L. Brink, R. R. Metsaev and M. A. Vasiliev, *How massless are massless fields in ads(d)*, *Nucl. Phys.* **B586** (2000) 183–205 [`hep-th/0005136`].

[83] E. Skvortsov, *Gauge fields in (A)dS(d) and Connections of its symmetry algebra*, *J.Phys.* **A42** (2009) 385401 [`0904.2919`].

[84] D. Kaparulin, S. Lyakhovich and A. Sharapov, *Consistent interactions and involution*, *JHEP* **1301** (2013) 097 [`1210.6821`].

[85] C. Aragone and S. Deser, *Consistency Problems of Hypergravity*, *Phys. Lett.* **B86** (1979) 161.

[86] S. W. MacDowell and F. Mansouri, *Unified geometric theory of gravity and supergravity*, *Phys. Rev. Lett.* **38** (1977) 739.

[87] K. S. Stelle and P. C. West, *Spontaneously broken de sitter symmetry and the gravitational holonomy group*, *Phys. Rev.* **D21** (1980) 1466.

[88] M. A. Vasiliev, *Cubic interactions of bosonic higher spin gauge fields in ads(5)*, *Nucl. Phys.* **B616** (2001) 106–162 [`hep-th/0106200`].

[89] T. Ortin, *Gravity and strings.* Cambridge, 2004.

[90] F. Hehl, P. Von Der Heyde, G. Kerlick and J. Nester, *General Relativity with Spin and Torsion: Foundations and Prospects*, *Rev.Mod.Phys.* **48** (1976) 393–416.

[91] Y. Ne'eman and D. Sijacki, *Unified Affine Gauge Theory of Gravity and Strong Interactions With Finite and Infinite Gl(4,r) Spinor Fields*, *Annals Phys.* **120** (1979) 292.

[92] M. Vasiliev, *On Conformal, SL(4,R) and Sp(8,R) Symmetries of 4d Massless Fields*, *Nucl.Phys.* **B793** (2008) 469–526 [`0707.1085`].

[93] D. Lovelock, *The Einstein tensor and its generalizations*, *J.Math.Phys.* **12** (1971) 498–501.

[94] M. A. Vasiliev, *'gauge' form of description of massless fields with arbitrary spin*, *Sov. J. Nucl. Phys.* **32** (1980) 439.

[95] V. E. Lopatin and M. A. Vasiliev, *Free massless bosonic fields of arbitrary spin in d-dimensional de sitter space*, *Mod. Phys. Lett.* **A3** (1988) 257.

[96] M. A. Vasiliev, *Actions, charges and off-shell fields in the unfolded dynamics approach*, *Int. J. Geom. Meth. Mod. Phys.* **3** (2006) 37–80 [`hep-th/0504090`].

[97] R. D'Auria and P. Fre, *Geometric supergravity in d = 11 and its hidden supergroup*, *Nucl. Phys.* **B201** (1982) 101–140.

[98] R. D'Auria, P. Fre, P. K. Townsend and P. van Nieuwenhuizen, *Invariance of actions, rheonomy and the new minimal n=1 supergravity in the group manifold approach*, *Ann. Phys.* **155** (1984) 423.

[99] P. van Nieuwenhuizen, *Free graded differential superalgebras, Invited talk given at 11th Int. Colloq. on Group Theoretical Methods in Physics, Istanbul, Turkey, Aug 23- 28, 1982.*

[100] D. Sullivan, *Infinitesimal computations in topology, Publ. Math. IHES* **47** (1977) 269–331.

[101] M. A. Vasiliev, *Holography, Unfolding and Higher-Spin Theory, J.Phys.* **A46** (2013) 214013 [`1203.5554`].

[102] M. A. Vasiliev, *Conformal higher spin symmetries of 4d massless supermultiplets and osp(l,2m) invariant equations in generalized (super)space, Phys. Rev.* **D66** (2002) 066006 [`hep-th/0106149`].

[103] O. V. Shaynkman and M. A. Vasiliev, *Higher spin conformal symmetry for matter fields in 2+1 dimensions, Theor. Math. Phys.* **128** (2001) 1155–1168 [`hep-th/0103208`].

[104] O. V. Shaynkman and M. A. Vasiliev, *Scalar field in any dimension from the higher spin gauge theory perspective, Theor. Math. Phys.* **123** (2000) 683–700 [`hep-th/0003123`].

[105] M. A. Vasiliev, *Extended higher spin superalgebras and their realizations in terms of quantum operators, Fortsch. Phys.* **36** (1988) 33–62.

[106] S. E. Konshtein and M. A. Vasiliev, *Massless representations and admissibility condition for higher spin superalgebras, Nucl. Phys.* **B312** (1989) 402.

[107] S. E. Konstein and M. A. Vasiliev, *Extended higher spin superalgebras and their massless representations, Nucl. Phys.* **B331** (1990) 475–499.

[108] K. Bolotin and M. A. Vasiliev, *Star product and massless free field dynamics in AdS(4), Phys.Lett.* **B479** (2000) 421–428 [`hep-th/0001031`].

[109] M. A. Vasiliev, *Dynamics of massless higher spins in the second order in curvatures, Phys. Lett.* **B238** (1990) 305–314.

[110] S. F. Prokushkin and M. A. Vasiliev, *Higher-spin gauge interactions for massive matter fields in 3d ads space-time, Nucl. Phys.* **B545** (1999) 385 [`hep-th/9806236`].

[111] E. P. Wigner, *Do the equations of motion determine the quantum mechanical commutation relations?, Phys. Rev.* **77** (Mar, 1950) 711–712.

[112] L. Brink, T. Hansson, S. Konstein and M. A. Vasiliev, *The Calogero model: Anyonic representation, fermionic extension and supersymmetry, Nucl.Phys.* **B401** (1993) 591–612 [`hep-th/9302023`].

[113] E. Sezgin and P. Sundell, *Analysis of higher spin field equations in four-dimensions, JHEP* **0207** (2002) 055 [`hep-th/0205132`].

[114] E. Sezgin and P. Sundell, *Geometry and Observables in Vasiliev's Higher Spin Gravity*, JHEP **1207** (2012) 121 [`1103.2360`].

[115] T. Damour, M. Henneaux and H. Nicolai, *Cosmological billiards*, *Class.Quant.Grav.* **20** (2003) R145–R200 [`hep-th/0212256`].

[116] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *Towards metric-like higher-spin gauge theories in three dimensions*, *J.Phys.* **A46** (2013) 214017 [`1208.1851`].

[117] M. A. Vasiliev, *Higher spin superalgebras in any dimension and their representations*, *JHEP* **12** (2004) 046 [`hep-th/0404124`].

[118] C. Iazeolla and P. Sundell, *A Fiber Approach to Harmonic Analysis of Unfolded Higher- Spin Field Equations*, *JHEP* **10** (2008) 022 [`0806.1942`].

[119] S. Fernando and M. Gunaydin, *Minimal unitary representation of SU(2,2) and its deformations as massless conformal fields and their supersymmetric extensions*, *J.Math.Phys.* **51** (2010) 082301 [`0908.3624`].

[120] K. Govil and M. Gunaydin, *Deformed Twistors and Higher Spin Conformal (Super-)Algebras in Four Dimensions*, `1312.2907`.

[121] V. Didenko and M. Vasiliev, *Free field dynamics in the generalized AdS (super)space*, *J.Math.Phys.* **45** (2004) 197–215 [`hep-th/0301054`].

[122] O. Gelfond and M. Vasiliev, *Sp(8) invariant higher spin theory, twistors and geometric BRST formulation of unfolded field equations*, *JHEP* **0912** (2009) 021 [`0901.2176`].

[123] O. Gelfond and M. Vasiliev, *Unfolding Versus BRST and Currents in Sp(2M) Invariant Higher-Spin Theory*, `1001.2585`.

[124] A. O. Barut and R. Raczka, *Theory of group representations and applications*, . Singapore, Singapore: World Scientific (1986) 717p.

[125] W. Fulton, *Young Tableaux: With Applications to Representation Theory and Geometry.* London Mathematical Society.

[126] J. Hong and S.-J. Kang, *Introduction to Quantum Groups and Crystal Bases.* Amer. Math. Soc., 2002.