2D conformal field theories and holography

Laurent Freidel[a]

*Laboratoire de Physique, Ecole Normale Supérieure de Lyon, 46, Allée d'Italie,
69364 Lyon Cedex 07, France and Perimeter Institute for Theoretical Physics,
35 King Street North, Waterloo N2J 2G9, Ontario, Canada*

Kirill Krasnov[b]

Albert Einstein Institute, Golm/Potsdam, 14476, Germany

It is known that the chiral part of any 2D conformal field theory defines a 3D topological quantum field theory: quantum states of this TQFT are the CFT conformal blocks. The main aim of this paper is to show that a similar CFT/TQFT relation exists also for the full CFT. The 3D topological theory that arises is a certain "square" of the chiral TQFT. Such topological theories were studied by Turaev and Viro; they are related to 3D gravity. We establish an operator/state correspondence in which operators in the chiral TQFT correspond to states in the Turaev–Viro theory. We use this correspondence to interpret CFT correlation functions as particular quantum states of the Turaev–Viro theory. We compute the components of these states in the basis in the Turaev–Viro Hilbert space given by colored 3-valent graphs. The formula we obtain is a generalization of the Verlinde formula. The later is obtained from our expression for a zero colored graph. Our results give an interesting "holographic" perspective on conformal field theories in two dimensions. © *2004 American Institute of Physics.*
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I. INTRODUCTION

To put results of this paper in a somewhat general context we recall that any conformal field theory (CFT) defines a topological quantum field theory (TQFT), see Refs. 1–4. The TQFT arises by extracting a modular tensor category from the CFT chiral vertex operator algebra. Then, as explained in Ref. 5, any modular category gives rise to a 3D TQFT. The TQFT can be (partially) described by saying that its Hilbert space is the space of (holomorphic) conformal blocks of the CFT. The canonical example of such CFT/TQFT correspondence is the well-known relation between Wess–Zumino–Witten (WZW) and Chern–Simons (CS) theories. Let us emphasize that this is always a relation between the holomorphic sector of the CFT (or its chiral part) and a TQFT. As such it is not an example of a holographic correspondence, in which correlation functions (comprising both the holomorphic and antiholomorphic sectors) of CFT on the boundary would be reproduced by some theory in bulk.

It is then natural to ask whether there is some 3D theory that corresponds to the *full* CFT. A proposal along these lines was put forward some time ago by Verlinde,[6] who argued that a relation must exist between the quantum Liouville theory (full, not just the chiral part) and 3D gravity. Recently one of us presented[7] some additional arguments in favor of this relation, hopefully somewhat clarifying the picture. The main goal of the present paper is to demonstrate that such a relation between the full CFT and a certain 3D theory exists for a large class of CFT's. Namely, we show that given a CFT there is a certain 3D field theory, which is a TQFT, and which is a rather natural spin-off of the corresponding "chiral" TQFT. The TQFT in question is not new, it is

[a] Electronic mail: freidel@ens-lyon.fr

[b] Address from September 1, 2004: School of Mathematical Sciences, University of Nottingham, University Park, Nottingham, NG7 2RD, United Kingdom. Electronic mail: krasnov@aei.mpg.de

the one defined by Turaev–Viro,[8] and described in great detail in Ref. 5. This paper is thus aimed at a clarification of the relation between the Turaev–Viro (TV) 3D TQFT's and CFT's in two dimensions.

The point that given a CFT there exists a relation between the full CFT and some 3D TQFT is to some extent contained in recent works on boundary conformal field theory, see Refs. 2, 3, 4, 9, and references therein, and also a more recent paper.[10] As is emphasized, e.g., in Ref. 4, the full CFT partition function on some Riemann surface X (possibly with a boundary) is equal to the chiral CFT partition function on the double $\tilde{X}$. There is then a certain "connecting" 3D manifold $\tilde{M}$ whose boundary $\partial\tilde{M}$ is the double $\tilde{X}$. Using the chiral CFT/TQFT relation one obtains a 3D TQFT in $\tilde{M}$ that reproduces the chiral partition function on $\tilde{X}$, and thus the full partition function on X. This formalism turns out to be very useful for analyzing the case when X has a boundary.

Our analysis was motivated by the above picture, but the logic is somewhat different. Instead of working with the chiral TQFT in the connecting 3-manifold $\tilde{M}$ we work directly with a 3-manifold M whose boundary is X, and the Turaev–Viro TQFT on M. The two approaches are clearly related as the TV theory is a "square" of the chiral TQFT. However, bringing the Turaev–Viro TQFT into the game suggests some new interpretations and provides new relations. Thus, most notably, we establish an operator/state correspondence in which the chiral TQFT operators correspond to states in the TV theory, and the trace of an operator product corresponds to the TV inner product. We use this to interpret the CFT correlators as quantum states of TV theory. Then, using the fact that a basis in the Hilbert space of TV theory on X is given by colored trivalent graph states, we will characterize the CFT correlation functions by finding their components in this basis. Thus, the relation that we demonstrate is about a 3D TQFT on a 3-manifold M and a CFT on the boundary X of M. It is therefore an example of a holographic correspondence, while this is not obviously so for the correspondence based on a chiral TQFT in the connecting manifold $\tilde{M}$.

The holography discussed may be viewed by some as trivial, because the three-dimensional theory is topological. What makes it interesting is that it provides a very large class of examples. Indeed, there is a relation of this type for any CFT. Importantly, this holography is not limited to any AdS type background, although a very interesting subclass of examples (not considered in this paper, but see Ref. 7) is exactly of this type.

As the relation chiral CFT/TQFT is best understood for the case of a rational CFT, we shall restrict our analysis to this case. Our constructions can also be expected to generalize to nonrational and even noncompact CFT's with a continuous spectrum, but such a generalization is non-trivial, and is not attempted in this paper. Even with noncompact CFT's excluded, the class of CFT's that is covered by our considerations, namely, rational CFT, is still very large. To describe the arising structure in its full generality we would need to introduce the apparatus of category theory, as it was done, e.g., in Ref. 5. In order to make the exposition as accessible as possible we shall not maintain the full generality. We demonstrate the CFT/TQFT holographic relation using a compact group WZW CFT (and CS theory as the corresponding chiral TQFT) as an example.

We shall often refer to the TV TQFT as "gravity." For the case of chiral TQFT being the Chern–Simons theory for a group $G = \mathrm{SU}(2)$ this "gravity" theory is just the usual 3D Euclidean gravity with positive cosmological constant. However, the theory can be associated to any CFT. The reader should keep in mind its rather general character.

In order to describe the holographic correspondence in detail we will need to review (and clarify) the relation between CS theory and gravity (or between the Reshetikhin–Turaev–Witten and Turaev–Viro invariants) for a 3-manifold with boundary. We found that the expositions of this relation available in the literature (see Refs. 5, 11) are rather brief and sketchy. This paper provides a more detailed account and obtains new results. In particular, the operator/state correspondence established in this paper is new.

Finally, we would like to emphasize that the approach presented in this paper is not equivalent to that of Refs. 2, 3, 4, 9, even though it was motivated by these papers. Thus, most of our discussion only concerns the diagonal-type partition functions, while Ref. 4 is applicable to the

more general case. It would be of interest to study the relation to Refs. 2, 3, 4, 9 in more detail, and also extend the approach presented here to modular invariant partition functions of other types. We shall not attempt this in the present paper.

The paper is organized as follows: In Sec. II we review the quantization of Chern–Simons theory. Section III is devoted to the Turaev–Viro theory. We then review the definition of 3-manifold invariants in Sec. IV, and some facts on the Verlinde formula in Sec. V. The new material starts in Sec. VI, where we discuss the CS/TV operator/state correspondence and the arising relation between the CS and TV Hilbert spaces. In Sec. VII we interpret the CFT partition function as a TV quantum state, and compute components of this state in a natural basis in the TV Hilbert space given by graphs. We conclude with a discussion.

II. CHERN–SIMONS THEORY

This section is a rather standard review of CS theory. We discuss the CS phase space, the Hilbert space that arises as its quantization, review the Verlinde formula, and a particular basis in the CS Hilbert space that arises from a pant decomposition. The reader may consult, e.g., Refs. 12 and 5 for more details.

A. Action

The Chern–Simons (CS) theory is a three-dimensional TQFT of Witten-type. The CS theory for a group G is defined by the following action functional:

$$S_{\mathrm{CS}}^{-}[A] = \frac{k}{4\pi} \int_{M} \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) - \frac{k}{4\pi} \int_{\partial M} dz \wedge d\bar{z}\, \mathrm{Tr}(A_{z} A_{\bar{z}}). \tag{2.1}$$

Here M is a three-dimensional manifold and A is a connection on the principal G-bundle over M. For the case of a compact G that we consider in this paper the action is gauge invariant (modulo 2π) when k is an integer. The second term in (2.1) is necessary to make the action principle well-defined on a manifold with boundary. To write it one needs to choose a complex structure on ∂M. As ∂M is a 2D Riemann surface, complex structures on it are same as conformal structures. Thus, one has to make a choice of the conformal structure. Then the term in (2.1) is the one relevant for fixing $A_{\bar{z}}$ on the boundary. Another possible choice of boundary condition is to fix A_{z}. The corresponding action is:

$$S_{\mathrm{CS}}^{+}[A] = \frac{k}{4\pi} \int_{M} \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) + \frac{k}{4\pi} \int_{\partial M} dz \wedge d\bar{z}\, \mathrm{Tr}(A_{z} A_{\bar{z}}). \tag{2.2}$$

B. Partition function

The partition function arises (formally) by considering the path integral for (2.1). For a closed M it can be given a precise meaning through the surgery representation of M and the Reshetikhin–Turaev–Witten (RTW) invariant of links. Before we review this construction, let us discuss the formal path integral for the case when M has a boundary. For example, let the manifold M be a handlebody H. Its boundary $X = \partial H$ is a (connected) Riemann surface. Recall that TQFT assigns a Hilbert space to each connected component of ∂M, and a map between these Hilbert spaces to M. The map can be heuristically thought of as given by the path integral. For a manifold with a single boundary component, which is the case for a handlebody H, TQFT on H gives a map $\mathcal{F}:\mathcal{H}_{X}^{\mathrm{CS}} \to \mathbb{C}$ mapping the CS Hilbert space of X into $\mathbb{C}$. This map can be obtained from the following Hartle–Hawking (HH) type state:

$$\mathcal{F}(\underline{A}) = \int_{A_{\bar{z}} = \underline{A}} \mathcal{D}A\; e^{iS_{\mathrm{CS}}^{-}[A]}. \tag{2.3}$$

The path integral is taken over connections in H with the restriction of A on X fixed. More precisely, with the choice of boundary term in the action as in (2.1), one fixes only the antiholomorphic part $\underline{A} = A_{\bar{z}}$ of the connection on X, as defined by an auxiliary complex structure. The result of the path integral (2.3) is the partition function of CS theory on H. It can be thought of as a particular quantum state $\mathcal{F}(\underline{A})$ in the CS Hilbert space $\mathcal{H}_X^{\text{CS}}$. The inner product in $\mathcal{H}_X^{\text{CS}}$ is (formally) defined as

$$\langle \Psi_1 | \Psi_2 \rangle = \int_{\mathcal{A}} \mathcal{D}\underline{A}\, \overline{\mathcal{D}\underline{A}}\,\, e^{-(k/\pi)\int_{\partial\mathcal{M}} d^2z\, \text{Tr}(\underline{A}_z \underline{A}_{\bar{z}})}\,\, \overline{\Psi_1(\underline{A})}\Psi_2(\underline{A}). \tag{2.4}$$

Here $d^2z = dz \wedge d\bar{z}/2i$ is the real measure on the boundary. The above mentioned map $\mathcal{F}: \mathcal{H}_X^{\text{CS}} \rightarrow \mathbb{C}$ is given by

$$\mathcal{F}(\Psi) = \langle \mathcal{F} | \Psi \rangle = \int_{\mathcal{A}} \mathcal{D}\underline{A}\, \overline{\mathcal{D}\underline{A}}\,\, e^{-(k/\pi)\int_{\partial\mathcal{M}} dz^2\, \text{Tr}(\underline{A}_z \underline{A}_{\bar{z}})}\,\, \overline{\mathcal{F}(\underline{A})}\Psi(\underline{A}). \tag{2.5}$$

The state $\mathcal{F}(\underline{A}) \in \mathcal{H}_X^{\text{CS}}$ depends only on the topological nature of the 3-manifold and a framing of M.

C. Phase space

To understand the structure of the CS Hilbert space $\mathcal{H}^{\text{CS}}$ it is natural to use the Hamiltonian description. Namely, near the boundary the manifold has the topology $X \times \mathbb{R}$. Then the phase space $\mathcal{P}^{\text{CS}}$ of CS theory based on a group G is the moduli space of flat G-connections on X modulo gauge transformations:

$$\mathcal{P}_X^{\text{CS}} \sim \mathcal{A}/\mathcal{G}. \tag{2.6}$$

It is finite dimensional.

Let X be a (connected) Riemann surface of type (g,n) with $g \geq 0$, $n > 0$, $2g+n-2 > 0$. Denote the fundamental group of X by $\pi(X)$. The moduli space $\mathcal{A}$ can then be parametrized by homomorphisms $\phi: \pi(X) \rightarrow G$. The phase space is, therefore, isomorphic to

$$\mathcal{P}_X^{\text{CS}} \sim \text{Hom}(\pi(X), G)/G, \tag{2.7}$$

where one mods out by the action of the group at the base point. The fundamental group is generated by $m_i, i=1,\ldots,n$ and $a_i, b_i, i=1,\ldots,g$ satisfying the following relation:

$$m_1 \cdots m_n [a_1,b_1] \cdots [a_g,b_g] = 1. \tag{2.8}$$

Here $[a,b] = aba^{-1}b^{-1}$. The dimension of the phase space can now be seen to be

$$\dim \mathcal{P}_X^{\text{CS}} = (2g+n-2)\dim G. \tag{2.9}$$

The fact that (2.7) is naturally a Poisson manifold was emphasized in Ref. 13. The Poisson structure described in Ref. 13 is the same as the one that comes from CS theory. For the case of a compact X the space (2.7) is actually a symplectic manifold. For the case when punctures are present the symplectic leaves are obtained by restricting the holonomy of $\underline{A}$ around punctures to lie in some conjugacy classes in the group. An appropriate power of the symplectic structure can be used as a volume form on the symplectic leaves. Their volume turns out to be finite. One thus expects to get finite dimensional Hilbert spaces upon quantization.

D. Hilbert space

The Hilbert space $\mathcal{H}_X^{\text{CS}}$ was understood[12,14] to be the same as the space of conformal blocks of the chiral Wess–Zumino–Witten (WZW) theory on a genus g-surface with n vertex operators

inserted. Let us fix irreducible representations $\mathbf{R} = \{\rho_1, \ldots, \rho_n\}$ of G labeling the punctures. The dimension of each of $\mathcal{H}_X^{\text{CS}}$ can be computed using the Verlinde formula:[15,16]

$$\dim \mathcal{H}_X^{\text{CS}} = \sum_\rho \frac{S_{\rho_1 \rho} \cdots S_{\rho_n \rho}}{S_{0\rho} \cdots S_{0\rho}} (S_{0\rho})^{2-2g}. \tag{2.10}$$

The sum is taken over irreducible representations ρ, $S_{\rho\rho'}$ is the modular S-matrix, see (4.3) below for the case of SU(2), and $S_{0\rho} = \eta \dim_\rho$, where η is given by (4.2).

E. Pant decomposition

The states from $\mathcal{H}_X^{\text{CS}}$ can be understood as the HH type states given by the path integral over a handlebody H with Wilson lines in representations $\mathbf{R}$ intersecting the boundary X transversally at n points. A convenient basis in $\mathcal{H}_X^{\text{CS}}$ can be obtained by choosing a pant decomposition of X. A pair of pants is a sphere with 3 holes (some of them can be punctures). A Riemann surface X of type (g, n) can be represented by $2g + n - 2$ pants glued together. For example, the surface of type $(0,4)$ with 4 punctures can be obtained by gluing together 2 spheres each with 2 punctures and one hole. Note that a pant decomposition is not unique. Different pant decompositions are related by simple "moves." A pant decomposition can be conveniently encoded in a tri-valent graph Δ with $2g + n - 2$ vertices and $3g + 2n - 3$ edges. Each vertex of Δ corresponds to a pair of pants, and each internal edge corresponds to two holes glued together. Open-ended edges of Δ end at punctures. We shall call such edges "loose." There are exactly n of them. The graph Δ can be thought of as a 1-skeleton of the Riemann surface X, or as a Feynman diagram that corresponds to the string world-sheet X. The handlebody H can be obtained from Δ as its regular neighborhood $U(\Delta)$, so that Δ is inside H and the loose edges of Δ end at the punctures. Let us label the loose edges by representations $\mathbf{R}$ and internal edges by some other (non-null) irreducible representations. It is convenient to formalize the labeling of Δ in a notion of *coloring* ϕ. A coloring ϕ is the map

$$\phi: E_\Delta \to \mathcal{I}, \quad \phi(e) = \rho_e \in \mathcal{I} \tag{2.11}$$

from the set E_Δ of edges of Δ to the set $\mathcal{I}$ of (non-null) irreducible representations of the quantum group G. The loose edges are colored by representations from $\mathbf{R}$. The CS path integral on H with the spin network Δ^ϕ inserted is a state in $\mathcal{H}_X^{\text{CS}}$. See below for a definition of spin networks. Changing the labels on the internal edges one gets states that span the whole $\mathcal{H}_X^{\text{CS}}$. Different choices of pant decomposition of H (and thus of Δ) lead to different bases in $\mathcal{H}_X^{\text{CS}}$.

F. Inner product

The inner product (2.4) of two states of the type described can be obtained by the following operation. Let one state be given by the path integral over H with Δ^ϕ inserted and the other by H with $\Delta^{\phi'}$ inserted, where both the graph and/or the coloring may be different in the two states. Let us invert orientation of the first copy of H and glue $-H$ to H across the boundary (using the identity homomorphism) to obtain some 3D space $\tilde{H}$ without boundary. We will refer to $\tilde{H}$ as the *double* of H. For H being a handlebody with g handles the double $\tilde{H}$ has the topology of a connected sum:

$$\tilde{H} \sim \#_{g-1} S^2 \times S^1. \tag{2.12}$$

The loose ends of Δ are connected at the punctures to the loose ends of Δ' to obtain a colored closed graph $\Delta^\phi \cup \Delta^{\phi'}$ inside $\tilde{H}$. The inner product (2.4) is given by the CS path integral over $\tilde{H}$ with the spin network $\Delta^\phi \cup \Delta^{\phi'}$ inserted. This path integral is given by the RTW evaluation of $\Delta^\phi \cup \Delta^{\phi'}$ in $\tilde{H}$, see below for a definition of the RTW evaluation.

III. GRAVITY

The material reviewed in this section is less familiar, although is contained in the literature. We give the action for Turaev–Viro theory, discuss the phase space, then introduce certain important graph coordinatization of it, define spin networks, and describe the TV Hilbert space. A useful reference for this section is the book of Turaev (Ref. 5 and Ref. 17).

A. Action

What we refer to as "gravity" arises as a certain "square" of CS theory. We will also refer to this gravity theory as Turaev–Viro (TV) theory, to have uniform notations (CS-TV).

To see how the TV theory (gravity) arises from CS theory, let us introduce two connection fields A and B. Consider the corresponding CS actions $S_{CS}[A], S_{CS}[B]$. Introduce the following parametrization of the fields:

$$A = \mathbf{w} + \left(\frac{\pi}{k}\right)\mathbf{e}, \quad B = \mathbf{w} - \left(\frac{\pi}{k}\right)\mathbf{e}. \tag{3.1}$$

Here $\mathbf{w}$ is a G-connection, and $\mathbf{e}$ is a one-form valued in the Lie algebra of G. The TV theory action is essentially given by the difference $S_{CS}^-[A] - S_{CS}^+[B]$, plus a boundary term such that the full action is

$$S_{TV}[\mathbf{w}, \mathbf{e}] = \int_M \text{Tr}\left(\mathbf{e} \wedge \mathbf{f}(\mathbf{w}) + \frac{\Lambda}{12}\mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e}\right). \tag{3.2}$$

The boundary condition for this action is that the restriction $\underline{\mathbf{w}}$ of $\mathbf{w}$ on $X = \partial M$ is kept fixed. Here Λ is the "cosmological constant" related to k as $k = 2\pi/\sqrt{\Lambda}$. For $G = \text{SU}(2)$ the TV theory is nothing else but the Euclidean gravity with positive cosmological constant Λ. We emphasize, however, that the theory is defined for other groups as well. Moreover, it also exists as a square of a chiral TQFT for any TQFT, that is even in cases when the chiral TQFT is not a CS theory.

B. Path integral

Similarly to CS theory, one can consider HH type states given by the path integral on a manifold with a single boundary component. Thus, for a manifold being a handlebody H we get the TV partition function:

$$\mathcal{T}(\underline{\mathbf{w}}) = \int_{\mathbf{w}|_X = \underline{\mathbf{w}}} \mathcal{D}\mathbf{w} \mathcal{D}\mathbf{e} \; e^{iS_{TV}[\mathbf{w}, \mathbf{e}]}. \tag{3.3}$$

The integral is taken over both $\mathbf{w}, \mathbf{e}$ fields in the bulk with the restriction $\underline{\mathbf{w}}$ of the connection fixed on the boundary. The TV partition function $\mathcal{T}(\underline{\mathbf{w}})$ is thus a functional of the boundary connection. It can also be interpreted as a particular state in the TV Hilbert space $\mathcal{H}_X^{TV}$.

States from $\mathcal{H}_X^{TV}$ are functionals of the boundary connection. The inner product on this space can be formally defined by the formula

$$\langle \Psi_1 | \Psi_2 \rangle = \frac{1}{\text{Vol}\,\mathcal{G}} \int_{\mathcal{A}} \mathcal{D}\underline{\mathbf{w}} \; \overline{\Psi_1(\underline{\mathbf{w}})} \Psi_2(\underline{\mathbf{w}}) \tag{3.4}$$

similar to (2.4). Note, however, that the measure in (3.4) is different from that in (2.4). We shall see this below when we describe how to compute TV inner products in practice.

C. Relation between TV and CS states

Formally, the following relation between TV and CS states exists. As one can easily check, the difference of two CS actions in the parametrization (3.1) is given by

$$S_{\mathrm{CS}}^{-}[A] - S_{\mathrm{CS}}^{+}[B] = S_{\mathrm{TV}}[\mathbf{w},\mathbf{e}] + \frac{1}{2}\int_{\partial\mathcal{M}}\mathrm{Tr}(\mathbf{e}\wedge\mathbf{w}) - \frac{k}{2\pi}\int_{\partial\mathcal{M}}dz\wedge d\bar{z}\ \mathrm{Tr}\left(\mathbf{w}_z\mathbf{w}_{\bar{z}} + \left(\frac{\pi}{k}\right)^2\mathbf{e}_z\mathbf{e}_{\bar{z}}\right).$$

(3.5)

Therefore,

$$e^{iS_{\mathrm{CS}}^{-}[A] - iS_{\mathrm{CS}}^{+}[B] - (k/\pi)\int_{\partial\mathcal{M}}d^2z\ \mathrm{Tr}(A_{\bar{z}}B_z)} = e^{iS_{\mathrm{TV}}[\mathbf{w},\mathbf{e}] + (2\pi/k)\int_{\partial\mathcal{M}}dz^2\ \mathrm{Tr}(\mathbf{e}_z\mathbf{e}_{\bar{z}})}.$$

(3.6)

Here $d^2z = dz\wedge d\bar{z}/2i$ is the real measure on $\partial\mathcal{M}$. Note that the last term in the exponential on the left hand side is the same as in the CS inner product (2.4). From this expression one can read off a prescription for obtaining the TV state $\mathcal{T}(\mathbf{w})$. Indeed, let us integrate the left-hand side over bulk A, B, keeping $A_{\bar{z}}, B_z$ fixed on the boundary. Let us denote the result by $\Phi(A_{\bar{z}}, B_z)$. We have

$$\Phi(A_{\bar{z}}, B_z) = \Psi(A_{\bar{z}})\bar{\Psi}(B_z)\ e^{-(k/\pi)\int_{\partial\mathcal{M}}d^2z\ \mathrm{Tr}(A_{\bar{z}}B_z)}.$$

(3.7)

To get $\mathcal{T}(\mathbf{w})$ one must take $\Phi(A_{\bar{z}}, B_z)$ in the parametrization (3.1), multiply the result by an exponential factor and integrate over the restriction $\underline{\mathbf{e}}$ of $\mathbf{e}$ on the boundary:

$$\mathcal{T}(\mathbf{w}) = \int\mathcal{D}\underline{\mathbf{e}}\ e^{-(2\pi/k)\int_{\partial\mathcal{M}}dz^2\ \mathrm{Tr}(\underline{\mathbf{e}_z}\underline{\mathbf{e}_{\bar{z}}})}\ \Phi\left(\mathbf{w}_{\bar{z}} + \left(\frac{\pi}{k}\right)\underline{\mathbf{e}}_{\bar{z}}, \mathbf{w}_z - \left(\frac{\pi}{k}\right)\underline{\mathbf{e}}_z\right).$$

(3.8)

The functional $\Phi(A_{\bar{z}}, B_z)$ is a vector in the Hilbert space $\mathcal{H}^{\mathrm{CS}}\otimes\overline{\mathcal{H}^{\mathrm{CS}}}$. We should view (3.8) as a transform between this Hilbert space and $\mathcal{H}^{\mathrm{TV}}$. This transform will play an important role in what follows. Below we shall see how the result of the transform (3.8) can be found in practice.

D. Phase space

The TV phase space is basically two copies of $\mathcal{P}^{\mathrm{CS}}$, but with an unusual polarization. The polarization on $\mathcal{P}^{\mathrm{TV}}$ is given by $\mathbf{e}, \mathbf{w}$, which are canonically conjugate variables. Note that there is no need to choose a complex structure in order to define this polarization.

It turns out to be very convenient to think of $\mathcal{P}^{\mathrm{TV}}$ as some deformation of the cotangent bundle $T^*(\mathcal{A}/\mathcal{G})$ over the moduli space $\mathcal{A}/\mathcal{G}$ of flat connections on X. Note, however, that the TV connection $\mathbf{w}$ on the boundary is not flat, so the configuration space for TV theory is not really the moduli space of flat connections. One does get $\mathcal{A}/\mathcal{G}$ as the configurational space in an important limit $k\to\infty$, in which the $\mathbf{e}^{\wedge 3}$ term drops from the action (3.2). Thus, it is only in this limit that the TV phase space is the cotangent bundle $T^*(\mathcal{A}/\mathcal{G})$. For a finite k the TV phase space is compact (as consisting of two copies of $\mathcal{P}^{\mathrm{CS}}$), while $T^*(\mathcal{A}/\mathcal{G})$ is not. We will see, however, that it is essentially correct to think of $\mathcal{P}^{\mathrm{TV}}$ as a deformation of $T^*(\mathcal{A}/\mathcal{G})$ even in the finite k case. The compactness of $\mathcal{P}^{\mathrm{TV}}$ will manifest itself in the fact that after the quantization the range of eigenvalues of $\mathbf{e}$ is bounded from above.

These remarks being made we write

$$\mathcal{P}^{\mathrm{TV}} \sim T_k^*(\mathcal{A}/\mathcal{G}),$$

(3.9)

where T_k^* is certain compact version of the cotangent bundle. The phase space becomes the usual cotangent bundle in the $k\to\infty$ limit. We will not need any further details on spaces T_k^*. As we shall see the quantization of $\mathcal{P}^{\mathrm{TV}}$ is rather straightforward once the quantization of the cotangent bundle is understood.

We note that the dimension

$$\dim\mathcal{P}^{\mathrm{TV}} = 2(2g + n - 2)\dim G$$

(3.10)

is twice the dimension of the phase space of the corresponding CS theory, as required. A convenient parametrization of the cotangent bundle phase space can be obtained by using graphs.
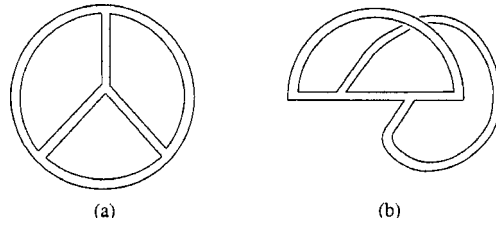
FIG. 1. A fat graph Γ for the (a) sphere with 4 punctures; (b) torus with one puncture.

E. Graphs

The graphs one considers are similar to those that arise in the Penner coordinatization[18] of the moduli space of punctured Riemann surfaces. Namely, given X, introduce a trivalent closed fat graph Γ with the number F of faces equal to the number n of punctures. Such a graph can be obtained by triangulating the surface X using punctures as vertices, and then constructing a dual graph. What arises is exactly a graph Γ. See Fig. 1 for examples of Γ. Note that different triangulations lead to different graphs, so Γ is by no means unique.

Because the graph is trivalent $3V = 2E$, where V is the number of vertices and E is the number of edges. We also have the Euler characteristics relation:

$$F - E + V = 2 - 2g. \tag{3.11}$$

We thus get that the number E of edges of Γ is $E = 3(2g + n - 2)$.

Note that the graph Γ does not coincide with the graph Δ introduced in the previous section. There is, however, a simple relation between them that is worth noting. Let us, as in the previous section, form the double $\tilde{H} = H \cup -H$. It is a closed 3-manifold obtained by gluing two copies of the handlebody H across the boundary X. Let us take a graph Δ in H, and another copy of Δ in $-H$. These graphs touch the boundary $\partial H = X$ at the punctures. Gluing these two copies of Δ at the punctures one obtains a closed graph $\Delta \cup \Delta$ in $\tilde{H}$. It is a trivalent graph with $2(2g + n - 2)$ vertices and $3(2g + n - 2)$ edges. Now consider the regular neighborhood $U(\Delta \cup \Delta)$ of $\Delta \cup \Delta$ in $H \cup -H$. This is a handlebody, whose boundary is of genus

$$G = 2g + n - 1. \tag{3.12}$$

The surface $\partial U(\Delta \cup \Delta)$ can be obtained by taking two copies of X, removing some small disks around the punctures, and identifying the resulting circular boundaries to get a closed surface without punctures. We have the following:

Lemma: The surface $\partial U(\Delta \cup \Delta)$ is a Heegard surface for $H \cup -H$. The complement of $U(\Delta \cup \Delta)$ in $H \cup -H$ is a handlebody that is the regular neighborhood $U(\Gamma)$ of the graph Γ on X.

Proof: The complement of $U(\Delta \cup \Delta)$ in $H \cup -H$ can be seen to be the cylinder $X \times [0,1]$ with n holes cut in it. So, it is indeed a handlebody of genus (3.12). Its 1-skeleton that can be obtained by choosing a pant decomposition is the trivalent graph Γ.

F. Graph connections

Given Γ equipped with an arbitrary orientation of all the edges, one can introduce what can be called graph connections. Denote the set of edges e of Γ by E. We use the same letter both for the set E of edges and for its dimension. A graph connection $\mathbf{A}$ is an assignment of a group element to every edge of the graph:

$$\mathbf{A}: E \to G, \quad \mathbf{A}(e) = \mathbf{g}_e \in G. \tag{3.13}$$

One can also introduce a notion of graph gauge transformations. These act at vertices of Γ. A gauge transformation is parametrized by V group elements. Let us introduce:

$$\mathbf{H}:V\rightarrow G, \quad \mathbf{H}(v)=\mathbf{h}_v \in G. \tag{3.14}$$

Here V is the set of vertices of Γ. For an edge $e \in E$ denote by $s(e)$ (source) the vertex from which e originates, and by $t(e)$ (target) the vertex where e ends. The action of a gauge transformation $\mathbf{H}$ on a graph connection $\mathbf{A}$ is now as follows:

$$\mathbf{A}^{\mathbf{H}}(e)=\mathbf{h}_{s(t)}^{-1}\,\mathbf{g}_e\,\mathbf{h}_{s(e)}. \tag{3.15}$$

The space of graph connections modulo graph gauge transformations can now be seen to be isomorphic to $G^{\otimes E}/G^{\otimes V}$. Its dimension is given by (2.9). We thus get a parametrization of the CS phase space $\mathcal{P}^{\text{CS}}$ based on a graph Γ:

$$\mathcal{P}^{\text{CS}}\sim G^{\otimes E}/G^{\otimes V}. \tag{3.16}$$

The TV phase space is the cotangent bundle

$$\mathcal{P}^{\text{TV}}\sim T_k^*(G^{\otimes E}/G^{\otimes V}). \tag{3.17}$$

As we shall see, it is rather straightforward to quantize the noncompact, $k\rightarrow\infty$ version of $\mathcal{P}^{\text{TV}}$, that is the cotangent bundle. The quantum states are given by spin networks.

G. Spin networks

To quantize the cotangent bundle $T^*(\mathcal{A}/\mathcal{G})$ one introduces a Hilbert space of functionals on the moduli space of flat connections. A complete set of such functionals is given by spin networks. These functions will thus form (an over-complete) basis in the Hilbert space of TV theory. They also serve as observables for CS quantum theory, see below.

Before we define these objects, let us introduce some convenient notations. Denote the set of irreducible representations ρ of the quantum group G by $\mathcal{I}$. Introduce a coloring $\psi:E\rightarrow\mathcal{I},\psi(e) =\rho_e$ of the edges of Γ with irreducible representations of G. A spin network Γ^ψ is a functional on the space of graph connections:

$$\Gamma^\psi:G^{\otimes E}\rightarrow\mathbb{C}. \tag{3.18}$$

Given a connection $\mathbf{A}$ the value of $\Gamma^\psi(\mathbf{A})$ is computed as follows. For every edge e take the group element $\mathbf{g}_e$ given by the graph connection in the irreducible representation ρ_e. One can think of this as a matrix with two indices: one for the source $s(e)$ and the other for the target $t(e)$. Multiply the matrices for all the edges of Γ. Then contract the indices at every tri-valent vertex using an intertwining operator. The normalization of intertwiners that we use is specified in the Appendix. The definition we gave is applicable to $G=\text{SU}(2)$. In this case the trivalent intertwiner is unique up to normalization. For other gauge groups one in addition has to label the vertices of Γ with intertwiners, so that a spin network explicitly depends on this labeling. The functional (3.18) so constructed is invariant under the graph gauge transformations (3.14) and is thus a functional on the moduli space of flat connections modulo gauge transformations. As such it is an element of the Hilbert space of TV theory. It is also an observable on the CS phase space (3.16).

H. Quantization

We can define the Hilbert space $\mathcal{H}^{\text{TV}}$ of Turaev–Viro theory to be the space of gauge-invariant functionals $\Psi(\mathbf{w})$ on the configurational space $G^{\otimes E}/G^{\otimes V}$. This gives a quantization of the $k\rightarrow\infty$ limit, but a modification for the case of finite k is straightforward. As we discussed above, a complete set of functionals on $G^{\otimes E}/G^{\otimes V}$ is given by spin networks. We denote the state corresponding to a spin network Γ^ψ by $|\Gamma^\psi\rangle$. They form a basis of states in $\mathcal{H}^{\text{TV}}$:

$$\mathcal{H}^{\text{TV}} = \text{Span}\{|\Gamma^\psi\rangle\}. \tag{3.19}$$

One can construct certain momenta operators, analogs of $\mathbf{e} \sim \partial/\partial \underline{\mathbf{w}}$ in the continuum theory. Spin networks are eigenfunctions of these momenta operators. To specialize to the case of finite k one has to replace all spin networks by quantum ones. That is, the coloring of edges of Γ must use irreducible representations of the quantum group, which there is only a finite set.

The spin network states $|\Gamma^\psi\rangle$ form an overcomplete basis in $\mathcal{H}^{\text{TV}}$, in that the TV inner product between differently colored states is nonzero. However, these states do give a partition of unity in that

$$\sum_\psi \left(\prod_{e \in E_\Gamma} \dim_{\rho_e} \right) |\Gamma^\psi\rangle \langle \Gamma^\psi| \tag{3.20}$$

is the identity operator in $\mathcal{H}^{\text{TV}}$. This will become clear from our definition of the TV inner product, and the definition of the TV invariant in the next section.

It seems from the way we have constructed the Hilbert space $\mathcal{H}^{\text{TV}}$ that it depends on the graph Γ. This is not so. Choosing Γ differently one gets a different basis in the same Hilbert space. To describe an effect of a change of Γ it is enough to give a rule for determining the inner products between states from two different bases.

I. Inner product

The inner product on $\mathcal{H}^{\text{TV}}$ is given (formally) by the integral (3.4) over boundary connections. To specify the measure in this integral, one has to consider the path integral for the theory. Namely, consider a 3-manifold $X \times [-1,1]$ over X, which is a 3-manifold with two boundary components, each of which is a copy of X. The TV path integral over $X \times [-1,1]$ gives a kernel that should be sandwiched between the two states whose inner product is to be computed. Thus, the measure in (3.4) is defined by the TV path integral. The measure, in particular, depends on the level k.

In practice the inner product of two states $\Gamma^\psi \ \Gamma^{\psi'}$, where both the graphs and the coloring may be different, is computed as the TV invariant for the manifold $X \times [-1,1]$ with Γ^ψ on $X \times \{-1\}$ and $\Gamma^{\psi'}$ on $X \times \{1\}$. Further details on the TV inner product are given in the next section.

IV. 3-MANIFOLD INVARIANTS

In this section we review the definition of RTW and TV invariants. The main references for this section are Refs. 19 and 20.

A. Reshetikhin–Turaev–Witten invariant

The RTW invariant of a closed 3-manifold (with, possibly, Wilson loops or spin networks inserted) gives a precise meaning to the CS path integral for this manifold. The definition we give is for M without insertions, and is different from, but equivalent to the original definition in Ref. 21. We follow Roberts.[19]

Any closed oriented 3-manifold M can be obtained from S^3 by a surgery on a link in S^3. Two framed links represent the same manifold M if and only if they are related by isotopy or a sequence of Kirby moves, that is either handle-slides or blow-ups, see Refs. 19 or 22 for more detail. Let L be a link giving the surgery representation of M. Define $\Omega L \in \mathbb{C}$ to be the evaluation of L in S^3 with a certain element Ω inserted along all the components of L, paying attention to the framing. The element Ω is defined as follows, see Ref. 19. It is an element of $\mathcal{H}_T^{\text{CS}}$, where T is the torus, and is given by

$$\Omega = \eta \sum_\rho \dim_\rho R_\rho. \tag{4.1}$$

The sum is taken over all irreducible representations $\rho \in \mathcal{I}$, the quantity $\dim_\rho$ is the quantum dimension, and R_ρ is the state in $\mathcal{H}_T^{\mathrm{CS}}$ obtained by inserting the 0-framed unknot in the ρ's representation along the cycle that is noncontractible inside the solid torus having T as its boundary. The quantity η is given by

$$\eta^{-2} = \sum_\rho \dim_\rho^2. \tag{4.2}$$

For example, for $G = \mathrm{SU}(2)$ $\eta = \sqrt{2/k}\, \sin(\pi/k) = S_{00}$, where

$$S_{ij} = \sqrt{\frac{2}{k}} \sin\left(\frac{(i+1)(j+1)\pi}{k} \right), \quad k \geqslant 3. \tag{4.3}$$

With the normalization chosen, the S^3 value of a 0-framed unknot with Ω attached is η^{-1}, while ± 1 framed unknots with Ω attached give certain unit modulus complex numbers $r^{\pm 1}$. For $G = \mathrm{SU}(2)$ $r = \exp(-i\pi/4 - 2\pi i(3+k^2)/4k)$.

Let us now continue with the definition of the RTW invariant. Define by $\sigma(L)$ the signature of the 4-manifold obtained by attaching 2-handles to the 4-ball B^4 along $L \subset S^3 = \partial B^4$. Define

$$I(M) = \eta r^{-\sigma(L)} \Omega L. \tag{4.4}$$

This is the RTW invariant of the manifold M presented by L. We use the normalization of Roberts,[19] in which the RTW invariant satisfies $I(S^3) = \eta, I(S^2 \times S^1) = 1$, as well as the connected sum rule $I(M_1 \# M_2) = \eta^{-1} I(M_1) I(M_2)$.

B. Turaev–Viro invariant

The original Turaev–Viro invariant is defined[8] for triangulated manifolds. A more convenient presentation[20] uses standard 2-polyhedra. Another definition is that of Roberts.[19] It uses a handle decomposition of M. We first give the original definition of Turaev and Viro.

Let T be a triangulation of 3D manifold M. We are mostly interested in case that M has a boundary. Denote by V_T the number of vertices of T, and by $\{e\}, \{f\}, \{t\}$ collections of edges, faces and tetrahedra of T. Choose a coloring μ of all the edges, so that $\mu(e) = \rho_e$ is the color assigned to an edge e. The Turaev–Viro invariant is defined as

$$\mathrm{TV}(M, T|_{\partial M}, \mu|_{\partial M}) = \eta^{2V_T} \sum_\mu \prod_{e \notin \partial M} \dim_{\rho_e} \prod_t (6j)_t. \tag{4.5}$$

Here $(6j)_t$ is the $6j$-symbol constructed out of 6 colors labeling the edges of a tetrahedron t, and the product is taken over all tetrahedra t of T. The product of dimensions of representations labelling the edges is taken over all edges that do not lie on the boundary. The sum is taken over all colorings μ keeping the coloring on the boundary fixed. The invariant depends on the restriction $T|_{\partial M}$ of the triangulation to the boundary ∂M, and on the coloring $\mu|_{\partial M}$ of this restriction. The invariant is independent of an extension of $T|_{\partial M}$ inside M.

Note that the TV invariant is constructed in such a way that for a closed 3-manifold $M = M_1 \cup M_2$ obtained by gluing two manifolds M_1, M_2 with a boundary across the boundary the invariant $\mathrm{TV}(M)$ is easily obtained once $\mathrm{TV}(M_{1,2}, T_{1,2}|_{\partial M_{1,2}}, \mu_{1,2}|_{\partial M_{1,2}})$ are known. One has to triangulate the boundary of $M_{1,2}$ in the same way $T_1|_{\partial M_1} = T_2|_{\partial M_2}$, multiply the invariants for $M_{1,2}$, multiply the result by the dimensions of the representations labelling the edges of $T_{1,2}|_{\partial M_{1,2}} = T|_{\partial M}$, and sum over these representations. The result is $\mathrm{TV}(M)$:

$$\mathrm{TV}(M) = \sum_{\mu|_{\partial M}} \left(\prod_{e \in \partial M} \dim_{\rho_e} \right) \mathrm{TV}(M_1, T|_{\partial M}, \mu|_{\partial M}) \mathrm{TV}(M_2, T|_{\partial M}, \mu|_{\partial M}). \tag{4.6}$$

This, together with the definition of the TV inner product as the TV invariant for $X \times I$ establishes that (3.20) is indeed the identity operator in $\mathcal{H}^{\text{TV}}$.

C. Roberts invariant

We shall now introduce the more general invariant of Roberts. We consider the case without boundary.

Consider a handle decomposition D of M. The canonical example to have in mind is the handle decomposition coming from a triangulation T of M. A thickening of the corresponding dual complex T^* then gives a handle decomposition. The vertices of the dual complex (baricenters of tetrahedra of the triangulation) correspond to 0-handles, edges of T^* (faces of T) correspond to 1-handles, faces of T^* (edges of T) give 2-handles, and 3-cells of T^* (vertices of T) give 3-handles. The union of 0- and 1-handles is a handlebody. Choose a system of meridian disks for it, one meridian disk for every 1-handle. Now specify the system of attaching curves for 2-handles. If the handle decomposition came from a triangulation there are exactly 3 attaching curves along each 1-handle. Frame all meridian and attaching curves using the orientation of the boundary of the handlebody. Denote the corresponding link by $C(M,D)$. Insert the element Ω on all the components of $C(M,D)$, paying attention to the framing, and evaluate $C(M,D)$ in S^3. This gives the Roberts invariant for M:

$$\mathrm{R}(M) = \eta^{d_3 + d_0} \Omega C(M,D). \tag{4.7}$$

Here d_3, d_0 are the numbers of 3- and 0-handles correspondingly. Note that to evaluate $\Omega C(M,D)$ in S^3 one needs to first specify an embedding. The result of the evaluation does not depend on the embedding, see Ref. 19. Moreover, the invariant does not depends on a handle decomposition D and is thus a true invariant of M.

When the handle decomposition D comes from a triangulation T the Roberts invariant (4.7) coincides with the Turaev–Viro invariant (4.5). An illustration of this fact is quite simple and uses the 3-fusion (A2), see Ref. 19 for more detail.

Lemma (Roberts): The described above system $C(M,D)$ of meridian and attaching curves for a handle decomposition D of M gives a surgery representation of $M \# - M$.

This immediately implies the theorem of Turaev and Walker:

$$\mathrm{TV}(M) = \eta\, I(M \# - M) = |I(M)|^2. \tag{4.8}$$

Below we shall see an analog of this relation for a manifold with boundary. All the facts mentioned make it clear that the TV invariant is a natural spin-off of the CS (RTW) invariant.

D. TV inner product

Recall that the Turaev–Viro inner product between the graph states $|\Gamma^\psi\rangle$ was defined in the previous section as the TV path integral on $X \times I$. The TV path integral is rigorously defined by the TV invariant (4.5). Here we describe how to compute the inner product in practice. The prescription we give is from Ref. 20, Sec. 4.d. We combine it with the chain-mail idea of Roberts[19] and give this chain-mail prescription.

The product $\langle \Gamma^\psi | \Gamma^{\psi'} \rangle$ is obtained by a certain face model on X. Namely, consider the 3-manifold $X \times I$, where I is the interval $[-1,1]$. Put Γ^ψ on $X \times \{-1\}$ and $\Gamma^{\psi'}$ on $X \times \{1\}$. Both graphs can be projected onto $X = X \times \{0\}$, keeping track of under- and upper-crossings. By using an isotopy of X the crossings can be brought into a generic position of double transversal crossing of edges. We thus get a graph on X, with both 3 and 4-valent vertices. The 3-valent vertices come from those of $\Gamma^\psi, \Gamma^{\psi'}$, and 4-valent vertices come from edge intersections between the two graphs. The inner product is given by evaluation in S^3 of a certain chain-mail that can be constructed from $\Gamma^\psi, \Gamma^{\psi'}$. Namely, let us take one 0-framed link for every face, and one 0-framed link around every edge of the graph $\Gamma^\psi \cup \Gamma^{\psi'}$ on X. We get the structure of links at vertices as is shown

in the following drawings:

(4.9)
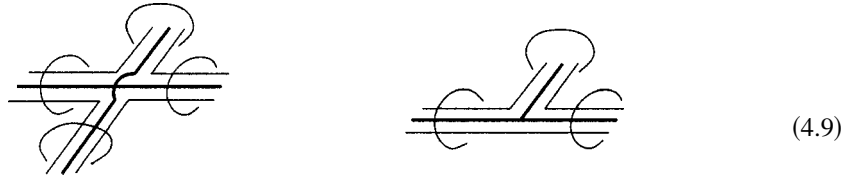
Denote by $C(\Gamma,\Gamma')$ the obtained collection of links. The inner product is given by

$$\langle \Gamma^\psi | \Gamma^{\psi'} \rangle = \eta^{V_\Gamma + V_{\Gamma'} + V_{\text{int}}}(\Gamma^\psi \cup \Gamma^{\psi'} . \Omega C(\Gamma \cup \Gamma')). \qquad (4.10)$$

Here $V_\Gamma, V_{\Gamma'}$ are the numbers of 3-valent vertices of graphs Γ, Γ' correspondingly, and V_{int} is the number of 4-valent vertices coming from intersections. The expression in brackets must be evaluated in S^3. Using the 3-fusion (A2) one can easily convince oneself that (4.10) coincides with the prescription given in Ref. 20.

We would also like to note an important relation for the TV inner product that expresses it as the RTW evaluation:

$$\langle \Gamma^\psi | \Gamma^{\psi'} \rangle = I(X \times S^1, \Gamma^\psi, \Gamma^{\psi'}). \qquad (4.11)$$

The evaluation is to be carried out in the 3-manifold $X \times S^1$. This relation that does not seem to have appeared in the literature. A justification for it comes from our operator/state correspondence, see below. Let us also note that a direct proof of a particular subcase of (4.11) corresponding to one of the graphs being zero colored is essentially given by our proof in the Appendix of the main theorem of Sec. VII. We decided not to attempt a direct proof of (4.11) in its full generality.

Turaev theorem: Let us note the theorem 7.2.1 from Ref. 5. It states that the TV invariant for H with the spin network Γ^ψ on $X = \partial H$ equals the RTW evaluation of Γ^ψ in $H \cup -H$:

$$TV(H, \Gamma^\psi) = I(H \cup -H, \Gamma^\psi). \qquad (4.12)$$

This is an analog of (4.8) for a manifold with a single boundary, and is somewhat analogous to our relation (4.11) for the TV inner product.

V. VERLINDE FORMULA

The purpose of this somewhat technical section is to review some facts about the Verlinde formula for the dimension of the CS Hilbert space. Considerations of this section will motivate a more general formula given in Sec. VII for the CFT partition function projected onto a spin network state. This section can be skipped on the first reading.

A. Dimension of the CS Hilbert space

Let us first obtain a formula for the dimension of the CS Hilbert space that explicitly sums over all different possible states. This can be obtained by computing the CS inner product. Indeed, as we have described in Sec. II, a basis in $\mathcal{H}_X^{\text{CS}}$ is given by spin networks Δ^ϕ. With our choice of the normalization of the 3-valent vertices the spin network states $|\Delta^\phi\rangle$ are orthogonal but not orthonormal. Below we will show that the dimension can be computed as

$$\dim \mathcal{H}_X^{\text{CS}} = \sum_\phi \left(\prod_{\text{int } e} \dim_{\rho_e} \right) \langle \Delta^\phi | \Delta^\phi \rangle = \sum_\phi \left(\prod_{\text{int } e} \dim_{\rho_e} \right) I(H \cup -H, \Delta^\phi \cup \Delta^\phi), \qquad (5.1)$$

where the sum is taken over the colorings of the internal edges. The coloring of the edges of Δ that end at punctures are fixed.

To evaluate $\Delta^{\phi} \cup \Delta^{\phi}$ we proceed as follows. Let us project the graph $\Delta \cup \Delta$ to X. We note that there is a canonical way to do this projection so that there are exactly two 3-valent vertices of $\Delta \cup \Delta$ on each pair of pants, and there are exactly two edges of $\Delta \cup \Delta$ going through each boundary circle of a pair of pants. For example, the part of $\Delta \cup \Delta$ projected on a pair of pants with no punctures looks like

$$(5.2)$$
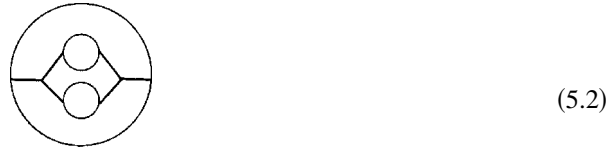
One gets a similar structure when projecting on a pair of pants with punctures. In that case the two holes in the center are replaced by punctures and loose edges of Δ are connected at the punctures to the loose edges of the other copy of Δ.

Let us now form a link L_{Δ} whose components are circles along which one glues the pant boundaries together. There are $3g + n - 3$ such circles, in one-to-one correspondence with internal edges of Δ. We push all components of L_{Δ} slightly out of X. Using the prescription of the Appendix of Ref. 20 for computing the RTW evaluation of M with a graph inserted, one obtains

$$I(H \cup - H, \Delta^{\phi} \cup \Delta^{\phi}) = \eta^{3g+n-3}(\Delta^{\phi} \cup \Delta^{\phi}. \Omega L_{\Delta}). \tag{5.3}$$

The evaluation on the right-hand side is to be taken in S^3. This relation establishes (5.1). Indeed, there are exactly two edges of $\Delta \cup \Delta$ linked by every component of L_{Δ}. Using the 2-fusion we get them connected at each pair of pants, times the factor of $\eta^{-1}/\dim_{\rho_e}$. The factors of η are canceled by the prefactor in (5.3), and the factors of $1/\dim_{\rho_e}$ are canceled by the product of dimensions in (5.1). What remains is the sum over the colorings of the internal edges of the product of N_{ijk} for every pair of pants. This gives the dimension. This argument also shows that the states $|\Delta^{\phi}\rangle$ with different coloring ϕ are orthogonal.

B. Computing the dimension: Verlinde formula

The sum over colorings of the internal edges in (5.3) can be computed. This gives the Verlinde formula. Let us sketch a simple proof of it, for further reference.

We first observe that, using the 3-fusion, the Verlinde formula for the 3-punctured sphere can be obtained as a chain-mail. Namely,

$$\eta^{-1} N_{ijk} = \quad \text{} \tag{5.4}$$

The Verlinde formula for N_{ijk} can be obtained by using the definition (4.1) of Ω and the recoupling identity (A4) of the Appendix. The computation is as follows:

$$\eta^{-1} N_{ijk} = \eta \sum_{l} \dim_{l} \left(\text{} \right) = \sum_{l} S_{il} \left(\text{} \right) = \sum_{l} \frac{S_{il} S_{jl}}{\eta \dim_{l}} \left(\text{} \right) = \eta^{-1} \sum_{l} \frac{S_{il} S_{jl} S_{kl}}{S_{0l}}. \tag{5.5}$$

This is the Verlinde formula (2.10) for the case of a 3-punctured sphere. We have used the fact that $\eta \dim_{l} = S_{0l}$. The above proof of the Verlinde formula for N_{ijk} is essentially that from Ref. 12.

The general Verlinde formula (2.10) can be obtained using a pant decomposition of X and taking a sum over labelings of the internal edges of Δ of the product of N_{ijk} one for every pair of pants. To get (2.10) one just has to use the unitarity $\Sigma_l S_{il} S_{jl} = \delta_{ij}$ of the S-matrix.

C. Verlinde formula using graph Γ: No punctures

Here we find a different representation of the Verlinde dimension. It was noticed in Ref. 23 that the Verlinde formula can be obtained using a certain gauge theory on a graph on X. Here we reinterpret this result using a chain-mail. We first derive a formula for a Riemann surface without punctures. It is obtained by starting from a graph Γ corresponding to a surface with some number n of punctures. Then a sum is carried over the labels at the punctures, so the end result depends only on the genus g, but not on n.

Consider a fat trivalent graph Γ that represents $X_{g,n}$. Let us form a chain-mail $C(\Gamma)$ as follows. Let us introduce a curve for every face of the fat graph Γ, and a linking curve around every $3(2g+n-2)$ edges of Γ, so that the obtained structure of curves at each 3-valent vertex is as in (4.9). Insert the element Ω along each component of $C(\Gamma)$, and evaluate the result in S^3. What is evaluated is just the chain-mail for Γ, no spin network corresponding to Γ is inserted. We get the following result:

Theorem (Boulatov): *The dimension of the Hilbert space of CS states on X_g is equal:*

$$\dim \mathcal{H}_{X_g}^{\text{CS}} = \eta^{V_\Gamma} \Omega C(\Gamma). \tag{5.6}$$

The expression on the right-hand side is independent of the graph Γ that is used to evaluate it.

To prove this formula we use the 2-strand fusion. We get that all of the n different colorings on the links of Γ become the same. Denote by ρ the corresponding representation. The result is then obtained by a simple counting. Each $3(2g+n-2)$ of links around edges introduces the factor of $\eta^{-1}/\dim_\rho$. Every $2(2g+n-2)$ vertices of Γ gives a factor of $\dim_\rho$. Each of n faces of Γ gives another factor of $\eta \dim_\rho$. All this combines, together with the prefactor to give

$$\dim \mathcal{H}_{X_g}^{\text{CS}} = \sum_\rho (\eta \dim_\rho)^{2-2g}, \tag{5.7}$$

which is the Verlinde formula (2.10) for the case with no punctures.

VI. OPERATOR/STATE CORRESPONDENCE

This section is central to the paper. Here we discuss a one-to-one correspondence between observables of CS theory and quantum states of TV theory. The fact that the algebra of observables in CS theory is given by graphs is due to Refs. 24 and 25, see also references below. The notion of the connecting 3-manifold $\tilde{M}$ is from Refs. 4 and 9. The operator/state correspondence of this section, as well as the arising relation between the CS and TV Hilbert spaces, although to some extent obvious, seems new.

A. CS observables and relation between the Hilbert spaces

We have seen that a convenient parametrization of the moduli space $\mathcal{A}/\mathcal{G}$ is given by the graph Γ connections. An expression for the CS Poisson structure in terms of graph connections was found in Ref. 24. A quantization of the corresponding algebra of observables was developed in Refs. 25–29 (see also Ref. 30 for a review). As we have seen in Sec. III a complete set of functionals on $\mathcal{A}/\mathcal{G}$ is given by spin networks. Spin networks thus become operators $\hat{\Gamma}^\psi$ in the CS Hilbert space $\mathcal{H}_X^{\text{CS}}$. We therefore get a version of an operator/state correspondence, in which TV states correspond to observables of CS theory.

The fact that a CS/TV operator/state correspondence must hold follows from the relation between the phase spaces of the two theories. Namely, as we have seen in Sec. III, the TV phase

space is given by two copies of the phase space of Chern–Simons theory: $\mathcal{P}^{\text{TV}} = \mathcal{P}^{\text{CS}} \otimes \bar{\mathcal{P}}^{\text{CS}}$, where the two copies have opposite Poisson structures. This means that in the quantum theory the following relation must hold:

$$\mathcal{H}_X^{\text{TV}} \sim \mathcal{H}_X^{\text{CS}} \otimes \mathcal{H}_{-X}^{\text{CS}} \sim \text{End}(\mathcal{H}_X^{\text{CS}}). \qquad (6.1)$$

Thus, the TV Hilbert space is isomorphic to the direct product of two copies of $\mathcal{H}^{\text{CS}}$. The above isomorphism is given formally by the formula (3.8). We shall denote it by $\mathbf{I}$. It identifies the TV spin network states $|\Gamma^{\psi}\rangle$ with the CS spin network observables $\hat{\Gamma}^{\psi}$. This statement deserves some explanation. The TV spin network states are wave functionals of the connection $\mathbf{w}$: $\mathcal{T}(\mathbf{w}) = \mathcal{T}(A_z + B_z, A_{\bar{z}} + B_{\bar{z}})$, whereas Chern–Simons states are functionals $\Psi(A_z, B_{\bar{z}})$. Thus, the isomorphism (6.1) can be understood as a change of polarization. Being a change of polarization it intertwines the operator algebras acting on the two sides of (6.1). The polarization we have chosen for the TV Viro model is the one for which $\hat{\mathbf{e}} \sim (\hat{A} - \hat{B})$ acts trivially on the TV vacuum state. Using the intertwinning property of $\mathbf{I}$ this means that $\mathbf{I}(|0\rangle_{\text{TV}})$ is commuting with all CS operators $\hat{\Gamma}^{\psi}$. It is therefore proportional to the identity in $\text{End}(\mathcal{H}_X^{\text{CS}})$. It follows from here that the operator that corresponds to the TV state $|\Gamma^{\psi}\rangle$ is the CS spin network operator:

$$\mathbf{I}(|\Gamma^{\psi}\rangle_{\text{TV}}) = \hat{\Gamma}^{\psi} \mathbf{I}(|0\rangle_{\text{TV}}) \propto \hat{\Gamma}^{\psi}. \qquad (6.2)$$

Thus, the described isomorphism (6.1) given by the change of polarization indeed identifies TV graph states with the CS spin network operators.

Another important fact is as follows. Being a change of polarization, the isomorphism (6.1) preserves the inner product. Since the inner product on the right-hand side of (6.1) is just the CS trace, we get an important relation:

$$\text{Tr}_{\text{CS}}(\hat{\Gamma}\hat{\Gamma}') = \langle \Gamma | \Gamma' \rangle_{\text{TV}}. \qquad (6.3)$$

In other words, the trace of the product of operators in the CS Hilbert space is the same as the inner product in the TV theory. This relation is central to the operator/state correspondence under consideration. Let us now describe the isomorphism (6.1) more explicitly.

B. Connecting manifold $\tilde{M}$

A very effective description of the above operator/state correspondence uses the "connecting manifold" $\tilde{M}$. It is a 3-manifold whose boundary is the Schottky double $\tilde{X}$ of the Riemann surface X. Recall that the Schottky double of a Riemann surface X is another Riemann surface $\tilde{X}$. For the case of a closed X, the surface $\tilde{X}$ consists of two disconnected copies of X, with all moduli replaced by their complex conjugates in the second copy. For X with a boundary (the case not considered in this paper, but of relevance to the subject of boundary CFT, see, e.g., Refs. 4, 9) the double $\tilde{X}$ is obtained by taking two copies of X and gluing them along the boundary. Consider a 3-manifold

$$\tilde{M} = \tilde{X} \times [0,1]/\sigma, \qquad (6.4)$$

where σ is an antiholomorphic map such that $\tilde{X}/\sigma = X$, and σ reverses the "time" direction. See, e.g., Ref. 4 for more detail on the construction of $\tilde{M}$. The manifold $\tilde{M}$ has a boundary $\partial\tilde{M} = \tilde{X}$, and the original surface X is embedded into $\tilde{M}$. For the case of a closed X, relevant for this paper, the manifold $\tilde{M}$ has the topology $X \times I$, where I is the interval $I = [0,1]$, see Fig. 2.

Consider the RTW evaluation of a spin network Γ^{ψ} in $\tilde{M}$. It gives a particular state in $\mathcal{H}_{\tilde{X}}^{\text{CS}}$:

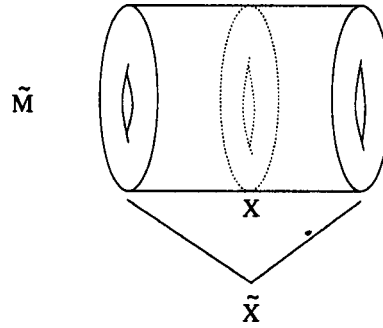$$I(\tilde{M}, \Gamma^{\psi}) \in \mathcal{H}_{\tilde{X}}^{\text{CS}}. \qquad (6.5)$$

FIG. 2. The manifold $\tilde{M}$.

However, we have

$$\mathcal{H}_{\tilde{X}}^{\mathrm{CS}} \sim \mathcal{H}_X^{\mathrm{CS}} \otimes \mathcal{H}_{-X}^{\mathrm{CS}} \sim \mathrm{End}(\mathcal{H}_X^{\mathrm{CS}}). \tag{6.6}$$

Thus (6.5) gives an operator in $\mathcal{H}_X^{\mathrm{CS}}$ for every graph state $|\Gamma^\psi\rangle \in \mathcal{H}_X^{\mathrm{TV}}$.

C. Operator product

In the realization described the product of two operators $\hat{\Gamma}^\psi, \hat{\Gamma}^{\psi'}$ is an element of $\mathcal{H}_{\tilde{X}}^{\mathrm{CS}}$ obtained by evaluating in $\tilde{M}$ both Γ^ψ and $\Gamma^{\psi'}$:

$$I(\tilde{M}, \Gamma^\psi, \Gamma^{\psi'}) \in \mathcal{H}_{\tilde{X}}^{\mathrm{CS}}. \tag{6.7}$$

D. Trace

The trace of an operator $\hat{\Gamma}^\psi$ is obtained by gluing the two boundaries of $\tilde{M}$ to form a closed manifold of the topology $X \times S^1$:

$$\mathrm{Tr}_{\mathrm{CS}}(\hat{\Gamma}^\psi) = I(X \times S^1, \Gamma^\psi). \tag{6.8}$$

One can similarly obtain the trace of an operator product:

$$\mathrm{Tr}_{\mathrm{CS}}(\hat{\Gamma}^\psi \hat{\Gamma}^{\psi'}) = I(X \times S^1, \Gamma^\psi, \Gamma^{\psi'}). \tag{6.9}$$

In view of (6.3), the above relation establishes (4.11).

E. Identity operator

It is easy to see that the operator/state correspondence defined by (6.5) is such that the zero colored graph Γ^0 corresponds to the identity operator in the CS Hilbert space:

$$\hat{\Gamma}^0 = \hat{I}. \tag{6.10}$$

Indeed, insertion of Γ^0 into $\tilde{M}$ is the same as $\tilde{M}$ with no insertion, whose RTW evaluation gives the identity operator in $\mathcal{H}^{\mathrm{CS}}$.

F. Matrix elements

We recall that a basis in $\mathcal{H}_X^{\mathrm{CS}}$ is obtained by choosing a pant decomposition of X, or, equivalently, choosing a trivalent graph Δ, with a coloring ϕ. The matrix elements $\langle \Delta^\phi | \hat{\Gamma}^\psi | \Delta^{\phi'} \rangle$ are obtained by the following procedure. Take a handlebody H with a graph Δ^ϕ in it, its loose ends

ending at the punctures. The boundary of H is X, so that we can glue H from the left to $\tilde{M}$. One similarly takes $-H$ with $\Delta^{\phi'}$ in it, and glues it to $\tilde{M}$ from the right. One connects the punctures on the boundary of H to those on the boundary of $-H$ by strands inside $\tilde{M}$. What one gets is a closed manifold of the topology $H \cup -H$, with closed graphs $\Delta^{\phi} \cup \Delta^{\phi'}$ and Γ^{ψ} sitting inside it. The matrix elements are obtained as the evaluation:

$$\langle \Delta^{\phi} | \hat{\Gamma}^{\psi} | \Delta^{\phi'} \rangle = I(H \cup -H, \Delta^{\phi} \cup \Delta^{\phi'}, \Gamma^{\psi}). \tag{6.11}$$

VII. CFT PARTITION FUNCTION AS A STATE

Here we interpret the CFT partition function (correlator) as a particular state in the Hilbert space of TV theory. We also compute components of this state in the basis of states given by spin networks.

A. CFT partition function

The partition function of any CFT holomorphically factorizes. To understand this holomorphic factorization, and the relation to the chiral TQFT, it is most instructive to consider the partition function as a function of an external connection. Namely, let CFT be the WZW model coupled to an external connection (gauged model), and consider its partition function $Z_X^{\mathrm{CFT}}[\mathbf{m}, \bar{\mathbf{m}}, \mathbf{z}, \bar{\mathbf{z}}, \underline{A}_z, \underline{A}_{\bar{z}}]$ on X. Note that no integration is carried over $\underline{A}$ yet. Thus, the above quantity is not what is usually called the gauged WZW partition function. The later is obtained by integrating over $\underline{A}$. The introduced partition function depends on the moduli (both holomorphic and antiholomorphic) $\mathbf{m}, \bar{\mathbf{m}}$, on positions of insertions of vertex operators coordinatized by $\mathbf{z}, \bar{\mathbf{z}}$, and on both the holomorphic and anti-holomorphic components of the connection $\underline{A}$ on X. The partition function holomorphically factorizes according to

$$Z_X^{\mathrm{CFT}}[\mathbf{m}, \bar{\mathbf{m}}, \mathbf{z}, \underline{z}, \underline{A}_z, \underline{A}_{\bar{z}}] = \sum_i \Psi_i[\mathbf{m}, \mathbf{z}, \underline{A}_z] \bar{\Psi}_i[\bar{\mathbf{m}}, \bar{\mathbf{z}}, \underline{A}_{\bar{z}}]. \tag{7.1}$$

Here $\Psi_i[\mathbf{m}, \mathbf{z}, \underline{A}_z]$ are the (holomorphic) conformal blocks with respect to the affine Lie algebra (in the case of WZW theories that we consider). The conformal blocks can be thought of as forming a basis in the Hilbert space $\mathcal{H}_X^{\mathrm{CS}}$ of CS theory on X. More precisely, there is a fiber bundle over the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of type (g,n) with fibers isomorphic to $\mathcal{H}_{X_{g,n}}^{\mathrm{CS}}$. The conformal blocks are (particular) sections of this bundle, see Ref. 31 for more detail. Note that the sum in (7.1) is finite as we consider a rational CFT. As was explained in Ref. 14, the usual CFT partition function is obtained by evaluating (7.1) on the "zero" connection. The formula (7.1) then gives the factorization of the usual partition function, with $\Psi_i[\mathbf{m}, \mathbf{z}, 0]$ being what is usually called the Virasoro conformal blocks.

Instead of evaluating (7.1) on the zero connection one can integrate over $\underline{A}$. The result is the partition function of the gauged model, which gives the dimension of the CS Hilbert space:

$$\dim \mathcal{H}_X^{\mathrm{CS}} = \frac{1}{\mathrm{Vol}\,\mathcal{G}} \int_{\mathcal{A}} \mathcal{D}\underline{A}\; Z_X^{\mathrm{CFT}}[\mathbf{m}, \bar{\mathbf{m}}, \mathbf{z}, \bar{\mathbf{z}}, \underline{A}_z, \underline{A}_{\bar{z}}]. \tag{7.2}$$

The value of the integral on the right-hand side is independent of moduli (or positions of insertion points).

A particular basis of states in $\mathcal{H}_X^{\mathrm{CS}}$ was described in Sec. II and is given by states $|\Delta^{\phi}\rangle$. Let us use these states in the holomorphic factorization formula (7.1). We can therefore think of the partition function (correlator) as an operator in the CS Hilbert space:

$$\hat{Z}_X^{\mathrm{CFT}} = \sum_{\phi} \left(\prod_{\mathrm{int}\, e} \dim_{\rho_e} \right) |\Delta^{\phi}\rangle \otimes \langle \Delta^{\phi}|. \tag{7.3}$$

The dimension of the CS Hilbert space is obtained by taking the CS trace of the above operator, which gives (5.1).

The CFT partition function (7.3) is the simplest possible modular invariant (the diagonal) that can be constructed out of the chiral CFT data. There are other possible modular invariants, and it is an ongoing effort to try to understand and classify different possibilities, see, e.g., the recent paper.[10] In this paper we only consider and give a TV interpretation of the simplest invariant (7.3). Our TV interpretation might prove useful also for the classification program, but we do not pursue this.

B. CFT partition function as a state

The formula (7.3) for the partition function, together with the operator/state correspondence of the previous section imply that Z_X^{CFT} can be interpreted as a particular state in the TV Hilbert space. We introduce a special notation for this state:

$$|Z_X^{\text{CFT}}\rangle \in \mathcal{H}_X^{\text{TV}}. \tag{7.4}$$

In order to characterize this state we first of all note that $\hat{Z}_X^{\text{CFT}}$ is just the identity operator in $\mathcal{H}_X^{\text{CS}}$:

$$\hat{Z}_X^{\text{CFT}} = \hat{I}. \tag{7.5}$$

The representation (7.3) gives the decomposition of the identity over a complete basis of states in $\mathcal{H}_X^{\text{CS}}$. Using (6.10) we see that the state $|Z_X^{\text{CFT}}\rangle$ is nothing else but the spin network state with zero coloring, together with a set of strands labeled with representations **R** and taking into account the punctures:

$$|Z_X^{\text{CFT}}\rangle = |\Gamma^0, \mathbf{R}\rangle. \tag{7.6}$$

Another thing that we are interested in is the components of $|Z_X^{\text{CFT}}\rangle$ in the basis of spin networks $|\Gamma^\psi\rangle$. In view of (4.11) we have

$$\langle \Gamma^\psi | Z_X^{\text{CFT}}\rangle = I(X \times S^1, \mathbf{R}, \Gamma^\psi). \tag{7.7}$$

The evaluation in $X \times S^1$ is taken in the presence of n links labeled by representations **R**. Note that all the dependence on the moduli of X is lost in (7.7). However, the coloring ψ of Γ can be thought of as specifying the "geometry" of X, see more on this below.

C. Zero colored punctures

Here, to motivate the general formula to be obtained below, we deduce an expression for (7.7) for the case where the colors at all punctures are zero. In this case there is no extra links to be inserted in $X \times S^1$, and (7.7) reduces to $\langle \Gamma^\psi | \Gamma^0 \rangle$. This can be evaluated using the prescription (4.10). One immediately obtains

$$\langle \Gamma^\psi | \Gamma^0 \rangle = \eta^{V_\Gamma}(\Gamma^\psi . \Omega C(\Gamma)) = \eta^{2-2g} \sum_{\{\rho_f\}} \prod_{f \in F_\Gamma} \dim_{\rho_f} \prod_{v \in V_\Gamma} (6j)_v. \tag{7.8}$$

Here $C(\Gamma)$ is the chain-mail for Γ, as defined in the formulation of the theorem (5.6). In the last formula the sum is taken over irreducible representations labeling the faces of the fat graph Γ, the product of $6j$-symbols is taken over all vertices of Γ, and the $6j$-symbols $(6j)_v$ are constructed out of three representations labeling the edges incident at v, and three representations labeling the faces adjacent at v. The last formula is obtained using the 3-fusion recoupling identity (A2).

D. Verlinde formula

The dimension of the CS Hilbert space can be obtained as the inner product of $|Z_X^{\text{CFT}}\rangle$ with the "vacuum" state $|\Gamma^0\rangle \in \mathcal{H}_X^{\text{TV}}$, which corresponds to the spin network with zero (trivial representation) coloring on all edges:

$$\dim \mathcal{H}_X^{\text{CS}} = \langle \Gamma^0 | Z_X^{\text{CFT}} \rangle. \tag{7.9}$$

Expression (7.9) gives an unusual perspective on the Verlinde formula: it appears as a particular case of a more general object (7.7).

E. General formula

Here we find the result of the evaluation (7.7). As we have just explained, (7.7) must reduce to the Verlinde formula (2.10) when the graph Γ has zero colors. We have seen in Sec. V that, at least for the case with no punctures, the Verlinde formula can be obtained from the chain-mail $C(\Gamma)$ with no graph Γ inserted. We have also seen in (7.8) that for the case with no punctures the quantity (7.7) is given by the evaluation of $C(\Gamma)$ together with the graph. Thus, a natural proposal for (7.7) is that it is given by the evaluation (5.6), with the graph Γ added, and with an additional set of curves taking into account the punctures. This results in:

Main Theorem: *The CFT partition function (correlator), interpreted as a state of TV theory, projected onto a spin network state is given by*

$$\langle \Gamma^\psi | Z_X^{\text{CFT}} \rangle = \eta^{2-2g-n} \sum_{\{\rho_f\}} \prod_{f \in F_\Gamma} S_{\rho_i \rho_{f_i}} \prod_{v \in V_\Gamma} (6j)_v. \tag{7.10}$$

A proof is given in the Appendix.

VIII. DISCUSSION

Thus, the CFT partition function (correlator) receives the interpretation of a state of TV theory. This state is the TV "vacuum" given in (7.6) by the graph with zero coloring. Thus, quite a nontrivial object from the point of view of the CFT, the partition function receives a rather simple interpretation in the TV theory.

We note that, apart from the partition function state $|Z_{\text{CFT}}\rangle$, there is another state in $\mathcal{H}^{\text{TV}}$ with a simple CS interpretation. This is the state that can be denoted as

$$|H\rangle \in \mathcal{H}^{\text{TV}}. \tag{8.1}$$

It arises as the TV partition function for a handlebody H. The TV invariant (4.5) for a manifold with boundary has the interpretation of the TV inner product of $|H\rangle$ with a spin network state:

$$\text{TV}(H, \Gamma^\psi) = \langle H | \Gamma^\psi \rangle. \tag{8.2}$$

In view of the Turaev theorem (4.12),

$$\langle H | \Gamma^\psi \rangle = I(H \cup -H, \Gamma^\psi). \tag{8.3}$$

From this, and the relation (6.11) for the matrix elements it can be seen that the state $|H\rangle$ corresponds in CS theory to the operator

$$\hat{H} = |\Delta^0\rangle \otimes \langle \Delta^0|, \tag{8.4}$$

which is just the projector on the CS "vacuum" state Δ^0, given by the zero colored pant decomposition graph Δ. We note that the TV state $|H\rangle$ has a rather nontrivial expression when decomposed into the spin network basis. Thus, the described relation between CS and TV theories (the operator/state correspondence) is a nontrivial duality in that simple objects on one side correspond

to nontrivial objects on the other: CFT correlators, nontrivial from the point of view of CS, are the TV "vacuum" states; the nontrivial TV handlebody state $|H\rangle$ is a rather trivial "vacuum" projector on the CS side.

We would like to emphasize that the CFT partition function state $|Z_X^{\text{CFT}}\rangle$ does not coincide with the TV partition function state $|H\rangle$ on a handlebody H. However, the state $|Z_X^{\text{CFT}}\rangle$ can be interpreted as a certain sum of TV handlebody states, provided a certain generalization of the Turaev theorem (4.12) holds. An attempt to prove this generalized Turaev theorem would lead us too far, so we shall formulate it as a conjecture:

Conjecture 1: Let $\text{TV}((H,\Delta^\phi),\Gamma^\psi)$ *denote the Turaev–Viro invariant for a manifold H with a spin network* Δ^ϕ *inserted in it, and a spin network* Γ^ψ *on the boundary. This invariant can be evaluated as the RTW invariant for the double* $H\cup -H$:

$$\text{TV}((H,\Delta^\phi),\Gamma^\psi)=\text{I}((H,\Delta^\phi)\cup -(H,\Delta^\phi),\Gamma^\psi). \tag{8.5}$$

The justification for this conjecture is the theorem (4.12), which is nothing but (8.5) with no graph Δ insertion. Let us now introduce a set of states $|H,\phi\rangle$ obtained as the TV partition function inside a handlebody H with a spin network Δ^ϕ inserted inside. From the point of view of CS theory these states are just the projection operators:

$$\hat{H}^\phi=|\Delta^\phi\rangle\langle\Delta^\phi|. \tag{8.6}$$

Indeed, let us compute the TV inner product between the state $|H,\Delta^\phi\rangle$ and a spin network state $|\Gamma^\psi\rangle$:

$$\langle H,\phi|\Gamma^\psi\rangle=\text{TV}((H,\Delta^\phi),\Gamma^\psi). \tag{8.7}$$

If the above conjecture holds this equals to the right hand side of (8.5). The later, on the other hand, is equal to

$$\langle\Delta^\phi|\hat{\Gamma}^\psi|\Delta^\phi\rangle. \tag{8.8}$$

This implies (8.6). Using relation (8.6) the CFT partition function as a state in TV Hilbert space can be represented as a sum of HH-type TV states obtained as the TV partition function over a handlebody with spin network insertion:

$$|Z_{\text{CFT}}\rangle=\sum_\phi \left(\prod_{\text{int } e} \dim_{\rho_e}\right)|H,\phi\rangle. \tag{8.9}$$

This formula should be contrasted with the usual AdS/CFT prescription, which states that the CFT partition function on the boundary is obtained as a sum of gravity partition functions on all manifolds H that have X as a boundary. In other words, one must sum over all ways to fill in the surface X so that the resulting 3-manifold H is non-singular inside. As we see, the "holography" arising in our context is different. Instead of taking a sum over all nonsingular manifolds H that have X as the boundary, in (8.9) one takes some fixed H, but sums over all labelings of the graph Δ sitting inside H. The graph Δ is a 1-skeleton of H; it can be thought of as a singularity inside the handlebody. Thus, to obtain the CFT partition function (modular invariant) one sums over labelings of the singularity inside H. This is a finite sum. The sum over different ways to fill in X is, on the other hand, infinite. It would be of interest to find if there is any relation between these two sums. If no such relation exists then the holographic prescription (8.9) predicted by our analysis is different from the AdS/CFT one.

Thus, we have seen that there are two TV states that correspond to CFT modular invariants: one is the TV vacuum (7.6) that gives the diagonal modular invariant, the other is the handlebody state $|H\rangle$ that gives the trivial modular invariant (8.4). An interesting question is what other states

in TV give CFT modular invariants. An answer to this question may be instrumental in understanding the structure of rational CFT's, see the recent paper[10] for a discussion along these lines.

Let us now discuss a physical interpretation of the formula (7.10). We note that the object (7.7) can be interpreted as the CFT partition function on a surface X whose "geometry" is specified by the state $|\Gamma^\psi\rangle$. This "geometry" should not be confused with the conformal geometry of X, on which the usual CFT partition function depends. Once the state $|Z_X^{CFT}\rangle$ is projected onto $|\Gamma^\psi\rangle$ the dependence on the moduli of X is traded for the dependence on the coloring ψ of Γ. All the dependence on the moduli is encoded in the spin network states. Let us first discuss the dependence on the "geometry" as specified by the colored graph Γ^ψ, and then make comments as to the dependence of $|\Gamma^\psi\rangle$ on the moduli.

To understand the spin network Γ^ψ as specifying the "geometry" of X we recall, see Sec. III, that $|\Gamma^\psi\rangle$ are eigenstates of the "momentum" operators $\mathbf{e} \sim \partial/\partial\mathbf{w}$. In this sense they are states of particular configuration of the $\mathbf{e}$ field on the boundary. To understand this in more detail let us consider the TV partition function $TV(H,\Gamma^\psi)$. Let us take the simple example of the 4-punctured sphere. Thus, we take $H = B^3$, a 3-ball. We will put all representations at the punctures to be trivial. In view of the Turaev theorem (4.12), $TV(B^3,\Gamma^\psi) = I(S^3,\Gamma^\psi)$. Thus, for $X = S^2$, the TV invariant is given simply by the evaluation of the spin network Γ^ψ in S^3. In our simple example of the 4-punctures sphere this evaluation is a single $6j$-symbol. Let us now restrict ourselves to the case $G = SU(2)$. As we have mentioned above, the TV theory in this case is nothing else but 3D gravity with positive cosmological constant. On the other hand, it is known that the quantum $(6j)$-symbol has, for large k and large spins, an asymptotic of the exponential of the classical Einstein–Hilbert action evaluated inside the tetrahedron:

$$(6j) \sim e^{iS_{TV}[\text{tet}]} + \text{c.c.} \tag{8.10}$$

This fact was first observed[32] by Ponzano and Regge for the classical $(6j)$-symbol. In that case one evaluates the classical gravity action inside a flat tetrahedron. The action reduces to a boundary term (the usual integral of the trace of the extrinsic curvature term), which for a tetrahedron is given by the so-called Regge action:

$$S_{TV}[\text{tet},\Lambda=0] \sim \sum_e l_e \theta_e, \tag{8.11}$$

where the sum is taken over the edges of the tetrahedron, and l_e, θ_e are the edge length and the dihedral angle at the edge correspondingly. Dihedral angles are fixed once all the edge length are specified. Ponzano and Regge observed that the $(6j)$-symbol has the asymptotic of (8.10) with the action given by (8.11) if spins labeling the edges are interpreted as the length of edges. A similar (8.10) interpretation is true for the $SU_q(2)$ $(6j)$-symbol, as was shown in Ref. 33. The gravity action in this case is that with a positive cosmological constant $\Lambda = (k/2\pi)^2$, and is evaluated in the interior of tetrahedron in S^3 whose edge length are given by spins. To summarize, in these examples the $(6j)$-symbol gets the interpretation of the exponential of the classical gravity action evaluated inside a tetrahedron embedded in either $\mathbb{R}^3$ or S^3, depending on whether one takes the classical limit $k \to \infty$ or considers a quantum group with finite k. The tetrahedron itself is fixed once all edge length are specified. The edge length are essentially given by the spins. We also note that the graph Γ in this example is the dual graph to the triangulated boundary of the tetrahedron in question.

Thus, the TV partition function (given by a single $(6j)$-symbol) inside a 4-punctured sphere (tetrahedron) has the interpretation of the gravity partition function inside the tetrahedron with its boundary geometry (edge length) fixed by the spins. This interpretation of Γ^ψ is valid also for other surfaces. One should think of Γ^ψ as specifying the geometry $\mathbf{e}$ on X. The TV invariant is, in the semi-classical limit of large representations, dominated by the exponential of the classical action evaluated inside the handlebody. The geometry inside is completely determined by the

geometry of the surface, in other words, the spins. The interpretation is valid not only for SU(2), but also for other groups. In such a general case the notion of "geometry" is more complicated, as described by the field **e** and the TV action (3.2).

The bottom line is that the TV spin network states $|\Gamma^\psi\rangle$ should be thought of as specifying the "geometry" of X. The quantity (7.10) then receives the interpretation of the CFT partition function on a surface X whose "geometry" is specified by Γ^ψ.

The other question is how the states $|\Gamma^\psi\rangle$ depend on the moduli of the surface. The fact that the graph Γ is the same as the one used in the Penner[18] coordinatization of the moduli space suggests that this dependence may be not very complicated. In fact, we believe that for the groups $SL(2,\mathbb{R})$ or $SL(2,\mathbb{C})$ that are relevant in the description of the moduli spaces, the dependence is rather simple: the described above "geometry" in this case must coincide with the usual conformal geometry of the surface. An argument for this is as follows. In the Penner coordinatization of the moduli space, or in any of its versions[34,35] the moduli are given by prescribing a set of real numbers: one for each edge of the graph Γ. The numbers specify how two ideal triangles are glued together across the edge, see Refs. 34, 35 for more detail. For the case when $G = SL_q(2,\mathbb{R})$, as is relevant for, e.g., Liouville theory, see Ref. 36, the representations are also labeled by a single real number. We believe that the Penner coordinates and the representations that label the edges are simply dual to each other, in the sense of duality between the conjugacy classes of elements in the group and its irreducible representations. A similar proposal for the relation between the SL(2) spin and length was made in Ref. 6. Thus, there is some hope that the dependence $|\Gamma^\psi\rangle$ on the moduli can be understood rather explicitly, at least for some groups. Having this said we note that considerations of the present paper do not immediately generalize to the case of noncompact groups, relevant for the description of the moduli spaces. It is an outstanding problem to develop a noncompact analog of the Verlinde formula, not speaking of the formula (7.10). Thus, at this stage of the development of the subject considerations of this paragraph remain mere guesses. However, progress along these lines may be instrumental in developing a better technique for integrating over the moduli spaces, and thus, eventually, for a better understanding of the structure of string theory.

ACKNOWLEDGMENTS

APPENDIX A: SOME RECOUPLING IDENTITIES

The 2-fusion identity:

$$(A1)$$

The 3-fusion identity:

$$(A2)$$

The 3-vertex is normalized so that

$$\left(\begin{array}{c} \text{i} \quad \text{j} \quad \text{k} \end{array}\right) = N_{ijk},$$ (A3)

where N_{ijk} is the multiplicity with which the trivial representation appears in the tensor product of i,j,k. For SU(2) this is either zero or one. In order to obtain (A1) from (A2) it is necessary to take into account the normalization (A3).

Another recoupling identity uses the modular S-matrix:

$$\left(\begin{array}{c} \text{i} \quad \text{l} \end{array}\right) \bullet\bullet\bullet = \frac{S_{il}}{\eta\,\dim_i \dim_l} \left(\begin{array}{c} \text{i} \quad \text{l} \end{array}\right) \bullet\bullet\bullet$$ (A4)

The dots on the right-hand side mean that the open ends can be connected (in an arbitrary way) to a larger graph.

APPENDIX B: PROOF

Here we give a proof of the main theorem.

1. Genus zero case

We start by working out the simplest case of the 3-punctured sphere. We choose Γ to be given by a dumbbell. We thus need to compute the following evaluation:

(B1)

Here we have used the observation (5.4) to replace two trivalent vertices of $\Delta \cup \Delta$ by a link with Ω inserted. Let us now slide the curve along which Ω is inserted to go all around the graph Γ, thus making one of the curves of the chain-mail $C(\Gamma)$. In the next step we add two more curves from $C(\Gamma)$ that go around punctures, and at the same time add two meridian curves with Ω inserted. This addition of two pairs of Ω linked does not change the evaluation in view of the killing property of Ω. The steps of sliding the Ω and adding two new pairs of curves is shown here:

(B2)

The last step is to use the sliding property of Ω to slide the links labeled i,j inside Γ:

$$\eta \qquad k \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{B3})$$
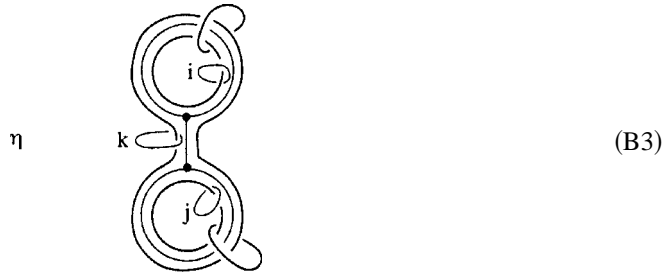
One can now use the recoupling identity (A4) to remove the curves i,j,k at the expense of introducing a factor of $\eta^{-1}S_{ii'}/\dim_{i'}$ and similarly for other loops. Here i' is the representation on the loop from $C(\Gamma)$ going around the puncture i. The element Ω on that loop must be expanded to (4.1). The factor $\eta\dim_{i'}$ from that expansion is canceling the factor we got when removing the loop i. What is left is the S-matrix element $S_{ii'}$, with no extra factors. One can now use the 3-fusion identity (A2) to get the formula (7.10). One uses the 3-fusion 2 times, which produces η^{-2}. This combines with the factor of η in (B1)–(B3) to give η^{-1}, as prescribed by (7.10) for the case $g=0,n=3$. One can easily extend this proof to the case $g=0$ arbitrary number of punctures. To understand the general case, we first find a surgery representation for $X\times S^1$.

2. Surgery representation for $X\times S^1$

Let us first understand the genus one case. A surgery representation for $X_{1,1}\times S^1$ is given by the following link:


$$(\text{B4})$$

One must insert the element Ω into all components, and evaluate in S^3. Representing all the Ω's as the sum (4.1) and using the recoupling identity (A4) it is easy to show that (B4) gives the correct expression $\eta I(L)=\Sigma_{\rho'}S_{\rho\rho'}/S_{0\rho'}$ for the dimension. The same surgery representation was noticed in Ref. 37. The generalization to higher genus and to a larger number of punctures is straightforward. It is given by the following link:


$$(\text{B5})$$
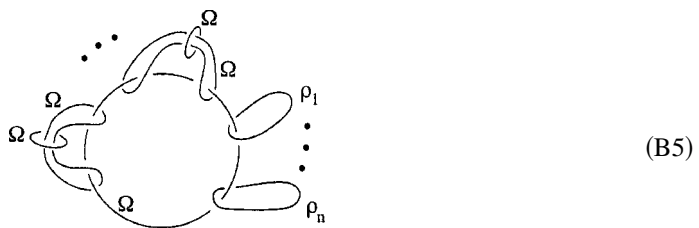
3. General case

We will work out only the (1,1) case. General case is treated similarly. We first note that the formula (7.10) for the (1,1) case can be obtained as the result of the following evaluation:

$$(B6)$$
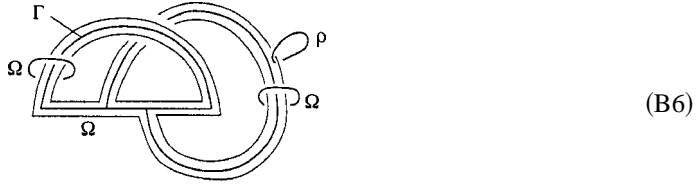
This link is to be evaluated in S^3 and, as usual, the result multiplied by the factor of η. This gives (7.10) specialized to the case (1,1). It is now a matter of patience to verify that by the isotropy moves in S^3 the above link can be brought to the form:

$$(B7)$$
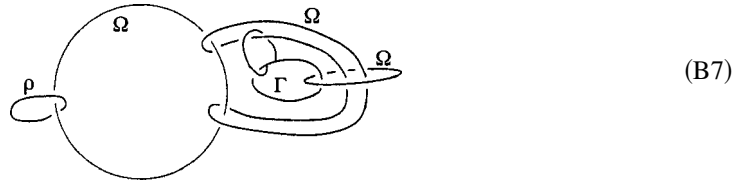
This is the correct surgery representation for $X_{1,1} \times S^1$ with the graph Γ inside. Thus, (7.10) indeed gives the evaluation $I(X \times S^1, \Gamma^\psi)$, which, in view of (7.7), proves the theorem.

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