# Towards More Practical Linear Programming-based Techniques for Algorithmic Mechanism Design 

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#### Abstract

R. Lavy and C. Swamy (FOCS 2005, J. ACM 2011) introduced a general method for obtaining truthful-in-expectation mechanisms from linear programming based approximation algorithms. Due to the use of the Ellipsoid method, a direct implementation of the method is unlikely to be efficient in practice. We propose to use the much simpler and usually faster multiplicative weights update method instead. The simplification comes at the cost of slightly weaker approximation and truthfulness guarantees.


## 1 Introduction

Algorithmic mechanism design studies optimization problems in which part of the input is not directly available to the algorithm; instead, this data is collected from self-interested agents who can manipulate the algorithm by mis-reporting their parts of the input, if that would improve their own objective functions. Algorithmic mechanism design quests for polynomial-time algorithms that (approximately) optimize a global objective function (usually called social welfare), subject to the strategic requirement that, with the assumption that agents have a prefect knowledge of the algorithm, their best strategy is to truthfully report their part of the input. Such algorithms are called truthful mechanisms. The celebrated VCG mechanism (see, e.g., [18]) achieves truthfulness and social-welfare optimization, but is, in general, not computationally efficient, since it may require solving an NP-hard optimization problem.

It is clearly desirable to have general constructions (frequently called blackbox reductions) of the form: if an optimization problem admits an $\alpha$-approximation algorithm, then it admits an $\alpha^{\prime}$-approximation truthful mechanism, where $\alpha^{\prime}=$ $f(\alpha)$. We concentrate here on the black-box reduction of Lavi and Swamy ([16]17) that turns a linear programming based approximation algorithm for the social welfare problem into a randomized mechanism that is truthful in expectation, i.e., reporting the truth maximizes the expected utility of an agent, and mention other work on black-box reductions only in passing [5]2]8. The LS-reduction is powerful (see $16|17| 6 \mid 3]$ for applications), but unlikely to be efficient in practice because of its use of the Ellipsoid method. We show how
to use the multiplicative weigths update method instead. This results in simpler algorithms at the cost of somewhat weaker approximation and truthfulness guarantees. We consider our paper an engineering paper.

We next give a high-level review of the LS-reduction. It is useful to read the following lines with a concrete example in mind ${ }_{3}^{3}$ The LS-reduction assumes access to an $\alpha$-approximation algorithm for maximizing the social welfare over integral points $x$ in a packing polytope $\mathcal{Q}$ and consists of three main steps:

1. Solve the LP-relaxation, which is obtained by dropping the integrality constraints on $x$; call the obtained optimum (fractional) solution $x^{*}$;
2. decompose $\alpha \cdot x^{*}$ as a convex combination of integral solutions in $\mathcal{Q}$;
3. given the decomposition in step 2, pick an integral solution with probability equal to the corresponding convex combination multiplier; coupled with VCG-type payments, this guarantees truthfulness-in-expectation and an approximation factor of $\alpha$.
With respect to practical applicability, steps 1 and 2 are the two major bottlenecks. Step 1 requires solving a linear program; an exact solution requires the use of the Ellipsoid method (see e.g. [12]) if the dimension is exponential. Furthermore, the only method known to perform the decomposition in Step 2 is through the Ellipsoid method.

Over the past 15 years, simple and fast methods 4|11|10|14|15|19|20 have been developed for solving packing linear (or even convex) programs within an arbitrarily small error guarantee $\epsilon$. These methods are based on the multiplicative weights update (MWU) method [1], in which a very simple update rule is repeatedly performed until a near-optimal solution is obtained. A natural question to ask is whether these methods can be used in the LS-reduction instead of the Ellipsoid method. There are two technical hurdles.

First, only an exact solution to the LP guarantees truthfulness of the resulting mechanism. We resort to ( $1-\epsilon$ )-truthfulness-in-expectation 9]: each player maximizes her expected utility within a factor of $(1-\epsilon)$ by reporting her valuation truthfully. A black-box reduction was given in [9] to obtain an $\left(1-\epsilon_{0}\right)$ truthfulness in expectation mechanism from a randomized $(1-\epsilon)$-approximation algorithm, where $n$ is the number of players and $\epsilon:=\Theta\left(\epsilon_{0} / n^{9}\right)$. We use similar ideas but improve this bound to $\epsilon:=\Theta\left(\epsilon_{0}^{5} / n^{4}\right)$.

Second, it is not known how to use these techniques to get an exact decomposition of the fractional solution obtained in step 1. We show that a (slight)
${ }^{3}$ In the combinatorial auction problem, there is a set of items to be sold to a set of agents. The (reported) value of a set $S$ of items to the $i$-th agent is $v_{i}(S)$ with $v_{i}(\emptyset)=0$ and $v_{i}(S) \leq v_{i}(T)$ whenever $S \subseteq T$. Let $x_{i, S}$ be a $0-1$ variable indicating that set $S$ is given to agent $i$. Then $\sum_{S} x_{i, S} \leq 1$ states that at most one set can be given to $i$, and $\sum_{i} \sum_{S ; j \in S} x_{i, S} \leq 1$ for every item $j$ states that any item can be given away only once. The social welfare is $\sum_{i, S} v_{i}(S) x_{i, S}$. For the moment, we dodge the issue of how the agents reveal their valuations in a compact form, i.e., polynomially in the number of items. An important special case are single-minded agents, i.e., for every $i$, there is a set $S_{i}$ such that $v_{i}(T)=v\left(S_{i}\right)$ if $S_{i} \subseteq T$ and $v_{i}(T)=0$ otherwise. Replacing the contraint $x_{i, S} \in\{0,1\}$ by $0 \leq x_{i, S} \leq 1$, results in an LP of the packing type. Let $\mathcal{Q}$ be the feasible region of the LP.
variation of the approach by Garg and Könemann [10 can be used to obtain a convex combination that dominates $\alpha \cdot x^{*}$. Then we use the packing property of the polytope to convert this to an exact equality.

Theorem 1 (Informal). Consider a combinatorial optimization problem whose $L P$ relaxation can be described by a packing polytope on $d$ variables, $m$ constraints, and n players, and given by a demand oracle. Assume that there is an $\alpha$-approximation algorithm for the problem that outputs a solution within a factor of $\alpha$ times the optimum fractional LP solution. Then, for any $\epsilon \in(0,1)$, there is a $(1-O(\epsilon))$-truthful-in-expectation mechanism $\mathcal{M}$ achieving a social welfare within a factor of $(1-O(\epsilon)) \alpha$ of the optimum. The mechanism makes $\tilde{O}\left(\frac{n^{9} m}{\epsilon^{10}}\right)$ calls to the oracle and $\tilde{O}\left(\frac{m}{\epsilon^{2}}\right)$ calls to the approximation algorithm.

Two remarks are in order. First, while the running time of the mechanism in Theorem 1 is high in terms of the number of players, it is almost linear in terms of the number constraints. Second, the factor $n^{9}$ is based on the assumption that the currently best algorithm for approximating, within a factor of $1-\epsilon_{0}$, the fractional packing LP problem (given by a demand oracle), runs in time $O\left(\frac{m \log m}{\epsilon^{2}}\right)$. As mentioned above, our mechanism requires setting $\epsilon$ to $\Theta\left(\frac{\epsilon_{0}^{5}}{n^{4}}\right)$. Thus, any improvement on the efficiency of the fractional packing algorithm will translate to an improvement on the running time given in Theorem 1

It is worth mentioning that MWU methods have been used recently for auction design in the bayesian settings, yielding optimal mechanisms for multi-unit auctions and some of their extensions 3.

## 2 Notation and preliminaries

There are $n$ players. $\Omega \subseteq \mathbb{R}_{\geq 0}^{d}$ is the set of outcomes; it contains $\mathbf{0}$. We have $d=d_{1}+\ldots+d_{n}$ and hence any outcome $x$ can be written as $x=\left(x_{1}, \ldots, x_{n}\right)$ with $x_{i} \in \mathbb{R}^{d_{i}}$. The valuations $v_{i}$ of player $i$ are linear, monotone, and homogenous functions depending only on $x_{i}$, i.e., $v_{i}(x)=V_{i}^{T} x_{i}$, where $V_{i} \in \mathbb{R}_{\geq 0}^{d_{i}}$ is a nonnegative vector. We use $\mathcal{V}_{i}$ to denote the possible valuations for $i$ and use $\mathcal{V}:=$ $\mathcal{V}_{1} \times \ldots \times \mathcal{V}_{n}$. Let $v(x):=\sum_{i} v_{i}(x)=\sum_{i} V_{i}^{T} x_{i}=V^{T} x$, where $V=\left(V_{1}, \ldots, V_{n}\right) \in$ $R^{d}$. The social welfare maximization problem is to find $z_{\Omega}^{*}(v)=\max _{x \in \Omega} v(x)$.

For a vector $x=\left(x_{1}, \ldots, x_{k}\right)$ and $i \in[k]$ we use $x_{-i}$ to denote $\left(x_{1}, \ldots, x_{i-1}\right.$, $\left.x_{i+1}, \ldots, x_{k}\right)$. A randomized mechanism $\mathcal{M}=(\mathcal{A}, \mathcal{P})$ for $(\Omega, \mathcal{V})$ is defined by an allocation rule $\mathcal{A}: \mathcal{V} \rightarrow \mathcal{D}(\Omega)$ and a payment rule $\mathcal{P}: \mathcal{V} \rightarrow \mathcal{D}\left(\mathbb{R}_{\geq 0}^{n}\right)$, where $\mathcal{D}(\mathcal{S})$ denotes the set of probability distributions over set $\mathcal{S}$. The utility of player $i$, under the mechanism, when it receives the vector of bids $v:=\left(v_{1} \ldots, v_{n}\right) \in \mathcal{V}$, is the random variable $U_{i}(v)=\bar{v}_{i}(x(v))-p_{i}(v)$, where $x(v) \sim \mathcal{A}(v)$, and $p(v)=$ $\left(p_{1}(v), \ldots, p_{n}(v)\right) \sim \mathcal{P}(v)$; here $\bar{v}_{i}$ denotes the true valuation of player $i$.

For $\beta \geq 0$, a randomized mechanism (for $(\Omega, \mathcal{V})$ ) is said to be $\beta$-absolutely (resp., relatively) truthful in expectation, denoted by $\beta$-abs-TIE (resp., $\beta$-rel-TIE) if for all $i$ and all $\bar{v}_{i}, v_{i} \in \mathcal{V}_{i}$, and $v_{-i} \in \mathcal{V}_{-i}$, it guarantees that $\mathbb{E}\left[U_{i}\left(\bar{v}_{i}, v_{-i}\right)\right] \geq$ $\mathbb{E}\left[U_{i}\left(v_{i}, v_{-i}\right)\right]-\beta$ (resp., $\left.\mathbb{E}\left[U_{i}\left(\bar{v}_{i}, v_{-i}\right)\right] \geq \beta \cdot \mathbb{E}\left[U_{i}\left(v_{i}, v_{-i}\right)\right]\right)$, when the true and
reported valuations of player $i$ are $\bar{v}_{i}$ and $v_{i}$, respectively, and where the expectation is taken over the random choices made by the mechanism: $\mathbb{E}\left[U_{i}(v)\right]=$ $\mathbb{E}_{x, p \sim \mathcal{M}(v)}\left[U_{i}(v)\right]:=\mathbb{E}_{x \sim \mathcal{A}(v)}\left[\bar{v}_{i}(x)\right]-\mathbb{E}_{p \sim \mathcal{P}(v)}\left[p_{i}\right]$. In words: the expected utility of any player is maximized within an absolute (resp., relative) error of $\beta$, when he/she reports the true valuation. We will drop $\beta$ when it is 0 (resp., when it is 1). The mechanism is individually rational (with probability $q$ ) if the utility of a truth-telling bidder under the mechanism is non-negative (with probability at least $q$ ). The mechanism is said to have no positive transfer if all payments are non-negative.

For a constant $\alpha \in(0,1]$, a randomized mechanism $(\mathcal{A}, \mathcal{P})$ is said to be $\alpha$ socially efficient if for any $v \in \mathcal{V}$ it guarantees that the expected social welfare obtained under the mechanism is at least an $\alpha$-fraction of the optimum social welfare, i.e., $\mathbb{E}_{x \sim \mathcal{A}(v)}[v(x)] \geq \alpha \cdot z_{\Omega}^{*}(v)$. An $\alpha$-socially efficient $\beta$-abs-TIE (resp., $\beta$-rel-TIE) will be called $(\alpha, \beta)$-abs-TIE (resp., $(\alpha, \beta)$-rel-TIE). When the specific values of $\alpha$ and $\beta$ are irrelevant, we talk about an almost-TIE.

Let $\mathcal{R} \subseteq \mathcal{D}(\Omega)$ be a compact subset of probability distributions over $\Omega$. For $\beta \geq 0$, a randomized allocation rule $\mathcal{A}: \mathcal{V} \rightarrow \mathcal{R}$ (for $(\Omega, \mathcal{V})$ ) is said to be $\beta$-absolutely (resp., $\beta$-relatively) maximal in distribution range [78], denoted $\beta$-abs-MIDR (resp., $\beta$-rel-MIDR) with respect to $\mathcal{R}$, if it guarantees that for all $v \in \mathcal{V}, \mathbb{E}_{x \sim \mathcal{A}(v)}[v(x)] \geq \max _{\mathcal{A}^{\prime} \in \mathcal{R}} \mathbb{E}_{x \sim \mathcal{A}^{\prime}}[v(x)]-\beta$ (resp., $\mathbb{E}_{x \sim \mathcal{A}(v)}[v(x)] \geq$ $\left.\beta \cdot \max _{\mathcal{A}^{\prime} \in \mathcal{R}} \mathbb{E}_{x \sim \mathcal{A}^{\prime}}[v(x)]\right)$, i.e., given the vector of reported valuations $v$, the mechanism always outputs a distribution in $\mathcal{R}$ maximizing, within an absolute (resp., relative) error of $\beta$, the expected social welfare, among all other distributions in $\mathcal{R}$. A randomized mechanism $(\mathcal{A}, \mathcal{P})$ is said to be $\beta$-abs-MIDR (resp., $\beta$-rel-MIDR) if its allocation rule $\mathcal{A}$ is $\beta$-abs-MIDR (resp., $\beta$-rel-MIDR). We will drop $\beta$ when it is 0 (resp., 1 ). An allocation rule is an $(\alpha, \beta)$-abs-MIDR (resp., $(\alpha, \beta)$-rel-MIDR) w.r.t. range $\mathcal{R}$, if it is $\alpha$-socially efficient, $\beta$-abs-MIDR (resp., $\beta$-rel-MIDR).

We are mainly interested in the case that the set of outcomes $\Omega$ is equal to the feasible (integral) solutions of a packing polytope

$$
\begin{equation*}
\mathcal{Q}:=\left\{x \in \mathbb{R}^{d} \mid A x \leq b, \quad \mathbf{0} \leq x \leq u\right\} \tag{1}
\end{equation*}
$$

where $A \in \mathbb{R}_{\geq 0}^{m \times d}, b \in \mathbb{R}_{>0}^{m}$, and $u \in \mathbb{R}_{>0}^{d}$. Packing polytopes have the packing property: $x \in \mathcal{Q}$ and $y \leq x$ implies $y \in \mathcal{Q}$. We denote by $\mathcal{Q}_{\mathcal{I}}:=\mathcal{Q} \cap \mathbb{Z}_{\geq 0}^{d}$ the set of integral points in $\mathcal{Q}$. When the dimension $d$ is exponential, we assume that the polytope $\mathcal{Q}$ is given by a (demand) oracle, which for any given $\lambda \in \mathbb{R}_{\geq 0}^{m}$, finds a column $A^{j}$ of $A$ that maximizes $V_{j}-\sum_{i} \lambda_{i} A_{i j} / b_{i}$. We denote by len $(\mathcal{Q})$ the size of the input description of (the oracle of) $\mathcal{Q}$.

For $\epsilon \in(0,1]$, we say that an algorithm $\mathcal{F}$ is a fully polynomial-time approximation scheme (FPTAS) for $\mathcal{Q}$ if, for any $V \in \mathbb{R}^{d}$, it returns in polynomial time an $x^{*} \in \mathcal{Q}$ such that $V^{T} x^{*} \geq(1-\epsilon) \max _{x \in \mathcal{Q}} V^{T} x$. We denote by $T_{\mathcal{F}}(\operatorname{len}(\mathcal{Q}), \epsilon)$

[^0]the running time of $\mathcal{F}$ on input $V$ (we assume it does not depend on $V$ ). A number of such FPTAS's exist in the literature (see, e.g., 4|11|10|14|15|19|20). We are interested mostly in FPTAS's that allow the use of an oracle to overcome the difficulty of dealing explicitly with an exponential dimension. Most of the currently known techniques (if not all) with this property have a quadratic dependence on $\frac{1}{\epsilon}$ (see, e.g., the Garg-Könemann routine [10] in Theorem ?? in Appendix D.

For $\alpha \in(0,1]$, we say that an algorithm $\mathcal{F}^{\prime}$ is an $\alpha$-integrality-gap-verifier for $\mathcal{Q}_{\mathcal{I}}$ if, for any $V \in \mathbb{R}^{d}$ and any $x^{*} \in \mathcal{Q}$, it returns in polynomial time an $x \in \mathcal{Q}_{\mathcal{I}}$ such that $V^{T} x \geq \alpha \cdot V^{T} x^{*}$. In other words, the algorithm returns an integral solution in $\mathcal{Q}$ whose objective value is at least $\alpha$ times the optimal fractional objective value. We denote by $T_{\mathcal{F}^{\prime}}(\operatorname{len}(\mathcal{Q}), s)$ the running time of $\mathcal{F}^{\prime}$ on input $\left(V, x^{*}\right)$, where $\operatorname{len}(\mathcal{Q})$ denotes the size of the input description of $\mathcal{Q}$ and $s$ is the size of the support of $x^{*}$.

Theorem 2. Consider a packing polytope $\mathcal{Q} \subseteq \mathbb{R}^{d}$, described as in (1) and admitting an FPTAS $\mathcal{F}$ returning a solution with support $s$ when asked for a $(1-\epsilon)$-optimal solution, and an $\alpha$-integrality-gap-verifier $\mathcal{F}^{\prime}$ for $\mathcal{Q}_{I}$. Then, for any $\epsilon_{0} \in\left(0, \frac{1}{2}\right]$, there is a $\left(\left(1-5 \epsilon_{0}\right) \alpha, 1-\epsilon_{0}\right)$-rel-TIE $\mathcal{M}$ for $\left(\mathcal{Q}_{\mathcal{I}}, \mathcal{V}\right)$ with running time $T_{\mathcal{M}}=\tilde{O}\left(n \cdot T_{\mathcal{F}}(\operatorname{len}(\mathcal{Q}), \epsilon)+\epsilon_{0}^{-2} s\left(T_{\mathcal{F}^{\prime}}(\operatorname{len}(\mathcal{Q}), s)+s\right)\right)$. Here, $\epsilon=\Theta\left(\epsilon_{0}^{5} / n^{4}\right)$ is defined as in Proposition 图.

Proof. By Corollary 2, there is a $\left(\left(1-\epsilon_{0}\right)(1-\epsilon), 1-\epsilon_{0}\right)$-rel-TIE $\mathcal{M}$ for $(\mathcal{Q}, \mathcal{V})$ with running time $T_{\mathcal{M}}=O\left(n \cdot T_{\mathcal{F}}(\operatorname{len}(\mathcal{Q}), \epsilon)\right)$. Let $(\hat{x}, \hat{p})$ be the allocation and payments output by $\mathcal{M}$. We then use $\mathcal{F}^{\prime}$ and Theorem 3 to obtain a convex representation of $\alpha \hat{x} /\left(1+3 \epsilon_{0}\right)$ in terms of integral solutions in $\mathcal{Q}_{\mathcal{I}}$. By Proposition 3 , we now have a a $\left(\left(1-\epsilon_{0}\right) \alpha(1-\epsilon) /\left(1+3 \epsilon_{0}\right), 1-\epsilon_{0}\right)$-rel TIE for $\left(\mathcal{Q}_{\mathcal{I}}, \mathcal{V}\right)$.

We turn to the running time. Depending on the random choice of the player's index $i$ in the first step of Algorithm $1, \mathcal{M}$ uses the FPTAS $\mathcal{F}$ either to compute a $(1-\epsilon)$-optimal solution $\hat{x}=x^{*} \in \mathcal{Q}$ for the fractional social welfare problem if $i=0$, or to compute a point $\hat{x}=u^{i} \in \mathcal{Q}$ maximizing within $(1-\epsilon)$ the fractional social welfare for only player $i \geq 1$. In either case, $\hat{x}$ has support of size $s$ and it takes time $T_{\mathcal{F}}(\operatorname{len}(\mathcal{Q}), \epsilon)$ to find it. Finally, in step 3 of Algorithm 1 we need to compute $\beta$ as defined in (6) and then the payments according to (2) and (3) of Proposition 1 This takes time $O(n)$ times $T_{\mathcal{F}}(\operatorname{len}(\mathcal{Q}), \epsilon)$. The computation of the convex decomposition also takes time $\epsilon_{0}^{-2} s \log s\left(T_{\mathcal{F}^{\prime}}(\operatorname{len}(\mathcal{Q}), s)+s \log s\right)$.

We remark that in Theorem 2 we can assume the value of $s=\epsilon^{-2} m \log m$ (see [10] and Theorem6in Appendix (D). We may also use the Simplex algorithm ${ }^{5}$

[^1]to reduce the size of the support to $s=m$, but this takes additional $O\left(m s^{2}\right)$ time.

## 3 Approximately truthful mechanisms

In this section we will study mechanisms that combine a $\beta$-abs-MIDR allocation rule and a randomized payment rule inspired by the VCG payment rule. We construct mechanisms that have no positive transfer, are individually rational, and $3 \beta$-truthful in expectation (Corollary 1). A VCG mechanism chooses an allocation $x^{*} \in \arg \max _{x \in \Omega} \sum_{1 \leq i \leq n} v_{i}(x)$ maximizing social welfare and charges the $j$-th player the price $\max _{x \in \Omega} \sum_{1 \leq i \leq n, i \neq j}\left(v_{i}(x)-v_{i}\left(x^{*}\right)\right)$, i.e., the maximum total value obtainable for the players different from $j$ minus their value obtained under the allocation $x^{*}$. This can be considered as the harm done by player $j$ to the other players. Let $\mathcal{A}$ be any randomized allocation rule and let $\beta$ be a parameter to be determined later (as in Proposition 1 below). We use the following randomized payment rule:

$$
\begin{align*}
p_{i}(v) & := \begin{cases}\frac{\max \left\{p_{i}^{V C G}(v)-\beta, 0\right\}}{\mathbb{E}\left[v_{i}(x)\right]} v_{i}(x) & \text { if } \mathbb{E}\left[v_{i}(x)\right]>0, \text { where } x \sim \mathcal{A}(v) \\
0 & \text { otherwise }\end{cases}  \tag{2}\\
\text { and } p_{i}^{V C G}(v) & :=\mathbb{E}_{x \sim \mathcal{A}\left(\mathbf{0}, v_{-i}\right)}\left[v_{-i}(x)\right]-\mathbb{E}_{x \sim \mathcal{A}(v)}\left[v_{-i}(x)\right], \tag{3}
\end{align*}
$$

where $v_{-i}(x)=\sum_{j \neq i} v_{j}(x)$. Observe the similarity in the definition of $p_{i}^{V C G}(v)$ to the VCG payment rule. In both cases, the payment is defined as the difference of the total value of two allocations to the players different from $i$. The first allocation ignores the influence of player $i\left(\mathcal{A}\left(0, v_{-i}\right)\right)$ and the second allocation takes it into account $(\mathcal{A}(v))$. Consider next the definition of $p_{i}(v)$. If $\beta=0$ and the allocation rule is deterministic and hence $v_{i}(x)=\mathbb{E}\left[v_{i}(x)\right]$, the first line simplifies to $p_{i}^{V C G}(v)$, i.e., $p_{i}(v)=p_{i}^{V C G}(v)$. Our definition takes into account that we will consider an allocation rule that only approximately maximizes social welfare (up to an error of $\beta$ ) and that our allocation rules are randomized; therefore the multiplication by $v_{i}(x) / \mathbb{E}\left[v_{i}(x)\right]$.
Proposition 1. Given an allocation rule $\mathcal{A}: \mathcal{V} \rightarrow \mathcal{D}(\Omega)$, a constant $\epsilon \in[0,1]$, and functions $\beta: \mathcal{V} \rightarrow \mathbb{R}_{\geq 0}$, such that for all $i, \tilde{v}:=\left(\tilde{v}_{i}, v_{-i}\right), v:=\left(v_{i}, v_{-i}\right) \in \mathcal{V}$,

$$
\begin{equation*}
\mathbb{E}_{x \sim \mathcal{A}(\tilde{v})}[\tilde{v}(x)] \geq \mathbb{E}_{x \sim \mathcal{A}(v)}[\tilde{v}(x)]-\beta\left(\mathbf{0}, v_{-i}\right)-\epsilon \cdot \mathbb{E}_{x \sim \mathcal{A}(v)}\left[\tilde{v}_{i}(x)\right] \tag{4}
\end{equation*}
$$

the above payment rule, where $\beta:=\beta\left(\mathbf{0}, v_{-i}\right)$ is used in (2), yields an individually rational mechanism with no positive transfer, such that for all $i$, and $v:=\left(v_{i}, v_{-i}\right) \in \mathcal{V}$,

$$
\begin{equation*}
\mathbb{E}_{x \sim \mathcal{A}(\bar{v})}\left[U_{i}(\bar{v})\right] \geq \mathbb{E}_{x \sim \mathcal{A}(v)}\left[U_{i}(v)\right]-\epsilon \cdot \mathbb{E}_{x \sim \mathcal{A}(v)}\left[\bar{v}_{i}(x)\right]-3 \beta\left(\mathbf{0}, v_{-i}\right) \tag{5}
\end{equation*}
$$

when $\bar{v}:=\left(\bar{v}_{i}, v_{-i}\right)$ and $\bar{v}_{i}$ is the true valuation of player $i$.
a square matrix. In at most $s+m$ iterations one arrives at a basic feasible solution. In each iteration one has to solve a $m \times s$ linear system (time $m^{2} s$; by rank-one updates, once ca reduce the time to ms ).

Proof. The quite technical proof can be found in Appendix A.
Corollary 1. Given an $(\alpha, \beta)$-abs-MIDR w.r.t. range $\mathcal{R}$ and some constants $\alpha$ and $\beta$, one can obtain an $(\alpha, 3 \beta)$-abs-TIE individually rational mechanism with no positive transfer.

Proof. This follows from proposition 1 by setting $\beta\left(0, v_{-i}\right)=\beta$ for all $i$ and $v_{-i} \in \mathcal{V}_{-i}$ and $\epsilon=0$.
[9] showed how to obtain a $\beta$-rel-TIE from an $(\alpha, \beta)$-rel-MIDR. We derive here a similar but more efficient mechanism from Proposition 1. Consider a $(1-\epsilon)$-rel-MIDR with allocation rule $\mathcal{A}$, and let $u^{i} \in \mathcal{Q}$ be s.t. $v_{i}\left(u^{i}\right) \geq$ $(1-\epsilon) \operatorname{argmax}_{x \in \mathcal{Q}} v_{i}(x)$. We start with the observation that condition (4) in Proposition 1 is satisfied with

$$
\begin{equation*}
\beta\left(0, v_{-i}\right):=\frac{\epsilon}{1-\epsilon} \sum_{j \neq i} v_{j}\left(u^{j}\right) \tag{6}
\end{equation*}
$$

Indeed, let $\tilde{v}=\left(\tilde{v}_{i}, v_{-i}\right)$. Then

$$
\begin{align*}
\mathbb{E}_{x \sim \mathcal{A}(\tilde{v})}[\tilde{v}(x)] & \geq(1-\epsilon) \max _{\mathcal{A}^{\prime} \in \mathcal{R}} \mathbb{E}_{x \sim \mathcal{A}^{\prime}}[\tilde{v}(x)] \geq(1-\epsilon) \mathbb{E}_{x \sim \mathcal{A}(v)}[\tilde{v}(x)] \\
& =\mathbb{E}_{x \sim \mathcal{A}(v)}[\tilde{v}(x)]-\epsilon \cdot \mathbb{E}_{x \sim \mathcal{A}(v)}\left[\sum_{j \neq i} \tilde{v}_{j}(x)\right]-\epsilon \cdot \mathbb{E}_{x \sim \mathcal{A}(v)}\left[\tilde{v}_{i}(x)\right] \\
& \geq \mathbb{E}_{x \sim \mathcal{A}(v)}[\tilde{v}(x)]-\beta\left(0, v_{-i}\right)-\epsilon \cdot \mathbb{E}_{x \sim \mathcal{A}(v)}\left[\tilde{v}_{i}(x)\right] . \tag{7}
\end{align*}
$$

Here, the first inequality follows from the fact that $\mathcal{A}$ is $(1-\epsilon)$-rel MIDR, the second inequality follows from $\mathcal{A} \in \mathcal{R}$, and the last inequality follows from $\mathbb{E}_{x \sim \mathcal{A}(v)}\left[\sum_{j \neq i} \tilde{v}_{j}(x)\right]=\mathbb{E}_{x \sim \mathcal{A}(v)}\left[\sum_{j \neq i} v_{j}(x)\right] \leq \frac{1}{1-\epsilon} \sum_{j \neq i} v_{j}\left(u^{j}\right)$.

Proposition 2. Let $\epsilon_{0} \in(0,1 / 2]$ and let $\epsilon=\left(1-\epsilon_{0}\right)^{2} \epsilon_{0}^{5} /\left(4 n^{4}\right)=\Theta\left(\epsilon_{0}^{5} / n^{4}\right)$. Given an $(\alpha, 1-\epsilon)$-rel-MIDR allocation rule $\mathcal{A}_{0}$ w.r.t. range $\mathcal{R}$, Algorithm 1 defines an $\left(\alpha^{\prime}, \beta^{\prime}\right)$-rel-TIE $\mathcal{M}=(\mathcal{A}, \mathcal{P})$ with no positive transfer, $\alpha^{\prime}:=\left(1-\epsilon_{0}\right) \alpha$ and $\beta^{\prime}:=1-\epsilon_{0}$, which is individually rational with probability at least $1-\epsilon_{0}$.

Proof. Define $\theta:=\frac{\epsilon_{0}}{n}, \gamma^{\prime}:=\frac{\epsilon_{0} \theta}{n}$ and $\gamma:=\theta(1-\theta)^{n} \gamma^{\prime}$. In step 5 of the algorithm, we choose $\tau\left(v_{-i}\right):=\gamma^{\prime} \sum_{j \neq i} v^{j}\left(u^{j}\right)$. Observe that $(1-\theta)^{n}+\sum_{i=1}^{n} \theta(1-\theta)^{n-i}=1$. Let $\mathbb{E}_{x, p \sim \mathcal{M}_{0}(v)}\left[U_{i}(v)\right]$ be the expected utility of player $i$ obtained from the input MIDR mechanism $\mathcal{M}_{0}:=\left(\mathcal{A}_{0}, \mathcal{P}_{0}\right)$ with payment rule $\mathcal{P}_{0}$ defined by (2) and (3), where $\beta$ is defined as in (6) (c.f. Proposition (1). Following [9, we call player $i$ active if the following two conditions hold:

$$
\begin{align*}
\mathbb{E}_{x, p \sim \mathcal{M}_{0}(v)}\left[U_{i}(v)\right]+\frac{\epsilon_{0} \theta}{(1-\theta)^{i}} v_{i}\left(u^{i}\right) & \geq \frac{\theta}{(1-\theta)^{i}} \tau\left(v_{-i}\right) \\
v_{i}\left(u^{i}\right) & \geq \gamma \sum_{j \neq i} v_{j}\left(u^{j}\right) \tag{8}
\end{align*}
$$

The proof that the algorithm has the stated properties is given in Appendix B

```
Algorithm \(1 \operatorname{rel-TIE}(v, \mathcal{Q}, \mathcal{M})\) : Constructing a rel-TIE \(\mathcal{M}=(\mathcal{A}, \mathcal{P})\) from an
( \(\alpha, 1-\epsilon\) )-rel-MIDR allocation rule \(\mathcal{A}_{0}\) (Proposition[2). The vectors \(u^{i}\) are defined
as above. The choice of \(\theta\) and the definition of active and \(\tau\left(v_{-j}\right)\) is given in
Proposition 2.
Require: A valuation vector \(v \in \mathcal{V}\), a packing convex set \(\mathcal{Q}\) and \((\alpha, 1-\epsilon)-\operatorname{MIDR} \mathcal{M}\)
Ensure: An allocation \(x \in \mathcal{Q}\) and a payment \(p \in \mathbb{R}^{n}\)
    1: Choose an index \(i \in\{0,1, \ldots, n\}\), where 0 is chosen with probability \((1-\theta)^{n}\) and
    \(i \geq 1\) is chosen with probability \(\theta(1-\theta)^{n-i}\).
    if \(i=0\) then
    Use the input MIDR \(\mathcal{A}_{0}\) with payment rule (2) and (3) of Proposition [1] where
    \(\beta\) is defined as in (6) to get an allocation \(x=\left(x_{1}, \ldots, x_{n}\right) \in \mathcal{Q}\) and payment
    \(p=\left(p_{1}, \ldots, p_{n}\right) \in \mathbb{R}^{n}\). For all inactive \(j\), change \(x_{j}\) and \(p_{j}\) to zero.
    else
            For \(j=1, \ldots, n\), find \(u^{j} \in \mathcal{Q}\) be s.t. \(v_{j}\left(u^{j}\right) \geq(1-\epsilon) \operatorname{argmax}_{x \in \mathcal{Q}} v_{j}(x)\).
            For every \(1 \leq j \leq n\), set
                    \(\begin{cases}x_{j}=u^{j}, p_{j}=\tau\left(v_{-j}\right) & \text { if } j=i \text { and } i \text { is active, } \\ x_{j}=u^{j}, p_{j}=0 & \text { if } j=i \text { and } i \text { is inactive, } \\ x_{j}=0, p_{j}=0 & \text { if } j \neq i .\end{cases}\)
    end if
    return \((x, p)\)
```


## 4 Obtaining truthful-in-expectation mechanisms

We come to the heart of the paper. In Section 4.1 we use Proposition 2 to obtain an almost-TIE from an FPTAS for $\mathcal{Q}$. This almost-TIE is fractional in the sense that it returns an arbitrary vector in $\mathcal{Q}$. In Section 4.2 we review the method of Lavi and Swamy for obtaining an integral TIE from a fractional TIE. A key ingredient for this method is an algorithm for writing any point in the convex hull of $\mathcal{Q}_{I}$ as a convex combination of polynomially many points in $\mathcal{Q}_{I}$. In Section 4.3, we describe a simple algorithm for approximately achieving such a decomposition.

### 4.1 Obtaining a fractional TIE

Corollary 2. Suppose that there is an FPTAS $\mathcal{F}$ for $\mathcal{Q}$ with running time $T_{\mathcal{F}}\left(\operatorname{len}(\mathcal{Q}), \frac{1}{\epsilon}\right)$. Then, for any $\epsilon \in\left(0, \frac{1}{2}\right]$, there is a $\left((1-\epsilon)\left(1-\Theta\left(\frac{\epsilon^{5}}{n_{5}^{4}}\right)\right), 1-\epsilon\right)$-relTIE $\mathcal{M}$ for $(\mathcal{Q}, \mathcal{V})$ with running time $T_{\mathcal{M}}=O\left(n \cdot T_{\mathcal{F}}\left(\operatorname{len}(\mathcal{Q}), \Theta\left(\frac{\epsilon^{5}}{n^{4}}\right)\right)\right)$.

Proof. We apply Proposition 2. The given algorithm $\mathcal{F}$ can be interpreted as a $(1-\epsilon, 1-\epsilon)$-rel-MIDR for $(\mathcal{Q}, \mathcal{V})$ as follows. Let $\mathcal{R}=\left\{\delta^{x}: x \in \mathcal{Q}\right\}$ be a set of one-point distributions $\sqrt{6}$, where $\delta^{x}$ chooses $x$ with probability 1 and

[^2]the other points with probability 0 . Suppose that, on input $v \in \mathcal{V}, \mathcal{F}$ returns $x^{*} \in\left\{x: v(x) \geq(1-\epsilon) \max _{x \in \mathcal{Q}} v(x)\right\}$. Then we use the allocation rule $v \mapsto \delta^{x^{*}}$. This is a $(1-\epsilon)$-rel MIDR since
$$
\mathbb{E}_{y \sim \delta^{x^{*}}}[v(y)]=v\left(x^{*}\right) \geq(1-\epsilon) \max _{x \in \mathcal{Q}} v(x)=(1-\epsilon) \max _{x \in \mathcal{Q}} \mathbb{E}_{y \sim \delta^{x}}[v(y)]
$$

The allocation rule is also $(1-\epsilon)$-socially efficient since $v\left(x^{*}\right) \geq(1-\epsilon) \max _{x \in \mathcal{Q}} v(x)$. An application of Proposition 2 finishes the proof.

### 4.2 Obtaining an integral TIE

This section is essentially a review of key parts of 16|17. For each $y \in \operatorname{conv}\left(\mathcal{Q}_{I}\right)$ we define a distribution $D_{\mathrm{ch}}^{y}$. Any such $y$ can be written as a convex combination of points in $\mathcal{Q}_{I}$, i.e., $y=\sum_{z \in \mathcal{Q}_{I}} \lambda_{z} z$ and $\sum_{z \in \mathcal{Q}_{I}} \lambda_{z}=1$ (this representation is not unique). For every $y \in \operatorname{conv}\left(\mathcal{Q}_{I}\right)$, we fix an arbitrary such presentation. Then $D_{\text {ch }}^{y}$ assigns probability $\lambda_{z}$ to $z$. For any linear function $f: \mathcal{Q}_{\mathcal{I}} \rightarrow \mathbb{R}$, we have by linearity of expectation $\mathbb{E}_{x \sim D_{c h}^{y}}[f(x)]=f\left(\mathbb{E}_{x \sim D_{c h}^{y}}[x]\right)=f(y)$.

Proposition 3. Let $\mathcal{M}=(\mathcal{A}, \mathcal{P})$ be a $(\gamma, \beta)$-abs-TIE (resp., $(\gamma, \beta)$-rel-TIE) for $(\mathcal{Q}, \mathcal{V})$. Suppose that there is an $\alpha \in(0,1]$ such that $\alpha \cdot x \in \operatorname{conv}\left(\mathcal{Q}_{\mathcal{I}}\right)$ for each $x \in \mathcal{Q}$. Then, the mechanism $\mathcal{M}^{\prime}=\left(\mathcal{A}^{\prime}, \mathcal{P}^{\prime}\right)$ is an $(\alpha \gamma, \alpha \beta)$-abs-TIE (resp., $(\alpha \gamma, \beta)$-rel-TIE) for $\left(\mathcal{Q}_{\mathcal{I}}, \mathcal{V}\right)$, where the rule $\mathcal{A}^{\prime}: \mathcal{V} \rightarrow \mathcal{D}\left(\mathcal{Q}_{\mathcal{I}}\right)$ is defined by $x^{\prime} \sim \mathcal{A}^{\prime}(v)$ if and only if $x^{\prime} \sim D_{c h}^{\alpha \cdot x}$ and $x \sim \mathcal{A}(v)$, and the payment $\mathcal{P}^{\prime}: \mathcal{V} \rightarrow \mathbb{R}_{+}^{n}$ is given by: $p_{i}^{\prime}(v):=p_{i}(v) \frac{v_{i}\left(x^{\prime}\right)}{v_{i}(x)}$, where $x \in \mathcal{Q}$ and $x^{\prime} \in \mathcal{Q}_{\mathcal{I}}$ are the points allocated by $\mathcal{A}(v)$ and $\mathcal{A}^{\prime}(v)$, respectively.

Proof. The range $\mathcal{R}^{\prime}$ of the rule $\mathcal{A}^{\prime}$ is the set of distributions $D^{\prime}$ such that $x^{\prime} \sim D^{\prime}$ if and only if $x^{\prime} \sim D_{c h}^{\alpha \cdot x}$ and $x \sim D$, for some $D \in \mathcal{R}$. For each $D^{\prime} \in \mathcal{R}^{\prime}$, we fix one such $D \in \mathcal{R}$ and denote it by $D\left(D^{\prime}\right)$. For any $v \in \mathcal{V}$, we have

$$
\begin{align*}
\mathbb{E}_{x^{\prime} \sim \mathcal{A}^{\prime}(v)}\left[v\left(x^{\prime}\right)\right] & =\mathbb{E}_{x \sim \mathcal{A}(v)}\left[\mathbb{E}_{x^{\prime} \sim D_{c h}^{\alpha \cdot x}}\left[v\left(x^{\prime}\right)\right]\right] \\
& =\mathbb{E}_{x \sim \mathcal{A}(v)}[\alpha \cdot v(x)] \geq \alpha \gamma \cdot z_{\mathcal{Q}}^{*}(v) \geq \alpha \gamma \cdot z_{\mathcal{Q}_{\mathcal{I}}}^{*}(v) \tag{9}
\end{align*}
$$

where the inequality holds because $\mathcal{A}$ is $\gamma$-socially efficient. Hence by (9), the allocation rule $\mathcal{A}^{\prime}$ is $\alpha \gamma$-socially efficient.

Let $U_{i}$ and $U_{i}^{\prime}$ be the utilities of player $i$ w.r.t. the $\mathcal{M}$ and $\mathcal{M}^{\prime}$, respectively. Suppose that $\mathcal{M}$ is a $\beta$-abs-TIE. Then, for any $v \in \mathcal{V}$, we have

$$
\begin{align*}
\mathbb{E}_{x^{\prime}, p^{\prime} \sim \mathcal{M}^{\prime}(v)}\left[U_{i}^{\prime}\left(x^{\prime}\right)\right] & =\mathbb{E}_{x \sim D\left(\mathcal{A}^{\prime}(v)\right)}\left[\mathbb{E}_{x^{\prime} \sim D_{c h}^{\alpha \cdot x}}\left[v_{i}\left(x^{\prime}\right)\right]\right]-\mathbb{E}_{x \sim D\left(\mathcal{A}^{\prime}(v)\right), p \sim \mathcal{P}}\left[\frac{p_{i}(v)}{v_{i}(x)} \mathbb{E}_{x^{\prime} \sim D_{c h}^{\alpha \cdot x}}\left[v_{i}\left(x^{\prime}\right)\right]\right] \\
& =\alpha \cdot \mathbb{E}_{x \sim D\left(\mathcal{A}^{\prime}(v)\right)}\left[v_{i}(x)\right]-\alpha \cdot \mathbb{E}_{p \sim \mathcal{P}}\left[p_{i}(v)\right]=\alpha \cdot \mathbb{E}_{x, p \sim \mathcal{M}(v)}\left[U_{i}(x)\right] . \tag{10}
\end{align*}
$$

The claim follows from (10). Note also that $\mathcal{M}^{\prime}$ is individually rational if $\mathcal{M}$ is.

### 4.3 A fast algorithm for Carr-Vempala decomposition

Let us $\mathcal{N}$ to index the elements in $\mathcal{Q}_{\mathcal{I}}$, let $\mathcal{F}^{\prime}$ be an $\alpha$-integrality-gap-verifier for $\mathcal{Q}_{\mathcal{I}}$, and let $x^{*} \in \mathcal{Q}$. Carr and Vempala [CV02] showed that in polynomial time one can express $\alpha \cdot x^{*}$ as a convex combination of integer points in $\mathcal{Q}$ : $\alpha \cdot x^{*}=\sum_{i \in \mathcal{N}} \lambda_{i} x^{i}, \sum_{i \in \mathcal{N}} \lambda_{i}=1, \lambda_{i} \geq 0$ for all $i \in \mathcal{N}$. Note that $\alpha x^{*} \in \mathcal{Q}$ since $x^{*} \in \mathcal{Q}$ and $\mathcal{Q}$ has the packing property. We show an approximate version that replaces the use of the Ellipsoid method by the MWU method.

Theorem 3. Given a fractional point $x^{*}$ of (1), and an $\alpha$-integrality-gap verifier $\mathcal{F}^{\prime}$ for $\mathcal{Q}_{I}$, then for any $\epsilon>0$, we can find a set $\mathcal{N}^{\prime} \subseteq \mathcal{N}$ of size $O\left(\epsilon^{-2} s \log s\right)$, where $s=\left|S^{+}\right|$and $S^{+}=\left\{j: x_{j}^{*}>0\right\}$, and numbers $\lambda_{i}$, for $i \in \mathcal{N}^{\prime}$, such that $\lambda_{i}>0$ for all $i \in \mathcal{N}^{\prime}, \sum_{i \in \mathcal{N}^{\prime}} \lambda_{i}=1$ and $\frac{\alpha}{1+3 \epsilon} \cdot x^{*}=\sum_{i \in \mathcal{N}^{\prime}} \lambda_{i} x^{i}$. The running time of the algorithm is $O\left(\epsilon^{-2} s \log s\left(T_{\mathcal{F}^{\prime}}(\operatorname{len}(\mathcal{Q}), s)+s \log s\right)\right)$, where $\mathcal{T}_{\mathcal{F}^{\prime}}(\operatorname{len}(\mathcal{Q}), s)$ is the running of $\mathcal{F}^{\prime}$ on an objective vector of support $s$.

Proof. The task of finding the multipliers $\lambda_{i}$ is naturally formulated as a covering LP. Minimize min $\sum_{i \in \mathcal{N}} \lambda_{i}$ subject to $\sum_{i \in \mathcal{N}} \lambda_{i} x_{j}^{i} \geq \alpha \cdot x_{j}^{*}$ for all $j, \sum_{i \in \mathcal{N}} \lambda_{i} \geq 1$, and $\lambda_{i} \geq 0$. In the language of Appendix [C] we have $m=s+1, n=|\mathcal{N}|, c=\mathbf{1}$, $b=\left(\alpha x^{*}, 1\right)$. The matrix $A=\left(a_{j, i}\right)$ is as follows:

$$
a_{j, i}:= \begin{cases}x_{j}^{i} & 1 \leq j \leq s, i \in \mathcal{N} \\ 1 & j=s+1, i \in \mathcal{N}\end{cases}
$$

Thus we can apply the algorithm for covering LP's in Appendix C provided we can efficiently implement the required oracle. We show that we can use $\mathcal{F}^{\prime}$ to implement an oracle $\mathcal{O}^{\prime}$. To do this, we first scale inequality $j$ in the LP by $\alpha x_{j}^{*}$. Oracle $\mathcal{O}^{\prime}$ has arguments $(A, \tilde{z})$ such that $1^{T} \tilde{z}=1$. Let us conveniently write $\tilde{z}=(w, z)$, where $w \in \mathbb{R}_{\geq 0}^{S^{+}}, z \in \mathbb{R}_{\geq 0}$, and $\sum_{j \in S^{+}} w_{j}+z=1$. Oracle $\mathcal{O}^{\prime}$ needs to find a column $i$ such that $\tilde{z}^{T} A \mathbf{1}_{i} \geq 1$. In our case $\tilde{z}^{T} A \mathbf{1}_{i}=\sum_{j=1}^{s} w_{j} x_{j}^{i} / \alpha x_{j}^{*}+z$, and we need to find a column $i$ for which this expression is at least one. Since $z$ does not depend on $i$, we concentrate on the first term. Define

$$
V_{j}:= \begin{cases}\frac{w_{j}}{\alpha x_{j}^{*}} & j \in S^{+} \\ 0 & \text { otherwise } .\end{cases}
$$

Call algorithm $\mathcal{F}^{\prime}$ with $x^{*} \in \mathcal{Q}$ and $V:=\left(V_{1}, \ldots, V_{d}\right) . \mathcal{F}^{\prime}$ returns an integer solution $x^{i} \in \mathcal{Q}_{\mathcal{I}}$ such that $\sum_{j \in S^{+}} \frac{w_{j}}{\alpha x_{j}^{*}} x_{j}^{i}=V^{T} x^{i} \geq \alpha \cdot V^{T} x^{*}=\sum_{j \in S^{+}} w_{j}$, and hence, $\sum_{j \in S^{+}} \frac{w_{j}}{\alpha \cdot x_{j}^{*}} x_{j}^{i}+z \geq \sum_{j \in S^{+}} w_{j}+z=1$, this $i$ is the desired column of A. It follows by Corollary 3 that Algorithm 2 finds a feasible solution $\lambda^{\prime} \in \mathbb{R}_{\geq 0}^{|\mathcal{N}|}$ to the cover LP, and a set vectors $\mathcal{Q}_{\mathcal{I}}^{\prime} \subseteq \mathcal{Q}_{\mathcal{I}}$ (returned by the oracle), such that $\lambda_{i}^{\prime}>0$ only for $i \in \mathcal{N}^{\prime},\left|\mathcal{N}^{\prime}\right| \leq \frac{s \ln s}{\epsilon^{2}}$, and $\sum_{i \in \mathcal{N}^{\prime}} \lambda_{i}^{\prime} \leq(1+3 \epsilon)$. Scaling $\lambda_{i}^{\prime}$ by $\sum_{i \in \mathcal{N}^{\prime}} \lambda_{i}^{\prime}$, we obtain a set of vectors with index set $\mathcal{N}^{\prime} \subseteq \mathcal{N}$, and convex multipliers $\left\{\lambda_{i}: i \in \mathcal{N}^{\prime}\right\}$, such that

$$
\begin{equation*}
\sum_{i \in \mathcal{N}^{\prime}} \lambda_{i} x^{i} \geq \frac{\alpha}{1+3 \epsilon} x^{*} \tag{11}
\end{equation*}
$$

We may assume $x_{j}^{i}=0$ for all $j \notin S^{+}$whenever $\lambda_{i}>0$; otherwise simply replace $x^{i}$ by a vector in which all components not in $S^{+}$are set to zero.

We will use the packing property of the polytope $\mathcal{Q}$ to modify the set $\mathcal{N}^{\prime}$ to a new set satisfying equality in (11) (such a construction seems to have been observed in [16], although not given explicitly). Since reducing a positive $x_{j}^{i}$ by one as long as (11) holds maintains feasibility, we may assume without loss of generality that the set of vectors indexed by $\mathcal{N}^{\prime}$ satisfy the following minimality condition,

$$
\begin{equation*}
\text { for all } i \in \mathcal{N}^{\prime}, j \in S^{+} \text {either } x_{j}^{i}=0, \text { or } \sum_{i^{\prime} \in \mathcal{N}^{\prime}, i^{\prime} \neq i} \lambda_{i^{\prime}} x_{j}^{i^{\prime}}+\lambda_{i} x_{j}^{i}-\lambda_{i}<\frac{\alpha}{1+3 \epsilon} x_{j}^{*} \tag{12}
\end{equation*}
$$

For $j \in S^{+}$, let $\Delta_{j}=\frac{\alpha}{1+3 \epsilon} x_{j}^{*}-\sum_{i \in \mathcal{N}^{\prime}} \lambda_{i} x_{j}^{i}$. Then $\Delta_{j} \geq 0$ and, by (12), for every $j \in S^{+}$, and $i \in \mathcal{N}^{\prime}$ either $\Delta_{j}<\lambda_{i} \leq \lambda_{i} x_{j}^{i}$ or $x_{j}^{i}=0$. If $\Delta_{j}=0$ for all $j$, we are done. Otherwise, choose $j^{*}$ and $i \in \mathcal{N}^{\prime}$ such that $\Delta_{j^{*}}>0$ and $x_{j^{*}}^{i}>0$. Let $B=\left\{j \in S^{+}: x_{j}^{i} \neq 0\right\}$ be the set of indices of non-zero components of $x^{i}$. We will change the left-hand side of (11) such that equality holds for all indices in $B$. The change will not destroy an already existing equality for an index outside $B$.

By renumbering the coordinates, we may assume $B=[1 . . b]$, where $b=|B|$, and $\Delta_{1} / x_{1}^{i} \leq \ldots \leq \Delta_{b} / x_{b}^{i}$. For $j \leq b$, we have

$$
\lambda_{i} x_{j}^{i}-\Delta_{j}=\left(\lambda_{i}-\frac{\Delta_{b}}{x_{b}^{i}}\right) x_{j}^{i}+\sum_{j \leq \ell<b}\left(\frac{\Delta_{\ell+1}}{x_{\ell+1}^{i}}-\frac{\Delta_{\ell}}{x_{\ell}^{i}}\right) x_{j}^{i}+\sum_{1 \leq \ell<j}\left(\frac{\Delta_{\ell+1}}{x_{\ell+1}^{i}}-\frac{\Delta_{\ell}}{x_{\ell}^{i}}\right) 0+\frac{\Delta_{1}}{x_{1}^{i}} 0
$$

For $0 \leq \ell<b$ define a vector $y^{\ell}$ by $y_{j}^{\ell}=x_{j}^{i}$ for $j \leq \ell$ and $y_{j}^{\ell}=0$ for $j>\ell$. Then

$$
\begin{equation*}
\lambda_{i} x_{j}^{i}-\Delta_{j}=\left(\lambda_{i}-\frac{\Delta_{b}}{x_{b}^{i}}\right) x_{j}^{i}+\sum_{1 \leq \ell<b}\left(\frac{\Delta_{\ell+1}}{x_{\ell+1}^{i}}-\frac{\Delta_{\ell}}{x_{\ell}^{i}}\right) y_{j}^{\ell}+\frac{\Delta_{1}}{x_{1}^{i}} y_{j}^{0} \quad \text { for all } j \leq b . \tag{13}
\end{equation*}
$$

Note that the coefficients on the right-hand side of (13) are non-negative and sum up to $\lambda_{i}$. Also, by the packing property of $\mathcal{Q}, y^{\ell} \in \mathcal{Q}_{\mathcal{I}}$ for $0 \leq \ell<b$. We now change the left-hand side of (11) as follows: we replace $\lambda_{i}$ by $\lambda_{i}-\Delta_{b} / x_{b}^{i}$, for $1 \leq \ell<b$, we increase the coefficient of $y^{\ell}$ by $\Delta_{\ell+1} / x_{\ell+1}^{i}-\Delta_{\ell} / x_{\ell}^{i}$ and we increase the coefficient of $y^{0}$ by $\Delta_{1} / x_{1}^{i}$. As a result, we now have equality for all indices in $B$. The $\Delta_{j}$ for $j \notin B$ are not affected by this change.

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## A The Proof of Proposition 1

Clearly, $p_{i}(v) \geq 0$ by definition, so this is a mechanism with no positive transfer. To see individual rationality, we need only to consider the case when $\mathbb{E}_{x \sim \mathcal{A}(\bar{v})}\left[\bar{v}_{i}(x)\right]>$ 0 and $p_{i}(\bar{v})=\left(p_{i}^{V C G}(\bar{v})-\beta\left(\mathbf{0}, v_{-i}\right)\right) \bar{v}_{i}(x) / \mathbb{E}_{x \sim \mathcal{A}(\bar{v})}\left[\bar{v}_{i}(x)\right]>0$, since in either case $\mathbb{E}_{x \sim \mathcal{A}(\bar{v})}\left[\bar{v}_{i}(x)\right]=0$ or $\left(\mathbb{E}_{x \sim \mathcal{A}(\bar{v})}\left[\bar{v}_{i}(x)\right]>0\right.$ and $\left.p_{i}(\bar{v})=0\right)$, we have $U_{i}\left(\bar{v}_{i}, v_{-i}\right)=$ $\bar{v}_{i}(x) \geq 0$. But then, for $x \sim \mathcal{A}(\bar{v})$,

$$
\begin{aligned}
U_{i}(\bar{v}) & =\bar{v}_{i}(x)-p_{i}(\bar{v}) \quad\left(\text { substitute } p_{i}(\bar{v})\right) \\
& =\left(\mathbb{E}_{x \sim \mathcal{A}(\bar{v})}\left[\bar{v}_{i}(x)\right]-p_{i}^{V C G}(\bar{v})+\beta\left(\mathbf{0}, v_{-i}\right)\right) \frac{\bar{v}_{i}(x)}{\mathbb{E}_{x \sim \mathcal{A}(\bar{v})}\left[\bar{v}_{i}(x)\right]} \\
& =\left(\mathbb{E}_{x \sim \mathcal{A}(\bar{v})}[\bar{v}(x)]-\mathbb{E}_{x \sim \mathcal{A}\left(\mathbf{0}, v_{-i}\right)}[\bar{v}(x)]+\mathbb{E}_{x \sim \mathcal{A}\left(\mathbf{0}, v_{-i}\right)}\left[\bar{v}_{i}(x)\right]+\beta\left(\mathbf{0}, v_{-i}\right)\right) \frac{\bar{v}_{i}(x)}{\mathbb{E}_{x \sim \mathcal{A}(\bar{v})}\left[\bar{v}_{i}(x)\right]} \\
& \geq(1-\epsilon) \mathbb{E}_{x \sim \mathcal{A}\left(\mathbf{0}, v_{-i}\right)}\left[\bar{v}_{i}(x)\right] \frac{\bar{v}_{i}(x)}{\mathbb{E}_{x \sim \mathcal{A}(\bar{v})}\left[\bar{v}_{i}(x)\right]} \geq 0, \\
& \left(\operatorname{set} \tilde{v}=\bar{v}, v=\left(0, v_{-i}\right), \text { then apply (4) }\right) .
\end{aligned}
$$

To argue about truthfulness, we first show that $p_{i}^{V C G}(v) \geq-\beta\left(\mathbf{0}, v_{-i}\right)$, for all $v \in \mathcal{V}$. By setting $\tilde{v}=\left(0, v_{-i}\right)$, we have $\mathbb{E}_{x \sim \mathcal{A}(\tilde{v})}[\tilde{v}(x)]=\mathbb{E}_{x \sim \mathcal{A}\left(\mathbf{0}, v_{-i}\right)}\left[v_{-i}(x)\right]$, $\mathbb{E}_{x \sim \mathcal{A}(v)}[\tilde{v}(x)]=\mathbb{E}_{x \sim \mathcal{A}(v)}\left[v_{-i}(x)\right]$ and $\mathbb{E}_{x \sim \mathcal{A}(v)}\left[\tilde{v}_{i}(x)\right]=0$. Now applying (4), we get

$$
\begin{align*}
P_{i}^{V C G}(v) & =\mathbb{E}_{x \sim \mathcal{A}(\tilde{v})}[\tilde{v}(x)]-\mathbb{E}_{x \sim \mathcal{A}(v)}[\tilde{v}(x)]+\epsilon \cdot \mathbb{E}_{x \sim \mathcal{A}(v)}\left[\tilde{v}_{i}(x)\right] \\
& \geq-\beta\left(\mathbf{0}, v_{-i}\right) \tag{14}
\end{align*}
$$

In the following we show that for all $v \in \mathcal{V}$,

$$
\begin{equation*}
\mathbb{E}_{x \sim \mathcal{A}(v)}\left[p_{i}(v)\right]-p_{i}^{V C G}(v) \in\left[-\beta\left(\mathbf{0}, v_{-i}\right), \beta\left(\mathbf{0}, v_{-i}\right)\right] \tag{15}
\end{equation*}
$$

To see (15), we consider two cases:
Case1. $\mathbb{E}_{x \sim \mathcal{A}(v)}\left[v_{i}(x)\right]>0$. Then using (14)

$$
\begin{aligned}
& \mathbb{E}_{x \sim \mathcal{A}(v)}\left[p_{i}(v)\right]-p_{i}^{V C G}(v)= \\
& \mathbb{E}\left[\frac{\max \left\{p_{i}^{V C G}(v)-\beta, 0\right\}}{\mathbb{E}\left[v_{i}(x)\right]} v_{i}(x)\right]-p_{i}^{V C G(v)}= \\
& \frac{\mathbb{E}\left[\max \left\{p_{i}^{V C G}(v)-\beta, 0\right\}\right]}{\mathbb{E}\left[v_{i}(x)\right]} \mathbb{E}\left[v_{i}(x)\right]-p_{i}^{V C G}(v)= \\
& \max \left\{-\beta\left(\mathbf{0}, v_{-i}\right),-p_{i}^{V C G}(v)\right\} \in\left[-\beta\left(\mathbf{0}, v_{-i}\right), \beta\left(\mathbf{0}, v_{-i}\right)\right]
\end{aligned}
$$

Case 2. $\mathbb{E}_{x \sim \mathcal{A}(v)}\left[v_{i}(x)\right]=0$. Then by (14) we have

$$
\mathbb{E}_{x \sim \mathcal{A}(v)}\left[p_{i}(v)\right]-p_{i}^{V C G}(v)=-p_{i}^{V C G}(v) \leq \beta\left(\mathbf{0}, v_{-i}\right)
$$

By definition of $p_{i}^{V C G}$ and (4) we conclude that

$$
\begin{aligned}
& -p_{i}^{V C G}(v)= \\
& \mathbb{E}_{x \sim \mathcal{A}(v)}[v(x)]-\mathbb{E}_{x \sim \mathcal{A}\left(\mathbf{0}, v_{-i}\right)}[v(x)]+\mathbb{E}_{x \sim \mathcal{A}\left(\mathbf{0}, v_{-i}\right)}\left[v_{i}(x)\right]-\mathbb{E}_{x \sim \mathcal{A}(v)}\left[v_{i}(x)\right]= \\
& \mathbb{E}_{x \sim \mathcal{A}(v)}[v(x)]-\mathbb{E}_{x \sim \mathcal{A}\left(\mathbf{0}, v_{-i}\right)}[v(x)]+\mathbb{E}_{x \sim \mathcal{A}\left(\mathbf{0}, v_{-i}\right)}\left[v_{i}(x)\right] \geq \\
& (1-\epsilon) \mathbb{E}_{x \sim \mathcal{A}\left(\mathbf{0}, v_{-i}\right)}\left[v_{i}(x)\right]-\beta\left(\mathbf{0}, v_{-i}\right) \geq-\beta\left(\mathbf{0}, v_{-i}\right) .
\end{aligned}
$$

Let $\bar{v}=\left(\bar{v}_{i}, v_{-i}\right)$ in which $\bar{v}_{i}$ is the true valuation of player $i$, and let $v=\left(v_{i}, v_{-i}\right)$ be any arbitrary valuation. By using (4) and (15), we have

$$
\mathbb{E}_{x \sim \mathcal{A}(\bar{v})}\left[U_{i}(\bar{v})\right]=\mathbb{E}_{x \sim \mathcal{A}(\bar{v})}\left[\bar{v}_{i}(x)\right]-\mathbb{E}_{x \sim \mathcal{A}(\bar{v})}\left[p_{i}(\bar{v})\right]
$$

(by definition of $U_{i}(\bar{v})$ )

$$
\begin{aligned}
& \geq \mathbb{E}_{x \sim \mathcal{A}(\bar{v})}\left[\bar{v}_{i}(x)\right]-p_{i}^{V C G}(\bar{v})-\beta\left(\mathbf{0}, v_{-i}\right) \\
&(\operatorname{apply} \text { (15) for } v=\bar{v}) \\
&=\mathbb{E}_{x \sim \mathcal{A}(\bar{v})}[\bar{v}(x)]-\mathbb{E}_{x \sim \mathcal{A}\left(\mathbf{0}, v_{-i}\right)}\left[v_{-i}(x)\right]-\beta\left(\mathbf{0}, v_{-i}\right)
\end{aligned}
$$

(by definition of $p_{i}^{V C G}(\bar{v})$ )

$$
\geq \mathbb{E}_{x \sim \mathcal{A}(v)}[\bar{v}(x)]-\mathbb{E}_{x \sim \mathcal{A}\left(\mathbf{0}, v_{-i}\right)}\left[v_{-i}(x)\right]-\epsilon \cdot \mathbb{E}_{x \sim \mathcal{A}(v)}\left[\bar{v}_{i}(x)\right]-2 \beta\left(\mathbf{0}, v_{-i}\right)
$$

(apply (4) for $\tilde{v}=\bar{v}$ )

$$
=\mathbb{E}_{x \sim \mathcal{A}(v)}\left[\bar{v}_{i}(x)\right]-p_{i}^{V C G}(v)-\epsilon \cdot \mathbb{E}_{x \sim \mathcal{A}(v)}\left[\bar{v}_{i}(x)\right]-2 \beta\left(\mathbf{0}, v_{-i}\right)
$$

(by definition of $p_{i}^{V C G}(v)$ )
$\geq \mathbb{E}_{x \sim \mathcal{A}(v)}\left[\bar{v}_{i}(x)\right]-\mathbb{E}_{x \sim \mathcal{A}(v)}\left[p_{i}(v)\right]-\epsilon \cdot \mathbb{E}_{x \sim \mathcal{A}(v)}\left[\bar{v}_{i}(x)\right]-3 \beta\left(\mathbf{0}, v_{-i}\right)$
(apply (15))
$=\mathbb{E}_{x \sim \mathcal{A}(v)}\left[U_{i}(v)\right]-\epsilon \cdot \mathbb{E}_{x \sim \mathcal{A}(v)}\left[\bar{v}_{i}(x)\right]-3 \beta\left(\mathbf{0}, v_{-i}\right)$
(by definition of $U_{i}(v)$ ).

## B The Proof that Algorithm 1 has the stated properties

We denote by $T=T(v)$ the set of active players when the valuation is $v=$ $\left(v_{1}, \ldots, v_{n}\right)$.

Non-negativity of payments is immediate from the definition of the mechanism, and Proposition 1. Moreover, the utility of a truth-telling bidder $i$ can be negative only if it is allocated in step 5 , which can happen with probability $\theta(1-\theta)^{n-i}$. It follows that the mechanism is individually rational with probability at least

$$
1-\theta \sum_{i=1}^{n}(1-\theta)^{n-i}=(1-\theta)^{n}=\left(1-\frac{\epsilon_{0}}{n}\right)^{n} \geq 1-\epsilon_{0} .
$$

We next argue that the mechanism is $\beta^{\prime}$-rel-TIE, for $\beta^{\prime}:=1-\epsilon_{0}$. Let $\mathbb{E}_{x, p \sim \mathcal{M}(v)}\left[U_{i}^{\prime}(v)\right]$ be the expected utility of player $i$ obtained from the mechanism in Algorithm 1 on input $v \in \mathcal{V}$. Then

$$
\mathbb{E}_{x, p \sim \mathcal{M}(v)}\left[U_{i}^{\prime}(v)\right]= \begin{cases}(1-\theta)^{n} \cdot \mathbb{E}_{x, p \sim \mathcal{M}_{0}(v)}\left[U_{i}(v)\right] &  \tag{16}\\ +\theta(1-\theta)^{n-i}\left(\bar{v}_{i}\left(u^{i}\right)-\tau\left(v_{-i}\right)\right) & \text { if } i \in T(v) \\ \theta(1-\theta)^{n-i} \cdot \bar{v}_{i}\left(u^{i}\right) & \text { otherwise. }\end{cases}
$$

Let $\bar{v}:=\left(\bar{v}_{i}, v_{-i}\right)$ be the valuation where $\overline{v_{i}}$ is the true valuation of player $i$ and $v:=\left(v_{i}, v_{-i}\right) \in \mathcal{V}$ be another valuation. Note that equation (16) implies that

$$
\begin{equation*}
\mathbb{E}_{x, p \sim \mathcal{M}(\bar{v})}\left[U_{i}^{\prime}(\bar{v})\right] \geq\left(1-\epsilon_{0}\right)(1-\theta)^{n-i} \theta \cdot \bar{v}_{i}\left(u^{i}\right) \tag{17}
\end{equation*}
$$

Indeed, the inequality is trivially satisfied if $i \notin T(\bar{v})$. On the other hand, if $i \in T(\bar{v})$, then the second inequality in (8) implies that $\mathbb{E}_{x, p \sim \mathcal{M}_{0}(\bar{v})}\left[U_{i}(\bar{v})\right] \geq$ $\frac{1}{(1-\theta)^{2}} \theta\left(\tau\left(v_{-i}\right)-\epsilon_{0} \bar{v}_{i}\left(u^{i}\right)\right)$, implying (17).

We consider four cases:
Case 1: $i \in T(\bar{v}) \cap T(v)$. Note that the first inequality in (18) for $\bar{v}$ implies that $\beta\left(0, v_{-i}\right) \leq \frac{\epsilon}{\gamma} \bar{v}_{i}\left(u^{i}\right)$. Thus, by (77) and Proposition (1)

$$
\mathbb{E}_{x, p \sim \mathcal{M}_{0}(\bar{v})}\left[U_{i}(\bar{v})\right] \geq \mathbb{E}_{x, p \sim \mathcal{M}_{0}(v)}\left[U_{i}(v)\right]-\epsilon\left(1+\frac{3}{\gamma}\right) \bar{v}_{i}\left(u^{i}\right)
$$

This together with (16) and (17) imply that

$$
\begin{aligned}
\mathbb{E}_{x, p \sim \mathcal{M}(v)}\left[U_{i}^{\prime}(v)\right] & \leq \mathbb{E}_{x, p \sim \mathcal{M}(\bar{v})}\left[U_{i}^{\prime}(\bar{v})\right]+\epsilon(1-\theta)^{n}\left(1+\frac{3}{\gamma}\right) \bar{v}_{i}\left(u^{i}\right) \\
& \leq\left(1+\frac{\epsilon(1-\theta)^{n}}{\left(1-\epsilon_{0}\right)(1-\theta)^{n-i} \theta}\left(1+\frac{3}{\gamma}\right)\right) \mathbb{E}_{x \sim \mathcal{M}(\bar{v})}\left[U_{i}^{\prime}(\bar{v})\right] \\
& \leq\left(1+\epsilon_{0}\right) \mathbb{E}_{x \sim \mathcal{M}(\bar{v})}\left[U_{i}^{\prime}(\bar{v})\right]
\end{aligned}
$$

where the last inequality follows from the definition of $\epsilon$.
Case 2: $i \notin T(v)$. By (17), we have

$$
\mathbb{E}_{x, p \sim \mathcal{M}(v)}\left[U_{i}^{\prime}(v)\right]=\theta(1-\theta)^{n-i} \bar{v}^{i}\left(u^{i}\right) \leq \frac{1}{1-\epsilon_{0}} \mathbb{E}_{x, p \sim \mathcal{M}(\bar{v})}\left[U_{i}^{\prime}(\bar{v})\right]
$$

Case 3: $i \in T(v) \backslash T(\bar{v})$ and $\bar{v}_{i}\left(u^{i}\right)<\gamma \sum_{j \neq i} v_{j}\left(u^{j}\right)=\gamma \frac{\tau\left(v_{-i}\right)}{\gamma^{\prime}}=\theta(1-\theta)^{n} \tau\left(v_{-i}\right)$. Then

$$
\begin{aligned}
\mathbb{E}_{x, p \sim \mathcal{M}(v)}\left[U_{i}^{\prime}(v)\right] & \leq\left((1-\theta)^{n}+\theta(1-\theta)^{n-i}\right) \bar{v}_{i}\left(u^{i}\right)-\theta(1-\theta)^{n-i} \tau\left(v_{-i}\right) \\
& <\left((1-\theta)^{n}+\theta(1-\theta)^{n-i}-1\right) \bar{v}_{i}\left(u^{i}\right) \leq 0 \leq \theta(1-\theta)^{n-i} \bar{v}_{i}\left(u^{i}\right) \\
& =\mathbb{E}_{x, p \sim \mathcal{M}(\bar{v})}\left[U_{i}^{\prime}(\bar{v})\right] .
\end{aligned}
$$

Case 4: $i \in T(v) \backslash T(\bar{v})$ and $\bar{v}_{i}\left(u^{i}\right) \geq \gamma \sum_{j \neq i} v_{j}\left(u^{j}\right)$. Then

$$
\mathbb{E}_{x, p \sim \mathcal{M}_{0}(\bar{v})}\left[U_{i}(\bar{v})\right]<\frac{\theta}{(1-\theta)^{i}}\left(\tau\left(v_{-i}\right)-\epsilon_{0} \bar{v}_{i}\left(u^{i}\right)\right)
$$

By (7) and Proposition $\mathbb{E}_{x, p \sim \mathcal{M}_{0}(\bar{v})}\left[U_{i}(\bar{v})\right] \geq \mathbb{E}_{x, p \sim \mathcal{M}_{0}(v)}\left[U_{i}(v)\right]-\epsilon\left(1+\frac{3}{\gamma}\right) \bar{v}_{i}\left(u^{i}\right)$. These inequalities together imply that

$$
\begin{aligned}
\mathbb{E}_{x, p \sim \mathcal{M}(v)}\left[U_{i}^{\prime}(v)\right] & \leq \\
& (1-\theta)^{n}\left(\mathbb{E}_{x, p \sim \mathcal{M}_{0}(\bar{v})}\left[U_{i}(\bar{v})\right]+\epsilon\left(1+\frac{3}{\gamma}\right) \bar{v}_{i}\left(u^{i}\right)\right) \\
& +(1-\theta)^{n-i} \theta\left(\bar{v}_{i}\left(u^{i}\right)-\tau\left(v_{-i}\right)\right) \\
& \leq(1-\theta)^{n-i} \theta \cdot \tau\left(v_{-i}\right)-(1-\theta)^{n-i} \theta \epsilon_{0} \bar{v}_{i}\left(u^{i}\right) \\
& +\epsilon\left(1+\frac{3}{\gamma}\right) \bar{v}_{i}\left(u^{i}\right)+(1-\theta)^{n-i} \theta\left(\bar{v}_{i}\left(u^{i}\right)-\tau\left(v_{-i}\right)\right) \\
& =\left(1-\epsilon_{0}\right)(1-\theta)^{n-i} \theta \cdot \bar{v}_{i}\left(u^{i}\right)+\epsilon\left(1+\frac{3}{\gamma}\right) \bar{v}_{i}\left(u^{i}\right) \\
& =\left(1-\epsilon_{0}\right) \mathbb{E}_{x, p \sim \mathcal{M}(\bar{v})}\left[U_{i}^{\prime}(\bar{v})\right]+\epsilon\left(1+\frac{3}{\gamma}\right) \bar{v}_{i}\left(u^{i}\right) \\
& \leq\left(1-\epsilon_{0}+\frac{\epsilon}{\left(1-\epsilon_{0}\right)(1-\theta)^{n-i} \theta}\left(1+\frac{3}{\gamma}\right)\right) \mathbb{E}_{x, p \sim \mathcal{M}(\bar{v})}\left[U_{i}^{\prime}(\bar{v})\right] \\
& \leq \frac{1}{1-\epsilon_{0}} \mathbb{E}_{x, p \sim \mathcal{M}(\bar{v})}\left[U_{i}^{\prime}(\bar{v})\right] .
\end{aligned}
$$

We argue now about the approximation ratio. Note that for $i \notin T(v)$, we have by (8) and the individual rationality of the mechanism $\mathcal{M}_{0}$ (c.f. Proposition (1) that $v_{i}\left(u^{i}\right)<\frac{\gamma^{\prime}}{\epsilon_{0}} \sum_{j \neq i} v_{j}\left(u^{j}\right)$. Since $\mathcal{M}_{0}$ is $\alpha$-socially efficient and $\theta(1-\theta)^{n-i}-$ $(1-\theta)^{n} n \frac{\gamma^{\prime}}{\epsilon_{0}} \geq 0$, it follows that for any $v \in \mathcal{V}$,

$$
\begin{aligned}
\mathbb{E}_{x \sim \mathcal{A}(v)}[v(x)] & =(1-\theta)^{n} \sum_{i \in T(v)} \mathbb{E}_{x \sim \mathcal{A}_{0}(v)}\left[v_{i}(x)\right]+\theta \sum_{i \in[n]}(1-\theta)^{n-i} v_{i}\left(u^{i}\right) \\
& \geq(1-\theta)^{n} \mathbb{E}_{x \sim \mathcal{A}_{0}(v)}[v(x)]-(1-\theta)^{n} \sum_{i \notin T(v)} v_{i}\left(u^{i}\right) \\
& +\theta \sum_{i \in[n]}(1-\theta)^{n-i} v_{i}\left(u^{i}\right) \\
& \geq(1-\theta)^{n} \mathbb{E}_{x \sim \mathcal{A}_{0}(v)}[v(x)]-(1-\theta)^{n} \frac{\gamma^{\prime}}{\epsilon_{0}} \sum_{i \notin T(v)} \sum_{j \neq i} v_{j}\left(u^{j}\right) \\
& +\theta \sum_{i \in[n]}(1-\theta)^{n-i} v_{i}\left(u^{i}\right) \\
& \geq(1-\theta)^{n} \mathbb{E}_{x \sim \mathcal{A}_{0}(v)}[v(x)]+\sum_{i \in[n]}\left(\theta(1-\theta)^{n-i}-(1-\theta)^{n} n \frac{\gamma^{\prime}}{\epsilon_{0}}\right) v_{i}\left(u^{i}\right) \\
& \geq(1-\theta)^{n} \alpha \cdot z_{\Omega}^{*} \geq\left(1-\epsilon_{0}\right) \alpha \cdot z_{\Omega}^{*} .
\end{aligned}
$$

The proposition follows.

## C An FPTAS for covering linear programs

Consider a covering linear program:

$$
\begin{align*}
\min & c^{T} x  \tag{18}\\
\text { s.t. } & A x \geq b \\
& x \geq 0
\end{align*}
$$

where $A \in \mathbb{R}_{\geq 0}^{m \times n}$ is an $m \times n$ matrix with non-negative entries and $c \in \mathbb{R}_{>0}^{n}$, $b \in \mathbb{R}_{>0}^{m}$ are positive vectors (all entries greater zero). We may assume that each column of $A$ contains at least one positive entry as otherwise the corresponding variable can be dropped and that each row of $A$ contains at least one positive entry as otherwise the problem is infeasible. Assume the availability of the following $\kappa$-approximation oracle:
$\mathcal{O}(A, b, c, z)$ : Given $z \in \mathbb{R}_{+}^{m}, \kappa \in(0,1]$, the oracle finds a column $j$ of $A$ that maximizes $\frac{1}{c_{j}} \sum_{i=1}^{m} \frac{z_{i} a_{i j}^{+}}{b_{i}}$ within a factor of $\kappa$ :

$$
\begin{equation*}
\frac{1}{c_{j}} \sum_{i=1}^{m} \frac{z_{i} a_{i j}}{b_{i}} \geq \kappa \cdot \max _{j^{\prime} \in[n]} \frac{1}{c_{j^{\prime}}} \sum_{i=1}^{m} \frac{z_{i} a_{i j^{\prime}}}{b_{i}} \tag{19}
\end{equation*}
$$

Denote by $z^{*}$ the value of the optimal solution to (18). In his thesis, Khandekar [14] showed the following theorem (for an exact oracle $\kappa=1$ ).
Theorem 4 ([14]). There is an algorithm that, given an error parameter $\epsilon \in$ $(0,1)$ and an oracle $\mathcal{O}(A, b, c, z)$, computes a feasible solution $\hat{x} \in \mathbb{R}_{\geq 0}^{n}$ to (18) such that $c^{T} \hat{x} \leq(1+3 \epsilon) z^{*}$. The algorithm makes $O\left(m \epsilon^{-2} \log m\right)$ calls to the oracle, where $m$ is the number of rows in $A$.

For completeness, we give a proof; our argument is slightly simpler than the original proof. We will next show (in Corollary 3) how to modify this proof to obtain the claim needed for the proof of Theorem 3

The algorithm is given as Algorithm 2 and can be thought of as the algorithmic dual of the FPTAS for multicommodity flows given in [10]. It keeps updating a vector $x(t) \in \mathbb{R}_{\geq 0}^{n}$, for $t=0,1, \ldots$, until $M(t):=\min _{i \in[m]}\left\{A_{i} x(t) / b_{i}\right\}$ becomes at least $T:=\frac{\ln m}{\varepsilon^{2}}$. Let $A_{i}$ denote the $i$ th row of $A$, and define, at any time $t$, the active list $L(t):=\left\{i \in[m]: A_{i} x(t-1)<T\right\}$. For $i \in L(t)$, define

$$
\begin{equation*}
p_{i}(t):=(1-\epsilon)^{A_{i} x(t-1) / b_{i}} \tag{20}
\end{equation*}
$$

and set $p_{i}(t)=0$ for $i \notin L(t)$. At each time $t$, the algorithm calls the oracle with the vector $z=p(t) /\|p(t)\|_{1}$, and increases the variable $x_{j(t)}$ by $\delta(t):=$ $\max _{i \in L(t)}$ and $a_{i, j(t) \neq 0} \frac{b_{i} c_{j(t)}}{a_{i, j(t)}}$, where $j(t)$ is the index returned by the oracle. Note that the RHS of (19) is positive and hence there is always an $i \in L(t)$ such that $a_{i, j(t)}>0$. In each iteration, some entry of $x$ is increased. Since $A$ is a nonnegative matrix and $b, c$ are positive vectors, this implies that the values $A_{i} x(t)$ are non-decreasing, and hence $L(t+1) \subseteq L(t)$ for all $t$.

```
Algorithm 2 Covering \((A, b, c, \mathcal{O})\)
Require: a covering system \((A, b, c)\) given by a \(\kappa\)-approximation oracle \(\mathcal{O}\), where
    \(A \in \mathbb{R}_{\geq 0}^{m \times n}, b \in \mathbb{R}_{>0}^{m}, c \in \mathbb{R}_{>0}^{n}, b>0, c>0\), and an accuracy parameter \(\epsilon \in(0,1)\)
Ensure: A feasible solution \(\hat{x} \in \mathbb{R}_{\geq 0}^{n}\) to (18) s.t. \(c^{T} \hat{x} \leq \frac{(1+3 \epsilon)}{\kappa} z^{*}\)
    \(x(0):=0 ; t:=0 ;\) and \(T:=\frac{\ln m}{\epsilon^{2}}\)
    while \(M(t)<T\) do
        \(t:=t+1\)
        Let \(j(t):=\mathcal{O}\left(A, b, c, \frac{p(t)}{\|p(t)\|_{1}}, \kappa\right)\)
        \(x_{j(t)}(t):=x_{j(t)}(t-1)+\delta(t) ;\)
    end while
    return \(\hat{x}=\frac{x(t)}{M(t)}\)
```

At the end, we scale $x(t)$ by $M(t)$ to guarantee feasibility.
By scaling $a_{i j}$ by $\frac{1}{b_{i} c_{j}}$, we may can assume, for the purpose of the analysis, that $b=\mathbf{1}$ and $c=\mathbf{1}$ are the vectors of all ones (of appropriate dimension). We may also assume that $\epsilon \in\left(0, \frac{1}{2}\right]$. Let $\mathbf{1}_{j}$ denote the $j$-th unit vector of dimension $n$.

Theorem 5. Procedure Covering $(A, b, c, \mathcal{O})$ terminates in at most mT iterations with a feasible solution $\hat{x}$ of at most $m T$ positive components. At termination, it holds that

$$
\begin{equation*}
c^{T} \hat{x} \leq \frac{(1+3 \epsilon)}{\kappa} z^{*} \tag{21}
\end{equation*}
$$

Proof. For simplicity assume $b=\mathbf{1}$ and $c=\mathbf{1}$. Feasibility is obvious since we scale by $M(t)$. Note also that the choice of $\delta(t)$ implies that

$$
\begin{equation*}
\max _{i \in L(t)}\left\{a_{i, j(t)} \delta(t)\right\}=1 \tag{22}
\end{equation*}
$$

This implies the bound in the number of iterations. Observe that in every iteration at least one of the $A_{i} x(t)$ increases by one. Since we remove $i$ from the active list once $A_{i} x(t) \geq T$, any $i \in[m]$ can be the index that is increased by one at most $T$ times. We conclude that the number of iterations is bounded by $m T$.

To show (21), we analyze the increase in the potential function

$$
\Phi(t):=\|p(t+1)\|_{1} .
$$

$$
\begin{align*}
\|p(t+1)\|_{1}= & \sum_{i \in L(t+1)} p_{i}(t+1)=\sum_{i \in L(t+1)}(1-\epsilon)^{A_{i} x_{i}(t)} \\
= & \sum_{i \in L(t+1)}(1-\epsilon)^{A_{i} x_{i}(t-1)+\delta(t) A_{i} \mathbf{1}_{j(t)}} \\
= & \sum_{i \in L(t+1)} p_{i}(t)(1-\epsilon)^{\delta(t) A_{i} \mathbf{1}_{j(t)}} \leq \sum_{i \in L(t+1)} p_{i}(t)\left(1-\epsilon \delta(t) A_{i} \mathbf{1}_{j(t)}\right) \\
& \left(\text { using (22) and }(1-\epsilon)^{x} \leq 1-\epsilon x \text { for all } \epsilon \in[0,1), x \in[0,1]\right) \\
\leq & \sum_{i \in L(t)} p_{i}(t)\left(1-\epsilon \delta(t) A_{i} \mathbf{1}_{j(t)}\right)=\|p(t)\|_{1}\left(1-\frac{\epsilon \delta(t) p(t)^{T} A \mathbf{1}_{j(t)}}{\|p(t)\|_{1}}\right) \\
& (\text { since } L(t+1) \subseteq L(t)) \\
\leq & \|p(t)\|_{1} e^{-\epsilon \delta(t) \frac{p(t) T}{T_{p}(t) \|_{1}} A \mathbf{1}_{j(t)}} .  \tag{23}\\
& \left(\text { since } 1-x \leq e^{-x}\right)
\end{align*}
$$

Iterating, we get

$$
\|p(t+1)\|_{1} \leq\|p(1)\|_{1} e^{-\epsilon \sum_{t^{\prime}=1}^{t} \delta\left(t^{\prime}\right) \frac{p\left(t^{\prime}\right)^{T}}{\left\|p\left(t^{\prime}\right)\right\|_{1}} A \mathbf{1}_{j\left(t^{\prime}\right)}}
$$

for which follows

$$
(1-\epsilon)^{A_{i} x(t)} \leq\|p(1)\|_{1} e^{-\epsilon \sum_{t^{\prime}=1}^{t} \delta\left(t^{\prime}\right) \frac{p\left(t^{\prime}\right)^{T}}{\left\|p\left(t^{\prime}\right)\right\|_{1}} A \mathbf{1}_{j\left(t^{\prime}\right)}} \quad \text { for all } i \in L(t+1)
$$

Taking logs and using $\|p(1)\|_{1}=m$, we conclude that

$$
\begin{equation*}
A_{i} x(t) \cdot \ln (1-\epsilon) \leq \ln m-\epsilon \sum_{t^{\prime}=1}^{t} \delta\left(t^{\prime}\right) \frac{p\left(t^{\prime}\right)^{T}}{\left\|p\left(t^{\prime}\right)\right\|_{1}} A \mathbf{1}_{j\left(t^{\prime}\right)} \quad \text { for all } i \in L(t+1) \tag{24}
\end{equation*}
$$

We will relate the objective value $\mathbf{1}^{T} x(t)=\sum_{t^{\prime}=1}^{t} \delta\left(t^{\prime}\right)$ at time $t$ to the optimal value $z^{*}$ by the following claim.
Claim. $\sum_{t^{\prime}=1}^{t} \delta\left(t^{\prime}\right) \frac{p\left(t^{\prime}\right)^{T}}{\left\|p\left(t^{\prime}\right)\right\|_{1}} A \mathbf{1}_{j\left(t^{\prime}\right)} \geq \frac{\kappa \cdot \mathbf{1}^{T} x(t)}{z^{*}}$.
Proof. Let $x^{*} \in \mathbb{R}_{\geq 0}^{n}$ be an optimal solution to (18). Then $z^{*}=\sum_{j \in[n]} x_{j}^{*}$ since we assumed $c^{T}=1$, also by feasibility of $x^{*}$, we have $A x^{*} \geq 1$, and thus for any $t^{\prime}$,

$$
\frac{p\left(t^{\prime}\right)^{T}}{\left\|p\left(t^{\prime}\right)\right\|_{1}} A x^{*} \geq 1
$$

By the choice of the index $j\left(t^{\prime}\right)$, we have that $\frac{p\left(t^{\prime}\right)^{T}}{\left\|p\left(t^{\prime}\right)\right\|_{1}} A \mathbf{1}_{j\left(t^{\prime}\right)} \geq \kappa \frac{p\left(t^{\prime}\right)^{T}}{\left\|p\left(t^{\prime}\right)\right\|_{1}} A \mathbf{1}_{j}$ for all $j \in[n]$. Thus,

$$
\begin{aligned}
z^{*} \frac{p\left(t^{\prime}\right)^{T}}{\left\|p\left(t^{\prime}\right)\right\|_{1}} A \mathbf{1}_{j\left(t^{\prime}\right)} & =\sum_{j \in[n]} x_{j}^{*} \frac{p\left(t^{\prime}\right)^{T}}{\left\|p\left(t^{\prime}\right)\right\|_{1}} A \mathbf{1}_{j\left(t^{\prime}\right)} \\
& \geq \sum_{j \in[n]} x_{j}^{*} \kappa \frac{p\left(t^{\prime}\right)^{T}}{\left\|p\left(t^{\prime}\right)\right\|_{1}} A \mathbf{1}_{j}=\kappa \frac{p\left(t^{\prime}\right)^{T}}{\left\|p\left(t^{\prime}\right)\right\|_{1}} A x^{*} \geq \kappa
\end{aligned}
$$

Multiplying both sides of this inequality by $\delta\left(t^{\prime}\right)$ and summing up over $1 \leq t^{\prime} \leq t$ finishes the proof.

Using claim (C), we can deduce from (24) that

$$
A_{i} x(t) \cdot \ln (1-\epsilon) \leq \ln m-\epsilon \cdot \frac{\kappa \cdot \mathbf{1}^{T} x(t)}{z^{*}} \quad \text { for all } i \in L(t+1)
$$

Dividing both sides by $M(t)$, arranging, and noting that at termination $M(t) \geq$ $T$, we get that for all $i \in L(t+1)$,

$$
\frac{\kappa \cdot \mathbf{1}^{T} x(t)}{M(t) z^{*}} \leq \frac{\ln \frac{1}{1-\epsilon}}{\epsilon} \cdot \frac{A_{i} x(t)}{M(t)}+\frac{\ln m}{\epsilon \cdot M(t)} \leq \frac{\ln \frac{1}{1-\epsilon}}{\epsilon} \cdot \frac{A_{i} x(t)}{M(t)}+\frac{\ln m}{\epsilon T}
$$

In particular for the index $i \in L(t+1)$ such that $M(t)=A_{i} x(t)$, we have at termination,

$$
\frac{\kappa \cdot \mathbf{1}^{T} \hat{x}}{z^{*}}=\frac{\kappa \cdot \mathbf{1}^{T} x(t)}{M(t) z^{*}} \leq \frac{\ln \frac{1}{1-\epsilon}}{\epsilon}+\frac{\ln m}{\epsilon T} .
$$

Using $T=\frac{\ln m}{\varepsilon^{2}}$, we finally get

$$
\frac{\kappa \cdot \mathbf{1}^{T} \hat{x}}{z^{*}} \leq \frac{\ln \frac{1}{1-\epsilon}}{\epsilon}+\epsilon \leq 1+3 \epsilon
$$

where the last inequality follows from the fact that $\frac{\ln \frac{1}{1-\epsilon}}{\epsilon} \leq 1+2 \epsilon$, valid for all $\epsilon \in\left(0, \frac{1}{2}\right]$.

Corollary 3. Suppose $b=1, c=1$, and we use the following oracle $\mathcal{O}^{\prime}$ instead of $\mathcal{O}$ in Algorithm 圆:
$\mathcal{O}^{\prime}(A, z):$ Given $z \in \mathbb{R}_{+}^{m}$, such that $\mathbf{1}^{T} z=1$, the oracle finds a column $j$ of $A$ such that $z^{T} A \mathbf{1}_{j} \geq 1$.

Then the algorithm terminates in at most $m T$ iterations with a feasible solution $\hat{x}$ having at most $m T$ positive components, such that $\mathbf{1}^{T} \hat{x} \leq 1+3 \epsilon$.
Proof. Using the assumption about $\mathcal{O}^{\prime}$, we can deduce from (24) that

$$
A_{i} x(t) \cdot \ln (1-\epsilon) \leq \ln m-\epsilon \cdot \mathbf{1}^{T} x(t) \quad \text { for all } i \in L(t+1)
$$

Proceeding as in the rest of the proof of Theorem 5, we get the result.

## D An FPTAS for packing linear programs

Consider a packing linear program:

$$
\begin{align*}
\max & c^{T} x  \tag{25}\\
\text { s.t. } & A x \leq b \\
& x \geq 0
\end{align*}
$$

```
Algorithm 3 Packing \((A, b, c, \mathcal{O})\)
Require: a packing system \((A, b, c)\) given by a demand oracle \(\mathcal{O}\), where \(A \in \mathbb{R}_{>0}^{m \times n}\),
    \(b \in \mathbb{R}_{>0}^{m}, c \in \mathbb{R}_{>0}^{n}, b>0\), and an accuracy parameter \(\epsilon \in(0,1)\)
Ensure: A feasible solution \(\hat{x} \in \mathbb{R}_{\geq 0}^{n}\) to (25) s.t. \(c^{T} \hat{x} \geq \frac{1-3 \epsilon}{\kappa} z^{*}\)
    \(x(0):=0 ; t:=0\); and \(T:=\frac{\ln m}{\epsilon^{2}}\)
    while \(M(t)<T\) do
        \(t:=t+1\)
        Let \(j(t):=\mathcal{O}\left(A, b, c, \frac{p(t)}{\|p(t)\|_{1}}\right)\)
        \(x_{j(t)}(t):=x_{j(t)}(t-1)+\delta(t) ;\)
    end while
    return \(\hat{x}=\frac{x(t)}{M(t)}\)
```

where $A \in \mathbb{R}_{\geq 0}^{m \times n}$ is an $m \times n$ matrix with non-negative entries and $c \in \mathbb{R}_{>0}^{n}$, $b \in \mathbb{R}_{>0}^{m}$ are positive vectors. We may assume that each column of $A$ contains a nonzero entry as otherwise the problem is trivially unbounded. Assume the availability of the following $\kappa$-approximation oracle:
$\mathcal{O}(A, b, c, z)$ : Given $z \in \mathbb{R}_{+}^{m}, \kappa \geq 1$, the oracle returns a $j$ such that

$$
\begin{equation*}
\frac{1}{c_{j}} \sum_{i=1}^{m} \frac{z_{i} a_{i j}}{b_{i}} \leq \kappa \cdot \min _{j^{\prime} \in[n]} \frac{1}{c_{j^{\prime}}} \sum_{i=1}^{m} \frac{z_{i} a_{i j^{\prime}}}{b_{i}} \tag{26}
\end{equation*}
$$

We note that an oracle of this form can be simulated by $O(\log U)$ calls to a demand oracle using binary search, where

$$
U:=\max \left\{\max _{j}\left\{V_{j}\right\}, \max _{i}\left\{b_{i}\right\}, 1 / \min _{i, j}\left\{a_{i j}\right\}\right\}
$$

Garg and Könemann proved Theorem 6 for the case $\kappa=1$. For $\kappa=1$, their algorithm is an FPTAS. We give the proof for completeness.
Denote by $z^{*}$ the value of the optimal solution to (25). Consider Algorithm 3. It constructs a sequence $x(1), x(2), \ldots, x(t), \ldots$ Let $M(t)=\max _{i \in[m]} A_{i} x(t) / b_{i}$, where $A_{i}$ is the $i$-th row of $A$. For $i \in[m]$ let

$$
\begin{equation*}
p_{i}(t):=(1+\epsilon)^{A_{i} x(t-1) / b_{i}} . \tag{27}
\end{equation*}
$$

At each time $t$, the algorithm calls the oracle with the vector $z=p(t) /\|p(t)\|_{1}$, and increases the variable $x_{j(t)}$ by $\delta(t):=\max _{i \in[m], a_{i, j(t)} \neq 0}\left\{\frac{b_{i} c_{j(t)}}{a_{i, j(t)}}\right\}$, where $j(t)$ is the index returned by the oracle. Note that there is always a $i$ such that $a_{i, j(t)}>0$ by our assumption on $A$. At the end, we scale $x(t)$ by $M(t)$ to guarantee feasibility.

Theorem 6 ([10]). Let $T=\epsilon^{-2} \ln m$. Procedure Packing $(A, b, c, \mathcal{O})$ terminates in at most $m T$ iterations with a feasible solution $\hat{x}$ of at most $m T$ positive components. At termination, it holds that

$$
\begin{equation*}
c^{T} \hat{x} \geq \frac{(1-3 \epsilon)}{\kappa} z^{*} \tag{28}
\end{equation*}
$$

Proof. We may assume $b=c=1$. As in the proof of Theorem 5, we can show that

$$
\begin{equation*}
A_{i} x(t) \cdot \ln (1+\epsilon) \leq \ln m+\epsilon \sum_{t^{\prime}=1}^{t} \delta\left(t^{\prime}\right) \frac{p\left(t^{\prime}\right)^{T}}{\left\|p\left(t^{\prime}\right)\right\|_{1}} A \mathbf{1}_{j(t)} \quad \text { for all } i \in[m] \tag{29}
\end{equation*}
$$

We will relate the objective value $\mathbf{1}^{T} x(t)=\sum_{t^{\prime}=1}^{t} \delta\left(t^{\prime}\right)$ at time $t$ to the optimal value $z^{*}$ by the following claim.
Claim. $\sum_{t^{\prime}=1}^{t} \delta\left(t^{\prime}\right) \frac{p\left(t^{\prime}\right)^{T}}{\left\|p\left(t^{\prime}\right)\right\|_{1}} A \mathbf{1}_{j\left(t^{\prime}\right)} \leq \frac{\kappa \cdot \mathbf{1}^{T} x(t)}{z^{*}}$.
Proof. Let $x^{*} \in \mathbb{R}_{\geq 0}^{n}$ be an optimal solution to (25). Then by feasibility of $x^{*}$, $A x^{*} \leq 1$, thus we have for any $t^{\prime}$,

$$
\frac{p\left(t^{\prime}\right)^{T}}{\left\|p\left(t^{\prime}\right)\right\|_{1}} A x^{*} \leq 1
$$

By the choice of the index $j\left(t^{\prime}\right)$, we have that $\frac{p\left(t^{\prime}\right)^{T}}{\left\|p\left(t^{\prime}\right)\right\|_{1}} A \mathbf{1}_{j\left(t^{\prime}\right)} \leq \kappa \frac{p\left(t^{\prime}\right)^{T}}{\left\|p\left(t^{\prime}\right)\right\|_{1}} A \mathbf{1}_{j}$ for all $j \in[n]$. Thus,

$$
\begin{aligned}
z^{*} \frac{p\left(t^{\prime}\right)^{T}}{\left\|p\left(t^{\prime}\right)\right\|_{1}} A \mathbf{1}_{j\left(t^{\prime}\right)} & =\sum_{j \in[n]} x_{j}^{*} \frac{p\left(t^{\prime}\right)^{T}}{\left\|p\left(t^{\prime}\right)\right\|_{1}} A \mathbf{1}_{j\left(t^{\prime}\right)} \\
& \leq \sum_{j \in[n]} x_{j}^{*} \kappa \frac{p\left(t^{\prime}\right)^{T}}{\left\|p\left(t^{\prime}\right)\right\|_{1}} A \mathbf{1}_{j}=\kappa \frac{p\left(t^{\prime}\right)^{T}}{\left\|p\left(t^{\prime}\right)\right\|_{1}} A x^{*} \leq \kappa
\end{aligned}
$$

Multiplying both sides of this inequality by $\delta\left(t^{\prime}\right)$ and summing up over $1 \leq t^{\prime} \leq t$ finishes the proof.

Using the above claim, we can deduce from (29) that

$$
A_{i} x(t) \cdot \ln (1+\epsilon) \leq \ln m+\epsilon \cdot \frac{\kappa \cdot \mathbf{1}^{T} x(t)}{z^{*}} \quad \text { for all } i \in[m]
$$

Dividing both sides by $M(t)$, arranging, and noting that at termination $M(t) \geq$ $T$, we get that for all $i \in[m]$,

$$
\frac{\kappa \cdot \mathbf{1}^{T} x(t)}{M(t) z^{*}} \geq \frac{\ln (1+\epsilon)}{\epsilon} \cdot \frac{A_{i} x(t)}{M(t)}-\frac{\ln m}{\epsilon \cdot M(t)} \geq \frac{\ln (1+\epsilon)}{\epsilon} \cdot \frac{A_{i} x(t)}{M(t)}-\frac{\ln m}{\epsilon T}
$$

In particular for the index $i \in[m]$ such that $M(t)=A_{i} x(t)$, we have at termination,

$$
\frac{\kappa \cdot \mathbf{1}^{T} \hat{x}}{z^{*}}=\frac{\kappa \cdot \mathbf{1}^{T} x(t)}{M(t) z^{*}} \geq \frac{\ln (1+\epsilon)}{\epsilon}-\frac{\ln m}{\epsilon T}
$$

Using $T=\frac{\ln m}{\varepsilon^{2}}$, we finally get

$$
\frac{\kappa \cdot \mathbf{1}^{T} \hat{x}}{z^{*}} \geq \frac{\ln (1+\epsilon)}{\epsilon}-\epsilon \geq 1-3 \epsilon
$$

where the last inequality follows from the fact that $\frac{\ln (1+\epsilon)}{\epsilon} \geq 1-2 \epsilon$, valid for all $\epsilon \in\left(0, \frac{1}{2}\right]$.

The bound on the number of iterations follows from the fact that $\sum_{i} A_{i} x(t)$ grows by at least one in each iteration.


[^0]:    ${ }^{4}$ In the combinatorial auction problem, there are variables $x_{i, S}$. Let $q_{i}$ be the multiplier for the constraint corresponding to agent $i$ and let $p_{j}$ be the multiplier (price) for item $j$. Then the demand oracle must find the pair $(i, S)$ maximizing $v_{i}(S)-q_{i}-\sum_{j \in S} p_{j}$, a question of direct economical interpretation.

[^1]:    ${ }^{5}$ Let $x$ be a feasible solution with all components positive (we may drop components that are zero). After adding slack variables and removing all columns corresponding to zero variables, we may assume that our system is of the form $A x=b$ and that $A$ has full row rank. Split $A$ into $A_{B}$ and $A_{N}$ where $A_{B}$ is square and invertible. Split the variables and the objective function $V$ accordingly. Then $x_{B}=A_{B}^{-1}\left(b-A_{N} x_{N}\right)$ and $z=V_{B} A_{B}^{-1} b+\left(V_{N}-V_{B}^{T} A_{B}^{-1} A_{N}\right) x_{N}$. Consider any variable in $x_{N}$. Its coefficient in $z$ tells us whether we should increase or decrease to improve the objective value. Do so until some variable becomes zero and eliminate the variable. Repeat until $A$ is

[^2]:    ${ }^{6}$ Formally, $\delta^{x}$ is the Dirac delta density satisfying: $\delta(x-y)=+\infty$, if $y=x$, and $\delta^{x}(y)=0$ for $y \neq x$ and $\int_{y \in \mathcal{Q}} \delta(x-y) d y=1$.

