Universal Abelian Covers for Surface Singularities $\{z^n = f(x, y)\}$

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ABSTRACT

ELIZABETH ANNE SELL: Universal Abelian Covers for Surface Singularities

$\{z^n = f(x, y)\}$

(Under the direction of Professor Jonathan Wahl)

In recent work, W. D. Neumann and J. Wahl construct explicit equations for many interesting normal surface singularities with rational homology sphere links, which they call splice quotients. The construction begins with the topological type of a normal surface singularity, that is, a good resolution graph Γ that is a tree of rational curves. If Γ satisfies certain combinatorial conditions, then there exist splice quotients with resolution graph Γ. Let $\{z^n = f(x, y)\}$ define a surface $X_{f,n}$ with an isolated singularity at the origin in $\mathbb{C}^3$. For f irreducible, we completely characterize, in terms of n and the Puiseux pairs of f, those $X_{f,n}$ for which the resolution graph satisfies the combinatorial conditions defined by Neumann and Wahl. Briefly stated, we find that the conditions are not often satisfied. Furthermore, given a splice quotient $(X, 0)$, it turns out that "equisingular deformations" of $(X, 0)$ are usually not splice quotients, as we demonstrate already for singularities of the form $\{z^2 = x^P + y^Q\}$ with rational homology sphere link.

To Professor Joseph Ferrar

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CHAPTER 1

Introduction

Overview. Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be the germ of a complex analytic normal surface singularity. The intersection of X with a sufficiently small sphere centered at the origin in $\mathbb{C}^n$ is a compact connected oriented three-manifold Σ that does not depend upon the embedding in $\mathbb{C}^n$. This manifold is called the link of $(X, 0)$. Since $X \setminus \{0\}$ is homeomorphic to the cone over Σ, the homeomorphism type of the link determines the topology of $(X, 0)$. The dual resolution graph Γ of a good resolution of the singularity is also a topological invariant; that is, the homeomorphism type of the link can be recovered from Γ, and W. Neumann proved that (aside from a few exceptions) the converse is true as well [**15**]. In general, there can be many different *analytic* types of singularities that have the same link.

Let Σ be the link of a normal surface singularity $(X, 0)$, and assume that Σ is a *rational homology sphere* ($\mathbb{Q}$HS). That is, $H_1(\Sigma, \mathbb{Q}) = 0$, or, equivalently, $H_1(\Sigma, \mathbb{Z})$ is a finite group. The link is a $\mathbb{Q}$HS if and only if any good resolution graph Γ of $(X, 0)$ is a tree of rational curves. The structure of the finite abelian group $H_1(\Sigma, \mathbb{Z})$ can be determined from the resolution graph Γ, and this group is also referred to as the *discriminant group* $D(\Gamma)$. The universal abelian cover (UAC) of Σ, in the topological sense, is a regular covering space with automorphism group $D(\Gamma) = H_1(\Sigma, \mathbb{Z})$. Coverings of Σ correspond to coverings of the manifold $X \smallsetminus \{0\}$. The analytic structure of $X \smallsetminus \{0\}$ lifts to an analytic structure on its UAC, and there is a unique way to add a point 0 to the UAC, resulting in a singularity $(Y, 0)$ and a finite map $(Y, 0) \to (X, 0)$ that is unramified away from the singular point. We refer to the singularity $(Y, 0)$ together with with this map as the *universal abelian cover of the singularity* $(X, 0)$.

In general, it is not easy to produce an analytic realization, i.e., defining equations, for a normal surface singularity with a given topological type. This is where universal abelian covers play a significant and surprising role. It turns out that in some cases it is easier to produce equations for the UAC than it is to produce them for the singularity itself.

Although the connection is seen only in retrospect, one can find this idea in the work of F. Klein [**4**], who determined defining equations for the quotient singularities $\mathbb{C}^2/G$, for finite non-cyclic subgroups G of SU(2). These equations are sometimes complicated, but equations for $\mathbb{C}^2/[G, G]$, where $[G, G]$ denotes the commutator subgroup of G, are simple. They are all of the form $x^p + y^q + z^r = 0$, and the values of p, q, and r are determined by G. Since $\mathbb{C}^2 \setminus \{0\}$ is the universal cover of $\mathbb{C}^2 \setminus \{0\}/G$, one can see that $\mathbb{C}^2/[G, G] \to \mathbb{C}^2/G$ is the universal abelian cover in the sense above. Klein's result was extended to several other groups by J. Milnor in [**10**].

In [**14**], Neumann produced equations for the universal abelian covers of weighted homogeneous surface singularities with $\mathbb{Q}$HS link; the equations are a generalization of those that appear in Klein's work for $\mathbb{C}^2/[G, G]$. Recently, the concept was extended by Neumann and J. Wahl [**18**] to a larger class of singularities. This work has led to a recent interest in universal abelian covers and related topics (see [**7**], [**13**], [**20**], [**21**], [**24**]), and there are still many unanswered questions.

The work of Neumann and Wahl (described in Chapter 2; see also [**17**] and [**26**]) provides a method for generating analytic data for singularities from topological data. Starting with a resolution graph Γ that satisfies certain combinatorial conditions, the "semigroup conditions", there is an algorithm that produces equations for a family of complete intersection surface singularities. The equations are referred to as *splice diagram equations*, since they are generated from the *splice diagram*, another combinatorial object that depends upon Γ. If Γ satisfies a further set of conditions, the "congruence conditions", then one can choose a set of splice diagram equations such that every singularity $(Y, 0)$ in the family is the universal abelian cover of a singularity with resolution graph Γ. The algorithm also produces an explicit action of the discriminant group $D(\Gamma)$ on Y such that the quotient of Y by this group action is a normal surface singularity with resolution graph Γ. The resulting quotient singularities are called *splice quotients*. If Σ is a $\mathbb{Z}$HS $(H_1(\Sigma, \mathbb{Z}) = 0)$, then only the semigroup conditions are relevant. When they are satisfied, the family of normal surface singularities produced by the algorithm are said to be *of splice type*.

The upshot is that the algorithm allows one to produce defining equations for singularities of a given topological type, and the key point is that the construction goes through the universal abelian cover, which in general has much simpler equations than those of the quotient.

Given a normal surface singularity $(X, 0)$ with $\mathbb{Q}$HS link and a good resolution graph Γ, one can ask the following questions:

(1) Does Γ satisfy the semigroup and congruence conditions needed for the Neumann-Wahl algorithm?

(2) If the answer to (1) is yes, is $(X, 0)$ itself a splice quotient? That is, does the Neumann-Wahl algorithm produce a singularity that is analytically isomorphic to $(X, 0)$?

(3) If $(X, 0)$ is a splice quotient, are "equisingular deformations" of $(X, 0)$ also splice quotients?

(4) If the answer to (1) or (2) is no, what is the UAC of $(X, 0)$?

It was originally conjectured that rational and $\mathbb{Q}$HS-link minimally elliptic singularities would be splice quotients, and one wondered whether all $\mathbb{Q}$-Gorenstein singularities with $\mathbb{Q}$HS link would turn out to be splice quotients. But, counterexamples were found in the paper of I. Luengo-Velasco, A. Melle-Hernández, and A. Némethi [**7**]. There, the authors give an example of a hypersurface singularity for which the resolution graph does not satisfy the semigroup conditions, and an example of a singularity for which the semigroup and congruence conditions are satisfied, but the analytic type is not a splice quotient. However, there are nice classes of singularities for which the answer to both (1) and (2) is yes, namely weighted homogeneous singularities, as shown by Neumann in [**14**], and rational and $\mathbb{Q}$HS-link minimally elliptic singularities, as shown by T. Okuma in [**21**]. Question (3) was raised by Neumann and Wahl in [**17**], and we show in Chapter 3 that the answer is generally no. As for question (4), very little is known. A recent preprint of J. Stevens [**24**] constructs the UAC (using a completely different method) for some particular examples of singularities for which the semigroup conditions are not satisfied.

Results. Suppose $\{f(x,y)=0\}$ defines a reduced curve with a singularity at the origin in $\mathbb{C}^2$. Then for $n>1$, the surface

$$X_{f,n} := \{z^n = f(x,y)\}$$

has an isolated (hence normal) singularity at the origin in $0 \in \mathbb{C}^3$. In this dissertation, we study singularities of the form $(X_{f,n}, 0)$ with $\mathbb{Q}$HS link. This is a natural class of singularities to study after weighted homogeneous, rational, and minimally elliptic singularities, and is not as difficult as the entire class of hypersurface singularities. One reason for this is that there is a well-known algorithm for constructing the resolution graph of such singularities from the topological data of the plane curve singularity defined by $f(x,y)=0$. One of the main results of this work is a complete characterization of the $(X_{f,n},0)$, with f irreducible, that have a resolution graph that satisfies the semigroup and congruence conditions (Theorem 6.0.1). In particular, even for $n=2$, it is rare that the conditions are satisfied.

For f irreducible, the construction of the resolution graph of $(X_{f,n},0)$ requires a finite set of pairs of positive integers associated to f, known as the *topological pairs* $(p_1,a_1),\ldots,(p_s,a_s)$ (a variant of the more commonly known *Puiseux pairs*), defined in [**3**]. These pairs satisfy the following properties: p_i and a_i are relatively prime for each i, $a_1>p_1$, and $a_{i+1}>a_ip_ip_{i+1}$. The topological pairs completely determine the topology of the plane curve singularity. More specifically, they describe the cabling of the iterated torus knot that results from the intersection of $\{f(x,y)=0\}$ with a small sphere around the origin in $\mathbb{C}^2$. If f_1 and f_2 have the same topological pairs, then $(X_{f_1,n},0)$ and $(X_{f_2,n},0)$ have the same link (equivalently the same resolution graph), but are not necessarily analytically isomorphic.

The $\mathbb{Z}$HS case has already been studied. In [**16**], Neumann and Wahl prove that the link of $(X_{f,n},0)$ is a $\mathbb{Z}$HS if and only if f is irreducible and all p_i and a_i are relatively prime to n, and in that case, they prove in [**19**] that any such $(X_{f,n},0)$ is of splice type. That is, not only are the semigroup conditions satisfied, but furthermore, every $(X_{f,n},0)$ with $\mathbb{Z}$HS link is isomorphic to one that results from the Neumann-Wahl algorithm.

For f irreducible, there is an explicit criterion, given by R. Mendris and Némethi in [**8**], in terms of n and the topological pairs that determines when the link of $(X_{f,n},0)$ is a $\mathbb{Q}$HS (see Proposition 4.1.2). Stated briefly, it says that the link is a $\mathbb{Q}$HS if and only if there does not exist an i for which both a_i and p_i have prime factors in common with $n/(n,p_{i+1}\cdots p_s)$ (n for $i=s$). So one can see that there are plenty of $(X_{f,n},0)$ for which the link is a $\mathbb{Q}$HS but not a $\mathbb{Z}$HS. For the rest of this discussion, we assume that the link of $(X_{f,n},0)$ is a $\mathbb{Q}$HS.

Let the resolution graph of $(X_{f,n},0)$ be denoted $\Gamma_{f,n}$, and let s be the number of topological pairs associated to f. If $s=1$, then $(X_{f,n},0)$ has the topological type of the weighted homogeneous singularity defined by $z^n = x^{a_1} + y^{p_1}$, which, as mentioned above, is a splice quotient. The analytic types for $s=1$ will be discussed further below. The following is one of our main results.

THEOREM (6.0.1). *Let f be irreducible with $s \geq 2$ topological pairs, and assume that $(X_{f,n},0)$ has $\mathbb{Q}$HS link. Then $\Gamma_{f,n}$ satisfies the semigroup and congruence conditions if and only if either*

(i) *$(n,p_s)=1,\ (n,p_i)=(n,a_i)=1$ for $1 \leq i \leq s-1$, and $a_s/(n,a_s)$ is in the semigroup generated by $\{a_{s-1},\ p_1\cdots p_{s-1},\ a_j p_{j+1}\cdots p_{s-1}\ :\ 1 \leq j \leq s-2\}$, or*

(ii) *$s=2,\ p_2=2,\ (n,p_2)=2$, and $(\frac{n}{2},p_1)=(\frac{n}{2},a_1)=1$.*

(Here, (a,b) denotes the greatest common divisor of a and b.)

Excluding Case (ii), which is rather restrictive, this result says that if any of the topological pairs (p_i,a_i) besides a_s have factors in common with n, then $(X_{f,n},0)$ does not have the topological type of a splice quotient. So, one could say that if $(X_{f,n},0)$ gets "too far" from the $\mathbb{Z}$HS case (in which all *analytic* types are splice quotients), it cannot even have the topology of a splice quotient.

Consider the following example:

$$X_n := \{z^n = y^5 - (x^3+y^2)^2\}.$$

The plane curve singularity defined by $y^5-(x^3+y^2)^2=0$ is irreducible with 2 topological pairs,

$$p_1=2,\ a_1=3,\ p_2=2,\ \text{and}\ a_2=15.$$

The link of $(X_n, 0)$ is a $\mathbb{Q}$HS if and only if neither 6 nor 10 divides n. We can say the following about X_n :

- If n is not divisible by 2, 3, or 5, then $(X_n, 0)$ has $\mathbb{Z}$HS link and hence is a splice quotient. In fact, we could replace $y^5 - (x^3 + y^2)^2$ by any curve with the same topological pairs, and we would still have a splice quotient.
- If n is divisible by 3, Theorem 6.0.1 says that $(X_n, 0)$ does not even have the topological type of a splice quotient.
- If $n = 5k$, where k is not divisible by 2 or 3, then $(X_n, 0)$ has the topology of a splice quotient (Case (i) of Theorem 6.0.1). The discriminant group has order 16. In Chapter 7, we show that $(X_n, 0)$ is itself a splice quotient. However, if we replace $y^5 - (x^3 + y^2)^2$ by another curve with the same topological pairs, it is unlikely that the new singularity will be a splice quotient.
- If $n = 2k$, where k is not divisible by 3 or 5, then $(X_n, 0)$ has the topology of a splice quotient (Case (ii) of Theorem 6.0.1). The discriminant group has order 15. It is unclear whether or not $(X_n, 0)$ is a splice quotient. However, if we replace $y^5 - (x^3 + y^2)^2$ by $(x^3 - y^2 - y^3)^2 - 4y^5$, which has the same topological pairs, it is a splice quotient.

This brings us to an important point. If a resolution graph satisfies the semigroup and congruence conditions, a priori we do not know what the equations of the splice quotients produced from the Neumann-Wahl algorithm look like. In Cases (i) and (ii) of Theorem 6.0.1, we know that there are splice quotients with the same topological type of an $(X_{f,n}, 0)$, but it is not immediately clear whether or not there even exist any splice quotients defined by an equation of the form $z^n = g(x, y)$. By studying the analytic types of the splice quotients, we are able to prove the following

Theorem (7.1.1 and 7.2.1). *Suppose $\Gamma_{f,n}$ satisfies the semigroup and congruence conditions. Then there exists a splice quotient that is defined by an equation of the form $z^n = g(x, y)$, where g is irreducible and has the same topological pairs as f.*

As we mentioned above, another question of interest is the behavior of splice quotients under deformation. Given a splice quotient $(X, 0)$, it is not necessarily true that any deformation of $(X, 0)$ with the same topological type is also a splice quotient. For example, the weighted homogeneous singularity defined by

$$z^2 = x^4 + y^9$$

is a splice quotient, but one can show that the deformation

$$z^2 = x^4 + y^9 + txy^7$$

is not, although it has the same link as the original.

For normal surface singularities in general, there is not a clear notion of what "equisingular deformations" should be. However, for weighted homogeneous surface singularities, there is one reasonable definition. Recall that a singularity $(X, 0)$ is *weighted homogeneous* if it is defined by weighted homogeneous polynomials. One can consider the equisingular deformations to be those given by adding higher weight terms to the defining equations of X. (For further discussion, see the beginning of Chapter 3.)

We thoroughly investigate this problem for singularities of the form $\{z^2 = x^P + y^Q\}$ with $\mathbb{Q}$HS link. There turn out to be three separate cases to consider, and in each case, we are able to give a versal splice quotient deformation (Propositions 3.2.3, 3.2.7, and 3.2.20). In two of the three cases, the versal equisingular deformation is in general not a family of splice quotients; that is, there exist equisingular deformations that are not splice quotients.

Outline of dissertation. In Chapter 2, we provide a summary of the Neumann-Wahl algorithm from [**18**], which we use throughout this work. In particular, the chapter contains explicit definitions of the semigroup conditions, splice diagram equations, action of the discriminant group, and congruence conditions.

Chapter 3 contains the discussion of the equisingular deformations of singularities of the form $\{z^2 = x^P + y^Q\}$ with $\mathbb{Q}$HS link. In the first part of the chapter, we outline our approach to the problem in general, and in the second part, we explicitly treat each of the three cases mentioned above. Chapter 3 is essentially independent of all of the chapters that follow.

Chapters 4, 5, and 6 are devoted to proving Theorem 6.0.1 on the characterization of the topological types of splice quotients of the form $(X_{f,n}, 0)$. Some of the computations that are necessary for the proof depend greatly upon work done by Mendris and Némethi in [**8**]. Section 4.1 contains a description of the minimal good resolution graph of $(X_{f,n}, 0)$; for the most part, the section is a reiteration of material that appears in [**8**]. In Section 4.2, we compute everything that is needed in order to explicitly describe the associated splice diagram. In Section 5.1, we describe the semigroup conditions for the splice diagram associated to $(X_{f,n}, 0)$ as completely as possible. Section 5.2 contains additional computations that we need in order to check the congruence conditions. Finally, in Chapter 6, we use the computations from the previous two chapters to prove Theorem 6.0.1.

After obtaining the characterization of the topological types of splice quotients, we investigate the analytic types of the splice quotients themselves in Chapter 7. The main results appear in Theorems 7.1.1 and 7.2.1, as stated above. We also give several concrete examples to illustrate the results.

CHAPTER 2

The Neumann-Wahl algorithm

This chapter contains a summary of the results in [**18**] that we apply throughout this work. The Neumann-Wahl algorithm begins with a negative-definite resolution graph Γ that is a tree of rational curves and the *splice diagram* Δ associated to Γ. Splice diagrams were introduced by Eisenbud and Neumann [**3**] for plane curve singularities (building on work of Siebenmann), and later generalized by Neumann and Wahl. The splice diagram is also a means for encoding topological data, but in general it carries less information than Γ; that is, one cannot always recover Γ from Δ. If Δ satisfies some combinatorial conditions, the "semigroup conditions" (Definition 2.0.1), then the algorithm produces a set of equations that defines a family of isolated complete intersection surface singularities. The algorithm also produces an action of the finite abelian group $D(\Gamma)$, the discriminant group of Γ, on the coordinates used for the splice diagram equations. If Γ satisfies further combinatorial conditions, the "congruence conditions" (Definition 2.0.7), then one can choose a set of splice diagram equations such that the discriminant group acts on every singularity $(Y, 0)$ in the family. Furthermore, the quotient of $(Y, 0)$ by $D(\Gamma)$ is an isolated normal surface singularity with resolution graph Γ, and the covering given by the quotient map is the universal abelian covering. The resulting quotient singularities are called *splice quotients*. If the discriminant group is trivial (i.e., the associated link Σ is a $\mathbb{Z}$HS), then only the semigroup conditions are relevant. When they are satisfied, the family of normal surface singularities produced by the algorithm are said to be *of splice type*. We describe this more completely below.

Terminology: In a weighted graph, the *valence* of a vertex is the number of adjacent edges. A *node* is a vertex of valence at least three, a *leaf* is a vertex of valence one, and a *string* is a connected subgraph that does not include a node.

The procedure for computing the splice diagram Δ associated to a resolution graph Γ that is a tree of rational curves is as follows. First, omit the self-intersection numbers of the vertices and contract all strings of valence two vertices in Γ. To each node v in the resulting diagram Δ, we attach a weight d_{ve} in the direction of each adjacent edge e. Let Γ_{ve} be the subgraph of Γ defined as follows. Remove the vertex that corresponds to the node v, and the edge that corresponds to e, and let Γ_{ve} be the remaining connected subgraph that was connected to v by e. Then the weight d_{ve} is $\det(-C_{ve})$, where C_{ve} is the intersection matrix of the graph Γ_{ve}. Figure 2.1 contains a simple example, the resolution graph of $\{z^4 = x^2y^8 - (x^5 + y^3)^3\}$ and the associated splice diagram.

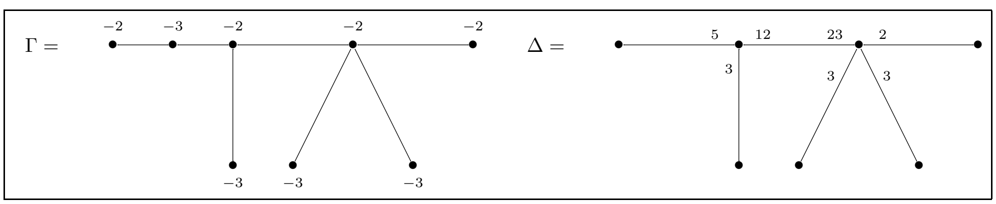


FIGURE 2.1. A resolution graph Γ and its associated splice diagram Δ.

Let Δ be the splice diagram associated to Γ. Let v be a node in Δ, e an edge adjacent to v, and d_{ve} the weight at v in the direction of e. We define a subgraph Δ_{ve} as follows. Remove v and e, and let Δ_{ve} be the remaining connected subgraph that was connected to v by e. For any two vertices v and w in Δ, the *linking number* ℓ_{vw} is the product of the weights adjacent to but not on the shortest path from v to w. Let ℓ'_{vw} be the linking number of v and w, excluding the weights around v and w.

Definition 2.0.1 (Semigroup Conditions). The *semigroup condition* at v in the direction of e is

$$d_{ve} \in \mathbb{N}\langle \ell'_{vw} \mid w \text{ is a leaf in } \Delta_{ve}\rangle.$$

We say that Δ *satisfies the semigroup conditions* if the semigroup condition for every node v and every adjacent edge e is satisfied. Note that for an edge leading to a leaf, the condition is trivial.

To each leaf w in Δ, associate a variable Z_w. If Δ satisfies the semigroup conditions, then for each v and e as above, we can write

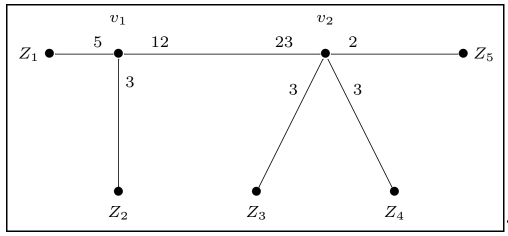
$$d_{ve} = \sum_{w \in \Delta_{ve}} \alpha_{vw} \ell'_{vw},$$

with $\alpha_{vw} \in \mathbb{N}$. Then a monomial

$$M_{ve} = \prod_{w \in \Delta_{ve}} Z_w^{\alpha_{vw}},$$

with α_{vw} as above is called an *admissible monomial* for e at v. For an edge e leading directly to a leaf w, the only choice of admissible monomial is $Z_w^{d_{ve}}$. Note that if one associates the weight ℓ_{vw} to Z_w, then for this weight system, the so-called *v-weighting*, M_{ve} is weighted-homogeneous of total weight $d_v = \prod_e d_{ve}$, where the product is taken over all edges e adjacent to v.

Example 2.0.2. In the example from Figure 2.1, Δ satisfies the semigroup conditions, since 23 is in the semigroup generated by 3 and 5, and 12 is the semigroup generated by 6 (and 9). Let the variables $Z_1, \dots, Z_5$ correspond to the leaves of Δ as follows:



Then there are three choices for an admissible monomial at the node v_1 for the edge with weight 12: Z_3^2, Z_4^2, and Z_3Z_4. There are two choices for an admissible monomial at v_2 for the edge with weight 23: $Z_1Z_2^4$ and $Z_1^6Z_2$.

Definition 2.0.3 (Splice Diagram Equations). Suppose Δ satisfies the semigroup conditions. For each node v and adjacent edge e, choose an admissible monomial M_{ve}. Let δ_v denote the valence of the vertex v. A a set of *splice diagram equations* for Δ is a set of equations of the form

$$\left\{\sum_e a_{vie}M_{ve} + H_{vi} = 0 \;:\; 1 \le i \le \delta_v - 2,\; v \text{ a node in } \Delta\right\},$$

where

- for each v, all maximal minors of the matrix (a_{vie}) have full rank,
- each H_{vi} is a convergent power series in the Z_w for which all of the terms have v-weight greater than or equal to d_v.

It is easy to see that one can always use the following matrix in place of (a_{vie}):

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & a_1 & b_1 \\ 0 & 1 & \cdots & 0 & a_2 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{\delta_v-2} & b_{\delta_v-2} \end{pmatrix},$$

where all a_i and b_i are nonzero, and $a_ib_j - a_jb_i \neq 0$ for all $i \neq j$.

A set of splice diagram equations for Example 2.0.2 is given by

$$\left\{ \begin{array}{rcl} Z_1^5 + aZ_2^3 + bZ_3Z_4 + H & = & 0 \\ Z_1Z_2^4 + c_1Z_4^3 + d_1Z_5^2 + G_1 & = & 0 \\ Z_3^3 + c_2Z_4^3 + d_2Z_5^2 + G_2 & = & 0 \end{array} \right\},$$

where $a,\ b,\ c_1,\ c_2,\ d_1$ and d_2 are all nonzero, and $c_1d_2 - c_2d_1 \neq 0$. The first equation belongs to the node v_1, and the second two equations belong to v_2. Therefore, H must be a convergent power series with v_1-weight greater than or equal to 180, and G_1 and G_2 must be convergent power series with v_2-weight greater than or equal to 414.

We now describe the aforementioned action of the discriminant group. Let Γ be a negative-definite tree of rational curves (equivalently, the dual resolution graph associated to a good resolution of a normal surface singularity with $\mathbb{Q}$HS link). Each vertex $v \in \Gamma$ corresponds to an exceptional curve E_v. Let

$$\mathbb{E} := \bigoplus_{v \in \Gamma} \mathbb{Z}E_v.$$

The intersection pairing defines a natural injection $\mathbb{E} \hookrightarrow \mathbb{E}^* = \mathrm{Hom}(\mathbb{E}, \mathbb{Z})$, and the discriminant group is the finite abelian group

$$D(\Gamma) := \mathbb{E}^*/\mathbb{E}.$$

The order of $D(\Gamma)$ is $\det(\Gamma) := \det(-C(\Gamma))$, where $C(\Gamma) : \mathbb{E} \times \mathbb{E} \to \mathbb{Z}$ is the intersection pairing. There are induced symmetric pairings of $\mathbb{E} \otimes \mathbb{Q}$ into $\mathbb{Q}$ and $D(\Gamma)$ into $\mathbb{Q}$.

Let $e_v \in \mathbb{E}^*$ be the dual basis element corresponding to E_v. That is, $e_v(E_w) = \delta_{vw}$. Neumann and Wahl proved the following propositions.

Proposition 2.0.4 ([**18**], Prop. 5.2). *Let $e_1, \dots, e_t$ be the dual basis elements of $\mathbb{E}^*$ corresponding to the t leaves of Γ. Then the homomorphism $\mathbb{E}^* \to \mathbb{Q}^t$ defined by*

$$e \mapsto (e \cdot e_1, \dots, e \cdot e_t)$$

induces an injection $D(\Gamma) \hookrightarrow (\mathbb{Q}/\mathbb{Z})^t$.

Consider the map $(\mathbb{Q}/\mathbb{Z})^t \hookrightarrow (\mathbb{C}^*)^t$ defined by

$$(r_1, \dots, r_t) \mapsto (\exp(2\pi i r_1), \dots, \exp(2\pi i r_t)) =: [r_1, \dots, r_t].$$

Proposition 2.0.5 ([**18**], Prop. 5.3). *Let the leaves of a resolution graph Γ be $w_1, \dots, w_t$. Then the discriminant group $D(\Gamma)$ is naturally represented by a diagonal action on $\mathbb{C}^t$, where the entries are t-tuples of* $\det(\Gamma)$*-th roots of unity. Each leaf w_j corresponds to an element*

$$[e_j \cdot e_1, \dots, e_j \cdot e_t] := (\exp(2\pi i(e_j \cdot e_1)), \dots, \exp(2\pi i(e_j \cdot e_t))).$$

Any $t-1$ of these elements generate $D(\Gamma)$.

Definition 2.0.6. This diagonal action on $\mathbb{C}^t$ induces a natural action of the discriminant group on the polynomial ring $\mathbb{C}[Z_1, \dots, Z_t]$ via

$$e \cdot Z_k = [-e \cdot e_k] Z_k = \exp(-2\pi i(e \cdot e_k)) Z_k.$$

Definition 2.0.7 (Congruence conditions). Let Γ be a graph for which the associated splice diagram Δ satisfies the semigroup conditions. Then we say that Γ satisfies the *congruence condition* at a node v if one can choose an admissible monomial for each adjacent edge e such that all of these monomials transform by the same character under the action of $D(\Gamma)$. If this condition is satisfied for every node v, then Γ *satisfies the congruence conditions.*

In other words, Γ satisfies the congruence conditions if and only if there is a set of splice diagram equations for Δ on which the discriminant group acts equivariantly.

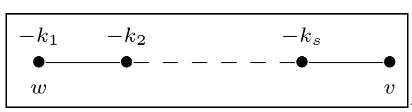
We should mention here that Okuma gives a single condition that is equivalent to the semigroup and congruence conditions together, "Condition 3.3" of [**21**]. That this condition is equivalent to the semigroup and congruence conditions is shown in [**18**].

From now on, we will often say "Γ satisfies the semigroup and congruence conditions", as opposed to "Δ satisfies the semigroup conditions and Γ satisfies the congruence conditions". The next three propositions show that the congruence conditions are, in theory, not difficult to check.

Proposition 2.0.8 ([**18**], Prop. 6.5). *Let w, w' be distinct leaves of Γ, corresponding to distinct leaves of Δ, and let $\ell_{ww'}$ denote their linking number. Then*

$$e_w \cdot e_{w'} = -\ell_{ww'} / \det(\Gamma).$$

Proposition 2.0.9 ([**18**], Prop. 6.6). *Suppose we have a string from a leaf w to an adjacent node v with associated continued fraction d/p, as in the following diagram,*



,

where

$$\frac{d}{p} = k_1 - \cfrac{1}{k_2 - \cfrac{1}{\ddots - \cfrac{1}{k_s}}}.$$

Then, if d_v is the product of weights at v,

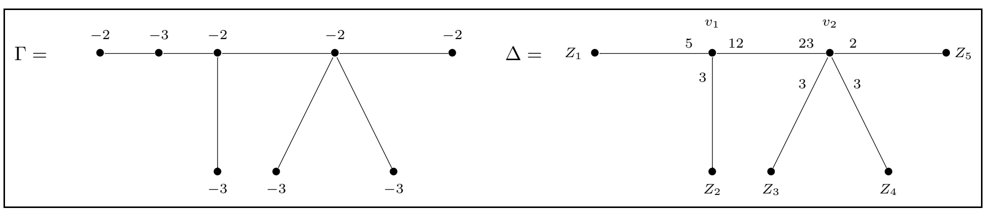
$$e_w \cdot e_w = -d_v/(d^2 \det(\Gamma)) - p/d.$$

Proposition 2.0.10 ([**18**], Prop. 6.8). *Let Γ be a graph for which the associated splice diagram Δ satisfies the semigroup conditions. Then the congruence conditions are equivalent to the following: For every node v and adjacent edge e, there is an admissible monomial $M_{ve} = \prod Z_w^{\alpha_w}$ (w running over the leaves in Δ_{ve}) such that for every leaf w' in Δ_{ve},*

$$\left[\sum_{w \neq w'} \alpha_w \frac{\ell_{ww'}}{\det(\Gamma)} - \alpha_{w'} e_{w'} \cdot e_{w'}\right] = \left[\frac{\ell_{vw'}}{\det(\Gamma)}\right]. \tag{2.1}$$

Remark 2.0.11. It is easy to check, using Proposition 2.0.9, that this condition is always satisfied for an edge leading directly to a leaf.

Let us use Proposition 2.0.10 to check the congruence conditions for Γ from Example 2.0.2.



The leaves of Δ are labelled 1 through 5, and the nodes v_1 and v_2. One can check that $\det(\Gamma) = 3$. For the node v_1, we must check the conditions for the edge with weight 12. The admissible monomials are Z_3^2, Z_4^2, and $Z_3 Z_4$.

Since $\ell_{v_1 w'}$ is divisible by 3 for $w' = 3$, 4, and 5, the right hand side of (2.1) is $[0]$ for each w' in question. It is easy to check, by Proposition 2.0.9, that

$$\begin{aligned} e_3 \cdot e_3 = e_4 \cdot e_4 &= -\frac{2 \cdot 3^2 \cdot 23}{3^2 \cdot 3} - \frac{1}{3} = -\frac{47}{3}, \text{ and} \\ e_5 \cdot e_5 &= -\frac{2 \cdot 3^2 \cdot 23}{2^2 \cdot 3} - \frac{1}{2} = -35. \end{aligned}$$

Furthermore, $\frac{\ell_{35}}{\det(\Gamma)} = \frac{\ell_{45}}{\det(\Gamma)} = 23$ and $\frac{\ell_{34}}{\det(\Gamma)} = \frac{46}{3}$. Therefore, for $w' = 3$, Equation (2.1) is

$$\left[\alpha_4 \frac{46}{3} + \alpha_3 \frac{47}{3}\right] = [0], \tag{2.2}$$

and for and $w' = 4$, it is

$$\left[\alpha_3 \frac{46}{3} + \alpha_4 \frac{47}{3}\right] = [0]\,. \tag{2.3}$$

For $w' = 5$, it is easy to see that the equation is always satisfied. The only possible values of α_3 and α_4 for which Equations (2.2) and (2.3) both hold are $\alpha_3 = 1$ and $\alpha_4 = 1$. Therefore, the congruence condition at v_1 holds, but we must use the admissible monomial $Z_3 Z_4$ and not the other two.

For node v_2, we must check the condition for the edge with weight 23. Again, $\ell_{v_2 w'}$ is divisible by 3 for $w' = 1$ and 2, so the right hand side of (2.1) is $[0]$. Furthermore, $\ell_{12} = 12$, and

$$\begin{aligned} e_1 \cdot e_1 &= -\frac{3 \cdot 5 \cdot 12}{5^2 \cdot 3} - \frac{3}{5} = -3, \text{ and} \\ e_2 \cdot e_2 &= -\frac{3 \cdot 5 \cdot 12}{3^2 \cdot 3} - \frac{1}{3} = -7. \end{aligned}$$

From here, it is easy to see that Equation (2.1) holds for both $w' = 1$ and 2 for any choice of admissible monomial. Hence, Γ satisfies the semigroup and congruence conditions.

Getting back to the general situation, suppose a resolution graph Γ satisfies the semigroup and congruence conditions. Then, by a set of splice diagram equations for Γ, we mean equations as in Definition 2.0.1 such that for each v, the admissible monomials M_{ve} and the power series H_{vi} transform equivariantly under $D(\Gamma)$.

We are almost prepared to state the main result concerning the Neumann-Wahl algorithm, but we need the following technical definition that comes up in the proof. A resolution tree Γ is *quasi-minimal* if any string in Γ either contains no (-1)-weighted vertex, or consists of a unique (-1)-weighted vertex.

Theorem 2.0.12 ([**18**], 7.2). *Suppose* Γ *is quasi-minimal and satisfies the semigroup and congruence conditions. Then*

(1) *A set of splice diagram equations for* Γ *defines an isolated complete intersection singularity* $(Y, 0)$.

(2) *The discriminant group* $D(\Gamma)$ *acts freely on a punctured neighborhood of* 0 *in* Y.

(3) *The quotient* $X := Y/D(\Gamma)$ *has an isolated normal surface singularity and a good resolution with dual resolution graph* Γ.

(4) $(Y, 0) \to (X, 0)$ *is the universal abelian cover.*

(5) *For any node* v*, the* v*-grading on functions on* X *(induced by the* v*-grading on functions on* Y*) is* $\det(\Gamma)$ *times the grading by order of vanishing on the exceptional curve* E_v.

(6) $Y \to X$ *maps the curve* $\{Z_w = 0\}$ *to an irreducible curve, whose proper transform on the good resolution of* X *is smooth and intersects the exceptional curve transversally, along* E_w*. In fact, the function* $Z_w^{\det(\Gamma)}$*, which is* $D(\Gamma)$*-invariant and hence defined on* X*, vanishes to order* $\det(\Gamma)$ *on this curve.*

Point (6) turns out to be very important for us; see the discussion surrounding the End-Curves Theorem (3.1.2) in Chapter 3.

CHAPTER 3

Deformations of $\{z^2 = x^P + y^Q\}$

"Equisingular deformations". For normal surface singularities, there are several notions that one could use to define "equisingular deformations" (e.g, deformations that admit a simultaneous resolution, μ-constant deformations), but in general the different definitions do not agree. However, for weighted homogeneous surface singularities, there is one reasonable definition. Recall that a singularity $(X, 0)$ is *weighted homogeneous* if it is defined by weighted homogeneous polynomials. One can consider deformations given by adding higher weight terms to the defining equations of X. H. Pinkham and J. Wahl proved ([**23**],[**25**]) that the higher weight deformations correspond precisely to deformations that admit a simultaneous resolution, i.e., a family of resolutions inducing a locally topologically trivial deformation of the exceptional set.

Another class of singularities for which there is a good definition of equisingularity are those defined by an equation of the form $z^2 = f(x, y)$. In this case, all deformations are given by deforming the plane curve singularity defined by $f(x, y) = 0$, so one may as well let the equisingular deformations be those given by deforming the plane curve singularity in an equisingular way. Although equisingularity is well-defined for plane curve singularities, we should note that it is in general very difficult to write down, say, a versal equisingular deformation.

Going back to the weighted homogeneous case, we know that any weighted homogeneous surface singularity with $\mathbb{Q}$HS link is a splice quotient [**14**]. However, as Neumann and Wahl mentioned in [**19**] (p.17), it is not clear whether or not equisingular deformations of such singularities are splice quotients as well. In this chapter, we investigate the equisingular deformations of the Brieskorn-type singularities defined by an equation of the form $z^2 = x^P + y^Q$. In particular, we show that there are very few such singularities for which the versal equisingular deformation is a family of splice quotients.

First of all, let us recall that in the $\mathbb{Z}$HS link case (P and Q odd and relatively prime), this problem has been solved. In fact, we have the much more general

THEOREM 3.0.1 (Neumann and Wahl ([**19**], Cor 4.2)). *Any normal surface singularity defined by an equation of the form $z^n = f(x, y)$ with $\mathbb{Z}$HS link is of splice type.*

(See Remark 3.1.3 for a word on the proof.) Since any deformation of $(\{z^2 = x^P + y^Q\}, 0)$ remains of the form $(\{z^2 = f(x, y)\}, 0)$, this says that in the $\mathbb{Z}$HS case, all equisingular deformations are of splice type.

3.1. General approach

For those $(X, 0) = (\{z^2 = x^P + y^Q\}, 0)$ for which the link is a $\mathbb{Q}$HS but not a $\mathbb{Z}$HS, we would like to determine which members of the versal equisingular deformation are also splice quotients. There are two different approaches that we employ.

Approach I. For $(X, 0)$ weighted homogeneous, Neumann [**14**] gave an explicit method for writing down splice diagram equations for the UAC $(Y, 0)$ and the action of the discriminant group on $(Y, 0)$. (This is a precursor to the Neumann-Wahl algorithm.) In Approach I, we write down a versal equisingular equivariant deformation (equivariant under the action of the discriminant group) of the UAC. We then compute the quotient by the action of the discriminant group. The resulting family of splice quotients is a "versal splice quotient deformation" of $(X, 0)$. However, this family may not present itself in the form $\{z^2 = f(x, y; \underline{t})\}$, which makes it difficult to compare this family with a versal equisingular deformation of $(X, 0)$. When this occurs, one can attempt to change coordinates to return to the desired form. Sometimes this is easily done, and sometimes it is not, as we will see in Section 3.2. (In particular, see Remark 3.2.19.)

Note that Approach I can be implemented when $(X, 0)$ is weighted homogeneous because the UAC is weighted homogeneous as well, and hence we know how to write down the versal equisingular deformation that is needed on the level of the UAC.

Approach II. In order to describe the second approach, we need to state the *End-Curves Theorem* of Neumann and Wahl.

Roughly speaking, the End-Curves Theorem says that a singularity is a splice quotient if and only if the UAC can be constructed by adjoining certain “roots of functions” to the local ring. Let $\mathcal{O}_X$ denote the local analytic ring of the germ of a normal surface singularity $(X, 0)$. Suppose there is a function f in the maximal ideal of $\mathcal{O}_X$ such that $(\{f = 0\} \cap X)$ is n copies of a reduced curve C. Adjoining to $\mathcal{O}_X$ a new element U such that $U^n = f$ and normalizing induces an n-fold cyclic cover of normal surface singularities, $(Y, 0) \to (X, 0)$, that is unramified away from 0. For, at any point on C away from 0, C is defined by, say, $\{v = 0\}$ in some local coordinates, so $f = v^n$ locally. Thus the cover consists of n smooth pieces over $\{f = 0\}$, since it is locally given by normalizing $U^n = v^n$. Clearly, if the link Σ of $(X, 0)$ is a $\mathbb{Q}$HS, then n must divide the order of the discriminant group.

Let $(X, 0)$ be the germ of a normal surface singularity with $\mathbb{Q}$HS link and minimal good resolution graph Γ. An irreducible exceptional curve E in the minimal good resolution of $(X, 0)$ is called an *end-curve* if it intersects the rest of the exceptional set in exactly one point. That is, E is an end-curve if it corresponds to a leaf (vertex of valence one) in Γ.

Definition 3.1.1. A function f in the maximal ideal of $\mathcal{O}_X$ is called an *end-curve function* (associated to E) if the proper transform of its zero locus in the minimal good resolution is nC, where C is a smooth irreducible curve that intersects the end-curve E transversally in exactly one point and does not intersect any other exceptional curve. Here, n is the order of the image of the dual basis element corresponding to E in the discriminant group.

Theorem 3.1.2 (Neumann/Wahl). *A normal surface singularity* $(X, 0)$ *with* $\mathbb{Q}$*HS link is a splice quotient if and only if for every end-curve* E *of the minimal good resolution of* $(X, 0)$ *there exists an end-curve function associated to* E *in the maximal ideal of* $\mathcal{O}_X$.

In the $\mathbb{Z}$HS case, this theorem is proved in [**19**]. For the $\mathbb{Q}$HS case, one direction, that splice quotients have end-curve functions, is point (6) of Theorem 2.0.12. In the other direction, a proof is announced but not yet published.

It is precisely these end-curve functions whose roots we adjoin to construct the UAC (see the Example below), and the newly adjoined variables can always be used as coordinates for the UAC. Upon adjoining *all* of the end-curve functions there is no need to normalize.

Remark 3.1.3. In the $\mathbb{Z}$HS case, the End-curves Theorem says $(X, 0)$ is of splice type if and only if there exist end-curve functions. Theorem 3.0.1, which says that any normal surface singularity defined by an equation of the form $z^n = f(x, y)$ with $\mathbb{Z}$HS link is of splice type, is a corollary of the End-curves Theorem. This is because one can produce end-curve functions for $(\{z^n = f(x, y)\}, 0)$ from end-curve functions that are known to exist for the splice diagram of the plane curve singularity $(\{f(x, y) = 0\}, 0)$. The argument does not work for the general $\mathbb{Q}$HS case; it is only in the $\mathbb{Z}$HS case that the functions are all guaranteed to lift to end-curve functions for $(\{z^n = f(x, y)\}, 0)$.

Example 3.1.4. Let $X \subset \mathbb{C}^3$ be defined by

$$X := \{z^2 = x^4 + y^9\}.$$

The minimal good resolution graph of $(X, 0)$, shown in Figure 3.1, has three leaves, labeled E_1, E_2 and E_3. The discriminant group has order 9, and the order of the images of the dual basis elements associated to the leaves are 1, 9, and 9, respectively. One can explicitly verify that there exist corresponding end-curve functions: x, $z - x^2$, and $z + x^2$. For instance, let $g = z^2 - x^4 - y^9$, so that $\mathcal{O}_X = \mathbb{C}[[x, y, z]]/(g)$. We have

$$g = (z - x^2)(z + x^2) - y^9.$$

That $(\{z - x^2 = 0\} \cap X)$ is a smooth irreducible curve counted 9 times is seen by considering the ring $\mathcal{O}_X/(z - x^2) \simeq \mathbb{C}[[x, y]]/(y^9)$.

We now construct the UAC of $(X, 0)$ by adjoining roots of the end-curve functions to $\mathcal{O}_X$. Adjoin U such that $U^9 = z - x^2$, and then normalize by adjoining $V := y/U$. (Equivalently, we could have adjoined V, the 9th root of the end-curve function $z + x^2$.) This results in a normal surface singularity $(Y, 0)$ with equation

$$V^9 = U^9 + 2x^2,$$

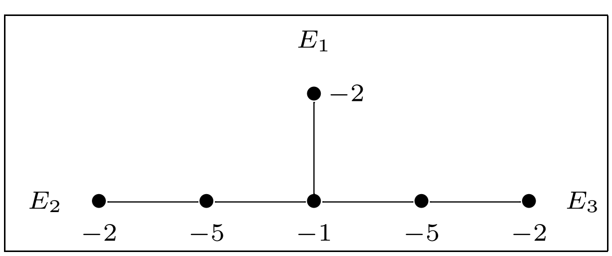


Figure 3.1. Minimal good resolution graph of $\{z^2 = x^4 + y^9\}$.

which is a 9-fold abelian cover, and hence the universal abelian cover, of $(X,0)$. The action of the discriminant group, $\mathbb{Z}/9\mathbb{Z}$, is generated by

$$(U,V,x) \mapsto (\zeta U, \zeta^{-1} V, x),$$

where ζ is a primitive 9th root of unity. □

We can now describe Approach II. Given a splice quotient $(X,0)$, we can search for deformations of $(X,0)$ that also have end-curve functions.

QUESTION 3.1.5. *When does an end-curve function continue to be an end-curve function under deformation?*

Consider the following particular situation. Let $(X,0) \subset (\mathbb{C}^3,0)$ be a hypersurface singularity with local ring $\mathcal{O}_X = \mathbb{C}[[x,y,z]]/(g)$. Suppose that g can be written as follows:

$$g = f_1 f_2 - h^n, \tag{3.1}$$

where f_1, f_2, and h are power series, and h is irreducible. Furthermore, suppose that the f_i are end-curve functions (necessarily associated to end-curves for which the image of the corresponding dual basis element has order n in the discriminant group). Let $g + \varepsilon\tilde{g}$ be an equisingular deformation of g. Then the f_i lift to end-curve functions for $g + \varepsilon\tilde{g}$ if and only if (3.1) lifts to

$$g + \varepsilon\tilde{g} = (f_1 + \varepsilon\tilde{f}_1)(f_2 + \varepsilon\tilde{f}_2) - (h + \varepsilon\tilde{h})^n. \tag{3.2}$$

If we consider this problem only to first order, i.e., $\varepsilon^2 = 0$, then (3.1) lifts to (3.2) if and only if $\tilde{g} = f_1\tilde{f}_2 + \tilde{f}_1 f_2 - nh^{n-1}\tilde{h}$.

Proposition 3.1.6. *To first-order, the end-curve functions f_i for g as given in (3.1) lift to end-curve functions for the deformation $g + \varepsilon\tilde{g}$ if and only if $\tilde{g}$ is in the ideal generated by $f_1,\ f_2$, and h^{n-1}.*

3.2. Specific computations

Let us first discuss equisingular deformations of weighted homogeneous hypersurface singularities. It is well known that for an isolated complete intersection singularity $(X, 0)$, any basis of the first Tjurina module T^1_X gives a (mini)versal deformation of $(X, 0)$. Recall that for a ring of the form $\mathcal{O}_X \simeq \mathbb{C}[[x, y, z]]/(g)$,

$$T^1_X = \mathcal{O}_X / \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right).$$

If g is weighted homogeneous, there is a natural way to define the weight of an element of T^1_X, namely, $h \in T^1_X$ is weighted homogeneous of weight k if and only if h is a weighted homogeneous polynomial such that

$$\text{weight}(h) = \text{weight}(g) + k.$$

Combining this with the result of Pinkham and Wahl mentioned in the first paragraph of this chapter, we have the following fact.

Fact 3.2.1 ([**23**],[**25**]). *Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be a hypersurface singularity that is weighted homogeneous of degree d and defined by $\{g(x, y, z) = 0\}$. Let $\{g_1, \ldots, g_m\}$ be a basis for the subspace of weighted homogeneous elements with nonnegative weight in the $\mathbb{C}$-vector space T^1_X. Then the family defined by*

$$\{g + t_1 g_1 + \cdots t_m g_m = 0\}$$

is a versal equisingular deformation of $(X, 0)$.

Now, it is clear that for singularities defined by equations of the form $z^2 = x^P + y^Q$, the versal equisingular deformation remains of the form $z^2 = f(x, y; \underline{t})$.

There is a well-known algorithm for computing the minimal good resolution graph of a weighted homogeneous surface singularity (e.g. [**22**]), and we use these results freely. The resolution graph is star-shaped, and all exceptional curves are rational except possibly the central curve. We begin by recalling when the singularities of interest have a $\mathbb{Q}$HS link.

Theorem 3.2.2 ([**22**]). *The link of* $(\{x^P + y^Q + z^R = 0\}, 0)$ *is a* $\mathbb{Q}$*HS if and only if either*

(1) $(P, QR) = 1$, *or*

(2) $(P, Q, R) = 2$ *and* $P/2$, $Q/2$, $R/2$ *are pairwise relatively prime.*

Proof. Orlik and Wagreich ([**22**], Prop. 3.5.1) show that if g is the genus of the central curve in the minimal good resolution graph of $(\{x^P + y^Q + z^R = 0\}, 0)$, then

$$2g = c^2 c_1 c_2 c_3 - c(c_1 + c_2 + c_3) + 2,$$

where

$$c = (P, Q, R),\ c_1 = (Q, R)/c,\ c_2 = (P, R)/c,\ \text{and } c_3 = (P, Q)/c.$$

It is easy to deduce that $g = 0$ if and only if either

(1) $c = 1$ and at least two of the c_i are 1, or

(2) $c = 2$ and $c_1 = c_2 = c_3 = 1$.

This is equivalent to the statement of the theorem. □

It is easy to determine that the link is a $\mathbb{Z}$HS if and only if P, Q, and R are pairwise relatively prime ($c = c_1 = c_2 = c_3 = 1$). In case (1), the order of the discriminant group is $P^{(c_1 - 1)}$. In case (2), the order of the discriminant group is $PQR/4$. All of these statements follow from the explicit formula for the determinant of the resolution graph that is given in [**22**], §2.5.

For $R = 2$, one can distinguish three cases for which the link is a $\mathbb{Q}$HS but not a $\mathbb{Z}$HS. They are:

(i) $z^2 = x^{2p} + y^q$, with $(2p, q) = 1$,

(ii) $z^2 = x^{2p} + y^{2q}$ with $(p, q) = 1$,

(iii) $z^2 = x^{pk} + y^{qk}$, with $p,\ q$, and k odd, $k > 1$, and $(p, q) = 1$.

We will study Cases (i) and (ii) in detail below, employing the two approaches outlined above. In these two cases, it is rare for all equisingular deformations to remain splice quotients.

However, we begin with Case (iii), which turns out to be easy to handle via the End-Curves Theorem; we will show that all equisingular deformations are splice quotients in this case. We are free to use

$$X := \{z^2 = x^{pk} - y^{qk}\}$$

instead for convenience.

Proposition 3.2.3. *All equisingular deformations of* $(X, 0)$ *are splice quotients.*

Proof. Clearly, we have

$$z^2 = \prod_{j=1}^{k} (x^p - \zeta^j y^q), \tag{3.3}$$

where ζ is a primitive kth root of unity. An equisingular deformation of $(X, 0)$ must preserve the factorization on the right hand side of (3.3). This is because all deformations of $(X, 0)$ come from deforming the plane curve singularity given by $x^{pk} - y^{qk} = 0$ in an equisingular way, and in an equisingular family of plane curve singularities, each of the branches must itself be deformed in an equisingular way.

We claim that the factors $f_j := x^p - \zeta^j y^q,\ 1 \le j \le k$, together with x and y, are the end-curve functions for $(X, 0)$. If so, any equisingular deformation of $(X, 0)$ also has end-curve functions ($x,\ y$, and the deformed f_j), and therefore is a splice quotient. To show that these functions are indeed end-curve functions, we construct the UAC of $(X, 0)$ via the Neumann-Wahl algorithm, and apply Theorem 2.0.12 (6). Alternatively, we could do an explicit construction of the resolution of $(X, 0)$, but it would ultimately be less efficient to do so here.

The splice diagram Δ for Case (iii) is shown in Figure 3.2. For simplicity we assume that neither p nor q is equal to 1, although the following argument goes through regardless.

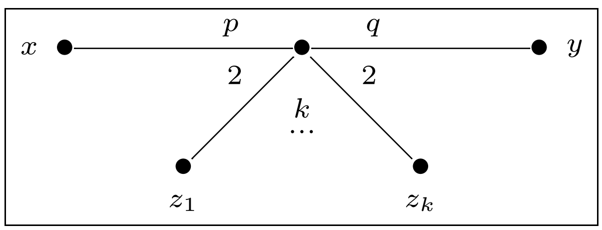


FIGURE 3.2. Splice diagram for Case (iii): $\{z^2 = x^{pk} + y^{qk}\}$.

A set of splice diagram equations for Δ, as described in Chapter 2, is

$$\left\{ \begin{array}{rcl} z_1^2 + a_1 x^p + b_1 y^q & = & 0 \\ z_2^2 + a_2 x^p + b_2 y^q & = & 0 \\ \vdots & & \vdots \\ z_k^2 + a_k x^p + b_k y^q & = & 0 \end{array} \right\},$$

where all a_i and b_i are nonzero, and $a_i b_j - a_j b_i \neq 0$ for all $i \neq j$.

The discriminant group has order 2^{k-1}. One can check, using Propositions 2.0.8 and 2.0.9, that the discriminant group elements e_x and e_y act trivially on $(z_1, \ldots, z_k, x, y)$, and e_{z_j} $(1 \le j \le k)$ acts as follows:

$$e_{z_j} \cdot (z_1, \ldots, z_k, x, y) = (-z_1, \ldots, z_j, \ldots, -z_k, x, y),$$

so that e_{z_j} has order 2. Therefore, x and y are invariant under the group action, as are

$$\begin{aligned} w_j &:= z_j^2,\ 1 \le j \le k,\ \text{and} \\ z &:= z_1 \cdots z_k, \end{aligned}$$

since k is odd. The splice diagram equations are invariant under the action of the discriminant group, and the quotient satisfies the following equations in $x,\ y,\ z$, and w_j:

$$\begin{aligned} z^2 &= \prod_{j=1}^{k} w_j,\ \text{and} \\ w_j &= -(a_1 x^p + b_1 y^q),\ 1 \le j \le k. \end{aligned}$$

Therefore,

$$z^2 = (-1)\prod_{j=1}^{k}(a_1 x^p + b_1 y^q). \tag{3.4}$$

If, for $1 \le j \le k$, we let $a_j = -1$ and $b_j = \zeta^j$, where ζ is a primitive k-th root of unity, then (3.4) is equal to (3.3), the defining equation of X.

We have shown that the splice diagram equations for the UAC of $(X, 0)$ are

$$\left\{ \begin{array}{rcl} z_1^2 - x^p + \zeta y^q &=& 0 \\ z_2^2 - x^p + \zeta^2 y^q &=& 0 \\ \vdots && \vdots \\ z_k^2 - x^p + y^q &=& 0 \end{array} \right\},$$

that is, $\{z_j^2 = f_j \ :\ 1 \le j \le k\}$. Recalling Definition 3.1.1 of end-curve functions, we see that Theorem 2.0.12 (6) implies that $x,\ y$, and the f_j are end-curve functions for $(X, 0)$. $\square$

Remark 3.2.4. Consider $X := \{z^n = x^{pk} - y^{qk}\}$, with $k > 1,\ (p, q) = 1$, and n relatively prime to $p,\ q$, and k. Then $(X, 0)$ has $\mathbb{Q}$HS link, and the discriminant group has order n^{k-1}. If we were to consider equisingular deformations of $(X, 0)$ given by adding higher weight terms only of the form x^iy^j, i.e., equisingular deformations of $(X,0)$ given by deforming the plane curve singularity $x^{pk} - y^{qk}$, then by the End-Curves Theorem, all such deformations are splice quotients.

Case (i): $\{\mathbf{z^2 = x^{2p} + y^q}\}$.

Let $X_1 \subset \mathbb{C}^3$ be defined by

$$X_1 := \{z^2 = x^{2p} + y^q\},$$

with $(2p, q) = 1$. For $q \geq 3$, the surface X_1 has an isolated singularity at the origin 0. We also assume that $p > 1$ since otherwise, $(X_1, 0)$ is a rational double point of type A_{q-1}, which is taut, and therefore uninteresting in this context. By Fact 3.2.1, a versal equisingular deformation of $(X_1, 0)$ is given by

$$\left\{ z^2 = x^{2p} + y^q + \sum_{(i,j)\in\mathbb{I}} t_{ij}x^iy^j \right\}, \text{ where} \tag{3.5}$$

$$\mathbb{I} = \left\{ (i,j) \in \mathbb{Z}^2 \mid 0 < i < 2p-1,\ 0 < j < q-1,\ \frac{i}{2p} + \frac{j}{q} \geq 1 \right\}.$$

We want to determine which members of this family are splice quotients.

Approach I. The goal is to write down a versal splice quotient deformation of $(X_1, 0)$. Just as in Example 3.1.4, we construct the UAC of $(X_1, 0)$ by adjoining roots of end-curve functions. We could use the algorithm given by Neumann in [**14**] to construct the UAC, but for our purposes, using the end-curve functions is more convenient. The difference is that Neumann's algorithm actually yields a family of singularities, and we would then have to search for the analytic type of the UAC of $(X_1, 0)$, as we did in Case (iii).

Adjoin U such that $U^q = z - x^p$ to $\mathcal{O}_{X_1}$, and then normalize by adjoining $V := y/U$. This results in a normal surface singularity $(Y_1, 0)$ with equation

$$V^q = U^q + 2x^p, \tag{3.6}$$

which is the universal abelian cover of $(X_1, 0)$. Note that since $(p, q) = 1$, the link of the UAC is always a $\mathbb{Q}$HS by Theorem 3.2.2, but never a $\mathbb{Z}$HS. The action of the discriminant group $\mathbb{Z}/q\mathbb{Z}$ on U must be given by $U \mapsto \zeta U$, where ζ is a primitive q-th root of unity; x must be invariant; and $V = x/U$ implies that $V \mapsto \zeta^{-1}V$. Therefore, the action of $\mathbb{Z}/q\mathbb{Z}$ on $(Y_1, 0)$ is generated by

$$(U, V, x) \mapsto (\zeta U, \zeta^{-1}V, x), \tag{3.7}$$

where ζ is a primitive q-th root of unity.

Proposition 3.2.5. *A first-order versal splice quotient deformation of $(X_1, 0)$ is given by*

$$\left\{ z^2 = x^{2p} + y^q + \sum_{(i,j)\in\mathbb{I}^+} \varepsilon_{ij} x^i y^j \right\}, \quad \textit{where} \tag{3.8}$$

$$\mathbb{I}^+ = \{(i, j) \in \mathbb{I} \mid i \geq p\}.$$

Proof. We begin by writing down a versal equisingular equivariant deformation of $(Y_1, 0)$. By Fact 3.2.1, we need a basis for the subspace of nonnegative weight elements of

$$T^1_{Y_1} = \mathcal{O}_{Y_1} / \left(V^{q-1},\ U^{q-1},\ x^{p-1}\right)$$

that are invariant under the action of $H := \mathbb{Z}/q\mathbb{Z}$ given by (3.7). (We are interested in H-invariant elements since H acts trivially on (3.6), the equation defining Y_1.) It is easy to see that a basis is given by $\{x^k(UV)^l\}$, where (k,l) runs over the set

$$\mathbb{K} = \left\{(k,l) \in \mathbb{Z}^2 \mid 0 \le k < p-1,\ 0 < l < q-1,\ \frac{k}{p} + \frac{2l}{q} \ge 1\right\}.$$

Therefore, the family

$$\left\{V^q = U^q + 2x^p + \sum_{(k,l)\in\mathbb{K}} \varepsilon_{kl} x^k (UV)^l\right\}, \tag{3.9}$$

is a versal first-order equisingular H-invariant deformation of $(Y_1, 0)$. For simplicity, we will just write Σ to indicate the sum over all $(k,l) \in \mathbb{K}$. The invariants under the group action are

$$A := U^q,\ B := V^q,\ y = UV,\ \text{and } x.$$

They satisfy the equations $y^q = AB$ and $B = A + 2x^p + \sum \varepsilon_{kl} x^k y^l$ (from (3.9)). So we have

$$y^q = A(A + 2x^p + \sum \varepsilon_{kl} x^k y^l).$$

This is not of the form $z^2 = f(x,y)$, but we can put it into that form by completing the square in A. Make a change of coordinates: $z := A + x^p + \frac{1}{2}\sum \varepsilon_{kl} x^k y^l$. Then

$$y^q = (z - x^p - \frac{1}{2}\sum \varepsilon_{kl} x^k y^l)(z + x^p + \frac{1}{2}\sum \varepsilon_{kl} x^k y^l). \tag{3.10}$$

Since we are looking at first-order deformations, $\varepsilon_{kl}\varepsilon_{mn} = 0$ for all (k,l) and (m,n) in $\mathbb{K}$, hence this is the same as

$$y^q = z^2 - x^{2p} - x^p \sum \varepsilon_{kl} x^k y^l,$$

i.e.,

$$z^2 = x^{2p} + y^q + \sum \varepsilon_{kl} x^{p+k} y^l.$$

It is easy to check that $(k,l) \in \mathbb{K}$ if and only if $(p+k, l) \in \mathbb{I}^+$. Thus (3.8) is a first-order versal splice quotient deformation of $(X_1, 0)$. □

On the level of tangent spaces, we see that if $(i,j) \in \mathbb{I}$ and $i < p$, then the deformation $z^2 = x^{2p} + y^q + \varepsilon_{ij} x^i y^j$ is not a splice quotient. In general, such deformations do exist in the versal equisingular family (3.5); that is, $\mathbb{I} \neq \mathbb{I}^+$ in general. For example, consider $z^2 = x^4 + y^9 + \varepsilon x y^7$. Therefore, we have shown that in general, not all equisingular deformations of $(X_1, 0)$ are splice quotients. More specifically,

Corollary 3.2.6. *If all elements in the versal equisingular deformation of $(X_1, 0)$ are splice quotients, then $q < 4p/(p-1)$.*

Proof. To see this, fix an integer i such that $1 \leq i \leq 2p-1$. If $(i,j) \in \mathbb{I}$, then

$$\frac{i}{2p} + \frac{j}{q} \geq 1 \Longleftrightarrow j \geq \left(1 - \frac{i}{2p}\right) q.$$

There exist integers j such that $(i,j) \in \mathbb{I}$ if and only if

$$q - 2 \geq \left(1 - \frac{i}{2p}\right) q \Longleftrightarrow q \geq \frac{4p}{i}.$$

Now suppose that $q \geq 4p/(p-1)$. Then there exists j such that $(p-1, j) \in \mathbb{I}$, and hence, not all elements of the versal equisingular deformation are splice quotients. □

So far, we have written down a first-order versal splice quotient deformation of $(X_1, 0)$. However, it is not difficult to write down the versal splice quotient deformation to all orders.

Theorem 3.2.7. *A versal splice quotient deformation of $(X_1, 0)$ is given by*

$$\left\{ z^2 = x^{2p} + y^q + \sum_{(i,j) \in \mathbb{I}^+} t_{ij} x^i y^j + \frac{1}{4} \left[\sum_{(i,j) \in \mathbb{I}^+} t_{ij} x^{i-p} y^j \right]^2 \right\}, \text{ where} \tag{3.11}$$

$$\mathbb{I}^+ = \{(i,j) \in \mathbb{I} \mid i \geq p\}.$$

Proof. The proof is the same as that of Proposition 3.2.5 up to Equation (3.10). Here, we will use t_{kl} as the deformation parameters, to distinguish from the first order parameters ε_{kl}. So Equation (3.10) becomes

$$y^q = (z - x^p - \frac{1}{2}\Sigma t_{kl} x^k y^l)(z + x^p + \frac{1}{2}\Sigma t_{kl} x^k y^l), \tag{3.12}$$

where Σ indicates the sum over all $(k, l) \in \mathbb{K}$. This is the same as

$$y^q = z^2 - x^{2p} - x^p \Sigma t_{kl} x^k y^l - \frac{1}{4}\left[\Sigma t_{kl} x^k y^l\right]^2,$$

i.e.,

$$z^2 = x^{2p} + y^q + \Sigma t_{kl} x^{p+k} y^l + \frac{1}{4}\left[\Sigma t_{kl} x^k y^l\right]^2.$$

Again, $(k, l) \in \mathbb{K}$ if and only if $(p + k, l) \in \mathbb{I}^+$. □

Remark 3.2.8. The end-curve functions for the family (3.11) can be seen in Equation (3.12).

Approach II. Now let us look at the problem from the point of view of end-curve functions. The minimal good resolution graph of $(X_1, 0)$ is star-shaped with three arms: two strings of type q/λ, and one string of type p/η, where λ satisfies $2p\lambda \equiv -1(q)$ and η satisfies $q\eta \equiv -1(p)$. (In this notation, the continued fraction expansions for the strings go from the central node to the leaf.) The central curve has self-intersection $-b$, with $b = (1 + 2p\lambda + q\eta)/pq$. The discriminant group is cyclic of order q. The associated splice diagram is shown in Figure 3.3. As in Example 3.1.4, the images of the dual basis elements corresponding to the E_i have order 1, q, and q, and the end-curve functions are x, $z - x^p$, and $z + x^p$.

We have $\mathcal{O}_X = \mathbb{C}[[x, y, z]]/(g)$, with

$$g = (z - x^p)(z + x^p) - y^q.$$

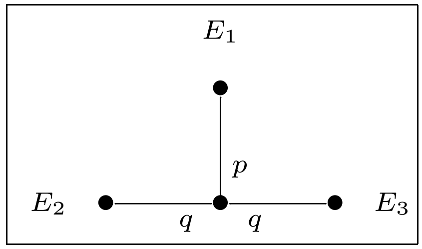


FIGURE 3.3. Splice diagram for Case (i): $\{z^2 = x^{2p} + y^q\}$.

By Proposition 3.1.6, a first-order deformation $g + \varepsilon\tilde{g}$ will admit a decomposition

$$g + \varepsilon\tilde{g} = (z - x^p - \varepsilon\tilde{f}_1)(z + x^p + \varepsilon\tilde{f}_2) - (y + \varepsilon\tilde{h})^q,$$

if and only if $\tilde{g}$ is in the ideal generated by $z - x^p$, $z + x^p$, and y^{q-1}, i.e., $(z,\ x^p,\ y^{q-1})$. Recall that

$$\mathbb{I} = \left\{(i,j) \in \mathbb{Z}^2 \mid 0 < i < 2p-1,\ 0 < j < q-1,\ \frac{i}{2p} + \frac{j}{q} \geq 1\right\}.$$

Therefore, if $\tilde{g} = x^i y^j$ for $(i,j) \in \mathbb{I}$, then $i \geq p$. On the other hand, if $i \geq p$, we have

$$g + \varepsilon x^i y^j = (z - x^p - \frac{\varepsilon}{2}x^{i-p}y^j)(z + x^p + \frac{\varepsilon}{2}x^{i-p}y^j) - y^q.$$

Therefore, Approach II yields the same answer as Proposition 3.2.5.

The versal splice quotient family (3.11) has terms that are quadratic in the deformation parameters t_{ij}. Perhaps there is a change of coordinates that would transform the family (3.11) into one with only *linear* terms in the deformation parameters, but it seems unlikely. However, we can point out one linear family of splice quotients, using end-curves.

PROPOSITION 3.2.9. *The linear family over* $\mathbb{C}[[t_{ij}]]$

$$\left\{z^2 = x^{2p} + y^q + \sum_{(i,j)\in\mathbb{I}^{++}} t_{ij}x^i y^j\right\},\ \text{where} \tag{3.13}$$

$$\mathbb{I}^{++} = \{(i,j) \in \mathbb{I} \mid i \geq p,\ j \geq q/2\},$$

is a family of splice quotients.

PROOF. This can be seen by the following end-curve decomposition of (3.13),

$$(z - x^p - \frac{1}{2}\Sigma t_{ij}x^{i-p}y^j)(z + x^p + \frac{1}{2}\Sigma t_{ij}x^{i-p}y^j) = y^q - \frac{1}{4}[\Sigma t_{ij}x^{i-p}y^j]^2,$$

in which the right side is y^q times a unit since all $j \geq q/2$. □

In general, this is not the entire versal splice quotient deformation restricted to linear terms in the deformation parameters. That is, $\mathbb{I}^+ \neq \mathbb{I}^{++}$ in general, even when $\mathbb{I} = \mathbb{I}^+$ (e.g., $z^2 = x^6 + y^5 + tx^4y^2$).

Finally, let us consider those $(X_1, 0)$ for which $\mathbb{I} = \mathbb{I}^+$. In this case, the entire versal equisingular deformation is a family of splice quotients. For, the versal splice quotient deformation (3.11) is itself a versal equisingular deformation since it agrees with the versal equisingular deformation (3.5) to first order. Upon closer inspection, it turns out that there are very few of these. Recall from Corollary 3.2.6 that

$$\mathbb{I} = \mathbb{I}^+ \Longrightarrow q < 4p/(p-1),$$

and note that $4p/(p-1) \leq 5$ for $p \geq 5$. The table in Figure 3.4 gives a complete list, excluding the rational double point $(\{z^2 = x^4 + y^3\}, 0)$, since it is taut (equivalently, $\mathbb{I}$ is empty).

Recall that a normal surface singularity is called *elliptic* (sometimes called "weakly" elliptic) if

$$\chi(Z_{min}) = 0,$$

where Z_{min} is the *minimal cycle*, also known as the *fundamental cycle of Artin*, associated to a good resolution graph Γ. The cycle Z_{min} is defined to be the minimal nonzero effective cycle Z such that $Z \cdot E_i \leq 0$ for all E_i in the exceptional set corresponding to Γ. A singularity is *minimally elliptic* if it is Gorenstein and has geometric genus equal to 1. Minimally elliptic singularities are elliptic. In the case we are studying, the geometric genus can easily be computed by the following

FACT 3.2.10 ([**11**], 4.23). *The geometric genus of* $(\{x^a + y^b + c^d = 0\}, 0)$ *is given by*

$$p_g = \#\left\{(i,j,k) \in \mathbb{N}^3 \ : \ \frac{i}{a} + \frac{j}{b} + \frac{k}{c} \leq 1\right\}.$$

CLAIM 3.2.11. *Those* $(X_1, 0)$ *for which the entire versal equisingular deformation is a family of splice quotients are all elliptic.*

PROOF OF CLAIM. The table in Figure 3.4 gives a complete list of the $X_1 = \{z^2 = x^{2p} + y^q\}$ in question, together with the corresponding geometric genus p_g. For those with $p_g = 1$, there is nothing to check. Furthermore, Gorenstein singularities with $p_g = 2$ are elliptic ([**11**], 4.18). Therefore, the only case that we must check is $\{z^2 = x^{2p} + y^3\}$, where $p > 5$ and $(p,3) = 1$. We directly show that $\chi(Z_{min}) = 0$.

$2p$	q	p_g
4	5	1
4	7	1
6	5	1
8	3	1
8	5	2
10	3	1
$> 10,\ (p,3) = 1$	3	$\lfloor \frac{p}{3} \rfloor$

FIGURE 3.4. List of all $\{z^2 = x^{2p} + y^q\}$ for which the entire versal equisingular deformation is a family of splice quotients.

If $p \equiv 1 \pmod 3$, the resolution graph is as in Figure 3.5, where the number of (-2)-curves on the right is $(p-4)/3$, i.e., $n = (p+8)/3$. (The string from E_4 to E_n has continued fraction expansion $\frac{p}{(p-1)/3}$.)

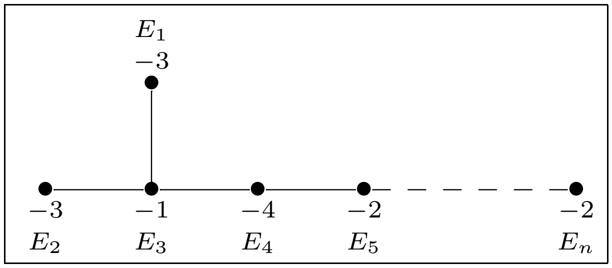


FIGURE 3.5. Minimal good resolution graph of $\{z^2 = x^{2p} + y^3\}$, for $p \equiv 1 \pmod 3$.

The minimal cycle is

$$Z_{min} = E_1 + E_2 + 3E_3 + E_4 + \cdots + E_n.$$

Recall that for a good resolution graph, the *canonical cycle* is the rational cycle Z_K such that

$$Z_K \cdot E_i = E_i^2 - 2g_i + 2$$

for all E_i in the exceptional set. (When $(X, 0)$ is Gorenstein, hence numerical Gorenstein, Z_K has integer coefficients.) In this case, the canonical cycle Z_K is

$$Z_K = \frac{p-1}{3}E_1 + \frac{p-1}{3}E_2 + (p-2)E_3 + \frac{p-1}{3}E_4 + \frac{p-4}{3}E_5 + \cdots + 2E_{n-1} + E_n.$$

Therefore,

$$Z_K - Z_{min} = \frac{p-4}{3}E_1 + \frac{p-4}{3}E_2 + (p-5)E_3 + \frac{p-4}{3}E_4 + \frac{p-7}{3}E_5 + \cdots + 2E_{n-2} + E_{n-1}.$$

From here, it is not difficult to check that

$$\chi(Z_{min}) = \frac{1}{2}Z_{min} \cdot (Z_K - Z_{min}) = 0.$$

If $p \equiv 2 \pmod 3$, the resolution graph is as in Figure 3.6, where the number of (-2)-

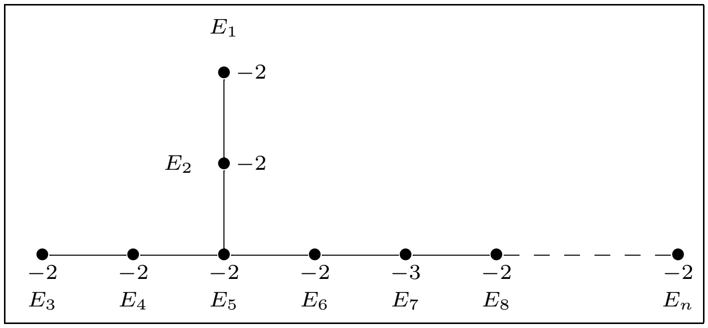


FIGURE 3.6. Minimal good resolution graph of $\{z^2 = x^{2p} + y^3\}$, for $p \equiv 2 \pmod 3$.

curves on the right is $(p-5)/3$, i.e., $n = (p+16)/3$. (The string from E_6 to E_n has continued fraction expansion $\frac{p}{(p-1)/3}$.) The minimal cycle is

$$Z_{min} = E_1 + 2E_2 + E_3 + 2E_4 + 3E_5 + 2E_6 + E_7 + E_8 + \cdots + E_n.$$

The canonical cycle Z_K is

$$\begin{aligned} Z_K &= \frac{p-2}{3}E_1 + \frac{2p-4}{3}E_2 + \frac{p-2}{3}E_3 + \frac{2p-4}{3}E_4 + (p-2)E_5 \\ &\quad + \frac{2p-4}{3}E_6 + \frac{p-2}{3}E_7 + \frac{p-5}{3}E_8 + \cdots + 2E_{n-1} + E_n. \end{aligned}$$

Therefore,

$$\begin{aligned} Z_K - Z_{min} &= \frac{p-5}{3}E_1 + \frac{2p-10}{3}E_2 + \frac{p-5}{3}E_3 + \frac{2p-10}{3}E_4 + (p-5)E_5 \\ &\quad + \frac{2p-10}{3}E_6 + \frac{p-5}{3}E_7 + \frac{p-8}{3}E_8 + \cdots + 2E_{n-2} + E_{n-1}. \end{aligned}$$

Again, it is easy to check that $\chi(Z_{min}) = \frac{1}{2}Z_{min} \cdot (Z_K - Z_{min}) = 0$. □

REMARK 3.2.12. The converse to this Claim is not true. The singularity $(\{z^2 = x^8 + y^7\}, 0)$ is elliptic, but not all elements of the versal equisingular deformation are splice quotients. The resolution graph is given in Figure 3.7.

One can check that

$$Z_{min} = 2E_1 + E_2 + 2E_3 + 3E_4 + 4E_5 + 5E_6 + 6E_7 + 7E_8 + 6E_9 + 5E_{10} + 4E_{11} + 3E_{12} + 2E_{13} + E_{14}$$

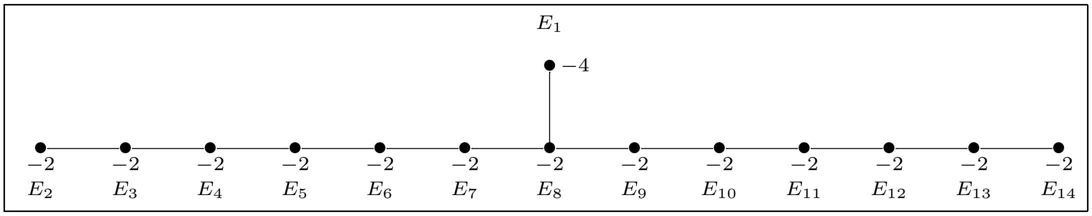


FIGURE 3.7. Minimal good resolution graph of $\{z^2 = x^8 + y^7\}$.

and $Z_K = 2Z_{min}$. Hence $Z_K - Z_{min} = Z_{min}$, and

$$\chi(Z_{min}) = \frac{1}{2} Z_{min}^2 = -1.$$

REMARK 3.2.13. We should also note that in the $\mathbb{Z}$HS case of $(\{z^2 = x^P + y^Q\}, 0)$, in which all deformations are splice quotients (Theorem 3.0.1), not all of the singularities are elliptic. For example, consider $(\{z^2 = x^5 + y^{11}\}, 0)$. The resolution graph is shown in Figure 3.8.

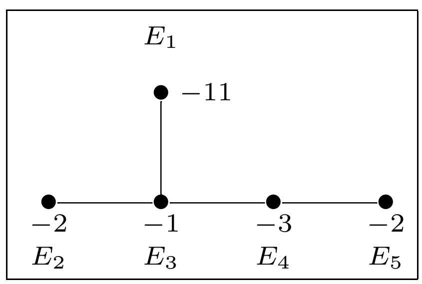


FIGURE 3.8. Minimal good resolution graph of $\{z^2 = x^5 + y^{11}\}$.

One can check that

$$Z_{min} = E_1 + 5E_2 + 10E_3 + 4E_4 + 2E_5$$

and

$$Z_K = 3E_1 + 12E_2 + 24E_3 + 10E_4 + 5E_5.$$

Hence $Z_K - Z_{min} = 2E_1 + 7E_2 + 14E_3 + 6E_4 + 3E_5$, and $\chi(Z_{min}) = -1$.

Case (ii): $\{\mathbf{z^2 = x^{2p} + y^{2q}}\}$.

Let $X_2 \subset \mathbb{C}^3$ be defined by

$$X_2 := \{z^2 = x^{2p} + y^{2q}\},$$

with $(p,q) = 1$. We also assume $p,\ q > 1$, since otherwise $(X_2, 0)$ is a rational double point.

By Fact 3.2.1, a versal equisingular deformation of $(X_2, 0)$ is given by

$$\left\{ z^2 = x^{2p} + y^{2q} + \sum_{(i,j)\in\mathbb{J}} t_{ij} x^i y^j \right\}, \text{ where} \tag{3.14}$$

$$\mathbb{J} = \left\{ (i,j) \in \mathbb{Z}^2 \mid 0 < i < 2p-1,\ 0 < j < 2q-1,\ \frac{i}{2p} + \frac{j}{2q} \geq 1 \right\}.$$

Note that the $x^p y^q$ term has weight 0; therefore,

$$\{z^2 = x^{2p} + y^{2q} + t_{pq} x^p y^q\}$$

with $t_{pq} \neq \pm 2$ gives the full weighted homogeneous family of singularities with this topological type.

Approach I. Recall that the goal is to write down a (first-order) versal family of splice quotients. Things are more complicated in this case than they were in Case (i). This time, it is no easier to construct the UAC of $(X_2, 0)$ using end-curve functions, so we will use the Neumann-Wahl algorithm.

The minimal good resolution graph of $(X_2, 0)$ is star-shaped with four arms: two strings of type q/λ, and two strings of type p/η, where λ satisfies $p\lambda \equiv -1(q)$ and η satisfies $q\eta \equiv -1(p)$. The central curve has self-intersection $-b$, with $b = (2 + 2p\lambda + 2q\eta)/pq$. The discriminant group is cyclic of order $2pq$. The associated splice diagram is shown in Figure 3.9.

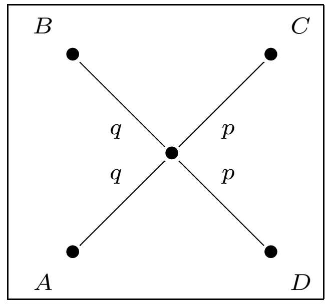


FIGURE 3.9. Splice diagram for Case (ii): $\{z^2 = x^{2p} + y^{2q}\}$.

Using the coordinates A, B, C, and D, the Neumann-Wahl algorithm yields equations of the form

$$\left\{ \begin{array}{rcl} C^p + a_1 A^q + b_1 B^q & = & 0 \\ D^p + a_2 A^q + b_2 B^q & = & 0 \end{array} \right\},$$

where the a_i and b_i are complex numbers such that $a_1 b_2 - a_2 b_1 \neq 0$. To get the actual UAC of $(X_2, 0)$, one must choose these coefficients carefully.

It turns out that the following equations define the UAC of $(X_2, 0)$:

$$(Y_2, 0) := \left\{ \begin{array}{rcl} C^p - kA^q - kB^q & = & 0 \\ D^p + kA^q - kB^q & = & 0 \end{array} \right\}, \tag{3.15}$$

where $k = \frac{1}{\sqrt{2}}$.

REMARK 3.2.14. The UAC of $(X_2, 0)$ is never a $\mathbb{Q}$HS. This can be determined, for example, by using the method described in [**25**] to show that the genus of the central curve in the minimal resolution graph of the weighted homogeneous singularity $(Y_2, 0)$ is nonzero. See also [**12**], Prop. 12.2.

The Neumann-Wahl algorithm also yields the action of the discriminant group $H := \mathbb{Z}/2pq\mathbb{Z}$ on the UAC $(Y_2, 0)$. The group action is generated by the following two elements:

$$(A, B, C, D) \mapsto \left(\left[-\frac{p}{2q} \right] A, \left[\frac{p}{2q} \right] B, -C, -D \right),$$

and

$$(A,B,C,D) \mapsto \left(-A, -B, \left[-\frac{q}{2p}\right] C, \left[\frac{q}{2p}\right] D\right),$$

where $[r] = \exp(2\pi i r)$. Taking care to distinguish between the cases when p and q are both odd and when one of them is even, one can show that this action is generated by

$$(A,B,C,D) \mapsto (\xi A, \xi^{-1} B, \zeta C, \zeta^{-1} D), \tag{3.16}$$

where ξ is a primitive $2q$-th root of unity and ζ is a primitive $2p$-th root of unity. (For p and q both odd, take the first element composed with the square of the second, and for p even, take the square of the first element composed with the second.)

The invariants under this action are $A^{2q},\ B^{2q},\ C^{2p},\ D^{2p},\ AB,\ CD,\ C^pA^q,\ C^pB^q,\ D^pA^q,$ and D^pB^q. However, modulo the equations in (3.15), they can all be written in terms of these four:

$$A^{2q},\ B^{2q},\ AB,\ CD.$$

The equations (3.15) of $(Y_2, 0)$ are weighted homogeneous of weight pq and transform by -1 under the generator (3.16) of the action of H. We must deform $(Y_2, 0)$ by adding equisingular terms that also transform by -1 under the generator, and then mod out by the action of H.

Recall that for a ring $\mathcal{O}_Y$ of the form $\mathbb{C}[[A,B,C,D]]/(f_1, f_2)$,

$$T^1_Y = \mathcal{O}^2_Y / J,$$

where J is the submodule generated by

$$\left\langle \frac{\partial f_1}{\partial A}, \frac{\partial f_2}{\partial A} \right\rangle,\ \left\langle \frac{\partial f_1}{\partial B}, \frac{\partial f_2}{\partial B} \right\rangle,\ \left\langle \frac{\partial f_1}{\partial C}, \frac{\partial f_2}{\partial C} \right\rangle,\ \text{and}\ \left\langle \frac{\partial f_1}{\partial D}, \frac{\partial f_2}{\partial D} \right\rangle.$$

When f_1 and f_2 are weighted homogeneous, there is a natural way to define the weight of an element of T^1_Y, namely $\langle g_1, g_2 \rangle$ is weighted homogeneous of weight k if and only if g_1 and

g_2 are weighted homogeneous polynomials such that

$$\text{weight}(g_i) = \text{weight}(f_i) + k, \text{ for } i = 1,\ 2.$$

Fact 3.2.15 ([**23**],[**25**]). *Let $(Y,0) \subset (\mathbb{C}^4,0)$ be a complete intersection singularity that is weighted homogeneous and defined by the ideal (f_1, f_2). Let*

$$\{\langle g_{11}, g_{12}\rangle, \ldots, \langle g_{m1}, g_{m2}\rangle\}$$

be a basis for the subspace of weighted homogeneous elements with nonnegative weight in T^1_Y. Then the family defined by

$$\left\{ \begin{array}{ll} f_1 + t_1 g_{11} + \cdots t_m g_{m1} & = 0 \\ f_2 + t_1 g_{12} + \cdots t_m g_{m2} & = 0 \end{array} \right\}$$

is a versal equisingular deformation of $(Y,0)$ (equisingular in the sense of having a simultaneous resolution).

Definition 3.2.16. Let T denote the subspace of weighted homogeneous elements in $T^1_{Y_2}$ that transform by -1 under the generator of the H-action and have nonnegative weight.

Lemma 3.2.17. *A weighted homogeneous basis of T is given by the vectors*

$$\{\langle (CD)^\alpha (AB)^\beta A^q, (CD)^\alpha (AB)^\beta A^q\rangle \mid 0 \leq \alpha \leq p-2,\ 0 \leq \beta \leq q-2\}.$$

Proof. Clearly, the elements listed in this set are weighted homogeneous of nonnegative weight and transform by -1 under the generator of the H-action.

Refer to the equations (3.15) that define $(Y_2, 0)$. $T^1_{Y_2} = \mathcal{O}^2_{Y_2}/J$, where J is the submodule generated by the vectors

$$\left\langle C^{p-1}, 0\right\rangle,\ \left\langle 0, D^{p-1}\right\rangle,\ \left\langle A^{q-1}, -A^{q-1}\right\rangle,\ \text{and}\ \left\langle B^{q-1}, B^{q-1}\right\rangle.$$

It is easy to check that the only monomials that transform by -1 under the action of the generator are $A^q,\ B^q,\ C^p,\ D^p$, plus any H-invariant monomial times one of these four. Recall that all of the H-invariant monomials can be written in terms of $A^{2q},\ B^{2q},\ AB$, and CD. Furthermore, C^p and D^p can clearly be written in terms of A^q and B^q modulo (3.15).

We have the following relations in $T^1_{Y_2}$:

$$\begin{aligned}\langle 0,0\rangle &= \langle C^p,0\rangle = \langle kA^q+kB^q,0\rangle = \langle kA^q,0\rangle+\langle kB^q,0\rangle;\\ \langle 0,0\rangle &= \langle 0,D^p\rangle = \langle 0,kB^q-kA^q\rangle = \langle 0,kB^q\rangle-\langle 0,kA^q\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\langle A^q,0\rangle &= \langle -B^q,0\rangle,\ \text{and}\\ \langle 0,A^q\rangle &= \langle 0,B^q\rangle.\end{aligned}$$

Multiplying each of these equations first by A^q and then by B^q implies

$$\begin{aligned}\langle A^{2q},0\rangle &= \langle B^{2q},0\rangle = \langle -(AB)^q,0\rangle,\ \text{and}\\ \langle 0,A^{2q}\rangle &= \langle 0,B^{2q}\rangle = \langle 0,(AB)^q\rangle.\end{aligned}$$

Furthermore,

$$\begin{aligned}B^{q-1}\langle A^{q-1},-A^{q-1}\rangle+A^{q-1}\langle B^{q-1},B^{q-1}\rangle &= \langle 2(AB)^{q-1},0\rangle,\ \text{and}\\ -B^{q-1}\langle A^{q-1},-A^{q-1}\rangle+A^{q-1}\langle B^{q-1},B^{q-1}\rangle &= \langle 0,2(AB)^{q-1}\rangle.\end{aligned}$$

Since the left hand side of each of these equations is trivial in $T^1_{Y_2}$ (i.e., is in the submodule J), we see that $\langle (AB)^{q-1},0\rangle$ and $\langle 0,(AB)^{q-1}\rangle$ are trivial, and therefore so are $\langle A^{2q},0\rangle,\ \langle 0,A^{2q}\rangle,\ \langle B^{2q},0\rangle$, and $\langle 0,B^{2q}\rangle$. Hence, the only nontrivial elements of T are vectors containing products of $AB,\ CD$, and A^q, or $AB,\ CD$, and B^q.

The nontrivial weighted homogeneous elements of T with nonnegative weight are of the form $\langle (CD)^\alpha (AB)^\beta A^q,(CD)^\alpha (AB)^\beta A^q\rangle$ (A^q could be replaced by $\pm B^q$ in either entry, as appropriate). Note that the values of α and β must be the same in both entries for weighted homogeneity. Furthermore, we must have $0 \leq \alpha \leq p-2,\ 0 \leq \beta \leq q-2$, since $(CD)^{p-1}$ and $(AB)^{q-1}$ are trivial in either entry. At this point, it should be clear that the elements listed indeed form a basis. ☐

PROPOSITION 3.2.18. *A first-order versal splice quotient deformation of* $(X_2, 0)$ *is given by*

$$\left\{ z^2 = x^{2p} + y^{2q} + \sum_{(i,j)\in\mathbb{J}^+} \varepsilon_{ij} x^i y^j \right\}, \text{ where} \tag{3.17}$$

$$\begin{aligned} \mathbb{J}^+ &= \{(i,j) \in \mathbb{Z}^2 \mid p \leq i < 2p-1,\ q \leq j < 2q-1\} \\ &= \{(i,j) \in \mathbb{J} \mid i \geq p \text{ and } j \geq q\}. \end{aligned}$$

PROOF. We consider the first-order family defined by

$$\left\{ \begin{aligned} C^p - kB^q - kA^q + k\Sigma\varepsilon_{\alpha\beta}(CD)^\alpha(AB)^\beta A^q &= 0 \\ D^p - kB^q + kA^q + k\Sigma\varepsilon_{\alpha\beta}(CD)^\alpha(AB)^\beta A^q &= 0 \end{aligned} \right\}, \tag{3.18}$$

where Σ denotes the sum over all $(\alpha, \beta) \in \mathbb{K}$, and

$$\mathbb{K} = \{(\alpha, \beta) \in \mathbb{Z}^2 \mid 0 \leq \alpha \leq p-2,\ 0 \leq \beta \leq q-2\}.$$

By first order, we mean $\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} = 0$ for all $(\alpha, \beta),\ (\gamma, \delta) \in \mathbb{K}$. Also, recall that k is defined to be $1/\sqrt{2}$.

By Fact 3.2.15 and Lemma 3.2.17, (3.18) is a versal equisingular H-equivariant deformation of $(Y_2, 0)$. We will compute the quotient of this family by H. The equations (3.18) imply

$$\begin{aligned} C^pD^p &= (kB^q + kA^q - k(\Sigma\varepsilon_{\alpha\beta}(CD)^\alpha(AB)^\beta)A^q)(kB^q - kA^q - k(\Sigma\varepsilon_{\alpha\beta}(CD)^\alpha(AB)^\beta)A^q) \\ &= k^2B^{2q} - k^2A^{2q} - 2k^2(\Sigma\varepsilon_{\alpha\beta}(CD)^\alpha(AB)^\beta)(AB)^q \\ &= \frac{1}{2}B^{2q} - \frac{1}{2}A^{2q} - \Sigma\varepsilon_{\alpha\beta}(CD)^\alpha(AB)^{\beta+q}. \end{aligned}$$

In terms of the invariants $S := A^{2q},\ T := B^{2q},\ y := AB$, and $x := CD$, we have

$$2x^p = T - S - \Sigma\varepsilon_{\alpha\beta}x^\alpha y^{\beta+q}.$$

Therefore,

$$T = S + 2x^p + \Sigma\varepsilon_{\alpha\beta}x^\alpha y^{\beta+q}.$$

The relation $y^{2q} = ST$ yields

$$y^{2q} = S \cdot \left\{S + 2x^p + \Sigma\varepsilon_{\alpha\beta}x^\alpha y^{\beta+q}\right\}.$$

This is a first order family of splice quotients; when all $\varepsilon_{\alpha\beta} = 0$, it is isomorphic to $(X_2, 0)$. We want to write this in the form $z^2 = x^{2p} + y^{2q}$ plus higher order terms, so we complete the square in S. Let $z = S + x^p + \frac{1}{2}\Sigma\varepsilon_{\alpha\beta}x^\alpha y^{\beta+q}$, and the family becomes

$$y^{2q} = \left(z - x^p - \frac{1}{2}\Sigma\varepsilon_{\alpha\beta}x^\alpha y^{\beta+q}\right)\left(z + x^p + \frac{1}{2}\Sigma\varepsilon_{\alpha\beta}x^\alpha y^{\beta+q}\right).$$

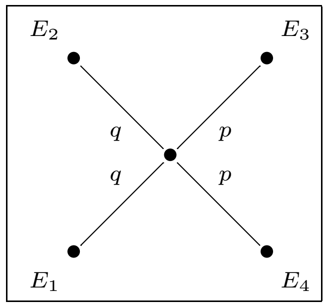
That is,

$$z^2 = x^{2p} + y^{2q} + \Sigma\varepsilon_{\alpha\beta}x^{\alpha+p}y^{\beta+q}.$$

Finally, it is clear that $(\alpha + p, \beta + q) \in \mathbb{J}^+$ if and only if $(\alpha, \beta) \in \mathbb{K}$. □

Remark 3.2.19. In this case, as opposed to Case (i) (Theorem 3.2.7), *at this point* it is unclear how to write down a versal splice quotient deformation in the form $z^2 = x^{2p} + y^{2q}$ plus higher order terms in $t_{\alpha\beta}$. There is a change of coordinates that will do this, but we have been unable to find a simple presentation. On the other hand, see Proposition 3.2.20 below.

Approach II. We now consider deformations of $(X_2, 0)$ from the point of view of end-curve functions. Recall that the resolution graph of $(X_2, 0)$ has four leaves. The associated splice diagram is



The dual basis elements corresponding to E_1 and E_2 have order $2q$ in the discriminant group, and those corresponding to E_3 and E_4 have order $2p$. Let $g = z^2 - x^{2p} - y^{2q}$, so that $\mathcal{O}_{X_2} = \mathbb{C}[[x, y, z]]/(g)$. Consider the following decompositions:

$$g = (z - x^p)(z + x^p) - y^{2q}; \tag{3.19}$$

$$g = (z - y^q)(z + y^q) - x^{2p}. \tag{3.20}$$

One can verify, e.g., using Theorem 2.0.12 (6), that the end-curve functions for the E_i are $z - x^p,\ z + x^p,\ z - y^q$, and $z + y^q$, respectively.

By Proposition 3.1.6, the decomposition (3.19) lifts to a decomposition of the first order deformation $g + \varepsilon\tilde{g}$ if and only if $\tilde{g}$ is in the ideal generated by $z - x^p,\ z + x^p$, and y^{2q-1}. Similarly, (3.20) lifts if and only if $\tilde{g}$ is in the ideal generated by $z - y^q,\ z + y^q$, and x^{2p-1}. For a splice quotient deformation, $\tilde{g}$ must be in the intersection of the two ideals. Therefore, $g + \varepsilon x^i y^j$ for $(i, j) \in \mathbb{J}$ is a splice quotient if and only if $i \geq p$ and $j \geq q$, i.e., if and only if $\mathbb{J} = \mathbb{J}^+$. This statement is in agreement with Proposition 3.2.18.

Finally, we can use end-curve functions to prove the following.

Proposition 3.2.20. *The linear family over* $\mathbb{C}[[t_{ij}]]$

$$\left\{ z^2 = x^{2p} + y^{2q} + \sum_{(i,j)\in\mathbb{J}^+} t_{ij}x^iy^j \right\}$$

is a versal family of splice quotients.

Proof. Consider the following decompositions of $z^2 = x^{2p} + y^{2q} + \sum t_{ij}x^iy^j$, where Σ indicates summation over all $(i,j) \in \mathbb{J}^+$:

$$\begin{aligned}(z - x^p - \frac{1}{2}\sum t_{ij}x^{i-p}y^j)(z + x^p + \frac{1}{2}\sum t_{ij}x^{i-p}y^j) &= y^{2q} - \frac{1}{4}\left[\sum t_{ij}x^{i-p}y^j\right]^2;\\ (z - y^q - \frac{1}{2}\sum t_{ij}x^iy^{j-q})(z + y^q + \frac{1}{2}\sum t_{ij}x^iy^{j-q}) &= x^{2p} - \frac{1}{4}\left[\sum t_{ij}x^iy^{j-q}\right]^2.\end{aligned}$$

In the first equation, the right side is divisible by y^{2q} since all $j \geq q$, so the two factors on the left side are the end-curve functions of order $2q$ analogous to (3.19). Similarly, in the second equation the right side is divisible by x^{2p} since all $i \geq p$, and we have the remaining two end-curve functions. This linear family must be versal since it agrees with the first-order versal family of splice quotients from Proposition 3.2.18. □

Remark 3.2.21. On the level of tangent spaces, we see that if $i < p$ or $j < q$, then the deformation $z^2 = x^{2p} + y^{2q} + \varepsilon x^iy^j$ is not a splice quotient. In fact, there is only one X_2 for which there are no terms in the versal equisingular deformation with $i < p$ or $j < q:$ $\{z^2 = x^4 + y^6\}$. It is easy to check that $(\{z^2 = x^4 + y^6\}, 0)$ is minimally elliptic.

CHAPTER 4

The topology of $\{z^n = f(x,y)\}$, for f irreducible

Let $\{f(x,y)=0\} \subset \mathbb{C}^2$ define an analytically irreducible plane curve with a singularity at the origin. Let $X_{f,n} \subset \mathbb{C}^3$ be defined by

$$X_{f,n} := \{z^n = f(x,y)\}.$$

The goal of Chapter 6 is to characterize those $(X_{f,n}, 0)$ for which the link has the topological type of a splice quotient; that is, we determine which topological types satisfy the semigroup and congruence conditions as defined by Neumann and Wahl in [**18**]. This chapter and the next contain all of the preliminary material that we will need.

In [**8**], Mendris and Némethi prove that for f irreducible, the link of $(X_{f,n}, 0)$ completely determines the Newton/topological pairs of f and the value of n, with two well-understood exceptions. In doing so, they give a presentation of the construction of the resolution graph of $(X_{f,n}, 0)$ that is very useful for our purposes. In particular, they give a criterion that determines when $(X_{f,n}, 0)$ has $\mathbb{Q}$HS link. Section 4.1 is mostly a reiteration of their work, and we use their notation whenever possible. For technical reasons, we will use a modification of the minimal good resolution graph, a quasi-minimal resolution graph, for many of the computations in the next few chapters. In §4.2, we set up the notation needed to describe the associated splice diagram and compute all of the weights at each node.

There is one case in which the resolution graph does not quite have the same structure as the general case. It is referred to as the "pathological case" by Mendris and Némethi, and we use this terminology as well. Most of the computations need to be done separately for the pathological case.

4.1. Resolution graph

In this section, we describe the construction of the minimal good resolution graph of $(X_{f,n}, 0)$ for f irreducible. As mentioned above, most of this section is a reiteration of work that appears in [**8**]. It begins with the plane curve singularity defined by $\{f = 0\}$. It has long been known (see, e.g. [**2**]) that in a neighborhood of the singular point, one can solve for one variable as a fractional power series in the other, leading to an expression of the following form, the so-called Puiseux expansion:

$$y = c_1 x^{m_1/p_1} + c_2 x^{m_2/p_1 p_2} + c_3 x^{m_3/p_1 p_2 p_3} + \dots,\ c_i \in \mathbb{C}.$$

Only finitely many of these terms are significant with respect to the topology of the singularity, and the pairs of positive integers (p_k, m_k) that appear in this expresssion are called the *Puiseux pairs* of f. Alternatively, if we write

$$y = x^{q_1/p_1}(c_1 + x^{q_2/p_1 p_2}(c_2 + x^{q_3/p_1 p_2 p_3}(c_3 + \cdots)\cdots)),$$

the pairs (p_k, q_k) are referred to as the *Newton pairs*. They satisfy the following properties:

$$\gcd(p_k, q_k) = 1,\ p_k \geq 2,\ q_k \geq 1,\ \text{and } q_1 > p_1.$$

Suppose that f has s Newton pairs $(p_k, q_k),\ 1 \leq k \leq s$. Define a_k by $a_1 = q_1$, and

$$a_k = q_k + a_{k-1} p_{k-1} p_k,\ 2 \leq k \leq s. \tag{4.1}$$

The pairs (p_k, a_k) are defined by Eisenbud and Neumann in [**3**], and are referred to as the *topological pairs* of f. These are the integers that appear in the splice diagram of the plane curve singularity $\{f=0\}$ in $\mathbb{C}^2$. Note that

$$\gcd(p_k, a_k) = 1,\ a_1 > p_1,\ \text{and } a_k > a_{k-1}p_{k-1}p_k.$$

Embedded resolutions. Let $(X,0)$ be a normal surface singularity, and let $g:(X,0)\to(\mathbb{C},0)$ be the germ of an analytic function. Then an *embedded resolution* of $\{g=0\}$ in $(X,0)$ is a manifold $\tilde{X}$ together with a proper map $\pi:\tilde{X}\to X$, such that π induces a biholomorphism of $\tilde{X}\setminus\pi^{-1}(0)$ and $X\setminus\{0\}$, and $\pi^{-1}(\{g=0\})$ is a divisor with only normal crossings. We also assume that no irreducible component of the exceptional set $E:=\pi^{-1}(0)$ intersects itself. The closure of $\pi^{-1}(\{g=0\}-\{0\})$ is called the *proper transform* of $\{g=0\}$.

The embedded resolution graph $\Gamma(X,g)$ of π is defined as follows. To each irreducible curve in the exceptional set E, we assign a vertex, and to each irreducible component of the proper transform of $\{g=0\}$ we assign an arrowhead. We draw an edge between two vertices or between a vertex and an arrowhead, if the corresponding components intersect in $\tilde{X}$. Each vertex v corresponding to the exceptional curve E_v is labeled with the self-intersection number E_v^2, the genus g_v of E_v, and the multiplicity m_v of g in E_v. The multiplicity m_v is the order of vanishing of $g\circ\pi$ along E_v. An arrowhead is labeled with the multiplicity of the corresponding irreducible component of the proper transform.

A *resolution* of $(X,0)$ is a manifold $\tilde{X}$ together with a proper map $\pi:\tilde{X}\to X$, such that π induces a biholomorphism of $\tilde{X}\setminus\pi^{-1}(0)$ and $X\setminus\{0\}$. A resolution is called *good* if each irreducible component of the exceptional divisor E is smooth, and E is a normal crossing divisor. Again, we assume that no irreducible component of the exceptional set E intersects itself. For a good resolution, the resolution graph of π is constructed in the same way as the embedded resolution graph above, only there are no arrowheads or multiplicities in this graph. A resolution graph or embedded resolution graph is called *minimal* if it does not contain any vertices with genus 0 and self-intersection -1. A minimal resolution graph is not necessarily good, but there does exist a unique minimal good (embedded) resolution graph.

The minimal good embedded resolution graph of $\{f = 0\}$ in $\mathbb{C}^2$ is a tree of rational curves, denoted $\Gamma(\mathbb{C}^2, f)$. This graph is constructed as follows (e.g., [**2**]). For $1 \leq k \leq s$, determine the continued fraction expansions

$$\frac{p_k}{q_k} = \mu_k^0 - \cfrac{1}{\mu_k^1 - \cfrac{1}{\mu_k^2 - \cfrac{1}{\ddots - \cfrac{1}{\mu_k^{t_k}}}}}, \text{ and } \frac{q_k}{p_k} = \nu_k^0 - \cfrac{1}{\nu_k^1 - \cfrac{1}{\nu_k^2 - \cfrac{1}{\ddots - \cfrac{1}{\nu_k^{r_k}}}}},$$

where $\mu_k^0,\ \nu_k^0 \geq 1$, and $\mu_k^j,\ \nu_k^j \geq 2$ for $j > 0$. Then $\Gamma(\mathbb{C}^2, f)$ (with multiplicities omitted) is given as in Figure 4.1.

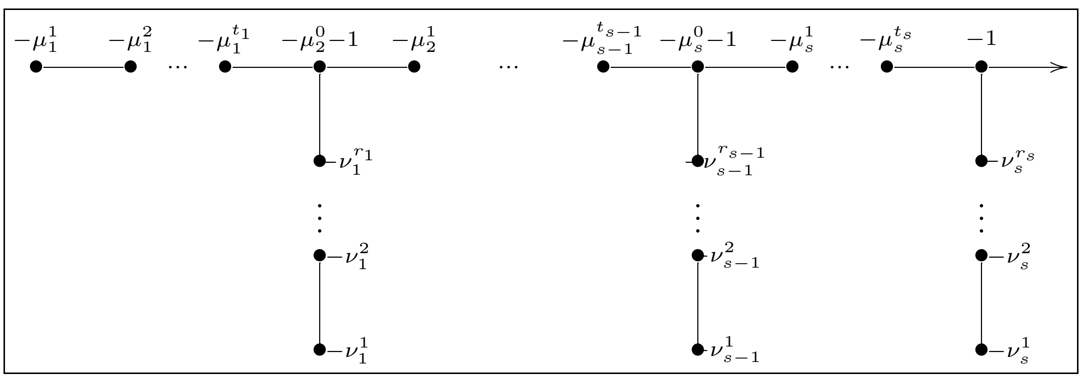


FIGURE 4.1. Minimal good embedded resolution graph of $\{f = 0\}$ in $\mathbb{C}^2$.

Following the notation of Mendris and Nèmethi [**8**], we consider this diagram in a more convenient schematic form (Figure 4.2), where the dashed lines represent strings of rational

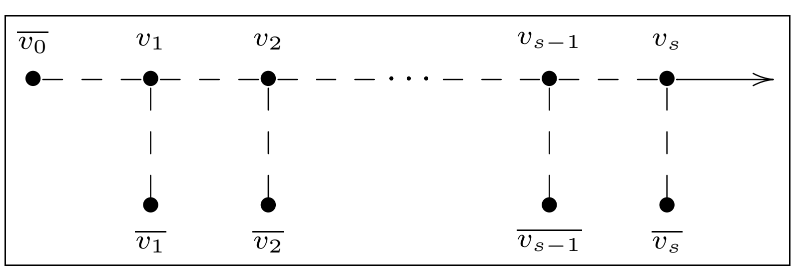


FIGURE 4.2. Schematic form of $\Gamma(\mathbb{C}^2, f)$.

curves (possibly empty). The multiplicities of the vertices v_k and $\overline{v_k}$ are as follows:

$$\begin{aligned} m_{v_k} &= a_k p_k p_{k+1} \cdots p_s, \quad \text{for } 1 \leq k \leq s, \\ m_{\overline{v_0}} &= p_1 p_2 \cdots p_s, \\ m_{\overline{v_k}} &= a_k p_{k+1} \cdots p_s, \qquad \text{for } 1 \leq k \leq s-1, \\ m_{\overline{v_s}} &= a_s. \end{aligned}$$

(We will need these multiplicities in §5.2.)

There is an algorithm for constructing an embedded resolution graph (not necessarily minimal) of the function $z : (X_{f,n}, 0) \to (\mathbb{C}, 0)$ from the graph $\Gamma(\mathbb{C}^2, f)$, for which the ideas first appeared in [**5**]. Here, we follow the presentation in §3 and 4 of [**8**]. The output of this algorithm, without any modifications by blow up or down, is referred to by the authors as the *canonical* embedded resolution graph of z in $(X_{f,n}, 0)$, and is denoted $\Gamma^{can}(X_{f,n}, z)$. The n-fold "covering" or "graph projection" produced in the algorithm is denoted $q : \Gamma^{can}(X_{f,n}, z) \to \Gamma(\mathbb{C}^2, f)$. We reproduce here only what is necessary for our purposes. For full details, the reader should see [**8**].

Definition 4.1.1. Define positive integers $d_k,\ h_k,\ \widetilde{h_k},\ p'_k$, and a'_k as follows.

- $d_k = (n, p_{k+1} p_{k+2} \cdots p_s)$ for $0 \leq k \leq s-1$, and $d_s = 1$;

and, for $1 \leq k \leq s$,

- $h_k = (p_k, n/d_k)$;
- $\widetilde{h_k} = (a_k, n/d_k)$;
- $p'_k = p_k / h_k$;
- $a'_k = a_k / \widetilde{h_k}$.

If w is a vertex in $\Gamma(\mathbb{C}^2, f)$, then all vertices in $q^{-1}(w)$ have the same multiplicity and genus, which we denote m_w and g_w, respectively.

PROPOSITION 4.1.2 ([**8**], Corollary 4.2). *Let* $q : \Gamma^{can}(X_{f,n}, z) \to \Gamma(\mathbb{C}^2, f)$ *be the "graph projection" mentioned above. Then* $\Gamma^{can}(X_{f,n}, z)$ *is a tree such that the following hold:*

$$
\begin{aligned}
&(a) \quad \#q^{-1}(v_s) = 1, && \#q^{-1}(v_k) = h_{k+1}\cdots h_s, && (1 \le k \le s-1)\\
&\quad\;\;\; \#q^{-1}(\overline{v_s}) = \widetilde{h}_s, && \#q^{-1}(\overline{v_k}) = \widetilde{h}_k h_{k+1}\cdots h_s, && (1 \le k \le s-1)\\
&\quad\;\;\; \#q^{-1}(\overline{v_0}) = h_1 \cdots h_s;
\end{aligned}
$$

$$
\begin{aligned}
(b) \quad m_{v_k} &= a'_k p'_k p'_{k+1} \cdots p'_s \quad (1 \le k \le s),\\
m_{\overline{v_0}} &= p'_1 p'_2 \cdots p'_s,\\
m_{\overline{v_k}} &= a'_k p'_{k+1} \cdots p'_s \qquad (1 \le k \le s-1),\\
m_{\overline{v_s}} &= a'_s;
\end{aligned}
$$

$$
\begin{aligned}
(c) \quad g_{\overline{v_k}} &= 0 && (0 \le k \le s),\\
g_{v_k} &= (h_k - 1)(\widetilde{h}_k - 1)/2 && (1 \le k \le s).
\end{aligned}
$$

In particular, the link of $(X_{f,n}, 0)$ *is a* $\mathbb{Q}$*HS if and only if* $(h_k - 1)(\widetilde{h}_k - 1) = 0$ *for all* $k,\ 1 \le k \le s$.

REMARK 4.1.3. We are only interested in those $(X_{f,n}, 0)$ that have $\mathbb{Q}$HS link. Part (c) of Proposition 4.1.2 tells us exactly when this occurs. Note that $\gcd(h_k, \widetilde{h}_k) = 1$ since $\gcd(p_k, a_k) = 1$ for all k. In particular, if n is prime, $(X_{f,n}, 0)$ automatically has $\mathbb{Q}$HS link.

From this point forward, we assume that $(X_{f,n}, 0)$ has $\mathbb{Q}$HS link; equivalently, either h_k or $\widetilde{h}_k$ is 1 for all k.

The appearance of $\Gamma^{can}(X_{f,n}, z)$ is displayed in Figure 4.3, which is reproduced from [**8**]. By abuse of notation, we have labelled any vertex in $q^{-1}(v_k)$ (respectively, $q^{-1}(\overline{v_k})$) with v_k (respectively, $\overline{v_k}$). The dashed lines represent strings of vertices. By the construction, each string must contain at least as many vertices as its image in $\Gamma(\mathbb{C}^2, f)$. It is important to remember that in $\Gamma^{can}(X_{f,n}, z)$, these strings are not necessarily minimal.

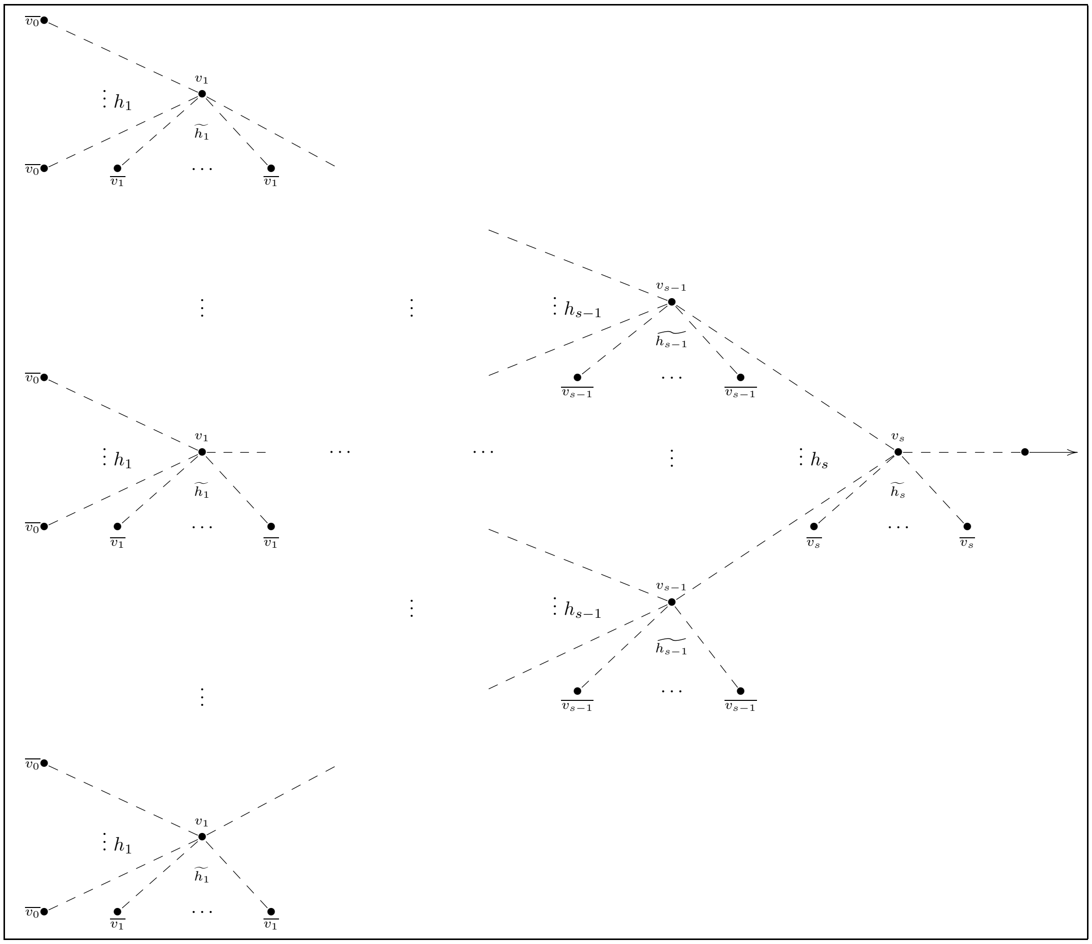


FIGURE 4.3. Schematic form of $\Gamma^{can}(X_{f,n}, z)$.

DEFINITION 4.1.4. A minimal good embedded resolution graph of z in $(X_{f,n}, 0)$, denoted $\Gamma^{min}(X_{f,n}, z)$, is obtained from $\Gamma^{can}(X_{f,n}, z)$ by blowing down any rational curves of self-intersection -1 for which the corresponding vertex has valence one or two. By dropping the arrowhead and multiplicities of $\Gamma^{min}(X_{f,n}, z)$ and then blowing down any appropriate rational curves of self-intersection -1, we obtain a minimal good resolution graph of $(X_{f,n}, 0)$, denoted $\Gamma^{min}(X_{f,n})$.

Recall that a *node* is a vertex of valence at least three, a *leaf* is a vertex of valence one, and a *string* is a connected subgraph that does not include a node.

PROPOSITION 4.1.5 ([**8**]). *All of the nodes in* $\Gamma^{can}(X_{f,n}, z)$ *survive as nodes in* $\Gamma^{min}(X_{f,n}, z)$. *That is, they are not blown down in the minimalization process, and after minimalization, they still have valence at least three.*

PROPOSITION 4.1.6 ([**8**]). *Assume that by deleting the arrowhead of* $\Gamma^{min}(X_{f,n}, z)$ *we obtain a non-minimal graph. This situation can happen if and only if* $n = p_s = 2$.

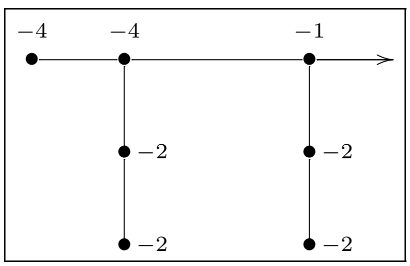
Because of this property, we refer to $n = p_s = 2$ as the *pathological case*, and it is treated separately.

A quasi-minimal resolution graph. Since the notation used to describe these graphs is cumbersome, we would like use the simplest version of the resolution graph to which the results of Neumann and Wahl can be applied. One complication that arises is that certain strings in $\Gamma^{can}(X_{f,n}, z)$ may completely collapse upon minimalization. Therefore, if we use the graph $\Gamma^{min}(X_{f,n})$ in what follows, we would constantly need to note that certain strings may not actually be there, and more importantly, that certain leaves in the splice diagram may not actually be there. We will avoid this by using a uniquely defined quasi-minimal resolution graph.

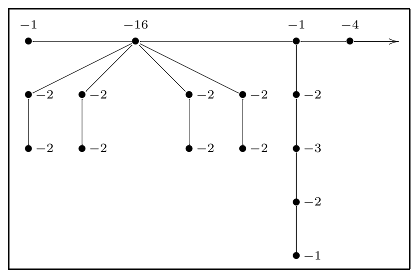
Recall that a resolution tree Γ is *quasi-minimal* if any string in Γ either contains no (-1)-weighted vertex, or consists of a unique (-1)-weighted vertex. Recall that Theorem 2.0.12, the main theorem of Neumann and Wahl concerning splice quotient singularities, applies to quasi-minimal resolution graphs that satisfy the semigroup and congruence conditions.

DEFINITION 4.1.7. We define quasi-minimal graphs $\Gamma^{qmin}(X_{f,n}, z)$ and $\Gamma^{qmin}(X_{f,n})$ uniquely as follows. Perform the minimalization process as usual on $\Gamma^{can}(X_{f,n}, z)$, except on any string from a leaf to a node that completely collapses. In that case, instead of blowing down the entire string, stop when the string has become one vertex of self-intersection -1. The resulting graph is $\Gamma^{qmin}(X_{f,n}, z)$. Drop the arrowhead and multiplicities to obtain $\Gamma^{qmin}(X_{f,n})$, which is indeed quasi-minimal (excluding the pathological case). In particular, if there does not exist a string from a leaf to a node that completely collapses, then $\Gamma^{qmin}(X_{f,n}, z) = \Gamma^{min}(X_{f,n}, z)$ and $\Gamma^{qmin}(X_{f,n}) = \Gamma^{min}(X_{f,n})$. We may refer to $\Gamma^{qmin}(X_{f,n})$ as *the* quasi-minimal resolution graph, although it is certainly not the only quasi-minimal modification of $\Gamma^{can}(X_{f,n}, z)$.

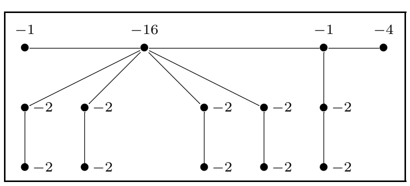
Example 4.1.8. Let $n = 4$, $p_1 = 3$, $p_2 = 3$, $q_1 = 4$, and $q_2 = 1$ ($a_2 = 37$). The various resolution graphs defined above are shown in Figure 4.4. Multiplicities are omitted.



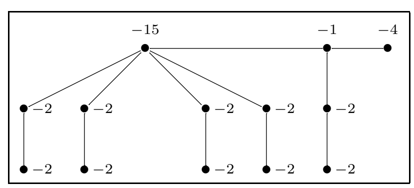
(a) $\Gamma(\mathbb{C}^2, f)$



(b) $\Gamma^{can}(X_{f,n}, z)$



(c) $\Gamma^{qmin}(X_{f,n})$



(d) $\Gamma^{min}(X_{f,n})$

Figure 4.4. Different versions of the resolution graph for Example 4.1.8.

The difference between $\Gamma^{min}(X_{f,n})$ and $\Gamma^{qmin}(X_{f,n})$ is not of great significance; we only use $\Gamma^{qmin}(X_{f,n})$ because it makes the arguments easier and the theorem of Neumann and Wahl can still be applied to it.

The pathological case. Here we describe the minimal resolution graph for the pathological case $n = p_s = 2$. Since $n/h_s = n/d_{s-1} = 1$, by definition $h_k = \widetilde{h_k} = 1$ for $1 \leq k \leq s-1$, and $\widetilde{h}_s = 1$ since $\gcd(p_s, a_s) = 1$. Therefore, the link is a $\mathbb{Q}$HS and $\Gamma^{min}(X_{f,n}, z)$ has the form shown in Figure 4.5. The minimal resolution graph $\Gamma^{min}(X_{f,n})$ is obtained from $\Gamma^{min}(X_{f,n}, z)$

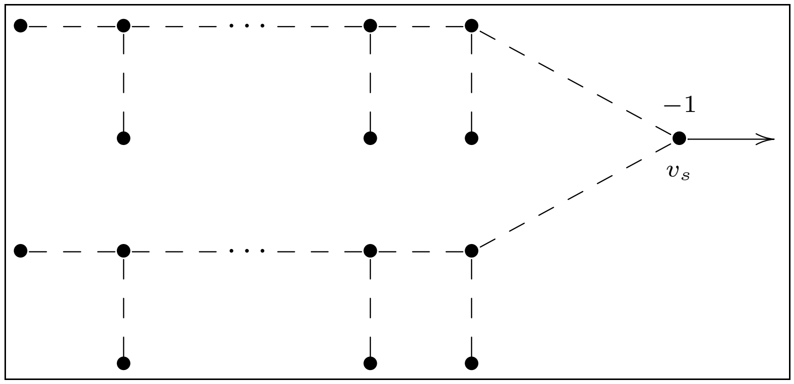


FIGURE 4.5. Schematic form of $\Gamma^{min}(X_{f,n}, z)$ in the pathological case.

by deleting the arrowhead and blowing down the vertex v_s. No other blow downs are necessary (see [**8**], Prop. 5.2).

4.2. Splice diagram

Recall the procedure for computing the splice diagram Δ associated to a resolution graph Γ that is a tree of rational curves; it is described in Chapter 2, and we repeat it here for convenience. First, omit the self-intersection numbers and contract all strings of valence two vertices in Γ. To each node v in the resulting diagram Δ, we attach a weight d_{ve} in the direction of each adjacent edge e. Let Γ_{ve} be the subgraph of Γ defined as follows. Remove the vertex that corresponds to the node v, and the edge that corresponds to e, and let Γ_{ve} be the remaining connected subgraph that was connected to v by e. Then the weight d_{ve} is $\det(-C_{ve})$, where C_{ve} is the intersection matrix of the graph Γ_{ve}.

DEFINITION 4.2.1. Let $\Delta_{f,n}$ be the splice diagram associated to $\Gamma^{qmin}(X_{f,n})$. Note, however, that for computing the weights of $\Delta_{f,n}$, we can use any modification of $\Gamma^{can}(X_{f,n}, z)$ that is a good resolution graph.

Almost all of the determinants that we need at each node for the splice diagram are explicitly computed in [**8**]. We again use their notation in what follows.

Subgraphs of $\Gamma^{qmin}(X_{f,n})$. We must label all of the subgraphs of $\Gamma^{qmin}(X_{f,n})$ whose determinants are the weights of $\Delta_{f,n}$.

By Proposition 4.1.5, the nodes of $\Gamma^{qmin}(X_{f,n})$ can be identified with the nodes of $\Gamma^{can}(X_{f,n}, z)$. If a vertex v in $\Gamma^{qmin}(X_{f,n})$ is identified with a vertex in $q^{-1}(v_k)$, we say that v is "of type v_k". Similarly, the leaves of $\Gamma^{qmin}(X_{f,n})$ can be identified with the leaves of $\Gamma^{can}(X_{f,n}, z)$, and we will say that a vertex v in $\Gamma^{qmin}(X_{f,n})$ is "of type $\overline{v_k}$" if it is the image (under minimalization) of a vertex in $q^{-1}(\overline{v_k})$. We will use the same terminology for the corresponding vertices of $\Delta_{f,n}$.

<u>$\Gamma(v_k)$ and $\Gamma(\overline{v_k})$</u>. Let w be a vertex in $\Gamma(\mathbb{C}^2, f)$, and let v' be any vertex of type w in $\Gamma^{qmin}(X_{f,n}, z)$. Consider the shortest path from v' to the arrowhead in $\Gamma^{qmin}(X_{f,n}, z)$. If $w \neq v_s$, then there is at least one node on this path; let v'' be the node on the path that is closest to v'. If $w = v_k$, $1 \leq k \leq s-1$, let $\Gamma(v')$ be the maximal string between v' and v'', excluding v' and v'' themselves. If $w = \overline{v_k}$, $0 \leq k \leq s$, let $\Gamma(v')$ be the maximal string between v' and v'', including v' but excluding v''. If $w = v_s$, let $\Gamma(v')$ be the maximal string between v' and the arrowhead, excluding v'.

Since these strings do not depend on the choice of v' of type w, it is be more convenient to refer to any such $\Gamma(v')$ by $\Gamma(w)$ instead. Hence we have defined $\Gamma(v_k)$ and $\Gamma(\overline{v_k})$.

<u>$\Gamma_-(v_k)$</u>. Fix an integer k, $1 \leq k \leq s$, and consider the collection of connected subgraphs that results from removing all of the vertices of type v_k in $\Gamma^{qmin}(X_{f,n}, z)$. There are $\widetilde{h_k} h_{k+1} \cdots h_s$ isomorphic components that contain vertices of type $\overline{v_k}$. These are all strings of type $\Gamma(\overline{v_k})$. There are also $h_k \cdots h_s$ isomorphic components that contain vertices of type $\overline{v_0}$. These subgraphs are denoted $\Gamma_-(v_k)$. Note that $\Gamma_-(v_1) = \Gamma(\overline{v_0})$.

$\underline{\Gamma_A(v_k)}$. Finally, fix an integer k, $1 \le k \le s$, and fix *one particular* vertex v' of type v_k. Delete v', and let $\Gamma_A(v_k)$ be the connected subgraph of $\Gamma^{qmin}(X_{f,n}, z) - v'$ that contains the arrowhead. In particular, $\Gamma_A(v_s) = \Gamma(v_s)$. (The subgraphs $\Gamma_A(v_k)$ do NOT appear in [**8**]; in particular, $\Gamma_A(v_k)$ is not the same as their $\Gamma_+(v_k)$.)

Computation of the weights. For any resolution graph Γ, let $\det(\Gamma) := \det(-C)$, where C is the intersection matrix of the exceptional curves in Γ. If Γ is empty, then we define $\det(\Gamma)$ to be 1. Now define

$$\begin{aligned}
D(v_k) &= \det(\Gamma(v_k)), & 1 \le k \le s;\\
D(\overline{v_k}) &= \det(\Gamma(\overline{v_k})), & 0 \le k \le s;\\
D_-(v_k) &= \det(\Gamma_-(v_k)), & 1 \le k \le s;\\
D_A(v_k) &= \det(\Gamma_A(v_k)), & 1 \le k \le s.
\end{aligned}$$

The next three lemmas give us formulas for these determinants.

LEMMA 4.2.2 ([**8**]). *Refer to Definition 4.1.1 for the notation.*

$$\begin{aligned}
D(\overline{v_0}) &= a_1', & \\
D(\overline{v_k}) &= p_k', & 1 \le k \le s,\\
D(v_s) &= n/(h_s\widetilde{h_s}), & \\
D(v_k) &= n \cdot q_{k+1}/(d_{k-1}\widetilde{h_k}\widetilde{h_{k+1}}), & 1 \le k \le s-1.
\end{aligned}$$

It follows from the construction of $\Gamma^{can}(X_{f,n}, z)$ that if $D(v_s) = 1$, this indicates that $\Gamma(v_s)$ is empty, and the arrowhead in $\Gamma^{can}(X_{f,n}, z)$ is connected directly to v_s. If $D(\overline{v_k}) = 1$, then the string $\Gamma(\overline{v_k})$ collapses to a single (-1)-curve in $\Gamma^{qmin}(X_{f,n})$, and completely collapses in the minimal resolution graph $\Gamma^{min}(X_{f,n})$.

The determinants $D(\overline{v_k})$ give us the weights of the splice diagram at any node of type v_k on the $\widetilde{h_k}$ edges towards a vertex of type $\overline{v_k}$. Similarly, $D(\overline{v_0})$ is the weight at the nodes of type v_1 on the h_1 edges towards a node of type $\overline{v_0}$.

For clarity, we picture the splice diagram $\Delta_{f,n}$ at the various nodes. For the nodes of type $v_k,\ 2 \le k \le s-1$, refer to Figure 4.6. For type v_1 and v_s, refer to Figure 4.7.

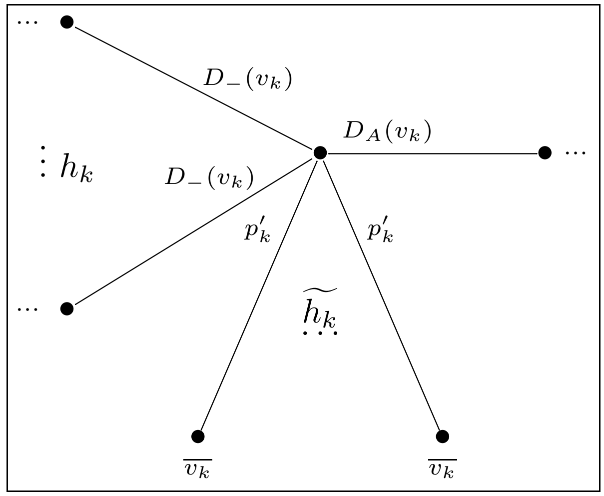
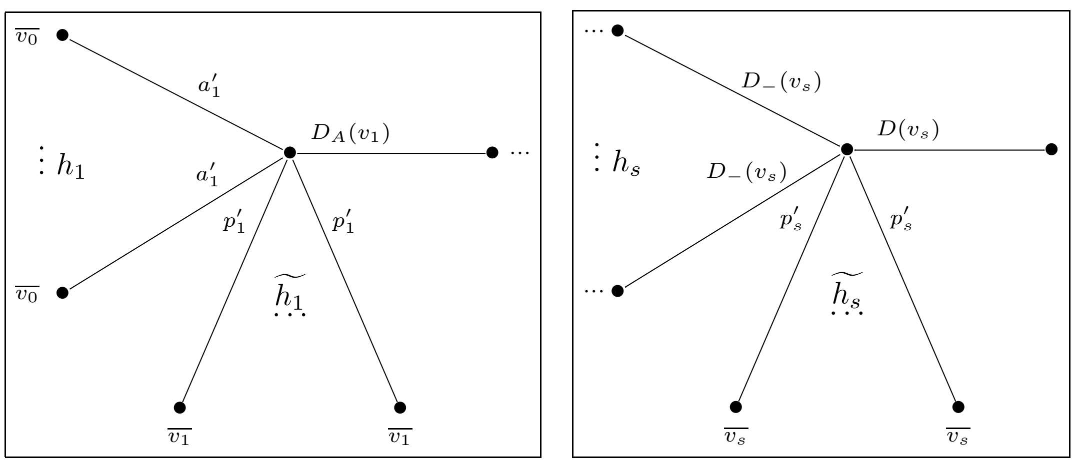


FIGURE 4.6. Splice diagram at a node of type $v_k,\ 2 \le k \le s-1$.



(a) Type v_1 (b) Type v_s

FIGURE 4.7. Splice diagram at a node of type v_1 and a node of type v_s.

Now all that remains is the computation of $D_-(v_k),\ 2 \le k \le s$, and $D_A(v_k),\ 1 \le k \le s-1$.

LEMMA 4.2.3 ([**8**]). *If* $s \ge 2$,

$$D_-(v_2) = a'_2 \cdot (a'_1)^{h_1-1}(p'_1)^{\widetilde{h_1}-1},$$

and if $s \geq 3$,

$$\frac{D_-(v_k)}{a'_k} = (a'_{k-1})^{h_{k-1}-1}(p'_{k-1})^{\widetilde{h_{k-1}}-1}\left[\frac{D_-(v_{k-1})}{a'_{k-1}}\right]^{h_{k-1}},$$

for $3 \leq k \leq s$.

The following expression for $D_-(v_k)$ is sometimes preferable:

$$D_-(v_k) = a'_k(p'_{k-1})^{\widetilde{h_{k-1}}-1}D_-(v_{k-1})^{h_{k-1}}/a'_{k-1}. \tag{4.2}$$

We briefly describe the method of proof (reproduced from [**8**], §2.5), since we need it to compute $D_A(v_k)$. Let $\mathcal{W}$ be the set of non-arrowhead vertices in a decorated graph Γ, and $\mathcal{A}$ the set of arrowhead vertices. By a decorated graph, we mean that each $w \in \mathcal{W}$ is decorated with a self-intersection number e_w, a genus g_w, and a multiplicity m_w, and each $a \in \mathcal{A}$ is decorated with a multiplicity m_a. Let $\mathcal{V} = \mathcal{W} \cup \mathcal{A}$. Let C be the intersection matrix $(E_w \cdot E_v)_{(w,v)\in\mathcal{W}\times\mathcal{W}}$. As defined above, $\det(\Gamma) = \det(-C)$.

For any vertex $w \in \mathcal{W}$, let $\mathcal{V}_w$ be the set of all $v \in \mathcal{V}$ that are adjacent to w. The set $\mathcal{V}$ is called a "compatible set" if

$$e_w m_w + \sum_{v\in\mathcal{V}_w} m_v = 0 \text{ for any } w \in \mathcal{W}.$$

In particular, the vertices and arrowhead in $\Gamma^{can}(X_{f,n}, z)$ are a compatible set (since $(z \circ \pi) \cdot E_w = 0$ for any w, where π is the embedded resolution). This set of relations can also be written in matrix form. Fix an ordering of the set $\mathcal{W}$. Let $\mathbf{m}_{\mathcal{W}}$ be the column vector with entries $\{m_w\}_{w\in\mathcal{W}}$, and let $\mathbf{m}_{\mathcal{A}}$ be the column vector with entries $\{\sum_{a\in\mathcal{A}\cap\mathcal{V}_w} m_a\}_{w\in\mathcal{W}}$. Then

$$\mathbf{m}_{\mathcal{W}} = -C^{-1} \cdot \mathbf{m}_{\mathcal{A}}. \tag{4.3}$$

Given any two vertices w_1 and w_2, let $\Gamma_{w_1w_2}$ be the maximal (non-connected) graph that is left after removing the shortest path (including w_1 and w_2) between the two vertices from Γ. Similarly, $\Gamma_{ww} := \Gamma - \{w\}$. Then, in the case that Γ is a tree,

$$-C^{-1}_{w_1w_2} = \frac{\det(\Gamma_{w_1w_2})}{\det(\Gamma)}. \tag{4.4}$$

Now we are prepared to compute $D_A(v_k)$.

LEMMA 4.2.4. *Assume $s \geq 2$. Let A_k be defined recursively by*

$$A_{s-1} = a_{s-1}p_{s-1}p'_s + q_s,$$

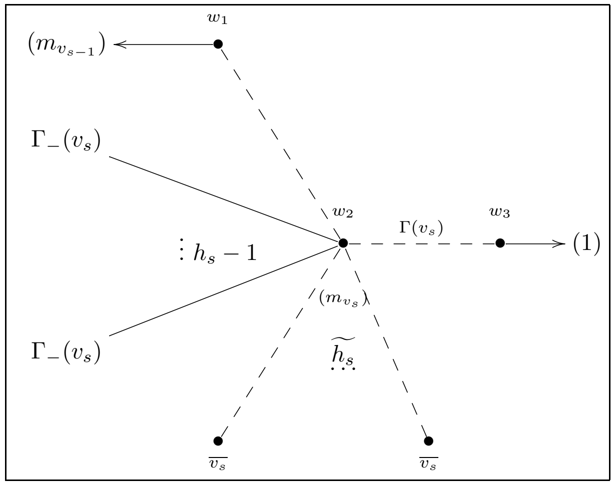
and, for $1 \leq k \leq s-2$,

$$A_k = a_kp_kp'_{k+1}A_{k+1} + q_{k+1}a_{k+2}\cdots a_s.$$

Then

$$D_A(v_k) = \frac{nA_k \cdot \left\{\prod_{j=k+1}^{s}(p'_j)^{\widetilde{h_j}-1}D_-(v_j)^{h_j-1}\right\}}{h_k\widetilde{h}_kd_ka_{k+1}\cdots a_s}, \text{ for } 1 \leq k \leq s-1. \tag{4.5}$$

PROOF. We will use the graph $\Gamma^{can}(X_{f,n}, z)$ to compute these determinants, since we have nice expressions for the multiplicities of the vertices in Proposition 4.1.2. Consider the augmentation of $\Gamma^{can}(X_{f,n}, z)$ obtained by removing one vertex v' of type v_{s-1} plus all of the connected subgraphs of $\Gamma^{can}(X_{f,n}, z) - v'$ except the one containing the arrowhead, and replacing v' by an arrowhead with multiplicity $m_{v_{s-1}}$. This graph forms a compatible set with the following schematic form.



The vertex w_2 is $q^{-1}(v_s)$. Note that if $\Gamma(v_{s-1})$ is empty, then $w_1 = w_2$, and similarly, if $\Gamma(v_s)$ is empty, $w_2 = w_3$. The following argument is valid whether these strings are empty or not. The determinant of this graph is $D_A(v_{s-1})$. By (4.3), we have

$$m_{v_s} = -C^{-1}_{w_2 w_1} \cdot m_{v_{s-1}} - C^{-1}_{w_2 w_3}.$$

By (4.4),

$$-C^{-1}_{w_2 w_1} = \frac{D_-(v_s)^{h_s - 1}(p'_s)^{\widetilde{h_s}} D(v_s)}{D_A(v_{s-1})},$$

and

$$-C^{-1}_{w_2 w_3} = \frac{D_-(v_s)^{h_s - 1}(p'_s)^{\widetilde{h_s}} D(v_{s-1})}{D_A(v_{s-1})}.$$

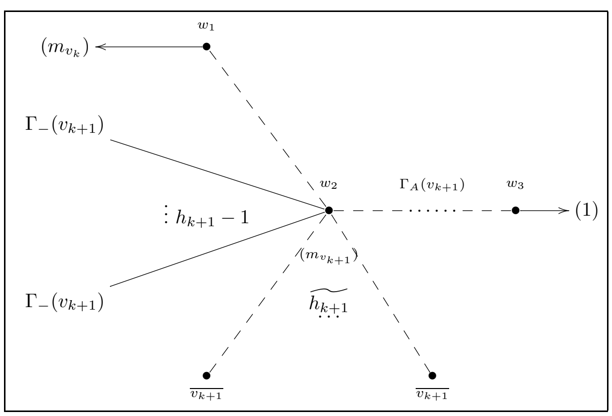
Therefore,

$$m_{v_s} D_A(v_{s-1}) = (m_{v_{s-1}} D(v_s) + D(v_{s-1})) D_-(v_s)^{h_s - 1}(p'_s)^{\widetilde{h_s}}.$$

Recall from Definition 4.1.1 that $p'_k h_k = p_k$, $a'_k \widetilde{h_k} = a_k$, and $d_k = h_{k+1} \cdots h_s$. Applying Lemma 4.2.2 and Proposition 4.1.2(b), we have

$$\begin{aligned}
D_A(v_{s-1}) &= \left(a'_{s-1} p'_{s-1} p'_s \cdot \frac{n}{h_s \widetilde{h_s}} + \frac{n \cdot q_s}{d_{s-2} \widetilde{h_{s-1}} \widetilde{h_s}} \right) \frac{D_-(v_s)^{h_s - 1} (p'_s)^{\widetilde{h_s}}}{a'_s p'_s} \\
&= \left(\frac{n(a'_{s-1} p'_{s-1} p'_s \widetilde{h_{s-1}} h_{s-1} + q_s)}{h_{s-1} \widetilde{h_{s-1}} h_s \widetilde{h_s}} \right) \frac{D_-(v_s)^{h_s - 1} (p'_s)^{\widetilde{h_s} - 1}}{a'_s} \\
&= \left(\frac{n(a_{s-1} p_{s-1} p'_s + q_s)}{h_{s-1} \widetilde{h_{s-1}} h_s \widetilde{h_s}} \right) \frac{D_-(v_s)^{h_s - 1} (p'_s)^{\widetilde{h_s} - 1}}{a'_s} \\
&= \frac{n A_{s-1} D_-(v_s)^{h_s - 1} (p'_s)^{\widetilde{h_s} - 1}}{h_{s-1} \widetilde{h_{s-1}} d_{s-1} a_s}.
\end{aligned}$$

We prove the general expression (4.5) by induction on k. Assume the expression is correct for all j such that $k+1 \leq j \leq s-1$, and consider the following augmentation of $\Gamma^{can}(X_{f,n}, z)$. Remove one vertex v' of type v_k plus all of the connected subgraphs of $\Gamma^{can}(X_{f,n}, z) - v'$ except the one containing the arrowhead, and replace v' by an arrowhead with multiplicity m_{v_k}. This graph forms a compatible set with the following schematic form.



The vertex w_2 is of type v_{k+1}. Here, w_1 and w_2 might be the same vertex, but $w_2 \neq w_3$. The determinant of this graph is $D_A(v_k)$. Again, by (4.4), we have

$$-C^{-1}_{w_2 w_1} = \frac{D_-(v_{k+1})^{h_{k+1} - 1} (p'_{k+1})^{\widetilde{h_{k+1}}} D_A(v_{k+1})}{D_A(v_k)}.$$

Upon removing the shortest path from w_2 (type v_{k+1}) to w_3, we are left with the following subgraphs: one of type $\Gamma(v_k)$, and for each j, $k+1 \le j \le s$, $h_j - 1$ of type $\Gamma_-(v_j)$ and $\widetilde{h_j}$ of type $\Gamma(\overline{v_j})$. Thus,

$$-C^{-1}_{w_2 w_3} = \frac{D(v_k)\prod_{j=k+1}^{s} D_-(v_j)^{h_j-1}(p'_j)^{\widetilde{h_j}}}{D_A(v_k)}.$$

From (4.3), we deduce

$$m_{v_{k+1}} D_A(v_k) = D_-(v_{k+1})^{h_{k+1}-1}(p'_{k+1})^{\widetilde{h_{k+1}}} D_A(v_{k+1}) m_{v_k} + D(v_k)\prod_{j=k+1}^{s} D_-(v_j)^{h_j-1}(p'_j)^{\widetilde{h_j}}.$$

By induction, $D_A(v_{k+1})$ is given by the corresponding expression in (4.5), and so the equation above is equivalent to

$$\begin{aligned}
D_A(v_k) &= \left(\frac{nA_{k+1}p'_{k+1}m_{v_k}}{h_{k+1}\widetilde{h_{k+1}}d_{k+1}a_{k+2}\cdots a_s} + \frac{nq_{k+1}p'_{k+1}\cdots p'_s}{d_{k-1}\widetilde{h_k}\widetilde{h_{k+1}}}\right)\frac{\prod_{j=k+1}^{s} D_-(v_j)^{h_j-1}(p'_j)^{\widetilde{h_j}-1}}{m_{v_{k+1}}}\\
&= \left(\frac{nA_{k+1}p'_{k+1}a'_k p'_k\cdots p'_s}{h_{k+1}\widetilde{h_{k+1}}d_{k+1}a_{k+2}\cdots a_s} + \frac{nq_{k+1}p'_{k+1}\cdots p'_s}{d_{k-1}\widetilde{h_k}\widetilde{h_{k+1}}}\right)\frac{\prod_{j=k+1}^{s} D_-(v_j)^{h_j-1}(p'_j)^{\widetilde{h_j}-1}}{a'_{k+1}p'_{k+1}\cdots p'_s}\\
&= \left(\frac{na'_k p'_k p'_{k+1}A_{k+1}}{h_{k+1}\widetilde{h_{k+1}}d_{k+1}a_{k+2}\cdots a_s} + \frac{n\cdot q_{k+1}}{d_{k-1}\widetilde{h_k}\widetilde{h_{k+1}}}\right)\frac{\prod_{j=k+1}^{s} D_-(v_j)^{h_j-1}(p'_j)^{\widetilde{h_j}-1}}{a'_{k+1}}\\
&= \left(\frac{n(a'_k p'_k p'_{k+1}A_{k+1}h_k\widetilde{h_k} + q_{k+1}a_{k+2}\cdots a_s)}{h_k\widetilde{h_k}h_{k+1}\widetilde{h_{k+1}}d_{k+1}a_{k+2}\cdots a_s}\right)\frac{\prod_{j=k+1}^{s} D_-(v_j)^{h_j-1}(p'_j)^{\widetilde{h_j}-1}}{a'_{k+1}}.
\end{aligned}$$

Since $a'_k p'_k p'_{k+1}A_{k+1}h_k\widetilde{h_k}+q_{k+1}a_{k+2}\cdots a_s = a_k p_k p'_{k+1}A_{k+1}+q_{k+1}a_{k+2}\cdots a_s = A_k$ by definition, and $h_k\widetilde{h_k}h_{k+1}\widetilde{h_{k+1}}d_{k+1}a'_{k+1} = h_k\widetilde{h_k}d_k a_{k+1}$, the proof is complete. □

REMARK 4.2.5. Although $D_A(v_k)$ is an integer, it is not always easy to see it from the expression (4.5). The product $h_k\widetilde{h_k}d_k$ divides n, but the a_j, $k+1 \le j \le s$, may divide either A_k or $D_-(v_j)^{h_j-1}$, depending on the situation.

By the same method used in the last proof, one can prove the following

LEMMA 4.2.6. *The determinant of* $\Gamma^{qmin}(X_{f,n})$ *is given by*

$$\det(\Gamma^{qmin}(X_{f,n})) = (a'_s)^{h_s-1}(p'_s)^{\widetilde{h_s}-1}\left[\frac{D_-(v_s)}{a'_s}\right]^{h_s},$$

which can also be written

$$\det(\Gamma^{qmin}(X_{f,n})) = (p'_s)^{\widetilde{h_s}-1}D_-(v_s)^{h_s}/a'_s.$$

Finally, we use the following technical computations so frequently in future sections that we list them here for convenience.

FACT DM. *Suppose that* $h_k\widetilde{h_k} = 1$ *for* $1 \le k \le s-1$. *Then*

$$\begin{aligned} D_-(v_k) &= a_k,\ 2 \le k \le s-1,\ \textit{and} \\ D_-(v_s) &= a'_s. \end{aligned}$$

FACT DA1. *Assume that* $h_s = 1$, *and suppose that for some* $k,\ 1 \le k \le s-2,\ h_i\widetilde{h_i} = 1$ *for* $k+1 \le i \le s-1$. *Then*

$$\begin{aligned} D_A(v_i) &= n(p_s)^{\widetilde{h_s}-1},\ \textit{for } k+1 \le i \le s-1,\ \textit{and} \\ D_A(v_k) &= \frac{n}{h_k\widetilde{h_k}}(p_s)^{\widetilde{h_s}-1}. \end{aligned}$$

In particular, if $h_k\widetilde{h_k} = 1$ *for* $1 \le k \le s-1$, *then all* $D_A(v_k) = n(p_s)^{\widetilde{h_s}-1}$.

FACT DA2. *Define integers* $\tilde{A}_i$ *as follows:*

$$\begin{aligned} \tilde{A}_i &:= a_s - a_ip_ip_{i+1}^2\cdots p_{s-1}^2(p_s - p'_s),\ \textit{for } 1 \le i \le s-2,\ \textit{and} \\ \tilde{A}_{s-1} &:= a_s - a_{s-1}p_{s-1}(p_s - p'_s). \end{aligned}$$

Assume that $h_s \neq 1$, *and suppose that for some* k, $1 \leq k \leq s-2$, $h_i\widetilde{h_i} = 1$ *for* $k+1 \leq i \leq s-1$. *Then*

$$\begin{aligned} D_A(v_i) &= \frac{n\tilde{A}_i D_-(v_s)^{h_s-1}}{h_s a_s}, \textit{ for } k+1 \leq i \leq s-1, \textit{ and} \\ D_A(v_k) &= \frac{n\tilde{A}_k D_-(v_s)^{h_s-1}}{h_k\widetilde{h_k} h_s a_s}. \end{aligned}$$

In particular, if $h_k\widetilde{h_k} = 1$ *for* $1 \leq k \leq s-1$, *then all* $D_A(v_k) = \frac{n}{h_s}\tilde{A}_k(a_s)^{h_s-2}$.

Fact DM is immediately clear from Lemma 4.2.3.

Proof of Fact DA1. Since $h_s = 1$ implies $p'_s = p_s$, we have

$$A_{s-1} = a_{s-1}p_{s-1}p_s + q_s = a_s.$$

We claim that for all i such that $k \leq i \leq s-2$, $A_i = a_{i+1}\cdots a_s$. This follows easily by induction, since $A_i = a_i p_i p'_{i+1} A_{i+1} + q_{i+1}a_{i+2}\cdots a_s$, $p'_{i+1} = p_{i+1}$, and $a_i p_i p_{i+1} + q_{i+1} = a_{i+1}$. Therefore, by Lemma 4.2.4,

$$D_A(v_i) = \frac{n a_{i+1}\cdots a_s (p_s)^{\widetilde{h_s}-1}}{h_i\widetilde{h_i} d_i a_{i+1}\cdots a_s}, \text{ for } k \leq i \leq s-1.$$

By definition, $d_i = h_{i+1}\cdots h_s = 1$, and thus $D_A(v_i)$ and $D_A(v_k)$ are as stated. □

Proof of Fact DA2. Since the link is a $\mathbb{Q}$HS, $h_s \neq 1$ implies $\widetilde{h_s} = 1$ and $a'_s = a_s$. Since $q_s = a_s - a_{s-1}p_{s-1}p_s$,

$$\begin{aligned} A_{s-1} &= a_{s-1}p_{s-1}p'_s + q_s \\ &= a_s - a_{s-1}p_{s-1}(p_s - p'_s). \end{aligned}$$

Then, if $h_{s-1} = 1$,

$$\begin{aligned} A_{s-2} &= a_{s-2}p_{s-2}p_{s-1}[a_s - a_{s-1}p_{s-1}(p_s - p'_s)] + q_{s-1}a_s \\ &= a_s[a_{s-2}p_{s-2}p_{s-1} + q_{s-1}] - a_{s-2}p_{s-2}a_{s-1}p_{s-1}^2(p_s - p'_s) \\ &= a_{s-1}[a_s - a_{s-2}p_{s-2}p_{s-1}^2(p_s - p'_s)] \\ &= a_{s-1}\tilde{A}_{s-2}. \end{aligned}$$

One can show, by induction, that $A_i = a_{i+1}\cdots a_{s-1}\tilde{A}_i$, for $k \leq i \leq s-2$. Therefore, by Lemma 4.2.4,

$$D_A(v_i) = \frac{na_{i+1}\cdots a_{s-1}\tilde{A}_i D_-(v_s)^{h_s-1}}{h_i\widetilde{h_i}d_i a_{i+1}\cdots a_s},\ \text{for } k \leq i \leq s-1.$$

By definition, $d_i = h_{i+1}\cdots h_s = h_s$, and thus $D_A(v_i)$ and $D_A(v_k)$ are as stated. ☐

The pathological case. We can use the non-minimal resolution graph $\Gamma^{can}(X_{f,n}, z)$ to compute the determinants for the splice diagram in the pathological case $n = p_s = 2$ as well. Therefore, the formulas given in the preceding section are true for the pathological case (when applicable). Since for $1 \leq k \leq s-1$, all $h_k\widetilde{h_k} = 1$, and $n/h_s = 1$, by Facts DM and DA2, we have

$$\begin{aligned} D_-(v_k) &= a_k,\ \text{for } 2 \leq k \leq s,\ \text{and} \\ D_A(v_k) &= \tilde{A}_k,\ \text{for } 1 \leq k \leq s-1. \end{aligned}$$

Finally, by Lemma 4.2.6, $\det(\Gamma) = a_s$. The splice diagram is given in Figure 4.8.

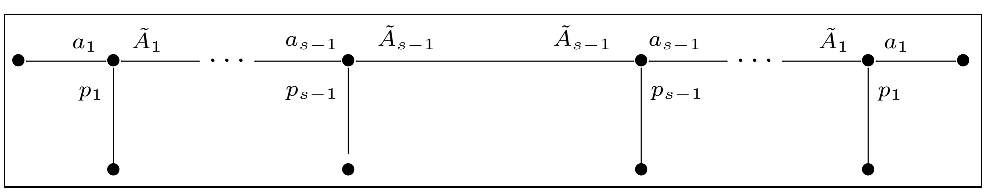


FIGURE 4.8. Splice diagram for the pathological case.

CHAPTER 5

The Neumann-Wahl algorithm for $\{z^n = f(x,y)\}$

In this chapter, we continue with the computations that we need to prove Theorem 6.0.1 concerning the topological types of $X_{f,n} = \{z^n = f(x,y)\}$. In §5.1, we describe the semigroup conditions for the splice diagram associated to the quasi-minimal resolution graph of $(X_{f,n}, 0)$ as completely as possible. As for the congruence conditions, the action of the discriminant group defined in the Neumann-Wahl algorithm can be computed via Propositions 2.0.8 and 2.0.9. The first of these propositions is generally easier to apply, because it requires only the splice diagram, whereas the second one requires the resolution graph as well. The purpose of §5.2 is to use Proposition 2.0.9 to compute explicitly the character given by $e_w \cdot e_w$, where w is a leaf in the splice diagram associated to $(X_{f,n}, 0)$.

5.1. Semigroup conditions

Assume that $(X_{f,n}, 0)$ has $\mathbb{Q}$HS link (see Prop. 4.1.2 (c)). In this section, we describe the semigroup conditions on the splice diagram $\Delta_{f,n}$ associated to $\Gamma_{f,n} := \Gamma^{qmin}(X_{f,n})$. Unless stated otherwise, we assume that we are not in the pathological case. Let us begin by fixing some notation.

Notation. Recall that for each node of type v_k in $\Gamma_{f,n}$, there are $h_k + \widetilde{h_k} + 1$ adjacent edges, connected to subgraphs of type $\Gamma_-(v_k)$, $\Gamma(\overline{v_k})$, and $\Gamma_A(v_k)$, respectively (see §4.2). The corresponding pieces of $\Delta_{f,n}$ associated to the subgraphs $\Gamma_-(v_k)$ and $\Gamma_A(v_k)$ are denoted $\Delta_-(v_k)$ and $\Delta_A(v_k)$, respectively. Recall that $\Gamma_-(v_1) = \Gamma(\overline{v_0})$, and $\Gamma_A(v_s) = \Gamma(v_s)$, and keep in mind that $\Gamma(v_s)$ may be empty. The corresponding determinants are $D_-(v_1) = a'_1$ and $D_A(v_s) = n/h_s\widetilde{h}_s$.

Semigroup conditions in the direction of $\Delta_-(v_k)$. We begin with the semigroup condition at a node of type v_k, $2 \le k \le s$, in the direction of any of the h_k edges that lead to subdiagrams of the form $\Delta_-(v_k)$. It is clear that this semigroup condition will be the same for any node v of type v_k. By Definition 2.0.1, the condition is

$$D_-(v_k) \in \mathbb{N}\langle \ell'_{vw} \mid w \text{ is a leaf in } \Delta_-(v_k)\rangle.$$

PROPOSITION 5.1.1. *At a node of type v_k, $2 \le k \le s$, the semigroup condition in the direction of any of the h_k edges that lead to a subdiagram of the form $\Delta_-(v_k)$ is equivalent to*

$$a'_k \in \mathbb{N}\langle a'_{k-1},\ p'_1p'_2\cdots p'_{k-1},\ a'_jp'_{j+1}\cdots p'_{k-1},\ 1 \le j \le k-2\rangle. \tag{5.1}$$

Furthermore, if $\widetilde{h}_k = 1$, this condition is automatically satisfied.

PROOF. We prove this by induction on k. Fix a node v of type v_k in $\Delta_{f,n}$. Let L_k be the semigroup

$$\mathrm{L}_k := \mathbb{N}\langle \ell'_{vw} \mid w \text{ is a leaf in } \Delta_-(v) = \Delta_-(v_k)\rangle.$$

The leaves in $\Delta_-(v_k)$ are of type $\overline{v_j}$, for j such that $0 \le j \le k-1$. Hence, there are k generators for L_k, namely, ℓ'_{vw_j}, $0 \le j \le k-1$, where w_j denotes any leaf in $\Delta_-(v)$ of type $\overline{v_j}$.

CLAIM 5.1.2. *Let v be a node of type v_k, $2 \le k \le s$, and let w_j be a leaf of type $\overline{v_j}$ in $\Delta_-(v)$, $0 \le j \le k-1$. Then*

$$\ell'_{vw_j} = \begin{cases} p'_1 \cdots p'_{k-1} \cdot D_-(v_k)/a'_k & \text{for } j = 0 \\ a'_j p'_{j+1} \cdots p'_{k-1} \cdot D_-(v_k)/a'_k & \text{for } 1 \le j \le k-2 \\ a'_{k-1} \cdot D_-(v_k)/a'_k & \text{for } j = k-1. \end{cases}$$

If this claim is true, then $D_-(v_k)$ and all generators of L_k are divisible by $D_-(v_k)/a'_k$, and the first statement of the Proposition is clearly true.

PROOF OF CLAIM 5.1.2. For $k = 2$, the claim is true, since if v is a node of type v_2,

$$\begin{aligned} \ell'_{vw_0} &= (a'_1)^{h_1-1}(p'_1)^{\widetilde{h_1}}, \\ \ell'_{vw_1} &= (a'_1)^{h_1}(p'_1)^{\widetilde{h_1}-1}, \text{ and} \\ D_-(v_2) &= a'_2(a'_1)^{h_1-1}(p'_1)^{\widetilde{h_1}-1}. \end{aligned}$$

(See Figure 4.7.) Clearly, the semigroup condition at v in the direction of $\Delta_-(v)$ is equivalent to $a'_2 \in \mathbb{N}\langle a'_1,\ p'_1\rangle$.

Now assume the claim is true for k such that $2 \le k \le i-1$; we will show that it is true for $k = i$. Refer to Figure 5.1 for what follows. Fix a node v of type v_i, and (abusing notation), let v_{i-1} denote the unique node of type v_{i-1} in $\Delta_-(v)$. Then

$$\ell'_{vw_j} = \begin{cases} \ell'_{v_{i-1}w_j} \cdot D_-(v_{i-1})^{h_{i-1}-1}(p'_{i-1})^{\widetilde{h_{i-1}}} & \text{for } 0 \le j \le i-2, \\ D_-(v_{i-1})^{h_{i-1}}(p'_{i-1})^{\widetilde{h_{i-1}}-1} & \text{for } j = i-1. \end{cases}$$

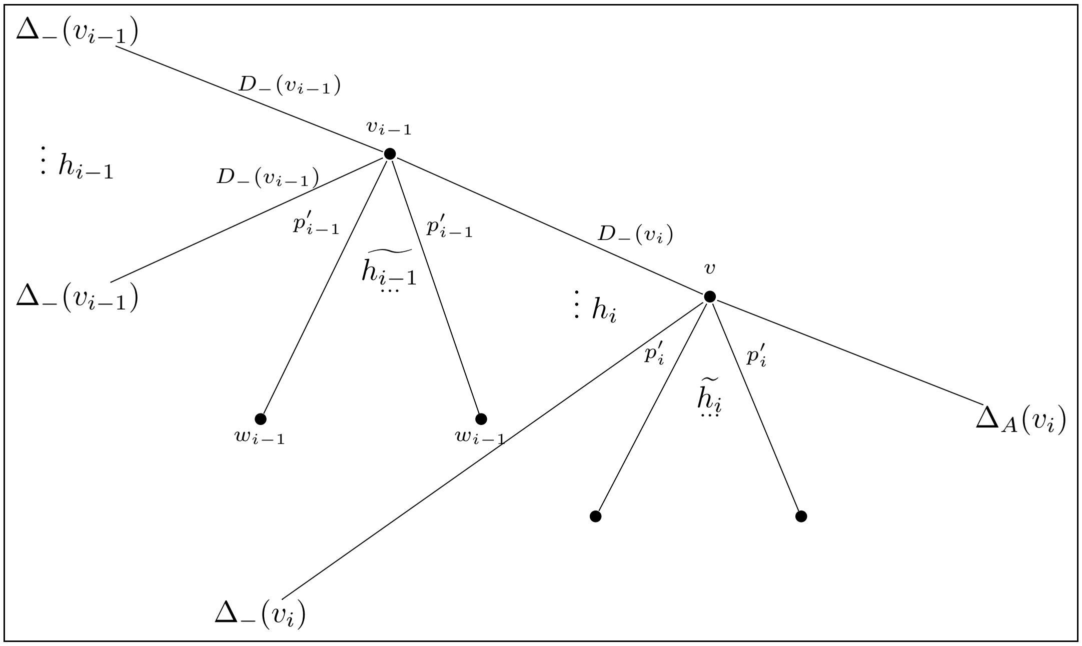


FIGURE 5.1. Relevant portion of $\Delta_{f,n}$ at a node v of type v_i.

Therefore, by induction,

$$\ell'_{vw_j} = \begin{cases} p'_1 \cdots p'_{i-2} \cdot D_-(v_{i-1})/a'_{i-1} \cdot D_-(v_{i-1})^{h_{i-1}-1}(p'_{i-1})^{\widetilde{h_{i-1}}} & \text{for } j = 0 \\ a'_j p'_{j+1} \cdots p'_{i-2} \cdot D_-(v_{i-1})/a'_{i-1} \cdot D_-(v_{i-1})^{h_{i-1}-1}(p'_{i-1})^{\widetilde{h_{i-1}}} & \text{for } 1 \le j \le i-3 \\ a'_{i-2} \cdot D_-(v_{i-1})/a'_{i-1} \cdot D_-(v_{i-1})^{h_{i-1}-1}(p'_{i-1})^{\widetilde{h_{i-1}}} & \text{for } j = i-2 \\ D_-(v_{i-1})^{h_{i-1}}(p'_{i-1})^{\widetilde{h_{i-1}}-1} & \text{for } j = i-1. \end{cases}$$

By Lemma 4.2.3,

$$\frac{D_-(v_i)}{a'_i} = (p'_{i-1})^{\widetilde{h_{i-1}}-1} D_-(v_{i-1})^{h_{i-1}-1} \cdot \frac{D_-(v_{i-1})}{a'_{i-1}},$$

and hence the claim is true. □

Thus, we have proved the first statement of the Proposition. The second statement is implied by a result in Teissier's appendix to [**27**] (see the Appendix of this work for an explanation). □

Semigroup conditions in the direction of $\Delta_A(v_k)$. Fix an integer k, $1 \le k \le s-1$, and fix a node v of type v_k. The semigroup condition at v in the direction of $\Delta_A(v)$ is a bit more complicated, and we do not prove a proposition that is analogous to Proposition 5.1.1. Rather, we describe the generators of the semigroup in question and prove a lemma that will be very useful in proof of Theorem 6.0.1. In practice, however, the fact that either h_i or $\widetilde{h_i}$ is equal to 1 (since the link is a $\mathbb{Q}$HS) for all i makes things much simpler than they appear here.

By definition, the semigroup condition is $D_A(v_k) \in \mathrm{R}_k$, where

$$\mathrm{R}_k := \mathbb{N}\langle \ell'_{vw} \mid w \text{ is a leaf in } \Delta_A(v) = \Delta_A(v_k) \rangle.$$

Refer to Figure 5.2 for what follows. There is at least one leaf w_s in $\Delta_A(v)$ of type $\overline{v_s}$ connected to v_s (the unique node of type v_s), and if $n/h_s\widetilde{h_s} \neq 1$, there is a leaf w_a resulting from the string $\Gamma(v_s)$ in $\Gamma_{f,n}$. These contribute ℓ'_{vw_s} and ℓ'_{vw_a} as generators of R_k.

Now travel along the shortest path from v to v_s. If $k < s-1$, this path contains one node of type v_m, for each m such that $k+1 \leq m \leq s-1$. Since there can be no confusion here, we will simply refer to the nodes along this path as v_m. Each of these nodes is directly

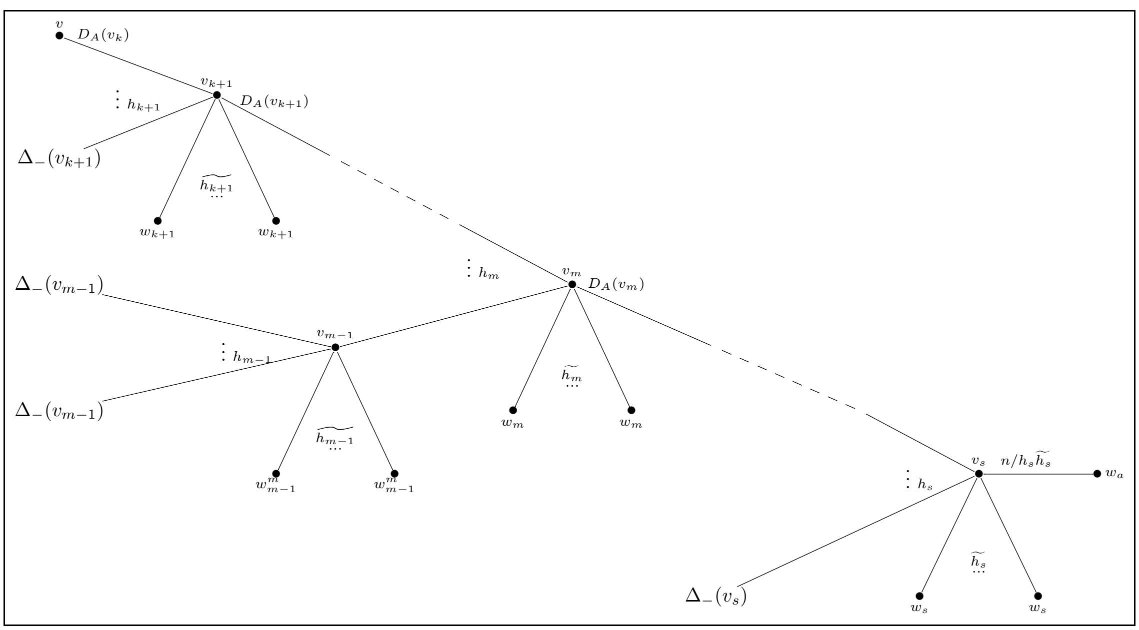


FIGURE 5.2. Relevant portion of $\Delta_{f,n}$ at a node v of type v_k.

connected to at least one leaf w_m of type $\overline{v_m}$. Each such leaf contributes the generator ℓ'_{vw_m} to R_k. If $h_i = 1$ for $k+1 \le i \le s$, there are no further leaves in $\Delta_A(v)$, and we have listed all the generators of R_k.

For each m such that $h_m \neq 1,\ k+1 \le m \le s$, there are more generators for R_k, namely ℓ'_{vw} for each type of leaf w in $\Delta_-(v_m)$. There are m different types of leaves that appear: type $\overline{v_j}$, for j such that $0 \le j \le m-1$. Let w_j^m be a leaf of type $\overline{v_j}$ in $\Delta_-(v_m)$ (clearly, the linking number with v will be the same for any such leaf). Now we have accounted for all possible generators of R_k.

To summarize, the generators of the semigroup R_k are:

$$\left\{ \begin{array}{ll} \ell'_{vw_a} & (\dagger), \\ \ell'_{vw_m} & \text{for any } m \text{ s.t. } k+1 \le m \le s, \\ \ell'_{vw_j^m}, & 0 \le j \le m-1, \text{for any } m \text{ s.t. } k+1 \le m \le s \text{ and } h_m \neq 1 \end{array} \right\}.$$

($\dagger$) Absent if $n/h_s\widetilde{h_s} = 1$.

The following Lemma is needed in Chapter 6, but it is convenient to prove it here.

Lemma 5.1.3. *Suppose that $(X_{f,n}, 0)$ has $\mathbb{Q}$HS link, and that $\Delta_{f,n}$ satisfies the semigroup conditions. Assume $s \ge 3$, and that $h_{s-1}\widetilde{h_{s-1}} = 1$. Then $h_k\widetilde{h_k} = 1$ for $1 \le k \le s-2$.*

Proof. We prove this by induction on k. The proof must be divided into two cases: (a) $h_s = (n, p_s) = 1$, and (b) $h_s = (n, p_s) \neq 1$.

Case (a). Assume that $h_s = 1$ and $h_{s-1}\widetilde{h_{s-1}} = 1$. First, we must show that $h_{s-2}\widetilde{h_{s-2}} = 1$. By Proposition 4.1.2(a), there is a unique node of type v_{s-1} in $\Delta_{f,n}$, which we will simply denote v_{s-1}, and a unique node of type v_{s-2}; call this node v. We will show that the semigroup condition for v in the direction of $\Delta_A(v)$ cannot be satisfied if $h_{s-2}\widetilde{h_{s-2}} \neq 1$. The splice diagram in this case is pictured in Figure 5.3. The semigroup condition is: $D_A(v_{s-2})$ is in

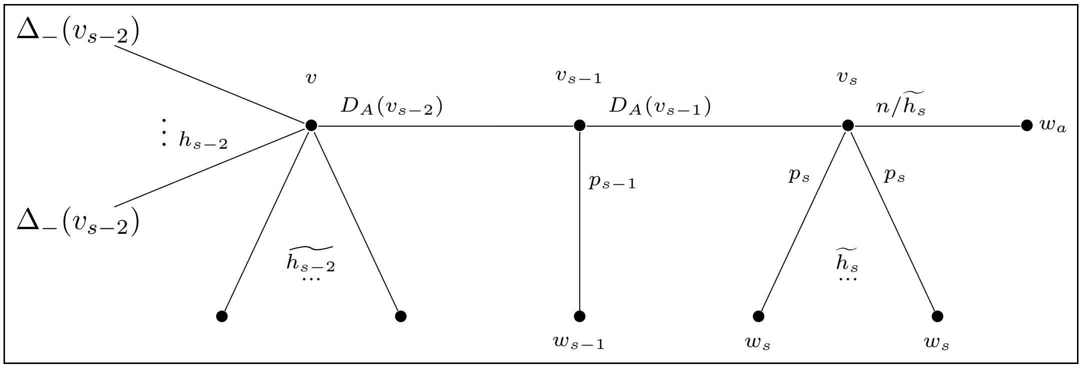


FIGURE 5.3. Splice diagram $\Delta_{f,n}$ for $h_s = h_{s-1}\widetilde{h_{s-1}} = 1$.

the semigroup generated by

$$\left\{D_A(v_{s-1}),\ p_{s-1}(p_s)^{\widetilde{h_s}-1}(n/\widetilde{h_s}),\ p_{s-1}(p_s)^{\widetilde{h_s}}\ (\dagger)\right\}.$$

(†) Absent if $n/\widetilde{h_s} = 1$.

By Fact DA1 in §4.2,

$$D_A(v_{s-1}) \;=\; n(p_s)^{\widetilde{h_s}-1}, \text{ and}$$

$$D_A(v_{s-2}) \;=\; \frac{n}{h_{s-2}\widetilde{h_{s-2}}}(p_s)^{\widetilde{h_s}-1}.$$

Clearly, $D_A(v_{s-2})$ and every generator of the semigroup are divisible by $(p_s)^{\widetilde{h_s}-1}$, and therefore the semigroup condition is equivalent to: $\dfrac{n}{h_{s-2}\widetilde{h_{s-2}}}$ is in the semigroup generated by

$$\left\{n,\ p_{s-1}(n/\widetilde{h_s}),\ p_{s-1}p_s\ (\dagger)\right\}.$$

If $h_{s-2}\widetilde{h_{s-2}} \neq 1$, then n is too large. But then the semigroup condition implies that $n/h_{s-2}\widetilde{h_{s-2}}$ is divisible by p_{s-1}, which is impossible since $h_{s-1} = (n, p_{s-1}) = 1$. Thus, we must have $h_{s-2}\widetilde{h_{s-2}} = 1$. (Note that the argument is valid for both $n/\widetilde{h_s} = 1$ and $\widetilde{h_s} = 1$.)

For the induction step, we assume that $h_i\widetilde{h_i} = 1$, for all i such that $k+1 \le i \le s-1$, and show that $h_k\widetilde{h_k} = 1$. The proof is very similar to that of the base step. By Proposition 4.1.2(a), there is a unique node of type v_k; call this node v. We show that the semigroup condition for v in the direction of $\Delta_A(v)$ cannot be satisfied if $h_k\widetilde{h_k} \neq 1$. The splice diagram in this case is pictured in Figure 5.4. The semigroup condition is $D_A(v_k) \in \mathrm{R}_k$, where R_k is defined as above. Since $h_s = 1$, and $h_{s-1}\widetilde{h_{s-1}} = \cdots = h_{k+1}\widetilde{h_{k+1}} = 1$, the generators of R_k are

$$\left\{\begin{array}{lcll} \ell'_{vw_s} & = & p_{k+1}\cdots p_{s-1}(p_s)^{\widetilde{h_s}-1}\cdot n/\widetilde{h_s}, & \\ \ell'_{vw_a} & = & p_{k+1}\cdots p_{s-1}(p_s)^{\widetilde{h_s}}, & (\dagger) \\ \ell'_{vw_{k+1}} & = & D_A(v_{k+1}), & \\ \ell'_{vw_m} & = & p_{k+1}\cdots p_{m-1}\cdot D_A(v_m), & k+2 \leq m \leq s-1. \end{array}\right\}$$

(†) Absent if $n/\widetilde{h_s} = 1$.

Again, by Fact DA1 in §4.2,

$$\begin{array}{rcl} D_A(v_m) & = & n(p_s)^{\widetilde{h_s}-1}, \text{ for } k+1 \leq m \leq s-1, \text{ and} \\ D_A(v_k) & = & \dfrac{n}{h_k\widetilde{h_k}}(p_s)^{\widetilde{h_s}-1}. \end{array}$$

Clearly, $D_A(v_k)$ and every generator of R_k are divisible by $(p_s)^{\widetilde{h_s}-1}$, and the semigroup condition is equivalent to: $\dfrac{n}{h_k\widetilde{h_k}}$ is in the semigroup generated by

$$\left\{\frac{n}{\widetilde{h_s}}p_{k+1}\cdots p_{s-1},\ p_{k+1}\cdots p_{s-1}p_s\ (\dagger),\ n,\ np_{k+1}\cdots p_{m-1}\ \middle|\ k+2 \leq m \leq s-1\right\}.$$

If $h_k\widetilde{h_k} \neq 1$, all of the generators of R_k are too large except for $\frac{n}{\widetilde{h_s}}p_{k+1}\cdots p_{s-1}$ and $p_{k+1}\cdots p_{s-1}p_s$ (†). Then the semigroup condition implies that $n/h_k\widetilde{h_k}$ is divisible by p_{k+1},

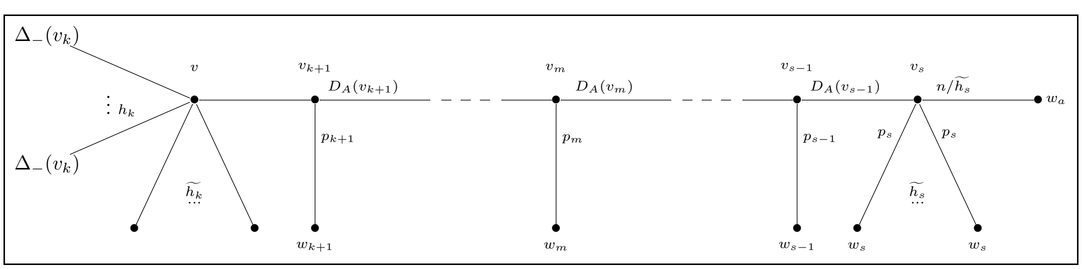


FIGURE 5.4. Splice diagram $\Delta_{f,n}$ for $h_s = h_{s-1}\widetilde{h_{s-1}} = \cdots = h_{k+1}\widetilde{h_{k+1}} = 1$.

which is impossible since $h_{k+1} = (n, p_{k+1}) = 1$. Thus, we must have $h_k \widetilde{h_k} = 1$. (Again, note that the argument is valid for both $n/\widetilde{h_s} = 1$ and $\widetilde{h_s} = 1$.)

Case (b). Assume that $h_s \neq 1$ and $h_{s-1}\widetilde{h_{s-1}} = 1$. First of all, note that if $n/h_s = 1$, then by Definition 4.1.1, $h_i = \widetilde{h_i} = 1$ for $1 \leq i \leq s-1$, so there is nothing to prove. Therefore, assume $n/h_s \neq 1$. We use the same strategy as in the proof of Case (a), and begin by showing that the semigroup conditions imply that $h_{s-2}\widetilde{h_{s-2}} = 1$.

By Proposition 4.1.2(a), there are h_s nodes of type v_{s-2}; let v be any such node. We will show that the semigroup condition for v in the direction of $\Delta_A(v)$ cannot be satisfied if $h_{s-2}\widetilde{h_{s-2}} \neq 1$. Refer to Figure 5.5.

Since $\widetilde{h_s} = 1$, and $h_{s-1}\widetilde{h_{s-1}} = 1$, the generators of R_{s-2} are

$$\left\{ \begin{array}{lcl} \ell'_{vw_s} & = & p_{s-1} \cdot D_-(v_s)^{h_s - 1} \cdot n/h_s, \\ \ell'_{vw_a} & = & p_{s-1} \cdot D_-(v_s)^{h_s-1} p'_s, \\ \ell'_{vw_{s-1}} & = & D_A(v_{s-1}), \\ \ell'_{vw^s_j} & = & p_{s-1} \cdot D_-(v_s)^{h_s-2}(p'_s) \cdot n/h_s \cdot \ell'_{v_s w^s_j}, \quad 0 \leq j \leq s-1. \end{array} \right\}.$$

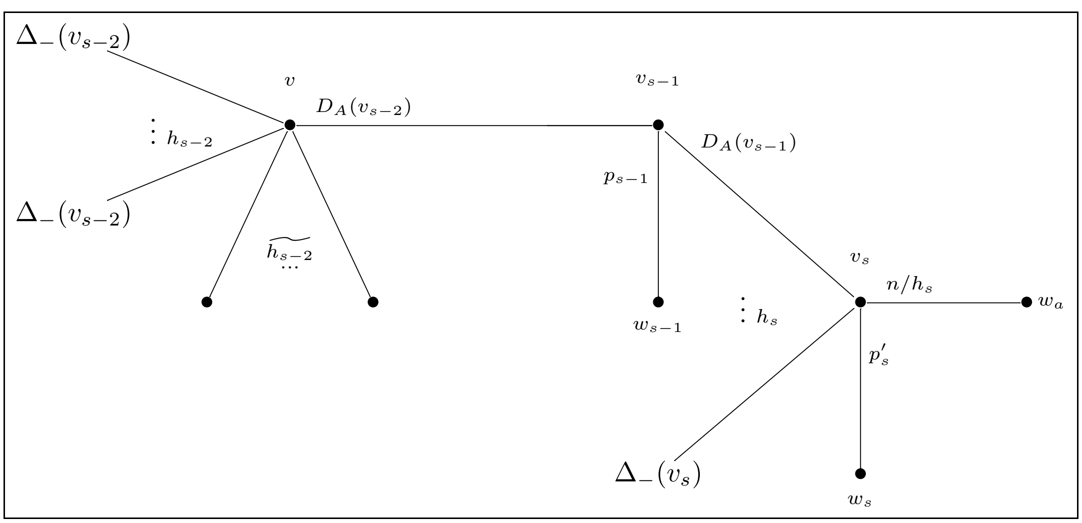


Figure 5.5. Splice diagram $\Delta_{f,n}$ for $h_s \neq 1$, $h_{s-1}\widetilde{h_{s-1}} = 1$.

By Fact DA2 in §4.2,

$$\begin{aligned} D_A(v_{s-1}) &= \frac{n\tilde{A}_{s-1}D_-(v_s)^{h_s-1}}{h_s a_s}, \text{ and} \\ D_A(v_{s-2}) &= \frac{n\tilde{A}_{s-2}D_-(v_s)^{h_s-1}}{h_{s-2}\widetilde{h_{s-2}}h_s a_s}. \end{aligned}$$

We claim that $D_A(v_{s-2})$ and all generators of R_{s-2} are divisible by $D_-(v_s)^{h_s-1}/a_s$, which is an integer by Lemma 4.2.3, since $a'_s = a_s$. (Also, note that $n/(h_{s-2}\widetilde{h_{s-2}}h_s)$ is an integer). This is clearly true for $D_A(v_{s-1})$, ℓ'_{vw_s}, and ℓ'_{vw_a}. To see that $\ell'_{vw^s_j}$ is also divisible by $D_-(v_s)^{h_s-1}/a_s$, recall from Claim 5.1.2 that we have

$$\ell'_{v_s w^s_j} = \begin{cases} p'_1 \cdots p'_{s-1} \cdot D_-(v_s)/a'_s & \text{for } j = 0 \\ a'_j p'_{j+1} \cdots p'_{s-1} \cdot D_-(v_s)/a'_s & \text{for } 1 \le j \le s-2 \\ a'_{s-1} \cdot D_-(v_s)/a'_s & \text{for } j = s-1. \end{cases}$$

Since $a'_s = a_s$ here, the statement is true.

Thus, the semigroup condition is equivalent to $\frac{n}{h_{s-2}\widetilde{h_{s-2}}h_s}\tilde{A}_{s-2}$ is in the semigroup generated by:

$$\left\{ \begin{array}{l} n/h_s \cdot a_s \cdot p_{s-1}, \\ a_s \cdot p'_s \cdot p_{s-1}, \\ n/h_s \cdot \tilde{A}_{s-1}, \\ n/h_s \cdot p'_s \cdot p'_1 p'_2 \cdots p'_{s-2} p^2_{s-1} \\ n/h_s \cdot p'_s \cdot a'_j p'_{j+1} \cdots p'_{s-2} p^2_{s-1}, \quad 1 \le j \le s-2, \\ n/h_s \cdot p'_s \cdot a_{s-1} \cdot p_{s-1} \end{array} \right\}.$$

We want to show that the semigroup condition implies $h_{s-2}\widetilde{h_{s-2}} = 1$. Since $\tilde{A}_{s-2} = a_s - a_{s-2}p_{s-2}p^2_{s-1}(p_s - p'_s) < a_s$, the first generator listed is clearly too large. All of the remaining generators besides $a_s p'_s p_{s-1}$ are divisible by $n/h_s \neq 1$. Write

$$\frac{n}{h_{s-2}\widetilde{h_{s-2}}h_s}\tilde{A}_{s-1} = M a_s p'_s p_{s-1} + \frac{n}{h_s}N,$$

where $M,\ N$ are in $\mathbb{N} \cup \{0\}$. Then we have

$$\begin{aligned}\left\{\frac{n}{h_{s-2}\widetilde{h_{s-2}}h_s} - Mp'_s p_{s-1}\right\} a_s &= \frac{n}{h_s}N + \frac{n}{h_{s-2}\widetilde{h_{s-2}}h_s} a_{s-2}p_{s-2}p_{s-1}^2(p_s - p'_s) \\ &= \frac{n}{h_s}\left\{N + a'_{s-2}p'_{s-2}p_{s-1}^2(p_s - p'_s)\right\}.\end{aligned}$$

Therefore, since $\widetilde{h_s} = (n, a_s) = 1$, n/h_s divides $\frac{n}{h_{s-2}\widetilde{h_{s-2}}h_s} - Mp'_s p_{s-1}$. But we have

$$0 < \frac{n}{h_{s-2}\widetilde{h_{s-2}}h_s} - Mp'_s p_{s-1} \leq \frac{n}{h_{s-2}\widetilde{h_{s-2}}h_s} \leq \frac{n}{h_s},$$

with equalities if and only if $M = 0$ and $h_{s-2}\widetilde{h_{s-2}} = 1$. Hence we must have $h_{s-2}\widetilde{h_{s-2}} = 1$.

The proof of the induction step is virtually identical to the argument just given. Assume that $h_i\widetilde{h_i} = 1$, for all i such that $k + 1 \leq i \leq s - 1$. By Proposition 4.1.2(a), there are h_s nodes of type v_k; let v be any such node. We will show that the semigroup condition for v in the direction of $\Delta_A(v)$ cannot be satisfied if $h_k\widetilde{h_k} \neq 1$. Refer to Figure 5.6.

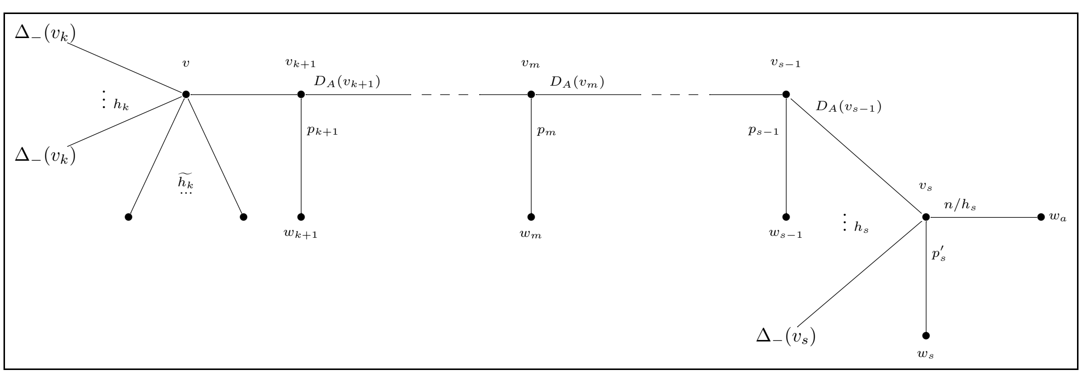


FIGURE 5.6. Splice diagram $\Delta_{f,n}$ for $h_s \neq 1,\ h_{s-1}\widetilde{h_{s-1}} = \cdots = h_{k+1}\widetilde{h_{k+1}} = 1$.

Since $\widetilde{h_s} = 1$, and $h_{s-1}\widetilde{h_{s-1}} = \cdots = h_{k+1}\widetilde{h_{k+1}} = 1$, the generators of R_k are

$$\left\{\begin{array}{lcll} \ell'_{vw_s} & = & p_{k+1}\cdots p_{s-1}\cdot D_-(v_s)^{h_s-1}\cdot n/h_s, & \\ \ell'_{vw_a} & = & p_{k+1}\cdots p_{s-1}\cdot D_-(v_s)^{h_s-1}p'_s, & \\ \ell'_{vw_{k+1}} & = & D_A(v_{k+1}), & \\ \ell'_{vw_m} & = & p_{k+1}\cdots p_{m-1}\cdot D_A(v_m), & k+2\le m\le s-1, \\ \ell'_{vw^s_j} & = & p_{k+1}\cdots p_{s-1}\cdot D_-(v_s)^{h_s-2}(p'_s)\cdot n/h_s\cdot \ell'_{v_s w^s_j}, & 0\le j\le s-1. \end{array}\right\}$$

By Fact DA2 in §4.2,

$$\begin{aligned} D_A(v_m) &= \frac{n\tilde{A}_m D_-(v_s)^{h_s-1}}{h_s a_s}, \text{ for } k+1\le m\le s-1, \text{ and} \\ D_A(v_k) &= \frac{n\tilde{A}_k D_-(v_s)^{h_s-1}}{h_k\widetilde{h_k}h_s a_s}. \end{aligned}$$

We claim that $D_A(v_k)$ and all generators of R_k are divisible by $D_-(v_s)^{h_s-1}/a_s$. This is clearly true for $D_A(v_k)$, ℓ'_{vw_s}, ℓ'_{vw_a}, and ℓ'_{vw_m}, $k+1\le m\le s-1$. That $\ell'_{vw^s_j}$ is also divisible by $D_-(v_s)^{h_s-1}/a_s$ follows from Claim 5.1.2 as above.

Thus, the semigroup condition is equivalent to $\frac{n}{h_k\widetilde{h_k}h_s}\tilde{A}_k$ is in the semigroup generated by:

$$\left\{\begin{array}{ll} n/h_s\cdot a_s\cdot p_{k+1}\cdots p_{s-1}, & \\ a_s\cdot p'_s\cdot p_{k+1}\cdots p_{s-1}, & \\ n/h_s\cdot \tilde{A}_{k+1}, & \\ n/h_s\cdot \tilde{A}_m\cdot p_{k+1}\cdots p_{m-1}, & k+2\le m\le s-1, \\ n/h_s\cdot p'_s\cdot p'_1p'_2\cdots p'_k p^2_{k+1}\cdots p^2_{s-1} & \\ n/h_s\cdot p'_s\cdot a'_l p'_{l+1}\cdots p'_k p^2_{k+1}\cdots p^2_{s-1}, & 1\le l\le k-1, \\ n/h_s\cdot p'_s\cdot a'_k\cdot p^2_{k+1}\cdots p^2_{s-1}, & \\ n/h_s\cdot p'_s\cdot a_l\cdot p_{k+1}\cdots p_l p^2_{l+1}\cdots p^2_{s-1}, & k+1\le l\le s-2, \\ n/h_s\cdot p'_s\cdot a_{s-1}p_{k+1}\cdots p_{s-1} & \end{array}\right\}.$$

We want to show that the semigroup condition implies $h_k\widetilde{h_k}=1$. Since

$$\tilde{A}_k = a_s - a_k p_k p_{k+1}^2 \cdots p_{s-1}^2 (p_s - p'_s) < a_s,$$

the first generator listed is too large. All of the remaining generators besides $a_s p'_s p_{k+1} \cdots p_{s-1}$ are divisible by $n/h_s \neq 1$. Write

$$\frac{n}{h_k \widetilde{h_k} h_s} \tilde{A}_k = M a_s p'_s p_{k+1} \cdots p_{s-1} + \frac{n}{h_s} N,$$

where $M,\ N$ are in $\mathbb{N} \cup \{0\}$. Just as in the proof of the base step, one can show that the only possibility is $M=0$ and $h_k\widetilde{h_k}=1$. □

The pathological case. For the pathological case $n = p_s = 2$, Proposition 5.1.1 is valid for $2 \leq k \leq s-1$, and since $\widetilde{h_k} = 1$ for $1 \leq k \leq s-1$, the semigroup condition at a node of type v_k in the direction of $\Delta_-(v_k)$ is always satisfied. Since $p_s - p'_s = 2 - 1 = 1$,

$$\begin{aligned} \tilde{A}_{s-1} &= a_s - a_{s-1}p_{s-1}, && \text{and} \\ \tilde{A}_k &= a_s - a_k p_k p_{k+1}^2 \cdots p_{s-1}^2, && \text{for } 1 \leq k \leq s-2. \end{aligned}$$

Refer to Figure 4.8 for the splice diagram. The semigroup condition for a node of type v_k in the direction of a subdiagram of the form $\Delta_A(v_k)$ for $1 \leq k \leq s-3$ is: $\tilde{A}_k$ is in the semigroup generated by

$$\left\{ \begin{array}{ll} \tilde{A}_{k+1}, & \\ p_{k+1} \cdots p_{m-1} \tilde{A}_m, & k+2 \leq m \leq s-1, \\ p_{k+1} \cdots p_{s-1} a_{s-1}, & \\ p_{k+1} \cdots p_{s-1} a_j p_{j+1} \cdots p_{s-1}, & 1 \leq j \leq s-2, \\ p_1 \cdots p_k p_{k+1}^2 \cdots p_{s-1}^2 & \end{array} \right\}.$$

For $k = s-2$, this semigroup condition is: $\tilde{A}_{s-2}$ is in the semigroup generated by

$$\left\{ \begin{array}{l} \tilde{A}_{s-1}, \\ \\ p_{s-1}a_{s-1}, \\ \\ a_j p_{j+1} \cdots p_{s-2} p_{s-1}^2, \quad 1 \le j \le s-2, \\ \\ p_1 \cdots p_{s-2} p_{s-1}^2 \end{array} \right\}.$$

Finally, for $k = s-1$, this semigroup condition is: $\tilde{A}_{s-1}$ is in the semigroup generated by

$$\left\{ \begin{array}{l} a_{s-1}, \\ \\ a_j p_{j+1} \cdots p_{s-1}, \quad 1 \le j \le s-2, \\ \\ p_1 \cdots p_{s-1} \end{array} \right\}.$$

5.2. Action of the discriminant group

In order to use Proposition 2.0.10 to check the congruence conditions for the resolution graph $\Gamma_{f,n}$ (when the semigroup conditions are satisfied), we need the continued fraction expansions of the strings from leaves to nodes. This is essentially done in Mendris and Némethi's paper ([**8**], proof of Prop. 4.5), but we will need a bit more detail than they included.

Background. We begin with a summary of facts that we need, which can be found in [**8**], 2.11-2.13. Let a, Q, and P be strictly positive integers with $\gcd(a, Q, P) = 1$. Then $(X(a, Q, P), 0)$ is defined to be the isolated surface singularity lying over the origin in the normalization of $(\{U^a V^Q = W^P\}, 0)$. For the rest of this section, (m, n) denotes $\gcd(m, n)$. Let λ be the unique integer such that $0 \le \lambda < P/(a, P)$ and

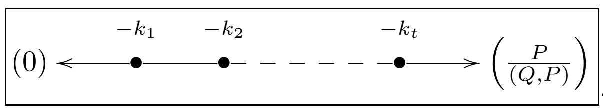
$$Q + \lambda \cdot \frac{a}{(a, P)} = m \cdot \frac{P}{(a, P)},$$

for some positive integer m. If $\lambda \neq 0$, then let $[k_1, \ldots, k_t]$ be the continued fraction expansion of $\frac{P/(a,P)}{\lambda}$. That is,

$$\frac{P/(a,P)}{\lambda} = k_1 - \cfrac{1}{k_2 - \cfrac{1}{\ddots - \cfrac{1}{k_t}}},$$

and all $k_i \geq 2$.

Then the minimal embedded resolution graph (see §4.1) of the coordinate function $V : X(a,Q,P) \to \mathbb{C}$ in $(X(a,Q,P),0)$ is given by the following string (omitting the multiplicities of the vertices):



If $\lambda = 0$, the string is empty. One can similarly describe the embedded resolution graphs of the functions U and W, but we do not need them here.

Now we compute the minimal resolution graph Γ of the singularity in the normalization of N-fold cover of $(X(1,Q,P),0)$ branched over $V^M = 0$, where $(Q,P) = 1$ and $(N,M) = 1$. The graph Γ is the resolution graph of the singularity in the normalization of

$$X_1 := \{UV^Q = W^P,\ T^N = V^M\} \subseteq \mathbb{C}^4.$$

We claim that we may assume $M = 1$, since this space has the same normalization as

$$X_2 := \{UV^Q = W^P,\ T^N = V\} \subseteq \mathbb{C}^4.$$

To see this, note that since $(N,M) = 1$, there exist positive integers r and s such that $rM - sN = 1$. Let $(\bar{X}_i, 0)$ denote the isolated singularity in the normalization of X_i, for $i = 1,\ 2$, and let $\mathcal{O}_{\bar{X}_i}$ be the local analytic ring of $(\bar{X}_i, 0)$. Now consider the element

$$t := \frac{T^r}{V^s}$$

of the quotient ring of $\mathcal{O}_{\bar{X}_1}$, which is integral over $\mathcal{O}_{\bar{X}_1}$, since

$$t^M = T \text{ and } t^N = V.$$

Adjoining t to $\mathcal{O}_{\bar{X}_1}$ results in the ring

$$\mathbb{C}[[U,t,W]]/(Ut^{QN} - W^P),$$

which is clearly isomorphic to $\mathcal{O}_{\bar{X}_2}$.

Therefore, Γ is the resolution graph of the singularity in the normalization of $\{UV^{QN} = W^P\}$, which is isomorphic to the singularity at the origin in

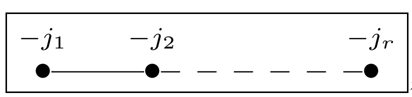
$$X\left(1, Q\frac{N}{(N,P)}, \frac{P}{(N,P)}\right) = \{UV^{QN/(N,P)} = W^{P/(N,P)}\}.$$

Hence, we have the following

Lemma 5.2.1. *Let Γ be the resolution graph of the singularity in the normalization of the N-fold cover of $(X(1,Q,P),0)$ branched over $V^M = 0$, where $(Q,P) = 1$ and $(N,M) = 1$. Let λ be the unique integer such that $0 \leq \lambda < P/(N,P)$ and*

$$Q\frac{N}{(N,P)} + \lambda = m \cdot \frac{P}{(N,P)}$$

for some positive integer m. Then Γ is a string of vertices with continued fraction expansion $\frac{P/(N,P)}{\lambda}$. That is, Γ is



,

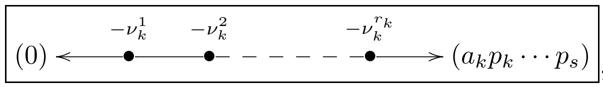
where $[j_1, \ldots, j_r]$ is the continued fraction expansion of $\frac{P/(N,P)}{\lambda}$.

Strings in $\Gamma_{f,n}$. Now we turn to the strings that we are interested in. We need the continued fraction expansion of the strings in $\Gamma_{f,n}$ from leaves of type $\overline{v_k}$, $0 \leq k \leq s$, to the corresponding node of type v_k (from type $\overline{v_0}$ to type v_1).

Recall the construction of $\Gamma(\mathbb{C}^2, f)$ from §4.1. The continued fraction expansion in $\Gamma(\mathbb{C}^2, f)$ from $\overline{v_k}$ to v_k for $1 \leq k \leq s$ is given by $\frac{p_k}{\eta_k}$, where $0 \leq \eta_k < p_k$ and

$$\frac{q_k}{p_k} = \nu_k^0 - \frac{\eta_k}{p_k} \Longleftrightarrow q_k + \eta_k = \nu_k^0 p_k.$$

The corresponding string in $\Gamma(\mathbb{C}^2, f)$ is (multiplicities of vertices omitted):



,

where $[\nu_k^1, \ldots, \nu_k^{r_k}] = \dfrac{p_k}{\eta_k}$. Let $X := X(1, q_k, p_k)$. Since $q_k + \eta_k = \nu_k^0 p_k$, the string above is the embedded resolution graph of $V^{a_k p_{k+1} \cdots p_s}$ in X. It follows from the construction of $\Gamma_{f,n}$ that the collection of strings that lies above this one in $\Gamma_{f,n}$ is the minimal (possibly non-connected) resolution graph of the singularity in the normalization of

$$\{UV^{q_k} = W^{p_k},\ T^n = V^{a_k p_{k+1} \cdots p_s}\}.$$

There are $(n, a_k p_{k+1} \cdots p_s) = \widetilde{h}_k d_k = \widetilde{h}_k h_{k+1} \cdots h_s$ connected components (recall from Definition 4.1.1 that $d_k = h_{k+1} \cdots h_s$ for $1 \leq k \leq s-1$, and $d_s = 1$), each being the resolution graph of the normalization of

$$\{UV^{q_k} = W^{p_k},\ T^{n/\widetilde{h}_k d_k} = V^{a'_k p'_{k+1} \cdots p'_s}\}.$$

Now we are in the situation of Lemma 5.2.1, with $Q = q_k$, $P = p_k$, and $N = n/\widetilde{h}_k d_k$. We have $(N, P) = (n/\widetilde{h}_k d_k, p_k) = h_k$ by definition of h_k, and so in this case $P/(N, P) = p'_k$ (as expected from Proposition 4.2.2). If $p'_k = 1$, then in $\Gamma_{f,n}$, the string of type $\overline{v_k}$ will consist of a single (-1)-curve.

Suppose $p'_k \neq 1$. By Lemma 5.2.1, the continued fraction expansion of the string(s) from a leaf of type $\overline{v_k}$ to the corresponding node of type v_k in the resolution graph $\Gamma_{f,n}$ is given by $\frac{p'_k}{\eta'_k}$, where η'_k is the unique integer such that $0 < \eta'_k < p'_k$ and

$$q_k \frac{n}{h_k \widetilde{h}_k d_k} + \eta'_k = m p'_k,$$

for some positive integer m. That is,

$$\eta'_k = -q_k \frac{n}{h_k \widetilde{h}_k d_k} + m p'_k.$$

Since $a_k = q_k + a_{k-1} p_{k-1} p_k$, we have

$$\eta'_k \equiv -a_k \cdot \frac{n}{h_k \widetilde{h}_k d_k} \pmod{p'_k}. \tag{5.2}$$

Knowing the congruence class of η'_k modulo p'_k is enough for our purposes.

The continued fraction expansion from $\overline{v_0}$ to v_1 is given by $\frac{q_1}{\eta_0} = \frac{a_1}{\eta_0}$, where $0 \leq \eta_0 < a_1$ and

$$\frac{p_1}{a_1} = \mu_1^0 - \frac{\eta_0}{a_1} \Longleftrightarrow p_1 + \eta_0 = \mu_1^0 a_1.$$

Using an argument analogous to the one above, we have that the continued fraction expansion of the string(s) from a leaf of type $\overline{v_0}$ to the corresponding node of type v_1 in $\Gamma_{f,n}$ is given as in Lemma 5.2.1, with $Q = p_1,\ P = a_1$, and $N = n/h_1 d_1$. The expansion is $\frac{a'_1}{\eta'_0}$, where

$$p_1 \frac{n}{h_1 \widetilde{h}_1 d_1} + \eta'_0 = m a'_1,$$

for some positive integer m. Thus,

$$\eta'_0 \equiv -p_1 \cdot \frac{n}{h_1 \widetilde{h}_1 d_1} \pmod{a'_1}. \tag{5.3}$$

Recall the notation defined in Chapter 2: for $r \in \mathbb{Q}$, $[r] = \exp(2\pi i r)$, and for a leaf $w \in \Gamma_{f,n}$, e_w denotes the image in the the discriminant group of the dual basis element in $\mathbb{E}^*$ corresponding to w.

Corollary 5.2.2. *Let w_k be any leaf of type $\overline{v_k}$ in $\Gamma_{f,n}$, $0 \le k \le s$. Then*

$$
\begin{aligned}
[e_{w_0} \cdot e_{w_0}] &= \left[\frac{(n/h_1\widetilde{h_1}d_1)(p_1a_2\cdots a_s - A_1p_1')}{a_1'a_2\cdots a_s} \right], \\
[e_{w_k} \cdot e_{w_k}] &= \left[\frac{(n/h_k\widetilde{h_k}d_k)(a_ka_{k+1}\cdots a_s - A_ka_k')}{p_k'a_{k+1}\cdots a_s} \right], \quad \textit{for } 1 \le k \le s-1, \textit{ and} \\
[e_{w_s} \cdot e_{w_s}] &= \left[\frac{(n/h_s\widetilde{h_s})(a_s - a_s')}{p_s'} \right].
\end{aligned}
$$

Proof. Proposition 2.0.9 says that for any leaf w connected by a string of vertices to a node v,

$$e_w \cdot e_w = -d_v/(d^2 \det(\Gamma)) - p/d,$$

where d_v is the product of weights at the node v, and d/p is the continued fraction expansion of the string from w to v. Let d_{v_k} be the product of the weights at any node of type v_k, $1 \le k \le s$ (Refer to Figures 4.6 and 4.7). Then

$$d_{v_k} = D_A(v_k)D_-(v_k)^{h_k}(p_k')^{\widetilde{h_k}}.$$

Furthermore, by Lemma 4.2.6, we have

$$\det(\Gamma) = (p_s')^{\widetilde{h_s}-1} \cdot \frac{D_-(v_s)^{h_s}}{a_s'}.$$

Since $D_A(v_s) = n/(h_s\widetilde{h_s})$, we have

$$
\begin{aligned}
e_{w_s} \cdot e_{w_s} &= -\frac{(n/h_s\widetilde{h_s})D_-(v_s)^{h_s}(p_s')^{\widetilde{h_s}}}{(p_s')^2\det(\Gamma)} - \frac{\eta_s'}{p_s'} \\
&= -\frac{(n/h_s\widetilde{h_s})a_s'}{p_s'} - \frac{\eta_s'}{p_s'}.
\end{aligned}
$$

By Equation (5.2), $\eta_s' \equiv -a_s \cdot \frac{n}{h_s\widetilde{h_s}} \pmod{p_s'}$ (recall that $d_s = 1$ by definition), and thus we have

$$[e_{w_s} \cdot e_{w_s}] = \left[\frac{(n/h_s\widetilde{h_s})(a_s - a_s')}{p_s'} \right].$$

We will need the following fact. For any k such that $1 \le k \le s-1$,

$$\det(\Gamma) = \frac{D_-(v_k)}{a'_k} \prod_{j=k}^{s} (p'_j)^{\widetilde{h_j}-1} \cdot D_-(v_j)^{h_j-1}.$$

This is clearly true, since

$$\det(\Gamma) = \frac{D_-(v_s)}{a'_s} (p'_s)^{\widetilde{h_s}-1} \cdot D_-(v_s)^{h_s-1},$$

and, by Lemma 4.2.3, for $1 \le j \le s$,

$$\frac{D_-(v_j)}{a'_j} = \frac{D_-(v_{j-1})}{a'_{j-1}} (p'_{j-1})^{\widetilde{h_{j-1}}-1} \cdot D_-(v_{j-1})^{h_{j-1}-1}.$$

Now, for $1 \le k \le s-1$,

$$\begin{aligned} e_{w_k} \cdot e_{w_k} &= -\frac{D_A(v_k) D_-(v_k)^{h_k} (p'_k)^{\widetilde{h_k}}}{(p'_k)^2 \det(\Gamma)} - \frac{\eta'_k}{p'_k} \\ &= -\frac{\left(\frac{nA_k \cdot \prod_{j=k+1}^{s} (p'_j)^{\widetilde{h_j}-1} D_-(v_j)^{h_j-1}}{h_k \widetilde{h_k} d_k a_{k+1} \cdots a_s}\right) D_-(v_k)^{h_k} (p'_k)^{\widetilde{h_k}}}{(p'_k)^2 \frac{D_-(v_k)}{a'_k} \left(\prod_{j=k}^{s} (p'_j)^{\widetilde{h_j}-1} \cdot D_-(v_j)^{h_j-1}\right)} - \frac{\eta'_k}{p'_k} \\ &= -\frac{\left(\frac{nA_k}{h_k \widetilde{h_k} d_k a_{k+1} \cdots a_s}\right) a'_k}{p'_k} - \frac{\eta'_k}{p'_k}. \end{aligned}$$

Hence, applying Equation (5.2),

$$[e_{w_k} \cdot e_{w_k}] = \left[\frac{(n/h_k \widetilde{h_k} d_k) a_k}{p'_k} - \frac{(n/h_k \widetilde{h_k} d_k) A_k a'_k}{p'_k a_{k+1} \cdots a_s}\right] = \left[\frac{(n/h_k \widetilde{h_k} d_k)(a_k a_{k+1} \cdots a_s - A_k a'_k)}{p'_k a_{k+1} \cdots a_s}\right].$$

Finally, we have

$$\begin{aligned} e_{w_0} \cdot e_{w_0} &= -\frac{D_A(v_1)(a'_1)^{h_1} (p'_1)^{\widetilde{h_1}}}{(a'_1)^2 \det(\Gamma)} - \frac{\eta'_0}{a'_1} \\ &= -\frac{\left(\frac{nA_1 \cdot \prod_{j=2}^{s} (p'_j)^{\widetilde{h_j}-1} D_-(v_j)^{h_j-1}}{h_1 \widetilde{h_1} d_1 a_2 \cdots a_s}\right) (a'_1)^{h_1} (p'_1)^{\widetilde{h_1}}}{(a'_1)^2 (a'_1)^{h_1-1} (p'_1)^{\widetilde{h_1}-1} \left(\prod_{j=2}^{s} (p'_j)^{\widetilde{h_j}-1} \cdot D_-(v_j)^{h_j-1}\right)} - \frac{\eta'_0}{a'_1} \\ &= -\frac{\left(\frac{nA_1}{h_1 \widetilde{h_1} d_1 a_2 \cdots a_s}\right) p'_1}{a'_1} - \frac{\eta'_0}{a'_1}. \end{aligned}$$

Therefore, by (5.3),

$$[e_{w_0} \cdot e_{w_0}] = \left[\frac{(n/h_1\widetilde{h}_1 d_1)p_1}{a_1'} - \frac{(n/h_1\widetilde{h}_1 d_1)A_1 p_1'}{a_1' a_2 \cdots a_s} \right] = \left[\frac{(n/h_1\widetilde{h}_1 d_1)(p_1 a_2 \cdots a_s - A_1 p_1')}{a_1' a_2 \cdots a_s} \right].$$

□

CHAPTER 6

Characterization of the topological types of splice quotients

In this chapter, we consider $X_{f,n} = \{z^n = f(x,y)\}$, where f is an irreducible plane curve with a singularity at the origin having $s \geq 2$ topological pairs (see §4.1). Let $\{(p_i, a_i)\}_{i=1}^{s}$ be the topological pairs. We determine which $(X_{f,n}, 0)$ have a splice diagram and resolution graph that satisfy both the semigroup conditions and the congruence conditions, as defined by Neumann and Wahl (see Chapter 2). Let $\Gamma_{f,n}$ denote the good quasi-minimal resolution graph $\Gamma^{qmin}(X_{f,n})$ defined in §4.1, and let $\Delta_{f,n}$ be the splice diagram associated to $\Gamma_{f,n}$. Recall that the semigroup conditions depend only on $\Delta_{f,n}$, whereas the congruence conditions depend on $\Gamma_{f,n}$ as well. With this in mind, we will often say "$\Gamma_{f,n}$ satisfies the semigroup and congruence conditions", as opposed to "$\Delta_{f,n}$ satisfies the semigroup conditions and $\Gamma_{f,n}$ satisfies the congruence conditions". If $\Gamma_{f,n}$ satisfies the semigroup and congruence conditions, then there exist splice quotients with resolution graph $\Gamma_{f,n}$, but whether or not there is a splice quotient of this topological type defined by an equation of the form $z^n = f(x,y)$ is another question. The analytic types of the resulting splice quotients are studied in Chapter 7. For the remainder of the chapter, (p,q) denotes $\gcd(p,q)$.

Theorem 6.0.1 (Main Theorem). *Assume $(X_{f,n}, 0)$ has $\mathbb{Q}HS$ link. Then $\Gamma_{f,n}$ satisfies the semigroup and congruence conditions if and only if either*

(i) $(n, p_s) = 1,\ (n, p_i) = (n, a_i) = 1$ *for* $1 \leq i \leq s-1$, *and* $a_s/(n, a_s)$ *is in the semigroup generated by* $\{a_{s-1},\ p_1 \cdots p_{s-1},\ a_j p_{j+1} \cdots p_{s-1}\ :\ 1 \leq j \leq s-2\}$, *or*

(ii) $s = 2,\ p_2 = 2,\ (n, p_2) = 2,$ *and* $(\frac{n}{2}, p_1) = (\frac{n}{2}, a_1) = 1.$

Remark 6.0.2. 1) The $\mathbb{Z}$HS link case belongs to (i) (see Proposition 6.0.3).

2) For the so-called pathological case $n = p_s = 2$, both semgroup and congruence conditions are satisfied only for $s = 2$.

3) There are classes of $(X_{f,n}, 0)$ for which the semigroup conditions are satisfied but the congruence conditions are not; we do not write up a complete list of these types. An example with this property is given by $n = 2,\ s = 2,\ p_1 = 2,\ a_1 = 3,\ p_2 = 3$, and $a_2 = 20$. The *minimal* good resolution graph and splice diagram for this example are given in Figure 6.1.

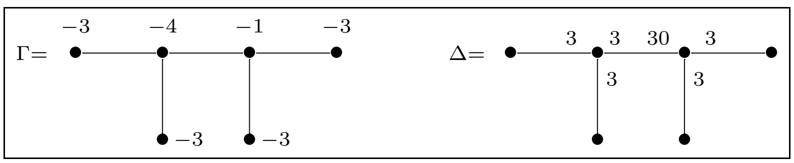


Figure 6.1. Example for which the semigroup conditions are satisfied but the congruence conditions are not.

Recall from Definition 4.1.1 that

- $d_i = (n, p_{i+1}p_{i+2}\cdots p_s)$ for $0 \le i \le s-1$, and $d_s = 1$;

and, for $1 \le i \le s$,

- $h_i = (p_i, n/d_i)$;
- $p'_i = p_i/h_i$;
- $\widetilde{h}_i = (a_i, n/d_i)$;
- $a'_i = a_i/\widetilde{h}_i$.

The link of $(X_{f,n}, 0)$ is a $\mathbb{Q}$HS if and only if for each i, either h_i or $\widetilde{h}_i$ is equal to 1 (Proposition 4.1.2(c)). We assume that this condition holds in all that follows.

The remainder of this chapter is devoted to proving Theorem 6.0.1. We treat the following cases separately:

- $h_s = \widetilde{h}_s = 1$,
- $\widetilde{h}_s \neq 1$ ($\Longrightarrow\ h_s = 1$),

- $h_s \neq 1$ ($\Longrightarrow$ $\widetilde{h}_s = 1$).

The first of these cases is the easiest to handle. The following result of Neumann and Wahl was mentioned in Chapter 1.

Proposition 6.0.3 ([**16**]). *The link of* $(X_{f,n}, 0)$ *is a* $\mathbb{Z}HS$ *if and only if the plane curve singularity* $\{f(x,y) = 0\}$ *has only one branch at* 0 *and* n *is relatively prime to each integer* p_i *and* a_i *in the topological pairs of* f.

Remark 6.0.4. In [**16**], the proposition is incorrectly stated. The pairs in question are identified as the Newton pairs instead of the topological pairs.

Clearly, n is relatively prime to each topological pair if and only if $h_i\widetilde{h}_i = 1$ for $1 \leq i \leq s$.

Proposition 6.0.5. *Suppose* $h_s\widetilde{h}_s = 1$. *Then* $\Delta_{f,n}$ *satisfies the semigroup conditions if and only if* $h_i\widetilde{h}_i = 1$ *for* $1 \leq i \leq s-1$, *that is, if and only if the link is a* $\mathbb{Z}HS$.

Proof. If $h_i\widetilde{h}_i = 1$ for all i, $1 \leq i \leq s$, then the semigroup conditions are satisfied. It is not difficult to see that this follows from the second statement of Proposition 5.1.1. A complete proof and discussion can be found in [**19**].

Now assume that the semigroup conditions are satisfied. Since $h_s = (n, p_s) = 1$, there is only one node of type v_{s-1} (see Prop. 4.1.2), which we denote v_{s-1}. The splice diagram $\Delta_{f,n}$ is shown in Figure 6.2.

The semigroup condition at the node v_{s-1} in the direction of $\Delta_A(v_{s-1})$ is

$$D_A(v_{s-1}) \in \mathbb{N}\langle n,\ p_s\rangle.$$

By Lemma 4.2.4,

$$D_A(v_{s-1}) = \frac{nA_{s-1}}{h_{s-1}\widetilde{h_{s-1}}d_{s-1}a_s},$$

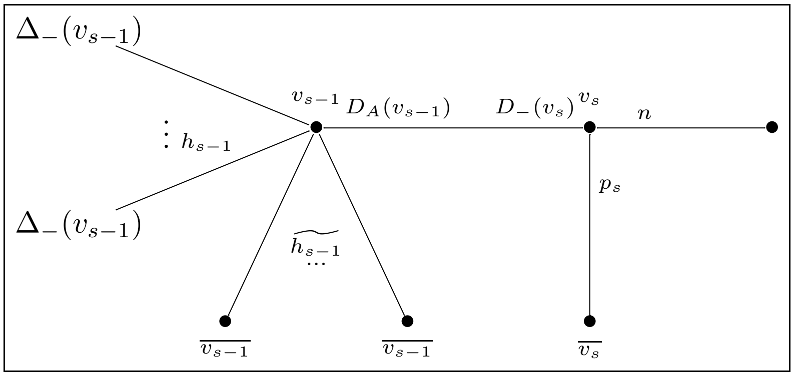


FIGURE 6.2. Splice diagram for $h_s = \widetilde{h}_s = 1$.

where $A_{s-1} = a_{s-1}p_{s-1}p'_s + q_s$. Since $p'_s = p_s$, we have $A_{s-1} = a_s$, and, by definition, $d_{s-1} = h_s = 1$. Thus,

$$D_A(v_{s-1}) = \frac{n}{h_{s-1}\widetilde{h_{s-1}}}.$$

The semigroup condition implies $h_{s-1}\widetilde{h_{s-1}} = 1$. For, if not, $n > n/h_{s-1}\widetilde{h_{s-1}}$, and hence, $n/h_{s-1}\widetilde{h_{s-1}} \in \mathbb{N}p_s$, which is impossible since $h_s = (n, p_s) = 1$.

If $s = 2$, we are done. If $s \geq 3$, then by Lemma 5.1.3, $h_i\widetilde{h}_i = 1$ for $1 \leq i \leq s$. □

The remaining $(X_{f,n}, 0)$ with $\mathbb{Q}$HS link are divided into two cases:

(i) $\widetilde{h}_s = (n, a_s) \neq 1$;

(ii) $h_s = (n, p_s) \neq 1$.

These lead to cases (i) and (ii) of the Main Theorem, respectively.

6.1. Case (i) $(n, a_s) \neq 1$

The purpose of this section is to prove the following

PROPOSITION 6.1.1. *Suppose* $\widetilde{h}_s = (n, a_s) \neq 1$. *Then* $\Gamma_{f,n}$ *satisfies the semigroup and congruence conditions if and only if both of the following hold:*

(I) $h_i\widetilde{h}_i = 1$ *for* $1 \leq i \leq s-1$,

(II) $a'_s = a_s/\widetilde{h}_s \in \mathbb{N}\langle a_{s-1},\ p_1 \cdots p_{s-1},\ a_j p_{j+1} \cdots p_{s-1}\ :\ 1 \leq j \leq s-2\rangle$.

Remark 6.1.2. The condition (II) is clearly not always satisfied. For example, take n divisible by a_s.

Claim 6.1.3. *Assume that $h_i\widetilde{h}_i = 1$ for $1 \le i \le s-1$. Then the following are equivalent:*

(a) $a'_s = a_s/\widetilde{h}_s \in \mathbb{N}\langle a_{s-1},\ p_1p_2\cdots p_{s-1},\ a_jp_{j+1}p_{j+2}\cdots p_{s-1}\ :\ 1 \le j \le s-2\rangle$,

(b) $\Delta_{f,n}$ *satisfies the semigroup conditions,*

(c) $\Gamma_{f,n}$ *satisfies the semigroup and congruence conditions.*

Once we have shown Claim 6.1.3, one direction of Proposition 6.1.1 is clear.

Proof of Claim 6.1.3. We have $h_i\widetilde{h}_i = 1$ for $1 \le i \le s-1$, and $h_s = 1$. Therefore, by Fact DM in §4.2,

$$D_-(v_k) = a_k, \quad 2 \le k \le s-1, \text{ and}$$

$$D_-(v_s) = a'_s,$$

and by Fact DA1,

$$D_A(v_k) = n(p_s)^{\widetilde{h}_s - 1}, \text{ for } 1 \le k \le s-1.$$

The splice diagram $\Delta_{f,n}$ is shown in Figure 6.3. There is exactly one node of type v_k in $\Delta_{f,n}$ for $1 \le k \le s$, which we denote v_k. The leaves are denoted $z_0, \dots, z_{s-1}$, $u_1, \dots, u_{\widetilde{h}_s}$, and y, as in Figure 6.3.

It is clear from Proposition 5.1.1 that the semigroup condition at the node v_k in the direction of $\Delta_-(v_k)$ is satisfied for $2 \le k \le s-1$, and at node v_s, the condition is equivalent

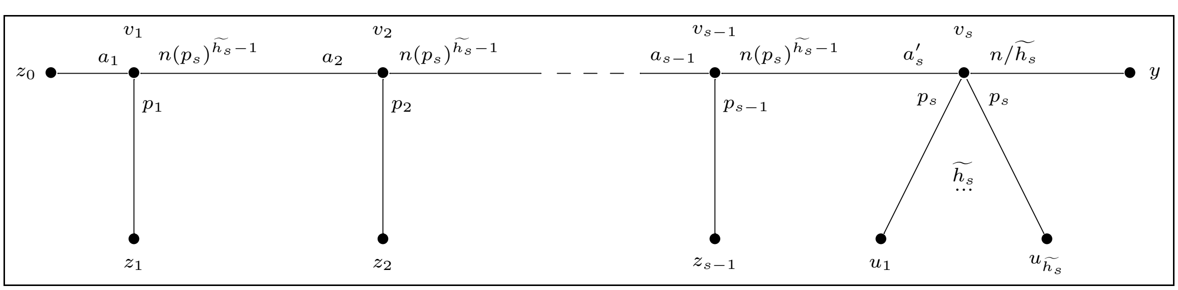


Figure 6.3. Splice diagram for $\widetilde{h}_s \ne 1$ and $h_i\widetilde{h}_i = 1,\ 1 \le i \le s-1$.

to (a). Furthermore, one can see by examination of the splice diagram that the semigroup condition at each v_k in the direction of $\Delta_A(v_k)$ (i.e., in the direction of v_s) is always satisfied (including in the case $n = \widetilde{h_s}$). Therefore, we have (a) $\Longleftrightarrow$ (b).

It remains to prove (b) $\Longrightarrow$ (c), i.e., $\Delta_{f,n}$ satisfies the semigroup conditions $\Longrightarrow$ $\Gamma_{f,n}$ satisfies the congruence conditions.

Assume that $\Delta_{f,n}$ satisfies the semigroup conditions. We use Proposition 2.0.10 to show that the congruence conditions are satisfied. We repeat this result here for convenience.

PROPOSITION 2.0.10. *Let Γ be a graph for which the associated splice diagram Δ satisfies the semigroup conditions. Then the congruence conditions are equivalent to the following: For every node v and adjacent edge e, there is an admissible monomial $M_{ve} = \prod Z_w^{\alpha_w}$ (w running over the leaves in Δ_{ve}) such that for every leaf w' in Δ_{ve},*

$$\left[\sum_{w \neq w'} \alpha_w \frac{\ell_{ww'}}{\det(\Gamma)} - \alpha_{w'} e_{w'} \cdot e_{w'}\right] = \left[\frac{\ell_{vw'}}{\det(\Gamma)}\right]. \tag{6.1}$$

Recall (Remark 2.0.11) that there is nothing to check for edges leading directly to leaves.

By Lemma 4.2.6, we have

$$\det(\Gamma_{f,n}) = (p_s)^{\widetilde{h_s}-1}.$$

In Figure 6.3, it is easy to see that for any node v and any leaf w' in $\Delta_{f,n}$, $\ell_{vw'}$ is always divisible by $(p_s)^{\widetilde{h_s}-1}$. That is,

$$\left[\frac{\ell_{vw'}}{\det(\Gamma_{f,n})}\right] = [0]$$

for any node v and any leaf w' in $\Delta_{f,n}$. Thus in Equation (6.1), the right hand side is always [0] in this setting.

First we show that for all v_k in $\Delta_{f,n}$, $2 \leq k \leq s$, the congruence condition for the edge in the direction of $\Delta_-(v_k)$ (i.e., the edge in the direction of v_1) is satisfied for any choice of admissible monomial. For any leaf z_j, $0 \leq j \leq s-1$, it is easy to see that $\ell_{z_j w'}$ is divisible by $(p_s)^{\widetilde{h_s}-1}$ for all leaves $w' \neq z_j$ in $\Delta_{f,n}$. Therefore, for any leaf $w' \in \Delta_-(v_k)$, the left hand side of Equation (6.1) is $[-\alpha_{w'}e_{w'} \cdot e_{w'}]$. The proof of Fact DA1 in §4.2 tells us that $A_i = a_{i+1} \cdots a_s$ for $1 \leq i \leq s-1$. Therefore, by Corollary 5.2.2, we have

$$[e_{z_0} \cdot e_{z_0}] = [0],$$

since $A_1 p_1' = a_2 \cdots a_s p_1$, and, for $1 \leq j \leq s-1$,

$$[e_{z_j} \cdot e_{z_j}] = [0],$$

since $A_j a_j' = a_{j+1} \cdots a_s a_j$. Since all of the subgraphs $\Delta_-(v_k)$ contain leaves only of the form z_j, $0 \leq j \leq k-1$, Equation (6.1) holds for all leaves in $\Delta_-(v_k)$ for any choice of admissible monomial. In fact, we have shown that the action of the discriminant group element e_{z_j} is trivial for $0 \leq j \leq s-1$.

Next, we show that for $1 \leq k \leq s-2$, the congruence condition at v_k in the direction of $\Delta_A(v_k)$ (i.e., in the direction of v_s) is satisfied for the admissible monomial Z_{k+1}, where Z_{k+1} the variable associated to the leaf z_{k+1}, as described in Chapter 2. So, let $\alpha_{z_{k+1}} = 1$ and $\alpha_w = 0$ for all leaves $w \neq z_{k+1}$. For any leaf $w' \neq z_{k+1}$ in $\Delta_A(v_k)$, Equation (6.1) is equivalent to

$$\left[\frac{\ell_{z_{k+1}w'}}{\det(\Gamma_{f,n})}\right] = [0],$$

which is clearly true. For $w' = z_{k+1}$, Equation (6.1) is equivalent to

$$[-e_{z_{k+1}} \cdot e_{z_{k+1}}] = [0],$$

which we have shown above. Therefore, the congruence condition is satisfied, using the admissible monomial Z_{k+1}.

All that remains is the congruence condition for the node v_{s-1} in the direction of v_s. We claim that the monomial $U_1 \cdots U_{\widetilde{h_s}}$ (which is easily seen to be an admissible monomial) satisfies the condition, where U_j is the variable associated to the leaf u_j, $1 \leq j \leq \widetilde{h_s}$. It is clear from the splice diagram that

$$\left[\frac{\ell_{u_i u_j}}{\det(\Gamma_{f,n})}\right] = \left[\frac{(n/\widetilde{h_s})a'_s}{p_s}\right] \text{ for } i \neq j,$$

and by Corollary 5.2.2, since each u_j is a leaf of type $\overline{v_s}$,

$$[e_{u_j} \cdot e_{u_j}] = \left[\frac{(n/\widetilde{h_s})(a_s - a'_s)}{p_s}\right] \text{ for all } j.$$

Hence, for each u_j, Equation (6.1) for the monomial $U_1 \cdots U_{\widetilde{h_s}}$ is

$$\left[(\widetilde{h_s} - 1)\frac{(n/\widetilde{h_s})a'_s}{p_s} - \frac{(n/\widetilde{h_s})(a_s - a'_s)}{p_s}\right] = [0].$$

This is clearly true, since $\widetilde{h_s}a'_s = a_s$. Finally, for the leaf y, Equation (6.1) for $U_1 \cdots U_{\widetilde{h_s}}$ is

$$\left[\sum_{j=1}^{\widetilde{h_s}} \frac{\ell_{yu_j}}{\det(\Gamma_{f,n})}\right] = [0]\,.$$

Since ℓ_{yu_j} is divisible by $(p_s)^{\widetilde{h_s}-1}$ for all j, the congruence condition is satisfied. This concludes the proof of Claim 6.1.3. □

Now we move on to the other direction of Proposition 6.1.1, that $\Gamma_{f,n}$ satisfies the semigroup and congruence conditions implies conditions (I) and (II). If we show that (I) must hold, then (II) is automatic by Claim 6.1.3. We prove that $h_i\widetilde{h_i} = 1$ for $1 \leq i \leq s-1$ by showing that the congruence conditions imply that $h_{s-1}\widetilde{h_{s-1}} = 1$, then applying Lemma 5.1.3. We really do need the congruence conditions here; see the example in Remark 6.0.2 3).

Claim 6.1.4. *Suppose that $\widetilde{h_s} \neq 1$. If $\Gamma_{f,n}$ satisfies the semigroup and congruence conditions, then $h_{s-1}\widetilde{h_{s-1}} = 1$.*

Proof. Since the link is a $\mathbb{Q}$HS and $\widetilde{h_s} \neq 1$, we know that $h_s = 1$. The splice diagram $\Delta_{f,n}$ is shown in Figure 6.4.

We will show that the congruence condition at the node v_{s-1} cannot be satisfied if $h_{s-1}\widetilde{h_{s-1}} \neq 1$. Let U_j be the variable associated to the leaf u_j (respectively, Y associated to y) as labelled in Figure 6.4, and let e_{u_j} be the element in the discriminant group associated to the leaf u_j. By Proposition 2.0.10, the congruence condition at v_{s-1} in the direction of $\Delta_A(v_{s-1})$ implies, in particular, that there exists an admissible monomial

$$H = U_1^{\alpha_1} \cdots U_{\widetilde{h_s}}^{\alpha_{\widetilde{h_s}}} Y^{\beta}$$

such that for every leaf $u_j,\ 1 \leq j \leq \widetilde{h_s}$,

$$\left[\beta \frac{\ell_{yu_j}}{\det(\Gamma_{f,n})} + \sum_{i \neq j} \alpha_i \frac{\ell_{u_i u_j}}{\det(\Gamma_{f,n})} - \alpha_j e_{u_j} \cdot e_{u_j}\right] = \left[\frac{\ell_{v_{s-1}u_j}}{\det(\Gamma_{f,n})}\right]. \tag{6.2}$$

We will show that this is impossible unless $h_{s-1}\widetilde{h_{s-1}} = 1$. By Lemmas 4.2.6 and 4.2.3,

$$\det(\Gamma_{f,n}) = (p_s)^{\widetilde{h_s}-1}\left(\frac{D_-(v_s)}{a'_s}\right) = (p_s)^{\widetilde{h_s}-1}(p'_{s-1})^{\widetilde{h_{s-1}}-1}\frac{D_-(v_{s-1})^{h_{s-1}}}{a'_{s-1}}.$$

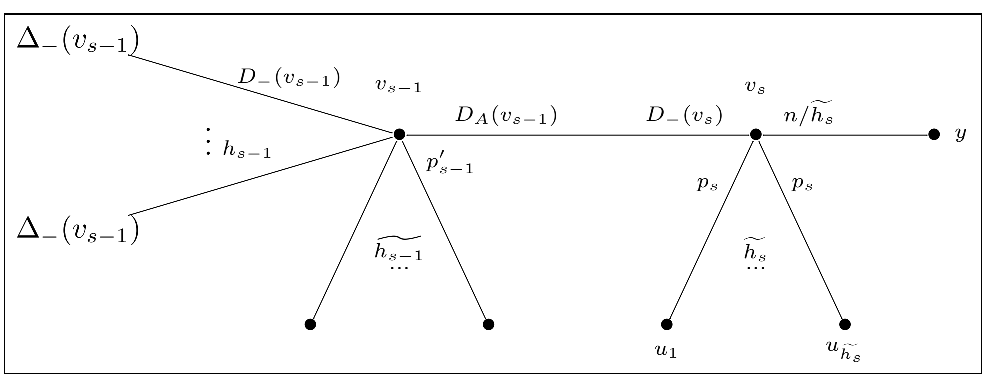


Figure 6.4. Splice diagram for $\widetilde{h_s} \neq 1$

One can easily check that

$$\begin{aligned}
\left[\ell_{v_{s-1}u_j}/\det(\Gamma_{f,n})\right] &= [0]\,, \\
\left[\ell_{yu_j}/\det(\Gamma_{f,n})\right] &= [0]\,, \\
\left[\ell_{u_iu_j}/\det(\Gamma_{f,n})\right] &= \left[(a'_s n/\widetilde{h_s})/p_s\right] \text{ (for } i \neq j),
\end{aligned}$$

and by Corollary 5.2.2,

$$[e_{u_j} \cdot e_{u_j}] = [((a_s - a'_s)n/\widetilde{h_s})/p_s].$$

Therefore, for each j,

$$\begin{aligned}
\left[\frac{\beta \ell_{yu_j}}{\det(\Gamma_{f,n})} + \sum_{i\neq j} \frac{\alpha_i \ell_{u_iu_j}}{\det(\Gamma_{f,n})} - \alpha_j e_{u_j} \cdot e_{u_j}\right] &= \left[\left(\sum_{i \neq j} \alpha_i\right) \frac{a'_s n/\widetilde{h_s}}{p_s} - \alpha_j \frac{(a_s - a'_s)n/\widetilde{h_s}}{p_s}\right] \\
&= \left[\left(\sum_{i=1}^{\widetilde{h_s}} \alpha_i\right) \frac{a'_s n/\widetilde{h_s}}{p_s} - \alpha_j \frac{a_s n/\widetilde{h_s}}{p_s}\right].
\end{aligned}$$

Hence the congruence condition (6.2) is equivalent to

$$\left[\left(\sum_{i=1}^{\widetilde{h_s}} \alpha_i\right) \frac{a'_s n/\widetilde{h_s}}{p_s} - \alpha_j \frac{a_s n/\widetilde{h_s}}{p_s}\right] = [0]\,.$$

The exponents of the admissible monomial H satisfy

$$D_A(v_{s-1}) = \left(\sum_{i=1}^{\widetilde{h_s}} \alpha_i\right) (p_s)^{\widetilde{h_s}-1} n/\widetilde{h_s} + \beta (p_s)^{\widetilde{h_s}}.$$

By Lemma 4.2.4, $A_{s-1} = a_s$, and $D_A(v_{s-1}) = (n/h_{s-1}\widetilde{h_{s-1}})(p_s)^{\widetilde{h_s}-1}$. Therefore we have

$$\frac{n}{h_{s-1}\widetilde{h_{s-1}}} = \left(\sum_{i=1}^{\widetilde{h_s}} \alpha_i\right) n/\widetilde{h_s} + \beta p_s. \tag{6.3}$$

If all $\alpha_j \geq 1$, then this implies that all α_j must equal 1, β must be 0, and $h_{s-1}\widetilde{h_{s-1}} = 1$. Therefore, if $h_{s-1}\widetilde{h_{s-1}} \neq 1$, there exists some j such that $\alpha_j = 0$. Then the congruence condition for the leaf u_j is

$$\left[\left(\sum_{i\neq j}\alpha_i\right)\frac{a'_s n/\widetilde{h}_s}{p_s}\right]=[0]\,,$$

that is,

$$\left(\sum_{i\neq j}\alpha_i\right)a'_s n/\widetilde{h}_s\in\mathbb{Z}p_s.$$

Since a'_s and $n/\widetilde{h}_s$ are relatively prime to p_s, this implies that $\sum_{i\neq j}\alpha_i\in\mathbb{Z}p_s$. But, by Equation (6.3), this implies that $n/(h_{s-1}\widetilde{h_{s-1}})$ is divisible by p_s, which is a contradiction. Therefore, the congruence conditions cannot be satisfied unless all $\alpha_j=1$, $\beta=1$, and $h_{s-1}\widetilde{h_{s-1}}=1$. □

This concludes the proof of Proposition 6.1.1.

6.2. Case (ii) $(n,p_s)\neq 1$

The goal of this section is to prove the following

PROPOSITION 6.2.1. *Suppose $h_s=(n,p_s)\neq 1$. Then $\Gamma_{f,n}$ satisfies the semigroup and congruence conditions if and only if*

$$(*)\ s=2,\ p_2=2,\ (n,p_2)=2,\ \textit{and}\ (\frac{n}{2},p_1)=(\frac{n}{2},a_1)=1.$$

Let us first assume that $\Gamma_{f,n}$ satisfies the semigroup and congruence conditions. We prove that $(*)$ must hold in four steps:

Step 1. The semigroup conditions imply that $h_s=p_s$, i.e., n is divisible by p_s.

Step 2. The semigroup conditions imply that $h_i\widetilde{h_i}=1$ for $1\leq i\leq s-1$.

Step 3. The congruence conditions imply that $p_s=2$.

Step 4. The congruence conditions imply that $s=2$.

Except where otherwise stated, we assume that we are not in the pathological case $n = p_s = 2$. Recall that since the link is a $\mathbb{Q}$HS, $\widetilde{h}_s = 1$ and $a'_s = a_s$.

Step 1. The semigroup conditions imply that $h_s = p_s$.

Claim 6.2.2. *The semigroup condition at a node of type v_{s-1} in the direction of $\Delta_A(v_{s-1})$ is equivalent to the following:* $\dfrac{n}{h_{s-1}\widetilde{h_{s-1}}h_s}[a_s - a_{s-1}p_{s-1}(p_s - p'_s)]$ *is in the semigroup generated by*

$$\left\{\frac{n}{h_s}a_s,\ a_s p'_s\ (\dagger),\ \frac{n}{h_s}p'_1\cdots p'_s,\ \frac{n}{h_s}a'_j p'_{j+1}\cdots p'_s\ :\ 1 \le j \le s-1\right\}.$$

($\dagger$) *Absent if* $\frac{n}{h_s} = 1$.

Proof of Claim. Let v be a node of type v_{s-1}, as pictured in Figure 6.5. The semi-

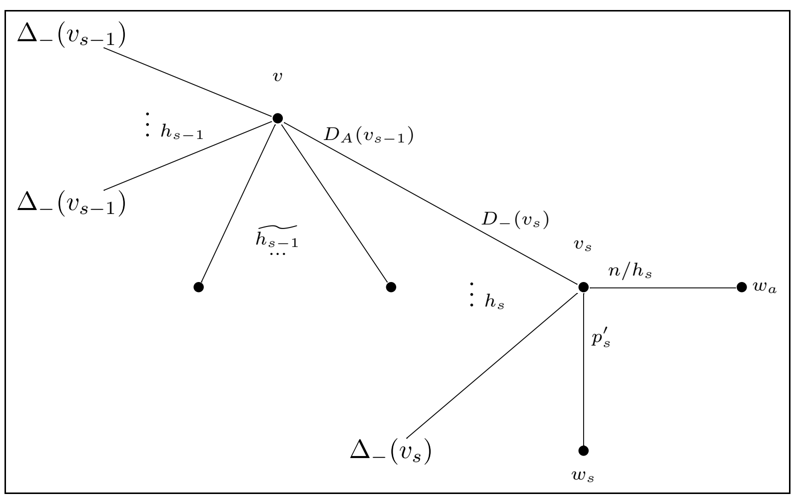


Figure 6.5. Splice diagram for $h_s \neq 1$.

group condition in question is: $D_A(v_{s-1})$ is in the semigroup

$$\mathrm{R}_{s-1} := \mathbb{N}\langle \ell'_{vw} \mid w \text{ is a leaf in } \Delta_A(v_{s-1})\rangle.$$

The generators of the semigroup R_{s-1} were computed in §5.1; they are ℓ'_{vw_s}, ℓ'_{vw_a} (†), and $\ell'_{vw_j^s}$, $0 \le j \le s-1$. It is easy to check that

$$\begin{aligned} \ell'_{vw_s} &= (n/h_s)D_-(v_s)^{h_s-1}, \\ \ell'_{vw_a} &= p'_s D_-(v_s)^{h_s-1}, \text{ and} \\ \ell'_{vw_j^s} &= (n/h_s)p'_s D_-(v_s)^{h_s-2}\ell'_{v_s w_j^s},\ 0 \le j \le s-1. \end{aligned}$$

By Claim 5.1.2, we have

$$\ell'_{v_s w_j^s} = \begin{cases} p'_1 \cdots p'_{s-1} \cdot D_-(v_s)/a_s & \text{for } j = 0 \\ a'_j p'_{j+1} \cdots p'_{s-1} \cdot D_-(v_s)/a_s & \text{for } 1 \le j \le s-2 \\ a'_{s-1} \cdot D_-(v_s)/a_s & \text{for } j = s-1. \end{cases}$$

Therefore, all generators of R_{s-1} are divisible by $D_-(v_s)^{h_s-1}/a_s$. Furthermore, by Lemma 4.2.4,

$$D_A(v_{s-1}) = \frac{nA_{s-1}D_-(v_s)^{h_s-1}}{h_{s-1}\widetilde{h_{s-1}}h_s a_s},$$

where

$$A_{s-1} = a_{s-1}p_{s-1}p'_s + q_s = a_s - a_{s-1}p_{s-1}(p_s - p'_s).$$

Upon factoring out $D_-(v_s)^{h_s-1}/a_s$ from all terms, the claim is clear. □

Now, since $p_s > p'_s$,

$$\frac{n}{h_{s-1}\widetilde{h_{s-1}}h_s}[a_s - a_{s-1}p_{s-1}(p_s - p'_s)] < \frac{n}{h_{s-1}\widetilde{h_{s-1}}h_s}a_s \le \frac{n}{h_s}a_s,$$

and hence the first term in the semigroup in Claim 6.2.2 is too large. All of the remaining generators of the semigroup are divisible by p'_s. Therefore, the semigroup condition implies that p'_s divides $(n/h_{s-1}\widetilde{h_{s-1}}h_s)[a_s - a_{s-1}p_{s-1}(p_s - p'_s)]$. Since p'_s divides $p_s - p'_s$, and $(a_s, p_s) = 1$, this implies that p'_s divides $n/(h_{s-1}\widetilde{h_{s-1}}h_s)$. If so, then p'_s must be 1, since by definition $p'_s = p_s/(n, p_s)$, and thus $(p'_s, n) = 1$. This concludes the proof of Step 1.

Step 2. The semigroup conditions imply that $h_i\widetilde{h_i} = 1$ for $1 \le i \le s-1$.

From now on, we assume $h_s = (n, p_s) = p_s$ (equivalently, $p'_s = 1$). Note that if $n/h_s = 1$, then Step 2 is automatically true by definition of the h_i and $\widetilde{h_i}$. Therefore, assume that $n/h_s \neq 1$. The strategy is to show that the semigroup conditions imply that $h_{s-1}\widetilde{h_{s-1}} = 1$, and then Step 2 follows from Lemma 5.1.3.

From the proof of Step 1, the semigroup condition at a node v of type v_{s-1} in the direction of $\Delta_A(v_{s-1})$ is equivalent to: $\dfrac{n}{h_{s-1}\widetilde{h_{s-1}}h_s}[a_s - a_{s-1}p_{s-1}(p_s-1)]$ is in the semigroup generated by

$$\left\{a_s,\ \frac{n}{h_s}p'_1\cdots p'_{s-1},\ \frac{n}{h_s}a'_{s-1},\ \frac{n}{h_s}a'_j p'_{j+1}\cdots p'_{s-1}\ :\ 1 \le j \le s-2\right\}.$$

Hence, we may write

$$\frac{n}{h_{s-1}\widetilde{h_{s-1}}h_s}[a_s - a_{s-1}p_{s-1}(p_s-1)] = \beta_0 a_s + \frac{n}{h_s}B,$$

where

$$B = \sum_{j=1}^{s-2}\beta_j a'_j p'_{j+1}\cdots p'_{s-1} + \beta_{s-1}a'_{s-1} + \beta_s p'_1\cdots p'_{s-1},$$

for some $\beta_j \in \mathbb{N}\cup\{0\},\ 0 \le j \le s$. Then we have

$$\begin{aligned}\left(\frac{n}{h_{s-1}\widetilde{h_{s-1}}h_s} - \beta_0\right)a_s &= \frac{n}{h_s}B + \frac{n}{h_{s-1}\widetilde{h_{s-1}}h_s}a_{s-1}p_{s-1}(p_s-1)\\ &= \frac{n}{h_s}\left(B + a'_{s-1}p'_{s-1}(p_s-1)\right).\end{aligned}$$

Obviously, the right hand side is divisible by $n/h_s \neq 1$. Since $(n, a_s) = 1$ by assumption, this implies that n/h_s divides $n/(h_{s-1}\widetilde{h_{s-1}}h_s) - \beta_0$. But

$$0 < \frac{n}{h_{s-1}\widetilde{h_{s-1}}h_s} - \beta_0 \le \frac{n}{h_{s-1}\widetilde{h_{s-1}}h_s} \le \frac{n}{h_s}.$$

Recall that $p_s \geq 2$ by definition, so the right hand side must be greater than 0. Therefore, the only possibility is $n/h_s = n/(h_{s-1}\widetilde{h_{s-1}}h_s) - \beta_0$, i.e., $\beta_0 = 0$ and $h_{s-1}\widetilde{h_{s-1}} = 1$.

Step 3. The congruence conditions imply that $p_s = 2$.

From now on, we assume that $h_i\widetilde{h_i} = 1$ for $1 \leq i \leq s-1$, and $h_s = p_s$. We show that the congruence condition in Proposition 2.0.10 for a node v of type v_{s-1} in the direction of $\Delta_A(v)$ cannot hold unless $p_s = 2$. The only difficulty is in notation.

By Fact DM in §4.2,

$$D_{-}(v_k) = a_k, \text{ for } 2 \leq k \leq s,$$

and by Fact DA2,

$$D_A(v_k) = \frac{n}{p_s}\tilde{A}_k(a_s)^{p_s-2}, \text{ for } 1 \leq k \leq s-1,$$

where

$$\begin{aligned} \tilde{A}_{s-1} &= a_s - a_{s-1}p_{s-1}(p_s-1), && \text{and} \\ \tilde{A}_k &= a_s - a_k p_k p_{k+1}^2 \cdots p_{s-1}^2(p_s-1), && 1 \leq k \leq s-2. \end{aligned}$$

Suppose that $p_s > 2$. The appearance of the splice diagram $\Delta_{f,n}$ associated to the *minimal* good resolution graph $\Gamma^{min}(X_{f,n})$ is shown in Figure 6.6. (Recall that $p'_s = 1$ implies that there is no leaf of type $\overline{v_s}$, since that string completely collapses in the minimal resolution graph.)

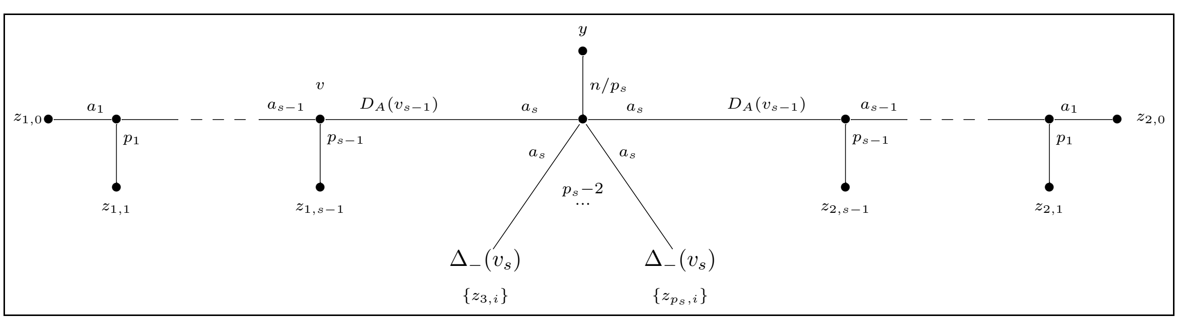


FIGURE 6.6. Splice diagram for $h_s = p_s$ and $h_i\widetilde{h_i} = 1$ for $1 \leq i \leq s-1$.

For each $i,\ 0 \le i \le s-1$, there are $h_s = p_s$ leaves of type $\overline{v_i}$. We label these leaves

$$\{z_{j,i} \ : \ 1 \le j \le p_s,\ 0 \le i \le s-1\},$$

as indicated in Figure 6.6. The leaf on the edge with weight n/p_s is denoted y, and is absent if $n/p_s = 1$. Let the corresponding variables for the splice diagram equations be $Z_{j,i}$, respectively, Y. Let G be an admissible monomial for v in the direction of $\Delta_A(v) = \Delta_A(v_{s-1})$. We know that the variable Y cannot appear in any admissible monomial G, by the argument in Step 2 ($\beta_0 = 0$). Therefore,

$$G = \prod_{j=2}^{p_s} (Z_{j,0})^{\alpha_{j,0}} \cdots (Z_{j,s-1})^{\alpha_{j,s-1}},$$

with all $\alpha_{j,k} \in \mathbb{N} \cup \{0\}$ and

$$D_A(v_{s-1}) = \sum_{k=0}^{s-1} \ell'_{vz_{2,k}} \left(\sum_{j=2}^{p_s} \alpha_{j,k} \right). \tag{6.4}$$

(It is clear that $\ell'_{vz_{j,k}} = \ell'_{vz_{2,k}}$ for all $j \neq 1$.)

For convenience of notation, we make the following definition:

$$\mathrm{m}(i) := \begin{cases} p_1 \cdots p_{s-1} & \text{for } i = 0 \\ a_i p_{i+1} \cdots p_{s-1} & \text{for } 1 \le i \le s-2 \\ a_{s-1} & \text{for } i = s-1. \end{cases}$$

Note that by Claim 5.1.2,

$$\mathrm{m}(i) = \ell'_{v_s z_{j,i}}$$

for any j, where v_s denotes the unique node of type v_s (the central node). It is clear from Figure 6.6 that

$$\begin{aligned} \ell'_{vz_{2,i}} &= (n/p_s)(a_s)^{p_s-2}\mathrm{m}(i), \text{ and} \\ \ell_{vz_{2,i}} &= (n/p_s)(a_s)^{p_s-2}\mathrm{m}(i) a_{s-1} p_{s-1}, \end{aligned}$$

for $1 \leq i \leq s-1$. Therefore, Equation (6.4) is equivalent to

$$D_A(v_{s-1}) = (n/p_s)(a_s)^{p_s-2} \sum_{k=0}^{s-1} \mathrm{m}(k) \left(\sum_{j=2}^{p_s} \alpha_{j,k} \right),$$

that is,

$$\tilde{A}_{s-1} = \sum_{k=0}^{s-1} \mathrm{m}(k) \left(\sum_{j=2}^{p_s} \alpha_{j,k} \right). \tag{6.5}$$

Let us consider the congruence condition in Proposition 2.0.10 for the node v in the direction of $\Delta_A(v)$ for each of the leaves $z_{2,i}$, $0 \leq i \leq s-1$. Note that

$$\ell_{z_{j,k} z_{2,i}} = \ell_{z_{1,k} z_{2,i}} \text{ for any } j \neq 2.$$

Thus the congruence condition for the leaf $w' = z_{2,i}$ for any admissible monomial G is equivalent to

$$\left[\sum_{k=0}^{s-1} \left(\sum_{j=3}^{p_s} \alpha_{j,k} \right) \frac{\ell_{z_{1,k} z_{2,i}}}{\det(\Gamma_{f,n})} + \sum_{k \neq i} \alpha_{2,k} \frac{\ell_{z_{2,k} z_{2,i}}}{\det(\Gamma_{f,n})} - \alpha_{2,i} e_{z_{2,i}} \cdot e_{z_{2,i}} \right] = \left[\frac{\ell_{v z_{2,i}}}{\det(\Gamma_{f,n})} \right]. \tag{6.6}$$

By Lemma 4.2.6,

$$\det(\Gamma_{f,n}) = (a_s)^{p_s-1}.$$

For $0 \leq i \leq s-1$,

$$\frac{\ell_{v z_{2,i}}}{\det(\Gamma_{f,n})} = \frac{(n/p_s) a_{s-1} p_{s-1} \mathrm{m}(i)}{a_s}. \tag{6.7}$$

Furthermore, it is clear that

$$\ell_{z_{1,k} z_{2,i}} = \frac{n}{p_s} (a_s)^{p_s-2} \mathrm{m}(i) \mathrm{m}(k), \text{ for } 0 \leq k, i \leq s-1. \tag{6.8}$$

Claim 6.2.3. *Fix i such that $0 \leq i \leq s-1$. Then*

- *For $0 \leq k \leq s-1$ and $k \neq i$,* $\left[\frac{\ell_{z_{2,k} z_{2,i}}}{\det(\Gamma_{f,n})} \right] = \left[\frac{-(n/p_s)\mathrm{m}(i)\mathrm{m}(k)(p_s-1)}{a_s} \right]$, *and*
- $[e_{z_{2,i}} \cdot e_{z_{2,i}}] = \left[\frac{(n/p_s)(\mathrm{m}(i))^2(p_s-1)}{a_s} \right]$.

Let us assume for now that the Claim is true and finish the proof of Step 3. By Equation (6.8) and the Claim, we can simplify the left side of Equation (6.6) as follows:

$$\begin{aligned}
\text{Left side of (6.6)} &= \left[\sum_{k=0}^{s-1}\left(\sum_{j=3}^{p_s}\alpha_{j,k}\right)\frac{\frac{n}{p_s}\mathrm{m}(i)\mathrm{m}(k)}{a_s} - \mathrm{m}(i)(p_s-1)\sum_{k=0}^{s-1}\alpha_{2,k}\frac{\frac{n}{p_s}\mathrm{m}(k)}{a_s}\right] \\
&= \left[\frac{(n/p_s)\mathrm{m}(i)}{a_s}\left\{\sum_{k=0}^{s-1}\left(\sum_{j=2}^{p_s}\alpha_{j,k}\right)\mathrm{m}(k) - p_s\sum_{k=0}^{s-1}\alpha_{2,k}\mathrm{m}(k)\right\}\right] \\
&= \left[\frac{(n/p_s)\mathrm{m}(i)}{a_s}\left\{\tilde{A}_{s-1} - p_s\sum_{k=0}^{s-1}\alpha_{2,k}\mathrm{m}(k)\right\}\right] \quad \text{(by (6.5))} \\
&= \left[\frac{(n/p_s)\mathrm{m}(i)}{a_s}\left\{a_s - a_{s-1}p_{s-1}(p_s-1) - p_s\sum_{k=0}^{s-1}\alpha_{2,k}\mathrm{m}(k)\right\}\right] \\
&= \left[\frac{(n/p_s)\mathrm{m}(i)}{a_s}\left\{-a_{s-1}p_{s-1}(p_s-1) - p_s\sum_{k=0}^{s-1}\alpha_{2,k}\mathrm{m}(k)\right\}\right].
\end{aligned}$$

Therefore, the congruence condition (6.6) is equivalent to

$$\left[\frac{(n/p_s)\mathrm{m}(i)}{a_s}\left\{-a_{s-1}p_{s-1}(p_s-1) - p_s\sum_{k=0}^{s-1}\alpha_{2,k}\mathrm{m}(k)\right\}\right] = \left[\frac{(n/p_s)a_{s-1}p_{s-1}\mathrm{m}(i)}{a_s}\right],$$

which is clearly equivalent to

$$\left[-\frac{(n/p_s)\mathrm{m}(i)p_s}{a_s}\left(a_{s-1}p_{s-1} + \sum_{k=0}^{s-1}\alpha_{2,k}\mathrm{m}(k)\right)\right] = [0].$$

Since $(a_s, n) = 1$, this is equivalent to

$$\text{(6.9)} \qquad \mathrm{m}(i)\left(a_{s-1}p_{s-1} + \sum_{k=0}^{s-1}\alpha_{2,k}\mathrm{m}(k)\right) \in \mathbb{Z}a_s.$$

Therefore, if the congruence conditions are satisfied, that implies, in particular, that (6.9) holds for all i such that $0 \leq i \leq s-1$. We must show that this is impossible.

First of all, we claim that if (6.9) holds for all i, this implies that a_s divides

$$\mathrm{S} := a_{s-1}p_{s-1} + \sum_{k=0}^{s-1}\alpha_{2,k}\mathrm{m}(k).$$

For, suppose there exists a prime factor q of a_s that does not divide S. Then q must divide m(i) for $0 \leq i \leq s-1$. In particular, q divides $\mathrm{m}(s-1) = a_{s-1}$, and since $(a_{s-1}, p_{s-1}) = 1$, this implies that q divides a_{s-2}, because $\mathrm{m}(s-2) = a_{s-2}p_{s-1}$. This, in turn, implies q divides a_{s-3}, and so forth, down to a_1. But $\mathrm{m}(0) = p_1 \cdots p_{s-1}$, which cannot possibly be divisible by q. Therefore, q divides S, and thus a_s divides S.

Finally, we claim that for $p_s > 2$, it is impossible for a_s to divide S, and therefore the congruence conditions cannot possibly be satisfied. Recall Equation (6.5), which is equivalent to

$$a_s - a_{s-1}p_{s-1}(p_s - 1) = \sum_{k=0}^{s-1} \mathrm{m}(k) \left(\sum_{j=2}^{p_s} \alpha_{j,k} \right).$$

This implies that

$$\sum_{k=0}^{s-1} \alpha_{2,k}\mathrm{m}(k) \leq a_s - a_{s-1}p_{s-1}(p_s - 1),$$

and hence

$$\mathrm{S} = a_{s-1}p_{s-1} + \sum_{k=0}^{s-1} \alpha_{2,k}\mathrm{m}(k) \leq a_s - a_{s-1}p_{s-1}(p_s - 2).$$

But, if $p_s > 2$, $a_s - a_{s-1}p_{s-1}(p_s - 2) < a_s$, which implies that $\mathrm{S} < a_s$, and hence S cannot be divisible by a_s. Therefore, we must have $p_s = 2$ for the congruence conditions to be satisfied.

Proof of Claim 6.2.3. We begin with the second statement. Recall that for $1 \leq i \leq s-2$,

$$\begin{aligned} A_i &= a_{i+1} \cdots a_{s-1}\tilde{A}_i, \text{ and} \\ \tilde{A}_i &= a_s - a_i p_i p_{i+1}^2 \cdots p_{s-1}^2 (p_s - 1), \end{aligned}$$

where A_i is as defined in Lemma 4.2.4. Since $z_{2,i}$ is a leaf of type $\overline{v_i}$, Corollary 5.2.2 implies that, for $1 \leq i \leq s-2$,

$$\begin{aligned}
[e_{z_{2,i}} \cdot e_{z_{2,i}}] &= \left[\frac{(n/p_s)(a_i a_s - a_i \tilde{A}_i)}{p_i a_s}\right] \\
&= \left[\frac{(n/p_s) a_i^2 p_{i+1}^2 \cdots p_{s-1}^2 (p_s - 1)}{a_s}\right] \\
&= \left[\frac{(n/p_s)(\mathrm{m}(i))^2 (p_s - 1)}{a_s}\right].
\end{aligned}$$

The equality is just as easily obtained for $i = 0$ and $i = s-1$.

It remains to show that

$$\left[\frac{\ell_{z_{2,k} z_{2,i}}}{\det(\Gamma_{f,n})}\right] = \left[\frac{-(n/p_s)\mathrm{m}(i)\mathrm{m}(k)(p_s - 1)}{a_s}\right],$$

for $0 \leq k \leq s-1$ and $k \neq i$. Without loss of generality, we can assume $i < k$. For $1 \leq i < k \leq s-2,\ i \neq k-1$, we have $\ell_{z_{2,k} z_{2,i}} = D_A(v_k) a_i p_{i+1} \cdots p_{k-1}$, and hence,

$$\begin{aligned}
\left[\frac{\ell_{z_{2,k} z_{2,i}}}{\det(\Gamma_{f,n})}\right] &= \left[\frac{(n/p_s)\tilde{A}_k a_i p_{i+1} \cdots p_{k-1}}{a_s}\right] \\
&= \left[\frac{(n/p_s)(a_s - a_k p_k p_{k+1}^2 \cdots p_{s-1}^2 (p_s - 1)) a_i p_{i+1} \cdots p_{k-1}}{a_s}\right] \\
&= \left[\frac{-(n/p_s)(p_s - 1) a_k p_k p_{k+1}^2 \cdots p_{s-1}^2 \cdot a_i p_{i+1} \cdots p_{k-1}}{a_s}\right] \\
&= \left[\frac{-(n/p_s)(p_s - 1) a_k p_{k+1} \cdots p_{s-1} \cdot a_i p_{i+1} \cdots p_{k-1} p_k p_{k+1} \cdots p_{s-1}}{a_s}\right] \\
&= \left[\frac{-(n/p_s)(p_s - 1)\mathrm{m}(k)\mathrm{m}(i)}{a_s}\right].
\end{aligned}$$

The remaining cases are all similar and easy to check. □

Note that the proof of Step 3 is valid whether or not $n/p_s = 1$.

Step 4. The congruence conditions imply that $s = 2$.

So far, we have that the semigroup and congruence conditions imply that $h_s = p_s = 2$, $h_i \widetilde{h_i} = 1$ for $1 \leq i \leq s-1$, and $n = 2n'$. Here, we assume $n' > 1$; the pathological case

$n' = 1$ is treated separately below. We will show that for $s \geq 3$, the congruence conditions at a node v of type v_{s-2} in the direction of $\Delta_A(v)$ cannot possibly be satisfied. Clearly, if $s = 2$, there is no such node. We should note that the congruence condition at a node of type v_{s-1} that we studied in Step 3 could be satisfied for $s \geq 3$. For example, take

$$a_1 = 3, \quad a_2 = 19, \quad a_3 = 117,$$
$$p_1 = 2, \quad p_2 = 3, \quad p_3 = 2,$$

and any $n = 2n'$ such that n' is relatively prime to 2, 3, 19, and 117.

Figure 6.7 depicts the splice diagram in the general situation. Recall the definition

$$\mathrm{m}(i) = \begin{cases} p_1 \cdots p_{s-1} & \text{for } i = 0 \\ a_i p_{i+1} \cdots p_{s-1} & \text{for } 1 \leq i \leq s-2 \\ a_{s-1} & \text{for } i = s-1. \end{cases}$$

The semigroup condition at v in the direction of $\Delta_A(v)$ is

$$D_A(v_{s-2}) \in \mathbb{N}\langle D_A(v_{s-1}),\ a_s p_{s-1},\ n' p_{s-1}\mathrm{m}(i),\ 0 \leq i \leq s-1\rangle.$$

Recall that

$$\begin{aligned} D_A(v_{s-1}) &= n'(a_s - a_{s-1}p_{s-1}), \text{ and} \\ D_A(v_k) &= n'(a_s - a_k p_k p_{k+1}^2 \cdots p_{s-1}^2), \text{ for } 1 \leq k \leq s-2. \end{aligned}$$

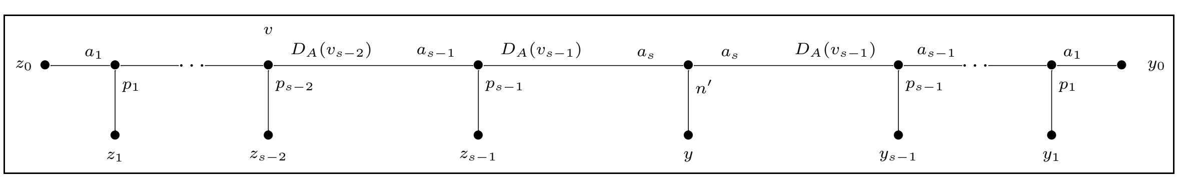


FIGURE 6.7. Splice diagram for $n > 2$, $h_s = p_s = 2$, and $h_i\widetilde{h_i} = 1$ for $1 \leq i \leq s-1$.

We claim that $D_A(v_{s-1})$ and $a_s p_{s-1}$ cannot appear in the expression for $D_A(v_{s-2})$ that comes from the semigroup condition. Suppose

$$n'(a_s - a_{s-2}p_{s-2}p_{s-1}^2) = \alpha n'(a_s - a_{s-1}p_{s-1}) + \beta a_s p_{s-1} + \sum_{i=0}^{s-1} \gamma_i n' \mathrm{m}(i) p_{s-1},$$

with $\alpha,\ \beta,\ \gamma_i \in \mathbb{N} \cup \{0\}$. If $\beta \neq 0$, then $\beta a_s p_{s-1}$ must be divisible by $n' > 1$. By assumption, $(a_s, n') = 1$ since $\widetilde{h}_s = 1$, and $(p_{s-1}, n') = h_{s-1} = 1$, and hence n' must divide β. But then $\beta a_s p_{s-1} \geq n' a_s p_{s-1} > n' a_s > n'(a_s - a_{s-2}p_{s-2}p_{s-1}^2) = D_A(v_{s-2})$, and this is impossible. Therefore, $\beta = 0$.

Hence, we can cancel n' from the equation above, leaving

$$a_s - a_{s-2}p_{s-2}p_{s-1}^2 = \alpha(a_s - a_{s-1}p_{s-1}) + \sum_{i=0}^{s-1} \gamma_i \mathrm{m}(i) p_{s-1}.$$

Since $\mathrm{m}(s-1) = a_{s-1}$, we have

$$(\alpha - \gamma_{s-1}) a_{s-1} p_{s-1} = (\alpha - 1) a_s + \sum_{i=0}^{s-2} \gamma_i \mathrm{m}(i) p_{s-1} + a_{s-2}p_{s-2}p_{s-1}^2,$$

which implies $(\alpha - \gamma_{s-1}) a_{s-1} p_{s-1} > (\alpha - 1) a_s$. Now suppose $\alpha > 1$. Then

$$(\alpha - \gamma_{s-1}) a_{s-1} p_{s-1} > (\alpha - 1) a_s > (\alpha - 1) 2 a_{s-1} p_{s-1}$$

(since $a_s = q_s + a_{s-1}p_{s-1}p_s$ and $p_s = 2$.) This implies $(\alpha - \gamma_{s-1}) - 2(\alpha - 1) > 0$, i.e.,

$$2 > \alpha + \gamma_{s-1}.$$

But this is impossible for $\alpha > 1$.

Now suppose $\alpha = 1$. Clearly, γ_{s-1} must be 0, and so we have

$$a_{s-1} p_{s-1} = \sum_{i=0}^{s-2} \gamma_i \mathrm{m}(i) p_{s-1} + a_{s-2}p_{s-2}p_{s-1}^2,$$

i.e.,

$$a_{s-1} = \sum_{i=0}^{s-2} \gamma_i \mathrm{m}(i) + a_{s-2}p_{s-2}p_{s-1}.$$

But $\mathrm{m}(i)$ is divisible by p_{s-1} for $0 \leq i \leq s-2$, so this would imply a_{s-1} is divisible by p_{s-1}, which is a contradiction. Therefore, $\alpha = 0$, and we have

$$a_s - a_{s-2}p_{s-2}p_{s-1}^2 = \sum_{i=0}^{s-1} \gamma_i \mathrm{m}(i) p_{s-1}. \tag{6.10}$$

(Note that this semigroup condition is already quite restrictive, because it requires a_s to be divisible by p_{s-1}.)

Now let us return to the congruence conditions for the node v in the direction of $\Delta_A(v) = \Delta_A(v_{s-2})$. An admissible monomial for v in that direction must be of the form

$$H = Y_0^{\gamma_0} \cdots Y_{s-1}^{\gamma_{s-1}},$$

for some $\gamma_i \in \mathbb{N} \cup \{0\}$. The congruence condition for the leaf y_{s-1} is

$$\left[\frac{\ell_{vy_{s-1}}}{\det(\Gamma_{f,n})}\right] = \left[\sum_{i=0}^{s-2} \gamma_i \frac{\ell_{y_{s-1}y_i}}{\det(\Gamma_{f,n})} - \gamma_{s-1} e_{y_{s-1}} \cdot e_{y_{s-1}}\right].$$

We have

$$\left[\frac{\ell_{vy_{s-1}}}{\det(\Gamma_{f,n})}\right] = \left[\frac{n' a_{s-2}p_{s-2}a_{s-1}p_{s-1}}{a_s}\right],$$

and, since Claim 6.2.3 is still valid in this setting,

$$\begin{aligned} \left[\frac{\ell_{y_{s-1}y_i}}{\det(\Gamma_{f,n})}\right] &= \left[\frac{-n'\mathrm{m}(s-1)\mathrm{m}(i)}{a_s}\right], \text{ for } 0 \leq i \leq s-2, \text{ and} \\ \left[e_{y_{s-1}} \cdot e_{y_{s-1}}\right] &= \left[\frac{n'(\mathrm{m}(s-1))^2}{a_s}\right]. \end{aligned}$$

Thus the congruence condition is equivalent to

$$\left[\frac{n' a_{s-2}p_{s-2}a_{s-1}p_{s-1}}{a_s}\right] = \left[-\frac{n' a_{s-1}}{a_s}\left(\sum_{i=0}^{s-1} \gamma_i \mathrm{m}(i)\right)\right],$$

that is,

$$n' a_{s-1}\left(a_{s-2}p_{s-2}p_{s-1} + \sum_{i=0}^{s-1} \gamma_i \mathrm{m}(i)\right) \in \mathbb{Z} a_s.$$

Since $(a_s, n') = 1$, we must have

$$a_{s-1}\left(a_{s-2}p_{s-2}p_{s-1} + \sum_{i=0}^{s-1}\gamma_i \mathrm{m}(i)\right) = Na_s$$

for some N in $\mathbb{Z}$. If we multiply both sides of this equation by p_{s-1} and apply Equation (6.10), we get

$$a_{s-1}a_{s-2}p_{s-2}p_{s-1}^2 + a_{s-1}(a_s - a_{s-2}p_{s-2}p_{s-1}^2) = Na_s p_{s-1},$$

i.e., $a_{s-1} = Np_{s-1}$. This implies p_{s-1} divides a_{s-1}, which is a contradiction.

Therefore, we have shown that if $s \geq 3$, then the congruence condition for the node v of type v_{s-2} in the direction of $\Delta_A(v)$ cannot be satisfied for the leaf y_{s-1}. Hence, the congruence conditions imply that $s = 2$. This concludes the proof (except for in the pathological case) that $\Gamma_{f,n}$ satisfies the semigroup and congruence conditions implies $(*)$.

Finally, we show that $(*)$ implies that the semigroup and congruence conditions are satisfied. Let $n = 2n'$ for $n' > 1$. We have $D_-(v_2) = a_2$ and

$$D_A(v_1) = n'(a_2 - a_1p_1) = n'(q_2 + a_1p_1).$$

The splice diagram in this situation is shown in Figure 6.8. The only semigroup condition that needs to be checked is

$$D_A(v_1) \in \mathbb{N}\langle a_2,\ n'a_1,\ n'p_1\rangle.$$

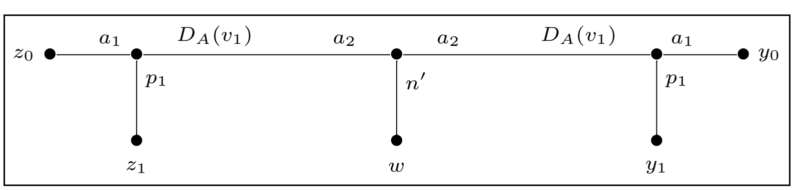


FIGURE 6.8. Splice diagram for $(*)$, $n > 2$.

Since a_1 and p_1 are relatively prime, the conductor of the semigroup generated by a_1 and p_1 is greater than a_1p_1, hence $a_1p_1 + q_2$ is in the semigroup generated by a_1 and p_1, and therefore this semigroup condition is satisfied. It is easy to argue that a_2 cannot occur in the semigroup representation of $D_A(v_1)$ (see the proof of Step 4 above).

Now we must check that the congruence conditions are satisfied. We need only check them for the central node and the left-most node, since the conditions for the right node will be the same as for the left. Recall that by Lemma 4.2.6,

$$\det(\Gamma_{f,n}) = a_2.$$

By Corollary 5.2.2, we have

$$[e_{z_0} \cdot e_{z_0}] = \left[\frac{n'(p_1a_2 - (a_2 - a_1p_1)p_1)}{a_1a_2}\right] = \left[\frac{n'p_1^2}{a_2}\right],$$

$$[e_{z_1} \cdot e_{z_1}] = \left[\frac{n'(a_1 - (a_2 - a_1p_1)a_1)}{p_1a_2}\right] = \left[\frac{n'a_1^2}{a_2}\right].$$

We also have

$$\left[\frac{\ell_{z_0z_1}}{\det(\Gamma_{f,n})}\right] = \left[\frac{n'(a_2 - a_1p_1)}{a_2}\right] = \left[\frac{-n'a_1p_1}{a_2}\right].$$

The linking of number of the central node with any leaf is divisible by a_2, so for the central node, the congruence conditions are equivalent to the following: there exist α_0 and α_1 in $\mathbb{N} \cup \{0\}$ such that $a_2 = \alpha_0p_1 + \alpha_1a_1$,

$$\left[\alpha_1\frac{-n'a_1p_1}{a_2} - \alpha_0\frac{n'p_1^2}{a_2}\right] = [0], \text{ and}$$

$$\left[\alpha_0\frac{-n'a_1p_1}{a_2} - \alpha_1\frac{n'a_1^2}{a_2}\right] = [0].$$

But these conditions are obviously both satisfied for any $\alpha_0,\ \alpha_1$ such that $a_2 = \alpha_0p_1 + \alpha_1a_1$.

Similarly, for the left-most node v, we know that there exist β_0 and β_1 such that $\beta_0 p_1 + \beta_1 a_1 = a_2 - a_1 p_1$. The congruence condition for the leaf w is trivially true, so the congruence conditions for the left node are equivalent to

$$\begin{aligned} \left[\beta_1 \frac{-n' a_1 p_1}{a_2} - \beta_0 \frac{n' p_1^2}{a_2}\right] &= \left[\frac{n' a_1 p_1^2}{a_2}\right], \text{ and} \\ \left[\beta_0 \frac{-n' a_1 p_1}{a_2} - \beta_1 \frac{n' a_1^2}{a_2}\right] &= \left[\frac{n' p_1 a_1^2}{a_2}\right]. \end{aligned}$$

These conditions are also both satisfied for any such β_0 and β_1.

Therefore, aside from the pathological case, we have finished the proof of the Main Theorem 6.0.1.

The pathological case.

For the pathological case $n = p_s = 2$, Steps 1 through 3 from the proof of Proposition 6.2.1 are automatically true. We have only to prove that if $\Gamma_{f,n}$ satisfies the semigroup and congruence conditions, then s must be 2. The splice diagram is pictured in Figure 6.9. We can use essentially the same argument as in Step 4 above to show that for $s \geq 3$, the congruence conditions at the node v of type v_{s-2} in the direction of $\Delta_A(v)$ cannot possibly be satisfied for the leaf y_{s-1}.

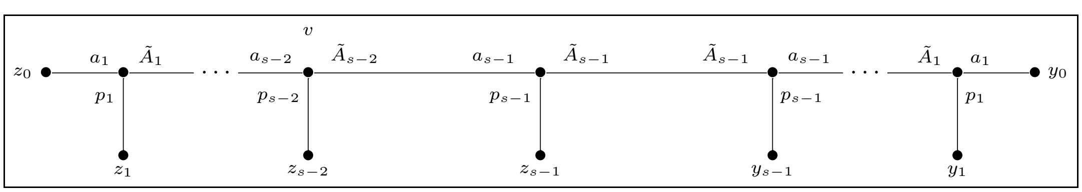


FIGURE 6.9. Splice diagram for the pathological case, $s > 2$.

Recall the definition

$$\mathrm{m}(i) = \begin{cases} p_1 \cdots p_{s-1} & \text{for } i = 0 \\ a_i p_{i+1} \cdots p_{s-1} & \text{for } 1 \leq i \leq s-2 \\ a_{s-1} & \text{for } i = s-1. \end{cases}$$

The semigroup condition at v in the direction of $\Delta_A(v)$ is

$$\tilde{A}_{s-2} \in \mathbb{N}\langle \tilde{A}_{s-1},\ p_{s-1}\mathrm{m}(i),\ 0 \leq i \leq s-1\rangle.$$

Recall that

$$\begin{aligned} \tilde{A}_{s-1} &= a_s - a_{s-1}p_{s-1}, \text{ and} \\ \tilde{A}_k &= a_s - a_k p_k p_{k+1}^2 \cdots p_{s-1}^2, \text{ for } 1 \leq k \leq s-2 \end{aligned}$$

(see the end of §5.1). We claim that $\tilde{A}_{s-1}$ cannot appear in the expression for $\tilde{A}_{s-2}$ that comes from the semigroup condition. We have

$$a_s - a_{s-2}p_{s-2}p_{s-1}^2 = \alpha(a_s - a_{s-1}p_{s-1}) + \sum_{i=0}^{s-1} \gamma_i \mathrm{m}(i) p_{s-1}.$$

Precisely the same argument as in Step 4 above shows that $\alpha = 0$, and we have

$$a_s - a_{s-2}p_{s-2}p_{s-1}^2 = \sum_{i=0}^{s-1} \gamma_i \mathrm{m}(i) p_{s-1}. \tag{6.11}$$

Now let us return to the congruence conditions for the node v in the direction of $\Delta_A(v)$. Suppose the semigroup conditions are satisfied, and let $H = Y_0^{\gamma_0} \cdots Y_{s-1}^{\gamma_{s-1}}$ be an admissible monomial for v in that direction. The congruence condition for the leaf y_{s-1} is

$$\left[\frac{\ell_{vy_{s-1}}}{\det(\Gamma_{f,n})}\right] = \left[\sum_{i=0}^{s-2} \gamma_i \frac{\ell_{y_{s-1}y_i}}{\det(\Gamma_{f,n})} - \gamma_{s-1} e_{y_{s-1}} \cdot e_{y_{s-1}}\right].$$

Clearly,

$$\left[\frac{\ell_{vy_{s-1}}}{\det(\Gamma_{f,n})}\right] = \left[\frac{a_{s-2}p_{s-2}a_{s-1}p_{s-1}}{a_s}\right],$$

and, for $0 \le i \le s-2$,

$$\begin{aligned}\left[\frac{\ell_{y_{s-1}y_i}}{\det(\Gamma_{f,n})}\right] &= \left[\frac{\tilde{A}_{s-1}\mathrm{m}(i)/p_{s-1}}{a_s}\right], \\ &= \left[\frac{-a_{s-1}p_{s-1}\mathrm{m}(i)/p_{s-1}}{a_s}\right] \\ &= \left[\frac{-a_{s-1}\mathrm{m}(i)}{a_s}\right].\end{aligned}$$

Corollary 5.2.2 holds for $0 \le k \le s-1$ in the pathological case, so

$$\begin{aligned}\left[e_{y_{s-1}} \cdot e_{y_{s-1}}\right] &= \left[\frac{a_{s-1}a_s - \tilde{A}_{s-1}a_{s-1}}{p_{s-1}a_s}\right] \text{ (since } A_{s-1} = \tilde{A}_{s-1}) \\ &= \left[\frac{a_{s-1}[a_{s-1}p_{s-1}]}{p_{s-1}a_s}\right] \\ &= \left[\frac{a_{s-1}^2}{a_s}\right].\end{aligned}$$

Thus the congruence condition is equivalent to

$$\left[\frac{a_{s-2}p_{s-2}a_{s-1}p_{s-1}}{a_s}\right] = \left[-\frac{a_{s-1}}{a_s}\left(\sum_{i=0}^{s-1}\gamma_i \mathrm{m}(i)\right)\right],$$

that is,

$$a_{s-1}\left(a_{s-2}p_{s-2}p_{s-1} + \sum_{i=0}^{s-1}\gamma_i \mathrm{m}(i)\right) = Na_s$$

for some N in $\mathbb{Z}$. If we multiply both sides of this equation by p_{s-1} and apply Equation (6.11), we get

$$a_{s-1}a_s = Na_s p_{s-1},$$

i.e., $a_{s-1} = Np_{s-1}$. This implies p_{s-1} divides a_{s-1}, which is a contradiction. Therefore, the congruence conditions cannot be satisfied for $s > 2$.

It only remains to check that the semigroup and congruence conditions are satisfied for $s = 2$. The splice diagram is shown in Figure 6.10.

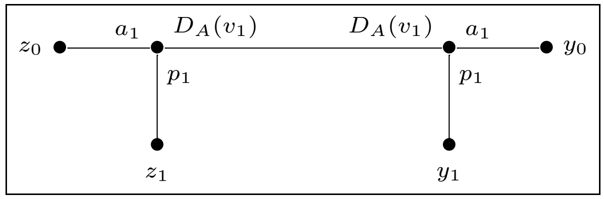


FIGURE 6.10. Splice diagram for the pathological case, $s = 2$.

Here, $D_A(v_1) = a_2 - a_1 p_1$, so the semigroup conditions are satisfied, as discussed above. It is easy to check that the congruence conditions are also satisfied.

CHAPTER 7

The splice quotients

Theorem 6.0.1 characterizes the $(X_{f,n}, 0) = (\{z^n = f(x, y)\}, 0)$, with f irreducible, that have the topological type of a splice quotient. That is, the Theorem characterizes those for which the resolution graph $\Gamma_{f,n}$ satisfies the semigroup and congruence conditions, and hence those for which there exist splice quotients with resolution graph $\Gamma_{f,n}$. In this chapter, we study the analytic types of the splice quotients that result from the Neumann-Wahl algorithm. Recall that the Neumann-Wahl algorithm produces a *family* of singularities via the splice diagram equations, and thus, upon taking the quotient by the action of the discriminant group, a family of splice quotients with resolution graph $\Gamma_{f,n}$. It is important to understand that if one begins with an arbitrary analytic type of $(X_{f,n}, 0)$, there need not be a splice quotient of that analytic type. By considering specific analytic types of the splice equations, we show that in both cases, there exist splice quotients defined by an equation of the form $z^n = g(x, y)$, which is not clear a priori.

In each case, we begin with the splice diagram $\Delta_{f,n}$ and write down a set of splice diagram equations that (automatically) defines an isolated complete intersection singularity $(Y, 0)$, and compute the action of the discriminant group $D(\Gamma_{f,n})$ on $(Y, 0)$, as described in Chapter 2. We then compute the quotient of $(Y, 0)$ under the group action; the resulting singularities are splice quotients with resolution graph $\Gamma_{f,n}$. We must choose the equations for Y carefully in order to end up with quotients of the form $\{z^n = g(x, y)\}$. In each case, we also give some concrete examples.

7.1. Case (i) $(n, a_s) \neq 1$

The goal of this section is to prove the following

THEOREM 7.1.1. *Let n be an integer greater than 1, and suppose that $f(x,y)=0$ defines an irreducible plane curve singularity with topological pairs $\{(p_i,a_i)\ :\ 1 \le i \le s\}$ that satisfy the following conditions: $(n,p_s)=1,\ (n,p_i)=(n,a_i)=1$ for $1 \le i \le s-1$, and $a_s/(n,a_s)$ is in the semigroup generated by $\{a_{s-1},\ p_1\cdots p_{s-1},\ a_j p_{j+1}\cdots p_{s-1}\ :\ 1 \le j \le s-2\}$. Then there exists a splice quotient that is defined by an equation of the form $z^n = g(x,y)$, where g is irreducible and has topological pairs $\{(p_i,a_i)\ :\ 1 \le i \le s\}$.*

For any $(X_{f,n},0)$ that satisfies these conditions, the link is a $\mathbb{Q}$HS and the resolution graph $\Gamma_{f,n}$ satisfies the semigroup and congruence conditions (Case (i) of Theorem 6.0.1). In the case that the link of $(X_{f,n},0)$ is a $\mathbb{Z}$HS (n relatively prime to all p_i and a_i), the splice equations were studied by Neumann and Wahl in [**19**] (see below for some details). Therefore, we assume that $\widetilde{h}_s = (n,a_s) \neq 1$.

The cases $n/\widetilde{h}_s \neq 1$ and $n/\widetilde{h}_s = 1$ are different enough to warrant separate treatment.

$n/\widetilde{h}_s \neq 1$. We must recall the notation used in the proof of Proposition 6.1.1. The splice diagram $\Delta_{f,n}$ for Case (i) is reproduced in Figure 7.1. Recall that e_{z_k}, respectively $e_{u_j},\ e_w$, denotes the image in the discriminant group $D(\Gamma_{f,n})$ of the dual basis element associated to the leaf z_k, respectively $u_j,\ w$ (see Chapter 2). The capital letters $Z_k,\ U_j$, and W are the

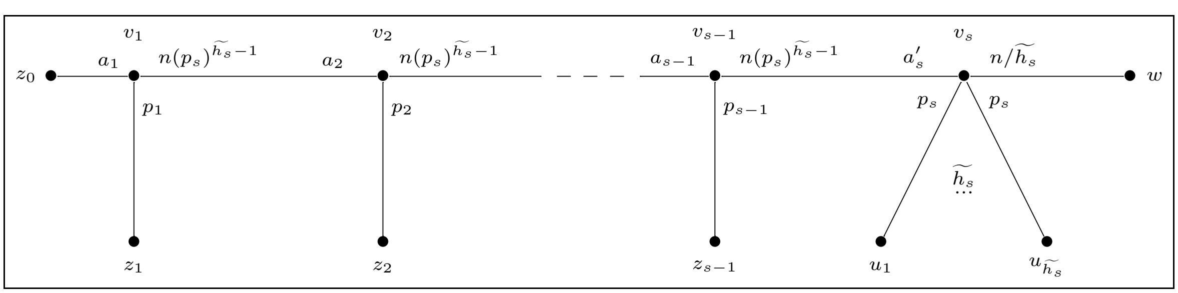


FIGURE 7.1. Splice diagram for Case (i), $n/\widetilde{h}_s \neq 1$.

variables for the splice equations associated to the corresponding leaves. Recall that $D(\Gamma_{f,n})$ has order $(p_s)^{\widetilde{h_s}-1}$.

It is easy to check (refer to the analysis of the congruence conditions in §6.1) that the discriminant group elements e_{z_k} act trivially on the all of the variables. Therefore, the action of $D(\Gamma_{f,n})$ is generated by the elements e_{u_j}, $1 \leq j \leq \widetilde{h_s}$. This action is given as follows:

$$
\begin{cases}
e_{u_j} \cdot U_i &= \left[\frac{(n/\widetilde{h_s})a'_s}{p_s}\right] U_i, \qquad\qquad i \neq j, \\
e_{u_j} \cdot U_j &= \left[-\frac{(n/\widetilde{h_s})(a_s - a'_s)}{p_s}\right] U_j, \\
e_{u_j} \cdot Z_k &= Z_k, \qquad\qquad\qquad\qquad 0 \leq k \leq s-1, \\
e_{u_j} \cdot W &= W.
\end{cases} \tag{7.1}
$$

In §6.1, we determined admissible monomials that satisfy the congruence conditions for every node and adjacent edge. The discriminant group acts trivially on every such admissible monomial. For $2 \leq k \leq s$, let

$$M_k(Z_0, \ldots, Z_{k-1}) = \prod_{j=0}^{k-1} Z_j^{\alpha_{j,k}}$$

be an admissible monomial for the node v_k at the edge leading towards $\Delta_-(v_k)$ (i.e., in the direction of the leaf z_0). Since all Z_k are invariant under the action of the discriminant group, we can choose any admissible monomial for the splice equations. For $1 \leq k \leq s-2$, there is only one admissible monomial at v_k in the direction of $\Delta_A(v_k)$, namely Z_{k+1}. Finally, there is only one choice of admissible monomial for v_{s-1} in the direction of $\Delta_A(v_{s-1})$ that will also satisfy the congruence conditions, and that is $U_1 \cdots U_{\widetilde{h_s}}$.

For the sake of simplicity, we will write h instead of $\widetilde{h_s}$ from now on.

The following is a set of splice equations for $\Delta_{f,n}$, which define an isolated complete intersection singularity $(Y, 0)$:

$$
(7.2) \qquad \left\{ \begin{array}{rcl}
Z_0^{a_1} + b_1 Z_1^{p_1} + c_1 Z_2 + H_1 & = & 0 \\
M_2(Z_0, Z_1) + b_2 Z_2^{p_2} + c_2 Z_3 + H_2 & = & 0 \\
\vdots & & \vdots \\
M_{s-2}(Z_0, \ldots, Z_{s-3}) + b_{s-2} Z_{s-2}^{p_{s-2}} + c_{s-2} Z_{s-1} + H_{s-2} & = & 0 \\
M_{s-1}(Z_0, \ldots, Z_{s-2}) + b_{s-1} Z_{s-1}^{p_{s-1}} + c_{s-1} U_1 \cdots U_h + H_{s-1} & = & 0 \\
U_1^{p_s} + d_1 M_s(Z_0, \cdots Z_{s-1}) + f_1 W^{n/h} + G_1 & = & 0 \\
\vdots & & \vdots \\
U_h^{p_s} + d_h M_s(Z_0, \cdots Z_{s-1}) + f_h W^{n/h} + G_h & = & 0
\end{array} \right\}
$$

where b_k and c_k $(1 \leq k \leq s-1)$ are nonzero complex numbers; d_j and f_j $(1 \leq j \leq h)$ are nonzero complex numbers such that $d_i f_j - d_j f_i \neq 0$ for all $i \neq j$; each H_k $(1 \leq k \leq s-1)$ is a convergent power series with all terms having v_k-weight greater than or equal to $a_k p_k n (p_s)^{h-1}$; and each G_j $(1 \leq j \leq h)$ is a convergent power series with all terms having v_s-weight greater than or equal to $a'_s (p_s)^h (n/h)$. Furthermore, we require that the H_k and G_j be invariant under the action of the discriminant group as well.

In order to compute the quotient of $(Y, 0)$ by $D(\Gamma_{f,n})$, we first need the subalgebra of $\mathbb{C}[[Z_0, \ldots, Z_{s-1}, U_1, \ldots, U_h, W]]$ that is invariant under the action. We have already stated that the Z_k and W are themselves invariant.

LEMMA 7.1.2. *The subalgebra of $\mathbb{C}[[U_1, \ldots, U_h]]$ of elements that are invariant under the action of the discriminant group $D(\Gamma_{f,n})$ given in (7.1) is generated by*

$$
\begin{aligned}
\overline{U}_j &:= U_j^{p_s}, \ 1 \leq j \leq h, \ \textit{and} \\
V &:= U_1 \cdots U_h.
\end{aligned}
$$

Proof. Suppose the monomial $M := U_1^{\alpha_1} \cdots U_h^{\alpha_h}$ is invariant under the action of $D(\Gamma_{f,n})$. We can assume all $\alpha_j < p_s$; otherwise, we could factor out a power of $\overline{U_j}$. We can also assume that at least one of the α_j is zero, because otherwise we could factor out a power of V. The element e_{u_j} in $D(\Gamma_{f,n})$ transforms M by the character

$$\left[\frac{a_s' \cdot n/h}{p_s}\left(\sum_{k \neq j} \alpha_k\right) - \frac{(a_s - a_s') \cdot n/h}{p_s}\alpha_j\right] = \left[\frac{a_s' \cdot n/h}{p_s}\left\{\sum_{k=1}^{h} \alpha_k - h\alpha_j\right\}\right]$$

(recall that $a_s = ha_s'$). Therefore, M is invariant under all e_{u_j} if and only if

$$a_s' \frac{n}{h}\left\{\sum_{k=1}^{h} \alpha_k - h\alpha_j\right\} \in \mathbb{Z} \cdot p_s \text{ for all } j.$$

Since $(a_s', p_s) = 1$ and, by assumption, $(n, p_s) = 1$, this is equivalent to

$$\sum_{k=1}^{h} \alpha_k - h\alpha_j \in \mathbb{Z} \cdot p_s \text{ for all } j.$$

Consider, for $1 \leq j \leq h$,

$$h\alpha_j = N_j p_s + \sum_{k=1}^{h} \alpha_k,$$

with $N_j \in \mathbb{Z}$. Recall that we can assume that at least one of the α_j is zero, say, $\alpha_1 = 0$. Then

$$0 = N_1 p_s + \sum_{k=1}^{h} \alpha_k.$$

Therefore, for any $\alpha_j > 0$, we have

$$h\alpha_j = N_j p_s - N_1 p_s = p_s(N_j - N_1).$$

Since $(h, p_s) = 1$, any nonzero α_j must be divisible by p_s. But we have assumed that all $\alpha_j < p_s$, and therefore all $\alpha_j = 0$ and $M = 1$. □

Clearly, these invariants satisfy

$$V^{p_s} = \overline{U}_1 \cdots \overline{U}_h.$$

We can now rewrite the splice equations (7.2) in terms of the invariant monomials. Since we are only looking for a particular analytic type of splice quotient, we can simplify things by setting all of the H_k and G_j equal to 0, and all of the b_k and c_k equal to 1. Then we have

$$\left\{ \begin{array}{rcl} Z_0^{a_1} + Z_1^{p_1} + Z_2 &=& 0 \\ M_2(Z_0, Z_1) + Z_2^{p_2} + Z_3 &=& 0 \\ \vdots & & \vdots \\ M_{s-2}(Z_0, \dots, Z_{s-3}) + Z_{s-2}^{p_{s-2}} + Z_{s-1} &=& 0 \\ M_{s-1}(Z_0, \dots, Z_{s-2}) + Z_{s-1}^{p_{s-1}} + V &=& 0 \\ \overline{U}_1 + d_1 M_s(Z_0, \cdots Z_{s-1}) + f_1 W^{n/h} &=& 0 \\ \vdots & & \vdots \\ \overline{U}_h + d_h M_s(Z_0, \cdots Z_{s-1}) + f_h W^{n/h} &=& 0 \end{array} \right\}. \tag{7.3}$$

The local analytic ring of the quotient of $(Y, 0)$ by $D(\Gamma_{f,n})$ is therefore $\mathbb{C}[[\{Z_k\}, \{\overline{U}_j\}, V, W]]$ modulo the ideal generated by $V^{p_s} = \overline{U}_1 \cdots \overline{U}_h$ and the equations (7.3).

It is clear that we can iteratively solve for $Z_2,\ Z_3, \dots, Z_{s-1}$ in terms of Z_0 and Z_1. That is,

$$\begin{aligned} Z_2 &= -(Z_0^{a_1} + Z_1^{p_1}) =: g_2(Z_0, Z_1), \\ Z_3 &= -(M_2(Z_0, Z_1) + Z_2^{p_2}) \\ &= -(M_2(Z_0, Z_1) + (g_2(Z_0, Z_1))^{p_2}) =: g_3(Z_0, Z_1), \end{aligned}$$

and so forth, until we have

$$Z_k = g_k(Z_0, Z_1),\ 2 \le k \le s - 1.$$

Therefore, we can rewrite the admissible monomials M_k as polynomials in Z_0 and Z_1 only. In particular, let

$$\begin{aligned}
\widetilde{M_{s-1}}(Z_0, Z_1) &:= M_{s-1}(Z_0, Z_1, g_2(Z_0, Z_1) \ldots, g_{s-2}(Z_0, Z_1)), \text{ and} \\
\widetilde{M_s}(Z_0, Z_1) &:= M_s(Z_0, Z_1, g_2(Z_0, Z_1), \ldots, g_{s-1}(Z_0, Z_1)).
\end{aligned}$$

Now, consider the equation

$$M_{s-1}(Z_0, \ldots, Z_{s-2}) + Z_{s-1}^{p_{s-1}} + V = 0$$

from (7.3). Clearly, we can solve for V in terms of Z_0 and Z_1, as follows:

$$\begin{aligned}
V &= -[M_{s-1}(Z_0, \ldots, Z_{s-2}) + Z_{s-1}^{p_{s-1}}] \\
&= -[\widetilde{M_{s-1}}(Z_0, Z_1) + (g_{s-1}(Z_0, Z_1))^{p_{s-1}}] \\
&:= \widetilde{V}(Z_0, Z_1).
\end{aligned}$$

Similarly, from the equations $\overline{U}_j + d_j M_s(Z_0, \cdots Z_{s-1}) + f_j W^{n/h} = 0$, we can solve for $\overline{U}_j$ in terms of $Z_0,\ Z_1$, and W:

$$\overline{U}_j = -[d_j \widetilde{M_s}(Z_0, Z_1) + f_j W^{n/h}].$$

Finally, since $V^{p_s} = \overline{U}_1 \cdots \overline{U}_h$, we have

$$[\widetilde{V}(Z_0, Z_1)]^{p_s} = (-1)^h \prod_{j=1}^{h} [d_j \widetilde{M_s}(Z_0, Z_1) + f_j W^{n/h}]. \tag{7.4}$$

This equation defines a hypersurface in $\mathbb{C}^3$ with coordinates Z_0, Z_1, and W. It is a splice quotient by definition, and it has the same resolution graph as the $(X_{f,n}, 0)$ whose data we began with.

Remark 7.1.3. The end curve functions can be seen in Equation (7.4).

Let $x := Z_0,\ y := Z_1,\ z := W$; let $f_j = 1$ for all j; and let $d_j = -\zeta^j$, where ζ is a primitive h-th root of unity. Then (7.4) becomes

$$[\widetilde{V}(x,y)]^{p_s} = (-1)^h[z^n - [\widetilde{M}_s(x,y)]^h], \tag{7.5}$$

and hence,

$$z^n = [\widetilde{M}_s(x,y)]^h + (-1)^h[\widetilde{V}(x,y)]^{p_s}.$$

We claim that

$$g(x,y) = [\widetilde{M}_s(x,y)]^h + (-1)^h[\widetilde{V}(x,y)]^{p_s}$$

is irreducible. If so, then g must have topological pairs $\{(p_i, a_i)\}$ by the following result of Mendris and Némethi. (Recall that the topological pairs can be recovered from the Newton pairs and vice versa.)

THEOREM 7.1.4 ([**8**]). *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an irreducible plane curve singularity with Newton pairs $\{(p_i, q_i)\}_{i=1}^s,\ s \geq 2$, and let $n \geq 2$ be an integer. Let L_X be the link of the hypersurface singularity $(X, 0) = (\{f(x,y) + z^n = 0\}, 0)$. If L_X is a rational homology sphere, then from the link L_X one can completely recover the Newton pairs of f and the integer n.*

The statement holds in greater generality, but we have stated only what we need in order to avoid complications.

In order to prove that g is irreducible, we will write down the splice equations in the $\mathbb{Z}$HS case, and show that the same polynomial g arises in the splice type equations. Take a positive integer N that is relatively prime to all p_i and a_i. The splice diagram Δ for $(X_{f,N}, 0)$ is in Figure 7.2 A set of splice equations for this Δ is given by

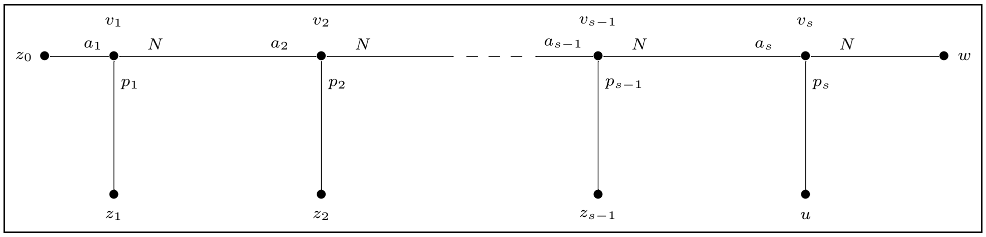


FIGURE 7.2. Splice diagram for the $\mathbb{Z}$HS case.

$$
\left\{
\begin{array}{rcl}
Z_0^{a_1} + Z_1^{p_1} + Z_2 & = & 0 \\
M_2(Z_0, Z_1) + Z_2^{p_2} + Z_3 & = & 0 \\
\vdots & & \vdots \\
M_{s-2}(Z_0, \dots, Z_{s-3}) + Z_{s-2}^{p_{s-2}} + Z_{s-1} & = & 0 \\
M_{s-1}(Z_0, \dots, Z_{s-2}) + Z_{s-1}^{p_{s-1}} + U & = & 0 \\
N_s(Z_0, \cdots Z_{s-1}) + U^{p_s} + W^N & = & 0
\end{array}
\right\}. \tag{7.6}
$$

It should be clear that we can use precisely the same admissible monomials M_k, $2 \leq k \leq s-1$, as we did in Equations (7.3) above. On the other hand, for an admissible monomial

$$N_s(Z_0, \cdots Z_{s-1}) = \prod_{k=0}^{s-1} Z_k^{\beta_k},$$

we cannot use M_s as in (7.3). The exponents for N_s satisfy

$$a_s = \sum_{k=0}^{s-1} \beta_k \ell'_{v_s z_k},$$

whereas the exponents α_k for M_s satisfy

$$a'_s = \sum_{k=0}^{s-1} \alpha_k \ell'_{v_s z_k}.$$

It should be clear that the $\ell'_{v_s z_k}$ are the same for both splice diagrams (Figure 7.1 and Figure 7.2). However, since $a'_s = a_s/h$,

$$[M_s(Z_0, \cdots Z_{s-1})]^h = \prod_{k=0}^{s-1} Z_k^{\alpha_k h}$$

is an admissible monomial for the node v_s in the $\mathbb{Z}$HS splice diagram, i.e., we can use $N_s = M_s^h$ in the last equation of (7.6). Now, solving iteratively for $Z_2,\ Z_3, \ldots, Z_{s-1}$, and then U, in terms of Z_0 and Z_1, one obtains

$$U = \widetilde{V}(Z_0, Z_1),$$

where $\widetilde{V}$ is exactly the same polynomial as before. The last equation in (7.6) is equivalent to

$$W^N = -\left([\widetilde{V}(Z_0, Z_1)]^{p_s} + \widetilde{N}_s(Z_0, Z_1) \right),$$

where $\widetilde{N}_s(Z_0, Z_1) = N_s(Z_0, Z_1, g_2(Z_0, Z_1), \ldots, g_{s-1}(Z_0, Z_1))$, with g_k as defined above. Finally, if we let $N_s = M_s^h$, we have

$$W^N = -\left([\widetilde{V}(Z_0, Z_1)]^{p_s} + [\widetilde{M}_s(Z_0, Z_1)]^h \right).$$

By construction, this equation defines a hypersurface with $\mathbb{Z}$HS link, and therefore the curve on the right hand side is irreducible by Theorem 6.0.3 of Neumann and Wahl.

This concludes the proof of Theorem 7.1.1 in the case that $n/\widetilde{h}_s \neq 1$, and we now present some concrete examples.

Example 7.1.5. We begin with a simple example:

$$s = 2, \quad p_1 = 2, \quad p_2 = 2,$$
$$n = 25, \quad a_1 = 3, \quad a_2 = 15.$$

The splice diagram Δ is shown in Figure 7.3. The following is a set of splice equations for

Δ, as in (7.3) :

$$\left\{\begin{array}{rcl} Z_0^3 + Z_1^2 + U_1U_2U_3U_4U_5 & = & 0 \\ U_1^2 + W^5 + d_1 Z_1 & = & 0 \\ \vdots & & \vdots \\ U_5^2 + W^5 + d_5 Z_1 & = & 0 \end{array}\right\}.$$

Note that Z_1 is the only admissible monomial at the node on the right in the direction of z_0. Let $x := Z_0,\ y := Z_1,\ z := W$, and $d_j = -\zeta^j$, where ζ is a primitive 5th root of unity. Equation (7.5) in this case is

$$(x^3 + y^2)^2 = -\prod_{j=1}^{5}(z^5 - \zeta^j y),$$

and therefore

$$z^{25} = y^5 - (x^3 + y^2)^2.$$

The computer algebra system SINGULAR verifies that the plane curve singularity defined by

$$y^5 - (x^3 + y^2)^2 = 0$$

is irreducible, and has 2 topological pairs, $(2,3)$ and $(2,15)$, as we expect. Note that changing n would not make any difference in the computation, as long as n remained divisible by 5 and relatively prime to 2 and 3. The resulting splice quotient would be $z^n = y^5 - (x^3 + y^2)^2$.

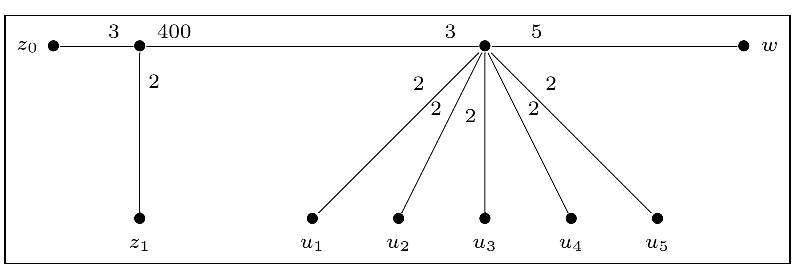


FIGURE 7.3. Splice diagram for Example 7.1.5

Example 7.1.6. Here is another simple example:

$$s = 2, \quad p_1 = 3, \quad p_2 = 3,$$

$$n = 4, \quad a_1 = 5, \quad a_2 = 46.$$

The splice diagram Δ is shown in Figure 7.4. The following is a set of splice equations for Δ:

$$\left\{ \begin{array}{rcl} Z_0^5 + Z_1^3 + U_1U_2 & = & 0 \\ U_1^3 + W^2 + Z_0Z_1^4 & = & 0 \\ U_2^3 + W^2 - Z_0Z_1^4 & = & 0 \end{array} \right\}.$$

In this case, we have a choice of admissible monomials for the node on the right in the direction of z_0; we could use $Z_0^6Z_1$ instead of $Z_0Z_1^4$. Let $x := Z_0,\ y := Z_1$, and $z := W$. Then

$$-(x^5 + y^3)^3 = (z^2 + xy^4)(z^2 - xy^4),$$

and therefore

$$z^4 = x^2y^8 - (x^5 + y^3)^3.$$

Again, SINGULAR verifies that the plane curve singularity defined by

$$x^2y^8 - (x^5 + y^3)^3 = 0$$

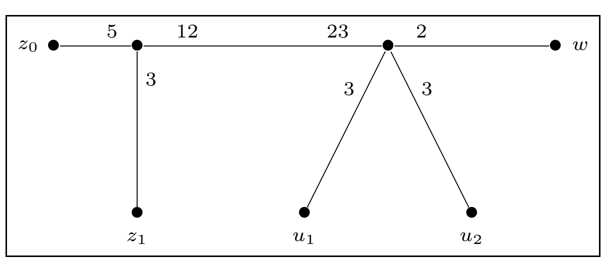


Figure 7.4. Splice diagram for Example 7.1.6

is irreducible, and has 2 topological pairs, $(3, 5)$ and $(3, 46)$, as expected. Using the other admissible monomial mentioned above yields

$$z^4 = x^{12}y^2 - (x^5 + y^3)^3.$$

EXAMPLE 7.1.7. Now we present a slightly more complicated example:

$$s = 3, \quad p_1 = 2, \quad p_2 = 2, \quad p_3 = 2,$$
$$n = 15, \quad a_1 = 7, \quad a_2 = 29, \quad a_3 = 129.$$

The splice diagram Δ is shown in Figure 7.5. The following is a set of splice equations for Δ:

$$\left\{ \begin{array}{rcl} Z_0^7 + Z_1^2 + Z_2 & = & 0 \\ Z_0^{11}Z_1 + Z_2^2 + U_1U_2U_3 & = & 0 \\ U_1^2 + W^5 + d_1Z_1Z_2 & = & 0 \\ U_2^2 + W^5 + d_2Z_1Z_2 & = & 0 \\ U_3^2 + W^5 + d_3Z_1Z_2 & = & 0 \end{array} \right\}.$$

Again, let $x := Z_0,\ y := Z_1$, and $z := W$. Then

$$Z_2 = -(x^7 + y^2),$$

and thus

$$Z_1Z_2 = -y(x^7 + y^2).$$

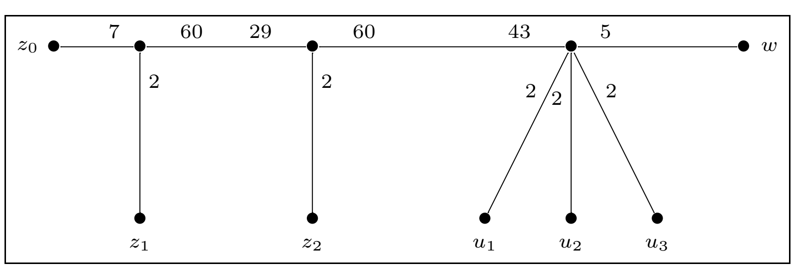


FIGURE 7.5. Splice diagram for Example 7.1.7

Therefore, Equation (7.5) in this case yields

$$[x^{11}y + (x^7 + y^2)^2]^2 = -\prod_{j=1}^{3}(z^5 - d_j y(x^7 + y^2)).$$

Let d_j be the jth power of a primitive third root of unity, and we have

$$z^{15} = y^3(x^7 + y^2)^3 - [x^{11}y + (x^7 + y^2)^2]^2.$$

Again, SINGULAR verifies that the plane curve singularity defined by

$$y^3(x^7 + y^2)^3 - [x^{11}y + (x^7 + y^2)^2]^2 = 0$$

is irreducible, and has the three expected topological pairs.

$\underline{n/\widetilde{h}_s = 1.}$ We proceed analogously in the case $n/\widetilde{h}_s = 1$. The splice diagram is shown in Figure 7.6. The action of the discriminant group is precisely the same as in (7.1), only in this case, there is no variable W. We begin with a set of splice equations, which are only

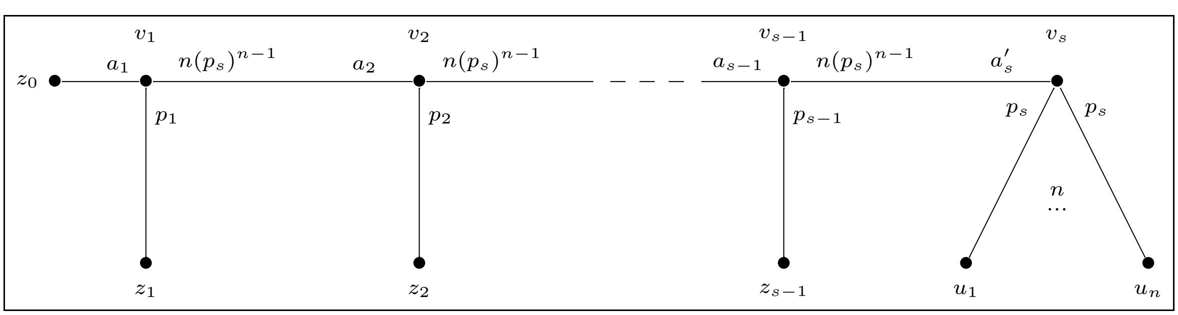


FIGURE 7.6. Splice diagram for Case (i), $n/\widetilde{h}_s = 1$.

slightly different from those in (7.3):

$$\left\{\begin{array}{rcl} Z_0^{a_1} + Z_1^{p_1} + Z_2 & = & 0 \\ M_2(Z_0, Z_1) + Z_2^{p_2} + Z_3 & = & 0 \\ \vdots & & \vdots \\ M_{s-2}(Z_0, \ldots, Z_{s-3}) + Z_{s-2}^{p_{s-2}} + Z_{s-1} & = & 0 \\ M_{s-1}(Z_0, \ldots, Z_{s-2}) + Z_{s-1}^{p_{s-1}} + U_1 \cdots U_n & = & 0 \\ U_1^{p_s} + d_1 M_s(Z_0, \cdots Z_{s-1}) + f_1 U_n^{p_s} & = & 0 \\ \vdots & & \vdots \\ U_{n-1}^{p_s} + d_{n-1} M_s(Z_0, \cdots Z_{s-1}) + f_{n-1} U_n^{p_s} & = & 0 \end{array}\right\}, \tag{7.7}$$

where, as before, d_j and f_j $(1 \leq j \leq n-1)$ are nonzero complex numbers such that $d_i f_j - d_j f_i \neq 0$ for all $i \neq j$.

Let $(Y, 0)$ be the isolated complete intersection singularity defined by (7.7). Lemma 7.1.2 still applies, and we can compute the quotient of $(Y, 0)$ by $D(\Gamma_{f,n})$ as above. In this case, there exists a splice quotient of the form

$$[\widetilde{V}(Z_0, Z_1)]^{p_s} = (-1)^{n-1} \overline{U}_n \prod_{j=1}^{n-1} [d_j \widetilde{M}_s(Z_0, Z_1) + f_j \overline{U}_n], \tag{7.8}$$

which defines a hypersurface in $\mathbb{C}^3$ with coordinates Z_0, Z_1, and $\overline{U_n}$.

Now, let $x := Z_0,\ y := Z_1$, and let $f_j = 1$ for all j. Then (7.8) becomes

$$[\widetilde{V}(x, y)]^{p_s} = (-1)^{n-1} \overline{U}_n \prod_{j=1}^{n-1} [\overline{U}_n + d_j \widetilde{M}_s(x, y)]. \tag{7.9}$$

Now, if we let $z := \overline{U}_n + \widetilde{M}_s(x, y)$, we have

$$(-1)^{n-1} [\widetilde{V}(x, y)]^{p_s} = [z - \widetilde{M}_s(x, y)] \prod_{j=1}^{n-1} [z - (1 - d_j) \widetilde{M}_s(x, y)].$$

Finally, let $d_j = 1 - \zeta^j$, where ζ is a primitive n-th root of unity, so that $1 - d_j = \zeta^j$, and hence

$$z^n = [\widetilde{M}_s(x,y)]^n + (-1)^{n-1}[\widetilde{V}(x,y)]^{p_s}.$$

The argument that we used above for $n/\widetilde{h}_s \neq 1$ can be used here to prove that

$$g(x,y) = [\widetilde{M}_s(x,y)]^n + (-1)^{n-1}[\widetilde{V}(x,y)]^{p_s}$$

is irreducible, and hence has topological pairs $\{(p_i, a_i)\}_{i=1}^{s}$. The following example will show that this case is hardly different from the case that $n/\widetilde{h}_s \neq 1$.

EXAMPLE 7.1.8. Consider this example:

$$s = 3, \quad p_1 = 2, \quad p_2 = 2, \quad p_3 = 2,$$
$$n = 3, \quad a_1 = 7, \quad a_2 = 29, \quad a_3 = 129.$$

This is the same as Example 7.1.7, except that here $n = 3$ instead of 15. The splice diagram Δ is shown in Figure 7.7. The following is a set of splice equations for Δ:

$$\left\{ \begin{array}{rcl} Z_0^7 + Z_1^2 + Z_2 & = & 0 \\ Z_0^{11} Z_1 + Z_2^2 + U_1 U_2 U_3 & = & 0 \\ U_1^2 + U_3^2 + d_1 Z_1 Z_2 & = & 0 \\ U_2^2 + U_3^2 + d_2 Z_1 Z_2 & = & 0 \end{array} \right\}.$$

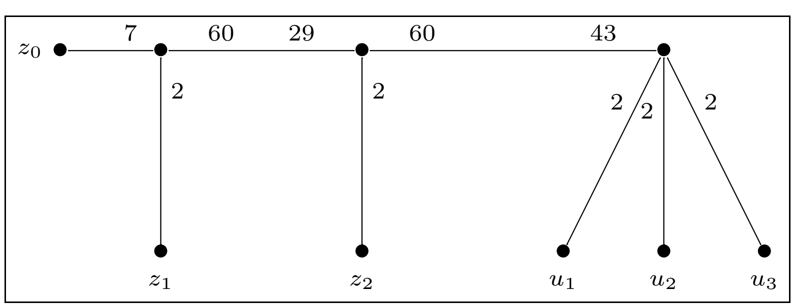


FIGURE 7.7. Splice diagram for Example 7.1.8

Let $x := Z_0$ and $y := Z_1$. Then

$$Z_2 = -(x^7 + y^2),$$

and thus

$$Z_1 Z_2 = -y(x^7 + y^2).$$

In this case, Equation (7.9) is

$$[x^{11}y + (x^7 + y^2)^2]^2 = \overline{U}_3(\overline{U}_3 - d_1 y(x^7 + y^2))(\overline{U}_3 - d_2 y(x^7 + y^2)).$$

If we make a change of coordinates, $z := \overline{U}_3 + y(x^7 + y^2)$, and Let $d_j = \zeta^j - 1$, where ζ is a primitive third root of unity, we have

$$[x^{11}y + (x^7 + y^2)^2]^2 = z^3 - y^3(x^7 + y^2)^3,$$

$$z^3 = y^3(x^7 + y^2)^3 + [x^{11}y + (x^7 + y^2)^2]^2.$$

We have already checked that the plane curve singularity defined by

$$y^3(x^7 + y^2)^3 + [x^{11}y + (x^7 + y^2)^2]^2 = 0$$

is irreducible, and has the three expected topological pairs.

7.2. Case (ii) $(n, p_s) \neq 1$

THEOREM 7.2.1. *Let n be a positive even integer and suppose that $f(x, y) = 0$ defines an irreducible plane curve singularity with two topological pairs, (p_1, a_1) and $(2, a_2)$, such that $(n, a_2) = (\frac{n}{2}, a_1) = (\frac{n}{2}, p_1) = 1$. Then there exists a splice quotient that is defined by an equation of the form $z^n = g(x, y)$, where g is irreducible and has topological pairs (p_1, a_1) and $(2, a_2)$.*

For any $(X_{f,n}, 0)$ that satisfies these conditions, the link is a $\mathbb{Q}$HS and the resolution graph $\Gamma_{f,n}$ satisfies the semigroup and congruence conditions (Case (ii) of Theorem 6.0.1). This time, let us begin with the pathological case. We have $s = 2,\ n = p_2 = 2$, and $h_1\widetilde{h_1} = 1$. The splice diagram $\Delta_{f,n}$ is given in Figure 7.8.

Recall that $D_A(v_1) = a_2 - a_1 p_1$, and the order of the discriminant group $D(\Gamma_{f,n})$ is a_2. As always, e_{y_j}, respectively e_{z_j}, denotes the image in $D(\Gamma_{f,n})$ of the dual basis element associated to the leaf y_j, respectively z_j. The capital letters $Y_0,\ Y_1,\ Z_0$ and Z_1 are the variables for the splice equations associated to the corresponding leaves.

Let us order the variables as follows: (Y_0, Y_1, Z_0, Z_1). Then for any leaf w in $\Delta_{f,n}$, we denote the diagonal action prescribed by the Neumann-Wahl algorithm of e_w on (Y_0, Y_1, Z_0, Z_1) by

$$e_w \mapsto [e_w \cdot e_{y_0},\ e_w \cdot e_{y_1},\ e_w \cdot e_{z_0},\ e_w \cdot e_{z_1}] .$$

It is easy to check the following (see §6.2):

$$\begin{aligned}
e_{y_0} &\mapsto \left[\frac{-p_1^2}{a_2}, \frac{-a_1p_1}{a_2}, \frac{p_1^2}{a_2}, \frac{a_1p_1}{a_2}\right],\\
e_{y_1} &\mapsto \left[\frac{-a_1p_1}{a_2}, \frac{-a_1^2}{a_2}, \frac{a_1p_1}{a_2}, \frac{a_1^2}{a_2}\right],\\
e_{z_0} &\mapsto \left[\frac{p_1^2}{a_2}, \frac{a_1p_1}{a_2}, \frac{-p_1^2}{a_2}, \frac{-a_1p_1}{a_2}\right],\\
e_{z_1} &\mapsto \left[\frac{a_1p_1}{a_2}, \frac{a_1^2}{a_2}, \frac{-a_1p_1}{a_2}, \frac{-a_1^2}{a_2}\right].
\end{aligned}$$

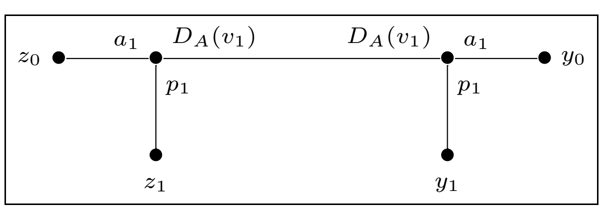


FIGURE 7.8. Splice diagram for the pathological case.

We claim that the action of discriminant group is actually generated by

$$e := \left[\frac{p_1}{a_2}, \frac{a_1}{a_2}, \frac{-p_1}{a_2}, \frac{-a_1}{a_2}\right].$$

Since we know that any three of the four elements above generate the discriminant group ([**18**], Prop. 5.3), and $e_{y_i} = e_{z_i}^{-1}$ for $i = 1,\ 2$, then clearly, e_{z_0} and e_{z_1} generate. Since a_1 and p_1 are relatively prime, there exist integers M and N such that $Mp_1 + Na_1 = 1$. Then

$$e_{z_0}^M \circ e_{z_1}^N \mapsto \left[\frac{p_1}{a_2}, \frac{a_1}{a_2}, \frac{-p_1}{a_2}, \frac{-a_1}{a_2}\right],$$

and it is not difficult to see that this element has order a_2.

A set of splice equations for $\Delta_{f,n}$ is

$$\left\{ \begin{array}{rcl} Z_0^{a_1} + b_1 Z_1^{p_1} + c_1 Y_0^{\alpha} Y_1^{\beta} + H_1 & = & 0 \\ Y_0^{a_1} + b_2 Y_1^{p_1} + c_2 Z_0^{\alpha} Z_1^{\beta} + H_2 & = & 0 \end{array} \right\},$$

where the b_i and c_i are nonzero complex numbers, H_1 and H_2 are higher weight power series in the appropriate sense, and

$$\alpha p_1 + \beta a_1 = a_2 - a_1 p_1.$$

It is easy to check that the congruence conditions hold for any such choice of α and β. We must also require that H_1 transforms by the character $\left[\frac{-a_1 p_1}{a_2}\right]$ under the action of e, and H_2 transforms by $\left[\frac{a_1 p_1}{a_2}\right]$.

Let $H_1 = H_2 = 0$, and $b_1 = b_2 = c_1 = c_2 = -1$, and let $(Y, 0)$ denote the singularity defined by

$$\left\{ \begin{array}{rcl} Z_0^{a_1} & = & Z_1^{p_1} + Y_0^{\alpha} Y_1^{\beta} \\ Y_0^{a_1} & = & Y_1^{p_1} + Z_0^{\alpha} Z_1^{\beta} \end{array} \right\}. \tag{7.10}$$

We will compute the quotient of $(Y, 0)$ by $D(\Gamma_{f,n})$. Note that the product of any admissible monomial from the first equation and any admissible monomial from the second equation is invariant under the group action. In fact, one can show that the subalgebra of the local analytic ring of $(Y, 0)$ of elements that are invariant under the action of the discriminant group $D(\Gamma_{f,n})$ given by e is generated by

$$A := Y_0 Z_0,\ B := Y_1 Z_1, C := Z_0^{\alpha} Z_1^{\beta+p_1},\ \text{and } D := Y_0^{\alpha} Y_1^{\beta+p_1} \tag{7.11}$$

For now, assume that this is true. The proof is at the end of the section. These invariants satisfy the following relations:

$$CD = A^{\alpha} B^{\beta+p_1} \text{ and } C + D = A^{a_1} - B^{p_1} - A^{\alpha} B^{\beta}.$$

Therefore,

$$C(C - A^{a_1} + B^{p_1} + A^{\alpha} B^{\beta}) = -A^{\alpha} B^{\beta+p_1},$$

that is,

$$\left\{C - \frac{1}{2}(A^{a_1} - B^{p_1} - A^{\alpha} B^{\beta})\right\}^2 = \frac{1}{4}(A^{a_1} - B^{p_1} - A^{\alpha} B^{\beta})^2 - A^{\alpha} B^{\beta+p_1}.$$

Hence, there exists a splice quotient of the form $\{z^2 = g(x, y)\}$. Let $x := A,\ y := B$, and $z =: 2(C - \frac{1}{2}(A^{a_1} - B^{p_1} - A^{\alpha} B^{\beta}))$, and we have

$$z^2 = (x^{a_1} - y^{p_1} - x^{\alpha} y^{\beta})^2 - 4x^{\alpha} y^{\beta+p_1}. \tag{7.12}$$

In [**6**], H. Laufer proves that if the link of a singularity defined by an equation of the form $z^2 = g(x, y)$, with g reduced, has $\mathbb{Q}$HS link, then the link uniquely determines the equisingularity type of g. Therefore, since the singularity defined by (7.12) has the same resolution graph as $(\{z^2 = f(x, y)\}, 0)$, where f is irreducible and has topological pairs (p_1, a_1) and $(2, a_2)$, then

$$g(x, y) = (x^{a_1} - y^{p_1} - x^{\alpha} y^{\beta})^2 - 4x^{\alpha} y^{\beta+p_1}$$

must also be irreducible with topological pairs (p_1, a_1) and $(2, a_2)$.

EXAMPLE 7.2.2. Consider the following example:

$$n = 2, \quad p_1 = 3, \quad p_2 = 2,$$

$$a_1 = 5, \quad a_2 = 31.$$

The splice diagram Δ is shown in Figure 7.9. The following is a set of splice equations for Δ as in (7.10):

$$\left\{ \begin{array}{ccc} Z_0^5 & = & Z_1^3 + Y_0^2 Y_1^2 \\ Y_0^5 & = & Y_1^3 + Z_0^2 Z_1^2 \end{array} \right\}.$$

The action of $\mathbb{Z}/31\mathbb{Z}$ is generated by

$$\left[\frac{3}{31}, \frac{5}{31}, \frac{-3}{31}, \frac{-5}{31} \right].$$

In this case, we have invariants

$$A := Z_0 Y_0, \ B := Z_1 Y_1, \ C := Z_0^2 Z_1^5, \text{ and } D := Y_0^2 Y_1^5,$$

and Equation (7.12) is

$$z^2 = (x^5 - y^3 - x^2 y^2)^2 - 4x^2 y^5.$$

For good measure, SINGULAR confirms that

$$(x^5 - y^3 - x^2 y^2)^2 - 4x^2 y^5 = 0$$

is irreducible, and has topological pairs $(3, 5)$ and $(2, 31)$.

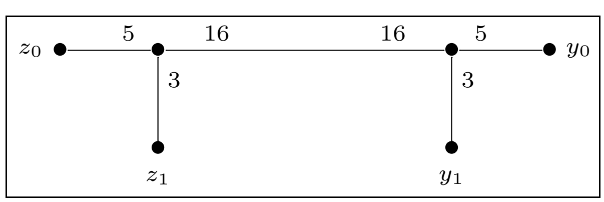


FIGURE 7.9. Splice diagram for Example 7.2.2

The rest of Case (ii), $s = 2$, $p_2 = 2$, $n = 2n'$, and $h_1\widetilde{h_1} = 1$, is a generalization of the pathological case. The splice diagram is shown in Figure 7.10. Here, $D_A(v_1) = n'(a_2 - a_1p_1)$, and the discriminant group still has order a_2. Let us order the variables as follows: (Y_0, Y_1, Z_0, Z_1, W). It is easy to check the following (see §6.2):

$$\begin{aligned} e_{y_0} &\mapsto \left[\frac{-n'p_1^2}{a_2}, \frac{-n'a_1p_1}{a_2}, \frac{n'p_1^2}{a_2}, \frac{n'a_1p_1}{a_2}, 0\right], \\ e_{y_1} &\mapsto \left[\frac{-n'a_1p_1}{a_2}, \frac{-n'a_1^2}{a_2}, \frac{n'a_1p_1}{a_2}, \frac{n'a_1^2}{a_2}, 0\right], \\ e_{z_0} &\mapsto \left[\frac{n'p_1^2}{a_2}, \frac{n'a_1p_1}{a_2}, \frac{-n'p_1^2}{a_2}, \frac{-n'a_1p_1}{a_2}, 0\right], \\ e_{z_1} &\mapsto \left[\frac{n'a_1p_1}{a_2}, \frac{n'a_1^2}{a_2}, \frac{-n'a_1p_1}{a_2}, \frac{-n'a_1^2}{a_2}, 0\right], \\ e_w &\mapsto [0,0,0,0,0]. \end{aligned}$$

Analogous to the pathological case, the action of discriminant group is generated by

$$e \mapsto \left[\frac{n'p_1}{a_2}, \frac{n'a_1}{a_2}, \frac{-n'p_1}{a_2}, \frac{-n'a_1}{a_2}, 0\right].$$

This element has order a_2, since n' is relatively prime to a_2, and n', a_1, and p_1 are pairwise relatively prime. Note that this e is "the same" as the generator in the pathological case above, meaning the two elements have the same action on $\mathbb{C}[[Y_0, Y_1, Z_0, Z_1]]$.

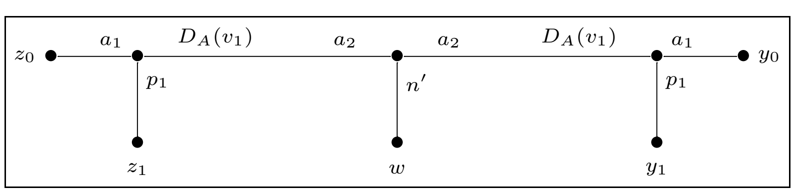


FIGURE 7.10. Splice diagram for Case (ii), $n > 2$.

One *particular* set of splice equations is

$$\left\{\begin{array}{lcl} Z_0^{a_1} & = & Z_1^{p_1} + Y_0^{\alpha} Y_1^{\beta} \\ Y_0^{a_1} & = & Y_1^{p_1} + Z_0^{\alpha} Z_1^{\beta} \\ W^{n'} & = & Z_0^{\alpha} Z_1^{\beta+p_1} - Y_0^{\alpha} Y_1^{\beta+p_1} \end{array}\right\}, \tag{7.13}$$

where

$$\alpha p_1 + \beta a_1 = a_2 - a_1 p_1.$$

(See the proof of Propostion 6.2.1.)

Since that the first two equations are the same as the splice equations we used in the pathological case, and that the third equation is invariant under the action of $D(\Gamma_{f,n})$, one can see that the subalgebra of invariants is generated by $A,\ B,\ C,\ D$ (as defined in (7.11)), and W.

We have the following relations between the invariants $A,\ B,\ C,\ D$, and W:

$$CD = A^{\alpha} B^{\beta+p_1},\ C - D = W^{n'},\ \text{and}$$

$$C + D = A^{a_1} - B^{p_1} - A^{\alpha} B^{\beta}.$$

We can easily write C and D in terms of $W,\ A$, and $B:$

$$\begin{array}{rcl} C & = & \frac{1}{2}(W^{n'} + A^{a_1} - B^{p_1} - A^{\alpha} B^{\beta}),\ \text{and} \\ D & = & -\frac{1}{2}(W^{n'} - (A^{a_1} - B^{p_1} - A^{\alpha} B^{\beta})). \end{array}$$

Therefore,

$$\frac{1}{4}(W^{n'} + A^{a_1} - B^{p_1} - A^{\alpha} B^{\beta})(W^{n'} - (A^{a_1} - B^{p_1} - A^{\alpha} B^{\beta})) = -A^{\alpha} B^{\beta+p_1},$$

and hence,

$$W^{2n'} = (A^{a_1} - B^{p_1} - A^{\alpha} B^{\beta})^2 - 4A^{\alpha} B^{\beta+p_1}.$$

If we let $x := A,\ y := B$, and $z := W$, then we have shown that there exists a splice quotient defined by an equation of the form

$$z^n = (x^{a_1} - y^{p_1} - x^\alpha y^\beta)^2 - 4x^\alpha y^{\beta+p_1},$$

which is precisely the equation that we got in the pathological case $n = 2$. We have already shown that

$$g(x, y) = (x^{a_1} - y^{p_1} - x^\alpha y^\beta)^2 - 4x^\alpha y^{\beta+p_1}$$

is irreducible with topological pairs (p_1, a_1) and $(2, a_2)$.

All that remains in the proof of Theorem 7.2.1 is to prove the following

LEMMA 7.2.3. *Consider the subalgebra of the local analytic ring of* $(Y, 0)$,

$$\mathcal{O}_Y = \mathbb{C}[[Y_0, Y_1, Z_0, Z_1]]/(Z_0^{a_1} - Z_1^{p_1} - Y_0^\alpha Y_1^\beta,\ Y_0^{a_1} - Y_1^{p_1} - Z_0^\alpha Z_1^\beta),$$

that is invariant under the action of $\mathbb{Z}/a_2\mathbb{Z}$ *generated by*

$$\left[\frac{p_1}{a_2}, \frac{a_1}{a_2}, \frac{-p_1}{a_2}, \frac{-a_1}{a_2}\right].$$

This subalgebra is generated by

$$A := Y_0Z_0,\ B := Y_1Z_1, C := Z_0^\alpha Z_1^{\beta+p_1},\ \textit{and}\ D := Y_0^\alpha Y_1^{\beta+p_1}.$$

PROOF. Consider the associated graded ring R of $\mathcal{O}_Y$ with respect to the weight filtration given by the right hand node in the splice diagram in Figure 7.8. The weights for each variable are as follows

Y_0	Y_1	Z_0	Z_1
$p_1(a_2 - a_1p_1)$	$a_1(a_2 - a_1p_1)$	$a_1p_1^2$	$a_1^2p_1$

.

Neumann and Wahl prove in [**18**] (Theorem 2.6(2)) that for any node v, the corresponding weight filtration has associated graded ring a reduced complete intersection, defined by the v-leading forms of the splice equations. Therefore, our associated graded ring R is defined by the equations

$$\begin{aligned} Y_0^{a_1} &= Y_1^{p_1} + Z_0^{\alpha} Z_1^{\beta}, \text{ and} \\ Z_0^{a_1} &= Z_1^{p_1}. \end{aligned}$$

The action of $G := \mathbb{Z}/a_2\mathbb{Z}$ obviously preserves the weight filtration, and since it is a diagonal action, the ring of invariants R^G is generated by monomials.

First of all, we show that $A,\ B,\ C$, and D generate the ring of invariants R^G. We do this by considering the invariant monomials case-by-case and showing that each such monomial can be written as a finite sum of terms, each of which is a monomial in $A,\ B,\ C$, and D times something of lower weight. Then, by downward induction on the weight, every invariant monomial is in the ring generated by $A,\ B,\ C$, and D. Obviously, for any monomial that contains both Y_0 and Z_0 or both Y_1 and Z_1, there is nothing to prove. Therefore, there are four cases of interest:

(a) $Z_0^I Z_1^J$ such that a_2 divides $p_1 I + a_1 J$,

(b) $Y_1^I Z_0^J$ such that a_2 divides $a_1 I - p_1 J$,

(c) $Y_0^I Z_1^J$ such that a_2 divides $p_1 I - a_1 J$,

(d) $Y_0^I Y_1^J$ such that a_2 divides $p_1 I + a_1 J$.

At first, we will assume that I and J are both nonzero in each case.

Case (a): $Z_0^I Z_1^J$ such that a_2 divides $p_1 I + a_1 J$. Let $p_1 I + a_1 J = a_2 m$, for some positive integer m. Recall that α and β are defined by

$$a_2 - a_1 p_1 = \alpha p_1 + \beta a_1.$$

So, we have $p_1 I + a_1 J = (\alpha p_1 + \beta a_1 + a_1 p_1)m$; that is,

$$p_1(\alpha m - I) = a_1(J - (\beta + p_1)m).$$

First, suppose that $\alpha m - I \geq 0$. Since a_1 and p_1 are relatively prime,

$$\begin{aligned} \alpha m - I &= a_1 d, \text{ and} \\ J - (\beta + p_1)m &= p_1 d, \end{aligned}$$

for some nonnegative integer d. Thus, $I + a_1 d = \alpha m$, and $J = p_1 d + (\beta + p_1)m$. Since $Z_0^{a_1} = Z_1^{p_1}$, we have

$$\begin{aligned} Z_0^I Z_1^J &= Z_0^I Z_1^{p_1 d + (\beta + p_1)m} \\ &= Z_0^I Z_0^{a_1 d} Z_1^{(\beta + p_1)m} \\ &= Z_0^{\alpha m} Z_1^{(\beta + p_1)m} \\ &= C^m. \end{aligned}$$

Note that $Z_0^I Z_1^J = C^m$ even for $d = 0$.

Now suppose that $\alpha m - I < 0$. Then $I - \alpha m = a_1 d$, and $(\beta + p_1)m - J = p_1 d$, for some positive integer d. Thus, $I = a_1 d + \alpha m$, $J + p_1 d = (\beta + p_1)m$, and

$$\begin{aligned} Z_0^I Z_1^J &= Z_0^{a_1 d + \alpha m} Z_1^J \\ &= Z_0^{\alpha m} Z_1^{p_1 d} Z_1^J \\ &= Z_0^{\alpha m} Z_1^{(\beta + p_1)m} \\ &= C^m. \end{aligned}$$

Case (b): $Y_1^I Z_0^J$ such that a_2 divides $a_1 I - p_1 J$. First of all, we claim that either $I \geq p_1$ or $J \geq a_1$. For, if not, then $|a_1 I - p_1 J| < a_1 p_1 < a_2$, and therefore, m must be 0. But that implies $a_1 I = p_1 J$, and hence p_1 divides I and a_1 divides J, which is a contradiction.

Suppose $J \geq a_1$. Then $Y_1^I Z_0^J$ is clearly a multiple of $B = Y_1 Z_1$. If $J < a_1$, then $I \geq p_1$, and therefore

$$Y_1^I Z_0^J = Y_1^{I-p_1}(Y_0^{a_1} - Z_0^{\alpha} Z_1^{\beta}) Z_0^J.$$

Let $M = \min(a_1, J)$, which we have assumed is greater than 0. Then

$$Y_1^I Z_0^J = A^M Y_1^{I-p_1} Y_0^{a_1 - M} Z_0^{J-M} - Y_1^{I-p_1} Z_1^{\beta} Z_0^{\alpha+J}.$$

If $\min(I - p_1, \beta) \neq 0$, the second term is a multiple of B, and we are finished. It is not possible for I to be p_1, for if so, we would have $J = a_1 - a_2 m/p_1$, and since $J < a_1$, this implies $J < 0$, since $a_2 > 2a_1 p_1$. Finally, if $\beta = 0$, then I must be divisible by p_1, and therefore

$$Y_1^I Z_0^J = (Y_0^{a_1} - Z_0^{\alpha})^d Z_0^J.$$

Every term in this expression is a multiple of A except for $Z_0^{\alpha d + J}$, which is treated below.

Case (c): $Y_0^I Z_1^J$ such that a_2 divides $p_1 I - a_1 J$. As above, it is clear that either $I \geq a_1$ or $J \geq p_1$. If $J \geq p_1$, then $Y_0^I Z_1^J$ is clearly a multiple of A. If $J < p_1$, then $I \geq a_1$, and therefore we have

$$Y_0^I Z_1^J = Y_0^{I-a_1}(Y_1^{p_1} + Z_0^{\alpha} Z_1^{\beta}) Z_1^J.$$

Let $M = \min(p_1, J)$, which we assume is greater than 0. Then

$$Y_0^I Z_1^J = B^M Y_0^{I-a_1} Y_1^{p_1 - M} Z_1^{J-M} - Y_0^{I-a_1} Z_0^{\alpha} Z_1^{\beta+J}.$$

If $\min(I - a_1, \alpha) \neq 0$, the second term is a multiple of A, and we are finished. Using the same argument as in Case (b), one can see that it is not possible for I to be a_1. Therefore the only problem is $\alpha = 0$. In that case, I must be divisible by a_1, and thus

$$Y_0^I Z_1^J = (Y_1^{p_1} - Z_1^{\beta})^d Z_1^J.$$

Every term in this expression is a multiple of B except for $Z_1^{\beta d+J}$, which is treated below.

Case (d): $Y_0^I Y_1^J$ such that a_2 divides $p_1 I + a_1 J$. Let $p_1 I + a_1 J = a_2 m$, where m is a positive integer. As in Case (a), we may write

$$p_1(\alpha m - I) = a_1(J - (\beta + p_1)m).$$

First, suppose that $\alpha m - I \geq 0$. Again, we have

$$\begin{aligned} \alpha m - I &= a_1 d, \text{ and} \\ J - (\beta + p_1)m &= p_1 d, \end{aligned}$$

for some nonnegative integer d. Thus, $I + a_1 d = \alpha m$, and $J = p_1 d + (\beta + p_1)m$. Since $Y_1^{p_1} = Y_0^{a_1} - Z_0^{\alpha} Z_1^{\beta}$, we have

$$Y_0^I Y_1^J = Y_0^I (Y_0^{a_1} - Z_0^{\alpha} Z_1^{\beta})^d Y_1^{(\beta+p_1)m}.$$

The terms on the right hand side are all of the form

$$Y_0^{I+a_1(d-k)} Z_0^{\alpha k} Z_1^{\beta k} Y_1^{(\beta+p_1)m}, \ 0 \leq k \leq d.$$

For $k = 0$, the term is D^m. The rest of the terms are divisible by either A or B unless $\min(I + a_1(d-k), \alpha k)$ and $\min(\beta k, (\beta + p_1)m)$ are both zero. But since we are assuming that I is nonzero, this cannot happen. For $\alpha m - I < 0$, the argument is virtually identical.

All that remains are the invariants of the form Y_0^I, Y_1^J, Z_0^I, and Z_1^J. If Z_0^I is invariant, then $p_1 I = a_2 m$, for some positive integer m. Then

$$p_1 I = (\alpha p_1 + \beta a_1 + a_1 p_1)m,$$

and hence $p_1(I - \alpha m) = a_1(\beta + p_1)m$. Clearly, $I - \alpha m > 0$, and

$$\begin{aligned} I - \alpha m &= a_1 d, \text{ and} \\ (\beta + p_1)m &= p_1 d, \end{aligned}$$

for some positive integer d. Then

$$Z_0^I = Z_0^{\alpha m + a_1 d} = Z_0^{\alpha m} Z_1^{(\beta + p_1)m} = C^m.$$

Note that the argument holds for $\alpha = 0$ as well, since in that case $C = Z_1^{\beta + p_1}$. In the same way, it is easy to show that any invariant of the form Z_1^J is equal to C^m as well.

Let Y_0^I be invariant, so that $p_1 I = a_2 m$. Then, just as above, we have $I - \alpha m = a_1 d$ and $(\beta + p_1)m = p_1 d$, for some positive integer d. Thus

$$Y_0^I = Y_0^{\alpha m + a_1 d} = Y_0^{\alpha m}(Y_1^{p_1} + Z_0^{\alpha} Z_1^{\beta})^d.$$

This expression consists of terms of the form

$$Y_0^{\alpha m} Z_0^{\alpha k} Y_1^{p_1(d-k)} Z_1^{\beta k}, \ 0 \leq k \leq d.$$

For $k = 0$, this is D^m (even if $\alpha = 0$), and the rest of the terms are multiples of A or B unless $\min(\alpha m, \alpha k)$ and $\min(\beta k, p_1(d-k))$ are both zero. The only possibility is $\alpha = 0$ and $k = d$. In this case, the monomial in question is $Z_1^{\beta d}$, and we covered the invariant powers of Z_1 above.

Finally, let Y_1^J be invariant, so that $a_1 J = a_2 m$. Then we have $J - (\beta + p_1)m = p_1 d$, and $\alpha m = a_1 d$, for some positive integer d. Thus,

$$Y_1^J = Y_1^{(\beta + p_1)m}(Y_0^{a_1} - Z_0^{\alpha} Z_1^{\beta})^d.$$

This expression consists of terms of the form

$$Y_0^{a_1(k-d)} Z_0^{\alpha k} Y_1^{(\beta+p_1)m} Z_1^{\beta k},\ 0 \le k \le d.$$

For $k = 0$, this is D^m (even if $\beta = 0$), and the rest of the terms are multiples of A or B unless $\min(a_1(d-k), \alpha k)$ and $\min((\beta + p_1)m, \beta k)$ are both zero. The only possibility is $\beta = 0$ and $k = d$. In this case, the monomial in question is $Z_0^{\alpha d} Y_1^{p_1 m}$, and this falls into Case (b).

Now, we have shown that the ring of invariants R^G in the associated graded ring is generated by A, B, C, and D. It is not hard to see that the ring of invariants for the associated graded is equal to the associated graded of the ring of invariants. That is, if I_n denotes the ideal of elements of weight greater than or equal to n (so that $R = \bigoplus I_n/I_{n+1}$), then, since the group action preserves the weight filtration, $(I_n/I_{n+1})^G \simeq I_n^G/I_{n+1}^G$. Finally, lifts of generators for the associated graded ring of the ring of invariants generate the entire algebra of invariants, since it is a complete local ring. □

Appendix

The goal of this appendix is to verify the proof of the second statement from Proposition 5.1.1, which we restate as follows.

PROPOSITION 7.2.4. *Suppose that* $(X_{f,n}, 0) := (\{z^n = f(x, y)\}, 0)$ *is a normal surface singularity with* $\mathbb{Q}$*HS link, and with* f *irreducible. Let* $\{(p_k, a_k) \ : \ 1 \leq k \leq s\}$ *be the topological pairs associated to the irreducible plane curve singularity, and let* p'_k, a'_k *be defined as in Definition 4.1.1. Assume that* $s \geq 2$*. Then, for all* k *such that* $2 \leq k \leq s$, a_k *is in the semigroup generated by*

$$\left\{a'_{k-1},\ p'_1 p'_2 \cdots p'_{k-1},\ a'_j p'_{j+1} \cdots p'_{k-1},\ 1 \leq j \leq k-2\right\}.$$

Clearly, it would be enough to show that a_k is in the semigroup generated by

$$\{a_{k-1},\ p_1 p_2 \cdots p_{k-1},\ a_j p_{j+1} \cdots p_{k-1},\ 1 \leq j \leq k-2\}.$$

For, if we had

$$a_k = \gamma_0 p_1 p_2 \cdots p_{k-1} + \sum_{j=1}^{k-2} \gamma_j a_j p_{j+1} \cdots p_{k-1} + \gamma_{k-1} a_{k-1},$$

then, since $p_j = h_j p'_j$ and $a_j = \widetilde{h}_j a'_j$ for all j,

$$\begin{aligned} a_k &= (\gamma_0 h_1 h_2 \cdots h_{k-1}) p'_1 p'_2 \cdots p'_{k-1} + \sum_{j=1}^{k-2} (\gamma_j \widetilde{h}_j h_{j+1} \cdots h_{k-1}) a'_j p'_{j+1} \cdots p'_{k-1} \\ &\quad + (\gamma_{k-1} \widetilde{h_{k-1}}) a'_{k-1}. \end{aligned}$$

The proof that a_k is in the semigroup generated by

$$\{a_{k-1},\ p_1 p_2 \cdots p_{k-1},\ a_j p_{j+1} \cdots p_{k-1},\ 1 \leq j \leq k-2\}$$

is equivalent to Lemma 2.2.1 in Teissier's appendix to [**27**] (attributed to Azevedo [**1**] and Merle [**9**]), but to show the equivalence requires some explanation.

To begin, we must give the precise definition of the Puiseux pairs $\{(p_k, m_k)\}$. Throughout, as a technical condition, we assume that $y = 0$ is not a tangent to $f = 0$ at the singular point.

Definition 7.2.5. (see, e.g. [**2**]) Suppose a Puiseux expansion of f is given by

$$y = \sum a_\kappa x^\kappa, \ a_\kappa \neq 0, \ \kappa \in \mathbb{Q}, \ \kappa \geq 1.$$

- Let κ_1 be the smallest exponent that is not an integer, and let $\kappa_1 = \frac{m_1}{p_1}$, with $m_1 > p_1$, and $\gcd(p_1, m_1) = 1$. Then (p_1, m_1) is the first Puiseux pair of f.
- Let κ_2 be the smallest exponent that cannot be written $\frac{q}{p_1}$ for some integer $q > p_1$. Write $\kappa_2 = \frac{m_2}{p_1 p_2}, \ p_2 > 1, \ \gcd(p_2, m_2) = 1$. Then (p_2, m_2) is the second Puiseux pair.
- Inductively, let κ_{i+1} be the smallest exponent that cannot be written $\frac{q}{p_1 \cdots p_i}$. Write $\kappa_{i+1} = \frac{m_{i+1}}{p_1 \cdots p_{i+1}}, \ p_{i+1} > 1, \ \gcd(p_{i+1}, m_{i+1}) = 1$. Then (p_{i+1}, m_{i+1}) is the $(i+1)$-st Puiseux pair.
- There exists s such that $p_1 \cdots p_s = p$ is a common denominator of all exponents in the Puiseux series. Then $(p_k, m_k), \ 1 \leq k \leq s$, are the Puiseux pairs.

The p_i are the same for the Puiseux pairs, Newton pairs, and topological pairs. It is easy to check (see §4.1) that the Puiseux pairs (p_k, m_k) are related to the topological pairs (p_k, a_k) as follows:

$$a_1 = m_1, \ \text{and} \ a_k = m_k - m_{k-1} p_k + a_{k-1} p_{k-1} p_k, \ 2 \leq k \leq s. \tag{7.1}$$

There is yet another finite set of positive integers that is equivalent to the set of Puiseux pairs of f called the "*characteristique*" ([**27**]) or *Puiseux characteristic* of f.

Definition 7.2.6. ([**27**], II.3) Suppose that f has Puiseux parameterization

$$\begin{cases} x &= t^p, \\ y &= \sum_{j\geq m} a_j t^j, \ a_j \neq 0, \end{cases}$$

where $m > p$ and $m \neq 0 \pmod p$. The Puiseux characteristic of f is $(p, \beta_1, \ldots, \beta_s)$, defined inductively as follows:

- p is the multiplicity of the singularity;
- β_1 is the smallest positive integer such that $\beta_1 \neq 0 \pmod p$, and $e_1 := (p, \beta_1)$;
- β_{i+1} is the smallest positive integer such that $\beta_{i+1} \neq 0 \pmod{e_i}$, and $e_{i+1} := (e_i, \beta_{i+1})$;
- the procedure ends once we reach s such that $e_s = 1$.

Note that $\beta_1 = m$.

By a careful reading of the two definitions, one can see that the Puiseux characteristic $(p, \beta_1, \ldots, \beta_s)$ and the Puiseux pairs (p_k, m_k) are related as follows:

$$\begin{aligned} p_1 &= \frac{p}{e_1}, \\ p_k &= \frac{e_{k-1}}{e_k}, \ 2 \leq k \leq s, \text{ and} \\ m_k &= \frac{\beta_k}{e_k}, \ 1 \leq k \leq s. \end{aligned}$$

Therefore, it is easy to see that

$$p = p_1 \cdots p_s, \tag{7.2}$$

$$\beta_k = m_k p_{k+1} \cdots p_s, \ 1 \leq k \leq s-1, \tag{7.3}$$

$$\beta_s = m_s. \tag{7.4}$$

Finally, we define integers $\bar{\beta}_k$ inductively as follows (see [**27**], Thm. 3.9):

$$\begin{aligned}\bar{\beta}_0 &= p\\ \bar{\beta}_1 &= \beta_1\\ \bar{\beta}_k &= p_{k-1}\bar{\beta}_{k-1} - \beta_{k-1} + \beta_k,\ 2 \leq k \leq s.\end{aligned}$$

The semigroup generated by the $\bar{\beta}_k$ is referred to as the semigroup associated to the irreducible plane curve singularity defined by f. There is much more that can be said about this semigroup, but we reproduce only what we require.

We are now ready to state the result from Teissier's appendix.

Lemma ([**27**], Lemma 2.2.1). *If $\langle\bar{\beta}_0, \ldots, \bar{\beta}_s\rangle$ is the semigroup of an irreducible plane curve singularity, one has*

$$p_k\bar{\beta}_k \in \mathbb{N}\langle\bar{\beta}_0, \ldots, \bar{\beta}_{k-1}\rangle,\ \textit{for } 1 \leq k \leq s.$$

To see that this is equivalent to the statement that a_k is in the semigroup generated by

$$\{a_{k-1},\ p_1p_2\cdots p_{k-1},\ a_jp_{j+1}\cdots p_{k-1},\ 1 \leq j \leq k-2\},$$

we need only rewrite the $\bar{\beta}_k$ in terms of the topological pairs (p_i, a_i).

Claim 7.2.7. *For $1 \leq k \leq s-1$, $\bar{\beta}_k = a_kp_{k+1}\cdots p_s$, and $\bar{\beta}_s = a_s$.*

Proof. For $k = 1$, this is true by (7.3) since $a_1 = m_1$. Assume the statement is true up to $k-1$. Then

$$\begin{aligned}\bar{\beta}_k &= p_{k-1}\bar{\beta}_{k-1} - \beta_{k-1} + \beta_k\\ &= p_{k-1}(a_{k-1}p_k\cdots p_s) - m_{k-1}p_k\cdots p_s + m_kp_{k+1}\cdots p_s\\ &= [a_{k-1}p_{k-1}p_k - m_{k-1}p_k + m_k]p_{k+1}\cdots p_s,\end{aligned}$$

and this is equal to $a_k p_{k+1} \cdots p_s$ by (7.1). $\square$

Therefore, for $1 \le k \le s$, the Lemma is equivalent to

$$a_k p_k \cdots p_s \in \mathbb{N}\langle p_1 \cdots p_s,\ a_j p_{j+1} \cdots p_s \mid 1 \le j \le k-1 \rangle.$$

If we divide each term by $p_k \cdots p_s$, we have

$$a_k \in \mathbb{N}\langle p_1 \cdots p_{k-1},\ a_{k-1},\ a_j p_{j+1} \cdots p_{k-1} \mid 1 \le j \le k-2 \rangle.$$

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