Determinants of the hypergeometric period matrices of an arrangement and its dual

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ABSTRACT

DEBABRATA MUKHERJEE: Determinants of the hypergeometric period matrices of an arrangement and its dual.
(Under the direction of Alexander Varchenko.)

We fix three natural numbers k, n, N, such that $n + k + 1 = N$, and introduce the notion of two dual arrangements of hyperplanes. One of the arrangements is an arrangement of N hyperplanes in a k-dimensional affine space, the other is an arrangement of N hyperplanes in an n-dimensional affine space. We assign weights $\alpha_1, \ldots, \alpha_N$ to the hyperplanes of the arrangements and for each of the arrangements consider the associated period matrices. The first is a matrix of k-dimensional hypergeometric integrals and the second is a matrix of n-dimensional hypergeometric integrals. The size of each matrix is equal to the number of bounded domains of the corresponding arrangement. We show that the dual arrangements have the same number of bounded domains and the product of the determinants of the period matrices is equal to an alternating product of certain values of Euler's gamma function multiplied by a product of exponentials of the weights.

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Contents

Chapter 1

Introduction

Let $\mathcal{V}$ be a real affine space of dimension k. Let $\mathcal{F} = \{ f^j \ : \ j \in \mathcal{J} \}$ be a finite set of degree one polynomials defined on $\mathcal{V}$. For $j \in \mathcal{J}$, let $\mathcal{H}^j$ be the hyperplane in $\mathcal{V}$ given by the zero-set of f^j. Consider the affine hyperplane arrangement $\mathcal{A} = \{ \mathcal{H}^j \ : \ j \in \mathcal{J} \}$. Assume that a positive number α_j is assigned to each hyperplane $\mathcal{H}^j$.

A bounded connected component of $\mathcal{V} \setminus \cup_{j \in \mathcal{J}} \mathcal{H}^j$ is called *a bounded domain* of $\mathcal{A}$. Let $\mathtt{Ch}(\mathcal{A})$ be the set of bounded domains of $\mathcal{A}$ and β the number of bounded domains.

A logarithmic differential k-form associated to $\mathcal{F}$ is a k-form of the type

$$\phi = \sum a_{j_1 \, j_2 \, \dots \, j_k} \frac{df^{j_1}}{f^{j_1}} \wedge \frac{df^{j_2}}{f^{j_2}} \wedge \cdots \wedge \frac{df^{j_k}}{f^{j_k}}$$

with $a_{j_1 \, j_2 \, \dots \, j_k} \in \mathbb{R}$. The form is regular on $\mathcal{V} \setminus \cup_{j \in \mathcal{J}} \mathcal{H}^j$.

If $\mathcal{F}$ is ordered, then, using constructions from [DT] and [V1], one obtains

- an ordered set $\Phi = \{\phi^1, \phi^2, \dots, \phi^\beta\}$ of logarithmic differential k-forms,
- an order on $\mathtt{Ch}(\mathcal{A}) = \{\Delta_1, \Delta_2, \dots, \Delta_\beta\}$,
- an orientation of each $\Delta \in \mathtt{Ch}(\mathcal{A})$.

Consider the multi-valued function

$$U^\alpha = \prod_{j \in \mathcal{J}} (f^j)^{\alpha_j}.$$

Fix a uni-valued branch of each function $(f^j)^{\alpha_j}$ on each bounded domain of the arrangement. This determines the uni-valued branch of the function U^α on each bounded

domain. The $\beta \times \beta$-matrix

$$\mathrm{PM}\,(\mathcal{A}, \alpha) = \left(\int_{\Delta_s} U^{\alpha} \phi^t \right), \qquad s, t = 1, 2, \ldots, \beta,$$

is called *the period matrix of the weighted ordered arrangement.*

Example 1. Let $a_1 < a_2 < a_3$ be real numbers. Consider the set of polynomials

$$\mathcal{F} = \{\, f^j \,=\, x - a_j \;:\; j = 1, 2, 3 \,\}$$

defined on $\mathbb{R}$. Then the arrangement $\mathcal{A} = \{\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3\}$ is the set of three points a_1, a_2, a_3 in $\mathbb{R}$. Let α_j be the weight of $\mathcal{H}^j$, $j = 1, 2, 3$.

There are two bounded domains: $\Delta_1 = (a_1, a_2), \Delta_2 = (a_2, a_3)$. The set of 1-forms is

$$\Phi \;=\; \{\; \phi^1 = \alpha_2 \frac{dx}{x - a_2} \;,\;\; \phi^2 = \alpha_3 \frac{dx}{x - a_3} \;\} \;.$$

Consider the function

$$U^{\alpha} \;=\; (x - a_1)^{\alpha_1} (x - a_2)^{\alpha_2} (x - a_3)^{\alpha_3} \;.$$

Fix a uni-valued branch of each function $(x - a_j)^{\alpha_j}$ on each interval Δ_s. The period matrix is

$$\begin{pmatrix} \int_{a_1}^{a_2} \alpha_2 \prod_{j=1}^{3} (x - a_j)^{\alpha_j} \frac{dx}{x - a_2} & \int_{a_1}^{a_2} \alpha_3 \prod_{j=1}^{3} (x - a_j)^{\alpha_j} \frac{dx}{x - a_3} \\ \\ \int_{a_2}^{a_3} \alpha_2 \prod_{j=1}^{3} (x - a_j)^{\alpha_j} \frac{dx}{x - a_2} & \int_{a_2}^{a_3} \alpha_3 \prod_{j=1}^{3} (x - a_j)^{\alpha_j} \frac{dx}{x - a_3} \end{pmatrix}.$$

In [V1] and [DT], the determinant of the period matrix was computed in terms of *critical values* $c\,((f^j)^{\alpha_j}, \Delta)$ of the chosen branches of the functions $(f^j)^{\alpha_j}$ on the bounded domains and a certain function, called *the beta function of the weighted arrangement*, see Sections 4.3.2 and 4.3.1. The beta function is an alternating product of values of Euler's gamma function whose arguments are appropriate linear combinations of the α_j's. It is proved in [V1] and [DT], that the determinant of the period matrix is given by the formula:

$$\det \left(\int_{\Delta_s} U^{\alpha} \phi^t \right) \;=\; \mathrm{B}(\mathcal{A}, \alpha) \cdot \prod_{\substack{\Delta \in\, \mathtt{Ch}(\mathcal{A}) \\ j \in \mathcal{J}}} c\,((f^j)^{\alpha_j}, \Delta) \;,$$

where $\mathrm{B}(\mathcal{A}, \alpha)$ is the beta function of the weighted arrangement.

Example 2. The beta function of the arrangement in Example 1 is

$$\mathrm{B}(\mathcal{A}, \alpha) = \frac{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)\Gamma(\alpha_3 + 1)}{\Gamma(\alpha_1 + \alpha_2 + \alpha_3 + 1)} \ .$$

The product of the critical values is

$$(f^1)^{\alpha_1}_{\Delta_1}(a_2) \cdot (f^1)^{\alpha_1}_{\Delta_2}(a_3) \cdot (f^2)^{\alpha_2}_{\Delta_1}(a_1) \cdot (f^2)^{\alpha_2}_{\Delta_2}(a_3) \cdot (f^3)^{\alpha_3}_{\Delta_1}(a_1) \cdot (f^3)^{\alpha_3}_{\Delta_2}(a_2) \ ,$$

where $(f^j)^{\alpha_j}_{\Delta_s}$ is the chosen branch of $(f^j)^{\alpha_j}$ on Δ_s, and $(f^j)^{\alpha_j}_{\Delta_s}(a_i)$ is the value of that branch at a_i.

In this paper, we introduce the notion of dual arrangements. We fix natural numbers k, n, N such that

$$k + n + 1 = N \ , \qquad 3 \leq N \ , \qquad 1 \leq k, n \leq N - 2 \ ,$$

and consider the vector space $\mathbb{R}^{N+1}$ and its dual space. Let $\{e_1, \ldots, e_{N+1}\}$ be the standard basis of $\mathbb{R}^{N+1}$ and $\{e^1, \ldots, e^{N+1}\}$ the dual basis of the dual space. We denote $\mathbb{R}^{N+1}$ by $\mathbb{X}$ and the dual space by $\mathbb{X}'$. Set $J = \{1, \ldots, N, N+1\}$ and $\mathcal{J} = \{1, \ldots, N\}$.

Let $\mathbb{W} \subset \mathbb{X}$ be a vector subspace of dimension $k+1$. Let $\mathbb{W}' \subset \mathbb{X}'$ be the annihilator of $\mathbb{W}$. The subspace $\mathbb{W}'$ is of dimension $n+1$. We assume that for any $a, b \in J$, $a \neq b$, the functions $e^a|_{\mathbb{W}}$ and $e^b|_{\mathbb{W}}$ are not proportional, and the functions $e_a|_{\mathbb{W}'}$ and $e_b|_{\mathbb{W}'}$ are not proportional.

The pair $\tau = (\mathbb{X}, \mathbb{W})$ with this property will be called *an admissible pair in* $\mathbb{X}$. Similarly, the pair $\tau' = (\mathbb{X}', \mathbb{W}')$ with this property will be called *an admissible pair in* $\mathbb{X}'$. The pairs τ and τ' will be called *dual*.

Let $V = \mathbb{P}(\mathbb{W})$ be the projective space associated with $\mathbb{W}$ and $\mathcal{V} \subset V$ the affine space defined by the condition $e^{N+1}|_{\mathbb{W}} \neq 0$. The spaces V and $\mathcal{V}$ are of dimension k. The functions e^j/e^{N+1}, $j \in \mathcal{J}$, define on $\mathcal{V}$ a set of degree one polynomials f^j and an arrangement of hyperplanes denoted by $\mathcal{A}[\tau]$.

Similarly, let $V' = \mathbb{P}(\mathbb{W}')$ be the projective space associated with $\mathbb{W}'$ and $\mathcal{V}' \subset V'$ the affine space defined by the condition $e_{N+1}|_{\mathbb{W}'} \neq 0$. These are spaces of dimension n. The functions e_j/e_{N+1}, $j \in \mathcal{J}$, define on $\mathcal{V}'$ a set of degree one polynomials f_j and an arrangement of hyperplanes denoted by $\mathcal{A}[\tau']$.

The arrangements $\mathcal{A}[\tau]$ and $\mathcal{A}[\tau']$ will be called *dual*. These are arrangements of

N hyperplanes in affine spaces of dimensions k and n, respectively. We prove that the number of bounded domains of $\mathcal{A}[\tau]$ is equal to the number of bounded domains of $\mathcal{A}[\tau']$ and study other combinatorial similarities between the dual arrangements.

Fix positive numbers $\{\alpha_j : j \in \mathcal{J}\}$, then arrangements $\mathcal{A}[\tau]$ and $\mathcal{A}[\tau']$ become weighted arrangements. Let $\mathrm{PM}\,(\mathcal{A}[\tau], \alpha)$ and $\mathrm{PM}\,(\mathcal{A}[\tau'], \alpha)$ be their period matrices. The period matrices depend on the choice of uni-valued branches of the corresponding functions on the corresponding bounded domains. In this paper, we give a construction of the choice of branches so that the determinants of the period matrices become related. For our choice of branches we prove that

$$\det \mathrm{PM}\,(\mathcal{A}[\tau], \alpha) \ \cdot \ \det \mathrm{PM}\,(\mathcal{A}[\tau'], \alpha) \ = \ \left[\frac{\prod_{j\in\mathcal{J}} e^{\pi i \alpha_j}\, \Gamma(\alpha_j + 1)}{\Gamma(\sum_{j\in\mathcal{J}} \alpha_j + 1)} \right]^{\beta} ,$$

see Theorem 4.5.1. This formula relates the determinants of matrices of k-dimensional and n-dimensional hypergeometric integrals and shows that the product of determinants of period matrices of dual arrangements is a combinatorial quantity, not depending on the size of the bounded domains or angles between hyperplanes. Theorem 4.5.1 is the main result of the paper.

Example 3. For the arrangement in Example 1, the dual arrangement is also an arrangement of three points on the real line. The points $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ of the dual arrangement are given by the zero-sets of the polynomials

$$f_1 = \frac{a_2 - a_3}{a_1 - a_2} \left(x - \frac{1}{a_3 - a_2} \right), \quad f_2 = \frac{a_3 - a_1}{a_1 - a_2} \left(x - \frac{1}{a_3 - a_1} \right), \quad f_3 = x \ ,$$

respectively. According to Theorem 4.5.1, the product of determinants of the period matrices of the arrangement of Example 1 and its dual is equal to

$$e^{2\pi i\,(\alpha_1+\alpha_2+\alpha_3)} \left[\frac{\Gamma(\alpha_1 + 1)\,\Gamma(\alpha_2 + 1)\,\Gamma(\alpha_3 + 1)}{\Gamma(\alpha_1 + \alpha_2 + \alpha_3 + 1)} \right]^2 .$$

The paper has the following structure. In Section 2 we discuss combinatorics of an arrangement of hyperplanes. In Section 3 we introduce the notion of dual arrangements and compare the combinatorics of dual arrangements. Section 4 is about period matrices. The section contains the statement of the main result of the paper, Theorem 4.5.1. In Section 5 we prove Theorem 4.5.1. In Appendix A, we introduce the notion of weak duality and show that natural constructions with dual arrangements lead to

weakly dual arrangements. In Appendix B, we formulate a statement which helps to determine if two given arrangements are dual.

The author thanks E. M. Rains for citing the paper by A. L. Dixon [D], published in 1905, in which certain hypergeometric integrals of different dimensions were equated; see an elliptic version of Dixon's identity in [R]. E. M. Rains suggested that there might be similar identities for determinants of periods of suitable arrangements of different dimensions.

A future direction of research would be to investigate if Dixon's result would lead to the understanding of other arrangements of hyperplanes in different dimensions and related objects.

Chapter 2

Preliminaries

This chapter introduces the basic notions, definitions, and relevant results of *arrangements of hyperplanes* and the theory of *matroids*.

The concept of a matroid generalizes the notion of linearly independent subsets of a set of vectors (Section 2.2). The *Tutte polynomial of a matroid* is a polynomial in two variables (see Section 2.2.2). This polynomial and its recursive properties lend easily to the definitions of several quantities associated to a matroid, see Section 2.2.2. These quantities defined with the aid of the coefficients of the Tutte polynomial assume great significance when specialized to the case of *matroids associated to arrangements*. These notions are the subject of discussion in the Sections 2.3—2.6.

All vector and affine spaces in this paper are over the field of real numbers. For a vector space U, $\mathbb{P}(U)$ denotes the projective space of one-dimensional vector subspaces of U.

2.1 Arrangement and edges

Let $\mathbb{W}$ be a vector space and $\Sigma = \{u^j : j \in J\}$ a finite collection of nonzero vectors in the dual space of $\mathbb{W}$. For $j \in J$, denote by $E^j \subset \mathbb{W}$ the hyperplane $\{z : u^j(z) = 0\}$ and by $H^j = \mathbb{P}(E^j) \subset \mathbb{P}(\mathbb{W})$ its projectivization. Then $A = \{H^j : j \in J\}$ is an arrangement of hyperplanes in $\mathbb{P}(\mathbb{W})$.

A non-empty intersection of some of the hyperplanes of the arrangement is called an *edge*. A *vertex* is a zero-dimensional edge.

To an edge L, we associate two arrangements:

- $A^L = \{H^j : L \subset H^j\}$, *the localization of* A *at* L,

- $A_L = \{H^j \cap L : L \not\subset H^j\}$, *the induced arrangement on* L.

An arrangement $A = \{H^j : j \in J\}$ is called *central* if the intersection $L = \cap_{j\in J} H^j$ is not empty.

For a central arrangement, consider the projective space $\mathbb{P}_L$ whose points are the $(\dim L + 1)$-dimensional projective subspaces of $\mathbb{P}(\mathbb{W})$ containing L. Any hyperplane in $\mathbb{P}(\mathbb{W})$, containing L, determines uniquely a hyperplane in $\mathbb{P}_L$. Thus, the central arrangement determines an arrangement of hyperplanes in $\mathbb{P}_L$ called *the projectivization of the central arrangement.*

The projectivization of the central arrangement A^L is called *the projective localization of* L and denoted by $P(A^L)$.

Let L be an edge. Let $a, b \in J$, $a \neq b$. We say that L is parallel to H^a in the affine space $V \setminus H^b$, if H^a does not coincide with H^b and L does not intersect H^a in the affine space. In that case we also say that the triple (L, H^a, H^b) is *a parallelism in the arrangement* A.

2.2 Matroids

This section introduces definitions and constructions from matroid theory. For more on matroid theory see [O] and [B].

Let J be a finite set and $\mathcal{I}$ a collection of subsets of J. The pair $M = (J, \mathcal{I})$ is called a *matroid* if the following properties hold.

$\mathcal{I}1$. $\emptyset \in \mathcal{I}$.

$\mathcal{I}2$. If X is in $\mathcal{I}$ and $Y \subset X$, then Y is also in $\mathcal{I}$.

$\mathcal{I}3$. If X and Y are in $\mathcal{I}$ and $|X| > |Y|$, then there is an element $x \in X \setminus Y$ such that $Y \cup \{x\}$ is in $\mathcal{I}$.

The set J is called *the ground set* of the matroid and elements of $\mathcal{I}$ are called *the independent sets* of the matroid.

Example 2.2.1. Let $\Sigma = \{u^j : j \in J\}$ be a finite collection of vectors in a vector space. Define the collection $\mathcal{I}$ of subsets of J: a subset $S \subset J$ belongs to $\mathcal{I}$ if and only if the vectors $\{u^j : j \in S\}$ are linearly independent. This defines *the matroid of the collection of vectors.*

Example 2.2.2. Let $\mathbb{W}$ be a vector space. Let $\Sigma = \{u^j : j \in J\}$ be a finite collection of nonzero vectors in the dual space of $\mathbb{W}$. The collection defines the arrangement A of hyperplanes in $\mathbb{P}(\mathbb{W})$. The matroid of Σ is called *the matroid of the arrangement*.

The arrangement defines vectors of Σ up to multiplication by nonzero numbers. This multiplication does not change the matroid of Σ. Hence, the matroid of the arrangement does not depend on the choice of the collection of vectors.

Let $M = (J, \mathcal{I})$ be a matroid. A maximal (with respect to inclusion) element of $\mathcal{I}$ is called a *basis*. The axiom $(\mathcal{I}3)$ implies that all bases have the same cardinality. More generally, for any subset X of J, the maximal independent subsets of X all have the same cardinality. Define

- $\operatorname{rank}_M X$ to be the cardinality of the largest independent subset of X,
- $\operatorname{corank}_M X = \operatorname{rank}_M J - \operatorname{rank}_M X$,
- $\operatorname{nullity}_M X = |X| - \operatorname{rank}_M X$,
- $\operatorname{rank} M = \operatorname{rank}_M J$.

2.2.1 Tutte polynomial

The *Tutte polynomial* of a matroid $M = (J, \mathcal{I})$ is the polynomial in x and y, given by the formula

$$\mathrm{T}(M; x, y) = \sum_{X \subset J} (x-1)^{\operatorname{corank}_M X} (y-1)^{\operatorname{nullity}_M X}.$$

Theorem 2.2.1 (B)**.** *Let M be a matroid on the ground set J. Let $\mathrm{T}(M; x, y) = \sum_{i,j} b_M^{ij} x^i y^j$.*

- *If $|J| \geq 2$, then $b_M^{10} = b_M^{01}$.*
- *If $|J| \geq 1$, then $b_M^{00} = 0$.*

2.2.2 Contraction and deletion

Let $M = (J, \mathcal{I})$ be a matroid. For a subset $X \subset J$, denote by $\hat{X} = J \backslash X$ its complement.

For a non-empty subset $X \subset J$, $|X| < |J|$, define the matroid $M/X = (\hat{X}, \mathcal{I}_{M/X})$ called the *contraction of* X. A subset $I \subset \hat{X}$ is in $\mathcal{I}_{M/X}$ if and only if for some maximal independent subset Y of X in M, the set $I \cup Y$ is independent in M.

For a non-empty subset $X \subset J$, $|X| < |J|$, define the matroid $M - X = (\hat{X}, \mathcal{I}_{M-X})$ called the *deletion of* X. A subset $I \subset \hat{X}$ is in $\mathcal{I}_{M-X}$ if and only if I is independent in M.

An element $j \in J$ is called a *loop* if it is not contained in any basis of M. Dually, an element j is called an *isthmus* if it is contained in every basis.

Theorem 2.2.2 (B)**.** *If j is neither a loop nor an isthmus, then*

$$\mathrm{T}(M; x, y) = \mathrm{T}(M - \{j\}; x, y) + \mathrm{T}(M/\{j\}; x, y).$$

If j is an isthmus, then

$$\mathrm{T}(M; x, y) = x\,\mathrm{T}(M/\{j\}; x, y).$$

A non-empty subset $X \subset J$ is called a *flat* if for every $y \in J \setminus X$,

$$\mathrm{rank}_M X \cup \{y\} > \mathrm{rank}_M X.$$

For a flat X, define its *discrete length, width, volume* as the numbers

$$\mathrm{l}_M X = b^{10}_{M/X}\ , \qquad \mathrm{w}_M X = b^{10}_{M-\hat{X}}\ , \qquad \mathrm{vol}_M X = \mathrm{l}_M X \cdot \mathrm{w}_M X\ ,$$

respectively. We say that a flat is *spacious* if it has a nonzero discrete volume.

Let X be a flat. Let $a, b \in J$, $a \neq b$. The triple (X, a, b) is called a *parallelism* in M if $a, b \notin X$, $\mathrm{rank}_M\{a, b\} = 2$, and $\mathrm{rank}_M X \cup \{a, b\} = \mathrm{rank}_M X + 1$.

Let (X, a, b) be a parallelism. Denote by $\hat{X}(a, b) = J \setminus (X \cup \{a, b\})$ the complement of $X \cup \{a, b\}$ in J.

For a parallelism (X, a, b), define its *discrete width, volume* as the numbers

$$\mathrm{w}_M(X, a, b) = b^{10}_{M-\hat{X}(a,b)}\ , \qquad \mathrm{vol}_M(X, a, b) = \mathrm{l}_M X \cdot \mathrm{w}_M(X, a, b)\ ,$$

respectively.

2.3 Matroid of an arrangement

Let M be the matroid of an arrangement $A = \{H^j : j \in J\}$ of hyperplanes in a projective space V. The flats in M are in one-to-one correspondence with edges of A.

If L is an edge, then $X = \{j : L \subset H^j\}$ is a flat.

Let L be an edge and X the corresponding flat. Let A^L and A_L be the localization and induced arrangements, respectively. Let $M[A^L]$ and $M[A_L]$ be the matroids associated to the arrangements A^L and A_L, respectively. Then

$$M[A^L] \;=\; M - \hat{X}\;, \qquad M[A_L] \;=\; M/X\;.$$

Let $P(A^L)$ be the projective localization of L. Then the matroid of A^L is also the matroid of $P(A^L)$.

Lemma 2.3.1. If rank $M = \dim V + 1$, then rank $M/X = \dim L + 1$. □

2.4 Edges and parallelisms in an arrangement

Define *the discrete length, width, and volume* of an edge L as the discrete length, width, and volume, respectively, of the flat X,

$$\mathrm{l}_A L \;=\; b^{10}_{M[A_L]}\;, \qquad \mathrm{w}_A L \;=\; b^{10}_{M[A^L]}\;, \qquad \mathrm{vol}_A L \;=\; \mathrm{l}_A L \cdot \mathrm{w}_A L\,,$$

cf. [V1]. An edge will be called *spacious* if it has a nonzero discrete volume.

Let L be an edge and $a, b \in J$. The edge L is parallel to H^a in $V \setminus H^b$, if and only if the triple (X, a, b) is a parallelism in the matroid M.

Define *the discrete width and volume of a parallelism* (L, H^a, H^b) in A as the discrete width and volume of the parallelism (X, a, b) in M. That is,

$$\mathrm{w}_A(L, H^a, H^b) = \mathrm{w}_M(X, a, b)\,, \qquad \mathrm{vol}_A(L, H^a, H^b) = \mathrm{l}_A L \cdot \mathrm{w}_A(L, H^a, H^b).$$

2.5 Bounded domains of an arrangement

Let $A = \{H^j : j \in J\}$ be an arrangement of hyperplanes in a projective space V. The connected components of the topological space $V \setminus \cup_{j\in J} H^j$ are called *domains*. For $j \in J$, a domain is called *bounded with respect to the hyperplane* H^j if the closure of the domain does not intersect the hyperplane.

Theorem 2.5.1 (Z)**.** *Assume that* rank $M = \dim V + 1$. *Then for $j \in J$, the number of domains of A bounded with respect to H^j is equal to b^{10}_M. In particular, the number of bounded domains does not depend on the choice of j.*

If rank $M = \dim V + 1$, then the discrete length and width of an edge L are the numbers of bounded domains in the arrangements A_L and $P(A^L)$, respectively. See Lemma 2.3.1, Theorem 2.5.1.

2.6 Geometric interpretation of the discrete volume of a parallelism

Assume that $H^a \neq H^b$. In the affine space $\mathcal{V} = V \setminus H^b$, consider the arrangement of hyperplanes $\mathcal{A} = \{\, \mathcal{H}^j \;:\; j \in J \setminus \{b\} \,\}$, where $\mathcal{H}^j = H^j \cap \mathcal{V}$. A domain of the arrangement A is called *bounded in* $\mathcal{V}$ if it is contained in a suitable ball in $\mathcal{V}$.

Let Δ be a bounded domain and $\bar{\Delta}$ its closure. Consider the subset $S \subset \bar{\Delta}$ of all maximally remote points from the hyperplane $\mathcal{H}^a$. This subset is the union of some open faces of $\bar{\Delta}$. The unique face $\Gamma \subset S$ of highest dimension is called *the $\mathcal{H}^a$-external supporting face* of Δ.

Let Γ be of dimension m, then there is a unique m-dimensional edge L of the projective arrangement A which contains Γ. The edge L is called *the $\mathcal{H}^a$-external supporting edge of* Δ. The triple (L, H^a, H^b) is a parallelism in A.

Lemma 2.6.1. Let (L, H^a, H^b) be a parallelism in A. Then the number of bounded domains with $\mathcal{H}^a$-external supporting edge L is equal to the discrete volume of the parallelism (L, H^a, H^b).

The lemma is proved in Section 5.1.

Chapter 3

Dual admissible pairs, associated arrangements and matroids

In this chapter we start with the notion of *admissible pairs* (Section 3.1) and introduce the notion of *duality* between two admissible pairs. To each of these admissible pairs we associate the following objects: an arrangement of projective hyperplanes in a projective space of appropriate dimension, a finite set of degree one polynomials defined on an affine space of the same dimension, and an arrangement of affine hyperplanes defined by the zero-set of these polynomials. We also associate to each admissible pair a matroid (Section 3.2). The corresponding objects (projective arrangement, set of polynomials, affine arrangement, and matroid) associated to two admissible pairs that are dual are also called dual.

The notion of *dual matroids*, which is basic to matroid theory is examined in Section 3.5 in the context of the definitions given in Chapter 2.

Important geometric and combinatorial consequences of duality between arrangements is examined in Sections 3.6 and 3.7.

3.1 Admissible pairs

Let N be a natural number, $N \geq 3$. Let k, n be natural numbers such that $k+n+1 = N$ and $1 \leq k, n \leq N-2$.

Consider the vector space $\mathbb{R}^{N+1}$ and its dual space. Let $\{e_1, \ldots, e_{N+1}\}$ be the standard basis of $\mathbb{R}^{N+1}$ and $\{e^1, \ldots, e^{N+1}\}$ the dual basis of the dual space. Denote $\mathbb{R}^{N+1}$ by $\mathbb{X}$, denote the dual space by $\mathbb{X}'$. Denote $J = \{1, \ldots, N+1\}$.

Let $\mathbb{W} \subset \mathbb{X}$ be a vector subspace of dimension $k+1$. Let $\mathbb{W}' \subset \mathbb{X}'$ be the annihilator of $\mathbb{W}$. The subspace $\mathbb{W}'$ is of dimension $n+1$.

The set of linear functions $\{ e^j|_{\mathbb{W}} \ : \ j \in J \}$ spans the dual space of $\mathbb{W}$. Similarly, the set of linear functions $\{ e_j|_{\mathbb{W}'} \ : \ j \in J \}$ spans the dual space of $\mathbb{W}'$.

Assume that for any $a, b \in J$, $a \neq b$, the functions $e^a|_{\mathbb{W}}$ and $e^b|_{\mathbb{W}}$ are not proportional, and the functions $e_a|_{\mathbb{W}'}$ and $e_b|_{\mathbb{W}'}$ are not proportional.

The pair $\tau = (\mathbb{X}, \mathbb{W})$ with this property will be called *an admissible pair in* $\mathbb{X}$. Similarly, the pair $\tau' = (\mathbb{X}', \mathbb{W}')$ with this property will be called *an admissible pair in* $\mathbb{X}'$. The pairs τ and τ' will be called *dual*.

3.2 The arrangements and matroid of an admissible pair, dual arrangements

Let $\tau = (\mathbb{X}, \mathbb{W})$ be an admissible pair. For $j \in J$, denote $E^j = \{ x \in \mathbb{W} \ : \ e^j(x) = 0 \}$. These are vector subspaces of $\mathbb{W}$ of codimension one.

Denote

- $V = \mathbb{P}(\mathbb{W})$, the projective space of dimension k,
- $H^j = \mathbb{P}(E^j)$, $j \in J$, projective hyperplanes in V,
- $A[\tau] = \{ H^j \ : \ j \in J \}$, the arrangement of projective hyperplanes in V.

Denote

$$\mathcal{J} \ = \ \{ 1, \dots, N \} \ = \ J \setminus \{ N+1 \} \ .$$

For $j \in \mathcal{J}$, the rational function $f^j = e^j / e^{N+1}$ restricted to $\mathbb{W}$ is regular on $\mathbb{W} \setminus E^{N+1}$ and homogeneous of degree zero. Thus, f^j is a well-defined degree one polynomial on the affine space

$$V \setminus H^{N+1} \ = \ \mathbb{P}(\mathbb{W}) \setminus \mathbb{P}(E^{N+1}) \ .$$

Denote

- $\mathcal{V} = V \setminus H^{N+1}$, the affine space of dimension k,
- $\mathcal{H}^j = \{ x \in \mathcal{V} \ : \ f^j(x) = 0 \}$, $j \in \mathcal{J}$, affine hyperplanes in $\mathcal{V}$,

- $\mathcal{A}[\tau] = \{\, \mathcal{H}^j \;:\; j \in \mathcal{J} \,\}$, the arrangement of affine hyperplanes in $\mathcal{V}$.

Observe that $\mathcal{H}^j = H^j \cap \mathcal{V}$, $j \in \mathcal{J}$.

The set $\mathcal{F}[\tau] = \{\, f^j \;:\; j \in \mathcal{J} \,\}$ of degree one polynomials on $\mathcal{V}$ will be called *the arrangement of polynomials associated to* τ.

For dual admissible pairs τ and τ', the corresponding pairs of objects: $A[\tau]$ and $A[\tau']$, $\mathcal{A}[\tau]$ and $\mathcal{A}[\tau']$, $\mathcal{F}[\tau]$ and $\mathcal{F}[\tau']$ - will be called *dual*.

Introduce *the matroid of* τ, denoted $M[\tau]$, as the matroid of the collection of vectors $\{\, e^j|_{\mathbb{W}} \;:\; j \in J \,\}$ in the dual space of $\mathbb{W}$.

Observe that the matroid of τ is the matroid of the arrangement $A[\tau]$.

3.2.1 The values of polynomials defining an arrangement at the vertices of the arrangement at a vertex

Let $P \in \mathcal{V}$ be a vertex of the arrangement $\mathcal{A}[\tau]$ and let X be the flat associated to P in the matroid $M[\tau]$. Let $Y = \{j_1, \dots, j_k\} \subset X$ be a maximal independent subset. Let $j \in \mathcal{J}$. Consider two vectors

$$v = (e^{j_1} \wedge e^{j_2} \wedge \cdots \wedge e^{j_k} \wedge e^{N+1})|_{\mathbb{W}}\,, \qquad v' = (e^{j_1} \wedge e^{j_2} \wedge \cdots \wedge e^{j_k} \wedge e^{j})|_{\mathbb{W}}$$

of the one-dimensional vector space $\bigwedge^{k+1} \mathbb{W}^*$, where $\mathbb{W}^*$ is the dual space of $\mathbb{W}$. The first vector is nonzero. Let $c \in \mathbb{R}$ be the coefficient of proportionality: $v' = cv$.

Lemma 3.2.1. The value of the polynomial f^j at P is equal to c. □

3.3 A coordinate description of dual arrangements

Let $\tau = (\mathbb{X}, \mathbb{W})$ and $\tau' = (\mathbb{X}', \mathbb{W}')$ be dual admissible pairs.

Let $w_1, \dots, w_k, w_{k+1}, w_{k+2}, \dots, w_{N+1}$ be any basis of $\mathbb{X}$ such that $w_1, \dots, w_k, w_{k+1}$ is a basis of $\mathbb{W}$. Consider the dual basis $w^1, \dots, w^k, w^{k+1}, w^{k+2}, \dots, w^{N+1}$ of $\mathbb{X}'$. Then $w^{k+2}, \dots, w^{N+1}$ is a basis of $\mathbb{W}'$.

Let $w_i = \sum_{l=1}^{N+1} b_i^l e_l$ and $w^i = \sum_{l=1}^{N+1} c_l^i e^l$, for $i = 1, 2, \dots, N+1$. Denote $\mathfrak{B} = (b_i^l)$, $\mathfrak{C} = (c_l^i)$. We have $\mathfrak{C} = (\mathfrak{B}^T)^{-1}$.

Introduce the $(k+1)\times(N+1)$-matrix B and the $(n+1)\times(N+1)$-matrix C by

$$B=\begin{pmatrix} b_1^1 & b_1^2 & \cdots & b_1^j & \cdots & b_1^N & b_1^{N+1} \\ b_2^1 & b_2^2 & \cdots & b_2^j & \cdots & b_2^N & b_2^{N+1} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ b_k^1 & b_k^2 & \cdots & b_k^j & \cdots & b_k^N & b_k^{N+1} \\ b_{k+1}^1 & b_{k+1}^2 & \cdots & b_{k+1}^j & \cdots & b_{k+1}^N & b_{k+1}^{N+1} \end{pmatrix}$$

and

$$C=\begin{pmatrix} c_1^{k+2} & c_2^{k+2} & \cdots & c_j^{k+2} & \cdots & c_N^{k+2} & c_{N+1}^{k+2} \\ c_1^{k+3} & c_2^{k+3} & \cdots & c_j^{k+3} & \cdots & c_N^{k+3} & c_{N+1}^{k+3} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ c_1^{N} & c_2^{N} & \cdots & c_j^{N} & \cdots & c_N^{N} & c_{N+1}^{N} \\ c_1^{N+1} & c_2^{N+1} & \cdots & c_j^{N+1} & \cdots & c_N^{N+1} & c_{N+1}^{N+1} \end{pmatrix}.$$

The matrices B and C are parts of the matrices $\mathfrak{B}$ and $\mathfrak{C}$, respectively. Clearly, rank $B=k+1$ and rank $C=n+1$.

Let $x^1,\dots,x^{k+1}:\mathbb{W}\longrightarrow\mathbb{R}$ be the coordinate functions with respect to the basis $w_1,\dots,w_{k+1}$, $x^i(w_j)=\delta^i_j,\quad i,j=1,\dots,k+1$. Observe that for any $j=1,\dots,k+1$, we have $x^j=w^j|_{\mathbb{W}}$. Then for any $j\in J$, we have

$$e^j|_{\mathbb{W}}=b^j_1x^1+\cdots+b^j_{k+1}x^{k+1}\ .$$

Similarly, let $x_1,\dots,x_{n+1}:\mathbb{W}'\longrightarrow\mathbb{R}$ be the coordinate functions with respect to the basis $w^{k+2},\dots,w^{N+1}$, $x_i(w^{k+j+1})=\delta^j_i,\quad i,j=1,\dots,n+1$. Observe that for any $j=1,\dots,n+1$, we have $x_j=w_{k+j+1}|_{\mathbb{W}}$. Then for any $j\in J$, we have

$$e_j|_{\mathbb{W}'}=c_j^{k+2}x_1+\cdots+c_j^{N+1}x_{n+1}\ .$$

Thus, the columns of B and C describe the coordinates of the functions $e^j|_{\mathbb{W}}$ and $e_j|_{\mathbb{W}'}$.

Denote by $B[j_1,j_2,\dots,j_{k+1}]$, the determinant of the $(k+1)\times(k+1)$-submatrix of B formed by the rows $1,2,\dots,k+1$ and columns $j_1<j_2<\cdots<j_{k+1}$. Denote by $C[l_1,l_2,\dots,l_{n+1}]$, the determinant of the $(n+1)\times(n+1)$-submatrix of C formed by the rows $1,2,\dots,n+1$ and columns $l_1<l_2<\cdots<l_{n+1}$.

The values of functions f^j, $j \in \mathcal{J}$, at the vertices of the affine arrangement $\mathcal{A}[\tau]$ in $\mathcal{V}$ can be computed in terms of the minors of the matrix B as follows.

Lemma 3.3.1. Let $P = \mathcal{H}^{j_1} \cap \cdots \cap \mathcal{H}^{j_k}$ be a vertex of the arrangement $\mathcal{A}[\tau]$ lying in $\mathcal{V}$, $j_1 < \cdots < j_k$. Let $j \in \mathcal{J} \setminus \{j_1, \ldots, j_k\}$. Assume that $j_1 < \cdots < j_m < j < j_{m+1} < \cdots < j_k$ where $m \leq k$. Then

$$f^j(P) = (-1)^{k+m} \frac{B[j_1, \ldots, j_m, j, j_{m+1}, \ldots, j_k]}{B[j_1, \ldots, j_k, N+1]} \ . \qquad \square$$

A similar statement holds for the arrangement $\mathcal{A}[\tau']$ and matrix C.

3.4 A result on determinants

Theorem 3.4.1 (M, pp. 165–169)**.** *Let $1 \leq j_1 < \cdots < j_{k+1} \leq N+1$ be a subset and let $1 \leq j_{k+2} < \cdots < j_{N+1} \leq N+1$ be the complementary subset. Then*

$$B[j_1, \ldots, j_{k+1}] = (-1)^\sigma \cdot \det \mathfrak{B} \cdot C[j_{k+2}, \ldots, j_{N+1}] \ ,$$

where

$$\sigma = 1 + 2 + \cdots + (k+1) + j_1 + j_2 + \cdots + j_{k+1} \ .$$

3.5 Dual matroids

Let $M = (J, \mathcal{I})$ be a matroid. A subset $X \subset J$ belongs to $\mathcal{I}$ if and only if X is contained in at least one basis of M. In this way, a matroid is characterized by its collection of bases.

The *dual* of M is the matroid M' on the same ground set J, whose bases are the complements in J of the bases of M. Evidently, $(M')' = M$.

Theorem 3.5.1 (O, B)**.** *Let M and M' be dual matroids. Then*

- *For any subset $X \subset J$,* $\operatorname{corank}_M X \ = \ \operatorname{nullity}_{M'} \hat{X}$,
- $\mathrm{T}(M; x, y) = \mathrm{T}(M'; y, x)$.

Lemma 3.5.2. Let M and M' be dual matroids, $|J| \geq 2$. Then $b_M^{10} = b_{M'}^{10}$. $\square$

The lemma follows from Theorems 2.2.1 and 3.5.1.

Lemma 3.5.3. Let M and M' be dual matroids. Let $X \subset J$, $|X| < |J|$. Then the matroids M/X and $M' - X$ are dual.

The lemma is proved in Section 5.2.

Theorem 3.5.4. *Let X be a spacious flat in M, $1 < |X| < |J| - 1$. Then the complement of X, the set $\hat{X} = J \setminus X$, is a spacious flat in M'. Furthermore,*

- $\mathrm{l}_M X = \mathrm{w}_{M'} \hat{X}$,
- $\mathrm{w}_M X = \mathrm{l}_{M'} \hat{X}$,
- $\mathrm{vol}_M X = \mathrm{vol}_{M'} \hat{X}$.

Theorem 3.5.4 is proved in Section 5.4.

Recall that for a parallelism (X, a, b) in M, we denote by $\hat{X}(a, b)$ the complement of $X \cup \{a, b\}$ in J.

Theorem 3.5.5. *Let (X, a, b) be a parallelism in M, $|X| < |J| - 2$. Assume that $\mathrm{corank}_M(J \setminus \{a, b\}) = 0$. Then $(\hat{X}(a, b), a, b)$ is a parallelism in M'. Furthermore,*

- $\mathrm{l}_M X = \mathrm{w}_{M'}(\hat{X}(a, b), a, b)$,
- $\mathrm{w}_M(X, a, b) = \mathrm{l}_{M'} \hat{X}(a, b)$,
- $\mathrm{vol}_M(X, a, b) = \mathrm{vol}_{M'}(\hat{X}(a, b), a, b)$.

Theorem 3.5.5 is proved in Section 5.5.

Lemma 3.5.6. Let τ and τ' be dual admissible pairs. Then the matroids $M[\tau]$ and $M[\tau']$ are dual. □

The lemma is a corollary of Theorem 3.4.1.

3.6 Bounded domains, edges, and parallelisms of dual arrangements

Lemma 3.6.1. The number of bounded domains of the arrangements $A[\tau]$ and $A[\tau']$ are equal. □

The lemma follows from Theorem 2.5.1 and Lemmas 3.5.2, 3.5.6.

For dual admissible pairs τ and τ', write

- $A[\tau] = \{H^j : j \in J\}, \qquad A[\tau'] = \{H_j : j \in J\},$
- $\mathcal{A}[\tau] = \{\mathcal{H}^j : j \in \mathcal{J}\}, \qquad \mathcal{A}[\tau'] = \{\mathcal{H}_j : j \in \mathcal{J}\},$
- $\mathcal{F}[\tau] = \{f^j : j \in \mathcal{J}\}, \qquad \mathcal{F}[\tau'] = \{f_j : j \in \mathcal{J}\}.$

Let L be a spacious edge of the arrangement $A[\tau]$ and X the associated flat in the matroid $M[\tau]$. Denote by $\hat{L}$ the edge $\cap_{j\in\hat{X}} H_j$ of the arrangement $A[\tau']$.

Lemma 3.6.2. Let L be a spacious edge of $A[\tau]$. Assume that L is not a hyperplane. Then $\hat{L}$ is a spacious edge in $A[\tau']$. Furthermore,

- $\mathrm{l}_{A[\tau]}L = \mathrm{w}_{A[\tau']}\hat{L}$,
- $\mathrm{w}_{A[\tau]}L = \mathrm{l}_{A[\tau']}\hat{L}$,
- $\mathrm{vol}_{A[\tau]}L = \mathrm{vol}_{A[\tau']}\hat{L}$. □

The lemma follows from Theorem 3.5.4.

The edges L and $\hat{L}$ will be called *dual*.

Let (L, H^a, H^b) be a parallelism in the arrangement $A[\tau]$. Let X be the flat in the matroid $M[\tau]$ associated to the edge L. Denote by $\hat{L}(H^a, H^b)$ the edge $\cap_{j\in\hat{X}(a,b)} H_j$ in the arrangement $A[\tau']$.

Lemma 3.6.3. Let (L, H^a, H^b) be a parallelism in $A[\tau]$. Then $(\hat{L}(H^a, H^b), H_a, H_b)$ is a parallelism in $A[\tau']$. Furthermore,

- $\mathrm{l}_{A[\tau]}L = \mathrm{w}_{A[\tau']}(\hat{L}(H^a, H^b), H_a, H_b)$,
- $\mathrm{w}_{A[\tau]}(L, H^a, H^b) = \mathrm{l}_{A[\tau']}\hat{L}(H^a, H^b)$,
- $\mathrm{vol}_{A[\tau]}(L, H^a, H^b) = \mathrm{vol}_{A[\tau']}(\hat{L}(H^a, H^b), H_a, H_b)$. □

The lemma follows from Theorem 3.5.5.

The parallelisms (L, H^a, H^b) and $(\hat{L}(H^a, H^b), H_a, H_b)$ will be called *dual*.

3.7 Relation between the values of polynomials defining dual arrangements

Lemma 3.7.1. Let (L, H^j, H^{N+1}) be a parallelism in $A[\tau]$ and $(\hat{L}(H^j, H^{N+1}), H_j, H_{N+1})$ the dual parallelism in $A[\tau']$. The product of the value of f^j on $L \setminus H^{N+1}$ and the value of f_j on $\hat{L}(H^j, H^{N+1}) \setminus H_{N+1}$ is -1.

Lemma 3.7.1 is proved in Section 5.6. Also, see Section 2.6.

Chapter 4

Determinant formula

Several constructions from the works [V1] and [DT] on the definition of *the period matrix* and the evaluation of its determinant are restated in brief in the Sections 4.1—4.3, ending with Theorem 4.3.1 evaluating the determinant of the period matrix in terms of the *beta function* and *critical values* (Section 4.3).

The Sections 4.4 and 4.5 focus on the definition of *associated period matrices of dual arrangements* finishing with the statement of the main result of the paper (Theorem 4.5.1) on the evaluation of the product of determinants of associated period matrices of dual arrangements.

4.1 Weighted arrangements

Let $A = \{H^j : j \in J\}$ be a projective arrangement. A set of numbers $\alpha = \{\alpha_j : j \in J\}$ with the property $\sum_{j\in J} \alpha_j = 0$ will be called *a set of weights*, where α_j is *the weight of* H^j. The pair (A, α) will be called *a weighted arrangement*.

The weight of an edge L of A is the sum, $\alpha(L)$, of the weights of the hyperplanes that contain L.

Let $A[\tau] = \{H^j : j \in J\}$ and $A[\tau'] = \{H_j : j \in J\}$ be dual arrangements. Let $\alpha = \{\alpha_j : j \in J\}$ be a set of numbers with the property $\sum_{j\in J} \alpha_j = 0$. Then $(A[\tau], \alpha)$ and $(A[\tau'], \alpha)$ will be called *dual weighted arrangements*.

Let $(A[\tau], \alpha)$ be a weighted arrangement. For $j \in \mathcal{J}$, the number α_j is also called the *weight of the affine hyperplane* $\mathcal{H}^j$. The associated affine arrangement $\mathcal{A}[\tau]$ with weight α_j assigned to the hyperplane $\mathcal{H}^j$ for any $j \in \mathcal{J}$, will be called *the associated weighted affine arrangement* and denoted $(\mathcal{A}[\tau], \alpha)$.

We always assume that for any $j \in \mathcal{J}$, the weight α_j of the hyperplane $\mathcal{H}^j$ is a positive real number.

4.2 Period matrix of a weighted arrangement

Let $(\mathcal{A}[\tau], \alpha)$ be a weighted affine arrangement. We have $\mathcal{A}[\tau] = \{\mathcal{H}^j : j \in \mathcal{J}\}$, where $\mathcal{J} = \{1, \dots, N\}$. Consider $\mathcal{J}$ as an ordered set with the standard order.

In [DT], for an ordered affine arrangement, a collection $\beta\mathbf{kbc}(\mathcal{A}[\tau])$ of ordered k-tuples $B = (\mathcal{H}^{j_1}, \mathcal{H}^{j_2}, \dots, \mathcal{H}^{j_k})$ is defined. For any tuple of that collection, the intersection $\cap_{l=1}^{k} \mathcal{H}^{j_l}$ is a vertex. The number of tuples in that collection is equal to the number $\beta = \beta(\mathcal{A}[\tau])$ of bounded domains of the affine arrangement.

Elements of the collection are called $\beta\mathbf{kbc}$-bases. The collection itself is ordered lexicographically, $\beta\mathbf{kbc}(\mathcal{A}[\tau]) = \{B_1, \dots, B_\beta\}$.

4.2.1 Logarithmic k-forms

For $B = (\mathcal{H}^{j_1}, \mathcal{H}^{j_2}, \dots, \mathcal{H}^{j_k}) \in \beta\mathbf{kbc}(\mathcal{A}[\tau])$, define the associated flag of edges

$$\xi(B) = (L_B^0 \subset L_B^1 \subset \cdots \subset L_B^k = \mathcal{V}) \ ,$$

where $L_B^i = \mathcal{H}^{j_{i+1}} \cap \mathcal{H}^{j_{i+2}} \cap \cdots \cap \mathcal{H}^{j_k}$ and $\dim L_B^i = i$.

Let $\mathcal{F}[\tau] = \{f^j : j \in \mathcal{J}\}$ be the polynomial arrangement associated to τ. Let $L \subset \mathcal{V}$ be an edge of $\mathcal{A}[\tau]$. Assign to L the differential 1-form

$$\omega(L) = \sum_{j,\ L \subset \mathcal{H}^j} \alpha_j \frac{df^j}{f^j} \ .$$

Assign to every $B = (\mathcal{H}^{j_1}, \mathcal{H}^{j_2}, \dots, \mathcal{H}^{j_k})$ the differential k-form

$$\phi(B) = \omega(L_B^0) \wedge \omega(L_B^1) \wedge \cdots \wedge \omega(L_B^{k-1}) \ .$$

Thus, we get an ordered set of differential k-forms

$$\Psi = \{\phi^1, \phi^2, \dots, \phi^\beta\} \ ,$$

where ϕ^i is the form corresponding to the i-th element of the collection $\beta\mathbf{kbc}(\mathcal{A}[\tau])$.

4.2.2 The βkbc-enumeration of bounded domains

Let $\xi = (L^0 \subset L^1 \subset \cdots \subset L^k)$ be a flag of edges of $\mathcal{A}[\tau]$ with $\dim L^j = j$ for all j. Let Δ be a bounded domain of $\mathcal{A}[\tau]$ and $\overline{\Delta}$ its closure. The flag ξ is said to be *adjacent to* Δ, if $\dim (L^j \cap \overline{\Delta}) = j$, for all j.

Denote by $\mathtt{Ch}(\mathcal{A}[\tau])$ the set of bounded domains of the arrangement $\mathcal{A}[\tau]$. In [DT], a bijection

$$C : \beta\mathbf{kbc}(\mathcal{A}[\tau]) \longrightarrow \mathtt{Ch}(\mathcal{A}[\tau])$$

is defined such that for any $B \in \beta\,\mathbf{kbc}(\mathcal{A}[\tau])$, the associated flag $\xi(B)$ is adjacent to the bounded domain $C(B)$.

Thus, one has $\mathtt{Ch}(\mathcal{A}[\tau]) = \{\Delta_1, \Delta_2, \ldots, \Delta_\beta\}$, where

$$\Delta_s = C(B_s)\ , \quad s = 1, 2, \ldots, \beta\ .$$

This is called the $\beta\mathbf{kbc}(\mathcal{A}[\tau])$-*ordering* of the bounded domains of $\mathcal{A}[\tau]$.

4.2.3 Orientation of bounded domains

Let $\Delta = C(B)$ and $\xi(B) = (L^0 \subset L^1 \subset \cdots \subset L^k)$. The flag $\xi(B)$ is adjacent to the domain $\Delta = C(B)$ and defines the *intrinsic orientation* of Δ, see Section 6.2 in [V2].

The intrinsic orientation is the orientation of the unique orthonormal frame $\{v_1, v_2, \ldots, v_k\}$ where v_i is the unit vector originating from the vertex L^0 in the direction of $L^i \cap \overline{\Delta}$.

4.2.4 The period matrix

Let $(\mathcal{A}[\tau], \alpha)$ be a weighted affine arrangement and $\mathcal{F}[\tau] = \{f^j : j \in \mathcal{J}\}$ the associated polynomial arrangement. Consider the multi-valued function

$$U^\alpha \ = \ \prod_{j \in \mathcal{J}} (f^j)^{\alpha_j} \ : \ \mathcal{V} \longrightarrow \mathbb{C}\ .$$

Fix a uni-valued branch of each $(f^j)^{\alpha_j}$ on each bounded domain Δ. The $\beta \times \beta$-matrix

$$\mathrm{PM}\,(\mathcal{A}[\tau], \alpha) \ = \ \left(\int_{\Delta_s} U^\alpha\, \phi^t \right)$$

is called *the period matrix* of the weighted affine arrangement $(\mathcal{A}[\tau], \alpha)$. In this matrix the differential forms are ordered as in Section 4.2.1, the domains are ordered as in Section 4.2.2, the domains are oriented as in Section 4.2.3, and in each integral of the matrix the uni-valued branch of U^α is chosen since the branches of each of its factors were chosen.

Denote the determinant of the period matrix by $\mathrm{D}(\mathcal{A}[\tau], \alpha)$.

Remark. The ordered set of differential k-forms described in Section 4.2.1, the order on the set of bounded domains as in Section 4.2.2, the orientation on bounded domains as in Section 4.2.3 will be called *canonical*.

4.3 Determinant of the period matrix

4.3.1 Beta function

Let $\mathcal{L}_-[\tau]$ be the set of all edges of $A[\tau]$ lying in H^{N+1} and $\mathcal{L}_+[\tau]$ the set of all other edges. The *beta function of the weighted affine arrangement* $(\mathcal{A}[\tau], \alpha)$ is defined in [V1] as

$$B(\mathcal{A}[\tau], \alpha) \;=\; \frac{\prod_{L\in\mathcal{L}_+[\tau]} \Gamma(\alpha(L)+1)^{\mathrm{vol}_{A[\tau]}(L)}}{\prod_{L\in\mathcal{L}_-[\tau]} \Gamma(-\alpha(L)+1)^{\mathrm{vol}_{A[\tau]}(L)}} \quad ,$$

where Γ is Euler's gamma function.

4.3.2 Critical values

Let Δ be a bounded domain of $\mathcal{A}[\tau]$. Let $j \in \mathcal{J}$. Let Σ be the $\mathcal{H}^j$-external supporting face of Δ. The value of the chosen branch of $(f^j)^{\alpha_j}$ on Σ is called *the critical value of* $(f^j)^{\alpha_j}$ *on* Δ and denoted by $c\,((f^j)^{\alpha_j}, \Delta)$.

4.3.3 Evaluation of the determinant

Theorem 4.3.1 (V1, DT)**.** *The determinant of the period matrix is given by the following formula:*

$$\mathrm{D}(\mathcal{A}[\tau], \alpha) = B(\mathcal{A}[\tau], \alpha) \;\cdot \prod_{\substack{j\in\mathcal{J} \\ \Delta\in \mathtt{Ch}\mathcal{A}[\tau]}} c\,((f^j)^{\alpha_j}, \Delta) \;.$$

4.4 Special choice of branches

The definition of the period matrix $\mathrm{PM}(\mathcal{A}[\tau], \alpha)$ involves the choice of branches of each $(f^j)^{\alpha_j}$ on each bounded domain of $\mathcal{A}[\tau]$. In this paper, we choose the branches as follows.

Let (L, H^j, H^{N+1}) be a parallelism in $A[\tau]$. On all bounded domains of $\mathcal{A}[\tau]$ whose $\mathcal{H}^j$-external supporting edge is L, choose the argument of f^j to be the same.

Then for any two bounded domains Δ and Δ' with the same $\mathcal{H}^j$-external supporting edge L, the values $c\left((f^j)^{\alpha_j}, \Delta\right)$ and $c\left((f^j)^{\alpha_j}, \Delta'\right)$ will be equal. Denote this common value by $c\left((f^j)^{\alpha_j}, L\right)$.

Repeating this process for all parallelisms (L, H^j, H^{N+1}) in $A[\tau]$ gives a choice of branches of each $(f^j)^{\alpha_j}$ on each bounded domain of $\mathcal{A}[\tau]$.

Such a choice of branches will be called *special*.

Lemma 4.4.1. For a special choice of branches, construct the period matrix $\mathrm{PM}(\mathcal{A}[\tau], \alpha)$. Then its determinant is given by the formula:

$$\mathrm{D}(\mathcal{A}[\tau], \alpha) = B(\mathcal{A}[\tau], \alpha) \cdot \prod c\left((f^j)^{\alpha_j}, L\right)^{\mathrm{vol}_{A[\tau]}(L, H^j, H^{N+1})} ,$$

where the product is taken over all parallelisms (L, H^j, H^{N+1}). □

The lemma follows from Lemma 2.6.1 and Theorem 4.3.1.

4.5 Associated period matrices of dual arrangements

Let $(\mathcal{A}[\tau], \alpha)$ and $(\mathcal{A}[\tau'], \alpha)$ be dual weighted affine arrangements. For each of them we can define period matrices and calculate their determinants. The period matrices depend on the choice of branches of functions $(f^j)^{\alpha_j}$ and $(f_j)^{\alpha_j}$ on bounded domains of those arrangements.

In this section we define the associated choices of branches in such a way that the determinants of period matrices will be related.

Consider $\mathcal{J} = \{1, \ldots, N\}$ with the standard order. Then we have the canonical ordered set of differential k-forms associated with $\mathcal{A}[\tau]$, ordering on the set of bounded domains of $\mathcal{A}[\tau]$, and orientation on each bounded domain. We also have the canonical ordered set of differential n-forms associated with $\mathcal{A}[\tau']$, ordering on the set of bounded domains of $\mathcal{A}[\tau']$, and orientation on each bounded domain.

Take an arbitrary special choice of branches of the functions $(f^j)^{\alpha_j}$ on bounded domains of the arrangement $\mathcal{A}[\tau]$. We will define now the associated special choice of branches of functions $(f_j)^{\alpha_j}$ on bounded domains of the arrangement $\mathcal{A}[\tau']$.

Let (L, H^j, H^{N+1}) be a parallelism in $A[\tau]$ and $(\hat{L}(H^j, H^{N+1}), H_j, H_{N+1})$ the dual parallelism in $A[\tau']$. Suppose that in the definition of $\mathrm{PM}(\mathcal{A}[\tau], \alpha)$, θ is the chosen argument of f^j on a bounded domain with $\mathcal{H}^j$-external supporting edge L. On each bounded domain of $\mathcal{A}[\tau']$ with $\mathcal{H}_j$-external supporting edge $\hat{L}(H^j, H^{N+1})$, choose the argument of f_j to be $-\theta + \pi$, see Lemma 3.7.1.

With this choice of branches, define the period matrix, $\mathrm{PM}(\mathcal{A}[\tau'], \alpha)$, of $(\mathcal{A}[\tau'], \alpha)$.

The period matrices $\mathrm{PM}(\mathcal{A}[\tau], \alpha)$ and $\mathrm{PM}(\mathcal{A}[\tau'], \alpha)$ will be called *associated*.

The main result of this paper is the following theorem.

Theorem 4.5.1. *The product of determinants of the associated period matrices is given by the formula:*

$$\mathrm{D}(\mathcal{A}[\tau], \alpha) \cdot \mathrm{D}(\mathcal{A}[\tau'], \alpha) = \left[\frac{\prod_{j \in \mathcal{J}} e^{\pi i \alpha_j} \Gamma(\alpha_j + 1)}{\Gamma(\sum_{j \in \mathcal{J}} \alpha_j + 1)} \right]^{\beta} ,$$

where β is the number of bounded domains in $\mathcal{A}[\tau]$.

Recall that the number of bounded domains in $\mathcal{A}[\tau]$ is equal to the number of bounded domains in $\mathcal{A}[\tau']$.

The theorem is proved in Section 5.7.

Chapter 5

Proofs

The topic of this chapter is the proof of all the results that were postponed hitherto. The chapter ends with the proof of the main result of this paper.

5.1 Proof of Lemma 2.6.1

Let (L, H^a, H^b) be a parallelism of the arrangement A in a projective space V. Let Σ be a domain of the induced arrangement A_L on L, bounded with respect to the hyperplane $H^b \cap L$. It is enough to show that the number of bounded domains with $\mathcal{H}^a$-external supporting face Σ is $\mathrm{w}_A(L, H^a, H^b)$, the discrete width of the parallelism (L, H^a, H^b).

Denote by X the flat associated to L in the matroid, M, of the arrangement A. By definition, $\mathrm{w}_A(L, H^a, H^b) = b^{10}_{M-\hat{X}(a,b)}$.

Suppose that L is a vertex. Then, by Theorem 2.5.1, $b^{10}_{M-\hat{X}(a,b)}$ is the number of bounded domains formed by the set of hyperplanes

$$A^{(L,H^a,H^b)} = \{H^j : j \in X \cup \{a, b\}\} \ .$$

Clearly, bounded domains of the arrangement $\mathcal{A}$ with $\mathcal{H}^a$-external supporting face Σ are in one-to-one correspondence with bounded domains formed by $A^{(L,H^a,H^b)}$. Thus, the number of bounded domains with $\mathcal{H}^a$-external supporting face Σ is $\mathrm{w}_A(L, H^a, H^b)$.

If L is not a vertex, consider a subspace $V' \subset V$, $\dim V' = \operatorname{codim} L$, such that V' intersects L transversally. Then V' also intersects H^a and H^b transversally. Consider the induced arrangement $\{H^j \cap V' : j \in X \cup \{a, b\}\}$ on V'. Each bounded domain of A with $\mathcal{H}^a$-external support Σ determines a unique bounded domain (with respect to

the hyperplane $H^b \cap V'$) of the new arrangement. The number of which is $b^{10}_{M-\hat{X}(a,b)}$ by Theorem 2.5.1. □

5.2 Proof of Lemma 3.5.3

Let $B \subset \hat{X}$ be a basis of M/X. Then there exists a maximal independent subset Y of X such that $B \cup Y$ is a basis of M. We want to show that $\hat{X} \setminus B$ is a basis of $M' - X$. That is, $\hat{X} \setminus B$ is a maximal independent subset of $\hat{X}$ in M'.

Since $B \cup Y$ is a basis of M, $(\hat{X} \setminus B) \cup (X \setminus Y)$ is a basis of M'. Hence, $\hat{X} \setminus B$ is independent. Now, $|B| = \text{corank}_M X = \text{nullity}_{M'} \hat{X} = |\hat{X}| - \text{rank}_{M'} \hat{X}$. Therefore, $|\hat{X} \setminus B| = \text{rank}_{M'} \hat{X}$. This shows that $\hat{X} \setminus B$ is maximally independent in $\hat{X}$ in M'.

Conversely, let B be a basis of $M' - X$. Then B is a maximal independent subset of $\hat{X}$ in M'. Hence there exists an independent subset Y of X in M' such that $B \cup Y$ is a basis of M'. Thus, $(\hat{X} \setminus B) \cup (X \setminus Y)$ is a basis of M. Hence, $X \setminus Y$ is independent in M. It remains to show that it is maximal independent.

Since $B \cup Y$ is a basis of M' and B is a maximal independent subset of $\hat{X}$ in M', $|Y| = \text{corank}_{M'} \hat{X} = \text{nullity}_M X$. Hence, $|X \setminus Y| = |X| - \text{nullity}_M X = \text{rank}_M X$. □

5.3 Flats of the dual matroid

Lemma 5.3.1. Let $M = (J, \mathcal{I})$ be a matroid. Let $X \subset J$ be a subset, $1 < |X| < |J| - 1$. Then $\hat{X}$ is a flat in the dual matroid M' if and only if the deletion $M - \hat{X}$ does not have an isthmus.

Proof. Suppose that $M - \hat{X}$ does not have an isthmus. Then for every $e \in X$, $\text{rank}_{M-\hat{X}} X \setminus \{e\} = \text{rank}_{M-\hat{X}} X$. Hence, for every $e \in X$, $\text{rank}_M X \setminus \{e\} = \text{rank}_M X$. That is, for every $e \in X$, $\text{nullity}_M X \setminus \{e\} = \text{nullity}_M X - 1$. Thus, for every $e \in X$, $\text{corank}_{M'} \hat{X} \cup \{e\} = \text{corank}_{M'} \hat{X} - 1$. Then for every $e \in X$, $\text{rank}_{M'} \hat{X} \cup \{e\} = \text{rank}_{M'} \hat{X} + 1$. This shows that the subset $\hat{X} \subset J$ is a flat in M'.

The converse follows by tracing the arguments backward. □

5.4 Proof of Theorem 3.5.4

Lemma 5.4.1. Let X be a spacious flat in M, $1 < |X| < |J| - 1$. Then the deletion $M - \hat{X}$ does not have an isthmus.

Proof. Let e be an isthmus in the matroid $M-\hat{X}$. By Theorem 2.2.2, $\mathrm{T}(M-\hat{X};x,y) = x\,\mathrm{T}((M-\hat{X})/\{e\};x,y)$. Hence, $\mathrm{w}_M(X) = b^{10}_{M-\hat{X}} = b^{00}_{(M-\hat{X})/\{e\}}$. By Theorem 2.2.1, $b^{00}_{(M-\hat{X})/\{e\}} = 0$ (since $|X| \geq 2$). This contradicts our assumption that the flat X is spacious. Hence $M-\hat{X}$ does not have an isthmus. □

Proof of Theorem 3.5.4. Let M and M' be dual matroids. Let X be a spacious flat in M, $1 < |X| < |J|-1$. It follows from Lemmas 5.3.1, 5.4.1 that $\hat{X}$ is a flat in M'.

Furthermore, $\mathrm{l}_M X = b^{10}_{M/X} = b^{10}_{M'-X} = \mathrm{w}_{M'}\hat{X}$ by Lemmas 3.5.2, 3.5.3. Hence, $\mathrm{vol}_M X = \mathrm{vol}_{M'}\hat{X}$. □

5.5 Proof of Theorem 3.5.5

Lemma 5.5.1. Let (X,a,b) be a parallelism in M, $|X| < |J|-2$. Then the deletion $M-\hat{X}(a,b)$ does not have an isthmus. □

The proof is similar to the proof of Lemma 5.4.1.

Proof of Theorem 3.5.5. Let (X,a,b) be a parallelism in M, $|X| < |J|-2$, $\mathrm{vol}_M(X,a,b) \neq 0$ and $\mathrm{corank}_M(J \setminus \{a,b\}) = 0$. It follows from Lemmas 5.3.1, 5.5.1 that $\hat{X}(a,b)$ is a flat in M'.

Clearly, $a,b \notin \hat{X}(a,b)$ and $\mathrm{rank}_{M'}\{a,b\} = 2$.

Since (X,a,b) is a parallelism, $\mathrm{rank}_M X \cup \{a,b\} = \mathrm{rank}_M X + 1$. So, $\mathrm{corank}_M X = \mathrm{corank}_M X \cup \{a,b\} + 1$. Thus, $\mathrm{nullity}_{M'}\hat{X}(a,b) \cup \{a,b\} = \mathrm{nullity}_{M'}\hat{X}(a,b) + 1$. Hence $\mathrm{rank}_{M'}\hat{X}(a,b) \cup \{a,b\} = \mathrm{rank}_{M'}\hat{X}(a,b) + 1$. This proves that $(\hat{X}(a,b),a,b)$ is a parallelism in M'.

Furthermore, $\mathrm{l}_M X = b^{10}_{M/X} = b^{10}_{M'-X} = \mathrm{w}_{M'}(\hat{X}(a,b),a,b)$ by Lemmas 3.5.2, 3.5.3. Hence, $\mathrm{vol}_M(X,a,b) = \mathrm{vol}_{M'}(\hat{X}(a,b),a,b)$. □

5.6 Proof of Lemma 3.7.1

Lemma 5.6.1. Let (L,H^a,H^b) be a parallelism in $A[\tau]$ and let $P = H^{j_1} \cap \cdots \cap H^{j_k}$ be a vertex on $L \setminus H^b$ for some $I = \{j_1,\ldots,j_k\} \subset J$. Let $\hat{I}(a,b) = J \setminus (I \cup \{a,b\})$. Then $|\hat{I}(a,b)| = n$ and $\check{P} = \cap_{j\in\hat{I}(a,b)} H_j$ is a vertex on $\hat{L}(H^a,H^b) \setminus H_b$ in the dual arrangement $A[\tau']$.

Proof. Since $\mathrm{corank}_{M[\tau]} I = 1$ and $a \notin X$, $\mathrm{rank}_{M[\tau]} I \cup \{a\} = \mathrm{rank}\, M$. Thus, $\mathrm{nullity}_{M[\tau]} I \cup \{a,b\} = 1$ and hence $\mathrm{corank}_{M[\tau]} \hat{I}(a,b) = 1$. So, $\check{P}$ is a vertex on $\hat{L}(H^a,H^b)$.

It remains to show that $\check{P} \notin H_b$. Assume that $\check{P} \in H_b$. Then $\text{nullity}_{M[\tau']}\hat{I}(a,b)+1 = \text{nullity}_{M[\tau']}\hat{I}(a,b) \cup \{b\}$. Thus, $\text{corank}_{M[\tau]}I \cup \{a,b\} + 1 = \text{corank}_{M[\tau]}I \cup \{a\}$. But, $\text{corank}_{M[\tau]}I \cup \{a\} = 0$. This is a contradiction. □

Proof of Lemma 3.7.1. Let (L, H^j, H^{N+1}) be a parallelism in $A[\tau]$ and $(\hat{L}(H^j, H^{N+1}), H_j, H_{N+1})$ the dual parallelism in $A[\tau']$. Let P and $\check{P}$ be vertices on $L \setminus H^{N+1}$ and $\hat{L}(H^j, H^{N+1}) \setminus H_{N+1}$, respectively, as in Lemma 5.6.1. We want to show that $f^j(P) \cdot f_j(\check{P}) = -1$.

Let I be as in Lemma 5.6.1. Assume that $I = \{j_1, j_2, \ldots, j_k\}$, $j_1 < j_2 < \cdots < j_k$, and $\hat{I}(j, N+1) = \{j_{k+1}, \ldots, j_{k+n}\}$, $j_{k+1} < \cdots < j_{k+n}$. Let m and m' be the number of elements less than j in I and $\hat{I}(a,b)$, respectively. Then $m + m' = j - 1$.

By Lemma 3.3.1,

$$f^j(P) = (-1)^{k+m} \frac{B[j_1, \ldots, j_m, j, j_{m+1}, \ldots, j_k]}{B[j_1, \ldots, j_k, N+1]}$$

and

$$f_j(\check{P}) = (-1)^{n+m'} \frac{C[j_{k+1}, \ldots, j_{k+m'}, j, j_{k+m'+1}, \ldots, j_{k+n}]}{C[j_{k+1}, \ldots, j_{k+n}, N+1]} .$$

By Theorem 3.4.1, $f^j(P) \cdot f_j(\check{P}) = (-1)^{(j+N+1)+(k+m)+(n+m')} = -1$. □

5.7 Proof of Theorem 4.5.1

Lemma 5.7.1. Let M and M' be dual matroids. Let X be a flat in M, $|X| = 1$.

- If X is also a flat in M', then $\text{l}_M X + \text{l}_{M'} X = b^{10}_M$.
- If X is not a flat in M', then $\text{l}_M X = b^{10}_M$.

Proof. Suppose that X is a flat in both M and M'. Then $\text{l}_M X = b^{10}_{M/X}$ and $\text{l}_{M'} X = b^{10}_{M'/X} = b^{10}_{M-X}$. Hence, $\text{l}_M X + \text{l}_{M'} X = b^{10}_{M/X} + b^{10}_{M-X} = b^{10}_M$, by Theorem 2.2.2.

Let X be a flat in M. Suppose that X is not a flat in M'. Then the matroid $M - X$ has an isthmus by Lemma 5.3.1. Hence, $b^{10}_{M-X} = 0$ by Theorem 2.2.2. Hence, $b^{10}_{M/X} = b^{10}_M$, by Theorem 2.2.2. That is, $\text{l}_M X = b^{10}_M$. □

Theorem 5.7.2. *Let $(\mathcal{A}[\tau], \alpha)$ and $(\mathcal{A}[\tau'], \alpha)$ be dual weighted affine arrangements. Then*

$$\text{B}(\mathcal{A}[\tau], \alpha) \cdot \text{B}(\mathcal{A}[\tau'], \alpha) = \left[\frac{\prod_{j \in \mathcal{J}} \Gamma(\alpha_j + 1)}{\Gamma(\sum_{j \in \mathcal{J}} \alpha_j + 1)} \right]^{\beta} .$$

Proof. Let L be a spacious edge of $A[\tau]$ that is not a hyperplane. By Lemma 3.6.2, the dual edge $\hat{L}$ is spacious in $A[\tau']$. Furthermore, $\mathrm{vol}_{A[\tau]}L = \mathrm{vol}_{A[\tau']}\hat{L}$.

Clearly, if $L \in \mathcal{L}_-[\tau]$, then $\hat{L} \in \mathcal{L}_+[\tau']$ and if $L \in \mathcal{L}_+[\tau]$, then $\hat{L} \in \mathcal{L}_-[\tau']$. We also have $\alpha(L) + \alpha(\hat{L}) = 0$.

If $L \in \mathcal{L}_+[\tau]$, then

$$\Gamma(\alpha(L)+1)^{\mathrm{vol}_{A[\tau]}L} = \Gamma(-\alpha(\hat{L})+1)^{\mathrm{vol}_{A[\tau']}\hat{L}} \ .$$

If $L \in \mathcal{L}_-[\tau]$, then

$$\Gamma(-\alpha(L)+1)^{\mathrm{vol}_{A[\tau]}L} = \Gamma(\alpha(\hat{L})+1)^{\mathrm{vol}_{A[\tau']}\hat{L}} \ .$$

Hence, by Lemma 5.7.1,

$$\begin{aligned} \mathrm{B}(\mathcal{A}[\tau],\alpha)\cdot \mathrm{B}(\mathcal{A}[\tau'],\alpha) &= \frac{\prod_{j=1}^N \Gamma(\alpha_j+1)^{\mathrm{l}_{A[\tau]}(H^j)}\cdot \Gamma(\alpha_j+1)^{\mathrm{l}_{A[\tau']}(H_j)}}{\Gamma(-\alpha_{N+1}+1)^{\mathrm{l}_{A[\tau]}(H^{N+1})}\cdot \Gamma(-\alpha_{N+1}+1)^{\mathrm{l}_{A[\tau']}(H_{N+1})}} \\ &= \left[\frac{\prod_{j=1}^N \Gamma(\alpha_j+1)}{\Gamma(\sum_{j=1}^N \alpha_j+1)}\right]^\beta \end{aligned}$$

as desired. □

Proof of Theorem 4.5.1. Let $(\hat{L}(H^j,H^{N+1}),H_j,H_{N+1})$ be the parallelism in $A[\tau']$ dual to the parallelism (L,H^j,H^{N+1}) in $A[\tau]$. Then

$$c\left((f^j)^{\alpha_j},L\right)\cdot c\left((f_j)^{\alpha_j},\hat{L}(H^j,H^{N+1})\right) = e^{i\pi\alpha_j}.$$

For each $j \in \mathcal{J}$, there are β critical values of $(f^j)^{\alpha_j}$. Hence

$$\prod_{\substack{j\in\mathcal{J}\\ \Delta\in \mathtt{Ch}\mathcal{A}[\tau]}} c\left((f^j)^{\alpha_j},\Delta\right)\cdot \prod_{\substack{j\in\mathcal{J}\\ \Delta'\in \mathtt{Ch}\mathcal{A}[\tau']}} c\left((f_j)^{\alpha_j},\Delta'\right) = e^{i\pi\beta\sum_{j\in\mathcal{J}}\alpha_j} \ .$$

The theorem now follows from Theorem 5.7.2 □

Appendix A

Weak duality

In this chapter we formulate the notion of *weak duality* starting with the definition of a *weakly admissible pair*. We associate to a weakly admissible pair an arrangement of projective hyperplanes in a projective space of appropriate dimension, and a matroid. The matroids associated to weakly dual pairs turn out to be dual. The corresponding projective arrangements are called weakly dual.

The Theorems A.1.1 and A.1.2 show that weakly dual arrangements arise out of natural constructions using dual arrangements.

A.1 Statement of results

Let $\mathbb{X}$, $\mathbb{W}$ and $\mathbb{X}'$, $\mathbb{W}'$ denote the same spaces as in Section 3.1. Assume that for any $j \in J$, the functions $e^j|_{\mathbb{W}}$ and $e_j|_{\mathbb{W}'}$ are not identically zero. The pair $\tau = (\mathbb{X}, \mathbb{W})$ with this property will be called a *weakly admissible* pair in $\mathbb{X}$. Similarly, the pair $\tau' = (\mathbb{X}', \mathbb{W}')$ with this property will be called a *weakly admissible* pair in $\mathbb{X}'$. The pairs τ and τ' will be called *weakly dual*.

Clearly, any dual pairs τ and τ' are weakly dual.

Let $\tau = (\mathbb{X}, \mathbb{W})$ be a weakly admissible pair. As in Section 3.2, define the following objects. For $j \in J$, denote $E^j = \{\, x \in \mathbb{W} \;:\; e^j(x) = 0 \,\}$. These are vector subspaces of $\mathbb{W}$ of codimension one.

Denote

- $V = \mathbb{P}(\mathbb{W})$, the projective space of dimension k,
- $H^j = \mathbb{P}(E^j)$, $j \in J$, projective hyperplanes in V,
- $A[\tau] = \{\, H^j \;:\; j \in J \,\}$, the arrangement of projective hyperplanes in V.

For weakly dual pairs τ and τ', the corresponding projective arrangements $A[\tau]$ and $A[\tau']$ will be called *weakly dual*.

For a weakly admissible pair τ, introduce the matroid of τ, denoted $M[\tau]$, as the matroid of the collection of vectors $\{ e^j|_{\mathbb{W}} \ : \ j \in J \}$ in the dual space of $\mathbb{W}$.

Let τ and τ' be weakly dual pairs. Then the matroids $M[\tau]$ and $M[\tau']$ are dual.

Theorem A.1.1. *Let $A[\tau]$ and $A[\tau']$ be dual projective arrangements. Let L be a spacious edge of $A[\tau]$ and $\hat{L}$ the dual spacious edge of $A[\tau']$. Then the induced arrangement on L, denoted by $A[\tau]_L$, is weakly dual to the projective localization of $A[\tau']$ at $\hat{L}$, denoted by $P(A[\tau']^{\hat{L}})$.*

We will prove a more general result. To formulate the result we need the notion of the projective localization of a sub-arrangement.

Let $A = \{ H^j \ : \ j \in J \}$ be a projective arrangement of hyperplanes. For a subset $I \subset J$, the set of hyperplanes $\{ H^j \ : \ j \in I \}$ will be called *a sub-arrangement* of A.

Let $I \subset J$ be such that $\cap_{j\in I} H^j \neq \emptyset$. Then the projectivization of the central arrangement $\{ H^j \ : \ j \in I \}$ will be called *the projective localization of the sub-arrangement* $\{ H^j \ : \ j \in I \}$, see Section 2.1.

Theorem A.1.2. *Let $A[\tau] = \{ H^j \ : \ j \in J \}$ and $A[\tau'] = \{ H_j \ : \ j \in J \}$ be weakly dual arrangements. Let L be an edge of $A[\tau]$ and let $X = \{ j \in J \ : \ L \subset H^j \}$ be the flat associated to L in the matroid $M[\tau]$. Assume that $|X| < |J|$.*

Let $\cap_{j\in\hat{X}} H_j = \emptyset$. Then the induced arrangement on L, $A[\tau]_L$, is weakly dual to the sub-arrangement $\{ H_j \ : \ j \in \hat{X} \}$ of $A[\tau']$.

Let $\cap_{j\in\hat{X}} H_j \neq \emptyset$. Then the induced arrangement on L, $A[\tau]_L$, is weakly dual to the projective localization of the sub-arrangement $\{ H_j \ : \ j \in \hat{X} \}$ of $A[\tau']$.

A.2 Proof of Theorem A.1.2

Let $\tau = (\mathbb{X}, \mathbb{W})$ and $\tau' = (\mathbb{X}', \mathbb{W}')$ be the weakly dual admissible tuples.

For $j \in J$, denote $F^j = \{e^j = 0\} \subset \mathbb{X}$ and $F_j = \{e_j = 0\} \subset \mathbb{X}'$.

Consider the vector space $\cap_{j\in X} F^j$ with the basis e_j, $j \in \hat{X}$. Then $\mathbb{X}'/\cap_{j\in\hat{X}} F_j$ may be identified with the dual space of $\cap_{j\in X} F^j$ with the dual basis $e^j + \cap_{j\in\hat{X}} F_j$, $j \in \hat{X}$.

The classes of the elements of $\mathbb{W}'$ in the vector space $\mathbb{X}'/\cap_{j\in\hat{X}} F_j$ form a subspace of $\mathbb{X}'/\cap_{j\in\hat{X}} F_j$. Denote this subspace by $\mathbb{W}'/\cap_{j\in\hat{X}} F_j$.

Consider the tuples

$$\sigma = \left(\cap_{j\in X} F^j \ , \ \cap_{j\in X} E^j\right), \quad \sigma' = \left(\mathbb{X}'/\cap_{j\in\hat{X}} F_j \ , \ \mathbb{W}'/\cap_{j\in\hat{X}} F_j\right).$$

Remark. To make sense of the definitions of σ and σ' the following identifications are required.

- The element $e^j + \cap_{j\in\hat{X}} F_j \in \mathbb{X}'/\cap_{j\in\hat{X}} F_j$ defines a linear function on $\cap_{j\in X} F^j$ given by $(e^j + \cap_{j\in\hat{X}} F_j)(x) = e^j(x)$ for every $x \in \cap_{j\in X} F^j$.
- The element $e_j \in \cap_{j\in\hat{X}} F_j$, $j \in \hat{X}$, defines a linear function on $\mathbb{X}'/\cap_{j\in\hat{X}} F_j$ given by $e_j(x' + \cap_{j\in\hat{X}} F_j) = e_j(x')$, for every $x' \in \mathbb{X}'$.

Lemma A.2.1. The tuples σ and σ' are weakly dual.

Proof. We first show that σ and σ' are weakly admissible. For $i \in \hat{X}$, $(e^i + \cap_{j\in\hat{X}} F_j)|_{\cap_{j\in X} E^j} \equiv 0$ implies that $e^i|_{\cap_{j\in X} E^j} \equiv 0$. Thus, $\cap_{j\in X} E^j \subset E^i$. This contradicts the assumption that X is a flat.

For any $i \in \hat{X}$, $e_i|_{\mathbb{W}'/\cap_{j\in\hat{X}} F_j}$ is a nonzero function since $e_i|_{\mathbb{W}'}$ is a nonzero linear function. This shows that σ and σ' are weakly admissible tuples.

The space $\mathbb{W}'/\cap_{j\in\hat{X}} F_j$ clearly annihilates $\cap_{j\in X} E^j$.

We have $\dim \cap_{j\in X} E^j = \dim \mathbb{W} \cap (\cap_{j\in X} F^j) = \operatorname{corank}_{M[\tau]} X$. Hence the annihilator of $\cap_{j\in X} E^j$ in $\mathbb{X}'/\cap_{j\in\hat{X}} F_j$ has dimension $|\hat{X}| - \operatorname{corank}_{M[\tau]} X = |\hat{X}| - \operatorname{nullity}_{M[\tau']} \hat{X} = \operatorname{rank}_{M[\tau']} \hat{X}$.

Observe that the spaces $\mathbb{W}'/\cap_{j\in\hat{X}} F_j$ and $\mathbb{W}'/\cap_{j\in\hat{X}} E_j$ are isomorphic. Hence, $\dim \mathbb{W}'/\cap_{j\in\hat{X}} F_j = \dim \mathbb{W}' - \dim \cap_{j\in\hat{X}} E_j = \operatorname{rank} M[\tau'] - \operatorname{corank}_{M[\tau']} \hat{X} = \operatorname{rank}_{M[\tau']} \hat{X}$.

This shows that $\mathbb{W}'/\cap_{j\in\hat{X}} F_j$ is the annihilator of $\cap_{j\in X} E^j$ in $\mathbb{X}'/\cap_{j\in\hat{X}} F_j$. Hence, σ and σ' are weakly dual. □

Lemma A.2.2.

(i) The arrangement $A[\sigma]$ is the induced arrangement $A[\tau]_L$.

(ii) If $\cap_{j\in\hat{X}} H_j = \emptyset$, then the arrangement $A[\sigma']$ is the sub-arrangement $\{H_j : j \in \hat{X}\}$ of $A[\tau']$.

(iii) If $\cap_{j\in\hat{X}} H_j \neq \emptyset$, then the arrangement $A[\sigma']$ is the projective localization of the sub-arrangement $\{H_j : j \in \hat{X}\}$ of $A[\tau']$.

Proof. Statement (i) is clear.

Let $\mathbb{U} \subset \mathbb{W}'$ be a subspace such that $\mathbb{U} \oplus \cap_{j\in\hat{X}} E_j = \mathbb{W}'$. Consider the isomorphism

$$\mathbb{U} \;\to\; \mathbb{W}'/\cap_{j\in\hat{X}} F_j\,, \qquad u \;\mapsto\; u + \cap_{j\in\hat{X}} F_j\,.$$

For every $j \in \hat{X}$, the subspace $E_j \cap \mathbb{U} \subset \mathbb{U}$ corresponds to the subspace $E_j / \cap_{j \in \hat{X}} F_j \subset \mathbb{W}' / \cap_{j \in \hat{X}} F_j$. The arrangement of hyperplanes $\mathbb{P}(\mathbb{U} \cap E_j) \subset \mathbb{P}(\mathbb{U})$, $j \in \hat{X}$, in the projectivization $\mathbb{P}(\mathbb{U})$ is the arrangement described in part (ii), if $\cap_{j \in \hat{X}} H_j = \emptyset$, and is the arrangement described in part (iii), if $\cap_{j \in \hat{X}} H_j \neq \emptyset$. Statements (ii) and (iii) are proved. □

Theorem A.1.2 is a corollary of Lemmas A.2.1 and A.2.2.

Appendix B

Plucker coordinates of dual arrangements

We formulate a statement which helps to determine if two given arrangements are dual.

Let k, n, N denote the same natural numbers as in Section 3.1.

Let $P(k+1, N+1)$ be the real projective space of dimension $\binom{N+1}{k+1} - 1$, whose projective coordinates (λ_L) are labeled by subsets $L = (l_1, \ldots, l_{k+1})$ such that $1 \leq l_1 < \cdots < l_{k+1} \leq N+1$.

Similarly, let $P(k+1, N+1)$ be the real projective space of (the same) dimension $\binom{N+1}{n+1} - 1$, whose projective coordinates (μ_M) are labeled by subsets $M = (m_1, \ldots, m_{n+1})$ such that $1 \leq m_1 < \cdots < m_{n+1} \leq N+1$.

Let

$$\delta \ : \ P(k+1, N+1) \ \rightarrow \ P(n+1, N+1) \ , \quad \lambda \ \mapsto \ \mu = \delta(\lambda) \ ,$$

be the isomorphism, where for any $M = (m_1, \ldots, m_{n+1})$, we set $\mu_M = (-1)^\sigma \lambda_L$, where $L = (l_1, \ldots, l_{k+1})$ is the subset complementary to M in $\{1, \ldots, N+1\}$, and $\sigma = 1 + 2 + \cdots + k + 1 + l_1 + \cdots + l_{k+1}$.

Let $\mathbb{X}$ and $\mathbb{X}'$ denote the same spaces as in Section 3.1. Let Gr denote the Grassmannian of all $k+1$-dimensional vector subspaces of $\mathbb{X}$ and Gr' the Grassmannian of all $n+1$-dimensional vector subspaces of $\mathbb{X}'$. Let

$$\pi \ : \ \mathrm{Gr} \ \rightarrow \ P(k+1, N+1) \ , \quad \mathbb{W} \ \mapsto \ (\lambda_L) = \pi(\mathbb{W}) \ ,$$

be the Plucker imbedding, $(\lambda_L) = (e^{l_1} \wedge \cdots \wedge e^{l_{k+1}}|_{\mathbb{W}})$. Let

$$\pi' \ : \ \mathrm{Gr}' \ \rightarrow \ P(n+1, N+1) \ , \quad \mathbb{W}' \ \mapsto \ (\mu_M) = \pi'(\mathbb{W}') \ ,$$

be the Plucker imbedding, $(\mu_M) = (e_{m_1} \wedge \cdots \wedge e_{m_{n+1}}|_{\mathbb{W}'})$.

Lemma B.0.3. For $\mathbb{W} \in \mathrm{Gr}$ and $\mathbb{W}' \in \mathrm{Gr}'$, the subspace $\mathbb{W}'$ is the annihilator of the subspace $\mathbb{W}$ if and only if $\pi'(\mathbb{W}') = \delta(\pi(\mathbb{W}))$.

The lemma follows from Lemma 3.3.1.

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