THE GEOMETRY OF RADIAL STATES IN NONLINEAR ELLIPTIC PROBLEMS

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ABSTRACT

Bevin Laurel Maultsby: The Geometry of Radial States in Nonlinear Elliptic Problems
(Under the direction of Christopher K.R.T. Jones)

In this dissertation we present a geometric approach to the study of nonlinear elliptic problems. In particular, we analyze radial solutions using techniques from dynamical systems. These techniques include a thorough study of the invariant manifolds that arise from the union of the solutions to the elliptic PDE in phase space, as well as computations involving two vector fields which are tangent to the invariant manifolds.

In Chapter 3, we consider radially symmetric positive solutions to $\Delta_p u + f(u) = 0$ on a ball centered at the origin in $\mathbb{R}^n$. The union of all radially symmetric solutions to this quasilinear elliptic equation forms an invariant manifold. We use two integral expressions that arise from vector fields on the manifold to show that for a certain class of f, there can be at most one such solution satisfying $\Delta_p u + f(u) = 0$ on a ball with Dirichlet boundary conditions.

In Chapter 4, we make a powerful connection between the Morse index of the operator $L_u g = \Delta g + f'(u)g$ linearized at a solution u of $\Delta u + f(u) = 0$ and a component of a tangent vector along the solution trajectory of u in phase space. We use this connection to give geometric proofs of the Morse indices of radial sign-changing solutions to $\Delta u + f(u) = 0$ on a ball in $\mathbb{R}^n$ with Dirichlet boundary conditions.

To Mom, Dad, and Galen.

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CHAPTER 1: INTRODUCTION

1.1 Overview

The goal of this dissertation is to study nonlinear elliptic PDEs by using techniques from geometric dynamical systems. Of particular interest is the p-Laplacian operator Δ_p, which may be singular or nondegenerate and which arises as an Euler-Lagrange equation to a Dirichlet integral.

In the 1980's, a great deal was discovered about positive solutions to semilinear elliptic equations of the form $\Delta u + f(u) = 0$ on a selected domain with appropriate boundary conditions; we will focus on a ball of radius $|\mathbf{x}| = R$ centered at the origin with Dirichlet boundary conditions. Kwong [22] proved that positive radial solutions of such an elliptic PDE on this domain were unique provided $f(u)$ was of a certain "superlinear" form; his results were extended by McLeod [25] to different $f(u)$. In Clemons' 1990 dissertation under Jones, he constructed a geometric argument for uniqueness of positive solutions satisfying $\Delta u + f(u) = 0$ on this domain by studying the invariant manifold created by the relevant radial solutions; see [7].

Our starting goal was to show uniqueness of *sign-changing solutions* to $\Delta u + f(u) = 0$ on a ball with Dirichlet boundary conditions. The question was: if u has k zeros on $\overline{B_R(0)}$, is it necessarily unique (perhaps with a small radius R)? In pursuit of this question, we supplemented the vector field that was used by Clemons and Jones [7] to show uniqueness in the case $k = 1$ in with another vector field whose geometry can be tracked as u changes signs and $|\mathbf{x}| \to R$. Although the original question remains open, we were able to provide results on the Morse index of a sign-changing solution to $\Delta u + f(u) = 0$. This material is the subject of Chapter 4.

If the Laplacian is the quintessential linear second-order elliptic operator, then the p-

Laplacian is its quasilinear counterpart. The p-Laplace equation $\Delta_p u + f(u) = 0$ is challenging as the p-Laplacian is non-uniformly elliptic if $p \neq 2$ and singular if $p \in (1, 2)$. Using two vector fields along solutions to $\Delta_p u + f(u) = 0$ in the ball $B_R(0)$, we prove that a positive radial solution satisfying a Dirichlet boundary condition must be unique. Most of Chapter 3 is dedicated to this proof.

In the rest of this introductory chapter, we provide the necessary background information to understand key concepts from dynamical systems and elliptic partial differential equations that will be used in subsequent chapters. The introduction concludes with a summary of Chapters 2-4.

1.2 Dynamical systems

Let us begin with a basic definition of a dynamical system, the flow it generates, and interesting structures that may arise; much of this material follows [33].

1.2.1 Basic definitions

Definition 1.2.1. A **dynamical system** is a smooth manifold (called the phase space) U endowed with a family of smooth functions $\Phi(\mathbf{x}, t) : \Omega \subset U \times I \to U$, where $I \subset \mathbb{R}$. Setting $\Phi_t(\mathbf{x}) = \Phi(\mathbf{x}, t)$, the Φ_t satisfy

- $\Phi_0(\mathbf{x}) = \mathbf{x}$, for all $\mathbf{x} \in U$, and
- $\Phi_t \circ \Phi_s(\mathbf{x}) = \Phi_{t+s}(\mathbf{x})$, if both sides are defined.

The group of functions $\Phi_t(\mathbf{x})$ is called a *flow* on U, and it evolves each point in U by time $t \in I$. Generally speaking, a dynamical system is a space U together with a rule for how points in that space evolve. This rule generates a vector field $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ in phase space; for an autonomous dynamical system, the vector field is often written

$$\dot{\mathbf{x}} = F(\mathbf{x}), \tag{1.1}$$

for all $\mathbf{x} \in U$. The vector field, in turn, generates a flow if F is locally Lipschitz. A point

$\mathbf{p} \in U$ is a *critical point* (or *fixed point*) of (1.1) if $F(\mathbf{p}) = 0$. Consequently $\Phi(\mathbf{p}, t) = \mathbf{p}$ for any $t \in I$.

1.2.2 Invariant manifolds

Invariant manifolds are special types of invariant sets for (1.1). While they may arise in relation to, say, a periodic orbit, we will focus below on invariant manifolds of a fixed point $\mathbf{p} \in U$. As the name suggests, an invariant manifold is "invariant" under the flow Φ, where we define invariant sets using the following definition.

Definition 1.2.2. A set $B \subset U$ is **positively invariant** if $B \cdot t \subset B$ for all $t \geq 0$, where

$$B \cdot t = \{\Phi_t(\mathbf{x}) \mid \mathbf{x} \in B\}.$$

B is **negatively invariant** if $B \cdot t \subset B$ for all $t \leq 0$. We say B is **invariant** if it is both positively and negatively invariant.

Basic examples of invariant sets in phase space include critical points, periodic orbits, and regions trapped by homoclinic orbits.

To construct an invariant manifold, we begin by linearizing the system (1.1) at a critical point. In particular, suppose $U \subset \mathbb{R}^n$ is open, and consider a C^1 vector field $F(\mathbf{x})$ for all $\mathbf{x} \in U$. Let $\mathbf{p} \in U$ be a critical point; the *linearization of (1.1) at* $\boldsymbol{p}$ is

$$\dot{\mathbf{y}} = DF(\mathbf{p})\mathbf{y} \tag{1.2}$$

where $\mathbf{y} \in \mathbb{R}^n$ and $DF(\mathbf{p})$ is an $n \times n$ matrix. To study the eigenvalues of $DF(\mathbf{p})$, let $\sigma(*)$ denote the spectrum of $*$. The set of eigenvalues of $DF(\mathbf{p})$ decomposes into subsets via

$$\sigma(DF(\mathbf{p})) = \sigma^- \cup \sigma^0 \cup \sigma^+$$

where σ^- corresponds to all eigenvalues of $DF(\mathbf{p})$ satisfying $\operatorname{Re}\lambda < 0$, σ^0 to eigenvalues with

$\operatorname{Re}\lambda = 0$ and σ^+ to eigenvalues with $\operatorname{Re}\lambda > 0$. Furthermore, the matrix $DF(\mathbf{p})$ can be diagonalized to the block form

$$DF(\mathbf{p}) = \begin{pmatrix} A^- & 0 & 0 \\ 0 & A^0 & 0 \\ 0 & 0 & A^+ \end{pmatrix}$$

with $\sigma(A^-) = \sigma^-$, etc. Spanned by each set of eigenvalues σ^-, σ^0, and σ^+ of $DF(\mathbf{p})$ are invariant subspaces E^-, E^0, and E^+ such that

$$\mathbb{R}^n = E^- \oplus E^0 \oplus E^+$$

and

$$\sigma\left(DF(\mathbf{p})|_{E^-}\right) = \sigma^-, \text{ etc.}$$

Each subspace E^-, E^0, and E^+ is an invariant set for (1.2), which is a linear dynamical system.

With each of the subspaces E^-, E^0, E^+ established, we define the invariant manifolds. These manifolds give a "nonlinear" version of the invariant subspaces. There are three classes of invariant manifolds: stable manifolds, unstable manifolds, and center manifolds, which are analogous to E^-, E^+, and E^0, respectively. Let N be an open neighborhood of the fixed point $\mathbf{p}$; the *stable manifold* is (locally) characterized as follows:

Definition 1.2.3. The **local stable manifold** is

$$W^s_{\text{loc}}(\mathbf{p}) = \{\mathbf{x} \in N \mid \mathbf{x} \cdot t \in N \text{ for all } t \geq 0, \mathbf{x} \cdot t \to \mathbf{p} \text{ exponentially as } t \to \infty\}. \qquad (1.3)$$

In other words, the stable manifold in a neighborhood of $\mathbf{p}$ consists of all the points which tend exponentially towards $\mathbf{p}$ as $t \to \infty$. Analogously, the *local unstable manifold* is the set

$$W^u_{\text{loc}}(\mathbf{p}) = \{\mathbf{x} \in N \mid \mathbf{x} \cdot t \in N \text{ for all } t \leq 0, \mathbf{x} \cdot t \to \mathbf{p} \text{ exponentially as } t \to -\infty\} \tag{1.4}$$

so that the local unstable manifold consists of all points which evolve to $\mathbf{p}$ as time is reversed. Notice both local manifolds are nonempty, as they both contain $\mathbf{p}$.

Theorem 1.2.4 (Stable and Unstable Manifold Theorem). *Assume $F \in C^1$ and $\boldsymbol{p}$ is a fixed point. Then there is a neighborhood N of $\boldsymbol{p}$ and a Lipschitz function*

$$h^s : (\boldsymbol{p} + E^-) \cap N \to E^0 \oplus E^+$$

so that the graph of h^s is W^s_{loc}. There exists also a neighborhood M of $\boldsymbol{p}$ and a Lipschitz function

$$h^u : (\boldsymbol{p} + E^+) \cap M \to E^- \oplus E^0$$

so that the graph of h^u is W^u_{loc}.

This theorem justifies calling the local stable and unstable manifolds defined in (1.3)-(1.4) "manifolds," as they are the graphs of Lipschitz functions.

Both local manifolds extend into global invariant manifolds. With $U \subset \mathbb{R}^n$ as before and for any choice of neighborhood N, the global versions are constructed by evolving $W^s_{\text{loc}}(\mathbf{p})$ and $W^u_{\text{loc}}(\mathbf{p})$ backwards and forwards in time, respectively.

Definition 1.2.5. The **(global) stable manifold** is

$$W^s(\mathbf{p}) = \{\Phi_t(\mathbf{x}) \mid \mathbf{x} \in W^s_{\text{loc}}(\mathbf{p}), t \leq 0\}. \tag{1.5}$$

Similarly, the *(global) unstable manifold* is defined as

$$W^u(\mathbf{p}) = \{\Phi_t(\mathbf{x}) \mid \mathbf{x} \in W^u_{\text{loc}}(\mathbf{p}), t \geq 0\}. \tag{1.6}$$

As discussed in [33], the stable and unstable manifolds are unique, and their tangent spaces at $\mathbf{p}$ are E^- and E^+, respectively.

A fixed point $\mathbf{p}$ is *hyperbolic* if the real part of each eigenvalue $\lambda \in \sigma(DF(\mathbf{p}))$ is nonzero. If $\mathbf{p}$ is not hyperbolic, then the center subspace E^0 is nontrivial. Associated with E^0 is the idea of a *center manifold* whose tangent space at $\mathbf{p}$ is E^0. For a neighborhood $N \ni \mathbf{p}$, trajectories that stay in N for all $t \geq 0$ tend to the center manifold as $t \to \infty$, while trajectories that stay in N for all $t \leq 0$ tend to the center manifold as $t \to -\infty$.

We note that global stable and unstable manifolds are always unique, but center manifolds are not necessarily so. In general, center manifolds are more difficult to define precisely than stable and unstable manifolds, and the reader should consult [33].

1.2.3 A review of the variational equation in differential form notation

To increase readability, we switch the notation in this section from $\dot{\mathbf{x}}$ to $\mathbf{x}'$. Consider an autonomous dynamical system

$$\mathbf{x}' = F(\mathbf{x}), \qquad \mathbf{x} \in U \subset \mathbb{R}^n$$

as before. If $\mathbf{x}(t)$ is a solution to $\mathbf{x}' = F(\mathbf{x})$, and δ_0 is a vector tangent to $\mathbf{x}(t)$ at $t = t_0$, then δ_0 satisfies the *variational equation*

$$\delta' = DF(\mathbf{x})\delta. \tag{1.7}$$

This equation generates a tangent vector field $\delta(t)$ with $\delta(t_0) = \delta_0$ and describes how these tangent vectors move under the flow. As $DF(\mathbf{x})$ is an $n \times n$ matrix and δ is an $n \times 1$ vector, the i^{th} coordinate of δ' is

$$\delta_i{}' = (dx_i(\delta))' = \sum_{j=1}^{n} \partial_j F_i(\mathbf{x}) dx_j(\delta).$$

We suppress the tangent vector δ and write

$$dx_i{}' = \sum_{j=1}^{n} \partial_j F_i(\mathbf{x}) dx_j.$$

However, it should be understood that $dx_i{}'$ applies to a tangent vector. This calculation establishes what it means to take the derivative of a 1-form.

Consider now a 2-form $dz_i \wedge dz_j$ applied to a pair of vectors (δ_1, δ_2). We claim that $(dz_i \wedge dz_j)'$, where the idea of a "derivative of a form" is the same as above, follows the product rule. To show this, we perform the following steps:

$$\begin{aligned}
(dz_i \wedge dz_j)' &= [dz_i \wedge dz_j(\delta_1, \delta_2)]' \\
&= \frac{d}{dt}\left[dz_i(\delta_1)dz_j(\delta_2) - dz_i(\delta_2)dz_j(\delta_1)\right] \\
&= dz_i(\delta_1)'dz_j(\delta_2) + dz_i(\delta_1)dz_j(\delta_2)' - dz_i(\delta_2)'dz_j(\delta_1) - dz_i(\delta_2)dz_j(\delta_1)' \\
&= dz_i(\delta_1)'dz_j(\delta_2) - dz_i(\delta_2)'dz_j(\delta_1) + dz_i(\delta_1)dz_j(\delta_2)' - dz_i(\delta_2)dz_j(\delta_1)' \\
&= dz_i' \wedge dz_j + dz_i \wedge dz_j'. \qquad (1.8)
\end{aligned}$$

In other words, if we construct a function of t given by

$$\omega(t) = du \wedge dv(\delta_1(t), \delta_2(t)),$$

then using the product rule above yields

$$\begin{aligned}
\omega'(t) &= \frac{d}{dt}\left[du \wedge dv(\delta_1(t), \delta_2(t))\right] \\
&= \cdots \\
&= (du \wedge dv)'(\delta_1(t), \delta_2(t)),
\end{aligned}$$

where $\cdots$ repeats the steps to attain (1.8). This construction is relevant to this dissertation, as we will repeatedly make use of 2-forms and their derivatives.

Suppose that for a given dynamical system in $\mathbb{R}^3$ described by $\mathbf{x}' = F(\mathbf{x})$, where $\mathbf{x} = (y, w, r)^T$, we have two linearly independent vector fields $(y', w', r')^T$ and $(\delta y, \delta w, \delta r)^T$ that are tangent to an invariant manifold. (The notation is relevant to Chapters 3-4, and rather than define these vector fields here we will simply assume that they are linearly independent as described.) As they are tangent vector fields, both satisfy (1.7). Their cross-product,

$$(\delta y^*, \delta w^*, \delta r^*)^T := (\dot{y}, \dot{w}, \dot{r})^T \times (\delta y, \delta w, \delta r)^T,$$

is a vector field normal to the invariant manifold, and it satisfies the following lemma.

Lemma 1.2.6. *For a dynamical system $\boldsymbol{x}' = F(\boldsymbol{x})$, let $A = DF(\boldsymbol{x})$ and let $(\delta y^*, \delta w^*, \delta r^*)^T$ be defined as above. Then this normal vector satisfies*

$$\begin{pmatrix} \delta y^* \\ \delta w^* \\ \delta r^* \end{pmatrix}' = (-A^* + (Tr\,A)I) \begin{pmatrix} \delta y^* \\ \delta w^* \\ \delta r^* \end{pmatrix},$$

where A^* is the transpose of A.

The proof is a straightforward matrix calculation. Lemma 1.2.6 will be consistently employed to easily compute time derivatives of 2-forms in Chapters 3 and 4.

1.3 Elliptic equations

We first recall some of the elementary definitions and results for elliptic partial differential equations and Sobolev spaces; see [14] and [32]. Through this section $\Omega \subset \mathbb{R}^n$ is a bounded domain (an open connected set) with smooth boundary $\partial\Omega$, and $u : \overline{\Omega} \to \mathbb{R}$ is in $C^2(\Omega) \cap C(\overline{\Omega})$. We remark that for Chapters 2-4, Ω will be the open ball

$$B_R(0) = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| < R\}, n \geq 2.$$

Definition 1.3.1 (Second Order Elliptic Equation)**.** An second-order partial differential operator L given by

$$Lu := \sum_{i,j=1}^{n} a_{ij}(\mathbf{x})u_{x_i x_j} + \sum_{i=1}^{n} b_i(\mathbf{x})u_{x_i} + c(\mathbf{x})u \tag{1.9}$$

is *elliptic* if there is some constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(\mathbf{x})\xi_i \xi_j \geq \theta |\xi|^2, \tag{1.10}$$

for a.e. $\mathbf{x} \in \Omega$ and all $\xi \in \mathbb{R}^n$.

In Chapter 2-4, we will consider quasilinear elliptic equations with the following second order operator.

Definition 1.3.2. The p-Laplacian Δ_p is defined by

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u \right), \tag{1.11}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$.

When $p = 2$, Δ_p is the regular uniformly-elliptic Laplacian. When $1 < p < 2$, (1.11) is a singular operator, as it is undefined whenever $\nabla u = 0$. Whenever $p > 2$, Δ_p is a degenerate elliptic operator; in other words, Δ_p satisfies (1.10) with the weaker condition obtained by setting $\theta = 0$.

Note setting (1.11) equal to zero is the Euler-Lagrange equation for the Dirichlet integral

$$J(u) = \int_\Omega |\nabla u|^p \, d\mathbf{x}.$$

The behavior of (1.11) depends on the particular value of p. Typically $p \in (1, \infty)$; we remark that setting $p = 1$ in (1.11) results in $-H$, where H is the mean curvature operator of differential geometry. There is a particular threshold that arises when $p = n$, in which case $\int_\Omega |\nabla u|^n \, dx$ is conformally invariant. Lastly, setting $p = \infty$ arises in optimal Lipschitz extensions. For an in-depth treatise of the p-Laplacian and p-harmonic functions, we advise the reader to consult [24].

The general form of the nonlinear elliptic problems studied in Chapters 2-4 is

$$\begin{cases} \Delta_p u + f(u) & = 0 \text{ on } B_R(0) \\ u & = 0 \text{ on } \partial B_R(0), \end{cases} \tag{1.12}$$

where $p \in (1, 2]$ and $f(u)$ is a nonlinear function.

1.3.1 Sobolev Spaces

The solutions to the elliptic equations such as (1.12) live naturally in Sobolev spaces. Let $\Omega \subset \mathbb{R}^n$ be a domain. The Sobolev space $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$||u||^p_{W_0^{1,p}(\Omega)} = \int_\Omega \left(|\nabla u|^p + |u|^p \right) \, d\mathbf{x}. \tag{1.13}$$

Thus, $W_0^{1,p}(\Omega)$ is a Banach space. In the case $p = 2$, $W_0^{1,2}(\Omega)$ is a Hilbert space with inner product

$$\langle u, v \rangle_{W_0^{1,2}(\Omega)} = \int_\Omega \left(\nabla u \cdot \nabla v + uv \right) \, d\mathbf{x}.$$

Consider the energy functional

$$J_p(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, d\mathbf{x} - \int_\Omega F(u(\mathbf{x})) \, d\mathbf{x}, \tag{1.14}$$

where $F(t) = \int_0^t f(s) \, ds$ for a function $f \in C^1([0, \infty))$. Critical points minimizing the functional must satisfy

$$\int_\Omega \left(|\nabla u|^{p-2} \nabla u \cdot \nabla \phi - f(u)\phi \right) \, d\mathbf{x} = 0,$$

for every $\phi \in C_0^\infty(\Omega)$ ([24], [28]). Thus such critical points are weak solutions in $W_0^{1,p}(\Omega)$ to the equation

$$\Delta_p u + f(u) = 0,$$

with Dirichlet boundary conditions.

Definition 1.3.3. For the Sobolev space $W^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$, the *Sobolev critical exponent* is defined by

$$\frac{1}{p^*} := \frac{1}{p} - \frac{1}{n}. \tag{1.15}$$

Hence $p^* = np/(n-p)$. The importance of (1.15) is that as we investigate equations of the form $\Delta_p u + f(u) = 0$, we choose nonlinearities $f(u)$ with the "correct" growth $|u|^{q-2}u$. In particular, we will choose the exponent term q so that q satisfies case (2) in the following theorem.

Theorem 1.3.4 (Sobolev, Rellich, Kondrachov)**.** *If $1 < p < n$, then*

1. *$W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ is a continuous embedding, and*

2. *if $q < p^* + 1$, $W^{1,p}(\Omega) \hookrightarrow L^{q-1}(\Omega)$ is a compact embedding.*

Notice that the hypothesis $1 < p < n$ is easily satisfied if $p \in (1, 2)$ and $n \geq 2$.

1.4 Sturm-Liouville Theory

A Sturm-Liouville (SL) equation is a type of ordinary differential equation in a finite domain with a well-understood set of eigenvalues and eigenfunctions. In particular, its eigenvalues are always real and discrete. Typically SL equations are described as having a smallest eigenvalue, from which the eigenvalues increase without bound. We shall cast SL equations slightly differently so that the eigenvalues have a largest member and decrease without limit; this change is convenient in the language of dynamical systems as positive eigenvalues are unstable.

Moreover, the eigenfunctions are orthogonal to each other and as the eigenvalue decreases, the corresponding eigenfunctions oscillate (have zeros) more rapidly. The beauty of SL results is that once we have identified a particular ODE as an SL system, then the results on eigenvalues and eigenfunctions described in this section are immediately applicable to the system.

We will state most of the major theorems on SL equations to provide the necessary background for Chapter 4; this presentation of SL equations follows [5]. Most sources, including [5], use Green's formula to prove the theorems in this section; we use a slight variation on the usual techniques.

An SL differential equation is of the form

$$(f_1(x)u'(x))' + f_2(x)u(x) = \lambda f_3(x)u(x), \tag{1.16}$$

where each $f_i(x)$ is real and continuous, $f_1'(x)$ is continuous, and $f_1(x)$ and $f_3(x)$ are positive on the open interval (a, b). This type of equation can be viewed as an eigenvalue problem for the linear operator L defined by

$$L : u(x) \mapsto (f_1(x)u'(x))' + f_2(x)u(x). \tag{1.17}$$

The boundary conditions for the types of regular SL systems we consider are either Dirichlet, Neumann, or Robin conditions at a and b. These conditions can be written

$$\begin{aligned} &(1)\ \ \alpha_1 u(a) + \alpha_2 u'(a) = 0, \text{ and} \\ &(2)\ \ \beta_1 u(b) + \beta_2 u'(b) = 0, \end{aligned} \tag{1.18}$$

where $\alpha_i, \beta_i \in \mathbb{R}$, with $\alpha_1^2 + \alpha_2^2 > 0$, $\beta_1^2 + \beta_2^2 > 0$. A eigenfunction solution $u_\lambda(x)$ to (1.16) corresponding to eigenvalue λ is *regular* if $u_\lambda(x) \in C^1([a, b])$.

Let us establish a series of basic results about SL equations which we will use in Chapter 4.

Theorem 1.4.1. *The eigenspace for each eigenvalue of a regular SL system is 1-dimensional.*

Proof. Suppose λ is an eigenvalue with eigenfunctions $u_1 \neq u_2$. To examine u_1 and u_2, let us compute the Wronskian of u_1 and u_2 at the left endpoint $x = a$:

$$W(u_1, u_2)(a) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}_{x=a} = u_1(a)u_2'(a) - u_1'(a)u_2(a) = 0, \tag{1.19}$$

by (1.21). Thus the two columns are proportional, and we may say $u_2(a) = Cu_1(a)$. By Abel's identity for second-order ordinary differential equations, for any $c \in [a, b]$,

$$W(u_1, u_2)(c) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}_{x=c} = W(u_1, u_2)(a)e^{-\int_a^c f_1'(s)/f_1(s)\,ds} = 0.$$

As $f_1(x) > 0$ for $a \leq x \leq b$, the above expression is defined. Hence the columns are proportional for any $x \in [a, b]$, and so $u_2(x) = Cu_1(x)$. Thus the eigenspace for λ is 1-dimensional. $\square$

The function $f_3(x)$ gives rise to the following inner product:

$$\left\langle u_{\lambda_i}(x), u_{\lambda_j}(x) \right\rangle = \int_a^b f_3(x)u_{\lambda_i}(x)u_{\lambda_j}(x)\,dx. \tag{1.20}$$

Theorem 1.4.2. *If $i \neq j$, then $u_{\lambda_i}(x)$ and $u_{\lambda_j}(x)$ are orthogonal with respect to (1.20).*

Proof. First, we claim that at either endpoint $x = a$ or $x = b$,

$$u_{\lambda_i}'u_{\lambda_j}|_a = u_{\lambda_i}u_{\lambda_j}'|_a, \qquad u_{\lambda_i}'u_{\lambda_j}|_b = u_{\lambda_i}u_{\lambda_j}'|_b. \tag{1.21}$$

The equalities in (1.21) stem from the mixed linear boundary conditions; if $\alpha_i, \beta_i \neq 0$, then

$$u_{\lambda_i}'(b)u_{\lambda_j}(b) = -\frac{\beta_1}{\beta_2}u_{\lambda_i}(b)\left(-\frac{\beta_2}{\beta_1}u_{\lambda_j}'(b)\right) = u_{\lambda_i}(b)u_{\lambda_j}'(b),$$

and similarly at $x = a$. If either $\beta_i = 0$ (or $\alpha_i = 0$ at $x = a$), then $u_{\lambda_i}'(b)u_{\lambda_j}(b) = 0 =$

$u_{\lambda_i}(b)u'_{\lambda_j}(b)$.

Assume without loss of generality that $\lambda_i \neq 0$ (in which case it may be possible for λ_j to be zero). Using the above statement and integration by parts, we obtain

$$\begin{aligned}
\left\langle u_{\lambda_i}, u_{\lambda_j} \right\rangle &= \int_a^b f_3(x) u_{\lambda_i} u_{\lambda_j}\, dx \\
&= \frac{1}{\lambda_i} \int_a^b \lambda_i f_3(x) u_{\lambda_i} u_{\lambda_j}\, dx \\
&= \frac{1}{\lambda_i} \left[\int_a^b u_{\lambda_j} \left(f_1(x) u'_{\lambda_i} \right)' \, dx + \int_a^b f_2(x) u_{\lambda_i} u_{\lambda_j}\, dx \right] \\
&= \frac{1}{\lambda_i} \left[u_{\lambda_j} f_1(x) u'_{\lambda_i} |_a^b - \int_a^b u'_{\lambda_j} f_1(x) u'_{\lambda_i}\, dx + \int_a^b f_2(x) u_{\lambda_i} u_{\lambda_j}\, dx \right].
\end{aligned}$$

By (1.21), we can write this as

$$\begin{aligned}
&= \frac{1}{\lambda_i} \left[u_{\lambda_i} f_1(x) u'_{\lambda_j} |_a^b - \int_a^b u'_{\lambda_i} f_1(x) u'_{\lambda_j}\, dx + \int_a^b f_2(x) u_{\lambda_i} u_{\lambda_j}\, dx \right] \\
&= \frac{1}{\lambda_i} \left[\int_a^b u_{\lambda_i} \left(f_1(x) u'_{\lambda_j} \right)' \, dx + \int_a^b f_2(x) u_{\lambda_i} u_{\lambda_j}\, dx \right] \\
&= \frac{1}{\lambda_i} \int_a^b f_3(x) u_{\lambda_i} L u_{\lambda_j}\, dx \\
&= \frac{\lambda_j}{\lambda_i} \int_a^b f_3(x) u_{\lambda_i} u_{\lambda_j}\, dx \\
&= \frac{\lambda_j}{\lambda_i} \left\langle u_{\lambda_i}, u_{\lambda_j} \right\rangle.
\end{aligned}$$

If $\lambda_i \neq \lambda_j$, then for the above conclusion to hold, it must be the case that $\left\langle u_{\lambda_i}, u_{\lambda_j} \right\rangle = 0$. $\square$

Theorem 1.4.3. *The eigenvalues of a regular SL system are real.*

Proof. Let u be a nondegenerate eigenfunction which solves (1.16) with eigenvalue λ. Then $\overline{u}$ solves (1.16) with eigenvalue $\overline{\lambda}$. As $0 \in \mathbb{R}$, assume $\lambda \neq 0$.

From the proof of Theorem 1.4.2, we know

$$|u|^2 = \langle u, \overline{u} \rangle = (\lambda/\overline{\lambda}) \langle u, \overline{u} \rangle = (\lambda/\overline{\lambda}) |u|^2.$$

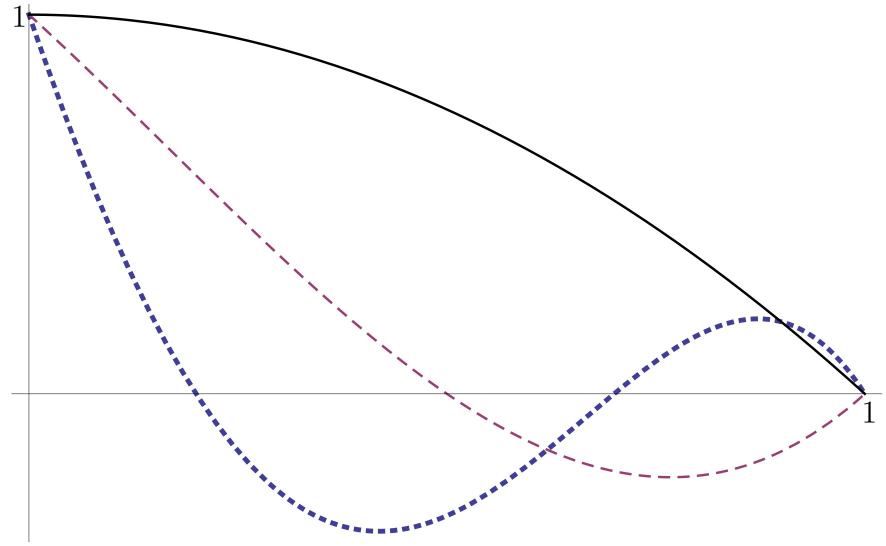


Figure 1.1: An illustration of the general oscillatory behavior of the first three eigenfunctions to a SL problem, with boundary conditions $u(0) = 1$ and $u(1) = 0$. In this picture, the solid line represents the eigenfunction u_{λ_0}, the dashed line u_{λ_1} and the dotted line u_{λ_2}.

Yet in order for this to be true, we must have $\lambda = \overline{\lambda}$. So $\lambda \in \mathbb{R}$. $\square$

By Theorem 1.4.1, eigenfunctions of an SL system are unique up to scalar multiplication. Therefore, together with Theorem 1.4.2, we can normalize the eigenfunctions to form an orthonormal set.

Theorem 1.4.4 (Sturm Comparison Theorem)**.** *Let λ_i and λ_j be two eigenvalues with eigenfunctions u_{λ_i} and u_{λ_j} satisfying (1.16). If $\lambda_i > \lambda_j$, then between consecutive zeros of u_{λ_i}, there lies at least one zero of u_{λ_j} .*

We omit the proof of Theorem 1.4.4, as it is very similar to the proof of Lemma 4.3.1 in Chapter 4. We will also omit the proof of the following theorem, as it is a standard result:

Theorem 1.4.5. *The SL problem (1.16) with regular boundary conditions (1.18) has an infinite set of real eigenvalues*

$$\lambda_0 > \lambda_1 > \lambda_2 > \cdots \quad \textit{with} \quad \lim_{n\to\infty} \lambda_n = -\infty,$$

and therefore an infinite number of eigenfunctions $u_{\lambda_n}(x)$. Moreover, each eigenfunction $u_{\lambda_n}(x)$ is unique (up to scalar multiplication) and has exactly n zeros on the open interval (a, b).

The consequences of Theorems 1.4.4 and 1.4.5 on the behavior of eigenfunctions is illustrated in Figure 1.1. In this illustration, the eigenfunction u_{λ_0} has 0 zeros in $(0, 1)$, u_{λ_1} has 1 zero in $(0, 1)$, and u_{λ_2} has 2 zeros in $(0, 1)$.

1.5 Applications

Let us conclude the introduction by mentioning one of the major applications of the p-Laplacian. In particular, we remark that the regular Laplacian Δu is a model for Newtonian fluids, which are characterized by having the viscous stress proportional to the strain rate at every point; see, e.g., [4]. The factor $|\nabla u|^{p-2}$ describes the speed that fluid particles travel in relation to each other; this term reduces to 1 in the linear Newtonian setting. For a non-Newtonian fluid, $|\nabla u|^{p-2}$ relates the effect of shear on the viscosity of the fluid. Hence the p-Laplacian Δ_p may be used to model non-Newtonian fluids.

In particular, $1 < p < 2$ models pseudoplastics, which are fluids that become less viscous as the shear increases (examples include blood, several types of paint, nail polish). The case $p > 2$ models dilatants, which become more viscous as the shear increases (the classic example is cornstarch in water).

1.6 Overview of dissertation

In Chapter 2, we describe previous work on symmetry, existence, and uniqueness for solutions to the elliptic equation (1.12). The bulk of Chapter 3 is dedicated to proving Theorem 3.0.2 on uniqueness of positive solutions to $\Delta_p u + f(u) = 0$ in a ball with Dirichlet boundary conditions for a certain class of f. In Chapter 4, we prove several results on Morse indices for sign-changing solutions for $p = 2$, notably Theorem 4.5.2, which states that there must be a solution on a ball with Dirichlet boundary conditions that has k zeros and Morse index k. Lastly, we mention in Chapter 5 two related problems of current interest.

CHAPTER 2: BACKGROUND

In this chapter we give an overview of the history and previous results on the semilinear elliptic equation

$$\Delta u + f(u) = 0, \tag{2.1}$$

and the quasilinear elliptic equation

$$\Delta_p u + f(u) = 0, \tag{2.2}$$

for different classes of nonlinearities f and in various domains with appropriate boundary conditions.

2.1 Results on uniformly elliptic partial differential equations

In this section we discuss the symmetry of positive solutions to (2.1), where the domain is typically a ball with Dirichlet boundary conditions, as well as results on existence and uniqueness for solutions to (2.1).

With the technique of moving parallel planes, Serrin showed in [30] that if Ω is a smooth bounded domain and u is a positive solution to $\Delta u + 1 = 0$ in Ω, with $u = 0$ on $\partial\Omega$ and with the outward normal vector $\frac{\partial u}{\partial \nu}$ constant on $\partial\Omega$, then Ω is necessarily a ball and u is a radial function. The method of moving parallel planes was originally used by Alexandroff to study surfaces of constant mean curvature in differential geometry. It was also used by Gidas, Ni and Nirenberg [18] to obtain the following famous result.

Theorem 2.1.1 (Gidas, Ni, Nirenberg)**.** *In the ball $\Omega = \{\boldsymbol{x} \in \mathbb{R}^n \mid |\boldsymbol{x}| < R\}$, let $u > 0$ be a positive solution in $C^2(\bar{\Omega})$ of*

$$\Delta u + f(u) = 0 \textit{ with } u = 0 \textit{ on } |\boldsymbol{x}| = R. \tag{2.3}$$

Here f is of class C^1. Then u is radially symmetric and

$$\frac{\partial u}{\partial r} < 0, \qquad \textit{for } 0 < r < R.$$

Part of the power of this result stems from the fact that they make no assumptions on the nonlinear term $f(u)$ except that $f \in C^1$.

In general, existence and uniqueness results for (2.1) and (2.2) do require more restrictive conditions on $f(u)$. The prototypical example is the Lane-Emden equation $f(u) = u^q$, for $q > 1$. If $q = (n+2)/(n-2) = 2^* - 1$, where 2^* is the Sobolev critical exponent for $p = 2$, then (2.1) is a version of the Yamabe problem from differential geometry. This particular exponent is a critical threshold for f, as demonstrated by the following result of Pohozaev [29]; see [31] for a discussion in English.

Theorem 2.1.2 (Pohozaev, 1965)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be an open, star-shaped (with respect to the origin) domain. The equation $\Delta u + u^q = 0$, $u|_{\partial\Omega} = 0$, has a positive solution only if $q < 2^* - 1$.*

We remark that the topology of the domain is important, and there may be a positive solution to $\Delta u + u^q = 0$ on a different domain, such as an annulus. To prove Theorem 2.1.2, Pohozaev proved that positive solutions to $\Delta u + u^q = 0$ must satisfy the Pohozaev identity

$$\int_\Omega \left(\frac{2n}{q+1} - (n-2) \right) u^{q+1}\, dx = \int_{\partial\Omega} |\nabla u|^2 \,(x \cdot \nu)\; dS. \tag{2.4}$$

If the domain is star-shaped, the right-hand side of (2.4) is always positive. The left-hand side, however, is always negative if $q > 2^* - 1$. Recall that according to Theorem 1.3.4, the Sobolev embedding theorem,

$$W^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$$

is a continuous embedding if $q \leq (n+2)(n-2)$, with strict inequality resulting in a compact embedding. Nonexistence of solutions in [29] stems from lack of compactness of the embedding.

Hence the nonlinearity f plays a big role in the existence or nonexistence of solutions to (2.1). To establish the existence of solutions to (2.1), authors frequently employ variational methods to show the existence of minimizers to certain functionals. For example, the critical exponent nonlinearity $f(u) = \lambda u + |u|^{p^*-2}u$ arises in the general Yamabe problem. For the semilinear case $p = 2$, Brezis and Nirenberg [6] used the energy functional

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\lambda}{2}\int_\Omega |u|^2\,dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx$$

to show a solution must exist if λ is smaller than the first eigenvalue of Δ.

When $f(u)$ satisfies $|f(u)| \le Cu^{q-1}$, $C > 0$, the question of whether f is subcritical ($q < p^*$), critical ($q = p^*$), or supercritical ($q > p^*$) may alter not only when a solution exists but whether or not is unique. For example, in the case $p = 2$, Ni and Nussbaum [27] determined that solutions to (2.1) with $f(u) = u^{q-1} + u$ are not necessarily unique in the supercritical case $q > p^*$.

Uniqueness of positive solutions to (2.1) for $p = 2$ has been addressed by many authors; the first was Coffman [8] for the subcritical case $n = 3$ and $f(u) = u^3 - u$. McLeod and Serrin [26] showed uniqueness results for $f(u) = u^q - u$ for certain q, which were generalized by Kwong [22] to $1 < q < (n+2)/(n-2)$; the method of Kwong was generalized and simplified by [25]. Other authors who investigated uniqueness of (2.1) with f subcritical, critical, or supercritical include Kwong and Zhang [23], who proved uniqueness in a ball by using a Sturm comparison principle.

Clemons and Jones illustrated this last uniqueness result with a geometric approach in [7] by recasting (2.1) as a dynamical system. The union of all solution forms a two-dimensional invariant manifold; showing uniqueness of a solution to the Dirichlet equation is interpreted as showing that the rotation of the manifold can be controlled. In Chapter 3, we will use similar geometric methods to illustrate recent existence results and prove a uniqueness results for the p-Laplacian for a large class of f.

2.2 Results for the p-Laplacian

In general, solutions to $\Delta_p u + f(u) = 0$ for $p \neq 2$ are considered in the weak sense because they belong to $C^{1,\alpha}(\Omega)$ for some $\alpha > 0$, see [11]. Many of the results on the uniqueness or symmetry properties of (2.1) rely on classical elliptic principles such as the maximum principle. These principles do not apply in a straightforward manner in (2.2), when the operator is singular (as in the case $p \in (1,2)$) or degenerate (as in the case $p > 2$). The principle of superposition is also lost when $p \neq 2$.

To underscore the importance of radial solutions to (2.2), we cite the following result from [9] and [10] on positive solutions to (2.2) on a ball with Dirichlet boundary conditions. The proof uses a modified moving plane method reminiscent of [18].

Theorem 2.2.1 (Damascelli and Pacella). *Suppose $p \in (1,2)$ and Ω is a ball about the origin. If f is locally Lipschitz continuous in $(0,\infty)$ and either*

- *$f(u) \geq 0$ for $u \geq 0$, or*
- *there exists $\beta_0 > 0$ and a continuous, positive (except at the origin), non-decreasing function $\beta : [0, \beta_0] \to \mathbb{R}$ with*

$$\beta(0) = 0, \qquad \text{and} \qquad \int_0^{\beta_0} \frac{1}{(s\beta(s))^{1/p}}\, ds = \infty,$$

  such that $f(u) + \beta(u) \geq 0$ for all $u \in [0, \beta_0]$,

then u must be radially symmetric and $\frac{\partial u}{\partial r} < 0$ for $r > 0$.

Several authors have examined existence and uniqueness questions for the p-Laplacian equation (2.2) with different choices of nonlinearity f, different domains (typically all of $\mathbb{R}^n$, a ball of radius R, or an annulus), and different boundary conditions (usually Dirichlet); we refer here to [20], [13], [19]. Guedda and Veron [20] determined criteria for existence of positive solutions to

$$\Delta_p u + u^{p^*-1} + a(x)u^{p-1} = 0,$$

in a bounded open subset of $\mathbb{R}^n$ with Dirichlet boundary conditions. Their result can be seen as an extension of the Brezis and Nirenberg [6] result. Erbe and Tang [13] proved uniqueness to (2.2) with $f(u) = u^q$ for q is subcritical. They also proved uniqueness holds for $f(u) = u^{q_1} + \lambda u^{q_2}$, with $q_1 > q_2$ and $\lambda > 0$, if the quantity

$$\xi x^2 + \lambda \sigma x + \lambda^2 v$$

is positive for all $x > 0$, where

$$\begin{aligned} \xi &= -\left(n - p - \frac{np}{q_1 + 1}\right) \\ \sigma &= v(q_2 - q_1 + 1) + \xi(q_1 - q_2 + 1) \\ v &= -\left(n - p - \frac{np}{q_2 + 1}\right). \end{aligned}$$

Gonçalves and Alves [19] used minimax arguments on an energy functional to study existence of solutions to (2.2) with $f(u) = u^{p^*-1} + h(x)u^q$ in $\mathbb{R}^n$.

Recently, existence of solutions has been studied in a geometric framework, notably by Franca ([15], [16], [17]). He used an Emden-Fowler transformation to show the existence or nonexistence of ground states and singular ground states (2.2) for positive solutions to (2.2) in $\mathbb{R}^n$. The function he studied is a Pohozaev function which is essentially related to the Hamiltonian structure that may arise in phase space; we will discuss this in Section 3.2.

Adimurtha and Yadava [1] investigated uniqueness of

$$-\Delta_p u = u^q + \lambda |u|^{p-2} u$$

in both a ball and an annulus in $\mathbb{R}^n$ by using a Pohozaev-type identity. In particular, we note that for the ball with Dirichlet boundary conditions, they established uniqueness with $\lambda \geq 0$, $1 < q + 1 \leq (np)(n - p)$, $p < n$.

Aftalion and Pacella [2] investigated uniqueness of positive radial solutions to (2.2) with

$f \in C^0[0,\infty) \cap C^1(0,\infty)$ satisfying the following conditions:

(AP1) $f(0) = 0$, with some $\theta > 0$ so that $f < 0$ in $(0, \theta)$ and $f > 0$ in (θ, ∞),

(AP2) the expression

$$K(u) := \frac{uf'(u)}{f(u)} \tag{2.5}$$

is nonincreasing in (θ, ∞), and

(AP3) The quantity

$$uf'(u) - (p-1)f(u) \tag{2.6}$$

is positive for $u > 0$.

The prototypical nonlinearity satisfying (AP1)-(AP3) is $f(u) = u^q - u^{p-1}$, $q > p - 1$. With an additional requirement on the growth of f near zero, [2] show that (2.2) has at most one weak radial solution if $p \leq 2$ by using a variant of the maximum principle and a suitable implicit function theorem.

The family of nonlinearities in Chapter 3 gives a weaker condition than (AP3) in Theorem 3.0.2; in fact, the quantity

$$uf'(u) - (p-1)f(u) \tag{2.7}$$

may change signs at some value of u_0, and f may be nonnegative. Moreover, our proof is geometric, and both quantities (2.5) and (2.6) will emerge in a physical interpretation of nonuniqueness as quantities that determine how a particular invariant manifold bends.

CHAPTER 3: UNIQUENESS OF POSITIVE SOLUTIONS FOR THE p-LAPLACIAN, $1 < p < 2$

In this chapter, we show the uniqueness of positive radial solutions to (2.2) on a ball with Dirichlet boundary conditions for a class of nonlinearities f that includes $f(u) = u^q$ and the model sign-changing function $f(u) = u^q - u$, for $q < p^* - 1$. The proof is in the spirit of the Clemons–Jones geometric proof of the case $p = 2$ ([7]) but must overcome extra difficulties arising from the singularity of the operator Δ_p.

The domain Ω is a ball about the origin of radius R in $\mathbb{R}^n$, $n \geq 2$, and we suppose $1 < p \leq 2$. We are interested in *regular* solutions, meaning $u(0) = d > 0$. The Dirichlet problem is

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and the nonlinearity $f \in C^0[0,\infty) \cap C^1(0,\infty)$ satisfies (F1), (F2), and (R) described below.

(F1) $f(0) = 0$, and either f is nonnegative, or there exists a $\theta > 0$ so that $f < 0$ for $(0, \theta)$ and $f > 0$ for (θ, ∞).

(F2) The quantity $K(u) = uf'(u)/f(u)$ is nonincreasing on $(0, \theta)$ and (θ, ∞), where θ is defined by (F1).

(R) There is a value $q_1 > p$ satisfying

$$\frac{p(n-1)}{n-p} < q_1 < p^* = \frac{pn}{n-p}, \tag{3.2}$$

and

$$\lim_{u\to\infty}\frac{f(u)}{u^{q_1-1}}=\ell>0. \tag{3.3}$$

If $f(u)$ changes signs at $\theta>0$, then there is a value q_2 with $q_1>q_2>1$, so that

$$\lim_{u\to 0}\frac{f(u)}{u^{q_2-1}}=\nu<0.$$

The requirement (R) says that if f changes signs, then f behaves asymptotically like $f(u)=u^{q_1-1}+\lambda u^{q_2-1}$, $q_1>q_2>0$, $\lambda<0$.

Let us remark on a few properties of $K(u)$. Under hypothesis (R),

$$\lim_{u\to\infty}\frac{uf'(u)}{f(u)}=\lim_{u\to\infty}u\frac{u^{q_1-1}}{f(u)}\cdot\frac{f'(u)}{(q_1-1)u^{q_1-2}}\cdot\frac{(q_1-1)u^{q_1-2}}{u^{q_1-1}}=q_1-1, \tag{3.4}$$

and similarly $K(u)\to q_2-1$ as $u\to 0$. By (F1), it follows that $K(u)\to-\infty$ as $u\to\theta^-$ and $K(u)\to\infty$ as $u\to\theta^+$. Hence if (F2) and (R) are satisfied, we obtain

- $K(u)\le q_2-1$ for $u\in(0,\theta)$,
- $K(u)\ge q_1-1$ for $u>\theta$,
- if $a<\theta<b$, then $K(b)>K(a)$,

where the possibility that f is nonnegative is addressed throughout by setting $\theta=0$.

We note that the conditions (F1) and (F2) are similar to hypotheses in [2]. However, we will not require f to change signs at θ, nor do we require (AP3), as the quantity (2.6) may be positive, negative, or zero for different values of u.

Hence any polynomial of the form

$$f(u)=u^{q_1-1}-\nu u^{q_2-1},\qquad \nu\ge 0,\qquad q_1>q_2>p \tag{3.5}$$

satisfies (F1), (F2) and (R), including the representative example $f(u)=u^q-u^{p-1}$, $q>p-1$, from [2]. However, many basic nonlinearities satisfy our requirements without satisfying (AP3), for example

- $f(u) = u^{q_1-1}$,
- $f(u) = u^3 - u^2$, where p and n must be chosen to satisfy (3.2), and
- $f(u) = \dfrac{u^{s_1}}{\nu + u^{s_2}}$, where $s_1, s_2, \nu > 0$, and the condition (R) is satisfied for $q_1 - 1 = s_1 - s_2$.

As we are interested in positive solutions solving the Dirichlet problem, we do not specify $f(u)$ for $u < 0$. However, to show existence using an Emden–Fowler approach, it is often necessary for f to be odd. One way to ensure this condition holds is to write (3.5) as

$$f(u) = |u|^{q_1-2}u - \nu|u|^{q_2-2}u, \qquad \nu \geq 0, \qquad q_1 > q_2 > p. \tag{3.6}$$

Lastly, requiring $q_1 < p^*$, where p^* is the Sobolev critical exponent, makes the growth rate of f subcritical; the inequalities in (3.2) will be elaborated on in Section 3.2. The bulk of this chapter is dedicated to proving the following theorem.

Theorem 3.0.2. *If $1 < p \leq 2$, (3.2) is satisfied, and f satisfies (F1), (F2), and (R), then for any positive radius R of Ω, there is at most one radial solution to (3.1).*

As the regular Laplacian operator corresponding to $p = 2$ is well studied, we will focus on $1 < p < 2$, when the p-Laplacian is singular. However, we do note that our proof covers the regular Laplacian case for f nonnegative; a class of nonlinearities that was not addressed by [7]. The proof follows these steps:

1. rewrite the PDE as an ODE using radial symmetry,
2. rescale solutions u to a new variable y using the Emden Fowler transformation,
3. consider existence of solutions by studying the flow in the $\{r = 0\}$ plane,
4. compute the winding number of a vector component δu along solutions,

5. compute the normal vector to the manifold formed by the union of solutions to (3.1) and vector normal to an $\{r = \text{constant}\}$-plane, and

6. show that the amount of winding of the third component of these two vectors can be controlled.

3.1 Set-up as a dynamical system

We recall that in the case $p = 2$, Gidas, Ni, Nirenberg [18] showed that if $f \in C^1$, then solutions to (3.1) must be radially symmetric and monotone decreasing. This result does not extend immediately to the case $p \neq 1$. However, we are interested in radially symmetric solutions so that we can consider u as a function of $r = |\mathbf{x}|$. The following lemma establishes that any solution to (3.1) with $f(u)$ in our class of nonlinearities *must* be radially symmetric and monotonically decreasing.

Lemma 3.1.1. *Any positive solution u to (3.1) with $f(u)$ satisfying (F1), (F2), and (R) is radially symmetric and monotonically decreasing.*

Proof. This is a corollary to the work of Damascelli and Pacella described in section 2.2. If f is nonnegative, then the result is automatic by Theorem 2.2.1. If f changes signs at $\theta > 0$, then let

$$\beta(u) = u^{q_1 - 1} - f(u),$$

where $q_1 > q_2 > p$ satisfy (R). Then $\beta(0) = 0$, $\beta(u) > 0$ for $0 < u < \theta$, and $\beta(u)$ is continuous. As $f(0) = 0$ and $f(u) < 0$ for $u < \theta$, there is some $\hat{u} \in (0, \theta)$ so that $f(u)$ is nonincreasing on $[0, \hat{u}]$. As a result, $\beta(u)$ is a nondecreasing function on $[0, \hat{u}]$. We note also that $f(u) + \beta(u) = u^{q_1 - 1} > 0$ for all $u > 0$.

It remains to check that there is some $\beta_0 > 0$ so that

$$\int_0^{\beta_0} \frac{1}{\left(s\beta(s)\right)^{1/p}}\, ds = \int_0^{\beta_0} \frac{1}{\left(s^{q_1} - sf(s)\right)^{1/p}}\, ds = \infty.$$

We use a comparison test. Notice all terms in the denominator are positive for any $s < \theta$. Moreover, if $s \le 1$, then $q_1 > q_2$ implies

$$\frac{1}{\left(s^{q_1} - sf(s)\right)^{1/p}} \ge \frac{1}{\left(s^{q_2} - sf(s)\right)^{1/p}} = \frac{1}{\left(s^{q_2}\left(1 - \frac{f(s)}{s^{q_1-1}}\right)\right)^{1/p}}.$$

By (R)

$$\lim_{s\to 0} \frac{-f(s)}{s^{q_1-1}} = |\nu| > 0.$$

Hence for any $\varepsilon > 0$ there is a $\delta > 0$ so that if $s < \delta$,

$$-\frac{f(s)}{s^{q_1-1}} < |\nu| + \varepsilon.$$

Thus for $s < \delta$,

$$\frac{1}{\left(s^{q_2}\left(1 - \frac{f(s)}{s^{q_1-1}}\right)\right)^{1/p}} > \frac{1}{\left(s^{q_2}\left(1 + |\nu| + \varepsilon\right)\right)^{1/p}}.$$

Let $\beta_0 = \min\{\hat{u}, 1, \delta\}$ so that the above inequalities are valid for all $s \in (0, \beta_0)$. Then

$$\int_0^{\beta_0} \frac{1}{\left(s\beta(s)\right)^{1/p}}\, ds > \frac{1}{\left(1 + |\nu| + \varepsilon\right)^{1/p}} \int_0^{\beta_0} \frac{1}{s^{q_2/p}}\, ds = \infty, \tag{3.7}$$

as $q_2/p > 1$. Hence all solutions must be radial and monotone decreasing. $\square$

3.1.1 Dynamical system in (u, ω, r)-coordinates

Radial solutions u to (3.1) can be rewritten in terms of $r = |\mathbf{x}|$ to obtain the following ODE:

$$\left(u'|u'|^{p-2}\right)' + \frac{n-1}{r} u'|u'|^{p-2} + f(u) = 0, \tag{3.8}$$

with $' = \frac{d}{dr}$. Radial symmetry implies $u'(0) = 0$, and the boundary condition $u|_{\partial\Omega} = 0$ can be written simply as $u(R) = 0$, where R is the radius of Ω. Setting $\omega = u'|u'|^{p-2} r^{p-1}$ yields a first-order ODE for ω,

$$\left(r^{1-p}\omega\right)' + r^{-p}(n-1)\omega + f(u) = 0, \tag{3.9}$$

and therefore the (u, ω, r) system can be written as

$$
\begin{aligned}
u' &= \frac{\omega}{r}|\omega|^{\frac{2-p}{p-1}}, && (3.10)\\
\omega' &= (p-n)\frac{\omega}{r} - r^{p-1}f(u), && (3.11)\\
r' &= 1. && (3.12)
\end{aligned}
$$

This system is undefined at the value $r = 0$; by introducing a new independent variable t and parametrizing r as $r(t) = e^t$, then we may choose any value $r_0 > 0$ and define time t so that $r(0) = r_0$. As a result, we blow up the singularity at $r = 0$ into an invariant plane $\{r = 0\}$. The resulting first-order system is

$$
\begin{aligned}
\dot{u} &= \omega|\omega|^{\frac{2-p}{p-1}}, && (3.13)\\
\dot{\omega} &= (p-n)\omega - r^p f(u), && (3.14)\\
\dot{r} &= r, && (3.15)
\end{aligned}
$$

with $\cdot = \frac{d}{dt}$. Solutions to (3.8) can now be viewed as trajectories in the phase space of (3.13)-(3.15). In phase space, an initial condition at $r = 0$ corresponds to the limit of a solution trajectory as $t \to -\infty$. Suppose a solution satisfies the boundary condition $u'(0) = 0$ and has an initial value $u(0) = a$, where $a > 0$; for this hypothesis to be satisfied in phase space, a trajectory must have as its limit the point $(a, 0, 0)$ on the u-axis. Each such point $(a, 0, 0)$ is a fixed point of (3.13)-(3.15); linearization about $(a, 0, 0)$ yields

$$
\left.\begin{pmatrix} 0 & \frac{1}{p-1}|\omega|^{\frac{2-p}{p-1}} & 0 \\ -r^p f'(u) & p-n & -pr^{p-1}f(u) \\ 0 & 0 & 1 \end{pmatrix}\right|_{(a,0,0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p-n & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (3.16)
$$

As $1 < p < 2$ and $n \geq 2$, there is one zero eigenvalue, one negative eigenvalue, and one positive eigenvalue. (We will not concern ourselves with (3.16) in the case $p = 2$ as

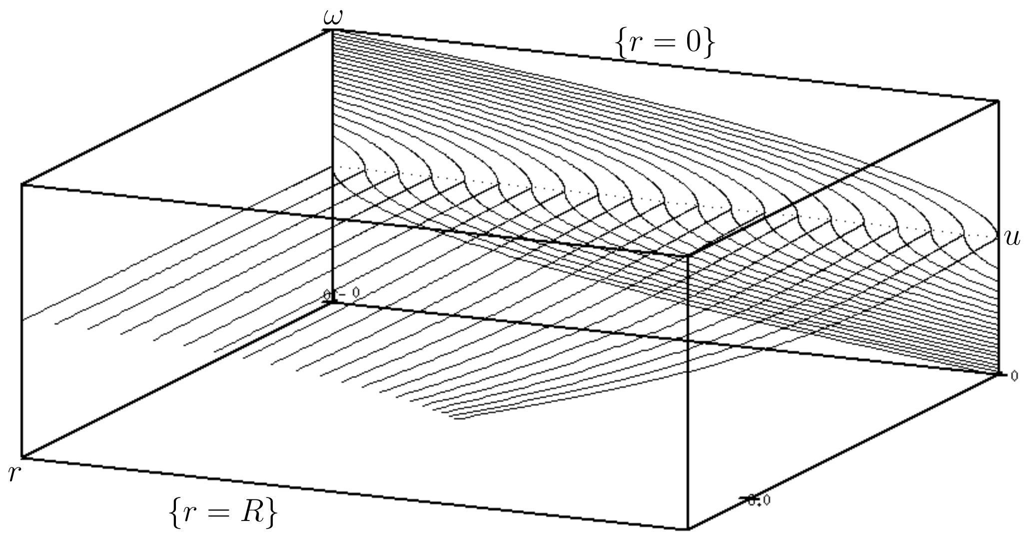


Figure 3.1: The center, stable, and unstable manifolds for twenty radial solutions to (3.13)-(3.15) with different initial values $u(0) = a > 0$. The dotted line in the plane $\{r = 0\}$ is the u-axis (the leftmost point along this axis is the origin); this forms the center manifold for each point $(a, 0, 0)$. The S-shaped curves are the stable manifolds in $\{r = 0\}$, and each curve moving into $r > 0$ space is an unstable manifold for one of the initial conditions $(a, 0, 0)$.

this is the well-understood case.) Hence each $(a, 0, 0)$ has a 1-dimensional stable manifold $W_p^s((a, 0, 0))$, a 1-dimensional unstable manifold $W_p^u((a, 0, 0))$, and a 1-dimensional center manifold $W_p^c((a, 0, 0))$.

An eigenvector for the eigenvalue 0 is parallel to the u-axis; by invariant manifold theory, the u-axis is the (global) center manifold $W_p^c((a, 0, 0))$ for each $a \in \mathbb{R}$. The global stable manifold to $(a, 0, 0)$ is the vertical line $\ell_a = \{(a, \omega, 0) \ : \omega \in \mathbb{R}\}$ in the case $p = 2$; for $1 < p < 2$, the stable manifold is tangent to the vertical vector $(0, 1, 0)$ at $(a, 0, 0)$. We note that the S-shape of the stable manifold in the plane $\{r = 0\}$ when $p \neq$ is due to the presence of $|\omega|^{\frac{2-p}{p-1}}$ in (3.13)-(3.15). See Figure 3.1 for a picture illustrating these manifolds at several values of $a > 0$.

Hence a plays the role of a parameter, and we are interested in examining how $W_p^u((a, 0, 0))$ behaves for different values of a. We define

$$W_p^{u,c} = \bigcup_{a>0} W_p^u((a, 0, 0)); \tag{3.17}$$

this union forms a two-dimensional "center-unstable" manifold whose winding behavior we will track in Section 3.4 until it intersects the plane $\{r = R\}$. This intersection is nonempty: as t grows large in (3.13)-(3.15), $r = e^t$ will grow large as well. In fact, $W_p^{u,c}$ must contain the point $(0, 0, R) \in W_p^u((0, 0, 0))$.

Any point lying on $W_p^{u,c}$ is part of a solution that tends to $(a, 0, 0)$, $a > 0$, as $t \to -\infty$. These solution trajectories, determined by the choice of a, foliate $W_p^{u,c}$. Thus we can denote any such solution by

$$(u(t, a), \omega(t, a), r(t)).$$

To be more concise, we will occasionally write the above solution as

$$(u(t), \omega(t), r(t))_a$$

to mean $(u(t), \omega(t), r(t))_a \to (a, 0, 0)$ as $t \to -\infty$.

3.1.2 Dynamical system in (y, w, r)-coordinates

We will make use of an Emden–Fowler transformation to exert extra control over the equations. Let $\lambda \in \mathbb{R}$; the Emden–Fowler transformation for (3.8) is

$$y = r^\lambda u, \tag{3.18}$$

where the parameter λ will be specified later to prove different cases. With the appropriately scaled replacement $w = r^{(p-1)\lambda}\omega$ for ω, one could pass immediately to a system of equations for $(\dot{y}, \dot{w}, \dot{r})$, where $r = e^t$ and $\cdot$ is differentiation with respect to t. To illustrate the construction of the missing w term as it may be useful for other variations of Laplace's equation, however, we show how to create such a w from (3.8) and (3.18) in the calculations below.

Substituting (3.18) into (3.8) yields

$$\begin{aligned} 0 &= (p-1)r^{\lambda(1-p)}\left|y' - \frac{\lambda}{r}y\right|^{p-2}\left(\frac{\lambda(\lambda+1)}{r^2}y - \frac{2\lambda}{r}y' + y''\right) \\ &\quad + \frac{n-1}{r}r^{\lambda(1-p)}\left|y' - \frac{\lambda}{r}y\right|^{p-2}\left(y' - \frac{\lambda}{r}y\right) + f(r^{-\lambda}y). \end{aligned} \tag{3.19}$$

Set $z = y' - \frac{\lambda}{r}y$ so that

$$z' = y'' + \frac{\lambda}{r^2}y - \frac{\lambda}{r}y'. \tag{3.20}$$

Then the differential equation (3.19) becomes

$$(p-1)r^{\lambda(1-p)}|z|^{p-2}\left(z' - \frac{\lambda}{r}z\right) + \frac{n-1}{r}r^{\lambda(1-p)}|z|^{p-2}z + f(r^{-\lambda}y) = 0. \tag{3.21}$$

We define w by $rz = w|w|^{\frac{2-p}{p-1}}$ to obtain the following relations:

$$z' = -\frac{1}{r^2}w|w|^{\frac{2-p}{p-1}} + \frac{1}{r}\frac{1}{p-1}|w|^{\frac{2-p}{p-1}}w', \tag{3.22}$$

$$|z|^{p-2}z = r^{1-p}w, \tag{3.23}$$

$$|z|^{p-2}\left(z' - \frac{\lambda}{r}z\right) = -r^{-p}w(1+\lambda) + \frac{1}{p-1}r^{1-p}w'. \tag{3.24}$$

Substituting (3.22)-(3.24) into (3.21) yields

$$(p-1)r^{\lambda(1-p)}\left(-r^{-p}w(1+\lambda) + \frac{1}{p-1}r^{1-p}w'\right) + (n-1)r^{\lambda(1-p)}r^{-p}w + f(r^{-\lambda}y) = 0, \tag{3.25}$$

which can be solved for w':

$$w' = ((p-1)\lambda - n + p)\frac{1}{r}w - r^{p+p\lambda-\lambda-1}f(r^{-\lambda}y). \tag{3.26}$$

To relate w to the original variable u, we note here that w can be written in terms of u as

$$w = u'|u'|^{p-2}r^{(p-1)(\lambda+1)}. \tag{3.27}$$

We note that w allows us to write (3.19) as a system of first-order equations. Defining w by (3.27) at the same step as (3.18) allows one to bypass the calculations in (3.19)-(3.25) to obtain the first-order system immediately. We remark that (3.27) does not reduce to the same w in the Emden–Fowler coordinate system in the proof by Clemons and Jones of the case $p = 2$. In particular, in [7] the first equation is $y' = w/r$, whereas we have the following system:

$$\begin{aligned} y' &= \frac{\lambda}{r}y + \frac{1}{r}w|w|^{\frac{2-p}{p-1}}, && (3.28)\\ w' &= ((p-1)\lambda - n + p)\frac{1}{r}w - r^{(p-1)(1+\lambda)}f(r^{-\lambda}y) && (3.29)\\ r' &= 1, && (3.30) \end{aligned}$$

where $' = \frac{\partial}{\partial r}$. As in the u-coordinate system, we rescale by parametrizing r as $r = e^t$; notice t and r have the same meaning before and after the Emden–Fowler transformation. The resulting equations are

$$\begin{aligned} \dot{y} &= \lambda y + w|w|^{\frac{2-p}{p-1}}, && (3.31)\\ \dot{w} &= ((p-1)\lambda - n + p)w - r^{p+\lambda(p-1)}f(r^{-\lambda}y) && (3.32)\\ \dot{r} &= r, && (3.33) \end{aligned}$$

where $\cdot = \frac{\partial}{\partial t}$. As before, the limit $r \to 0$ is equivalent to $t \to -\infty$; for the limit of (3.31)-(3.33) to exist as $t \to -\infty$, we need to ensure that the quantity $r^{p+\lambda(p-1)}f(r^{-\lambda}y)$ exists in the limit. Let us treat y as independent of λ and set $u = r^{-\lambda}y$. Then if $\lambda > 0$ and $y \to 0$ such that $u = r^{-\lambda}y \to u_0 > 0$, where u_0 is finite, then

$$\lim_{r\to 0} r^{p+\lambda(p-1)}f(r^{-\lambda}y)$$

exists if $\lambda \geq -p/(p-1)$. This condition happens automatically if $\lambda \geq 0$.

If we continue to consider the case $\lambda > 0$ and suppose that as $r \to 0$, $y \to y_0 > 0$, where

$y_0 < \infty$, then $u = r^{-\lambda} y \to \infty$. Using (R), we can compare f to $r^{p-\lambda(q_1-p)} y^{q_1-1}$ and obtain

$$\lim_{r\to 0} \frac{r^{p+\lambda(p-1)} f(r^{-\lambda}y)}{r^{p-\lambda(q_1-p)} y^{q_1-1}} = \lim_{r\to 0} \frac{r^{p+\lambda(p-1)} f(r^{-\lambda}y)}{r^{p+\lambda(p-1)} (r^{-\lambda}y)^{q_1-1}} = \ell.$$

Thus if the limit of $r^{p-\lambda(q_1-p)} y^{q_1-1}$ is finite as $r \to 0$, then limit of $r^{p+\lambda(p-1)} f(r^{-\lambda}y)$ is finite as well. We therefore require

$$\lambda \le \frac{p}{q_1 - p} =: \hat{\lambda}, \tag{3.34}$$

and we treat this number as an upper bound for λ. We will not require λ to be nonnegative, however, and we make a note of the effect that setting $\lambda < 0$ has on the dynamics of (3.31)-(3.33) in Section 3.2.

In the $(\dot{u}, \dot{\omega}, \dot{r})$ system, each point $(a, 0, 0)$ is a fixed point. The analog under the Emden–Fowler transformation is the origin; linearization of (3.31)-(3.33) at the origin yields

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & (p-1)\lambda - n + p & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } p \neq 2, \qquad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda - n + 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } p = 2, \tag{3.35}$$

with eigenvalues $\{\lambda, (p-1)\lambda - n + p, 1\}$. Notice that if $p \in (1, 2)$, the eigenvectors are parallel to the axes; see Figure 3.1.

3.2 Critical exponents and existence of solutions

Studying uniqueness amounts to understanding how the manifold $W_p^{u,c}$ evolves as the radius r increases to some chosen value R; in particular we study how many times $W_p^{u,c}$ can intersect the plane $\{u = 0\}$. The beauty of the Emden–Fowler approach is that the calculations to show uniqueness are greatly simplified. First, we must understand the effect of the Emden–Fowler parameter λ on the manifold of interest.

3.2.1 $W_p^{u,c}$ under the Emden–Fowler transformation

Let $T[\cdot]$ be the Emden–Fowler transformation (3.18) from (u, ω, r) to (y, w, r), and let $T\left[W_p^{u,c}\right] = \widetilde{W}_p$. Whenever $r > 0$ and the Emden–Fowler parameter λ satisfies $0 < \lambda \leq \hat{\lambda}$, then any point $T\left[(u(t), v(t), r(t))_a\right]$ on $\widetilde{W}_p$ now satisfies

$$(y(t), w(t), r(t)) \to (0, 0, 0) \text{ as } t \to -\infty.$$

Thus choosing $\lambda \in (0, \hat{\lambda}]$ has the effect of "blowing down" the u-axis. As a consequence, it no longer makes sense to parametrize solutions via their limit as $t \to -\infty$. To employ a similar notion, the notation $y(t, a)$ and $(y(t), w(t), r(t))_a$ will mean that the solution $(u(t), \omega(t), r(t))$ obtained from $y = r^\lambda u$ satisfies $(u(t), \omega(t), r(t)) \to (a, 0, 0)$ as $t \to -\infty$.

Assuming the critical exponent inequalities (3.2) are satisfied and $p \in (1, 2)$, we describe below the behavior of the invariant manifold $\widetilde{W}_p$.

Invariant manifold structure if $\lambda > \hat{\lambda}$. We will never consider this case as we require $\lambda \leq \hat{\lambda}$. As it may be interesting in future problems, we note here that should one choose $\lambda > \hat{\lambda}$, then the limit of $(\dot{y}, \dot{w}, \dot{r})$ as $t \to -\infty$ is undefined. However, the manifold $W_p^{u,c}$ derives from the (u, ω, r)-system and therefore exists independently of λ. For any $\varepsilon > 0$, $T[W_p^{u,c} \cap \{\varepsilon \leq r \leq R\}]$ is a two-dimensional manifold with boundary. Selecting $\lambda > \hat{\lambda}$ and defining $\widetilde{W}_p$ as

$$\widetilde{W}_p = T[W_p^{u,c} \cap \{\varepsilon \leq r \leq R\}] \tag{3.36}$$

yields a well-defined two-dimensional manifold with boundary in (y, w, r)-space.

Invariant manifold structure if $0 < \lambda \leq \hat{\lambda}$. If $\lambda \in (0, \hat{\lambda}]$ then (3.35) has one negative eigenvalue and two positive eigenvalues. Under the Emden–Fowler transformation, $\widetilde{W}_p$ is a two-dimensional unstable manifold of the origin.

Invariant manifold structure if $\lambda = 0$. If $\lambda = 0$, then the Emden–Fowler transformation is simply $u = y$. Hence $\widetilde{W}_p$ is identical to $W_p^{u,c}$, a two-dimensional center-unstable manifold.

Invariant manifold structure if $\lambda < 0$. If $\lambda < 0$, then there is one positive eigenvalue of (3.35) and two negative eigenvalues,

$$\{\lambda, (p-1)\lambda - n + p\}\,.$$

Thus all trajectories on the plane $\{r = 0\}$ tend to the origin as $t \to \infty$. If $\lambda \neq \frac{n-p}{p-2}$, these eigenvalues are distinct. In this case, $\widetilde{W}_p$ transforms to a two-dimensional stable-unstable manifold composed of the unstable manifold of the origin and the 1-dimensional subspace of the origin in $\{r = 0\}$ associated with the eigenvalue λ; this is the subspace tangent to the y-axis at the origin.

As in Section 3.2.1, if we select $\lambda < 0$, then we define $\widetilde{W}_p$ by (3.36) for an appropriately small $\varepsilon > 0$.

3.2.2 Existence of solutions

In this section, we list two expressions for each $\dot{w}$ equation: the first leaves f in its general form (where we assume f is odd), while the second uses (3.6). Recall equations (3.31)-(3.33):

$$\begin{aligned}
\dot{y} &= \lambda y + w|w|^{\frac{2-p}{p-1}} \\
\dot{w} &= ((p-1)\lambda - n + p)w - r^{p+\lambda(p-1)} f(r^{-\lambda}y) \\
&= ((p-1)\lambda - n + p)w - r^{p+\lambda(p-q_1)}|y|^{q_1-2}y + \nu r^{p+\lambda(p-q_2)}|y|^{q_2-2}y, \\
\dot{r} &= r.
\end{aligned}$$

At this point, we have not specified any particular λ; the choice we make to demonstrate the existence of solutions is $\lambda = \hat{\lambda}$. This selection characterizes $\widetilde{W}_p$ as described above in Section 3.2.1, and moreover, this choice is ideal as (3.31)-(3.33) simplifies to

$$\begin{aligned}
\dot{y} &= \hat{\lambda} y + w|w|^{\frac{2-p}{p-1}} \\
\dot{w} &= ((p-1)\hat{\lambda} - n + p)w - r^{p+\hat{\lambda}(p-1)} f(r^{-\hat{\lambda}} y) \\
&= ((p-1)\hat{\lambda} - n + p)w - |y|^{q_1-2} y + \nu r^{p - \frac{p(p-q_2)}{p-q_1}} |y|^{q_2-2} y, \\
\dot{r} &= r,
\end{aligned}$$

which in the invariant plane $\{r = 0\}$ reduces to

$$\begin{aligned}
\dot{y} &= \hat{\lambda} y + w|w|^{\frac{2-p}{p-1}} && (3.37) \\
\dot{w} &= ((p-1)\hat{\lambda} - n + p)w - r^{p+\hat{\lambda}(p-1)} f(r^{-\hat{\lambda}} y)\Big|_{r=0} && (3.38) \\
&= (p\hat{\lambda} - n + p)w - \hat{\lambda} w - |y|^{q_1-2} y. && (3.39)
\end{aligned}$$

If it were the case that $p\hat{\lambda} - n + p = 0$, then this system would be Hamiltonian in the plane $\{r = 0\}$ with

$$\begin{aligned}
H(y, w) &= \hat{\lambda} y w + \frac{p-1}{p}|w|^{\frac{p}{p-1}} + \int r^{p+\hat{\lambda}(p-1)} f(r^{-\hat{\lambda}} y)\Big|_{r=0} \, dy && (3.40) \\
&= \hat{\lambda} y w + \frac{p-1}{p}|w|^{\frac{p}{p-1}} + \frac{1}{q_1}|y|^{q_1}.
\end{aligned}$$

However, whenever $\lambda = \hat{\lambda}$,

$$p\hat{\lambda} - n + p > \frac{-p^2}{p - q_1} + \frac{p q_1}{p - q_1} + p = 0. \tag{3.41}$$

The resulting behavior of $W^u((0, 0))$ in the system (3.37)-(3.39) produces a "bowtie" as seen in Figure 3.2. Existence of solutions to (3.1) follows whenever the structure of the stable and unstable manifolds is in the configuration of in Figure 3.2; the stable manifold is trapped

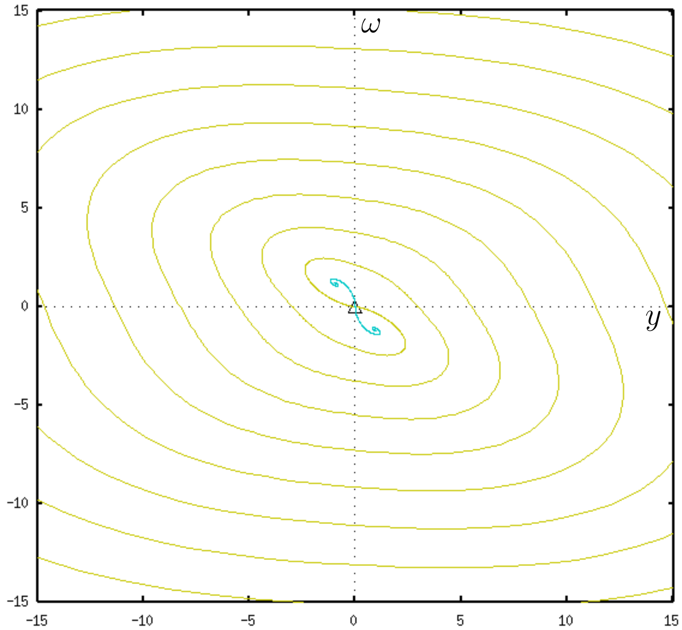


Figure 3.2: The stable and unstable manifolds of $(y, w, r) = (0, 0, 0)$ in the $\{r = 0\}$ plane, with $p = 1.5$, $d = 3$, and $q_1 = 2.5$. The manifold spiraling outward is the unstable manifold.

inside of the curve $H(y, w) = 0$ while the unstable manifold appears to spiral outwards. It is a result of Franca [17] that this occurs for a large class of nonlinearities.

We can now explore precisely why we require (3.2) to be satisfied. The different dynamics corresponding to different values of q_1, with $(n, p) = (3, 1.8)$, in the $\{r = 0\}$-plane are pictured Figure 3.3. Notice the switching of roles between the stable and unstable manifolds of $(0, 0, 0)$ as q_1 is varied to be below, in and above the inequalities in (3.2).

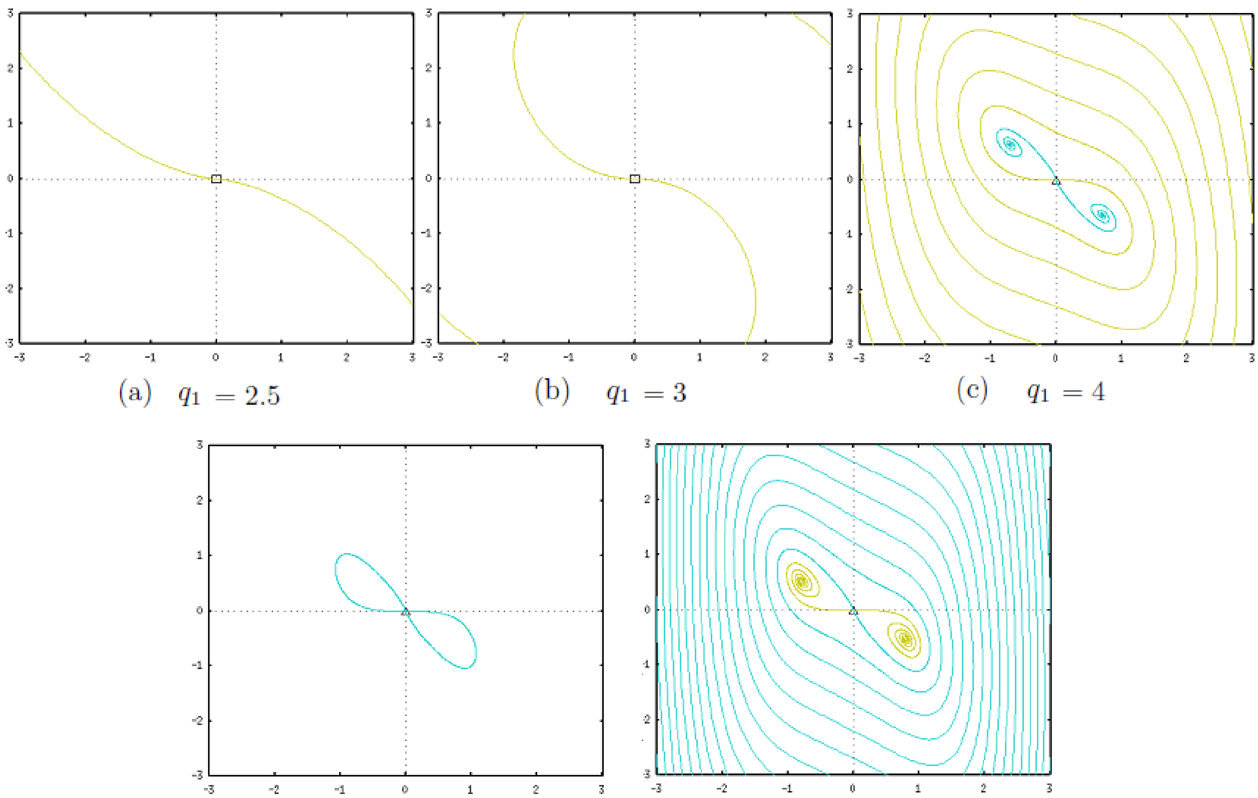
As Theorem 3.0.2 is concerned with uniqueness, rather than existence, we will not explore existence further in this chapter.

3.3 Variational equations

3.3.1 Definitions

For any time $\tau \in \mathbb{R}$, we define the *intersection curve*

$$C(\tau) = W_p^{u,c} \cap \{r = r(\tau)\}. \tag{3.42}$$



(a) $q_1 = 2.5$ (b) $q_1 = 3$ (c) $q_1 = 4$

(d) $q_1 = 4.5$ (e) $q_1 = 5$

Figure 3.3: Varying q_1 with $(n, p) = (3, 1.8)$ to show the stable and unstable manifolds of $(y, w, r) = (0, 0, 0)$ in the $\{r = 0\}$ plane. (a) is less than the lower bound, (b) is the lower bound, (c) is within the bounds, (d) is at the Sobolev critical exponent p^*, and (e) is above p^*. In (a) and (b), the origin is a source; for (c)-(e), the origin is a hyperbolic saddle point. Figures (c) and (e) show the behavior switching between the stable and unstable manifolds.

For any chosen u with initial condition $\alpha > 0$, let $C(\tau, \alpha)$ be the *truncated intersection curve* defined by

$$C(\tau, \alpha) = \{(u(\tau), \omega(\tau), r(\tau))_a \in C(\tau) \; : \; a \in (0, \alpha]\}.$$

Notice $C(\tau, \alpha) \subset C(\tau)$ lie in the $\{r = r(\tau)\}$-plane. The curve defined by

$$\gamma(\tau, \alpha) = \{(u(t), \omega(t), r(t))_\alpha \; : \; t \in (-\infty, \tau]\},$$

which we will refer to as a *solution trajectory*, limits to $(\alpha, 0, 0)$ as $t \to -\infty$ and intersects $C(\tau, \alpha)$ at $u(\tau, \alpha)$. Examples of both of these curves are sketched in Figure 3.4.

3.3.2 Variational equations

For any choice of $t \in \mathbb{R}$, the curve $C(t)$ in (3.42) can be parametrized by the u-coordinate initial condition a via

$$c_t(a) = (u(t, a), \omega(t, a), r(t)). \tag{3.43}$$

Taking the derivative along $c_t(a)$ with respect to a yields a family of tangent vectors with $\delta r \equiv 0$:

$$\frac{dc_t}{da}(\alpha) = \left(\left.\frac{\partial u(t, a)}{\partial a}\right|_{a=\alpha}, \left.\frac{\partial \omega(t, a)}{\partial a}\right|_{a=\alpha}, \left.\frac{\partial r(t)}{\partial a}\right|_{a=\alpha} \right) =: (\delta u(t, \alpha), \delta\omega(t, \alpha), 0). \tag{3.44}$$

Notice that in the $\{r = 0\}$-plane, we can parametrize the center manifold in the same fashion as $c_{\{r=0\}}(a) = (a, 0, 0)$, and thus

$$\frac{dc_{\{r=0\}}}{da}(\alpha) = (1, 0, 0).$$

Hence

$$\lim_{t \to -\infty} \delta u(t, \alpha) = 1, \qquad \lim_{t \to -\infty} \delta\omega(t, \alpha) = 0.$$

The variational equations to describe how such a family of tangent vectors in the (u, ω, r)-system is carried under the flow are given by

$$
\begin{aligned}
\dot{\delta u} &= \frac{1}{p-1}|\omega|^{\frac{2-p}{p-1}}\,\delta\omega, && (3.45)\\
\dot{\delta\omega} &= (p-n)\,\delta\omega - \frac{\partial}{\partial u}\left(r^p f(u)\right)\,\delta u - \frac{\partial}{\partial r}\left(r^p f(u)\right)\,\delta r && (3.46)\\
\dot{\delta r} &= \delta r. && (3.47)
\end{aligned}
$$

In particular, the tangent vector field in (3.44) satisfies

$$
\begin{aligned}
\dot{\delta u} &= \frac{1}{p-1}|\omega|^{\frac{2-p}{p-1}}\,\delta\omega, && (3.48)\\
\dot{\delta\omega} &= (p-n)\,\delta\omega - r^p f'(u)\,\delta u && (3.49)\\
\dot{\delta r} &= 0. && (3.50)
\end{aligned}
$$

We define two curves in the tangent bundle to $W_p^{u,c}$ as follows: for any point $(u(\tau), \omega(\tau), r(\tau))_a \in C(\tau, \alpha)$, we find the tangent vector from (3.44) and form the following curve:

$$S_{C(\tau,\alpha)} = \{\delta u(\tau, a), \delta\omega(\tau, a), 0) \;:\; a \in (0, \alpha]\}.$$

Similarly, for each point $(u(t), \omega(t), r(t))_\alpha$ along a single solution trajectory $\gamma(\tau, \alpha)$, we find the tangent vector

$$(\delta u(t, \alpha), \delta\omega(t, \alpha), 0)$$

defined by (3.44) and then construct the following curve:

$$S_{\gamma(\tau,\alpha)} = \{(\delta u(t, \alpha), \delta\omega(t, \alpha), 0) \;:\; t \in (-\infty, \tau]\}.$$

Figure 3.4 illustrates the tangent vectors that define these curves.

As in Jones and Küpper [21], we will let I denote the winding number of the admissible

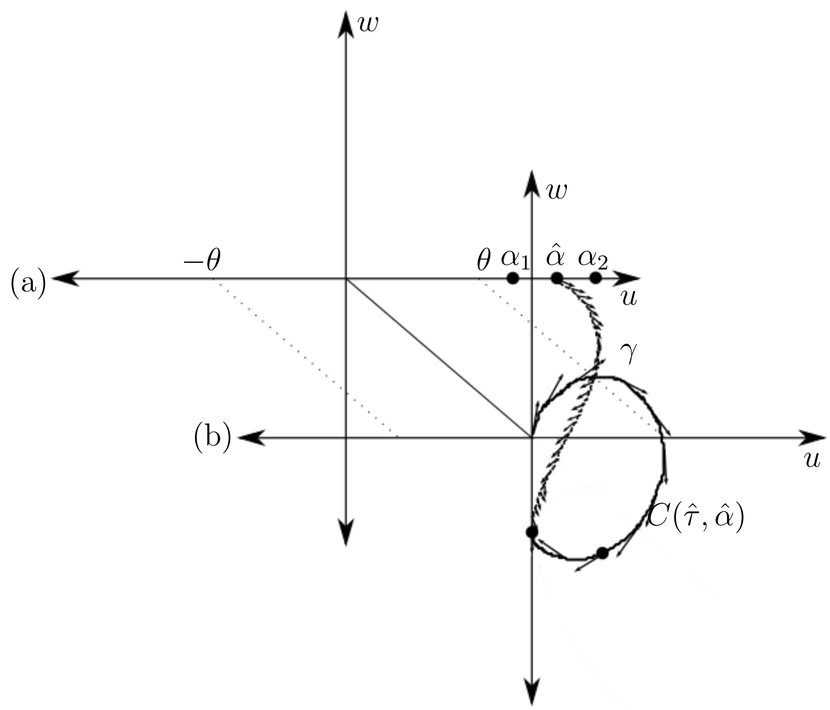


Figure 3.4: The plane (a) is the $\{r = 0\}$, and (b) is the plane $\{r = r(\hat{\tau})\}$ from the proof of Lemma 3.4.1 illustrated for the case $k = 1$. The curves $\gamma = \gamma(\hat{\tau}, \hat{\alpha})$ and $C(\hat{\tau}, \hat{\alpha})$ are labeled, while their corresponding vector fields, S_γ and S_C, respectively, are sketched as well.

curve $c_\tau(a) : [0, \alpha] \to \mathbb{R}^2$. To state the definition of I, we first define a *continuous* angle measure $\vartheta : c_\tau(a) \to \mathbb{R}$ so that $\vartheta(c_\tau(a))$ is on the appropriate branch of $\arctan(\delta\omega(\tau, a)/\delta u(\tau, a))$, where $(\delta u(\tau, a), \delta\omega(\tau, a))$ is the tangent vector along $c_\tau(a)$ at the point $(u(\tau, a), \omega(\tau, a))$. Moreover,

$$\vartheta(c_\tau(0)) \equiv \arctan\left(\frac{\delta\omega(\tau, 0)}{\delta u(\tau, 0)}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{3.51}$$

We remark that for $1 < p < 2$, the angle $\vartheta(c_\tau(0))$ is strictly between $-\pi/2$ and $\pi/2$, as along the invariant line $\{(0, 0, r) \mid r \geq 0\}$, the first component δu has $\dot{\delta u} \equiv 0$ by (3.45). Hence $\delta u \equiv 1$ along $\{(0, 0, r) \mid r \geq 0\}$, and therefore (3.51) is defined for every τ. (The case $p = 2$ is done in [21].)

The winding number I along the intersection curve is defined by

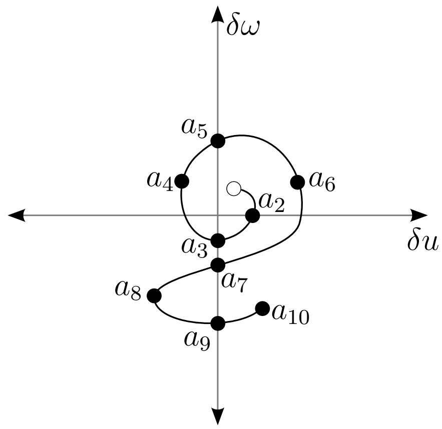
$$I(c_\tau(a)) = \left\lfloor \frac{1}{2}\left(\frac{-2\vartheta(c_\tau(\alpha))}{\pi} + 1\right) \right\rfloor - \left\lfloor \frac{1}{2}\left(\frac{-2\vartheta(c_\tau(0))}{\pi} + 1\right) \right\rfloor = \left\lfloor \frac{1}{2}\left(\frac{-2\vartheta(c_\tau(\alpha))}{\pi} + 1\right) \right\rfloor;$$

the symbol $\lfloor \cdot \rfloor$ denotes the greatest integer function. To show that it does indeed reduce to the right-hand side and demonstrate how this calculation works, notice

$$\begin{aligned}
\vartheta(c_\tau(0)) &\in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\
-2\vartheta(c_\tau(0)) &\in (-\pi, \pi) \\
\frac{-2\vartheta(c_\tau(0))}{\pi} &\in (-1, 1) \\
\frac{-2\vartheta(c_\tau(0))}{\pi} + 1 &\in (0, 2) \\
\frac{1}{2}\left(\frac{-2\vartheta(c_\tau(0))}{\pi} + 1\right) &\in (0, 1) \\
\left\lfloor \frac{1}{2}\left(\frac{-2\vartheta(c_\tau(0))}{\pi} + 1\right) \right\rfloor &= 0.
\end{aligned}$$

This quantity counts the number of net crossings (with clockwise about the origin crossings positive and counterclockwise about the origin crossings negative) of the $\delta\omega$-axis in the $(\delta u, \delta\omega)$

(a)



(b)

Point	$\vartheta(c_\tau(a))$	$I(c_\tau(a))$
$a = 0$	$\pi/3$	0
$a = a_2$	0	0
$a = a_3$	$-\pi/2$	1
$a = a_4$	$-5\pi/4$	1
$a = a_5$	$-3\pi/2$	2
$a = a_6$	$-5\pi/3$	2
$a = a_7$	$-5\pi/2$	3
$a = a_8$	$-11\pi/4$	3
$a = a_9$	$-5\pi/2$	3
$a = a_{10}$	$-7\pi/3$	2

Figure 3.5: (a) An imagined $S_{C(\tau,\hat{\alpha})}$ with 10 selected points. In (b), we estimate the angle measure ϑ for each of the 10 points, beginning with the open circle and moving in the direction of increasing initial condition a. For each ϑ, we compute the winding number I in the third column.

plane. See Figure 3.5 for a demonstration; this demonstration is particularly important as it shows how the winding number is calculated when the curve $S_{C(\tau,a)}\big|_{a\leq\alpha}$ stops on the $\delta\omega$-axis. We establish a convention in this chapter that in winding number drawings, we mark the beginning of the curve, $c_\tau(0)$, with an open circle.

We use the word "homotopic" for curves to refer to the notion of being pathwise homotopic into the punctured plane $\mathbb{R}^2\backslash\{0\}$. The winding number I is then invariant for homotopic curves. Let us consider the piecewise-defined curves $\{(0,0,r) \mid 0 \leq r \leq r(\tau)\} \cup C(\tau,\alpha)$ and $\{(a,0,0) \mid 0 \leq a \leq \alpha\} \cup \gamma(\tau,\alpha)$. As they form the boundary of the region

$$\{0 \leq r \leq r(\tau)\} \bigcap \left\{ \bigcup_{0<a<\alpha} W_p^u((a,0,0)) \right\},$$

there is a piecewise smooth path homotopy between these two curves. Thus the winding number along them must be the same. However, $\delta u \equiv 1$ along both pieces $\{(a,0,0) \mid 0 \leq a \leq \alpha\}$ and $\{(0,0,r) \mid 0 \leq r \leq r(\tau)\}$. Thus any winding behavior happens along $C(\tau,\alpha)$ and $\gamma(\tau,\alpha)$. Hence we conclude that $I(S_{C(\tau,\alpha)}) = I(S_{\gamma(\tau,\alpha)})$.

With this construction, we can now state a result connecting the algebraic winding number of δu and the number of zeros of δu along $\gamma(\tau,\alpha)$. The following lemma is similar to

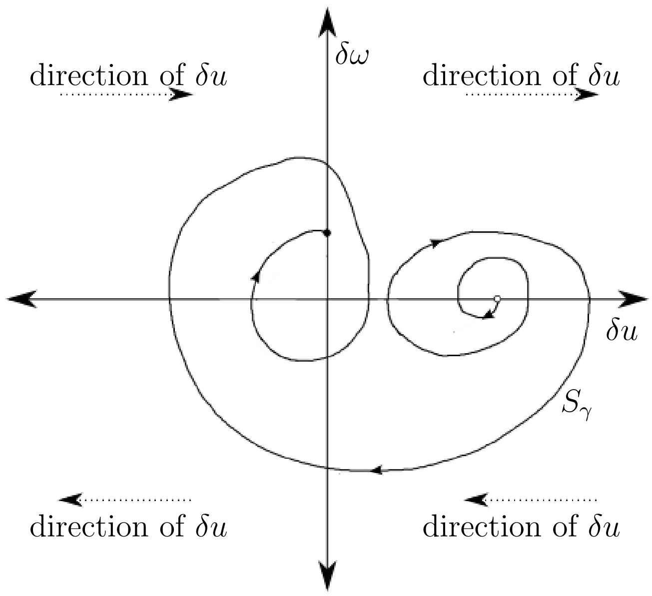


Figure 3.6: An imagined $S_{\gamma(\tau,\alpha)}$ from Lemma 3.3.1 satisfying $\dot{\delta u} = \frac{1}{p-1}|\omega|^{\frac{2-p}{p-1}}$. The open circle on the δu-axis indicates that the limit as $t \to -\infty$ of $S_{\gamma(\tau,\alpha)}$ is $(1,0)$. The closed circle on the $\delta\omega$-axis indicates that at the moment this winding number is computed for this particular trajectory, $\delta u(\tau,\alpha) = 0$ with $\delta\omega(\tau,\alpha) > 0$. The winding number of the curve $S_{\gamma(\tau,\alpha)}$ in this case is 4.

Proposition 3.5 from [21].

Lemma 3.3.1. *For any trajectory $(u(t),\omega(t),r(t))_\alpha$ at time $t = \tau$, $I(S_{\gamma(\tau,\alpha)})$ is the exact number of zeros of $\delta u(t,\alpha)$ for $-\infty < t \leq \tau$.*

This is not immediate: in a winding number calculation, it is possible for crossings (instances where $\delta u = 0$) to cancel each other out if they cross with $\delta\omega > 0$ or $\delta\omega < 0$ in the opposite direction. We see, therefore, that $I(S_{\gamma(\tau,\alpha)})$ is at the very least a lower bound on the number of times $\delta u = 0$. To prove this lemma, therefore, we must show that along $\gamma(\tau,\alpha)$, the winding curve can only cross the axis $\{\delta u = 0\}$ in one direction, namely in a manner clockwise about the origin.

Remark 3.3.2. When examining Figures 3.6-3.9, it is important to remember that $\dot{\delta u}$ is differentiation of δu with respect to *time*, and not with respect to the initial condition, a. Therefore, it is generally not possible to determine $\dot{\delta u}$ when examining an r = constant plane.

Proof. This lemma relies on the fact that $\dot{\delta u} = \frac{1}{p-1}|\omega|^{\frac{2-p}{p-1}}\,\delta\omega$. Consider the $(\delta u, \delta\omega)$-plane as pictured in Figure 3.6. We notice immediately that $S_{\gamma(\tau,\alpha)}$ can never intersect the origin, as $\delta u = \delta\omega = 0$ is invariant in (3.48)-(3.48), and the limit of $(\delta u, \delta\omega)$ as $t \to -\infty$ is $(1, 0)$. The fact that whenever $r > 0$, the relation

$$\delta\omega = 0 \qquad \Longleftrightarrow \qquad \dot{\delta u} = 0 \tag{3.52}$$

implies that any time $S_{\gamma(\tau,\alpha)}$ crosses the δu-axis (i.e., the line $\{\delta\omega = 0\}$), then $\dot{\delta u} = 0$. Hence the $S_{\gamma(\tau,\alpha)}$ must be perpendicular to the δu-axis at any such crossing. Conversely, the curve can only turn vertical if it is crossing the δu-axis. Thus there are no tangential intersections of either axes.

Furthermore, as the sign of $\delta\omega$ and $\dot{\delta u}$ must be the same, then δu must be increasing in the first and second quadrants, and decreasing in the third and fourth quadrants. Thus if it crosses the $\delta\omega$-axis with $\delta\omega < 0$, it must be crossing from the fourth quadrant to the third quadrant, and if it crosses the $\delta\omega$-axis with $\delta\omega > 0$, it must be crossing from the second quadrant to the first quadrant. Hence each crossing of the line $\{\delta u = 0\}$ must be in the clockwise direction. Therefore, the winding number $I(S_{\gamma(\tau,\alpha)})$ is equal to the exact number of zeros of δu. A whimsical example of an $S_{\gamma(\tau,\alpha)}$ that follows these guidelines is pictured in Figure 3.6. $\square$

Recalling that T is the Emden–Fowler transformation, we consider the intersection curve in Emden–Fowler coordinates by computing

$$D(t) = T(C(t)).$$

By construction, $D(t)$ is a curve lying in $\widetilde{W}_p$ in the $\{r = r(t)\}$ plane. As before, this is parametrized by a. Define

$$\frac{dD_t}{\partial a}(\alpha) = \left(\left.\frac{\partial y(t,a)}{\partial a}\right|_{a=\alpha}, \left.\frac{\partial w(t,a)}{\partial a}\right|_{a=\alpha}, \left.\frac{\partial r(t)}{\partial a}\right|_{a=\alpha} \right) =: (\delta y(t,\alpha), \delta w(t,\alpha), 0). \tag{3.53}$$

Under the Emden–Fowler transformation, the variational equations for the (y, w, r)-system are given by

$$\begin{aligned}
\dot{\delta y} &= \lambda\,\delta y + \frac{1}{p-1}|w|^{\frac{2-p}{p-1}}\,\delta w, && (3.54)\\
\dot{\delta w} &= ((p-1)\lambda - n + p)\,\delta w - \frac{\partial}{\partial y}\left(r^{p+\lambda(p-1)}f(r^{-\lambda}y)\right)\,\delta y && (3.55)\\
&\quad -\frac{\partial}{\partial r}\left(r^{p+\lambda(p-1)}f(r^{-\lambda}y)\right)\,\delta r\\
\dot{\delta r} &= \delta r. && (3.56)
\end{aligned}$$

Notice in particular that the vector field $(\delta y(t), \delta w(t), 0)_\alpha$ satisfies (3.54)-(3.56) with $\delta r \equiv 0$. As $y = r^\lambda u$, along this vector field we can write

$$\delta y|_{\delta r=0} = \lambda r^{\lambda-1}u\,\delta r + r^\lambda\,\delta u\big|_{\delta r=0} = r^\lambda\,\delta u. \tag{3.57}$$

There are therefore three cases for $\lim_{t\to-\infty}\delta y$:

1. if $\lambda > 0$, then $\delta y \to 0$,

2. if $\lambda = 0$, then $\delta y \to 1$, and

3. if $\lambda < 0$, then the limit of δy is undefined.

In case (3), although δy is undefined in the limit (more precisely, $|\delta y| \to \infty$), we recall that the tangent vector field δu exists independently of λ, and for any $\varepsilon > 0$, $T[\delta u|_{r\geq\varepsilon}] = \delta y|_{r\geq\varepsilon}$ is a well-defined vector field.

3.4 Proof of uniqueness

Nonuniqueness implies that $C(T)$ contains two points that can be written as

$$(u(T),\omega(T),r(T))_{\alpha_1}=(0,\beta_1,r(T)) \qquad \text{and} \qquad (u(T),\omega(T),r(T))_{\alpha_2}=(0,\beta_2,r(T))$$

with $\beta_1\neq\beta_2$, such that $u(t,\alpha_i)>0$, $i=1,2$, for all $t<T$.

With this terminology, we set up two possible cases to be ruled out. We will assume that $C(T)$ intersects $\{u=0\}$ transversally at $a=\alpha_1$ and $a=\alpha_2$ so that $\delta u(T,\alpha_i)\neq 0$, $i=1,2$. We remark that this assumption is safe because the proof of uniqueness actually rules out the possibility that a positive solution could intersect $\{u=0\}$ tangentially. In other words, a consequence of the proof calculations is that any solution $u(t,a)$ which is positive for $0\leq t<t$ with $u(T,a)=0$ cannot have $\delta u(T,a)=0$.

The following lemma is similar to Lemma 1 from [7], with the notable difference that we do not assume δu must be zero before $u=0$ (in other words, we will rule out the "underrotation" case with a specific calculation later in Section 3.4.1).

Lemma 3.4.1. *Nonuniqueness at $r(T)=R$ means that there exists τ and $\hat{\alpha}$, with $-\infty<\tau<T$, such that $u(\tau,\hat{\alpha})=0$ and $\delta u(\tau,\hat{\alpha})=0$. Moreover, exactly one of the following two statements must hold:*

1. *It is the case that $\delta u(t,\hat{\alpha})>0$ for all $t\in(-\infty,\tau)$, and $\delta\omega(\tau,\hat{\alpha})<0$. This is referred to as "underrotation."*

2. *There exists τ_0, with $-\infty<\tau_0<\tau<T$, such that $\delta u(\tau_0,\hat{\alpha})=0$ and $\delta u(t,\hat{\alpha})\neq 0$ for $t\in(-\infty,\tau_0)\cup(\tau_0,\tau)$. Moreover, $\delta\omega(\tau,\hat{\alpha})>0$. This is referred to as "overrotation."*

Proof. Consider the interval $I=[\alpha_1,\alpha_2]$. By the transversality assumption preceding the statement of the lemma, we may choose α_1 and α_2 so that for each $a\in I$, $u(\tau,a)\neq 0$.

Moreover, by selecting α_1 as

$$\alpha_1 = \min\{a > 0 \mid u(T, a) = 0\}$$

and α_2 as

$$\alpha_2 = \min\{a > \alpha_1 \mid u(T, a) = 0\},$$

then for each $a \in I$, $u(\tau, a) \leq 0$. By the intermediate value theorem, for each $a \in I$ there is some time $T_a \leq T$ such that $u(T_a, a) = 0$. We define a continuous map associating to each $a \in I$ the corresponding time T_a using the following.

Let $t_I : I \to \mathbb{R}$ be the map that sends each $a \in I$ to the first time $t_I(a) \leq T$ such that $u(t_I(a), a) = 0$. This map is well-defined and continuous, and as I is compact, $t_I(I)$ is compact. Therefore, $t_I(I)$ attains its minimum; let $\tau = \min\{t_I(a)\}_{a \in I}$, and let $\hat{\alpha} \in I$ denote a solution trajectory that satisfies $u(\tau, \hat{\alpha}) = 0$.

As $\hat{\alpha}$ must be an isolated zero, there is a neighborhood $B_{\hat{\alpha}} \subset I$ about $\hat{\alpha}$ so that $u(\tau, a)|_{B_{\hat{\alpha}}} > 0$. By continuity, we conclude

$$\delta u(\tau, \hat{\alpha}) = \left.\frac{\partial u(\tau, a)}{\partial a}\right|_{a=\hat{\alpha}} = 0,$$

as u has a local minimum at $\hat{\alpha}$. Figure 3.7 illustrates this for the overrotation case.

Either $\omega(T, \alpha_1) > \omega(T, \alpha_2)$ or $\omega(T, \alpha_1) < \omega(T, \alpha_2)$. (The first possibility is the underrotation case while the second is the overrotation case.) If $\omega(T, \alpha_1) > \omega(T, \alpha_2)$, then

$$I(S_\gamma) = I(C(\tau, \hat{\alpha})) = 1,$$

as $I(C(\tau, \hat{\alpha}))$ is homotopic to the curve pictured in Figure 3.8(a). The winding number of this curve is computed in Figure 3.8(b).

This underrotation scenario implies that the first zero of $\delta u(t, \hat{\alpha})$ occurs when $t = \tau$. In this case, $\delta u(t, \hat{\alpha})$ would be positive for all $t < \tau$, as the limit of δu as $t \to -\infty$ is 1. The

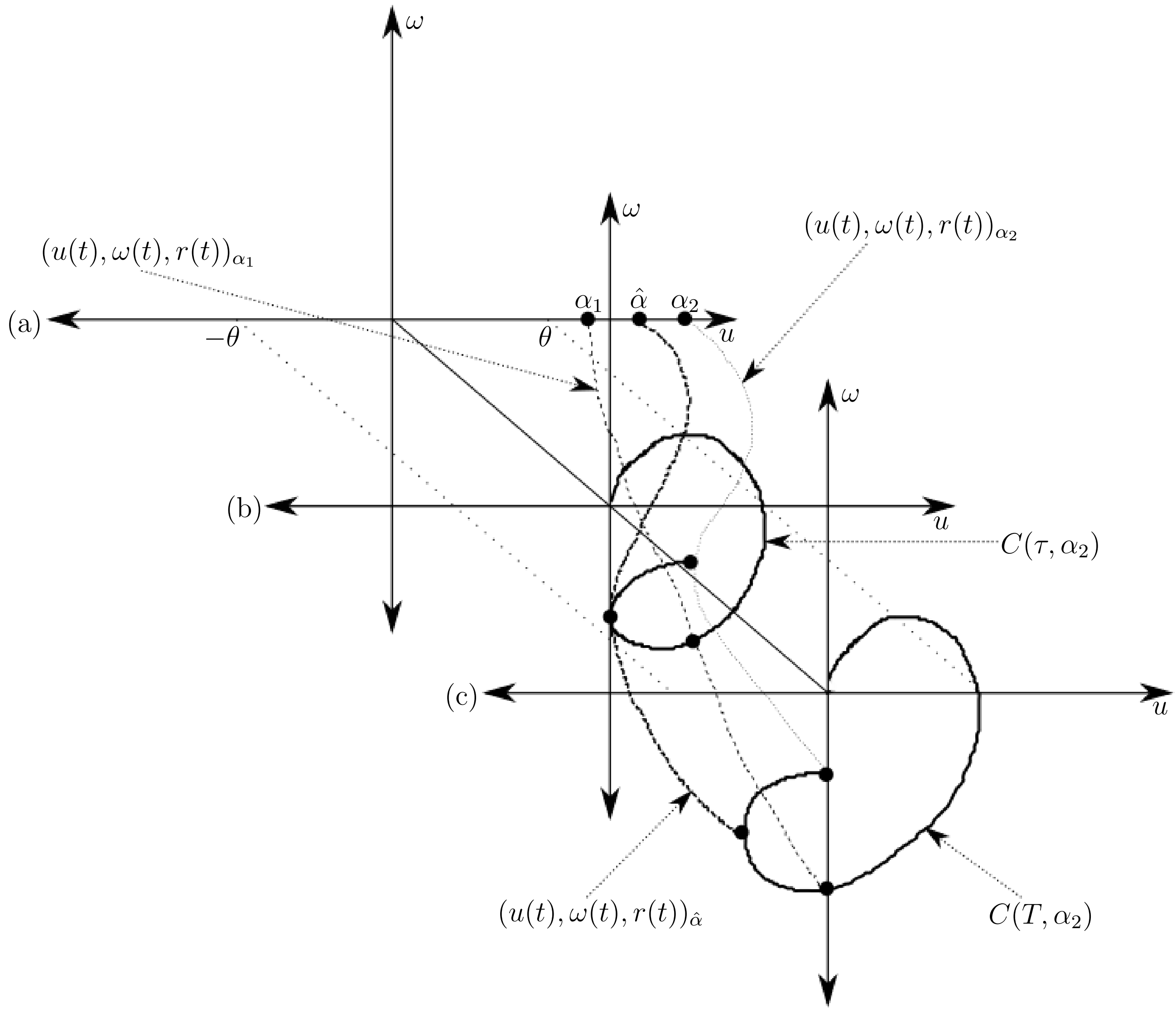
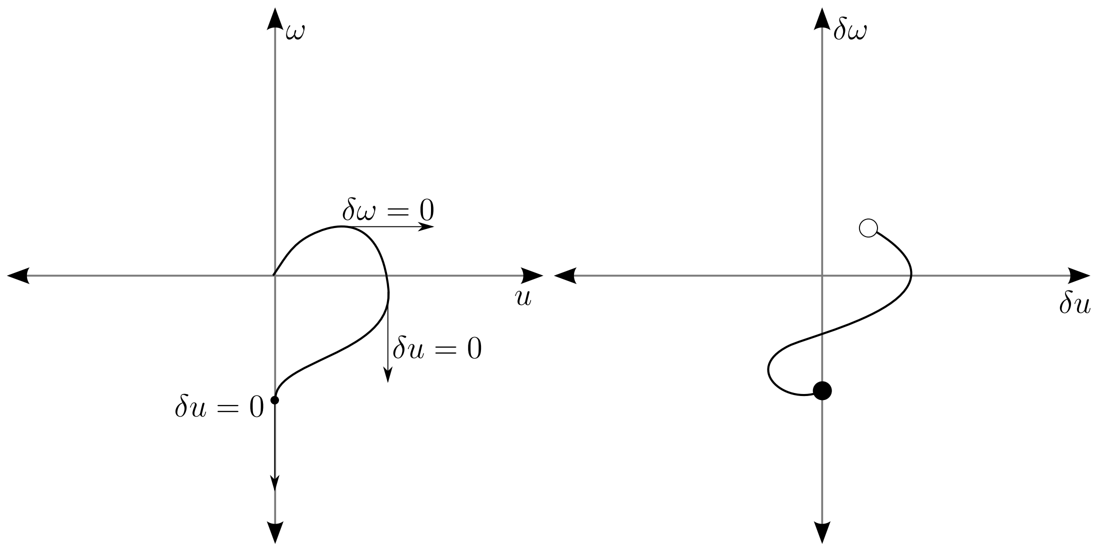


Figure 3.7: This figure illustrates the overrotation setup for Lemma 3.4.1. (a) is the plane $\{r = 0\}$, (b) is the plane $\{r = r(\tau)\}$, and (c) is the plane $\{r = R\}$, where $r(T) = R$. Pictured are the curve $C(\tau) \subset W_+^0 \cap \{r = r(\tau)\}$ and the curve $C(T) \subset W_+^0 \cap \{r = R\}$. The trajectories $(u(t), \omega(t), r(t))_{\alpha_1}$ and $(u(t), \omega(t), r(t))_{\alpha_2}$ each have their first intersection with the plane $\{u = 0\}$ when $r = R$. Lemma 3.4.1 guarantees the existence of $\hat{\alpha}$ and τ with $\tau < T$ such that the curve $(u(t), \omega(t), r(t))_{\hat{\alpha}}$ intersects $\{u = 0\}$ when $r = r(\tau)$, as pictured. See Lemma 3.4.1 for the precise statement.



(a) $C(\tau, \hat{\alpha})$ (b) $S_{C(\tau,\hat{\alpha})}$

Figure 3.8: (a) The general form of $C(\tau, \hat{\alpha})$ from Claim 3 in Lemma 3.4.1 under the assumption that $\omega(\tau, \alpha_1) > \omega(\tau, \alpha_2)$. The winding number of this curve is 1. Unlike in Figure 3.6, this curve may be vertical without crossing the δu-axis, as $C(\tau, \hat{\alpha})$ is parametrized by a, whereas γ in Figure 3.6 is parametrized by t. (b) The general shape of $S_{C(\tau,\hat{\alpha})}$ based on (a).

concludes the proof for case (1) of the lemma.

If $\omega(\tau, \alpha_1) < \omega(\tau, \alpha_2)$, then

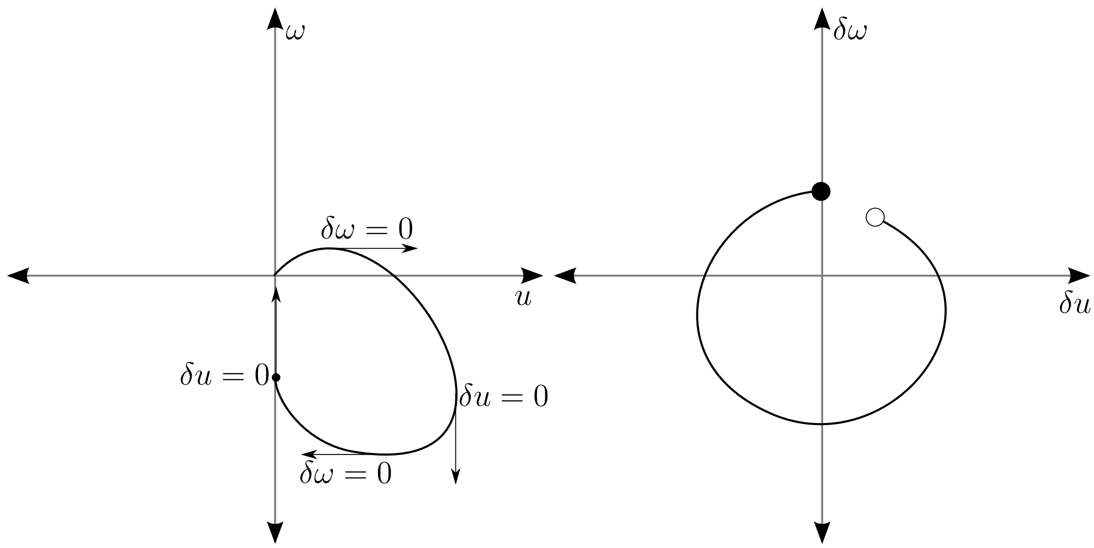
$$I(S_\gamma) = I(C(\tau, \hat{\alpha})) = 2.$$

In this case, $I(C(\tau, \hat{\alpha})$ is homotopic to the curve pictured in Figure 3.9(a), whose winding number is computed in Figure 3.9(b).

This calculation finishes the proof: for the overrotation case there is exactly one value $\tau_0 \in (-\infty, \tau)$ such that $\delta u(\tau_0, \hat{\alpha}) = 0$. Notice that in this case, we also discover that $\delta\omega(\tau, \hat{\alpha}) > 0$, as pictured in Figure 3.9(a). $\square$

Now we can state the analog of Lemma 3.4.1 in Emden–Fowler coordinates.

Lemma 3.4.2. *Assume the same hypotheses of Lemma 3.4.1 with the same $T = \ln R$. Then nonuniqueness means that there exists τ and $\hat{\alpha}$, with $-\infty < \tau < T$, such that $y(\tau, \hat{\alpha}) = 0$, $\delta y(\tau, \hat{\alpha}) = 0$. Moreover, one of the following two must hold:*



(a) $C(\tau, \hat{\alpha})$ (b) $S_{C(\tau,\hat{\alpha})}$

Figure 3.9: (a) The general form of $C(\tau, \hat{\alpha})$ from Claim 4 in Lemma 3.4.1 under the assumption that $\omega(\tau, \alpha_1) < \omega(\tau, \alpha_2)$. (b) The general form of $S_{C(\tau,\hat{\alpha})}$ based on (a). Unlike in Figure 3.6, this curve may be vertical without crossing the δu-axis, as $C(\tau, \hat{\alpha})$ is parametrized by a, whereas γ in Figure 3.6 is parametrized by t. The winding number of this curve is 2.

1. *In the underrotation case, $\delta y(t, \hat{\alpha}) > 0$ for all $t \in (-\infty, \tau)$. Moreover, $\delta w(\tau, \hat{\alpha}) < 0$.*

2. *In the overrotation case, there is τ_0, with $-\infty < \tau_0 < \tau < T$, such that $\delta y(\tau_0, \hat{\alpha}) = 0$, and $\delta y(t, \hat{\alpha}) \neq 0$ for $t \in (-\infty, \tau_0) \cup (\tau_0, \tau)$. Moreover, $\delta w(\tau, \hat{\alpha}) > 0$.*

Proof. First it clear that the signs and zeros of $u(t, \hat{\alpha})$ and $y(t, \hat{\alpha})$ agree for $r > 0$ since $u = r^{-\lambda} y$. Furthermore, by (3.57), we know that the signs and zeros of δu and δy agree. Lastly, by (3.27), we know that at $a = \hat{\alpha}$, $t = \tau$, we have

$$\delta\omega(\tau, \hat{\alpha}) = \left.\frac{\partial w(\tau, a)}{\partial a}\right|_{\substack{a=\hat{\alpha}\\ \lambda=0}} = \left.\left((p-1)|u'|^{p-2} r^{(p-1)(\lambda+1)}\, \delta u'\right)\right|_{\lambda=0}.$$

Therefore, the sign of $\delta w(\tau, \hat{\alpha})$ is the same as the sign of $\delta\omega(\tau, \hat{\alpha})$. □

Following the methodology in [7], we define the normal vector in the dual space to $T_{(y(t),w(t),r(t))_a} \widetilde{W_p}$ by

$$(\delta y^*(t, a), \delta w^*(t, a), \delta r^*(t, a)) = (\dot{y}, \dot{w}, \dot{r}(t)) \times (\delta y, \delta w, \delta r)\big|_{(t,a)}. \tag{3.58}$$

Unless the normal vector is being evaluated at a specific time, we will often omit the dependence on time t and write the third component as δr^* to mean $\delta r^*(t)$. When discussing δr^* for a specific trajectory, we will also suppress the dependence on the initial condition a.

Lemma 3.4.3. *On the trajectory $(y(t), w(t), r(t))_{\hat{\alpha}}$ of Lemma 3.4.2, the third component of (3.58),*

$$\delta r^*(\tau) = \dot{y}(\tau)\, \delta w(\tau) - \dot{w}(\tau)\, \delta y(\tau),$$

is positive in the underrotation case and negative in the overrotation case.

Proof. This lemma is an immediate consequence of Lemma 3.4.2. For both cases, we know $\delta y(\tau, \hat{\alpha}) = 0$, $w < 0$, and as $y(t)$ is decreasing from $y > 0$ to $y < 0$, $\dot{y} < 0$. In the underrotation case $\delta w(\tau, \hat{\alpha}) < 0$ while for the overrotation case, $\delta w(\tau, \hat{\alpha}) > 0$. $\square$

By Lemma 1.2.6, the normal vector $(\delta y^*, \delta w^*, \delta r^*)$ has derivative

$$(\delta r^*)^{\cdot} = (p\lambda - n + p)\, \delta r^* + r^{p+\lambda(p-1)} \left((p + \lambda(p-1))f(u) - \lambda u f'(u)\right) \delta y. \tag{3.59}$$

Lemma 3.4.4. *As $t \to -\infty$, the quantity $\delta r^* \cdot e^{(n-p-p\lambda)t} \to 0$*

The proof of Lemma 3.4.4 requires a careful examination of u, y, and λ, so we include the calculation of the limit below in full detail.

Proof. Employing the expression $y = r^\lambda u$ to write $\dot{y} = \lambda r^\lambda u + r^{\lambda+1} u'$, and recalling that

$r = e^t$ and $\delta y = r^\lambda\,\delta u$, we obtain

$$\begin{aligned}
\lim_{t\to-\infty} e^{(n-p-p\lambda)t}\,\delta r^* &= \frac{\dot y\,\delta w - \dot w\,\delta y}{r^{-(n-p-p\lambda)}} \\
&= \frac{\left(\lambda r^\lambda u + r^{\lambda+1}u'\right)\,\delta w}{r^{-(n-p-p\lambda)}} - \frac{\left[((p-1)\lambda - n + p)w - r^{p+\lambda(p-1)}f(u)\right]\,\delta y}{r^{-(n-p-p\lambda)}} \\
&= \frac{\left(\lambda r^\lambda u + r^{\lambda+1}u'\right)\,\delta w}{r^{-(n-p-p\lambda)}} \\
&\quad - \frac{\left[((p-1)\lambda - n + p)u'|u'|^{p-2}r^{(p-1)(\lambda+1)} - r^{p+\lambda(p-1)}f(u)\right]\,r^\lambda\,\delta u}{r^{-(n-p-p\lambda)}} \\
&= \frac{\left(\lambda r^\lambda u + r^{\lambda+1}u'\right)\,\delta w}{r^{-(n-p-p\lambda)}} \\
&\quad - \frac{\left[((p-1)\lambda - n + p)u'|u'|^{p-2}r^{p+p\lambda-1} - r^{p+p\lambda)}f(u)\right]\,\delta u}{r^{-(n-p-p\lambda)}}.
\end{aligned}$$

Hence

$$\begin{aligned}
\lim_{t\to-\infty} e^{(n-p-p\lambda)t}\,\delta r^* &= (\lambda r^{\lambda+n-p-p\lambda}u + r^{\lambda+n-p-p\lambda}ru')\,\delta w \\
&\quad - \left[((p-1)\lambda - n + p)u'|u'|^{p-2}r^{n-1} - r^n f(u)\right]\,\delta u.
\end{aligned}$$

For the second term, $u'|u'|^{p-2}, r^{n-1}, r^n \to 0$, $f(u) \to f(a)$, and $\delta u \to 1$. Hence

$$\left[((p-1)\lambda - n + p)u'|u'|^{p-2}r^{n-1} - r^n f(u)\right]\,\delta u \to 0$$

independently of the chosen parameter $\lambda \in \mathbb{R}$. For the first term, if $0 < \lambda \le \frac{-p}{p-q_1}$, we may conclude that $\lambda + n - p - p\lambda > 0$ and $\delta w \to 0$, forcing the first term to vanish. Notice

$$\begin{aligned}
(\lambda r^{\lambda+n-p-p\lambda}u + r^{\lambda+n-p-p\lambda}ru')\,\delta w &= \lambda r^{\lambda+n-p-p\lambda}(u + ru')(p-1)|u'|^{p-2}r^{p\lambda+p-\lambda-1}\,\delta u' \\
&= (p-1)\left(r^{n-1}u|u'|^{p-2} + r^n|u'|^{p-2}u'\right)\,\delta u',
\end{aligned}$$

which is *independent* of λ. Thus we conclude the first term must vanish independently of λ.

Therefore, for any $\lambda \in \mathbb{R}$,

$$e^{(n-p-p\lambda)t}\delta r^* \to 0 \qquad \text{as } t \to -\infty.$$

$\square$

Referring back to the linear differential equation for δr^* in (3.59), let

$$I(u,\lambda) = (p + \lambda(p-1))f(u) - \lambda u f'(u). \tag{3.60}$$

Then along a given trajectory $(y(t,a), w(t,a), r(t))$, we can compute δr^* at any time t for that trajectory with the following integral,

$$\delta r^*(t,a) = e^{(p\lambda-n+p)t}\int_{-\infty}^{t} e^{-(p\lambda-n+p)s} r(s)^{p+\lambda(p-1)} I(u(s,a),\lambda)\,\delta y(s,a)\,ds.$$

As $r(t) = e^t$ and $y = r^\lambda u$, this can be rewritten as

$$\delta r^* = r(t)^{p\lambda-n+p}\int_{-\infty}^{t} r(s)^n I(u(s,a),\lambda)\,\delta u(s,a)\,ds. \tag{3.61}$$

Once a particular trajectory has been identified, we will frequently suppress the dependence on the initial condition $u(0) = a$ and write $u\,\delta u$ to mean $u(s,a)\,\delta u(s,a)$. As we have converted from y back to u, notice the Emden–Fowler transformation only manifests itself in the λ-dependence of $I(u,\lambda)$.

Let us now define a second 2-form that does not appear in [7]:

$$W_p(y(a), w(a))(t) := \begin{vmatrix} y(t,a) & (p-1)w(t,a) \\ \delta y(t,a) & \delta w(t,a) \end{vmatrix} = y\,\delta w - (p-1)w\,\delta y\big|_{(t,a)}. \tag{3.62}$$

This expression W_p will also appear in the Morse index calculations in Chapter 4. For our purposes here, we select $\lambda = \hat{\lambda}$. Omitting in the notation below the dependence on time and the initial condition $u(0) = a$, we obtain

$$\dot{W}_p = \left(p\hat{\lambda} - n + p\right) W_p + r^{p+\hat{\lambda}(p-1)} \left((p-1)f(u) - uf'(u)\right) \delta y, \tag{3.63}$$

which is a linear differential equation. We compute

$$\begin{aligned}
\lim_{t\to-\infty} W_p(t)e^{(n-p-p\hat{\lambda})t} &= \lim_{t\to-\infty} r^{(n-p-p\hat{\lambda})t} \left(y\,\delta w - (p-1)w\,\delta y\right) &(3.64)\\
&= \lim_{t\to-\infty} r^{(n-p-p\hat{\lambda})t} \left(r^{\hat{\lambda}}u\,\delta w - (p-1)wr^{\hat{\lambda}}\,\delta u\right) &(3.65)\\
&= \lim_{t\to-\infty} \left[r^{\hat{\lambda}+n-p-p\hat{\lambda}}u\,\delta w - (p-1)wr^{\hat{\lambda}+n-p-p\hat{\lambda}}\,\delta u\right]. &(3.66)
\end{aligned}$$

The quantity $\hat{\lambda} + n - p - p\hat{\lambda}$ is positive; thus $r^{\hat{\lambda}+n-p-p\hat{\lambda}} \to 0$. Moreover, $w \to 0$, $u \to a$, $\delta w \to 0$, and $\delta u \to 1$, and we conclude

$$\lim_{t\to-\infty} W_p \cdot e^{(n-p-p\hat{\lambda})t} = 0. \tag{3.67}$$

Hence we may use an integrating factor to solve (3.63), and obtain

$$W_p(t) = r(t)^{-(n-p-p\hat{\lambda})} \int_{-\infty}^{t} r(s)^n \left[(p-1)f(u) - uf'(u)\right] \delta u(s)\,ds.$$

In terms of $K(u)$ from hypothesis (F2), we may write

$$W_p(t) = r(t)^{-(n-p-p\hat{\lambda})} \int_{-\infty}^{t} r(s)^n f(u) \left[(p-1) - K(u)\right] \delta u(s)\,ds. \tag{3.68}$$

Remark 3.4.5. Lastly, we note that for either the underrotation or overrotation case for the trajectory identified in Lemma 3.4.2, then by the definition of $W_p(\tau) = 0$ given in (3.62) we know $W_p(\tau) = 0$, as the vectors $(y, (p-1)w)$ and $(\delta y, \delta w)$ are parallel at $t = \tau$.

To show uniqueness in general, there are two cases to consider: underrotation and overrotation.

3.4.1 Eliminate Underrotation

To eliminate the possibility that underrotation of the invariant manifold leads to nonuniqueness, we set $\lambda = \hat{\lambda}$ in the expressions for δr^*. For the solution identified in Lemmas 3.4.1 and 3.4.2, we have established that $t = \tau$ is the time when $u(t) = 0$ with $\delta u = 0$ and $\delta v < 0$ (by the assumption of underrotation). Then

$$\delta r^*(\tau) = r(\tau)^{p\hat{\lambda}-n+p} \int_{-\infty}^{\tau} r(s)^n \frac{p}{q_1 - p} f(u) \left((q_1 - 1) - K(u) \right) \delta u \, ds. \tag{3.69}$$

By the remarks on $K(u)$ (see (3.4)), we know that $(q_1 - 1) - K(u)$ is positive for $u < \theta$ and negative for $u > \theta$, hence

$$f(u) \left((q_1 - 1) - K(u) \right) \leq 0, \qquad u > 0.$$

(This statement holds for f nonnegative by replacing θ with 0). In underrotation, $\delta u > 0$ for all $t < \tau$; thus the integrand of (3.69) is nonpositive. Hence $\delta r^*(\tau)$ is nonpositive. However, $\delta r^*(\tau) \leq 0$ contradicts Lemma 3.4.3. Therefore, the underrotation case is impossible.

3.4.2 The overrotation cases

Suppose that for the solution trajectory identified by lemmas 3.4.1 and 3.4.2 underrotation does not occur; by Lemma 3.4.2, this hypothesis implies that $\delta u = 0$ exactly once (at $t = \tau_0$) before $u = 0$. Let $u(\tau_0) = u_0 > 0$. Recall that θ from condition (F1) satisfies $f(u) < 0$ for $u < \theta$ and $f(u) > 0$ for $u > \theta$ if f is sign-changing. The possibility that f is nonnegative is satisfied by setting $\theta = 0$.

If $u_0 = \theta$. Let us first consider a very specific case: suppose that the value u_0 described above corresponds exactly to θ, at which point $f(u) = 0$. Then we set $\lambda = 0$ and find

$$I(u, 0) = pf(u).$$

Equation (3.61) becomes

$$\delta r^*(\tau) = r(\tau)^{p-n} \int_{-\infty}^{\tau} r(s)^n p f(u) \delta u \, ds > 0,$$

as $f(u), \delta u > 0$ for $u > u_0 = \theta$ and $f(u), \delta u < 0$ for $u < u_0 = \theta$. However, this contradicts Lemma 3.4.3. Hence we cannot have nonuniqueness if $u_0 = \theta$.

Therefore, for the remainder of the proof, we will address overrotation with the basic assumption that $u_0 \neq \theta$. Imagine that λ_0 can be chosen to force $I(u, \lambda_0)$ to be zero at some time, in particular at $t = \tau_0$ when $u = u_0$. Then we can explicitly compute what λ_0 must be and obtain

$$\lambda_0 = \frac{p f(u_0)}{u_0 f'(u_0) - (p-1) f(u_0)}. \tag{3.70}$$

We recognize the denominator from (AP3); unlike [2], we will not require it to be nonzero or have a particular sign. Assuming for the moment that this λ_0 is defined and satisfies the upper bound $\hat{\lambda}$, then $I(u, \lambda_0)$ simplifies dramatically to the following expression:

$$I(u) := I(u, \lambda_0) = \lambda_0 \left(K(u_0) - K(u) \right) f(u). \tag{3.71}$$

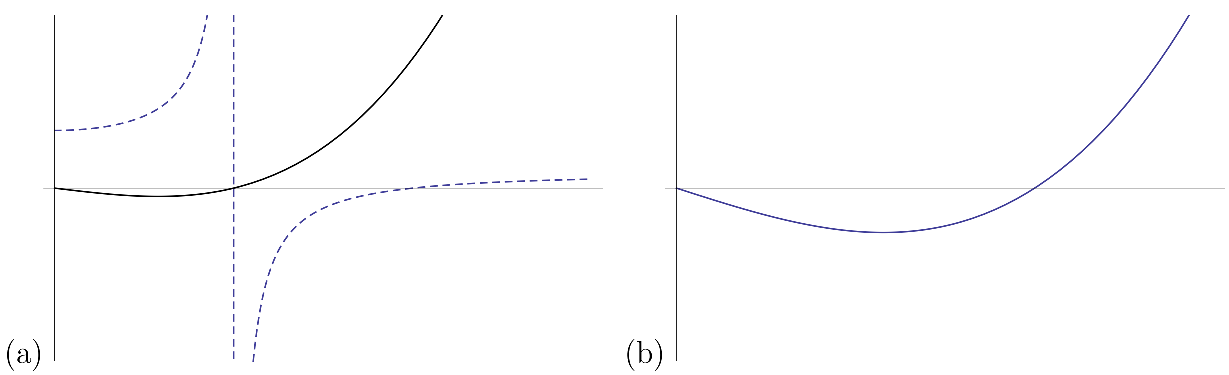


Figure 3.10: This figure assumes $u_0 > \theta$, with $f(u) = u^3 - u$. (a) is a plot of $K(u_0) - K(u)$ (dashed, with its vertical asymptote at $u = \theta$ included), and $f(u)$, solid. The product of $(K(u_0) - K(u))f(u)$ is in (b). Notice this product only changes signs at $u = u_0$.

Suppose f changes signs at $u = u_0$. At first glance, $I(u)$ appears to change signs twice: at

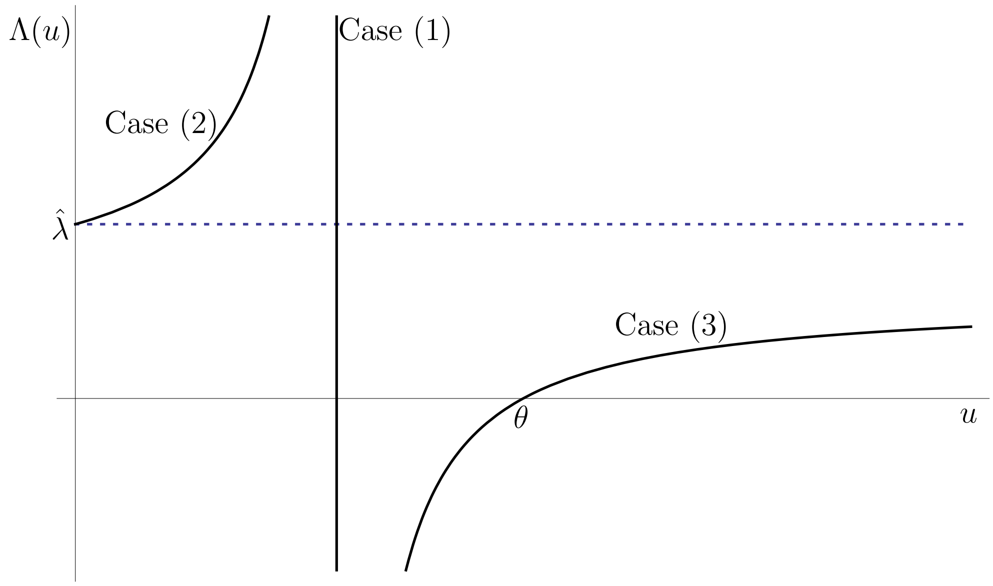


Figure 3.11: The graph of $\Lambda(u)$ from (3.72), with $p = 1.6$, $n = 3$ and $f(u) = u^3 - u^2$. This curve graphs the appropriate choice of λ_0 given u_0. The vertical asymptote occurs when $uf'(u) - (p-1)f(u) = 0$, while the horizontal asymptote is the line $\Lambda = \hat{\lambda}$. The x-intercept occurs at $u_0 = \theta$.

$u = u_0$ and $u = \theta$. If $u_0 > \theta$, however, by (F2) the expression $(K(u_0) - K(u))\, f(u)$ is negative for $u < u_0$ and positive for $u > u_0$; see Figure 3.10. If $u_0 < \theta$, then $(K(u_0) - K(u))\, f(u)$ is positive for $u < u_0$ and negative for $u > u_0$. Hence it changes signs exactly once, at u_0.

If f is nonnegative, then $I(u)$ is either zero, or $(K(u_0) - K(u))\, f(u)$ satisfies the same statement as $u_0 > \theta$ above. We will prove this case as part of Case (3), below.

To determine whether or not λ_0 is a valid choice for λ, we must understand the quantity $u_0 f'(u_0) - (p-1)f(u_0)$ in the expression for λ_0. In particular, as u_0 is not a quantity that is easy to determine, let

$$\Lambda(u) = \frac{pf(u)}{uf'(u) - (p-1)f(u)}, \tag{3.72}$$

so that $\Lambda(u_0) = \lambda_0$. This curve is illustrated in Figure 3.11. There are three cases to consider:

1. $u_0 f'(u_0) - (p-1)f(u_0) = 0$. Notice this case is equivalent to $K(u_0) = p - 1$. Since $p < q_1$, then by the remarks on $K(u)$, we determine that $u_0 < \theta$. We consider this scenario in Case (1) below.

2. $u_0 f'(u_0) - (p-1)f(u_0) < 0$. We will prove overrotation cannot occur in Case (2) below.

3. $u_0 f'(u_0) - (p-1)f(u_0) > 0$. We will consider this situation in Case (3) below. This case will also cover f nonnegative.

These three possibilities are shown in Figure 3.11. In Figure 3.11, the region to the right, which corresponds to case (3), shows that the appropriate value of λ_0 is strictly less than the upper bound $\hat{\lambda}$. This fact is the subject of the next lemma.

Lemma 3.4.6. *If $u_0 > \theta$, then $\lambda_0 < \hat{\lambda}$.*

Proof. We first note that $\lambda_0 \neq 0$ if $u \neq \theta$. Moreover, if $\lambda < 0$, then it satisfies $\lambda < \hat{\lambda}$, so assume $\lambda > 0$. Notice

$$\frac{1}{\lambda_0} = \frac{u_0 f'(u_0)}{p f(u_0)} - \frac{p-1}{p} = \frac{K(u_0) - (p-1)}{p}.$$

If $u_0 > \theta$, then we know that $K(u_0) > q_1 - 1$. Hence

$$\frac{1}{\lambda_0} > \frac{q_1 - 1 - (p-1)}{p}, \qquad \Rightarrow \qquad \lambda < \frac{-p}{p - q_1} = \hat{\lambda}.$$

$\square$

3.4.3 Proof for Case (1) (the asymptote case)

In the asymptote case, we consider $u_0 f'(u_0) - (p-1)f(u_0) = 0$, or equivalently, $K(u_0) = p - 1$. As u_0 must be smaller than θ, for all $u \in (0, u_0)$, $f(u) < 0$. Moreover, by (F2), for all $t \in (\tau_0, \tau)$, $K(u) \geq K(u_0)$. Hence for all $t \in (\tau_0, \tau)$,

$$\frac{u f'(u)}{f(u)} > p - 1 \qquad \Longrightarrow \qquad (p-1)f(u) - u f'(u) > 0. \tag{3.73}$$

Recall

$$W_p(t) = r(t)^{-(n-p-p\hat{\lambda})} \int_{-\infty}^{t} r(s)^n f(u(s,a)) \left[(p-1) - K(u(s,a))\right] \delta u(s)\, ds.$$

For this particular region, we write

$$\begin{aligned} W_p(\tau) &= r(\tau)^{-(n-p-p\hat{\lambda})} \left(\int_{-\infty}^{\tau_0} r(s)^n f(u) \left[(p-1) - K(u) \right] \delta u \, ds \right. \\ & \quad \left. + \int_{\tau_0}^{\tau} r(s)^n f(u) \left[(p-1) - K(u) \right] \delta u \, ds \right) \\ &= \left(\frac{r(\tau_0)}{r(\tau)} \right)^{(n-p-p\hat{\lambda})} W_p(\tau_0) + r(\tau)^{-(n-p-p\hat{\lambda})} \int_{\tau_0}^{\tau} r^n f(u) \left[(p-1) - K(u) \right] \delta u \, ds. \end{aligned}$$

The last integral is negative, as $\delta u < 0$ for $t \in (\tau_0, \tau)$ and (3.73) must hold. We can compute $W_p(\tau_0)$ directly:

$$W_p(\tau_0) = y(\tau_0, \hat{\alpha}) \, \delta w(\tau_0, \hat{\alpha}) - (p-1) w(\tau_0, \hat{\alpha}) \, \delta y(\tau_0, \hat{\alpha}) = y(\tau_0, \hat{\alpha}) \, \delta w(\tau_0, \hat{\alpha}),$$

which must be negative as $y > 0$ and $\delta w < 0$. We conclude $W_p(\tau) < 0$. Yet at τ, we know $y(\tau, \hat{\alpha}) = \delta y(\tau, \hat{\alpha}) = 0$, and therefore

$$W_p(\tau) = y(\tau, \hat{\alpha}) \, \delta w(\tau, \hat{\alpha}) - (p-1) w(\tau, \hat{\alpha}) \, \delta y(\tau, \hat{\alpha}) = 0. \tag{3.74}$$

Thus Case (1) is impossible.

3.4.4 Proof for Case (2)

To show that the invariant manifold cannot overrotate to cause nonuniqueness in the case $u_0 f'(u_0) - (p-1) f(u_0) < 0$, we again use $W_p(y, w)(t)$ with $\lambda = \hat{\lambda}$ to write

$$W_p(\tau) = \left(\frac{r(\tau_0)}{r(\tau)} \right)^{(n-p-p\hat{\lambda})} W_p(\tau_0) + r(\tau)^{-(n-p-p\hat{\lambda})} \int_{\tau_0}^{\tau} r^n f(u) \left[(p-1) - K(u) \right] \delta u \, ds.$$

As before, $W_p(\tau_0) < 0$. Suppose $u_0 > \theta$, which implies $f(u_0) > \theta$. Then

$$u_0 f'(u_0) - (p-1) f(u_0) < 0 \qquad \Rightarrow \qquad K(u_0) < p - 1 < q_1 - 1, \tag{3.75}$$

which is a contradiction for $u > \theta$; thus $u < \theta$. As u is monotone decreasing and K is nonincreasing (or nondecreasing as $u \searrow$), we conclude that the expression

$$(p-1)f(u) - uf'(u) = f(u)\left[(p-1) - K(u)\right] \tag{3.76}$$

is positive for all $t \in (\tau_0, \tau)$. As in the Case (1) region, we know that for $t \in (\tau_0, \tau)$, the component $\delta u < 0$. Hence the last integrand expression is negative. We know that for $t \in (\tau_0, \tau)$, $\delta u < 0$; this fact together with (3.73) allows us to conclude that $W_p(\tau) < 0$. However, this is a contradiction, as $W_p(\tau)$ must be zero.

3.4.5 Proof for Case (3)

f sign-changing at $u = \theta$. We conclude with the case $u_0 f'(u_0) - (p-1)f(u_0) > 0$. In this case, we cannot arrive at any contradictions in the style of (3.75). Therefore, we must consider two possibilities: $u_0 > \theta$ and $u_0 < \theta$.

If $u_0 > \theta$, then by Lemma 3.4.6, λ_0 is well-defined and satisfies $\lambda < \hat{\lambda}$. Moreover, we see that $f(u_0) > 0$ in (3.70) implies $\lambda_0 > 0$. Hence $I(u)$ in (3.71) changes sign once from positive to negative as u decreases through u_0. As a result, the expression for δr^*,

$$\delta r^*(\tau) = r(\tau)^{p\lambda - n + p} \int_{-\infty}^{\tau} r(s)^n I(u(s,a), \lambda)\, \delta u(s,a)\, ds,$$

is positive. But this contradicts Lemma 3.4.3.

Now suppose $u_0 < \theta$. We choose λ_0 to be whatever value of λ forces $I(u, \lambda)$ to change signs at u_0. Since $u_0 < \theta$, then as $f(u_0) < 0$, then we conclude $\lambda_0 < 0 < \hat{\lambda}$. Recall the integral definition of δr^* from (3.61):

$$\delta r^*(\tau, a) = r(\tau)^{p\lambda - n + p} \int_{-\infty}^{\tau} r(s)^n I(u(s,a), \lambda)\, \delta u(s,a)\, ds.$$

Although the integrand contains u and δu, which are independent of λ, we take extra care here to calculate δr^* because we passed through the Emden–Fowler transformation to arrive at (3.61). This precaution reconciles the two different ways to calculate δy: the first way is simply $\delta y = r^{\lambda}\,\delta u$, which if $r > 0$ exists for all λ. However, $\delta y(t)$ is also defined by the intersection curve on $\widetilde{W_p}$, which does not necessarily exist for $\lambda < 0$ in the limit $r \to 0$, as described in Section 3.2.1.

We remark first that by Lemma 3.4.4, the integral definition above does solve (3.59) for any $\lambda \in \mathbb{R}$. Let $\varepsilon > 0$ be sufficiently small so that the first zero of $\delta u = r^{-\lambda}\,\delta y$ occurs at $t = \tau_0 > \ln(\varepsilon)$. Then the manifold $\widetilde{W_p} = T[W_p^{u,c} \cap \{\varepsilon \le r \le R\}]$ described in Section 3.2.1 captures the sign-switching behavior at τ_0. However, for any $\varepsilon_0 \in (0, \varepsilon)$, we can just as easily construct $\widetilde{W_p'} = T[W_p^{u,c} \cap \{\varepsilon_0 \le r \le R\}]$ so that $\widetilde{W_p} \subset \widetilde{W_p'}$. As $\delta u > 0$ for all $t < \tau_0$,

$$\int_{\varepsilon_0}^{\varepsilon} r(s)^n I(u(s,a),\lambda)\,\delta u(s,a)\,ds > 0,$$

as $I(u(t,a))$ must be positive for all $t \in [\ln(\varepsilon_0), \ln(\varepsilon)]$. Therefore,

$$\delta r^*(\tau) > r(\tau)^{p\lambda - n + p} \int_{\varepsilon}^{\tau} r(s)^n I(u(s,a),\lambda)\,\delta u(s,a)\,ds.$$

However, as $I(u,\lambda)$ and δu both change signs from positive to negative at τ_0, we conclude that the integrand must always be nonnegative. Hence

$$\delta r^*(\tau) > 0.$$

To verify that this conclusion contradicts Lemma 3.4.3, we must make sure that Lemma 3.4.3 is still true for $\lambda < 0$. If we convert δr^* to an expression containing u, ω, δu, and $\delta\omega$ rather than y and w, we obtain

$$\delta r^*(\tau) = r^{p\lambda}\,(\lambda u + \dot{u})\,\delta\omega - (\lambda(p-1)\omega + \dot{\omega})\,\delta u\Big|_{t=\tau} = r^{p\lambda}\,(\dot{u}\,\delta\omega)\,|_{t=\tau} < 0.$$

Thus Case (3) is impossible for sign-changing f.

f **nonnegative.** We first show that f nonnegative must fall under Case (3). Notice

$$u_0 f'(u_0) - (p-1)f(u_0) \leq 0, \qquad \Rightarrow \qquad K(u_0) \leq p - 1 < q_1 - 1.$$

By the remarks on $K(u)$, this statement implies that $u_0 < \theta$, which is a contradiction, as $\theta = 0$ for f nonnegative.

If $f(u) \geq 0$ for all $u > 0$, then λ_0 in (3.70) must be positive, and by Lemma 3.4.6, we know that λ_0 is a valid choice for λ. It is possible that $I(u) \equiv 0$; this would be the case if $f(u) = u^q$. In this scenario, we conclude by (3.61) that $\delta r^*(\tau) \equiv 0$; this contradicts Lemma 3.4.3.

If $I(u)$ is not identically zero, then the nonincreasing behavior of $K(u)$ implies

$$(K(u_0) - K(u)) f(u)\, \delta u \geq 0.$$

Hence the sign of

$$\delta r^*(\tau) = r(\tau)^{p\lambda - n + p} \int_{-\infty}^{\tau} r(s)^n I(u(s,a), \lambda)\, \delta u(s,a)\, ds,$$

is determined by λ_0. As $\lambda_0 > 0$, we conclude $\delta r^*(\tau) > 0$. However, this contradicts Lemma 3.4.3. Thus Case (3) is impossible.

3.5 Summary

Over the past several pages, we have performed the following:

1. Converted $\Delta_p u + f(u) = 0$ with radial symmetry and appropriate boundary conditions into an ODE.

2. Established existence of solutions by using the Emden–Fowler transformation and setting $\lambda = \hat{\lambda}$.

3. Defined the winding number for the tangent vector field $(\delta u, \delta v, 0)$, and used this to establish the number of zeros for underrotation and overrotation for a trajectory that intersects $\{u = 0\}$ tangentially at $t = \tau$.

4. Computed two vector components, the third component of the normal vector and the third component of $(y, w, r) \times (\delta y, \delta w, \delta r)$, in two different ways.

5. Showed how a careful selection of λ reveals a contradiction between the definitions of δr^* and W_p.

Hence we conclude that it is impossible for a positive solution $u(t, a)$ to intersect $\{u = 0\}$ tangentially. As this type of intersection must occur to violate uniqueness, we have proven Theorem 3.0.2.

CHAPTER 4: MORSE INDICES OF SIGN-CHANGING SOLUTIONS

In this chapter we relate two different properties for a radial solution u of the semilinear elliptic equation

$$\begin{cases} \Delta u + f(u) & = 0 \text{ on } B_R(0) \\ u & = 0 \text{ on } \partial B_R(0). \end{cases} \tag{4.1}$$

On one hand, one naturally associates (4.1) with an Euler functional and finds that solutions of (4.1) are critical points of this functional. In Section 4.2, we use Morse theory to define the Morse index of the functional at a solution u to measure the directions of decrease of the functional. We restrict our attention to radial functions and compute the Morse index of the functional on the subspace of radial functions. On the other hand, we associate u with a tangent vector field $(\delta u, \delta v)$ whose behavior is governed by the variational equations of (4.1). In particular, we may count the number of zeros of δu on $\overline{B_R(0)}$. We use Sturm-Liouville theory in Section 4.3 to prove Theorem 4.3.4, which demonstrates how the two approaches—the Morse index of a radial solution u and the number of zeros of δu—agree.

With this proposition, we then proceed in Sections 4.4 and 4.5 to prove a series of results about the Morse indices of solutions to (4.1).

Remark 4.0.1. A note on notation. There are several integer-valued quantities referred to below. For clarity's sake, k generally refers to the number of zeros a solution u may be attaining on $[0, R]$ for some particular radius R, while μ refers to the number of zeros of δu on $[0, R]$, and M denotes the Morse index of u (restricted to the subspace of radial functions in $W_0^{1,2}(\Omega)$). Occasionally the Morse index will be the exact number of zeros of both u and δu, in which case the distinction becomes blurred.

4.1 Statement of theorems

We establish in Section 4.4 a series of results on the tangent vector component δu defined in (3.44). The results on δu are valid for both the regular Laplacian and the p-Laplacian. As these results may be useful for future work on the p-Laplacian, we leave the propositions about δu in their most general form. Therefore, the PDE we consider in Section 4.4 is

$$\begin{cases} \Delta_p u + f(u) & = 0 \text{ on } B_R(0) \\ u & = 0 \text{ on } \partial B_R(0). \end{cases} \tag{4.2}$$

where $1 < p \leq 2$ and $f(u) \in C^1$ satisfies

(H1) $f(-u) = -f(u)$, and

(H2) $u\,[uf'(u) - (p-1)f(u)]$ is nonnegative.

We recognize condition (H2) as similar to (AP3) from [2]. Unlike in Chapter 3, we require $uf'(u) - (p-1)f(u)$ and u to have the same sign. Examples include

(A) the regular Laplacian ($p = 2$) with $f(u) = |u|^{q-2}u - \nu u$, $q > 2$, $\nu \in \mathbb{R}$ and

(B) $f(u) = |u|^{q_1-2}u + \nu|u|^{q_2-2}u$, with $p \in (1, 2]$, $q_1 > 2$, $q_2 \geq 2$, and $\nu \geq 0$.

Class example (B) is new, as it does not satisfy the requirements of the uniqueness proof in Chapter 3. With these hypotheses, the main results on the tangent vector component δu are the following:

1. (Proposition 4.4.5) The number of zeros of δu along any solution u solving the Dirichlet problem on $B_R(0)$ with k zeros is greater than or equal to k.

2. (Proposition 4.4.8) For any k, there exists a solution with k zeros whose vector component δu has k or $k+1$ zeros.

3. (Proposition 4.4.9) If a solution with k zeros on $B_{R_1}(0)$ has $\mu > k$ (where μ is the number of zeros of the vector component δu), then if that solution u extends to a solution with $j > k$ zeros on $B_{R_2}(0)$, $R_2 > R_1$, it cannot possibly be a unique solution (i.e. there is another solution to the Dirichlet problem on $B_{R_2}(0)$ with j zeros).

4. (Proposition 4.4.10) If there are at least two regular solutions whose δu components have k zeros on $\overline{B_R(0)}$, then there must be a solution whose δu component has more than k zeros on $\overline{B_R(0)}$.

To consider the Morse index of sign-changing solutions, we will restrict our attention to the regular Laplacian; i.e. we will set $p = 2$ in (4.2). The main results of Section 4.5-4.6 are the following:

1. (Theorem 4.5.1) The Morse index of any solution u solving the Dirichlet problem on $B_R(0)$ with k zeros is greater than or equal to k.

2. (Theorem 4.5.2) For any k, there exists a solution with k zeros whose Morse index is k.

3. (Theorem 4.5.3) If a solution with k zeros on $B_{R_1}(0)$ has Morse index $> k$, or if its Morse index is k and δu is an eigenfunction, then if extends to a solution with $j > k$ zeros on $B_{R_2}(0)$, $R_2 > R_1$, it cannot possibly be a unique solution (i.e. there is another solution to the Dirichlet problem on $B_{R_2}(0)$ with j zeros).

4. (Theorem 4.6.1) If there is a solution u with k zeros on $\overline{B_R(0)}$ and $M = k + \ell$, $\ell \geq 1$, then for any integer $j \in \{k, k+1, \ldots, k+\ell\}$, there is a solution u_j with k zeros on $\overline{B_R(0)}$ and $M = j$.

We remark that in each theorem, the Morse index refers to the Morse index restricted to the subspace of radial functions.

4.2 Morse index

Let $\Omega \subset \mathbb{R}^n$ be a domain, and consider the Sobolev space $W_0^{1,2}(\Omega)$ as described in Section 1.3.1. Weak solutions of

$$\Delta u + f(u) = 0, \qquad u = 0 \text{ on } \partial B \tag{4.3}$$

are critical points of the Euler functional $J_2(u)$ defined in (1.14). For the regular Laplacian, the functional J_2 yields an evolution equation whose linearized operator at a solution u of (4.3) is given by

$$L_u g = \Delta g + f'(u)g. \tag{4.4}$$

The Morse index of the operator L_u is the number of unstable (positive) eigenvalues of L_u; it could theoretically be infinite. It is well known in this case ([3]), however, that the linearized operator L_u is a compact perturbation of the Laplacian whose spectrum is real with only a finite number of unstable eigenvalues. Therefore, the critical points of J_2 (the solutions to (4.3)) have a finite Morse index. We define a bilinear form B by

$$B(v,w) = \int_v (\nabla v \cdot \nabla w - f'(u)vw)\; dx, \tag{4.5}$$

where $v, w \in W_0^{1,2}(\Omega)$. The Morse index is then the supremum of the dimensions of the subspaces on which B is negative definite.

We will consider the Morse index of L_u restricted to the subspace of *radial* functions in $W_0^{1,2}(\Omega)$. Throughout this chapter, we will refer to the Morse index of L_u restricted to the subspace of radial solutions as the Morse index of the solution u. In general, as the space of radial solutions is a subspace of $W_0^{1,2}(\Omega)$, the Morse index M that we compute is less than or equal to the Morse index of L_u defined in (4.4).

4.3 Sturm-Liouville theory

We consider the space of radial solutions to

$$\Delta u + f(u) = 0$$

on $B_R(0)$ with Dirichlet boundary conditions; thus the eigenvalue problem for the linearized operator in (4.4) can be written

$$L_u g = g'' + \frac{n-1}{r} g' + f'(u) g = \lambda g. \tag{4.6}$$

Observe that, by abuse of language, we use the same notation L_u in (4.6) as in (4.4); however, this operator is the operator in (4.4) restricted to the space of radial solutions. Thus the Morse index M that we compute is in fact the Morse index of the operator in (4.6).

By multiplying each side by r^{n-1}, we rewrite (4.6) as

$$\left(r^{n-1} g'\right)' + r^{n-1} f'(u) g = \lambda r^{n-1} g. \tag{4.7}$$

Equation (4.7) is in SL form as described in Section 1.4. The boundary conditions on the eigenfunctions g are the same as those for u, namely $g'(0) = 0$ and $g(R) = 0$. From the discussion in Section 1.4 (as well as the discussion in Section 4.2), we know that L must have a finite number of positive eigenvalues λ, and the eigenvalues form a discrete and real sequence

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots \to -\infty.$$

Moreover, on the interval $(0, R)$, the eigenfunction g_j with eigenvalue λ_j must have $j - 1$ zeros; together with the boundary condition $g_j(R) = 0$, we conclude that on $\overline{B_R(0)}$, the eigenfunction g_j has j zeros.

To calculate the amount of oscillation of the eigenfunction with the smallest positive eigenvalue, we will make use of the tangent vector component δu of the vector field $(\delta u, \delta\omega, \delta r)$ defined in (3.44). To construct an ODE for δu in the regular Laplacian setting, let $p = 2$ and consider the vector field $(\delta u, \delta v, \delta r)$, where $v = (\omega/r)|_{p=2}$. The variational equations for this vector field are

$$\begin{aligned} \delta u' &= \delta v \\ \delta v' &= -\frac{d-1}{r}\,\delta v - f'(u)\delta u \\ \delta r' &= 0. \end{aligned}$$

Here we use $' = \frac{d}{dr}$ rather than passing through $r = e^t$ to write $\cdot = \frac{d}{dt}$. Notice

$$\delta u'' = \delta v' = -\frac{n-1}{r}\,\delta v - f'(u)\delta u,$$

or equivalently

$$\delta u'' + \frac{n-1}{r}\,\delta v + f'(u)\delta u = 0.$$

Hence

$$\left(r^{n-1}\delta u'\right)' + r^{n-1}f'(u)\delta u = 0, \tag{4.8}$$

which suggests that δu solves (4.7) with eigenvalue 0. However, although $\delta u'(0) = 0$ it is not necessarily the case (and will frequently not be the case) that $\delta u(R) = 0$. Despite the differing boundary conditions, we apply below the same techniques from the Sturm Comparison Theorem to demonstrate that the oscillatory behavior of δu determines the Morse index of u.

If δu satisfies $\delta u(R) = 0$ then δu is an eigenfunction on $[0, R]$ with eigenvalue 0. In this case, the Morse index of u is immediate: if δu has μ zeros on $[0, R]$, then the Morse index of u is $\mu - 1$. As $\delta u(R)$ is not necessarily zero, however, we cannot assume that δu is an eigenfunction with eigenvalue 0. Let g_M be the eigenfunction whose eigenvalue λ_M is the smallest positive eigenvalue; then the Morse index of u on $[0, R]$ is M. The first lemma shows that δu oscillates in $[0, R]$ at least as much as g_M.

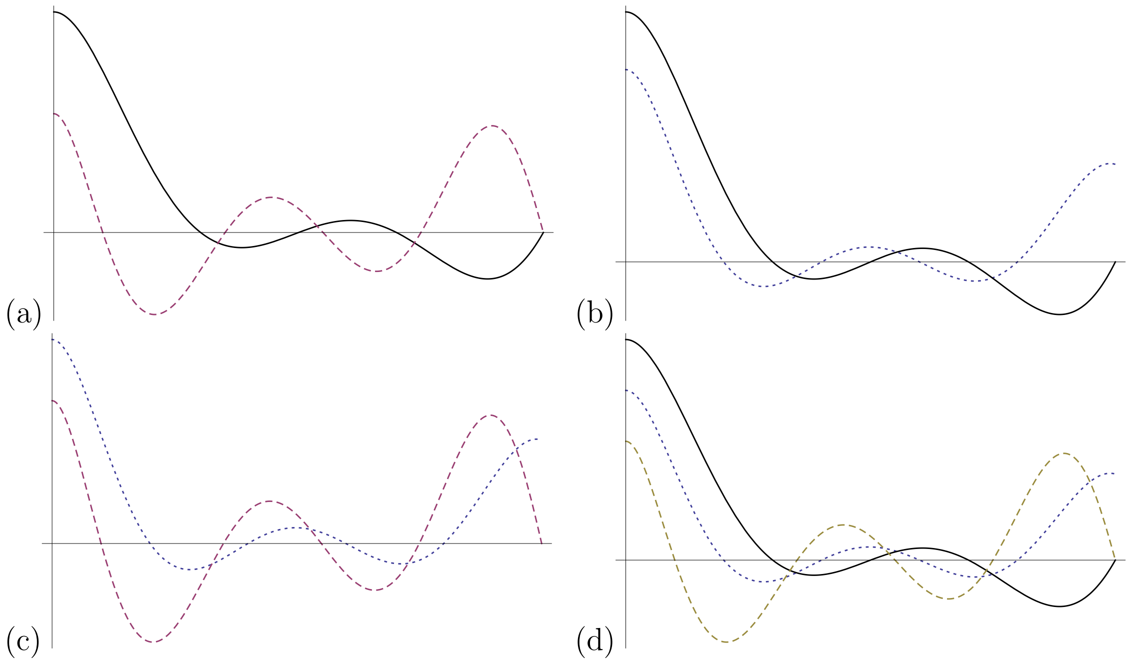


Figure 4.1: In (a)-(d), the solid black curve is the eigenfunction u_4 with the smallest positive eigenvalue (in this example, the smallest positive eigenvalue is λ_4, as there are three interior zeros on $(0, R)$). The dashed curve is the eigenfunction u_5 for λ_5, the largest negative eigenvalue, and the dotted curve is the tangent vector component δu. In this figure, δu does not satisfy $\delta u(R) = 0$; thus it is not an eigenfunction. (a) illustrates the Theorem 1.4.4, the Sturm Comparison Theorem, as well as Theorem 1.4.5. (b) compares u_4 with δu; we note that δu must have at least 4 zeros according to Lemma 4.3.1. (c) compares u_5 with δu, and by Lemma 4.3.2, we know that δu cannot have more than 5 zeros. (d) combines all of the information about δu based on u_4 and u_5; as a result, δu must have exactly 4 zeros.

Lemma 4.3.1. *The number of zeros of δu on $[0, R]$ is greater than or equal to M.*

Proof. We write the expressions (4.8) and (4.7) for δu and g_M, respectively, multiply each expression by the other function, and subtract:

$$\begin{array}{rcl} \left(r^{n-1}\delta u'\right)' g_M + r^{n-1} f'(u)\delta u \, g_M & = & 0 \\ -\left(r^{n-1}g_M'\right)' \delta u - r^{n-1} f'(u) g_M \, \delta u & = & -\lambda_M r^{n-1} g_M \, \delta u \\ \hline \left(r^{n-1}\delta u'\right)' g_M - \left(r^{n-1}g_M'\right)' \delta u & = & -\lambda_M r^{n-1} g_M \, \delta u. \end{array}$$

The left-hand side can be rewritten to attain

$$\left[r^{n-1}\left(g_M\,\delta u' - g_M'\,\delta u\right)\right]' = -\lambda_M r^{n-1} g_M\,\delta u. \tag{4.9}$$

Integrating (4.9) from 0 to $x \in [0, R]$ yields

$$x^{n-1}\left(g_M(x)\delta u'(x) - g_M'(x)\delta u(x)\right) = -\lambda_M \int_0^x r^{n-1} g_M\,\delta u\,dr, \tag{4.10}$$

where the left-hand side uses the fact that $g'(0) = \delta u'(0) = 0$.

Now suppose x_1 is the first zero in $[0, R]$ for g_M. Then $g_M(x)$ is either positive or negative for all $x < x_1$. Then (4.10) becomes

$$-x_1^{n-1} g_M'(x_1)\delta u(x_1) = -\lambda_M \int_0^{x_1} r^{n-1} g_M\,\delta u(r)\,dr,$$

We proceed with a sign argument. Notice that if $g_M(x) > 0$ for $x < x_1$, then as $g_M(x_1)$ is decreasing, $g_M'(x_1) < 0$. If $g_M(x) < 0$ for $x < x_1$, then $g_M'(x_1) > 0$. As either case results in opposite signs for g_M and g_M', we therefore assume without loss of generality that $g_M(x) > 0$ for $x < x_1$. Hence the signs are

$$(+)\delta u(x_1) = (-)\int_0^{x_1} (+)\delta u\,dr.$$

If δu did not change signs on $(0, x_1)$, there would be a contradiction. Hence δu must have a zero before the first zero of g_M. Let us denote by y_1 the first zero of δu, then $y_1 < x_1$.

Now let $x_j < x_{j+1}$ be any two values in $[0, R]$ that yield consecutive zeros of g_M. We integrate (4.9) from x_j to x_{j+1} to obtain

$$x_{j+1}^{n-1}\left(-g_M'(x_{j+1})\delta u(x_{j+1})\right) - x_j^{n-1}\left(-g_M'(x_j)\delta u(x_j)\right) = -\lambda_M \int_{x_j}^{x_{j+1}} r^{n-1} g_M\,\delta u\,dr,$$

where the left-hand side uses the fact that $g_M(x_j) = g_M(x_{j+1}) = 0$. Choosing either $g_M > 0$

or $g_M < 0$ in (x_j, x_{j+1}) forces the same sign-changing behavior in δu, so suppose without loss of generality that $g_M > 0$ in (x_j, x_{j+1}). Then $g_M'(x_j) > 0$ and $g_M'(x_{j+1}) < 0$, and the signs become

$$(+)\left((+)\delta u(x_{j+1})\right) - (+)\left((-)\delta u(x_j)\right) = (-)\int_{x_j}^{x_{j+1}} (+)\, \delta u\, dr.$$

Again δu must change signs to avoid contradiction.

Hence before the first zero of g_M there must be a zero of δu, and between any two zeros of g_M there must be a zero of δu. Therefore, δu cannot have fewer zeros that g_M. $\square$

The next lemma has as analogous proof as the previous one, so we omit it.

Lemma 4.3.2. *The number of zeros of δu on $[0, R]$ is less than or equal to $M + 1$.*

If δu is an eigenfunction, then it would have exactly $M + 1$ zeros on $[0, R]$. To prove Lemma 4.3.2 if δu is not an eigenfunction, it suffices to compare δu with g_{M+1}, where g_{M+1} is the eigenfunction with the largest negative eigenvalue. Similar computation to the preceding proof shows that g_{M+1} must have a zero before the first zero of δu and between any consecutive zeros of δu.

Lemma 4.3.3. *If δu is not an eigenfunction, then the number of zeros of δu on $[0, R]$ is equal to M, the Morse index of u.*

Proof. We compare δu to g_{M+1}, which has $M + 1$ zeros denoted $\{x_1, x_2, \ldots, x_{M+1}\}$; k of its zeros occur in $[0, R)$ and the $(M + 1)$'th occurs when $r = R$. Suppose δu had $M + 1$ zeros, $\{y_1, y_2, \ldots, y_{M+1}\}$. Then by the previous lemmas, we can compare the sets and conclude that $x_i \leq y_i$ for each $i = \{1, \ldots, M + 1\}$. However, $x_{j+1} = R$, and as δu is not an eigenfunction, then $\delta u(R) \neq 0$. Therefore, y_{M+1} must be smaller than R, which is a contradiction. $\square$

The results of these lemmas can now be summarized by the following theorem relating μ, the number of zeros of δu, to the Morse index of u on $\overline{B_R(0)}$.

Theorem 4.3.4. *Let u be a radial solution to (4.1), and let μ be the number of zeros of δu on $\overline{B_R(0)}$. If $\delta u(R) = 0$, then the Morse index of u on $\overline{B_R(0)}$ is $\mu - 1$. If $\delta u(R) \neq 0$, then the Morse index of u on $\overline{B_R(0)}$ is μ.*

If $\delta u(R) = 0$, then δu is an eigenfunction with eigenvalue 0, proving the first statement of Theorem 4.3.4. If $\delta u(R) \neq 0$, then Lemma 4.3.3 proves the second statement.

4.4 Proofs of results on δu, $1 < p \leq 2$

In this section, we focus on the behavior of the tangent vector component δu. As these results are true for radial solutions for $\Delta_p u$ for $1 < p \leq 2$, and not solely the case $p = 2$, we prove each statement in the more general p-Laplacian setting.

We recall the center-unstable manifold $W_p^{u,c}$ defined in (3.17) with intersection curve $C(\tau)$ defined in (3.42). The *winding number* of δu has the same meaning as in Section 3.3.2. In particular, we recall the following principle, a consequence of Lemma 3.3.1:

The algebraic winding number of δu on $[0, R]$ is equal to μ.

In the subsequent section, Section 4.5, we will connect the results on δu for the regular Laplacian ($p = 2$) to results on the Morse indices of sign-changing solutions.

Let us first recall the variational equations (3.48)-(3.50) for the tangent vector $(\delta u, \delta\omega, 0)$ from Section 3.3.2:

$$
\begin{aligned}
\dot{\delta u} &= \frac{1}{p-1}|\omega|^{\frac{2-p}{p-1}}\,\delta\omega, \\
\dot{\delta\omega} &= (p-n)\,\delta\omega - r^p f'(u)\,\delta u \\
\dot{\delta r} &= 0.
\end{aligned}
$$

From the construction of the winding number, the following lemma is immediate.

Lemma 4.4.1. *Suppose a solution has $u(t, a) = 0$ at some time $t = \tau$. If μ is even at $t = \tau$, then either $\delta u(\tau, a) > 0$, or $\delta u(\tau, a) = 0$ with $\delta\omega(\tau, a) > 0$.*

If μ is odd at $t = \tau$, then either $\delta u(\tau, a) < 0$, or $\delta u(\tau, a) = 0$ with $\delta\omega(\tau, a) < 0$.

Proof. As the initial condition of the tangent vector $(\delta u, \delta\omega, \delta r)$ is $(1, 0, 0)$, the winding begins in the right-half of the $(\delta u, \delta\omega)$-plane. An even net number of crossings means either a return to the right-half plane, where $\delta u > 0$, or $(\delta u, \delta\omega)$ curve ends on the positive $\delta\omega$-axis. An odd net number of crossings means the curve ends either in the left-hand plane or on the negative $\delta\omega$-axis. □

Recall the 2-form $W_p(y, w)(t)$ from Section 3.4.4. We will use the same construction for (u, ω) below.

Definition 4.4.2. The 2-form $W_p(u, \omega)(t)$ along a solution curve $u(t, a)$ is defined by the determinant

$$W_p(u,\omega)(t) = \begin{vmatrix} u & (p-1)\omega \\ \delta u & \delta\omega \end{vmatrix} = u\,\delta\omega - (p-1)\omega\,\delta u.$$

We will often suppress the dependence on (u, ω). Although $W_p(t)$ is real-valued, we notice that $W_p(t)$ can be viewed as the length of the vector $(u, \omega, 0) \times (\delta u, \delta\omega, 0)$ which is parallel to the r-axis. We use $W_p(t)$ to understand how a particular solution $u(t, a)$ must wind about the r-axis as it attains zeros. In particular, notice the rule

$$\text{“}\delta u = 0 \text{ when } u = 0 \Longrightarrow W_p = 0,\text{”} \tag{4.11}$$

as the two vectors $(u, (p-1)\omega)$ and $(\delta u, \delta\omega)$ are parallel at that moment.

Lemma 4.4.3. *The quantity $W_p(t)$ for a solution $(u(t), \omega(t), r(t))_a$ has an explicit analytical expression as*

$$W_p(t) = r^{p-n} \int_{-\infty}^{t} r^d \left((p-1)f(u) - u\frac{df}{du} \right) \delta u\, ds.$$

Proof. For any radial solution $u(t, a)$ solving (4.1) on W^u, as $t \to -\infty$, $(u, (p-1)\omega) \to (a, 0)$

and $(\delta u, \delta\omega) \to (1,0)$. Therefore, $W_p(t) \to 0$. Notice further that

$$\begin{aligned}
\dot{W}_p &= \dot{u}\,\delta\omega + u\,\dot{\delta\omega} - (p-1)\dot{\omega}\,\delta u - (p-1)\omega\,\dot{\delta u} \\
&= \omega|\omega|^{\frac{2-p}{p-1}}\,\delta\omega + u\left((p-n)\delta\omega - r^p\frac{df}{du}\delta u\right) - (p-1)\left((p-n))\omega - r^p f(u)\right)\,\delta u - \omega|\omega|^{\frac{2-p}{p-1}}\,\delta\omega \\
&= (p-n)W_p + r^p\left((p-1)f(u) - u\frac{df}{du}\right)\,\delta u.
\end{aligned}$$

Rearranging terms gives the following linear differential equation:

$$\dot{W}_p + (n-p)W_p = r^p\left((p-1)f(u) - u\frac{df}{du}\right)\,\delta u. \tag{4.12}$$

Notice

$$\lim_{t\to-\infty} W_p(t)e^{(n-p)t} = 0$$

as $W_p \to 0$ and $n-p \geq 0$. Recalling that $r = e^t$, we now solve (4.12) using an integrating factor to obtain

$$W_p(t) = r^{p-n}\int_{-\infty}^{t} r(s)^n\left((p-1)f(u(s,a)) - u(s,a)\frac{df}{du}(s,a)\right)\,\delta u(s,a)\,ds.$$

We often suppress the (s,a) when the initial condition is understood, and write

$$W_p(t) = r^{p-n}\int_{-\infty}^{t} r^n\left((p-1)f(u) - u\frac{df}{du}\right)\,\delta u\,ds.$$

□

The above lemma is true for any nonlinearity $f \in C^1$ satisfying (H2). For an explicit example, in case (A), with $p=2$ and $f(u) = |u|^{q-2}u - \nu u$, we can calculate

$$(p-1)f(u) - uf'(u) = (2-q)|u|^{q-2}u.$$

For any particular solution $u(t,a)$, let t_k refer to the time when that solution attains its kth zero.

Lemma 4.4.4. *Suppose a solution has μ_1 at t_{k_1} and μ_2 at $t_{k_2} > t_{k_1}$. Then $\mu_2 > \mu_1$.*

Proof. By Lemma 3.3.1, it is clear that $\mu_2 \geq \mu_1$, as a solution trajectory $u(t,a)$ cannot lose instances where $\delta u = 0$ as t increases .

It suffices to show that if a solution $u(t,a)$ has $\mu = N$ at its $(k-1)$th zero, then it must have $\mu > N$ at its kth zero. By Lemma 3.3.1, verifying this claim amounts to showing that δu must change signs between t_{k-1} and t_k.

Suppose without loss of generality that $k > 1$ is even. So $u(t,a) < 0$ for $t \in (t_{k-1}, t_k)$, which by (H2) implies $(p-1)f(u) - uf'(u) > 0$. To achieve a contradiction, assume that $\mu = N$ at both $t = t_{k-1}$ and $t = t_k$. Then as δu cannot signs between t_{k-1} and t_k, it must be the case that either $\delta u < 0$ or $\delta u > 0$ for all $t \in (t_{k-1}, t_k)$.

If $\delta u < 0$, then at the $(k-1)$th zero, the sign of $W_p(t_{k-1})$ is negative since

$$W_p(t_{k-1}) = \begin{vmatrix} u & (p-1)\omega \\ \delta u & \delta\omega \end{vmatrix} = -\omega\,\delta u \leq 0, \tag{4.13}$$

as ω is negative for odd zero. At the kth zero, $\omega > 0$ and so

$$W_p(t_k) \geq 0. \tag{4.14}$$

Returning to the analytical expression for $W_p(t)$, as we have assumed that $\delta u < 0$ for all $t \in (t_{k-1}, t_k)$, then by (H2) in this interval we must have

$$\left((p-1)f(u) - u\frac{df}{du}\right)\delta u < 0.$$

This fact together with (4.13) and Lemma 4.4.3 allows us to write

$$W_p(t_k) = r(t_k)^{p-d}\left[r(t_{k-1})^{d-p}W_p(t_{k-1}) + \int_{t_{k-1}}^{t_k} r^d\left((p-1)f(u) - u\frac{df}{du}\right)\delta u\,ds\right] < 0,$$

which contradicts (4.14). Instead, if $\delta u > 0$, then

$$W_p(t_{k-1}) \geq 0, \qquad \text{and} \qquad W_p(t_k) \leq 0,$$

contradicting

$$W_p(t_k) = r(t_k)^{p-n}\left[r(t_{k-1})^{n-p}W_p(t_{k-1}) + \int_{t_{k-1}}^{t_k} r^n\left((p-1)f(u) - u\frac{df}{du}\right)\delta u\, ds\right] > 0.$$

Hence δu must be zero between t_{k-1} and t_k. As the winding number counts the exact number of zeros of δu along this trajectory, $\mu(t_k)$ must be larger than N. □

The following proposition is now immediate.

Proposition 4.4.5. *For any solution with k zeros, $\mu \geq k$.*

Proof. From the proof of Theorem 3.0.2, we know that the number of zeros of δu of any solution at its first zero is 1 (as $\mu = 0$ implies underrotation). Hence, the proof follows by induction. Assume that for any solution at its $(k-1)$th zero, $\mu \geq k-1$. Then by Lemma 4.4.4, the number of zeros of δu at the kth zero of any solution satisfies

$$\mu \geq (k-1) + 1 = k.$$

□

Lemma 4.4.6. *If $\mu(u(t_k, a)) = k$, then $(-1)^k \delta u(t_k, a) > 0$.*

Proof. If the winding number of $\delta u(t_k, a)$ is k, then Lemmas 4.4.4 and 4.4.5 imply that winding number of u at t_{k-1} is $k-1$. Therefore, if $\delta u(t_k) = 0$, then δu cannot change signs for $t \in (t_{k-1}, t_k)$, as it had attained $k-1$ zeros by t_{k-1}, and it attains its kth zero at t_k. The same argument as in Lemma 4.4.4 now yields a contradiction again.

In particular, let us again assume without loss of generality that $\mu = k$ is even. Then by Lemma 4.4.1 we must rule out the possibility that $\delta u(t_k, a) = 0$ with $\delta\omega(t_k, a) > 0$.

Combining the hypothesis that $\delta u = 0$ with $\delta\omega > 0$ and the relation $\dot{\delta u} = \frac{1}{p-1}|\omega|^{\frac{2-p}{p-1}}\,\delta\omega$ allows us to conclude that $\delta u < 0$ in the interval (t_{k-1}, t_k). Therefore, $\delta u \leq 0$ at $t = t_{k-1}$, whence

$$W_p(t_{k-1}) = u\,\delta\omega - v\,\delta u = -v\,\delta u \leq 0.$$

Hence

$$W_p(t_k) = r(t_k)^{-(n-p)}\left[r(t_{k-1})^{n-p}W_p(t_{k-1}) + \int_{t_{k-1}}^{t_k} r^n\left((p-1)f(u) - u\frac{df}{du}\right)\delta u\,ds\right] < 0.$$

However, if we assume that $\delta u(t_k) = 0$, then $W_p(t_k) = 0$. Therefore, we cannot have $\delta u = 0$, and so $\delta u > 0$. In general, we must have $(-1)^k\delta u(t_k, a) > 0$. $\square$

Let $\mathcal{S}_{t_k,k}$ be the set of all solutions on W_+^0 attaining their kth zero at time t_k. Denote by $u(t, \alpha_k)$ the first such solution to have its kth zero on $C(t_k)$; in other words, define α_k by

$$\alpha_k = \min\{a > 0 \ : \ u(t, a) \in \mathcal{S}_{t_k,k}\}.$$

We can now rule out the analogous underrotation formation discussed in the proof of Lemma 3.4.1; see Figure 4.2(a) for an illustration.

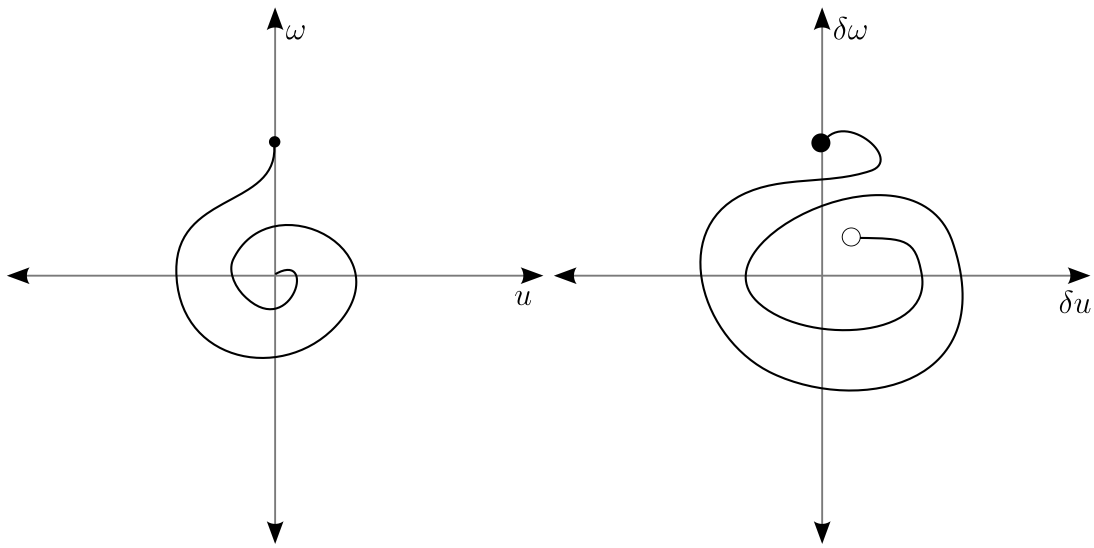
Corollary 4.4.7 (to Lemma 4.4.6)**.** *With terms defined as above, $C(t_k)$ cannot underrotate at $u(t_k, \alpha_k)$. In other words, if $\delta u(t_k, \alpha_k) = 0$, then $(-1)^k\delta\omega(t_k, \alpha_k) < 0$.*

Proof. Underrotation would imply that $\mu = k$ with $\delta u = 0$, contradicting Lemma 4.4.6, see Figure 4.2(a) with the relevant winding number computed in Figure 4.2(b). $\square$

We have now proved the following lower bound for δu:

Proposition 4.4.8. *With α_k defined as in Lemma 4.4.7, the number of zeros μ of $\delta u(t_k, \alpha_k))$ is k or $k+1$.*

This theorem results from Proposition 4.4.5, Corollary 4.4.7.



(a) $C(t_4, \alpha_4)$ (b) $S_{(t_4,\alpha_4)}$

Figure 4.2: Figure (a) shows the curve homotopic to $C(t_k, \alpha_k)$ in the plane $\{r = r(t_k)\}$ described in Corollary 4.4.7 in the case $k = 4$. The winding number for this case is computed in (b).

Proof. Either $(-1)^k \delta u > 0$, in which case $\mu = k$, or $\delta u = 0$ with $(-1)^k \delta\omega < 0$, in which case $\mu = k + 1$. $\square$

As in the proof of Lemma 4.4.6, the following assertion relies on the fact that $\dot{\delta u} = \frac{1}{p-1} |\omega|^{\frac{2-p}{p-1}} \delta\omega$.

Proposition 4.4.9. *With α_k defined as in Lemma 4.4.7, suppose $u(t, \alpha_k)$ has $\mu(u(t_k, \alpha_k)) \geq k + 1$. Then if $u(t, \alpha_k)$ attains any jth zero with $j > k$ at $t_j > t_k$, it cannot be unique. In other words, $\mathcal{S}_{t_j,j} \backslash u(t, \alpha_k) \neq \emptyset$.*

Proof. Let $d = j - k$. By Lemma 4.4.4, if $\mu(u(t_k, \alpha_k)) \geq k + 2$, then

$$\mu(u(t_j, \alpha_k)) \geq k + 2 + d = j + 2.$$

By Proposition 4.4.8, the set $\mathcal{S}_{t_j,j}$ must contain a solution with $\mu = j$ or $j + 1$.

Now suppose $\mu(u(t_k, \alpha_k)) = k + 1$. Then by Proposition 4.4.8, $\delta u(t_k, \alpha_k) = 0$ and $(-1)^k \delta\omega(t_k, \alpha_k) < 0$. Suppose without loss of generality that k is even, whence $\delta\omega(t_k, \alpha_k) < 0$.

Recall that

$$\dot{\delta u} = \frac{1}{p-1}|\omega|^{\frac{2-p}{p-1}}\,\delta\omega,$$

which constrains the winding of δu along $u(t,\alpha_k)$, as illustrated in Figure 3.6. Hence as t increases by a small enough amount, δu must be decreasing. By Lemma 4.4.4, δu must change signs between $t = t_k$ and $t = t_j$, and thus at some time t_0 between $t = t_k$ and $t = t_{k+1}$, we must have $\delta u > 0$.

Lemma 4.4.4 asserts that δu must change signs at least once along $u(t,\alpha_k)$ between each zero. Should δu along $u(t,\alpha_k)$ change signs exactly once between subsequent zeros up to and including j, then δu will point "backwards" in each case, and thus $u(t,\alpha_k)$ cannot be a unique kth, $(k+1)$th, ..., jth zero at any of these times $t_k, t_{k+1}, \ldots, t_j$.

(For illustration purposes, if δu changes signs exactly once between t_k and t_{k+1} (with k even), then $\delta u(t_{k+1},\alpha_k) > 0$. As $k+1$ is odd by our assumption on k, there must be an initial condition $\hat{\alpha_k} < \alpha_k$ with $u(t_{k+1},\hat{\alpha_k}) = 0$ for the $(k+1)$th time and $\delta u(t_{k+1},\hat{\alpha_k}) \leq 0$.)

If δu changes signs $m > 1$ times between t_ℓ and $t_{\ell+1}$, $k \leq \ell < j$, then the winding number at $t_{\ell+1}$ becomes

$$\mu \geq \ell + 1 + m > \ell + 2.$$

Thus if $d' = j - (\ell+1)$, then the winding number at t_j satisfies

$$\mu > \ell + 2 + d' = j + 1.$$

However, we again refer to Proposition 4.4.8 and assert that the set $\mathcal{S}_{t_j,j}$ must contain a solution with $\mu = j$ or $j+1$. □

Proposition 4.4.10. *If there are two (or more) nondegenerate solutions $u_1(t,\alpha_1)$ and $u_2(t,\alpha_2)$ with k zeros on $\overline{B_R(0)}$ and $\mu(t_k,\alpha_i) = k$, $i = 1,2$, then there must be at least one solution $u_0(t,\hat{\alpha})$, with $\alpha_1 < \hat{\alpha} < \alpha_2$, with k zeros on $\overline{B_R(0)}$ and $\mu(t_k,\hat{\alpha}) \geq k+1$.*

Proof. Without loss of generality, suppose k is even. Further suppose that $(u(t), v(t), r(t))_{\alpha_1}$

and $(u(t), v(t), r(t))_{\alpha_2}$, with $\alpha_1 < \alpha_2$, are nondegenerate solutions with $\mu = k$ at $t = t_k$. By Lemma 4.4.6, $\delta u(t_k, \alpha_i) > 0$. Therefore, for some $\varepsilon > 0$ small enough,

$$u(t_k, \alpha_1 + \varepsilon) > 0, \qquad \text{and} \qquad u(t_k, \alpha_2 - \varepsilon) < 0.$$

By the Intermediate Value Theorem, there is some $\hat{\alpha} \in (\alpha_1 + \varepsilon, \alpha_2 - \varepsilon)$, such that $u(t_k, \hat{\alpha}) = 0$. Moreover, we can choose $\hat{\alpha}$ so that $\delta u(t, \hat{\alpha}) \leq 0$, as Lemma 4.4.6 requires that $C(t_k)$ must cross from $\{u > 0\}$ to $\{u < 0\}$ as a increases from α_1 to α_2.

By the contrapositive of Lemma 4.4.6, as $\delta u(t_k, \hat{\alpha}) \leq 0$, then $\mu \geq k + 1$. $\square$

4.5 Proofs of theorems on the Morse index of $\Delta u + f(u) = 0$

We now apply Theorem 4.3.4 to the results of Section 4.4 to calculate the Morse index of solutions u to (4.1). The first result is an immediate application of Proposition 4.4.5.

Theorem 4.5.1. *For any solution to (4.1) with k zeros on $\overline{B_R(0)}$, the Morse index is at least k.*

For Theorem 4.5.1, we note that in general, $\mu(u(t_k, \alpha_k))$ is equal to the Morse index. The exception is if $\delta u(t_k, \alpha_k) = 0$, in which case μ is the $M + 1$, by Proposition 4.3.4. However, by Corollary 4.4.7, we cannot have $\delta u(t_k, \alpha_k) = 0$ with $\mu(u(t_k, \alpha_k)) = k$. Thus Theorem 4.5.1 holds.

The following result is an application of Proposition 4.4.8 to the case $p = 2$:

Theorem 4.5.2. *With α_k defined as in Lemma 4.4.7, the Morse index of $\mu(u(t_k, \alpha_k))$ is k.*

Proof. Either $(-1)^k \delta u > 0$, in which case the Morse index is $\mu = k$, or $\delta u = 0$ with $(-1)^k \delta\omega < 0$, in which $\mu = k + 1$. However, by Theorem 4.3.4, $\mu = k + 1$ and $\delta u = 0$ implies the Morse index is k. $\square$

The following theorem is a repeat of Proposition 4.5.3 for the case $p = 2$. Because of the relationship between δu and M when $\delta u = 0$ established by Theorem 4.3.4, it is stated slightly differently.

Theorem 4.5.3. *With α_k defined as in Lemma 4.4.7, suppose $u(t, \alpha_k)$ has $M(u(t_k, \alpha_k)) \geq k + 1$, or suppose $M = k$ and δu is an eigenfunction. Then if $u(t, \alpha_k)$ attains any jth zero with $j > k$ at $t_j > t_k$, it cannot be unique. In other words, $\mathcal{S}_{t_j,j} \backslash u(t, \alpha_k) \neq \emptyset$.*

4.6 Proof of Theorem 4.6.1

The next theorem is similar to Proposition 4.4.10, but not identical, see Figure 4.3 (A2) for an example of and intersection curve that produces two solutions with Morse index k, without necessarily producing a solution with Morse index $M \geq k + 1$. However, the next theorem is a stronger result. Its proof requires an exhaustive examination of many different cases.

Theorem 4.6.1. *Suppose $u_\ell(t, \alpha)$ has k zeros on $\overline{B_R(0)}$, for $R = e^{t_k}$. Suppose the Morse index of u_ℓ on $\overline{B_R(0)}$ is $M = k + \ell$, $\ell > 1$. Then there are solutions $u_1, u_2, \ldots, u_{\ell-1}$ so that for any $i \in \{1, \ldots, \ell - 1\}$, the solution u_i has Morse index $k + i$ at $t = t_k$.*

For example, if $k = 4$ and there is a solution with k zeros and Morse index $M = 12$, then there must be a solution with Morse index $M = 5$, another with $M = 6$, etc., up to 12. (The existence of a solution with $M = 4$ is guaranteed by Theorem 4.5.2.)

Proof. By hypothesis, we suppose there is a solution $u_\ell(t, \alpha)$ with k zeros on $\overline{B_R(0)}$ ($R = e^{t_k}$) with Morse index $k + \ell$, $\ell > 1$. By Theorem 4.5.2, we know there is a solution $u_0(t, \alpha_0)$ with $\alpha_0 < \alpha$ so that u_0 has Morse index k at $t = t_k$. We wish to find a set of solutions

$$\{u_0, u_1, \ldots, u_{k+\ell-1}, u_{k+\ell}\}$$

so that u_i has k zeros and Morse index $k + i$ at $t = t_k$. It suffices to show that if there is such a solution u_M, $0 < M < k + \ell - 1$, with Morse index M then there must be a solution u_{M+1} with Morse index $M + 1$. Let α_M be the initial condition of the solution u_M.

We assume without loss of generality that k is even and divide the proof into four main cases A, B, C, D below based on the sign of $\delta u_M(t_k, \alpha_M)$. Each case has several subcases;

we supplement the proofs of all of the cases with Figures 4.3-4.8. Each large graph in Figures 4.3-4.8 depicts a curve homotopic to $C(t_k, \widetilde{\alpha})$ for $k = 2$, while the accompanying inset figure illustrates the winding along $C(t_k, \widetilde{\alpha})$.

The curve $C(t_k, \alpha_M)$ is dashed, the solutions $u_M(t_k, \alpha_M)$ and $u(t_k, \widetilde{\alpha})$ are depicted as dots on $C(t_k, \widetilde{\alpha})$, and the portion of the intersection curve that connects them, $C(t_k)|_{\alpha_M \leq a \leq \widetilde{\alpha}}$, is solid. As the dashed portion of $C(t_k)$ is irrelevant to determining the change in Morse index from u_M to $u(t, \widetilde{\alpha})$, these figures give the general picture for any k.

Let

$$\widetilde{\alpha} = \min\{a > \alpha_M \mid u(t_k, a) = 0 \text{ and } u \text{ has } k \text{ zeros on } B_R(0)\}.$$

We consider how the curve $C(t_k, \widetilde{\alpha})$ looks between the solutions $u_M(t, \alpha_M)$ and $u(t, \widetilde{\alpha})$ to track the winding of δu. Then we can determine how the Morse index of $u(t, \widetilde{\alpha})$ compares to u_M.

The outcome for each case is that the Morse index of $u(t, \widetilde{\alpha})$ either

(O1) stays the same,

(O2) decreases,

(O3) increase by 1, which demonstrates the existence of a solution with Morse index $M + 1$, or

(O4) increases by 2.

In outcomes (O1) or (O2), we restart the process by replacing u_M and α_M with $u(t, \widetilde{\alpha})$ and $\widetilde{\alpha}$. In outcome (O4), we will show that the configuration of the intersection curve guarantees that there is a solution $u(t, \alpha_*)$ with $\alpha_* > \widetilde{\alpha}$ that must have Morse index M or $M + 1$. The process then restarts with $u(t, \alpha_*)$.

Case A

For Case A, we suppose $\delta u(t_k, \alpha_M) > 0$.

Suppose $\omega(t_k, \alpha_M) > \omega(t_k, \widetilde{\alpha})$. Then $\delta u(t_k, \widetilde{\alpha})$ is either negative (Case (A1)), zero with $\delta\omega(t_k, \widetilde{\alpha}) < 0$ (Case (A2)), or zero with $\delta\omega(t_k, \widetilde{\alpha}) > 0$ (Case (A3)). In Case (A1), the vector $(\delta u(t_k, a), \delta\omega(t_k, a))$ must rotate clockwise from the right half of the $(\delta u, \delta\omega)$ to the left half; therefore, the index of δu along $C(t_k)$ increases by 1. Hence the Morse index of $u(t_k, \widetilde{\alpha}) = M + 1$. In Cases (A2) and (A3), the index of δu along $C(t_k)$ increases by 1 and 2, respectively. As $\delta u(t_k, \widetilde{\alpha}) = 0$, then by Theorem 4.3.4 the Morse index of $u(t_k, \widetilde{\alpha})$ is M and $M + 1$, respectively.

The part of $C(t, \widetilde{\alpha})$ for $\alpha_M \leq a \leq \widetilde{\alpha}$ is path homotopic to the curves in Figure 4.3 (A1), (A2), and (A3).

Now suppose $\omega(t_k, \alpha_M) < \omega(t_k, \widetilde{\alpha})$. Then $\delta u(t_k, \widetilde{\alpha})$ is either negative (Case (A4)), zero with $\delta\omega(t_k, \widetilde{\alpha}) < 0$ (Case (A5)), or zero with $\delta\omega(t_k, \widetilde{\alpha}) > 0$ (Case (A6)). If $\omega(t_k, \alpha_M) < \omega(t_k, \widetilde{\alpha})$, then $C(t_k)$ underrotates for $\alpha_M \leq a \leq \widetilde{\alpha}$. Hence the index of δu decreases over this interval, and thus the Morse index of any solution decreases, placing us in outcome (O2). See Figure 4.3 (A4), (A5), and (A6).

Case B

For Case B, we suppose $\delta u(t_k, \alpha_M) > 0$. There are again six cases: three for $\omega(t_k, \alpha_M) > \omega(t_k, \widetilde{\alpha})$ and three for $\omega(t_k, \alpha_M) < \omega(t_k, \widetilde{\alpha})$. If $\omega(t_k, \alpha_M) > \omega(t_k, \widetilde{\alpha})$, then the curve is underrotating, and the Morse index decreases; see Figure 4.4 (B1), (B2), and (B3).

If $\omega(t_k, \alpha_M) < \omega(t_k, \widetilde{\alpha})$, then the same arguments as in Cases (A1), (A2) and (A3) shows that the Morse index of $u(t, \widetilde{\alpha})$ at t_k must be either M or $M + 1$; see Figure 4.4 (B4), (B5), and (B6).

In cases C and D, we suppose that $\delta u(t_k, \alpha_M)$ is zero with $\delta\omega > 0$ or $\delta\omega < 0$, respectively.

Case C

Suppose $\delta u(t_k, \alpha_M)$ is zero with $\delta\omega(t_k, \alpha_M) > 0$. Again we consider two possibilities: $\omega(t_k, \alpha_M) > \omega(t_k, \widetilde{\alpha})$, or $\omega(t_k, \alpha_M) < \omega(t_k, \widetilde{\alpha})$. Each of these possibilities then has six individual cases, based on whether choosing either $\delta u(t_k, \widetilde{\alpha})$ negative, zero with $\delta\omega(t_k, \widetilde{\alpha}) < 0$, or zero with $\delta\omega(t_k, \widetilde{\alpha}) > 0$, and either $C(t_k)|_{\alpha_M < a < \widetilde{\alpha}}$ on half plane $u > 0$ or on $u < 0$. Again for each case, we can track how the index of δu changes along $C(t_k)$ from $u(t_k, \alpha_M)$ to $u(t_k, \widetilde{\alpha})$. The twelve possible cases are shown in Figures 4.5 and 4.6.

In cases (C8) and (C10), the Morse index of $u(t, \widetilde{\alpha})$ at $t = t_k$ is $M + 2$, placing us in outcome (O4). In both (C8) and (C10), we have $\omega(t_k, \alpha_M) > \omega(t_k, \widetilde{\alpha})$ with $C(t_k)|_{\alpha_M < a < \widetilde{\alpha}}$ on the right half plane. Thus the curve $C(t_k)$ must intersect the ω axis for some $a > \widetilde{\alpha}$ with $\omega(t_k, a) < \omega(t_k, \widetilde{\alpha})$. Hence we define α_* by

$$\alpha_* = \min\{a > \widetilde{\alpha} \mid u(t_k, a) = 0 \text{ and } u \text{ has } k \text{ zeros on } B_R(0), \omega(t_k, a) < \omega(t_k, \widetilde{\alpha})\}. \tag{4.15}$$

In case (C8), the solutions $u(t_k, \widetilde{\alpha})$ and $u(t_k, \alpha_*)$ are in configurations (B1), (B2), or (B3). Thus the Morse index of $u(t, \alpha_*)$ at $t = t_k$ must be M or $M + 1$.

In case (C10), the solutions $u(t_k, \widetilde{\alpha})$ and $u(t_k, \alpha_*)$ are in configuration (C7), (C9), or (C11). Thus the Morse index of $u(t, \alpha_*)$ at $t = t_k$ must be M or $M + 1$.

Case D

Case D is similar to Case C. We suppose $\delta u(t_k, \alpha_M)$ is zero with $\delta\omega(t_k, \alpha_M) < 0$. There are again two possibilities for $\omega(t_k, \widetilde{\alpha})$: either $\omega(t_k, \alpha_M) > \omega(t_k, \widetilde{\alpha})$, or $\omega(t_k, \alpha_M) < \omega(t_k, \widetilde{\alpha})$, with six individual cases each. We track how the index of δu changes along $C(t_k)$ from $u(t_k, \alpha_M)$ to $u(t_k, \widetilde{\alpha})$ in Figures 4.7 and 4.8.

In cases (D1) and (D3), the Morse index of $u(t, \widetilde{\alpha})$ at $t = t_k$ is $M + 2$, placing us in outcome (O4). In both (D1) and (D3), we have $\omega(t_k, \alpha_M) < \omega(t_k, \widetilde{\alpha})$ with $C(t_k)|_{\alpha_M < a < \widetilde{\alpha}}$ on the left half plane. Thus the curve $C(t_k)$ must intersect the ω axis for some $a > \widetilde{\alpha}$ with $\omega(t_k, a) > \omega(t_k, \widetilde{\alpha})$. Hence we define α_* by

$$\alpha_* = \min\{a > \widetilde{\alpha} \mid u(t_k, a) = 0 \text{ and } u \text{ has } k \text{ zeros on } B_R(0), \omega(t_k, a) > \omega(t_k, \widetilde{\alpha})\}. \tag{4.16}$$

In case (D1), the solutions $u(t_k, \widetilde{\alpha})$ and $u(t_k, \alpha_*)$ are in configuration (A4), (A5), or (A6).

Thus the Morse index of $u(t, \alpha_*)$ at $t = t_k$ must be M or $M + 1$.

In case (D3), the solutions $u(t_k, \widetilde{\alpha})$ and $u(t_k, \alpha_*)$ are in configuration (D2), (D4), or (D6). Thus the Morse index of $u(t, \alpha_*)$ at $t = t_k$ must be M or $M + 1$. $\square$

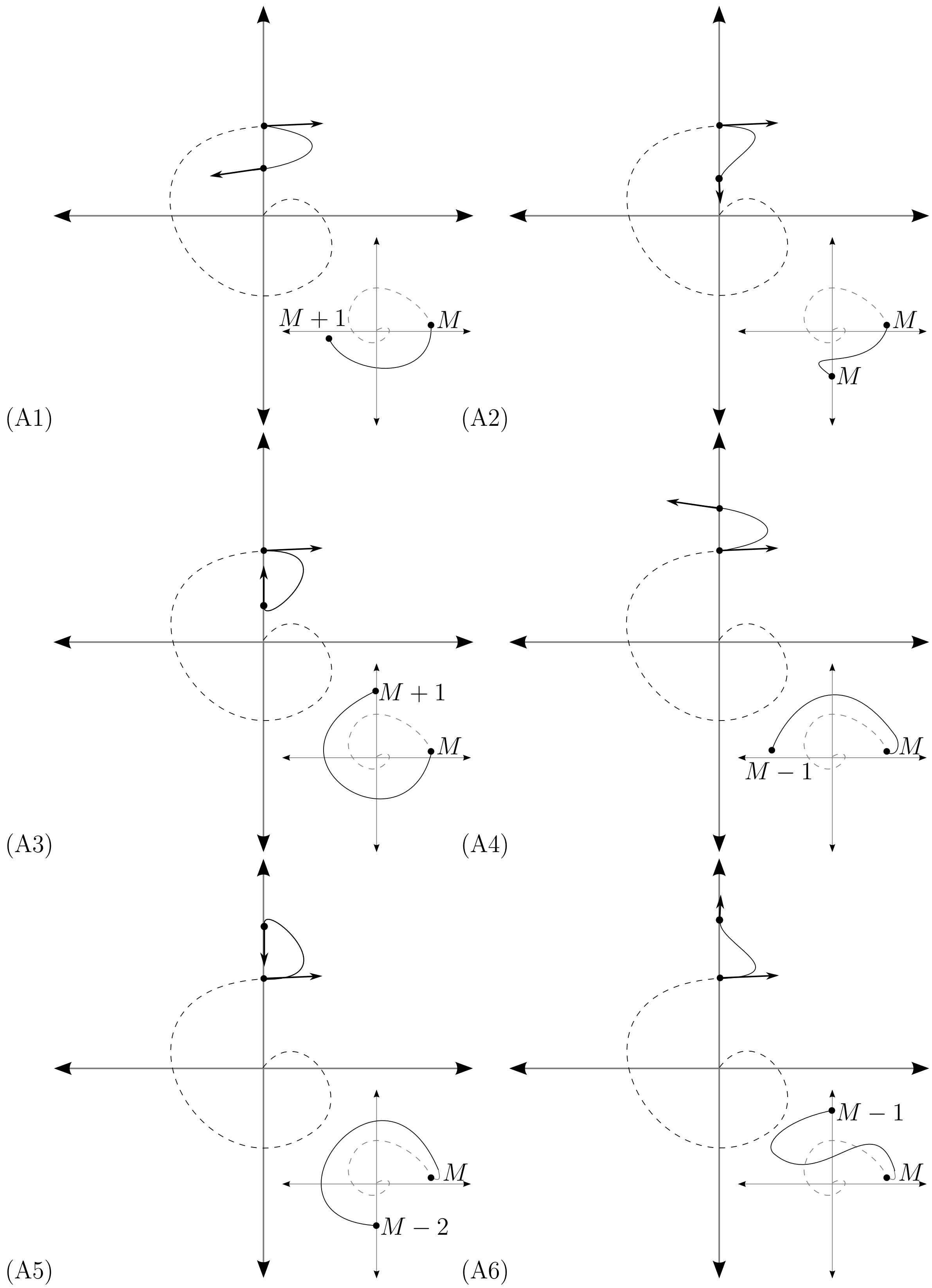


Figure 4.3: Figures (A1)-(A6) illustrate the six possible subcases of Case A in the proof of Theorem 4.6.1.

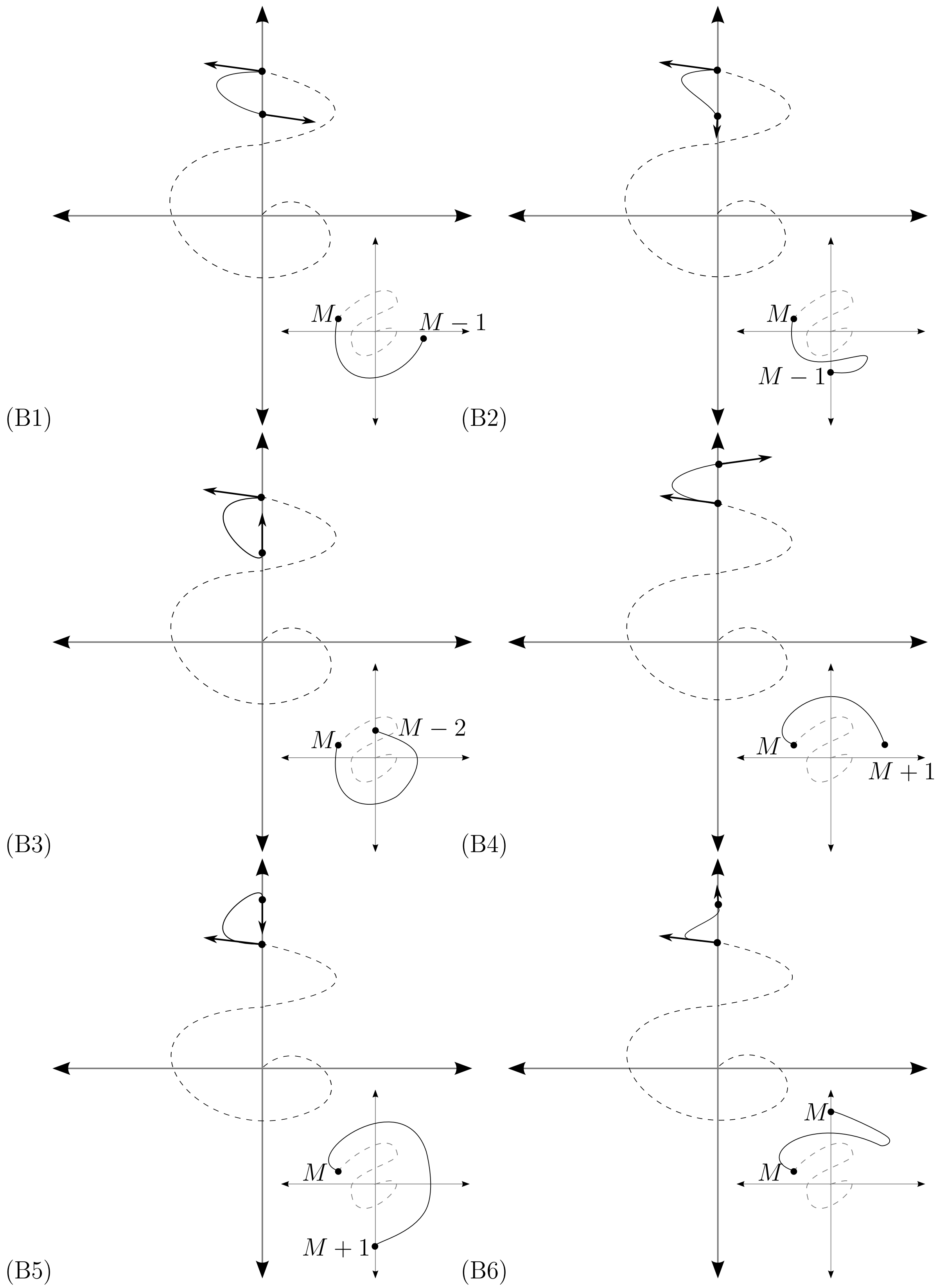


Figure 4.4: Figures (B1)-(B6) illustrate the six possible subcases of Case B in the proof of Theorem 4.6.1.

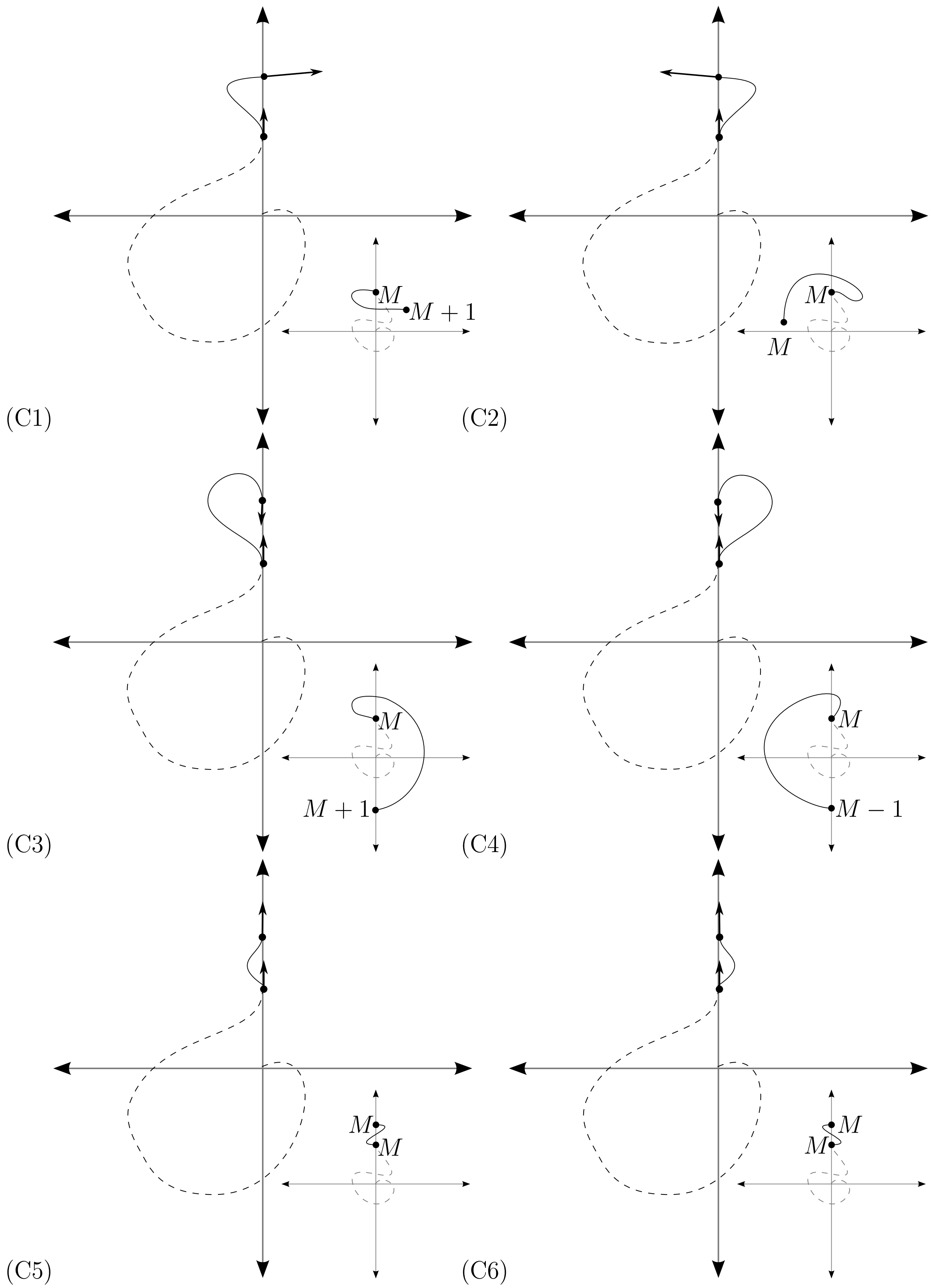


Figure 4.5: Figures (C1)-(C6) illustrate six of the possible twelve subcases of Case C in the proof of Theorem 4.6.1.

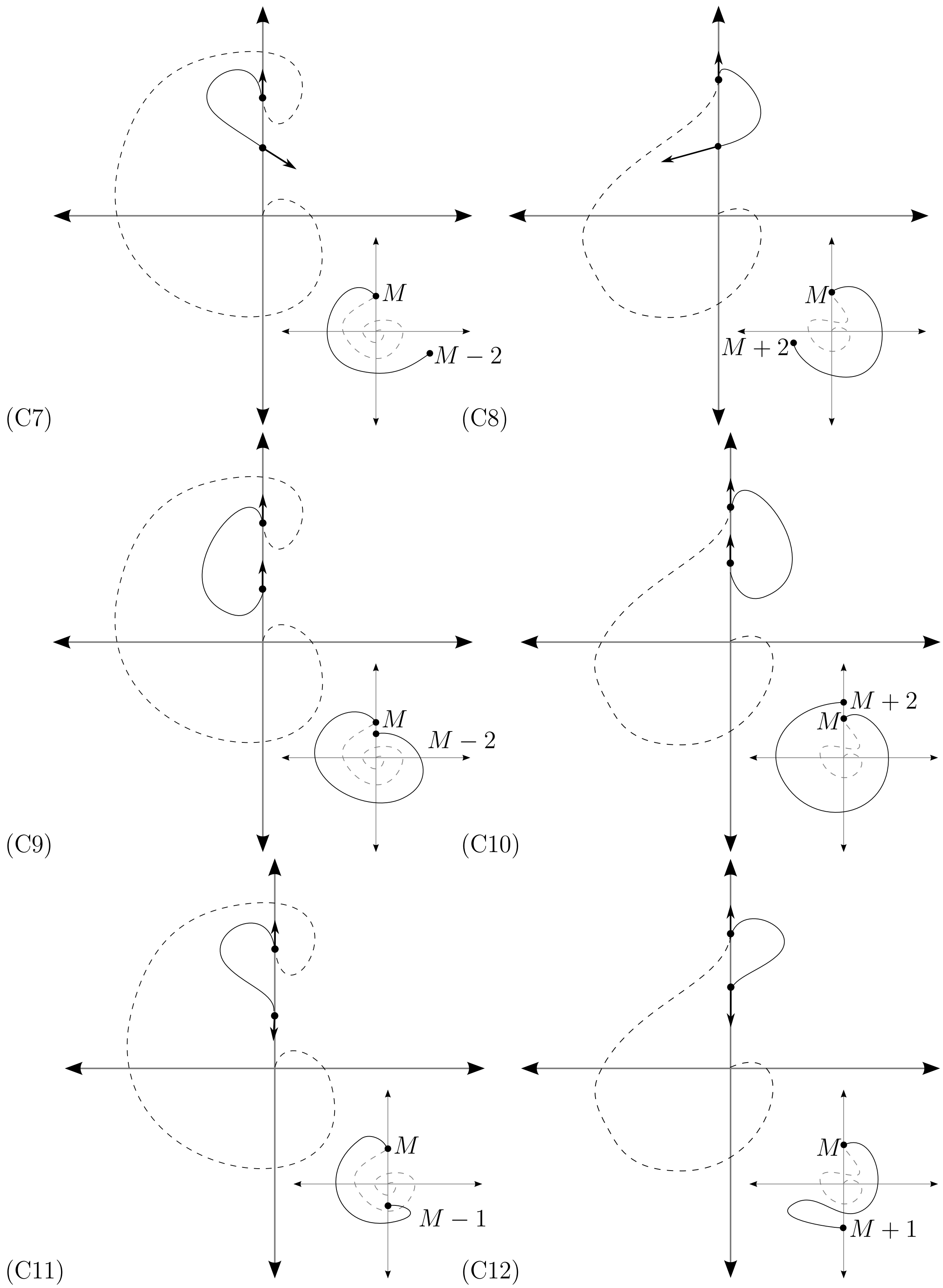


Figure 4.6: Figures (C7)-(C12) illustrate six of the possible twelve subcases of Case C in the proof of Theorem 4.6.1.

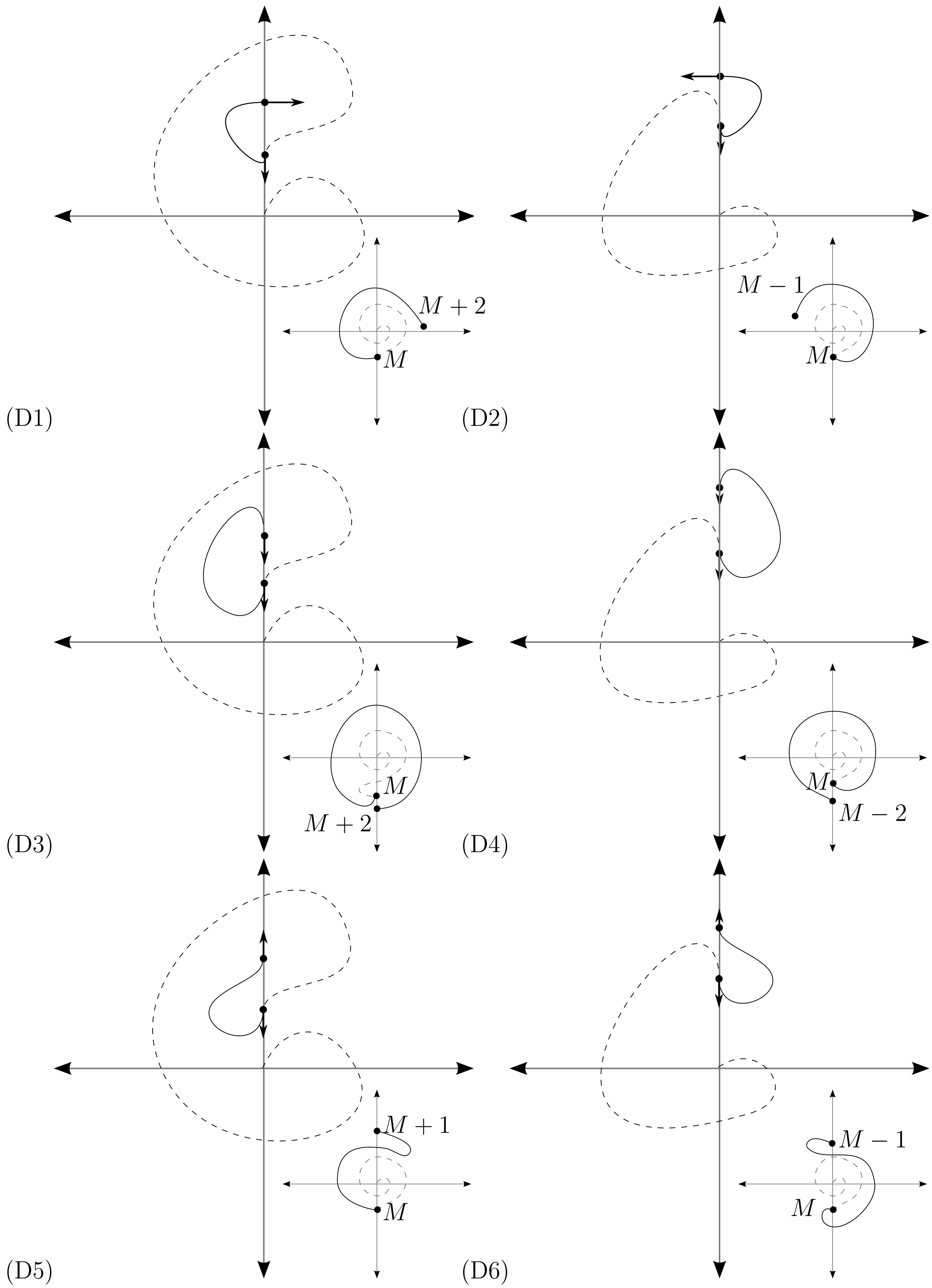


Figure 4.7: Figures (D1)-(D6) illustrate six of the possible twelve subcases of Case D in the proof of Theorem 4.6.1.

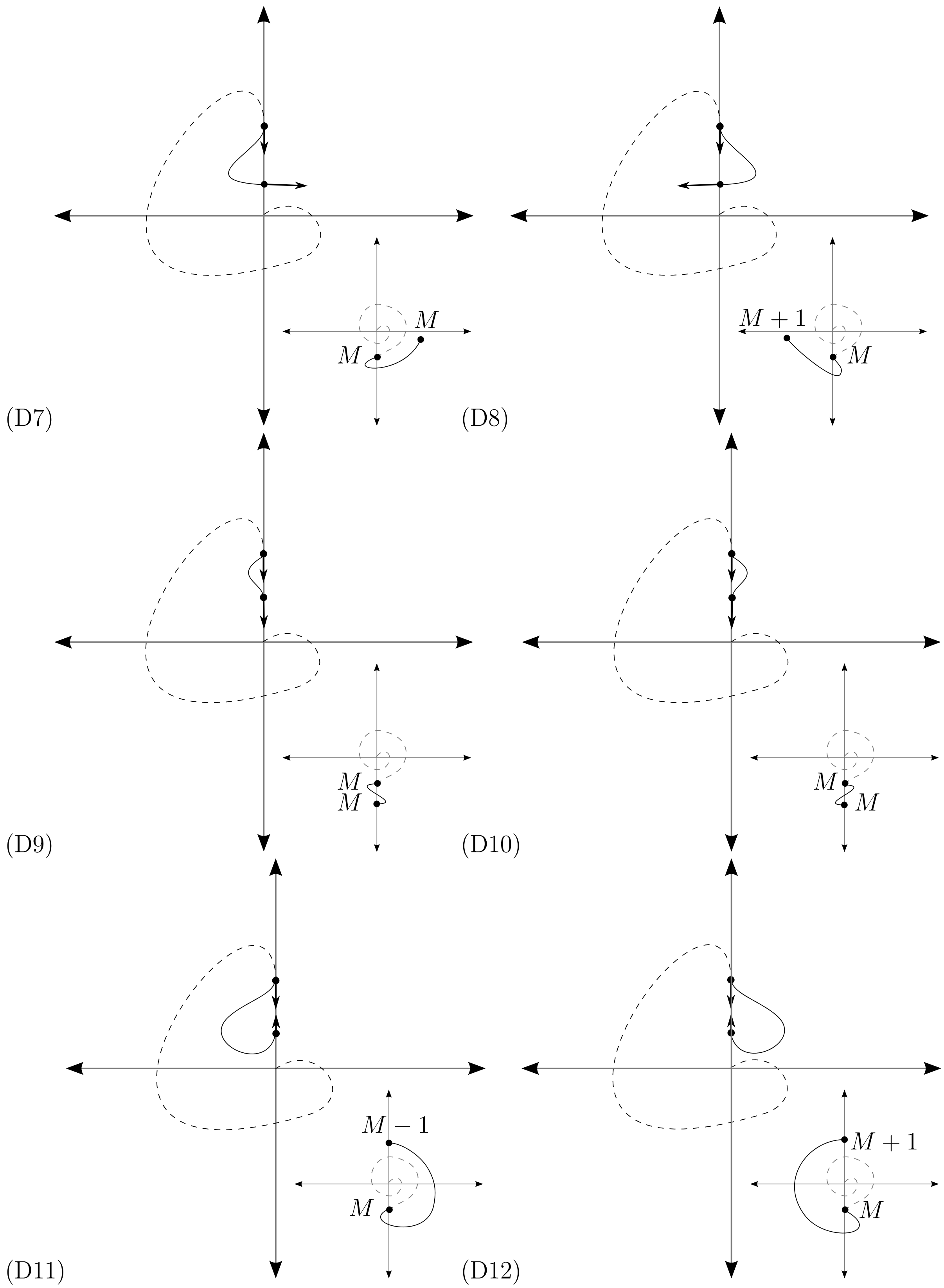


Figure 4.8: Figures (D1)-(D6) illustrate six of the possible twelve subcases of Case D in the proof of Theorem 4.6.1.

CHAPTER 5: REMARKS AND FUTURE DIRECTIONS

Let us summarize what we have achieved. In Chapter 3, we proved uniqueness of positive radial solutions of the p-Laplacian problem $\Delta_p u + f(u) = 0$ for a large class of nonlinearities f. The approach was geometric and we interpreted solutions of the equation as curves on a manifold in phase space. In Chapter 4, we established connections between geometric properties of solutions and the Morse index of a linearized operator restricted to the subspace of radial solutions in $W_0^{1,2}(\Omega)$. In this chapter, we describe some continuations of these results and other applications of the geometric methods to two related problems.

5.1 Hyperbolic metric

A possible future direction is to extend uniqueness results further to different metrics. The hyperbolic metric is particularly challenging because this metric has the effect of bringing increased r-dependence into the dynamics. With the hyperbolic metric, radial solutions in $B_R(0) \subset \mathbb{R}^n$ solve the following ordinary differential equation:

$$u_{rr} + (n-1)\coth(r)u_r + f(u) = 0, \tag{5.1}$$

with $f \in C^1([0,\infty))$. The approach to this equation is similar to the p-Laplacian and uses an Emden-Fowler transformation. In particular, rather than defining y as $y = r^\lambda u$, we use the transformation $y = \sinh^\lambda(r)u$.

With this transformation, the definitions

$$\begin{aligned} z(r) &= y' - \lambda\coth(r)y \\ w(r) &= \tanh(r)z, \end{aligned}$$

and the reparametrization of r as a function of t so that $\dot{r} = \tanh(r)$, we can rewrite (5.1) as the system

$$\dot{y} = \lambda y + w \tag{5.2}$$

$$\dot{w} = \operatorname{sech}^2(r)w - (n-1-\lambda)w - \tanh^2(r)\sinh^\lambda(r)f \tag{5.3}$$

$$\dot{r} = \tanh(r), \tag{5.4}$$

where $\cdot$ is differentiation with respect to t. The variational equations for (5.2)-(5.4) with $\delta r \equiv 0$ are

$$\dot{\delta y} = \lambda\,\delta y + \delta w \tag{5.5}$$

$$\dot{\delta w} = \operatorname{sech}^2(r)\delta w - (n-1-\lambda)\,\delta w - \tanh^2(r)f'(u)\,\delta y. \tag{5.6}$$

Let us consider the two 2-forms from Chapter 3 for (5.2)-(5.6). The 2-form

$$\delta r^* = \dot{y}\,\delta w - \dot{w}\,\delta y$$

satisfies

$$\begin{aligned}(\delta r^*)^{\cdot} &= (2\lambda - n + 1 + \operatorname{sech}^2(r))\,\delta r^* + 2\operatorname{sech}^2(r)\tanh^2(r) w\,\delta y \\ &\quad + \tanh^2(r)\sinh^\lambda(r)\left[(\lambda + 2\operatorname{sech}^2(r))f(u) - \lambda u f'(u)\right]\,\delta y.\end{aligned} \tag{5.7}$$

This linear differential equation contains the expression

$$I_H(u,\lambda) := (\lambda + 2\operatorname{sech}^2(r))f(u) - \lambda u f'(u),$$

which is reminiscent of (3.60) in the p-Laplacian calculation. However, the term $w\,\delta y$ from

(5.7) must be dealt with differently. The other form,

$$W_H(t) = \begin{vmatrix} y & w \\ \delta y & \delta w \end{vmatrix},$$

yields

$$\dot{W}_H = (2\lambda - n + 1 + \operatorname{sech}^2(r))W_H + \tanh^2(r)\sinh^\lambda(r)\,(uf'(u) - f(u))\;\delta y.$$

Hence if $f(u)$ satisfies $uf'(u) - f(u) \neq 0$, then this form might eliminate underrotation.

5.2 Algal bloom model

In this section, we mention a separate uniqueness result which uses a very similar approach to Chapter 3. The steady state algal bloom model derived by Ebert et al. [12] is a second-order ordinary differential equation. It models density ρ of an algal bloom population as a function of the depth in the water layer x, where $x = 0$ is the surface and $x = L$ is the bottom of the layer. The model is

$$\rho'' - C\rho' + A\left(e^{-x-\int_0^x \rho(y)\,dy} - B\right)\rho = 0, \tag{5.8}$$

where $\rho(x) \geq 0$ for all $0 \leq x \leq L$. The boundary conditions are the Robin conditions

$$[\rho' - C\rho]_{x=0,L} = 0. \tag{5.9}$$

For the algal blooms of interest, the parameters A, B, C, L satisfy $0 < A < \infty$, $0 < B < 1$, $C \in \mathbb{R}$, and $0 < L < \infty$.

We can recast (5.8) as a dynamical system with three first-order equations to view solutions as curves in phase space. Using a 2-form equivalent to W_p with the appropriate variational equations, we have performed a sign argument similar to the approach in Section 3.4.4 to show that (5.8) has a unique solution.

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