Solutions to the Nonlinear Schrodinger Equation with Dirac Mass Initial Data

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ABSTRACT

JASON NEWPORT: Solutions to the Nonlinear Schrodinger Equation with Dirac Mass Initial Data

(Under the direction of Kenneth T-R McLaughlin)

We study the Nonlinear Schrodinger Equation Dirac mass initial data. We use scattering and inverse scattering theory to pose a Riemann Hilbert problem with a regularized reflection coefficient. We study the asymptotic behaviour of this RHP as the regularizing parameter tends to zero. We also establish asymptotic descriptions of solutions for sequences of initial data that converge to a Dirac mass, using a connection to previously known long time asymptotics.

To my entire family who has helped me through every step of this process.

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CHAPTER 1

Introduction

The goal of this paper is to analyze solutions of the defocussing nonlinear Schrodinger (NLS) equation with Dirac mass intial data:

$$i\varphi_t + \varphi_{xx} - 2|\varphi|^2\varphi = 0 \tag{1.1}$$

$$\varphi(x, t = 0) = \delta(x). \tag{1.2}$$

In chapter 1.1 we investigate the linear Schrodinger equation with Dirac mass initial data:

$$i\varphi_t + \varphi_{xx} = 0 \tag{1.3}$$

$$\varphi(x, t = 0) = \delta(x). \tag{1.4}$$

We will study this via a straightforward regularization which will motivate the regularization that we need to use when solving the nonlinear problem.

In chapter 2 we will describe the Scattering and Inverse Scattering Theory associated to the NLS equation. It is a nonlinear analogue of the Fourier theory used to solve the linear problem. We will study this theory for Schwartz class initial data.

In chapter 3 we will find the scattering data corresponding to Dirac mass initial data. We will formulate a Riemann Hilbert Problem (RHP), and see why the theory breaks down with constant scattering data. Then we introduce a regularization similar to the one we used in section 1.1.

In chapter 4 we preform asymptotic calculations on the RHP with the regularized reflection coefficient to find the asymptotic behaviour of the potential.

In chapter 5 we find a connection between long time asymptotics and sequences of initial data that converge to a Dirac mass.

1.1. Linear Schrodinger Equation

To motivate the need for a regularization of the scattering data in the nonlinear case, we will start with the simpler linear equation. The Linear Schrodinger Equation is defined as

$$iq_t + q_{xx} = 0, \tag{1.5}$$

where we have chosen the initial data as the Dirac mass:

$$q(t = 0, x) = \delta(x). \tag{1.6}$$

Using the fourier transform

$$\mathbb{F}\{f\}(k) = \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx,$$

we see that, formally, the solution to our equation should be

$$q(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-ik^2t}dk. \tag{1.7}$$

However, the integrand in (1.7) is purely oscillitory, and we must decide upon a proper interpretation of the integral.

At this point we introduce a regularization to our transformed data to ensure that the inverse transform of our initial data converges. Let

$$\hat{q}_0^\epsilon = \frac{1}{\sqrt{2\pi}}e^{-\epsilon k^2}. \tag{1.8}$$

Now, our solution (1.7) becomes

$$q^\epsilon(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx-ik^2t-\epsilon k^2}dk.$$

Straightforward calculations yield:

$$q^\epsilon(x,t) = \frac{1}{2\pi}e^{\frac{ix^2}{4t}}e^{-i\frac{\pi}{4}}\int_{\mathbb{R}} e^{-(t-i\epsilon)z^2}dz. \tag{1.9}$$

In the limit as $\epsilon \to 0$ we find the solution is

$$q(t,x) = \frac{1}{2\sqrt{\pi t}}e^{-i\frac{\pi}{4}}e^{\frac{ix^2}{4t}}. \tag{1.10}$$

Consider the asymptotics of our solution, as $t \to 0$. We will show that it converges weakly to the Dirac delta. Let $h(x)$ be any smooth test function, and define

$$\mathrm{I}(x,t) = \int_{\mathbb{R}} q(x,t)h(x)dx = \frac{1}{2\sqrt{\pi t}}e^{-i\frac{\pi}{4}}\int_{\mathbb{R}} e^{\frac{ix^2}{4t}}h(x)dx.$$

As $t \to 0$, the dominant contribution from the integral is at $x = 0$. The stationary phase method yields

$$\mathrm{I}(x;\ s) = h(0) + O(\sqrt{t}).$$

Thus, our solution converges weakly to the Dirac delta function.

Remark: We have shown that if $\epsilon \to 0$ the formula (1.9), converges to a Dirac mass as $t \to 0$. A modification of this calculation shows that the order of limits does not matter. Indeed, if the vector $(t, \epsilon) \to 0$ in formula (1.9), then the result converges to a Dirac mass as $\epsilon \to 0$. In the nonlinear case, we will see that we cannot interchange the order of limits.

1.2. Results

Formal considerations in chapter 3 lead us to a one parameter family of reflection coefficients:

$$r(z,t) = \frac{1}{1 + \gamma - \gamma^2}\, e^{4iz^2 t}.$$

For ease of calculation we will choose $\gamma = \frac{1}{2}$. We consider the *regularized* reflection coefficient

$$r(z,t) = \frac{4}{5} e^{-2\epsilon z^2 + 4iz^2 t}. \tag{1.11}$$

Our interest is the behaviour of the solution to the NLS equation with initial data corresponding to (1.11), as $\epsilon \to 0$. The following is our main result, proven in chapter 4.

Theorem 1.2.1. *Let $\varphi(x,t;\epsilon)$ represent the solution to the NLS equation corresponding to the reflection coefficient (1.11). Then for each T_1, T_2 and X such that*

$0 < T_1 < T_2 < \infty$ and $0 < X < \infty$, the following expansion holds true for all $T_1 < t < T_2$ and $|x| < X$:

$$\varphi = \frac{1}{\sqrt{t}} e^{\frac{ix^2}{4t} - 2i\nu log(8t)} \epsilon^{-2i\nu} e^{h(z_0)} \sqrt{\pi} \frac{5e^{i\pi/4 - \pi\nu/2}}{4\Gamma(-i\nu)} + O\left(\sqrt{\epsilon} log(\epsilon)\right),$$

with $\nu(z)$ and $h(z)$ defined in (A.1) and (A.3) respectively.

The meaning of the error term is that there exists a constant C depending on T_1, T_2 and X so that the error is bounded by $C|\sqrt{\epsilon}(log\epsilon)^2|$ for all allowable x and t.

THEOREM 1.2.2. *Let $\tilde{r}(\lambda)$ be the reflection coefficient corresponding to the initial data $f(x)$. Consider initial conditions of the form*

$$\varphi(x, t = 0; \epsilon) = \frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right),$$

and let M be a positive constant. Then for all values x and t where $|z_0| = \left|\frac{-\epsilon x}{4t}\right| \leq M$, the solution has the following asymptotic expansion:

$$\varphi(x, t; \epsilon) = \frac{1}{\sqrt{t}} e^{\frac{ix^2}{4t} - i\nu log(8t)} \epsilon^{2i\nu} u(z_0) + O\left(\epsilon log\epsilon\right)$$

with

$$i\nu(z_0) = \frac{1}{2\pi i} log\left(1 - |\tilde{r}(z_0)|^2\right).$$

The function u is defined in terms of its modulus and phase:

$$|u(z_0)|^2 = \nu/2 = -\frac{1}{4\pi} log(1 - |\tilde{r}(z_0)|^2)$$

$$arg\ u(z_0) = \frac{1}{\pi} \int_{-\infty}^{z_0} log(z_0 - s) d\ log(1 - |\tilde{r}(s)|^2) + \frac{\pi}{4} + arg\Gamma(i\nu) - arg\ \tilde{r}(z_0).$$

Remark: The question which led to the first theorem was 'What is the behaviour of solutions associated with regularizations of constant reflection coefficients.' The sort of regularizations we considered were in the general class of delta sequences discussed in Theorem 1.2.2. In addition, the method used to prove the two theorems are quite different. The former involves Riemann Hilbert Analysis of the inverse spectral problem, while the latter relies on established long time asymptotics.

A slightly stronger version of Theorem 1.2.2 follows.

THEOREM 1.2.3. *Let $\tilde{r}(\lambda)$ be the reflection coefficient corresponding to the initial data $f(x)$. Consider iinitial conditions of the form*

$$\varphi(x, t = 0; \epsilon) = \frac{1}{\epsilon} e^{-2i\alpha x} f\left(\frac{x}{\epsilon}\right),$$

and let M be a positive real constant. For any constant $\alpha \in \mathbb{R}$ and for all values x and t where $|z_0| = \left|\frac{-\epsilon x}{4t}\right| \leq M$ the solution has the following asymptotic expansion:

$$\varphi(x, t; \epsilon) = e^{-2i\alpha x - 2i\alpha^2 t} \left[\frac{1}{\sqrt{t}} e^{\frac{ix^2}{4t} - i\nu log(8t)} \epsilon^{2i\nu} u(z_0) + O\left(\epsilon log \epsilon\right)\right]$$

with

$$i\nu(z_0) = \frac{1}{2\pi i} log\left(1 - |\tilde{r}(z_0)|^2\right).$$

The function u is defined in terms of its modulus and phase:

$$|u(z_0)|^2 = \nu/2 = -\frac{1}{4\pi} log(1 - |\tilde{r}(z_0)|^2)$$

$$arg\ u(z_0) = \frac{1}{\pi} \int_{-\infty}^{z_0} log(z_0 - s) d\ log(1 - |\tilde{r}(s)|^2) + \frac{\pi}{4} + arg\Gamma(i\nu) - arg\ \tilde{r}(z_0).$$

1.3. Motivation For Studying Singular Limits of the NLS Equation

We are studying the behaviour of the NLS equation with Dirac mass initial data for several reasons. The scattering and inverse scattering theory found herein is well established. Existence and uniqueness of long time asymptotics are known for initial data in a weighted Sobolev space [**4**]. The Dirac mass is a distribution that is outside this class of functions. In this paper we find that solutions exist for sequences of initial conditions that converge to a Dirac mass. One interesting part of analysis is that the solutions are not unique.

Another reason we studied these problems is to figure out how the NLS equation regularizes singular data. In order to study this we had to regularize the reflection coefficient and develop the machinery to handle the limit when this smoothing parameter went to zero. Once we had found solutions, one could ask what happened when $t \to 0$. The limit as the smoothing parameter ϵ tends to zero does not necessarily commute with the limit when $t \to 0$.

The NLS equation can also be used to model laser pulses in optical fibres. Our work could be used to understand the behaviour of ultra short high intensity pulses.

CHAPTER 2

Scattering and Inverse Scattering Theory

In this chapter we will study the defocusing nonlinear Schrodinger equation:

$$i\varphi_t + \varphi_{xx} - 2|\varphi|^2\varphi = 0$$

with Schwartz class initial data.

The scattering and inverse scattering theory we will discuss is a nonlinear version of the Fourier method for solving linear partial differential equations. We find scattering data via the direct spectral transform. This scattering data has a very simple evolution in time. In order to reconstruct the solution at later times we must go from the evolved scattering data back to the potential, which is achieved using Riemann Hilbert methods. This scattering and inverse scattering theory have been studied in great detail; in [**1**] and [**6**], for example.

The Lax pair associated with the NLS equation is the pair of linear operators:

$$L = i\sigma_3\frac{\partial}{\partial x} + i\begin{pmatrix} 0 & -\varphi \\ \overline{\varphi} & 0 \end{pmatrix} \tag{2.1}$$

$$B = 2zI\frac{\partial}{\partial x} + i \begin{pmatrix} -|\varphi|^2 & \varphi_x \\ -\overline{\varphi_x} & |\varphi|^2 \end{pmatrix}. \tag{2.2}$$

If a 2×2 matrix function $\psi = \psi(x, t, z; \varphi)$ exists so that

$$L\psi = z\psi \tag{2.3}$$

$$\frac{\partial}{\partial t}\psi = B\psi, \tag{2.4}$$

the compatibility of partial derivatives implies that φ solves the NLS equation.

Let $z \in \mathbb{C}_+$ and assume $t = 0$. Then the differential equation

$$i\sigma_3\frac{\partial}{\partial x}\Psi + i \begin{pmatrix} 0 & -\varphi \\ \overline{\varphi} & 0 \end{pmatrix} \Psi = z\Psi,$$

together with the asymptotic conditions:

$$\Psi(x, z) \begin{pmatrix} e^{izx} & 0 \\ 0 & e^{-izx} \end{pmatrix} \rightarrow I \text{ as } x \rightarrow \infty; \tag{2.5}$$

and

$$\Psi(x, z) \begin{pmatrix} e^{izx} & 0 \\ 0 & e^{-izx} \end{pmatrix} \text{ bounded as } x \rightarrow -\infty. \tag{2.6}$$

possesses a unique solution.

We will use the notation

$$e^{izx\sigma_3} = \begin{pmatrix} e^{izx} & 0 \\ 0 & e^{-izx} \end{pmatrix}$$

Define

$$M = \Psi(x,z)e^{izx\sigma_3} \tag{2.7}$$

so that

$$M \to I \qquad \text{when } x \to \infty. \tag{2.8}$$

2.1. Potentials with Compact Support

In this section we will assume that $supp(\varphi) = (x_l, x_r)$. Many of the analytical issues are greatly simplified in this setting. Our goal is to provide a well-known (see [**1**]) description of the reflection coefficient.

We will begin by building unique solutions to (2.3) and (2.4) at $t = 0$. For $x > x_r$, (2.3) simplifies to

$$i\sigma_3 \frac{\partial}{\partial x}\Psi = z\Psi.$$

Thus,

$$\Psi(x,z) = e^{izx\sigma_3}C = \begin{pmatrix} e^{izx} & 0 \\ 0 & e^{-izx} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

for $x > x_r$, where the matrix C is constant in x.

Let $z \in \mathbb{C}_+$. By the asymptotics in (2.5), we know $c_{22} = c_{11} = 1, c_{12} = 0$. However, c_{21} is not determined. If we do a similar analysis as $x \to -\infty$, we find that

$$\Psi(x,z) = \begin{pmatrix} e^{-izx} & 0 \\ 0 & e^{izx} \end{pmatrix} \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}.$$

In order for Ψ to be bounded, we need $d_{21} = 0$. However, d_{11}, d_{12}, d_{22} have yet to be determined. Thus, we have

$$\Psi(x,z) = e^{-izx\sigma_3} \begin{pmatrix} 1 & 0 \\ c_{21} & 1 \end{pmatrix} \qquad x > x_r \tag{2.9}$$

$$\Psi(x,z) = e^{-izx\sigma_3} \begin{pmatrix} d_{11} & d_{12} \\ 0 & d_{22} \end{pmatrix} \qquad x < x_l. \tag{2.10}$$

For $z \in \mathbb{C}_-$, a similar analysis shows that our solution is

$$\Psi(x,z) = e^{-izx\sigma_3} \begin{pmatrix} 1 & \hat{c}_{12} \\ 0 & 1 \end{pmatrix} \qquad x > x_r \tag{2.11}$$

$$\Psi(x,z) = e^{-izx\sigma_3} \begin{pmatrix} \hat{d}_{11} & 0 \\ \hat{d}_{21} & \hat{d}_{22} \end{pmatrix} \qquad x < x_l. \tag{2.12}$$

Proving Ψ exists in (x_l, x_r) follows from standard ODE theory. We find that there is a unique $c_{21}(z)$ that forces $d_{21} = 0$.

The function Ψ has boundary values as z tends towards the real axis. Let

$$\Psi_+ = \lim_{Im(z)\searrow 0} \Psi$$

$$\Psi_- = \lim_{Im(z)\nearrow 0} \Psi.$$

These both exist since $\Psi(z)$ is analytic off the real axis and the support of φ is compact. Now we have 2 fundamental solutions to the same ODE on the real axis, and so they must be related as follows:

$$\Psi_+(x,z) = \Psi_-(x,z)\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}(z). \tag{2.13}$$

For $x > x_r$, using (2.9) and (2.11), we get

$$\begin{pmatrix} 1 & 0 \\ c_{21} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \hat{c}_{12} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}(\xi).$$

Simplifying, we get

$$\begin{pmatrix} 1 - c_{21}\hat{c}_{12} & -\hat{c}_{12} \\ c_{21} & 1 \end{pmatrix} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}(z). \tag{2.14}$$

The mapping $\varphi(x) \longrightarrow c_{12}(z)$ is our nonlinear transform. The function c_{21} is called the reflection coefficient and is denoted $r(z,t)$.

2.2. Evolution in Time

For $t = 0$ we have a reflection coefficient. We want to find how it evolves in time as the NLS equation evolves. In the previous section we showed that Ψ exists when $t = 0$. Since compact support is not preserved by the NLS equation, the previous section does not yield existence for potentials φ with non compact support. For the remainder of this section we will assume that Ψ exists for each $t > 0$. A general existence theory is found in appendix A.2. See also [**4**]. It should be noted that due to this theory, we know the solution Ψ (and therefore, M) is differentiable.

Straightforward algebra shows that $\frac{\partial}{\partial t}L = [B \ \ L]$. Recall, $L\Psi = z\Psi$ with the asymptotic conditions (2.5) and (2.6). Differentiating with respect to t, we obtain

$$\frac{\partial}{\partial t}(z\Psi) = BL\Psi - LB\Psi + L\Psi_t.$$

Rearranging, yields

$$L(\Psi_t - B\Psi) = z(\Psi_t - B\Psi).$$

Therefore, both $\Psi_t - B\Psi$, and Ψ are eigenfunctions of L with the same eigenvalue. Thus they are related by:

$$\Psi_t - B\Psi = \Psi E,$$

where E is a matrix whose entries are constant in x. We can rewrite this equation in terms of M:

$$M_t - BM = Me^{-izx\sigma_3}Ee^{izx\sigma_3}.$$

In A.2 we show some properties of M, including the fact that it is bounded. It is easily shown that, $M^{-1}(M_t - BM)$ is bounded as $x \to \pm\infty$. In addition, it follows that $det\ M = 1$. Therefore, $E_{12} = E_{21} = 0$. Using the asymptotics for M, derived from the asymptotics for Ψ (2.5) and (2.6), we find that

$$M^{-1}(M_t - BM) \to 2iz^2\sigma_3,$$

as $x \to \infty$.

Thus, we find $E = 2iz^2\sigma_3$. In terms of Ψ, we have

$$\Psi_t - B\Psi = 2iz^2\sigma_3\Psi. \tag{2.15}$$

Differentiating (2.13) with respect to time, and using (2.15), we find

$$\begin{aligned} V_t &= [V\ ,\ E] \\ &= 2iz^2[V\ ,\ \sigma_3]. \end{aligned}$$

Using the definition (2.13) of the jump matrix, we have

$$r(z,t) = e^{4iz^2t}r(z,0) = e^{4iz^2t}v_{12} \qquad \text{for } z \in \mathbb{R}. \tag{2.16}$$

A symmetry of the Lax pair allows us to write the jump matrix for Ψ as

$$V = \begin{pmatrix} 1-|r|^2 & -\overline{r}(z,t) \\ r(z,t) & 1 \end{pmatrix}. \tag{2.17}$$

2.3. Formulation of RHP and Inverse Scattering Theory

We have computed the reflection coefficient and how it evolves in time. We want a procedure to reconstruct the potential as it evolves with the NLS equation. We use the inverse scattering transform solved with Riemann Hilbert approach. For this section we assume M exists and is differentiable. The proof that M exists relies heavily on the knowledge of the Lax pair and the function φ, and can be found in appendix A.2.

We know M is analytic off of the real axis and that it has identity asymptotics as $z \to \infty$. We will assume $r(z)$ is analytic, therefore, M is continuous in the closed upper half plane, and in the closed lower half plane. In this sense, M achieves boundary values on the real axis. Using the jump matrix for Ψ and the definition of M, (2.17) and (2.7), we can find the jump relation for M:

$$M_+(z) = M_-(z) \begin{pmatrix} 1-|r|^2 & -\overline{r(z)}e^{-2izx-4iz^2t} \\ r(z)e^{2izx+4iz^2t} & 1 \end{pmatrix} \qquad z \in \mathbb{R}.$$

These properties can be combined to formulate a Riemann Hilbert Problem: We wish to find a matrix M that has the following properties:

$$\begin{aligned} &M(z) && \text{is analytic off the real axis} \\ &M(z) && = I + \tfrac{M_1}{z} + \tfrac{M_2}{z^2} + \cdots \qquad z \to \infty \\ &M_+(z) && = M_-(z) \begin{pmatrix} 1-|r|^2 & -\overline{r(z)}e^{-2izx-4iz^2t} \\ r(z)e^{2izx+4iz^2t} & 1 \end{pmatrix} \qquad z \in \mathbb{R}. \end{aligned}$$

If we can find such an M, the solution to the NLS equation is embedded in M as shown in the following theorem.

THEOREM 2.3.1. *Assume φ is Schwartz class for $t = 0$. Then the solution φ to the NLS equation is:*

$$\varphi = 2i(M_1(x,t))_{12},$$

where $(M_1)_{12}$ is the upper right entry of the first moment of M as $z \to \infty$.

PROOF. We know that M solves the differential equation

$$\frac{\partial}{\partial x}M = iz[M, \sigma_3] + QM,$$

where

$$Q = \begin{pmatrix} 0 & \varphi \\ \overline{\varphi} & 0 \end{pmatrix}.$$

If $M = I + z^{-1}M_1 + \cdots$, then

$$z^{-1}\frac{\partial}{\partial x}M_1 + \cdots = iz[I + z^{-1}M_1 + \cdots, \sigma_3] + Q(I + z^{-1}M_1 + \cdots).$$

Solving for the leading order term yields $Q = -i[M_1, \sigma_3]$. In terms of the matrix entries:

$$\begin{pmatrix} 0 & \varphi \\ \overline{\varphi} & 0 \end{pmatrix} = 2i \begin{pmatrix} 0 & (M_1)_{12} \\ -(M_1)_{21} & 0 \end{pmatrix}.$$

Thus,

$$\varphi = 2i(M_1)_{12} = 2i \lim_{z\to\infty} z(M(z) - I)_{12}.$$

□

Remark: Theorem 2.3.1 is a well known fact and is true under weaker conditions on the potential. In [**4**] Deift and Zhou require only that the intial data be in the weighted sobolev space $H^{1,1}$.

Retrieving the solution to the NLS equation from the solution to our RHP is called Inverse Scattering Theory. There are several ways to achieve this retrieval; the more classical approach involves the Gelfand-Levitan-Marchenko equations, but we use a more recently developed Riemann–Hilbert approach, which is more suited to our subsequent asymptotic analysis. In chapter 4 we will solve the RHP, and then the inverse scattering theory will allow us to find the solution to the NLS equation.

CHAPTER 3

Obtaining the Reflection Coefficient with Dirac Mass Initial Data

Our goal is to find the reflection coefficient corresponding to NLS equation with Dirac mass initial data:

$$i\varphi_t + \varphi_{xx} - 2|\varphi|^2\varphi = 0 \tag{3.1}$$

$$\varphi(x, t = 0) = \delta(x),$$

where a is a constant.

Seeking a solution to equation (2.3), satisfying (2.5) and (2.6), with a Dirac mass potential is somewhat problematic. At $t = 0$, we have $\varphi = \overline{\varphi} = \delta(x)$. For this reason, Ψ will have a jump discontinuity at $x = 0$. We must append a rule for how a Dirac mass acts on a function with a jump discontinuity. We will proceed formally by imposing rules to evaluate

$$\int_{\mathbb{R}} \Psi\delta(x)dx.$$

We will use a weighted average of limiting values from the left and right. Using (3.3), we define

$$\int_{\mathbb{R}} \Psi\delta(x)dx = \gamma\left(\lim_{x\to 0^-} \Psi(x)\right) + (1-\gamma)\left(\lim_{x\to 0^+} \Psi(x)\right). \tag{3.2}$$

Now we will compute our reflection coefficient. Note that $supp(\varphi(x,t=0)) = \{0\}$. Thus, (2.10) and (2.9) become

$$\Psi(x,z) = e^{-izx\sigma_3}\begin{pmatrix} 1 & 0 \\ c_{21} & 1 \end{pmatrix} \qquad x > 0$$

$$\Psi(x,z) = e^{-izx\sigma_3}\begin{pmatrix} d_{11} & d_{12} \\ 0 & d_{22} \end{pmatrix} \qquad x < 0.$$

For brevity, let

$$C = \begin{pmatrix} 1 & 0 \\ c_{21} & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} d_{11} & d_{12} \\ 0 & d_{22} \end{pmatrix}.$$

Then Ψ has the following representation:

$$\Psi = e^{-izx\sigma_3}D + H(x)e^{-izx\sigma_3}\left(C - D\right), \tag{3.3}$$

with H being the heavyside function defined as

$$H(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0. \end{cases}$$

Suppose $t = 0$. To find our reflection coefficient we integrate the equation $L\Psi - z\Psi = 0$ against a test function $h(x)$ over the entire real axis. The weak form of this equation is:

$$-i\sigma_3\int_{\mathbb{R}}\Psi h'(x)dx - i\int_{\mathbb{R}}\begin{pmatrix} 0 & -\varphi \\ \overline{\varphi} & 0 \end{pmatrix}\Psi h(x)dx - z\int_{\mathbb{R}}\Psi h(x)dx = 0. \tag{3.4}$$

Since we already know Ψ solves this equation for $x \neq 0$, most of the terms in the equation will cancel. Then (3.4) can be rewritten as

$$\begin{aligned} 0 &= i\sigma_3 e^{-iza\sigma_3}(C-D)h(0) + i\int_{\mathbb{R}} \begin{pmatrix} 0 & -\varphi \\ \overline{\varphi} & 0 \end{pmatrix} \Psi h(x)dx \\ &= h(0)\left[i\sigma_3(C-D) + i \begin{pmatrix} -\gamma c_{21} & -\gamma-(1-\gamma)d_{22} \\ \gamma+(1-\gamma)d_{11} & (1-\gamma)d_{12} \end{pmatrix} \right] \end{aligned}$$

This is valid for any test function h, so

$$\begin{pmatrix} (1-d_{11}) & -d_{12} \\ -c_{21} & -(1-d_{22}) \end{pmatrix} + \begin{pmatrix} -\gamma c_{21} & -\gamma-(1-\gamma)d_{22} \\ \gamma+(1-\gamma)d_{11} & (1-\gamma)d_{12} \end{pmatrix} = 0.$$

We can easily solve these equations for c_{21}, yielding

$$c_{21} = \frac{1}{1+\gamma-\gamma^2}.$$

We can consider the Dirac delta function as a limit of piecewise constant functions whose integral is one:

$$q(x) = \begin{cases} \frac{1}{\delta} & |x| \leq \delta \\ 0 & |x| > \delta \end{cases}. \tag{3.5}$$

Using the work of DiFranco and McLaughlin in [**5**], we can find the reflection coefficient for initial conditions of the form (3.5). In the limit when $\delta \to 0$ the reflection coefficient is a constant:

$$\lim_{\delta\to 0} r(z,\delta) = \frac{e^4-1}{e^4+1}.$$

We can choose γ so that our reflection coefficient matches this limiting case. For simplicity, we choose $\gamma = \frac{1}{2}$ yielding our reflection coefficient

$$r(z, t = 0) = c_{21} = \frac{4}{5}, \tag{3.6}$$

as this will not effect the analysis in the upcoming calculations.

Recall the RHP for M. The problem is to find a 2 matrix M that satisfies:

$$\begin{aligned}
&(3.7) && M(z) && \text{is analytic off the real axis} \\
&(3.8) && M(z) &= \;& I + \frac{M_1}{z} + \frac{M_2}{z^2} + \cdots && z \to \infty \\
&(3.9) && M_+(z) &= \;& M_-(z) \begin{pmatrix} 1 - |r|^2 & -\overline{r(z)}e^{-2izx-4iz^2t} \\ r(z)e^{2izx+4iz^2t} & 1 \end{pmatrix} && z \in \mathbb{R}.
\end{aligned}$$

Unfortunately, with Dirac mass initial data, this problem is ill-posed. Similar to the linear problem we looked at, our transformed data $r(z) = \frac{4}{5}$ is a constant. We know $M \to I$ as $z \to \infty$. However, if we look at the third condition of our RHP, we see that $M^+, M^- \to I$ for $z \to \infty$. But our jump matrix will not converge since r is constant, and $z \in \mathbb{R}$.

In order to solve our problem, we will introduce a regularization in the same way that we did for the linear case. Let

$$r^\epsilon(z) := r(z)e^{-2\epsilon z^2} = \frac{4}{5}e^{-2\epsilon z^2}. \tag{3.10}$$

Using this regularization our reflection coefficient now decays as $z \to \infty$, and we can find the solution to our RHP. It is the goal of the next chapter to study the behavior as $\epsilon \to 0$.

Remark: The derivation that the reflection coefficient is constant in z was formal. However, it is important to observe that the asymptotic analysis of the inverse problem with this regularized reflection coefficient, appearing in chapter (4), is completely rigorous.

CHAPTER 4

Asymptotic Analysis of the RHP

In this chapter we will find an explicit approximation for our Riemann Hilbert Problem with regularized data. We will make use of a solution to a similar RHP, solved by Deift and Zhou in [**3**]. Using a series of explicit transformations, we will show that an equivalent form of our RHP is sufficiently close to theirs. We will then use Neumann series to compute the error.

We will use the Lie algebra notation

$$\lambda^{ad\sigma_3} v = \lambda^{\sigma_3} v \lambda^{-\sigma_3},$$

to make the transformations simpler to read.

Define a RHP with regularized data to be

$$(4.1) \qquad M^{\epsilon}(z) \qquad \text{is analytic off the real axis}$$

$$(4.2) \qquad M^{\epsilon}(z) = I + \frac{M_1^{\epsilon}}{z} + \frac{M_2^{\epsilon}}{z^2} + \cdots \qquad z \to \infty$$

$$(4.3) \qquad M_+^{\epsilon}(z) = M_-^{\epsilon}(z)\, e^{(-izx-2iz^2t)\; ad\sigma_3} \begin{pmatrix} 1-|r^{\epsilon}(z)|^2 & -\overline{r^{\epsilon}(z)} \\ r^{\epsilon}(z) & 1 \end{pmatrix} \qquad z \in \mathbb{R}.$$

The solution to the NLS equation is

$$(4.4) \qquad \varphi = 2i \lim_{\epsilon \to 0} \left(M_1^{\epsilon}\right)_{12}.$$

The RHP solved by Deift and Zhou in [**3**] is:

$$M^{DZ}(\omega) \qquad \text{is analytic off the contour } \Sigma \tag{4.5}$$

$$M^{DZ}(\omega) = I + \frac{M_1^{DZ}}{\omega} + \cdots \qquad \omega \to \infty \tag{4.6}$$

$$M_+^{DZ}(\omega) = M_-^{DZ}(\omega)\, e^{-i\omega^2 ad\sigma_3/4}\omega^{i\nu ad\sigma_3}V^{DZ} \qquad \omega \in \Sigma, \tag{4.7}$$

where $i\nu = \frac{1}{2\pi i} log\,(1 - |r(z_0)|^2)$ with $z_0 = \frac{-x}{4t}$ the stationary phase point. The contour

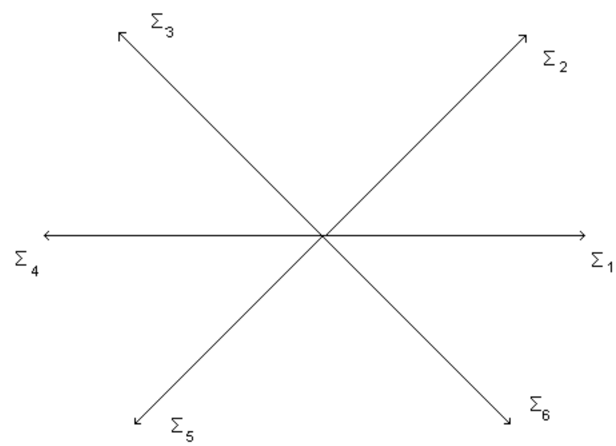

Figure 4.1. The Contour Σ

Σ is shown in Figure 4.1. The jump matrix is defined as:

$$V^{DZ}(\omega) = \begin{cases} V_r(z_0) & \omega \in \Sigma_2 \\ \hat{V}_r(z_0) & \omega \in \Sigma_3 \\ \hat{V}_l(z_0) & \omega \in \Sigma_5 \\ V_l(z_0) & \omega \in \Sigma_6\ . \end{cases} ,$$

using the definitions of V_l, V_r, $\hat{V}_l$ and $\hat{V}_r$ in (4.10), (4.11), (4.13) and (4.15). Note that in section 4.3 we will refer back to this RHP.

As in (4.4), the information we need is in the $(1,2)$ entry of the the first moment of M^{DZ} at infinity. From their paper,

$$\left(M_1^{DZ}\right)_{12} = \frac{-i\sqrt{2\pi}e^{i\pi/4}e^{-\pi\nu/2}}{r(z_0)\Gamma(-i\nu)}. \tag{4.8}$$

This is the solution to a model RHP, obtained through a series of transformations.

4.1. Transformations to an Equivalent RHP

In this section we will show the explicit transformations we use to find a RHP equivalent to ours, which is close to the RHP stated in (4.5) - (4.7).

It will be useful to define V_0 as

$$V_0 = \begin{pmatrix} 1 - |r^\epsilon(z)|^2 & -\overline{r^\epsilon(z)} \\ r^\epsilon(z) & 1 \end{pmatrix}.$$

The jump matrix can be written as $V = e^{-it\theta ad\sigma_3}V_0$, where

$$\theta = 2z^2 + \frac{zx}{t} = 2(z - z_0)^2 - 2z_0.$$

The stationary phase point is defined to be $z_0 = -\frac{x}{4t}$.

Remark: We will drop the superscript ϵ from the reflection coefficient. For the remainder of this chapter it is assumed that we are working with the regularized reflection coefficient.

Now we define two factorizations of V_0:

$$V_0 = V_l\, V_r \tag{4.9}$$

with

$$V_l = \begin{pmatrix} 1 & -\overline{r(z)} \\ 0 & 1 \end{pmatrix}, \tag{4.10}$$

$$V_r = \begin{pmatrix} 1 & 0 \\ r(z) & 1 \end{pmatrix}; \tag{4.11}$$

and

$$V_0 = \hat{V}_l \, \hat{V}_c \, \hat{V}_r \tag{4.12}$$

with

$$\hat{V}_l = \begin{pmatrix} 1 & 0 \\ \frac{r(z)}{1-|r(z)|^2} & 1 \end{pmatrix}, \tag{4.13}$$

$$\hat{V}_c = \begin{pmatrix} 1 - |r(z)|^2 & 0 \\ 0 & (1 - |r(z)|^2)^{-1} \end{pmatrix}, \tag{4.14}$$

$$\hat{V}_r = \begin{pmatrix} 1 & -\frac{\overline{r(z)}}{1-|r(z)|^2} \\ 0 & 1 \end{pmatrix}. \tag{4.15}$$

Let V_0 be factored according to:

$$V_0 = \begin{cases} V_l \, V_r & z > z_0 \\ \hat{V}_l \, \hat{V}_c \, \hat{V}_r & z < z_0. \end{cases}$$

The matrices V_r and V_l can be analytically extended into the upper and lower right quadrants, respectively. Similarily, $\hat{V}_r$ and $\hat{V}_l$ can be extended into the upper and lower left quadrants, respectively. The term $\hat{V}_c$ can be removed from the factorization (4.12) via the transformation

$$L = M\delta^{-\sigma_3} = M \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}, \tag{4.16}$$

where δ solves the one dimensional RHP defined in (4.17) - (4.20).

The jump relation for L can be found as follows

$$\begin{aligned} M_+ &= M_- V \\ M_+\delta^{-\sigma_3} &= M_- V\delta^{-\sigma_3} \\ M_+\delta_+^{-\sigma_3} &= M_-\delta_-^{-\sigma_3}\delta_-^{\sigma_3} V\delta_+^{-\sigma_3} \\ L_+ &= L_-\delta_-^{\sigma_3} V\delta_+^{-\sigma_3}, \end{aligned}$$

where $\delta_\pm$ denotes the boundary values for δ as z approaches the real axis from the plus and minus sides.

Now, the jump matrix for L can be written in a factored form. For $z < z_0$ we have

$$\begin{aligned} \delta_-^{\sigma_3} V\delta_+^{-\sigma_3} &= \begin{pmatrix} 1 & 0 \\ \frac{r(z)}{1-|r(z)|^2} e^{it\theta}\delta_-^{-2} & 1 \end{pmatrix} \times \\ &\begin{pmatrix} \delta_-\delta_+^{-1}\left(1-|r(z)|^2\right) & 0 \\ 0 & \delta_+\delta_-^{-1}\left(1-|r(z)|^2\right)^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\frac{\overline{r(z)}}{1-|r(z)|^2} e^{-it\theta}\delta_+^2 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

The desired conditions for δ are

$$\delta \text{ is analytic off the real axis} \tag{4.17}$$

$$\delta \to 1 \qquad \text{as } z \to \infty \tag{4.18}$$

$$\delta_+ = \delta_- \left(1 - |r^\epsilon|^2\right) \qquad z < z_0 \tag{4.19}$$

$$\delta_+ = \delta_- \qquad z > z_0 \tag{4.20}$$

The solution to this scalar RHP (see [[**3**]]), is

$$\delta = exp\left(\frac{1}{2\pi i}\int_{-\infty}^{z_0} \frac{log\,(1-|r(s)|^2)}{s-z}ds\right). \tag{4.21}$$

The form of δ is shown in the following lemma.

Lemma 1. *δ can be written in the form:*

$$\delta = (z-z_0)^{i\nu}\,\epsilon^{i\nu/2}e^{h(z)},$$

with

$$i\nu = \frac{1}{2\pi i}log\left(1-|r(z_0)|^2\right),$$

and

$$h(z) = -\frac{1}{2\pi i}\int_{-\infty}^{\sqrt{\epsilon}z_0} log(\sqrt{\epsilon}z-\lambda)\frac{\partial}{\partial\lambda}log\left(1-\frac{16}{25}e^{-4\lambda^2}\right)d\lambda.$$

The function $h(z)$ satisfies the following properties:

- *$e^{h(z)-h(z_0)}$ is bounded uniformly in the complex z plane*
- *Suppose $z_0 \in \mathbb{R}$ is fixed and $z \in \Sigma_2 \cup \Sigma_3 \cup \Sigma_5 \cup \Sigma_6$ (as defined in Figure 4.1) such that $|z| \le \sqrt{2log\epsilon}$. Then*

$$h(z) - h(z_0) = O(\sqrt{\epsilon}(log\epsilon)^2),$$

as $\epsilon \to 0$.

The proof of this lemma is shown in appendix (A.1).

The next transformation is the following change of variables: $\omega = \sqrt{8t}(z - z_0)$. Define

$$N(\omega) = e^{-2itz_0^2\sigma_3} L(z_0 + \omega/\sqrt{8t}) e^{2itz_0^2\sigma_3}. \tag{4.22}$$

Under this latest transformation, we have

$$N_+(\omega) = N_-(\omega) e^{-i\omega^2 ad\sigma_3/4} \delta^{-ad\sigma_3} V_0(z_0 + \omega/\sqrt{8t}).$$

Note that in the ω variable, V_0 has the factorization

$$V_0(z_0 + \omega/\sqrt{8t}) = \begin{cases} V_l(z_0 + \omega/\sqrt{8t})\ V_r(z_0 + \omega/\sqrt{8t}) & \omega > 0 \\ \hat{V}_l(z_0 + \omega/\sqrt{8t})\ \hat{V}_r(z_0 + \omega/\sqrt{8t}) & \omega < 0. \end{cases}$$

The next transformation involves opening sectors, and changing the contour on which N has a jump. Define P so that

$$P(\omega) = N(\omega) \begin{cases} I & arg\ \omega \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right) \cup \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right) \\ V_r^{-1}(z_0 + \omega/\sqrt{8t}) & arg\ \omega \in \left(0, \frac{\pi}{4}\right) \\ V_l(z_0 + \omega/\sqrt{8t}) & arg\ \omega \in \left(\frac{7\pi}{4}, 2\pi\right) \\ \hat{V}_r^{-1}(z_0 + \omega/\sqrt{8t}) & arg\ \omega \in \left(\frac{3\pi}{4}, \pi\right) \\ \hat{V}_l(z_0 + \omega/\sqrt{8t}) & arg\ \omega \in \left(\pi, \frac{5\pi}{4}\right) \end{cases}. \tag{4.23}$$

This new matrix P will have jump matrices defined on the contour Σ (see, 4.1).

The matrix P still has identity asymptotics as $\omega \to \infty$ since N had identity asymptotics, and each of the matrices $V_r, V_l, \hat{V}_r$, and $\hat{V}_l$ have identity asymptotics in the appropriate region.

It can be shown that P has the following jumps

$$P_+ = P_- e^{-i\omega^2 ad\sigma_3/4} \delta^{-ad\sigma_3} \begin{cases} I & \omega \in \Sigma_1 \cup \Sigma_4 \\ V_r & \omega \in \Sigma_2 \\ \hat{V}_r & \omega \in \Sigma_3 \\ \hat{V}_l & \omega \in \Sigma_5 \\ V_l & \omega \in \Sigma_6 \end{cases} .$$

The next transformation uses the form of δ that was described in Theorem 1. In the z variable, δ has the form:

$$\delta = (z - z_0)^{i\nu} \epsilon^{i\nu/2} e^{h(z)}.$$

In the ω variable, this becomes

$$\delta = \omega^{i\nu} (8t)^{-i\nu/2} \epsilon^{i\nu/2} e^{h(z_0 + \omega/\sqrt{8t})}.$$

We now decompose δ into two parts. One of which, can be factored out of our problem as a constant (in ω).

$$\delta(\omega) = \delta_0 \, \delta_1(\omega),$$

with

$$\delta_0 = (8t)^{-i\nu/2} e^{h(z_0)} \epsilon^{i\nu/2},$$

and

$$\delta_1(\omega) = \omega^{i\nu} e^{h\left(z_0 + \omega/\sqrt{8t}\right) - h(z_0)}.$$

The factor δ_0 is constant in ω and can be factored out of our problem by defining

$$T = \delta_0^{-\sigma_3} P \delta_0^{\sigma_3}.$$

Notice that all of our transformations have preserved analyticity, and the identity asymptotics at infinity. The jump relation for T is as follows:

$$T_+(\omega) = T_-(\omega)\omega^{i\nu ad\sigma_3} e^{-i\omega^2 ad\sigma_3/4} e^{[h(z_0+\omega/(8t))-h(z_0)]ad\sigma_3} V_\Sigma(\omega),$$

where V_Σ is defined according to

$$V_\Sigma(\omega) = \begin{cases} I & \omega \in \Sigma_1 \cup \Sigma_4 \\ V_r(z_0 + \omega/\sqrt{8t}) & \omega \in \Sigma_2 \\ \hat{V}_r(z_0 + \omega/\sqrt{8t}) & \omega \in \Sigma_3 \\ \hat{V}_l(z_0 + \omega/\sqrt{8t}) & \omega \in \Sigma_5 \\ V_l(z_0 + \omega/\sqrt{8t}) & \omega \in \Sigma_6 \end{cases}.$$

The RHP for T is now very similar to the RHP defined by (4.5), (4.6), and (4.7); and solved in [**3**].

This completes the transformations we perform on M. In the next section, we will prove that T and M^{DZ} differ by something which is $O(\sqrt{\epsilon}(log\epsilon)^2)$. Following that we will outline a solution.

4.2. Comparison of Two RHP's

In this section we will prove that the jump matrices for T and M^{DZ} differ by $O(\sqrt{\epsilon}(log\epsilon)^2)$.

Recall, the RHP for T has been constructed as:

$$T(\omega) \quad \text{is analytic off of } \Sigma \tag{4.24}$$

$$T(\omega) = I + O\left(\frac{1}{\omega}\right) \qquad \omega \to \infty \tag{4.25}$$

$$T_+(\omega) = T_-(\omega) e^{-i\omega^2 ad\sigma_3/4} \omega^{i\nu ad\sigma_3} e^{[h(z_0+\omega/(8t))-h(z_0)]ad\sigma_3} V_\Sigma. \tag{4.26}$$

Both T and M^{DZ} have the same asymptotics, and are both analytic in the same regions. We will need to show that their jump matrices are close in norm.

THEOREM 4.2.1. *For $||\cdot||$ representing the L^1, L^2, or L^∞ norms, we have*

$$\left|\left|V_T - V^{MDZ}\right|\right| < C\sqrt{\epsilon}(log\epsilon)^2, \tag{4.27}$$

with

$$V_T = e^{-i\omega^2 ad\sigma_3/4} \omega^{i\nu ad\sigma_3} e^{\frac{1}{2\pi i}[h(z_0+\omega/(8t))-h(z_0)]ad\sigma_3} V_\Sigma,$$

and

$$V^{MDZ} = e^{-i\omega^2 ad\sigma_3/4} \omega^{i\nu ad\sigma_3} V^{DZ}.$$

PROOF. Suppose $z \in \Sigma_2$. Our reflection coefficient is $r(z) = \frac{4}{5} e^{-2\epsilon(z_0+\omega/\sqrt{8t})^2}$. Define $r^{DZ} = r(z_0)$, the constant used in [**3**]. The particular solution constructed depends on a parameter $r(z_0)$ which may be freely chosen, as long as $|r(z_0)| \leq 1$. We choose $r^{DZ} = \frac{4}{5}$.

It is sufficient to show (4.27) along one of the rays of Σ. The inequalities along the other rays follow by analogous reasoning. For $z \in \Sigma_1$ we have

$$V_T - V^{MDZ} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \eta,$$

where $\eta = e^{i\omega^2/2}\omega^{2i\nu}\left(e^{2h(z_0+\omega/\sqrt{8t})-2h(z_0)}r(\omega) - r^{DZ}\right)$.

The appendix is devoted to establishing several useful properties of $h(z)$. One such property is the approximation

$$e^{2\left(\left(h(z_0+\omega/\sqrt{8t})-h(z_0)\right)\right)} = 1 + O(\sqrt{\epsilon}(log\epsilon)^2), \tag{4.28}$$

for $\omega \in \Sigma_2 \cup \Sigma_3 \cup \Sigma_5 \cup \Sigma_6$ such that $|\omega| \leq \sqrt{2log\epsilon}$. If ω is unbounded, we only know that this quantity is bounded.

Since $\omega \in \Sigma_2$, we can substitute $\omega = e^{i\pi/4}s$, with $s > 0$ yielding

$$\eta = e^{-s^2/2}\omega^{2i\nu}\frac{4}{5}\left[e^{2\left(\left(h(z_0+e^{i\pi/4}s/\sqrt{8t})-h(z_0)\right)\right)}e^{-2i\epsilon s^2/8t-2\epsilon z_0^2-2\epsilon e^{i\pi/4}z_0 s/\sqrt{8t}} - 1\right]. \tag{4.29}$$

The matrix norm of $V_T - V^{MDZ}$ is bounded by

$$\begin{aligned} \left|\left|V_T - V^{MDZ}\right|\right| &\leq ||\eta|| \\ &\leq C\left|\left|e^{-s^2/2}\left[e^{2\left(\left(h(z_0+e^{i\pi/4}s/\sqrt{8t})-h(z_0)\right)\right)}e^{-2i\epsilon s^2/8t-2\epsilon z_0^2-2\epsilon e^{i\pi/4}z_0 s/\sqrt{8t}} - 1\right]\right|\right|. \end{aligned}$$

We choose $R(\epsilon)$ so that for $s > R(\epsilon)$ we know $|e^{-s^2/2}|$ is small. We know the terms in the square brackets in (4.29) are bounded. Recall, we want (4.27) for the L^1, L^2, and

L^∞ norms. For the L^∞ norm, we can choose $R(\epsilon)$ so that $e^{-R(\epsilon)^2/2} = \epsilon$. For the other norms, we choose $R(\epsilon)$ so that $\int_{s>R} |e^{-s^2/2}|^q ds < \epsilon$, where $q = 1, 2$.

If $\int_R^\infty |e^{-s^2/2}|^q ds < \epsilon$, then

$$\begin{aligned} \int_R^\infty e^{-s^2/2} ds &\leq \int_R^\infty s e^{-s^2/2} ds \\ &\leq -e^{-s^2/2}\Big|_{s=R}^{s=\infty} \\ &\leq e^{-R^2/2}. \end{aligned}$$

We again can choose $R(\epsilon)$ so that $e^{-R^2/2} = \epsilon$. Note that R is growing with ϵ like

$$R = \sqrt{2log\epsilon}.$$

Therefore, we have established that

$$||\eta||_{L^q(\mathbb{R}\backslash(-R(\epsilon),R(\epsilon)))} \leq \epsilon$$

for $q = 1, 2, \infty$. Our next task is to establish the bounds for η when $|s| < R(\epsilon)$.

For any of these norms, we have chosen $R(\epsilon)$ so that the Gaussian factor in (4.29) controlled the asymptotics for large s. For $|s| < R(\epsilon)$ the factor of (4.29) in square brackets will control the asymptotics.

For $|s| < R(\epsilon)$, we can use the bound (4.28). Note that $|e^{-s^2/2} \leq 1|$. Then we have

$$\begin{aligned} ||\eta|| &\leq C \left|\left| \left(1 + O(\sqrt{\epsilon}(log\epsilon)^2)\right) e^{-2i\epsilon s^2/8t - 2\epsilon z_0^2 - 2\epsilon e^{i\pi/4} z_0 s/\sqrt{8t}} - 1 \right|\right| \\ &\leq C \left|\left| e^{-2i\epsilon s^2/8t - 2\epsilon z_0^2 - 2\epsilon e^{i\pi/4} z_0 s/\sqrt{8t}} - 1 \right|\right| \\ &+ \hat{C}\sqrt{\epsilon}(log\epsilon)^2 \left|\left| e^{-2i\epsilon s^2/8t - 2\epsilon z_0^2 - 2\epsilon e^{i\pi/4} z_0 s/\sqrt{8t}} \right|\right|. \end{aligned}$$

Define

$$p = \left|\left| (e^{-2i\epsilon s^2/8t - 2\epsilon z_0^2 - 2\epsilon e^{i\pi/4} z_0 s/\sqrt{8t}} - 1) \right|\right| \ , \tag{4.30}$$

and

$$\zeta = -2i\epsilon s^2/8t - 2\epsilon z_0^2 - 2\epsilon e^{i\pi/4} z_0 s/\sqrt{8t}.$$

Thus, we can rewrite p as

$$p = \left|\left| \int_0^\zeta e^x dx \right|\right|.$$

We can bound p as follows

$$\begin{aligned} p &= \left|\left| \int_0^\zeta e^x dx \right|\right| \\ &\leq \left|\left| \, |\zeta| \sup_{|x|<|\zeta|} |e^x| \, \right|\right|. \end{aligned}$$

We know that there is a constant H so that

$$|\zeta| \leq H\epsilon R^2.$$

We can bound p by

$$p \leq \hat{H} \left|\left| \epsilon \ log\epsilon \ e^{\epsilon log\epsilon} \right|\right|.$$

Thus, we have that

$$||\eta|| \leq \hat{C}\sqrt{\epsilon}(log\epsilon)^2 + C\epsilon log\epsilon.$$

This completes the proof of our theorem. $\square$

4.3. Asymptotic Estimate of the RHP Yielding the Evolved Potential

In the previous section we proved that the difference of the jump matrices for T and M^{DZ} was $O(\sqrt{\epsilon}(log\epsilon)^2)$. We will now use the solution for M^{DZ} to find our solution.

We start by defining a new quantity

$$E := T(M^{DZ})^{-1}. \tag{4.31}$$

E solves a RHP:

$$E(\omega) \qquad \text{is analytic off of } \Sigma \tag{4.32}$$

$$E(\omega) \rightarrow I \qquad \omega \rightarrow \infty \tag{4.33}$$

$$E_+(\omega) = E_-(\omega)J. \tag{4.34}$$

To find J, we use the definition of E

$$\begin{aligned} J &= E_-^{-1}E_+ \\ &= I + \left(M_-^{DZ} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \left(M_-^{DZ}\right)^{-1} \right) \eta, \end{aligned}$$

where $\eta = e^{i\omega^2/2}\omega^{2i\nu}\left(e^{h(z_0+\omega/\sqrt{8t})-h(z_0)}r(z_0+\omega/\sqrt{8t}) - r^{DZ}\right)$.

We already know that M_-^{DZ} is bounded. Theorem 4.2.1 proved that $\eta = O(\sqrt{\epsilon}(log\epsilon)^2)$. Therefore, we have established the following inequalities:

$$||J - I||_{L^\infty} \leq C\sqrt{\epsilon}(log\epsilon)^2 \tag{4.35}$$

$$||J - I||_{L^1} \leq C\sqrt{\epsilon}(log\epsilon)^2 \tag{4.36}$$

$$||J - I||_{L^2} \leq C\sqrt{\epsilon}(log\epsilon)^2. \tag{4.37}$$

A standard procedure involving small norm Riemann Hilbert problems establishes the following result. (For a discussion of small norm RHP's, see [**3**])

THEOREM 4.3.1. *There exists E solving the RHP (4.32) - (4.34) in the L^2 sense. As $\omega \to \infty$ bounded away from the contour Σ, E has the following expansion:*

$$E = I + O\left(\frac{\sqrt{\epsilon}(log\epsilon)^2}{1 + |\omega|}\right).$$

Equipped with the solution E, we can find an explicit representation for T. Recall,

$$T = EM^{DZ}.$$

Deift and Zhou found an explicit solution for M^{DZ} in [**3**], and we can find E by Neumann series. Compiling all this information, we can find an explicit approximation of T.

We will invert each of our transformations to find the solution M^ϵ. For ω bounded away from the sectors $arg(\omega) \in (0, \pi/4) \cup (3\pi/4, 5\pi/4) \cup (7\pi/4, 2\pi)$ (see (4.31), (4.23), (4.22) and (4.16)) the following identities hold true:

$$
\begin{aligned}
T(\omega) &= E(\omega)M^{DZ}(\omega), \\
P(\omega) &= \delta_0^{\sigma_3}\left(E(\omega)M^{DZ}(\omega)\right)\delta_0^{-\sigma_3}, \\
N(\omega) &= \delta_0^{\sigma_3}\left(E(\omega)M^{DZ}(\omega)\right)\delta_0^{-\sigma_3}, \\
L(z) &= e^{2iz_0^2t\sigma_3}\delta_0^{\sigma_3}\left(E(\sqrt{8t}(z-z_0))M^{DZ}(\sqrt{8t}(z-z_0))\right)\delta_0^{-\sigma_3}e^{-2iz_0^2t\sigma_3}.
\end{aligned}
$$

It then follows, for ω bounded away from the boundaries of these sectors, that:

$$
M^\epsilon(z) = e^{2iz_0^2t\sigma_3}\delta_0^{\sigma_3}\left(E(\sqrt{8t}(z-z_0))M^{DZ}(\sqrt{8t}(z-z_0))\right)\delta_0^{-\sigma_3}e^{-2iz_0^2t\sigma_3}\delta^{\sigma_3} \tag{4.38}
$$

The solution to the NLS equation is given by

$$
\varphi = 2i \lim_{z\to\infty} z(M^\epsilon - I)_{12}.
$$

Thus, we need to find the $O(\frac{1}{z})$ term in our expansion. We made the variable change $z = z_0 + \omega/\sqrt{8t}$. As we have defined them, E and M^{DZ} are properly functions of ω. Extracting the $O(\frac{1}{z})$ term, we get

$$
(M_1^\epsilon)_{12} = \frac{1}{\sqrt{8t}}e^{4iz_0^2t}\delta_0^2(M_1^{DZ})_{12} + O\left(\sqrt{\epsilon}(log\epsilon)^2\right).
$$

Notice that δ^{σ_3} is diagonal, and $\delta = I + O\left(\frac{1}{z}\right)$. Therefore, it does not appear in this term.

Substituting the values for z_0 and δ_0, we get

$$2i(M_1^{\epsilon})_{12} = 2i\frac{1}{\sqrt{8t}}e^{\frac{ix^2}{4t}}(8t)^{-i\nu}e^{2h(z_0)}\epsilon^{i\nu}(M_1^{DZ})_{12} + O\left(\sqrt{\epsilon}(log\epsilon)^2\right).$$

After substituting (4.8) we arrive at our solutions:

$$\varphi(x,t;\epsilon) = 2i(M_1^{\epsilon})_{12} = \frac{1}{\sqrt{t}}e^{\frac{ix^2}{4t}-i\nu log(8t)}\epsilon^{i\nu}e^{2h(z_0)}\sqrt{\pi}\frac{5e^{i\pi/4-\pi\nu/2}}{4\Gamma(-i\nu)} + O\left(\sqrt{\epsilon}(log\epsilon)^2\right). \tag{4.39}$$

4.4. Remarks on the Solution

Since $\nu > 0$, the factor $\epsilon^{i\nu} = e^{i\nu log(\epsilon)}$ oscillates rapidly. In contrast to the linear case, there does not exist a limit when $\epsilon \to 0$ for the nonlinear case. However, the factor $\epsilon^{i\nu}$ in (4.39) is independent of x and t at first order. If we divide our solution by $\epsilon^{i\nu_0}$ we will remove the fast oscillations, arriving at

$$\tilde{\varphi}(x,t) = \frac{1}{\sqrt{t}}e^{\frac{ix^2}{4t}-i\nu log(8t)}e^{2h(z_0)}\sqrt{\pi}\frac{5e^{i\pi/4-\pi\nu/2}}{4\Gamma(-i\nu)} + O\left(\sqrt{\epsilon}(log\epsilon)^2\right). \tag{4.40}$$

Thus, we have arrived at a solution to the NLS equation that will have a limit as $\epsilon \to 0$. Suppose $\epsilon = 0$, and denote the leading order behavior of this solution by

$$\varphi_{asy}(x,t) = \frac{1}{\sqrt{t}}e^{\frac{ix^2}{4t}-i\nu_0 log(8t)}e^{2h_0}\sqrt{\pi}\frac{5e^{i\pi/4-\pi\nu_0/2}}{4\Gamma(-\alpha_0)}, \tag{4.41}$$

with

$$\nu_0 = \lim_{\epsilon\to 0}\nu = -\frac{1}{2\pi}log\left(\frac{9}{25}\right),$$

and

$$h_0 = \lim_{\epsilon\to 0}h(z).$$

It can be shown that φ_{asy} does not solve the NLS equation. A brief explanation of this surprising fact follows.

PROPOSITION 1.

$$\frac{\partial}{\partial x}\left(\lim_{\epsilon\to 0}\tilde{\varphi}\right) = \frac{\partial}{\partial x}\varphi_{asy} \quad \textit{is bounded.} \tag{4.42}$$

$$\lim_{\epsilon\to 0}\left(\frac{\partial}{\partial x}\tilde{\varphi}\right) \quad \textit{is unbounded.} \tag{4.43}$$

This proposition tell us that the operations of differentiation and the limit when epsilon goes to zero do not commute. It is for precisely this reason that $\tilde{\varphi}$ solves NLS equation (by construction) but φ_{asy} does not solve the NLS equation. We will outline a proof of the proposition (4.43) below. Straightforward calculations show that $\frac{\partial}{\partial x}\varphi_{asy}$ is bounded.

We will outline a proof of the (4.43). Recall in Theorem 2.3.1 we substituted the asymptotics for M into the equation

$$\frac{\partial}{\partial x}M = iz[M,\sigma_3] + \begin{pmatrix} 0 & \varphi \\ \overline{\varphi} & 0 \end{pmatrix} M$$

to get information about φ. Plugging in $M = I + z^{-1}M_1 + z^{-2}M_2 + \cdots$ into the above equation yielded (at first order) that $\varphi = 2i(M_1)_{12}$. The $O(z^{-2})$ term in this equation yields

$$\varphi_x = 4(M_2)_{12} + \varphi\int^x |\varphi(x')|^2 dx'. \tag{4.44}$$

This means we can extract information about φ_x directly from our Lax pair, without taking a derivative of our asymptotic result. We already know that φ is bounded. We need to see if $(M_2)_{12}$ is bounded or not. Recall the exact formula (4.38) for M:

$$M^\epsilon(z) = e^{2iz_0^2 t\sigma_3}\delta_0^{\sigma_3}\left(E(\sqrt{8t}(z-z_0))M^{DZ}(\sqrt{8t}(z-z_0))\right)\delta_0^{-\sigma_3}e^{-2iz_0^2 t\sigma_3}\delta^{\sigma_3}$$

To find $(M_2)_{12}$, we need to find all the $O(z^{-2})$ terms:

(4.45)

$$M_2^\epsilon(z) = e^{2iz_0^2 t\sigma_3}\delta_0^{\sigma_3}\left(E_2 + M_2^{DZ} + \delta_2^{\sigma_3} + E_1 M_1^{DZ} + E_1\delta_1^{\sigma_3} + M_1^{DZ}\delta_1^{\sigma_3}\right)\delta_0^{-\sigma_3}e^{-2iz_0^2 t\sigma_3},$$

where δ has the expansion

$$\delta(z) = 1 + \frac{\delta_1}{z} + \frac{\delta_2}{z}\cdots.$$

M_2^{DZ} is bounded. To see that E_2 is bounded, we can write

$$\begin{aligned} E &= I + \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{(I+\gamma)(J-I)}{s-z}ds \\ &= I - \frac{1}{2\pi i z}\int_{\mathbb{R}}(I+\gamma)(J-I)(1+s/z+s^2/z^2+\cdots)ds \end{aligned}$$

where γ has a small norm Neumann expansion. Then

$$E_2 = -\frac{1}{2\pi i}\int_{\mathbb{R}}(I+\gamma)(J-I)s\,ds.$$

We can factor the sup norm of $I+\gamma$ out of the integral above and so we need to prove that the quantity

$$||(J-I)s||_{L^p(ds)}$$

is small for $p = 1, 2, \infty$. This can be done in a manner analogous to the proof of Theorem 4.2.1 since the exponential decay will dominate the algebraically growing term, s, we've introduced.

Since $\delta_2^{\sigma_3}$ is diagonal, so it will not contribute to $(M_2)_{12}$. Consider the term involving $M_1^{DZ}\delta_1$. We know M_1^{DZ} is constant in z. We need to examine the δ_1 as it $\epsilon \to 0$. We can rewrite the integral in

$$\delta(z;\epsilon) = exp\left[\frac{1}{2\pi i}\int_{-\infty}^{z_0} \frac{log\left(1 - \frac{16}{25}e^{-4\epsilon s^2}\right)}{s-z} ds\right].$$

as:

$$\int_{-\infty}^{z_0} \frac{log\left(1 - \frac{16}{25}e^{-4\epsilon s^2}\right)}{s-z} ds = -\frac{1}{z}\int_{-\infty}^{z_0} log\left(1 - \frac{16}{25}e^{-4\epsilon s^2}\right)(1 + s/z + s^2/z^2 + \cdots)ds.$$

Thus,

$$\delta(z;\epsilon) = 1 - \frac{1}{2\pi i z}\int_{-\infty}^{z_0} log\left(1 - \frac{16}{25}e^{-4\epsilon s^2}\right) ds + \cdots .$$

Clearly the $O(z^{-1})$ term is blowing up when $\epsilon \to 0$. Thus $(M_2)_{12}$ is not bounded. Then equation (4.44) implies that $\tilde{\varphi}_x$ is not bounded. This explains precisely why our asymptotic approximation for φ does not solve the NLS equation.

It should also be noted that the expansion for φ is valid for ϵ small, and $t > 0$ fixed. It is not necessarily valid uniformly for $t \to 0$. But it can be shown to be valid for $t = \tau\epsilon^{\frac{1}{2}-\beta}$ with $0 < \beta < 1/2$. If we rescale t so that $t = \tau\epsilon^{\frac{1}{2}-\beta}$ then the solution becomes:

$$\varphi = \frac{\epsilon^{\beta/2-1/4}}{\tau} e^{\frac{ix^2}{4\tau\epsilon^{1/2-\beta}}} e^{i\nu log(\frac{\epsilon^{1/2+\beta}}{8\tau})} e^{2h\left(\frac{x}{4\tau\epsilon^{1/2-\beta}}\right)} \frac{5\sqrt{\pi}e^{i\pi/4-\pi\nu/2}}{4\Gamma(-i\nu)}.$$

The factor $\frac{\epsilon^{\beta/2-1/4}}{\tau}e^{\frac{ix^2}{4\tau\epsilon^{1/2-\beta}}}$ above converges to a Dirac mass when $\epsilon \to 0$ (up to a constant). However, the next factor will oscillate rapidly when $\epsilon \to 0$. The remaining factors tend towards a constant as $\epsilon \to 0$.

CHAPTER 5

Several Approaches to Finding Evolution Under the NLS Equation for Sequences Approximating Dirac Masses

In computing our reflection coefficient (see chapter 3), we imposed a formal rule to evaluate certain integrals. We then arrived at a sequence of reflection coefficients with parameter ϵ. These correspond to an unknown sequence of initial data.

In this chapter we will study particular sequences of initial data that converge to a Dirac mass. This will be done in two separate ways. The first is to study the NLS equation directly. This is done in section 5.1 using a series of transformations. We can also study the NLS equation using the associated Lax pair and the reflection coefficient. We study this in section 5.2. Each of these methods reveals a connection between long time asymptotics and sequences of initial conditions that converge to a Dirac mass.

In this chapter we will make use of long time asymptotics for the NLS (see [**2**], and the references therein). The behaviour of solutions is given as follows:

$$\psi(y,\tau) = \frac{1}{\sqrt{\tau}} e^{\frac{iy^2}{4\tau} - i\nu(z_0) log(8\tau)} u(z_0) + O\left(\frac{(log\tau)}{\tau}\right) \tag{5.1}$$

with

$$i\nu(z_0) = \frac{1}{2\pi i} log\left(1 - |\tilde{r}(z_0)|^2\right)$$

and where u is a function of the reflection coefficient, $\tilde{r}(\lambda)$, associated with initial conditions $f(y)$ and $z_0 = \frac{-y}{4\tau}$. The function u can be written in terms of its modulus and phase:

$$|u(z_0)|^2 = \nu/2 = -\frac{1}{4\pi} log(1 - |\tilde{r}(z_0)|^2)$$

$$arg\ u(z_0) = \frac{1}{\pi}\int_{-\infty}^{z_0} log(z_0 - s)d\ log(1 - |\tilde{r}(s)|^2) + \frac{\pi}{4} + arg\Gamma(i\nu) - arg\ \tilde{r}(z_0).$$

These asymptotics are valid for $\tau \to \infty$ and $|z_0| \leq C$ where C is a fixed constant.

5.1. Solutions via Long Time Asymptotics for a Particular Sequence of Initial Data

We want to find solutions to the NLS equation for a sequence of initial conditions that converge to a Dirac mass.

Consider a Schwartz class function, $f(x) \in S(\mathbb{R})$. If $\int_{\mathbb{R}} f(x)dx = 1$, then

$$\frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right)$$

converges to a Dirac mass (as a distribution) when $\epsilon \to 0$. We will look for solutions to the NLS equation with this type of initial condition:

$$i\varphi_t + \varphi_{xx} - 2|\varphi|^2\varphi = 0$$

$$\varphi(x, t = 0; \epsilon) = \frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right).$$

THEOREM 5.1.1. *For a sequence of initials conditions of the form* $\frac{1}{\epsilon}f\left(\frac{x}{\epsilon}\right)$*, the evolution under NLS gives the solutions:*

$$\varphi(x,t;\epsilon) = \frac{1}{\sqrt{t}}e^{\frac{ix^2}{4t}-i\nu log(8t)}\epsilon^{2i\nu}u(z_0) + O\left(\epsilon log(\epsilon)\right)$$

PROOF. Denote φ as the solution to the initial value problem:

$$\varphi_t + \varphi_{xx} - 2|\varphi|\varphi = 0$$

$$\varphi(t=0,x) = \frac{1}{\epsilon}f\left(\frac{x}{\epsilon}\right)$$

Using the variable changes

$$t = \epsilon^2\tau$$

$$x = \epsilon y$$

$$\epsilon\varphi = \psi$$

one can see that ψ solves the initial value problem

$$i\psi_\tau + \psi_{yy} - 2|\psi|^2\psi = 0$$

$$\psi(y,\tau=0) = f(y).$$

Recall, the long time asymptotics (5.1), yield

$$\psi(y,\tau) = \frac{1}{\sqrt{\tau}}e^{\frac{iy^2}{4\tau}-i\nu(z_0)log(8\tau)}u(z_0) + O\left(\frac{(log\tau)}{\tau}\right).$$

These asymptotics are valid when $\tau \to \infty$. If we change back to the x and t variables, we see that $|z_0| = \left|\frac{\epsilon x}{4t}\right|$. Thus, these asymptotics hold when $\epsilon \to 0$, for x and t in a compact set. Moreover, we can use ψ and it's asymptotics to find φ:

$$\begin{aligned}\varphi(x,t;\epsilon) &= \frac{1}{\epsilon}\psi\left(\frac{x}{\epsilon},\frac{t}{\epsilon^2}\right)\\ &= \frac{1}{\sqrt{t}}e^{\frac{ix^2}{4t}-i\nu log(8t/\epsilon^2)}u(z_0)+O\left(\frac{(log\tau}{\tau}\right)\\ &= \frac{1}{\sqrt{t}}e^{\frac{ix^2}{4t}-i\nu log(8t)}\epsilon^{2i\nu}u(z_0)+O\left(\epsilon log(\epsilon)\right).\end{aligned}$$

$\square$

Recall, in section (1.1), we considered the asymptotics when $t \to 0$. The solution above was found using long time asymptotics. These asymptotics are valid when $\left|\frac{\epsilon x}{4t}\right| \le C$. If we try to let $t = 0$ the point z_0 will no longer be bounded, and our asymptotics will be invalid. If we let $t = \epsilon$ and then let $\epsilon \to 0$, these asymptotics our valid, and are given by:

$$\varphi(x,t=\epsilon;\epsilon) = \frac{1}{\sqrt{\epsilon}}e^{\frac{ix^2}{4\epsilon}}e^{-i\nu log(8\epsilon)+2i\nu log(\epsilon)}u(z_0)+O\left(\epsilon log(\epsilon)\right)$$

Due to the fast oscillations that are present, φ will not converge to a Dirac mass if $t = \epsilon$ and $\epsilon \to 0$.

Remark: In this section we have established asymptotic descriptions of solutions for sequences of initial data converging to a Dirac mass. It should be noted that the asymptotic descriptions are not unique. Suppose we have two functions $f(x)$ and $g(x)$ with unit mass. Consider initial conditions of the form $\frac{1}{\epsilon}f\left(\frac{x}{\epsilon}\right)$ and $\frac{1}{\epsilon}g\left(\frac{x}{\epsilon}\right)$. These both converge to a Dirac mass as $\epsilon \to 0$. However, they yield two different solutions when evolved using the NLS equation. The solutions will be similar, but the function $u(z_0)$ is a function of the reflection coefficient. Since $f(x)$ and $g(x)$ will have different reflection coefficients associated to them, their corresponding solutions will differ. The factor $e^{-i\nu log(8\epsilon)+2i\nu log(\epsilon)}$ will be different because ν is a function of the reflection coefficient as well.

Note that in the (x,t) variables, the point z_0 scales with epsilon. In the next section we will see why this is the case.

5.2. Sequences Approximating Dirac Mass Initial Data, Studied Through the Reflection Coefficient

Using the existence theory in Appendix A.1 it can be shown that if the initial data is Schwartz class, then the reflection coefficient will also be Schwartz class. We will consider how the reflection coefficient scales when the initial data scales with ϵ as in the previous section.

LEMMA 2. *Suppose $\tilde{r}(\lambda)$ is the reflection coefficient associated with the initial condition $f(y)$. Then the reflection coefficient associated with the initial condition $\frac{1}{\epsilon}f\left(\frac{x}{\epsilon}\right)$ is $r(z) = \tilde{r}(\epsilon z)$.*

PROOF. Let $t = 0$, and consider the first Lax equation $L\Psi = z\Psi$. Rewritten this is:

$$i\sigma_3 \frac{\partial}{\partial x}\Psi + i\begin{pmatrix} 0 & -\varphi \\ \overline{\varphi} & 0 \end{pmatrix}\Psi = z\Psi.$$

Assume that the potential is of the form $\varphi(x) = \frac{1}{\epsilon}\psi\left(\frac{x}{\epsilon}, \frac{t}{\epsilon^2}\right)$ as in the previous section. If we define $y = x/\epsilon$ and $\tau = t/\epsilon^2$ then the first Lax equation becomes

$$i\sigma_3 \frac{\partial}{\partial y}\Psi + i\begin{pmatrix} 0 & -\psi(y,\tau) \\ \overline{\psi(y,\tau)} & 0 \end{pmatrix}\Psi = \epsilon z\Psi.$$

Recall the second Lax equation:

$$\Psi_t = 2z\Psi_x + i\begin{pmatrix} -|\varphi|^2 & \varphi_x \\ -\overline{\varphi_x} & |\varphi|^2 \end{pmatrix}\Psi + 2iz^2\Psi\sigma_3.$$

Using the same change of variables one sees that

$$\Psi_\tau = 2(\epsilon z)\Psi_y + i \begin{pmatrix} -|\psi|^2 & \psi_y \\ -\overline{\psi_y} & |\psi|^2 \end{pmatrix} \Psi + 2i(\epsilon z)^2 \Psi \sigma_3.$$

Thus, when the initial condition scales like $\frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right)$ this amounts to re-scaling of spectral variable so that the reflection coefficient is $r(z) = \tilde{r}(\epsilon z)$. □

Formal considerations, in chapter 3, led us to define a regularized reflection coefficient. This family of reflection coefficients corresponds to a family of initial conditions, however until now we did not know the form of the initial data. The following is a direct consequence of Lemma 2:

Corollary 1. *The regularized reflection coefficient*

$$r(z) = \frac{4}{5} e^{-\epsilon z^2}.$$

corresponds to initial data of the form $\frac{1}{\sqrt{\epsilon}} f\left(\frac{x}{\sqrt{\epsilon}}\right)$ *for some* $f \in \mathbb{S}(\mathbb{R})$.

5.3. Sequences of Initial Data with Variable Phase

In the previous section we considered scaled initial data that led to a scaled reflection coefficient. We will now consider what affect a shift will have on the spectral variable z.

Lemma 3. *Suppose* $\tilde{r}(\lambda)$ *is the reflection coefficient associated with the initial conditions* $f(y)$. *Then the reflection coefficient associated with the initial condition*

$$\varphi(x, t = 0) = \frac{1}{\epsilon} e^{-2i\alpha x} f\left(\frac{x}{\epsilon}\right),$$

with $\alpha \in \mathbb{R}$ *a constant, is* $r(z) = \tilde{r}(\epsilon(z - \alpha))$.

Proof. Suppose φ solves the NLS equation with

$$\varphi(x, t=0) = \frac{1}{\epsilon} e^{-2i\alpha x} f\left(\frac{x}{\epsilon}\right).$$

The solution has the following form

$$\varphi(x,t) = \frac{1}{\epsilon} e^{-2i\alpha x - 2i\omega t/\epsilon^2} \psi\left(\frac{x+4\alpha t}{\epsilon}, \frac{t}{\epsilon^2}\right)$$

where α is a real parameter, and ω is real, but yet to be determined and $\psi(\hat{y}, \hat{\tau})$ will be found using below, using long time asymptotics. Indeed, let $y = x/\epsilon$ and $\tau = t/\epsilon^2$. Inserting this form of φ into the NLS equation yields an equation for ψ:

$$i\psi_\tau - 4i\alpha\epsilon\psi_y + \psi_{yy} - 2|\psi|^2\psi = 0.$$

We chose $\omega = 2\alpha^2\epsilon^2$.

Let $y = \hat{y} - a\hat{\tau}$ and $\tau = \hat{\tau}$. Then $\frac{\partial}{\partial y} = \frac{\partial}{\partial \hat{y}}$ and $\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \hat{\tau}} + a\frac{\partial}{\partial \hat{y}}$. Choosing $a = 4\alpha\epsilon$, we find that $\psi(\hat{y}, \hat{\tau})$ solves the NLS initial value problem:

$$i\psi_{\hat{\tau}} + \psi_{\hat{y}\hat{y}} - 2|\psi|^2\psi = 0$$

$$\psi(\hat{y}, \hat{\tau} = 0) = f(\hat{y}).$$

Now that we know the form of the potential, we study the Lax equations to see how the spectral variable scales. Since φ solves the NLS equation there exists a Ψ solving the equations

$$i\sigma_3 \frac{\partial}{\partial x}\Psi + i\begin{pmatrix} 0 & -\varphi \\ \overline{\varphi} & 0 \end{pmatrix}\Psi = z\Psi$$

and

$$\Psi_t = 2z\Psi_x + i \begin{pmatrix} -|\varphi|^2 & \varphi_x \\ -\overline{\varphi_x} & |\varphi|^2 \end{pmatrix} \Psi + 2iz^2\Psi\sigma_3.$$

Define $\hat{\Psi} = e^{i\alpha x\sigma_3 + i\omega(t/\epsilon^2)\sigma_3}\Psi$. After making the transformations

$$\begin{aligned} y &= x/\epsilon \\ \tau &= t/\epsilon^2 \\ \varphi &= \frac{1}{\epsilon}e^{-2i\alpha x - 2i\omega t/\epsilon^2}\psi\left(\frac{x+4\alpha t}{\epsilon}, \frac{t}{\epsilon^2}\right) \end{aligned}$$

then $\hat{\Psi}$ solves the equation

$$i\sigma_3\frac{\partial}{\partial y}\hat{\Psi} + i \begin{pmatrix} 0 & -\psi \\ \overline{\psi} & 0 \end{pmatrix} \hat{\Psi} = \epsilon(z-\alpha)\hat{\Psi}.$$

We know Ψ solves the second Lax equation:

$$\Psi_t = 2z\Psi_x + i \begin{pmatrix} -|\varphi|^2 & \varphi_x \\ -\overline{\varphi_x} & |\varphi|^2 \end{pmatrix} \Psi + 2iz^2\Psi\sigma_3.$$

Making the same transformations, we see that

$$\hat{\Psi}_\tau = 2\epsilon z\hat{\Psi}_y + i \begin{pmatrix} -|\psi|^2 & \psi_y \\ -\overline{\psi}_y & |\psi|^2 \end{pmatrix} \hat{\Psi} + 2i\epsilon^2 z^2\hat{\Psi}\sigma_3 - 2i\epsilon^2 z\alpha\sigma_3\hat{\Psi} + i\omega\sigma_3\hat{\Psi} - 2\alpha\epsilon\sigma_3 \begin{pmatrix} 0 & -\psi \\ \overline{\psi} & 0 \end{pmatrix} \hat{\Psi}$$

Let

$$\begin{aligned} y &= \hat{y} - a\hat{\tau} \\ \tau &= \hat{\tau} \end{aligned}$$

with $a = 4\epsilon\alpha$. Recall that $\omega = 2\epsilon^2\alpha^2$. After these transformations we see that $\hat{\Psi}$ solves the equation:

$$\hat{\Psi}_{\hat{\tau}} = 2\epsilon(z-\alpha)\hat{\Psi}_{\hat{y}} + i\begin{pmatrix} -|\psi|^2 & \psi_{\hat{y}} \\ -\overline{\psi}_{\hat{y}} & |\psi|^2 \end{pmatrix}\hat{\Psi} + 2i(\epsilon(z-\alpha))^2\hat{\Psi}\sigma_3 + (4i\epsilon^2 z\alpha - 2i\epsilon^2\alpha^2)\hat{\Psi}\sigma_3.$$

Now, define

$$\tilde{\Psi} = \hat{\Psi}e^{-(4i\epsilon^2 z\alpha - 2i\epsilon^2\alpha^2)\sigma_3\hat{\tau}},$$

so that $\tilde{\Psi}$ solves the second Lax equation:

$$\tilde{\Psi}_{\hat{\tau}} = 2\epsilon(z-\alpha)\tilde{\Psi}_{\hat{y}} + i\begin{pmatrix} -|\psi|^2 & \psi_{\hat{y}} \\ -\overline{\psi}_{\hat{y}} & |\psi|^2 \end{pmatrix}\tilde{\Psi} + 2i(\epsilon(z-\alpha))^2\tilde{\Psi}\sigma_3.$$

Because of the nature of the transformations to $(\hat{y}, \hat{\tau})$ and $\tilde{\Psi}$ it can be easily shown that $\tilde{\Psi}$ solves the first Lax equation:

$$i\sigma_3\frac{\partial}{\partial y}\tilde{\Psi} + i\begin{pmatrix} 0 & -\psi \\ \overline{\psi} & 0 \end{pmatrix}\tilde{\Psi} = \epsilon(z-\alpha)\tilde{\Psi}.$$

with spectral parameter $\epsilon(z-\alpha)$.

Summarizing, we started with the Lax solution, Ψ, with φ as the potential in the (x,t) variables, and spectral parameter z. We derived a Lax solution with ψ as the potential in the $(\hat{y}, \hat{\tau})$ variables with spectral parameter $\lambda = \epsilon(z-\alpha)$. However, in order to conclude that ψ solves the NLS equation, we need $\tilde{\Psi}$ to have the proper asymptotics.

We show this by recalling Ψ solves the Lax pair, having the asymptotics:

$$\Psi e^{izx\sigma_3} \to I$$

as $x \to \infty$.

It follows that

$$\tilde{\Psi} e^{i\lambda\hat{y}\sigma_3} \to I$$

as $\hat{y} \to \infty$.

Indeed, if we reverse our transformations, we see that

$$\begin{aligned}
\tilde{\Psi} e^{i\lambda\hat{y}\sigma_3} &= exp\left[i\alpha x\sigma_3 + 2i\alpha^2 t\sigma_3\right] \hat{\Psi}\ exp\left[-4i\alpha z t\sigma_3 + 2i\alpha^2 t\sigma_3 + i\epsilon(z-\alpha)(y + 4\alpha\epsilon t/\epsilon^2)\sigma_3\right] \\
&= e^{i\alpha x\sigma_3 + 2i\alpha^2 t\sigma_3} \Psi e^{izx\sigma_3} e^{-i\alpha x\sigma_3 - 2i\alpha^2 t\sigma_3}.
\end{aligned}$$

Now letting $\hat{y} \to \infty$ we see that $\tilde{\Psi} e^{i\lambda\hat{y}\sigma_3} \to I$.

Now, in the variables $\hat{y}, \hat{\tau}$ and λ, we will denote the reflection coefficient corresponding to ψ by $\tilde{r}(\lambda)$. The explicit transformations presented above now show that the reflection coefficient corresponding to ϕ is given by $r(z) = \tilde{r}(\epsilon(z-\alpha))$. □

Using the long time asymptotics (5.1) yields the following theorem, as a direct result of the above lemma.

THEOREM 5.3.1. *For a sequence of initials conditions of the form* $\frac{1}{\epsilon} e^{-2i\alpha x} f\left(\frac{x}{\epsilon}\right)$, *the evolution under NLS gives the solutions:*

$$\varphi(x,t) = \frac{1}{\sqrt{t}} e^{-2i\alpha x - 4i\alpha^2 t} e^{\frac{ix^2}{4t} - i\nu(z_0) log(8t/\epsilon^2)} u(z_0) + O\left(\epsilon log(\epsilon)\right),$$

where $z_0 = \frac{-y}{4\tau} = -\frac{\epsilon x}{4t}$. *This is valid when* $\epsilon \to 0$, *for* x *and* t *in a compact set.*

5.4. Remarks on the Solution

Recall, in the original problem we remarked (see section 4.4) that the asymptotic description of the solution did not solve the NLS equation, whereas the full expansion did (by construction). In this chapter we are considering a generalized version of the original problem, thus one expects that the asymptotic description of φ does not solve the NLS equation.

PROPOSITION 2. *Let φ represent the asymptotic description of a solution from theorem 5.1.1,*

$$\varphi(x,t;\epsilon) = \frac{1}{\sqrt{t}} e^{\frac{ix^2}{4t} - i\nu log(8t)} \epsilon^{2i\nu} u(z_0) + O\left(\epsilon log(\epsilon)\right).$$

Then

$$\lim_{\epsilon \to 0} \frac{\partial}{\partial t} \varphi \neq \frac{\partial}{\partial t} \lim_{\epsilon \to 0} \varphi$$

PROOF. If we let $\epsilon \to 0$, and then differentiate the asymptotic description of φ it is clearly bounded.

However, if we differentiate the full expansion, we get terms that are unbounded as $\epsilon \to 0$. Recall, we wrote the solution using the (y, τ) variables, as:

$$\psi(y,\tau) = \frac{1}{\sqrt{\tau}} e^{\frac{iy^2}{4\tau} - i\nu(z_0) log(8\tau)} u(z_0) + f_1(y,\tau) \frac{\log \tau}{\tau},$$

where $f_1(y, \tau)$ represents the higher order terms in the asymptotic expansion. Now, when we take a derivative in t, we should note

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} \tau \frac{\partial}{\partial \tau} \tag{5.2}$$

$$= \frac{1}{\epsilon^2} \frac{\partial}{\partial \tau} \tag{5.3}$$

When we differentiate the corrections above, one term will be

$$\left(\frac{\partial}{\partial t} f_1(y,\tau)\right)\frac{\log\tau}{\tau} = \frac{1}{\epsilon^2}\left(\frac{\partial}{\partial\tau} f_1(y,\tau)\right)\epsilon^2\log\epsilon + O(\epsilon log \epsilon) \tag{5.4}$$

$$= \log\epsilon\frac{\partial}{\partial\tau} f_1(y,\tau) + O(\epsilon\log\epsilon). \tag{5.5}$$

Since this term is unbounded when $\epsilon \to 0$, the full expansion is unbounded when a derivative (in t) is performed and then we let $\epsilon \to 0$. □

One should note that a similar calculation can be shown for derivatives in x.

APPENDIX A

Appendix

A.1. Properties of h(z)

Recall, in section 4.1 we stated the solution of a scalar RHP to be

$$\delta = exp\left[\frac{1}{2\pi i}\int_{-\infty}^{z_0}\frac{log\left(1-|r(s)|^2\right)}{s-z}ds\right].$$

In this appendix we will prove various properties of δ.

The function δ can be written in the form:

$$\delta = (z-z_0)^{i\nu}\,\epsilon^{i\nu/2}e^{h(z)},$$

with

$$i\nu = \frac{1}{2\pi i}log\left(1-|r(z_0)|^2\right), \tag{A.1}$$

and

$$\begin{aligned} h &= h(z) = h(z,z_0;\epsilon) \quad &\text{(A.2)}\\ &= -\frac{1}{2\pi i}\int_{-\infty}^{\sqrt{\epsilon}z_0} log(\sqrt{\epsilon}z-\lambda)\frac{\partial}{\partial\lambda}log\left(1-\frac{16}{25}e^{-4\lambda^2}\right)d\lambda. \quad &\text{(A.3)}\end{aligned}$$

The form of δ and the function $h(z)$ can be shown directly using integration by parts and straightforward calculus. We require certain bounds on the function h for small values of ϵ. These are described in the following theorem.

THEOREM A.1.1. *Let $z_0 \in \mathbb{R}$ be fixed. Suppose $z \in \Sigma$ such that $|z| \leq \sqrt{2log\epsilon}$. Then*

$$h(z) - h(z_0) = O(\sqrt{\epsilon}(log\epsilon)^2),$$

as $\epsilon \to 0$. Furthermore, $e^{h(z)-h(z_0)}$ is bounded uniformly in the complex z plane.

PROOF. For simplicity of notation, we make the substitutions $u = \sqrt{\epsilon}z$, and $v = \sqrt{\epsilon}z_0$.

The integrand has a logarithmic singularity at $\lambda = u$ which needs to be removed. We rewrite the desired difference as

$$\begin{aligned} h(z) - h(z_0) &= -\frac{1}{2\pi i}\int_{-\infty}^{v} log(u-\lambda)\frac{\partial}{\partial\lambda}log\left(1-\frac{16}{25}e^{-4\lambda^2}\right)d\lambda \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{v} log(v-\lambda)\frac{\partial}{\partial\lambda}log\left(1-\frac{16}{25}e^{-4\lambda^2}\right)d\lambda. \end{aligned}$$

Let c be a fixed negative parameter. We split the domain of integration into two separate intervals, $(-\infty, c)$ and (c, v). Integrating by parts on the bounded interval yields

$$h(u) - h(v) = -\frac{1}{2\pi i}\int_{-\infty}^{c} log\left(\frac{u-\lambda}{v-\lambda}\right)\frac{\partial}{\partial\lambda}log\left(1-\frac{16}{25}e^{-4\lambda^2}\right)d\lambda \tag{A.4}$$

$$+ C\frac{\lambda e^{-4\lambda^2}}{1-\frac{16}{25}e^{-4\lambda^2}}\left[(u-\lambda)log(u-\lambda)-(u-\lambda)\right]_c^{\lambda=v} \tag{A.5}$$

$$+ C\frac{\lambda e^{-4\lambda^2}}{1-\frac{16}{25}e^{-4\lambda^2}}\left[-(v-\lambda)log(v-\lambda)+(v-\lambda)\right]_c^{\lambda=v} \tag{A.6}$$

$$+ C\int_c^v\int_u^v log(s-\lambda)ds\frac{\partial^2}{\partial\lambda^2}log\left(1-\frac{16}{25}e^{-4\lambda^2}\right)d\lambda, \tag{A.7}$$

with

$$C = -\frac{64}{25\pi i}.$$

We can analyze this quantity term by term. Notice in the first term, we can rewrite the argument of the logarithm as

$$\frac{u-\lambda}{v-\lambda} = \frac{\lambda-\sqrt{\epsilon}z}{\lambda-\sqrt{\epsilon}z_0}.$$

An expansion in powers of $\sqrt{\epsilon}$ yields

$$\frac{u-\lambda}{v-\lambda} = 1-\sqrt{\epsilon}\frac{\sqrt{2log\epsilon}-z_0}{\lambda}+O(\epsilon log\epsilon). \tag{A.8}$$

This expansion is valid since $|\lambda| \geq |c|$.

If we substitue (A.8) into (A.4) and use the Taylor expansion of the logarithm, we get

$$\sqrt{\epsilon}\,C\int_{-\infty}^{c}\lambda^{-1}(\sqrt{2log\epsilon}-z_0)\frac{\partial}{\partial\lambda}log\left(1-\frac{16}{25}e^{-4\lambda^2}\right)d\lambda+O(\epsilon log\epsilon).$$

The integral exists and is bounded, therefore the first term (A.4) is $O(\sqrt{\epsilon log\epsilon})$.

The boundary terms, (A.5) and (A.6), arising from integration by parts, are $O(\sqrt{\epsilon})$. This is easily shown by evaluating these terms at $\lambda = v$ and $\lambda = c$, the endpoints of the interval of integration. After simplifying we arrive at

$$\begin{aligned}
& C\sqrt{\epsilon}\frac{z_0 e^{-4\epsilon z_0^2}}{1-\frac{16}{25}e^{-4\epsilon z_0^2}}\left(\sqrt{\epsilon}(z-z_0)log(\sqrt{\epsilon}(z-z_0)) - \sqrt{\epsilon}(z-z_0)\right) \\
+ \;\; & C\left[(u-c)log(u-c) - (u-c) - (v-c)log(v-c) + (v-c)\right] \\
\leq \;\; & \hat{C}\epsilon log\epsilon + C\left[u\; log(u-c) - v\; log(v-c) + (u-v) - c\; log\left(\frac{u-c}{v-c}\right)\right] \\
\leq \;\; & \hat{C}\epsilon\; log\epsilon + C\sqrt{\epsilon}z\; log(\sqrt{\epsilon}z - c) - \sqrt{\epsilon}z_0\; log(\sqrt{\epsilon}z_0 - c) + \sqrt{\epsilon}(z-z_0) - c\sqrt{\epsilon}\frac{z-z_0}{c} \\
\leq \;\; & O(\sqrt{\epsilon log\epsilon}).
\end{aligned}$$

We also used the expansion (A.8), which is again valid since $|\lambda| \geq |c|$.

The fourth term (A.7) encompasses the logarithmic singularity. Note that the logarithm and $V(\lambda)$ are integrable functions. The following lemma will prove the desired bounds for (A.7).

Lemma 4. *For $u = \sqrt{\epsilon}z$, $v = \sqrt{\epsilon}z_0$ and $\lambda \in (-\infty, c)$ we have that*

$$\left|\int_u^v log(s-\lambda)ds\right| \leq O(\sqrt{\epsilon}(log\epsilon)^2).$$

Proof. We can assume that $\lambda = \lambda_0\sqrt{\epsilon}$. If λ were any larger, $|s-\lambda|$ will be bounded away from the origin; thus, the logarithm will be bounded.

Using the definition of the complex logarithm, we have

$$log(s-\lambda) = log|s-\lambda| + i\theta, \tag{A.9}$$

where θ is an angle chosen in a suitable branch.

We now rewrite our integral using (A.9) as follows

$$\left|\int_u^v log(s-\lambda)ds\right| \leq \int_u^v |log(s-\lambda)| \; |ds| \tag{A.10}$$

$$\leq \int_u^v |log|s-\lambda|| \; |ds| + |\theta| \int_u^v |ds|. \tag{A.11}$$

The second term in (A.11) is bounded by

$$\begin{aligned} |\theta| \int_u^v |ds| &\leq |\theta| \; |u-v| \\ &\leq |\theta|\sqrt{\epsilon}|z-z_0| \\ &= O(\sqrt{\epsilon log \epsilon}). \end{aligned}$$

The integrand in the first term of (A.11) is decreasing since $s, \lambda = O(\sqrt{\epsilon})$. We know that $\lambda \in (c, v) \subset \mathbb{R}$ and $s \in (v, u) \subset \Sigma$. Using this geometry, we find

$$|s-\lambda| \geq |\Re(s) - \lambda|. \tag{A.12}$$

Parameterizing the contour using $s = v + t(u-v)$, and using (A.12) we have

$$\begin{aligned} \int_u^v |log|s-\lambda|| \, |ds| &\leq \int_u^v |log|\Re(s)-\lambda|| \; |ds| \\ &\leq |u-v| \int_0^1 |log|v + t(\Re(u) - v) - \lambda|| \, dt. \end{aligned}$$

Because we assumed $z \neq z_0$, then we know $\Re u \neq v$. Integrating yields

$$\begin{aligned}
\int_u^v |log|s-\lambda||\,|ds| &\leq |u-v| \int_0^1 |log|v + t(\Re(u) - v) - \lambda||\,dt \\
&\leq \frac{|u-v|}{|\Re(u)-v|}\left[(\Re(u)-\lambda)log(\Re(u)-\lambda) - (\Re(u)-\lambda)\right] \\
&- \frac{|u-v|}{|\Re(u)-v|}\left[(v-\lambda)log(v-\lambda) - (v-\lambda)\right].
\end{aligned}$$

Replacing $u = \sqrt{\epsilon}z,\ v = \sqrt{\epsilon}z_0$, we have that

$$\begin{aligned}
\int_u^v log|s-\lambda| &\leq \sqrt{\epsilon}\frac{|z-z_0|}{|\Re(z)-z_0|}\left[(\Re(z)-\lambda_0)log(\sqrt{\epsilon}(\Re(z)-\lambda_0)) - (\Re(z)-\lambda_0)\right] \\
&- \sqrt{\epsilon}\frac{|z-z_0|}{|\Re(z)-z_0|}\left[(z_0-\lambda_0)log(\sqrt{\epsilon}(z_0-\lambda_0)) - (z_0-\lambda_0)\right] \\
&\leq O(\sqrt{\epsilon}(log\epsilon)^2)
\end{aligned}$$

□

Thus, (A.7) is $O(\sqrt{\epsilon}(log\epsilon)^2)$. Previously, we showed that (A.4) - (A.6) were of higher order. This proves the desired result. □

A.2. General Existence Theory

In subsection (2.1) we assumed the support of φ was compact in order to prove Ψ existed. In this subsection we will prove that Ψ exists under more general assumptions. Proving that $M = \Psi e^{izx\sigma_3}$ exists is equivalent to proving that Ψ exists. We will prove the existence of M since the identity asymptotics that M possesses at infinity are more readily turned into integral equations normalized at infinity.

Suppose that $t \geq 0$ and $z \in \mathbb{R}$. Furthermore, suppose $\varphi \in \mathrm{L}^2(\mathbb{R})$ and the support of φ is non-compact. M solves the ODE

$$M_x = iz[M, \sigma_3] + \begin{pmatrix} 0 & \varphi \\ \overline{\varphi} & 0 \end{pmatrix} M. \tag{A.13}$$

Using the fact that

$$\frac{\partial}{\partial x}\left(e^{izx\sigma_3} M e^{-izx\sigma_3}\right) = e^{izx\sigma_3} M_x e^{-izx\sigma_3} - ize^{izx\sigma_3}[M, \sigma_3]e^{-izx\sigma_3},$$

one finds

$$\frac{\partial}{\partial x}\left(e^{izx\sigma_3} M e^{-izx\sigma_3}\right) = \begin{pmatrix} 0 & \varphi e^{2izx} \\ \overline{\varphi} e^{-2izx} & 0 \end{pmatrix} \left(e^{izx\sigma_3} M e^{-izx\sigma_3}\right). \tag{A.14}$$

From the normalization of M at infinity, and (A.14), we find the integral equation

$$M = I + \int_{\infty}^{x} \begin{pmatrix} 0 & \varphi e^{2iz(x'-x)} \\ \overline{\varphi} e^{-2iz(x'-x)} & 0 \end{pmatrix} M e^{-2iz(x'-x)} dx'. \tag{A.15}$$

To build the solution M in the complex z plane, we will construct M^+ and M^- for $z \in \mathbb{R}$. Then we will analytically extend columns of these matrices into the upper and lower half planes. M^+ and M^- are defined to solve (A.13) with the following normalization conditions:

$$M^+ \to I \qquad \text{as } x \to +\infty \tag{A.16}$$

$$M^- \to I \qquad \text{as } x \to -\infty. \tag{A.17}$$

This yields a pair of integral equations

$$M^{(\pm)} = I + \int_{\pm\infty}^{x} \begin{pmatrix} 0 & \varphi e^{2iz(x'-x)} \\ \overline{\varphi} e^{-2iz(x'-x)} & 0 \end{pmatrix} M^{(\pm)} e^{-2iz(x'-x)} dx'.$$

To prove that such solutions exist, we will use a standard contraction mapping argument. Consider

$$M^{(\pm)} = \begin{pmatrix} M_{11}^{\pm} & M_{12}^{\pm} \\ M_{21}^{\pm} & M_{22}^{\pm} \end{pmatrix} = (M_1|M_2)^{(\pm)}$$

with $M_1^{(\pm)}$, and $M_2^{(\pm)}$ being the columns of $M^{(\pm)}$. Then, M_2^{+} satisfies the following equation

$$M_2^{+} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{\infty}^{x} \begin{pmatrix} 0 & \varphi e^{4iz(x'-x)} \\ \overline{\varphi} & 0 \end{pmatrix} M_2^{+}\, dx'.$$

Since $z \in \mathbb{R}$, we know $|e^{2iz(x'-x)}| = 1$. We can analytically extend M_2^{+} into $\mathbb{C}_+$ since $x' > x$ implies $|e^{2iz(x'-x)}| \leq 1$ for $z \in \mathbb{C}$. For simplicity, let $u = M_2^{+}$ and

$$Q = \begin{pmatrix} 0 & \varphi e^{2iz(x'-x)} \\ \overline{\varphi} & 0 \end{pmatrix}.$$

Note that $||Q|| = const|\varphi|$.

Let u_0 be the zero vector, and define

$$u_{j+1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{\infty}^{x} Q u_j dx'.$$

Then, $u_n = \sum_{j=1}^{n} (u_j - u_{j-1})$. We claim that if $n \to \infty$ then u_n will converge. We can show this by controlling the size of $|u_j - u_{j-1}|$.

The following list of results will be required but not proven in this paper:

- For $j \geq 1$ we have $|u_j - u_{j-1}| \leq \frac{c}{j!}\left(\int_x^\infty \varphi dx'\right)^j$
- The series $u_n = \sum_{j=0}^n (u_j - u_{j-1})$ converges exponentially, and

$$|u| = |M_2^+| \leq Ce^{\int_\infty^x |\varphi|\, dx'}$$

- M_1^- exists and is bounded for $z \in \mathbb{R}$. M_1^- can also be extended analytically into the upper half plane.

One then constructs the columns of M, for $z \in \mathbb{C}_+$, from M_2^+ and M_1^-, since they can both be extended into the upper half plane. However, we need to make sure that the asymptotics in (2.8) are satisfied.

Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = M_1^-$. We need to show that $v_2 \to 0$ as $x \to \infty$. We already know v is bounded as $x \to \infty$.

$$v_2 = \int_{-\infty}^x \overline{\varphi}(x')e^{-2iz(x'-x)}v_1 dx'.$$

Assume that x is large, and let $\epsilon > 0$. There exists a $\mathbb{X}$ such that $|\int_{\mathbb{X}}^x \overline{\varphi}e^{-2iz(x'-x)}v_1 dx'| < \epsilon$. We know v_1 is bounded and $|e^{-2iz(x'-x)}| \leq 1$ since $z \in \mathbb{C}_+$ and $x' \leq x$. Thus we can choose such an $\mathbb{X}$ so that

$$\int_{\mathbb{X}}^\infty |\overline{\varphi}|dx' < \epsilon. \tag{A.18}$$

We can split the integral equation for v_2 into 2 parts as follows

$$v_2 = \int_{-\infty}^{\mathbb{X}} \overline{\varphi}e^{-2iz(x'-x)}v_1 dx' + \int_{\mathbb{X}}^x \overline{\varphi}e^{-2iz(x'-x)}v_1 dx'.$$

By the Lebesgue Dominated Convergence theorem we have $\int_{-\infty}^{\mathbb{X}} \overline{\varphi} e^{-2iz(x'-x)} v_1 dx' \to 0$ as $x \to \infty$. More precisely, there exists a $\mathbb{Y}$ so that

$$\left| \int_{-\infty}^{\mathbb{X}} \overline{\varphi} e^{-2iz(x'-x)} v_1 dx' \right| < \epsilon, \tag{A.19}$$

when $x > \mathbb{Y}$. Let $\hat{\mathbb{X}} = max(\mathbb{X}, \mathbb{Y})$. Then (A.18) and (A.19) yields

$$\begin{aligned} |v_2| &\leq \left| \int_{-\infty}^{\mathbb{X}} \overline{\varphi} e^{-2iz(x'-x)} v_1 dx' \right| + \left| \int_{\mathbb{X}}^{x} \overline{\varphi} e^{-2iz(x'-x)} v_1 dx' \right| \\ &\leq \left| \int_{-\infty}^{\mathbb{X}} \overline{\varphi} e^{-2iz(x'-x)} v_1 dx' \right| + \int_{\mathbb{X}}^{\infty} |\overline{\varphi}| dx' \\ &< 2\epsilon, \end{aligned}$$

when $x > \hat{\mathbb{X}}$. Thus $v_2 \to 0$ when $x \to \infty$.

We already know that v_1 is bounded. It will converge to a constant as $x \to \infty$. Thus, we have shown that

$$\left(M_1^- | M_2^+ \right) \to \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix},$$

when $x \to \infty$.

Now, we define our solution

$$M := \left(\frac{M_1^-}{a(z)} \middle| M_2^+ \right), \tag{A.20}$$

where $a(z) = det \left(M_1^- | M_2^+ \right) = 1 + \int_{\mathbb{R}} \varphi M_{21}^+ dx'$. We will show that $a(z)$ is non-vanishing so that this construction of M is valid.

THEOREM A.2.1. *Let φ be Schwartz class at $t = 0$. Then there exists a unique solution M to (A.15).*

PROOF. We've already proven that under the above assumptions $M^{\pm}$ exist. We need to show that $a(z)$ is non-vanishing.

The following properties hold for all $z \in \mathbb{R}$, and can be shown using (2.3),

$$detM^{+} = detM^{-} = 1, \tag{A.21}$$

$$M^{\pm}(z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overline{M^{\pm}(\overline{z})} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{A.22}$$

$$M^{+} = M^{-} \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}. \tag{A.23}$$

LEMMA 5. *For all $z \in \mathbb{C}$, $a(z) \neq 0$.*

PROOF. First, let $z \in \mathbb{C} \setminus \mathbb{R}$, and suppose $a(z) = 0$. Since $a(z) = det\ \left(M_1^{-}|M_2^{+}\right)$, these columns are linearly dependent. This means we can write $M_1^{-} = const(M_2^{+})$. Now M_1^{-} is bounded as $x \to \pm\infty$. Therefore, M_1^{-} is an eigenfunction of L, with eigenvalue z. Since L is a self adjoint operator we have a contradiction. Therefore, $a(z) \neq 0$ for $z \in \mathbb{C} \setminus \mathbb{R}$.

Let $z \in \mathbb{R}$. By (A.23) M_2^{+} is related to M^{-} as follows:

$$M_2^{+} = \begin{pmatrix} \beta M_{11}^{-} + \overline{\alpha} M_{12}^{-} \\ \beta M_{21}^{-} + \overline{\alpha} M_{22}^{-} \end{pmatrix}.$$

Now, $a(z)$ can be rewritten in terms of M^-

$$\begin{aligned} a(z) &= det\left(M_1^- \,\middle|\, M_2^+\right) \\ &= \overline{\alpha}\, det\, M^-. \end{aligned}$$

From (A.21) we know $det\ M^- = 1$, so $a(z) = \overline{\alpha}$.

Using (A.23) and (A.21) we have that

$$\begin{aligned} 1 &= det \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \\ &= |\alpha|^2 - |\beta|^2. \end{aligned}$$

Therefore

$$|\alpha|^2 = 1 + |\beta|^2 \geq 1,$$

and we know $a(z)$ is non-vanishing for $z \in \mathbb{R}$. □

Thus, the construction of M is valid and unique, and so we have proved our theorem.

□

Remark: We did not need to use the fact that φ was Schwartz class. We could therefore relax the conditions on φ, as in [**4**] where Deift and Zhou assume only that φ is in a weighted Sobolev space.

Now, since $a(z) \neq 0$, it follows that $det M = 1$. Furthermore, we have constructed a solution to (A.13) that is valid in the upper half plane. We can construct a solution valid in the lower half plane by defining

$$M := \left(M_1^+ \left| \frac{M_2^-}{b(z)} \right. \right),$$

where $b(z) = det\left(M_1^+|M_2^-\right)$.

Since M has the same symmetry as $M^{\pm}$:

$$M(z) = \begin{pmatrix} 0 & 1 \\ & \\ 1 & 0 \end{pmatrix} \overline{M(\overline{z})} \begin{pmatrix} 0 & 1 \\ & \\ 1 & 0 \end{pmatrix},$$

we find an alternate way of defining M in the lower half plane.

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