RANK REDUCTION OF CONFORMAL BLOCKS

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ABSTRACT

Michael Schuster: Rank Reduction of Conformal Blocks
(Under the direction of Prakash Belkale)

Let X be a smooth, pointed Riemann surface of genus zero, and G a simple, simply-connected complex algebraic group. Associated to a finite number of weights of G and a level is a vector space called the space of conformal blocks, and a vector bundle over $\overline{\mathrm{M}}_{0,n}$. We show that, assuming the weights are on a regular facet of the multiplicative polytope, the space of conformal blocks is isomorphic to a product of conformal blocks over groups of lower rank. If the weights are on a classical wall, then we also show that there is an isomorphism of conformal blocks bundles, giving an explicit relation between the associated nef divisors. The methods of the proof are geometric, and use the identification of conformal blocks with spaces of generalized theta functions, and the moduli stacks of parahoric bundles recently studied by Balaji and Seshadri. We conclude this dissertation with a number of examples in types A and C.

To my family.

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INTRODUCTION

Let G be a simple, simply connected algebraic group over $\mathbb{C}$ of rank r. Choose a maximal torus T and Borel subgroup B of G. Let $K \subseteq G$ be a maximal compact subgroup. Let $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{h} \subseteq \mathfrak{g}$ the Cartan subalgebra associated to T. Let $\alpha_1, \ldots, \alpha_r \in \mathfrak{h}^*$ be the simple roots of $\mathfrak{g}$, and let θ be the highest root. Then the *fundamental alcove* $\mathcal{A}$ is defined as follows:

$$\mathcal{A} = \{\mu \in \mathfrak{h} | \theta(\mu) \leq 1, \alpha_i(\mu) \geq 0 \text{ for all } i\}.$$

The *multiplicative polytope* is defined as

$$\Delta_n = \{\vec{\mu} \in \mathcal{A}^n | \mathrm{Id} \in C(\mu_1) \cdots C(\mu_n)\},$$

where $C(\mu)$ denotes the conjugacy class of $\exp(2\pi i \mu)$ in K. This subset was shown to be a convex polytope in [39] using symplectic methods.

The set $\mathcal{A}^n$ also parametrizes other objects, such the holonomy of flat K-bundles, line bundles over certain moduli spaces, and vector spaces from physics called *conformal blocks*. For each of these there is a corresponding existence problem solved by the multiplicative polytope. Conformal blocks also form vector bundles over families of stable pointed curves, and in genus zero these vector bundles correspond to nef divisors on the moduli space $\overline{\mathrm{M}}_{0,n}$ of genus zero stable pointed curves. Fakhruddin recently proved formulas for conformal blocks divisors in terms of boundary divisors, generating interest in using conformal blocks divisors to study the geometry of $\overline{\mathrm{M}}_{0,n}$ [17]. The goal of this dissertation is to study conformal blocks and their associated divisors when $\vec{\mu}$ is on a face of the multiplicative polytope.

0.1 Reduction rules on the regular faces

The effect of weights being on a facet of the multiplicative polytope is a reduction of the problem to a lower rank group. For example, if $\vec{\mu} \in \Delta_n$ is on a regular facet, then one can find $A_i \in C(\mu_i)$

such that $A_1 \cdots A_n = \mathrm{Id}$ and each A_i is block diagonal of the same dimensions. See Knutson's proof [31] of a similar result for Hermitian matrices.

More recently, factorization results have been proven for Littlewood-Richardson coefficients, first for SL_{n+1} by King, Tollu, and Toumazet [30], then in all types by Roth [44]. The assumption on the weights is that they lie on a face of the *additive eigencone*, which is an analogue of the multiplicative polytope for Lie algebras. Ressayre has generalized Roth's result to general branching coefficients [42].

In this dissertation we show that a similar result holds for *conformal blocks* in all Lie types. Conformal blocks are vector spaces originally arising in conformal field theory. They were later shown to be isomorphic to spaces of global sections of line bundles on the moduli stack of *parabolic bundles* over a smooth curve X - denoted Parbun_G - therefore giving a description of conformal blocks as *generalized theta functions*.

0.1.1 Regular faces of the multiplicative polytope

The problem of finding the inequalities defining the faces of Δ_n has a long history. The general form of the answer is that the inequalities are parametrized by certain cohomology products in the cohomology ring of Grassmannians G/P. It was first solved for SL_2 by Biswas [12], then by Agnihotri and Woodward for SL_n [1] and independently by Belkale in [6]; Belkale furthermore reduced the inequalities to an irredundant set. Teleman and Woodward [53] found inequalities defining Δ_n in general type, and more recently Belkale and Kumar [10] reduced these inequalities to an irredundant set, building on their work on the additive eigencone [9] and Ressayre's proof of the irredundancy of Belkale and Kumar's inequalities in [43].

More precisely, the cited works study the *regular* faces of Δ_n, that is, the faces that pass through the interior of $\mathcal{A}^n$. These faces correspond to a set of irredundant inequalities determining Δ_n. Teleman and Woodward showed that Δ_n is determined by a set of inequalities parametrized by (small) quantum cohomology products in $\mathrm{QH}^*(G/P)$ of the form $\sigma_{u_1} * \cdots * \sigma_{u_n} = q^d[pt]$ for all maximal parabolics P. In [10] Belkale and Kumar define a degeneration of the quantum product $\circledast_0$ that selects a subset of the inequalities determining Δ_n. Belkale and Kumar show that this set of inequalities is exactly the irredundant set of inequalities determining Δ_n:

Theorem 0.1.1. *([10]) The multiplicative polytope $\Delta_n \subseteq \mathcal{A}^n$ is determined by the following inequalities: for any maximal parabolic P, and Schubert classes $\sigma_{u_1}, \ldots, \sigma_{u_n}$ such that*

$$\sigma_{u_1} \circledast_0 \cdots \circledast_0 \sigma_{u_n} = q^d[pt]$$

any $\vec{\mu} \in \Delta_n$ must satisfy

$$\sum_{i=1}^{n} \omega_P(u_i^{-1}\mu_i) \leq d,$$

where ω_P denotes the fundamental weight associated to P. This set of inequalities is irredundant.

For a precise definition of $\circledast_0$ and further discussion of its relationship with the multiplicative polytope, see section 1.1.

0.1.2 Main theorem for degree zero walls

For the remainder of the introduction, assume $X \cong \mathbb{P}^1$. Conformal blocks take as input a choice of distinct points $p_1, \ldots, p_n \in X$, a finite number of dominant, integral weights $\lambda_1, \ldots, \lambda_n \in \mathfrak{h}^*$, and an integer ℓ called the level, and produce a finite dimensional vector space denoted $\mathcal{V}^\dagger_{\mathfrak{g},\vec{\lambda},\ell}(X, \vec{p})$. The dimension of the space of conformal blocks does not depend on the choice of points, and so generally we will suppress the pointed curve in our notation. Conformal blocks are defined as certain invariants related to the pointed curve of a tensor representation of the affine Kac-Moody Lie algebra associated to G (see section 1.2).

Now to weights $\lambda_1, \ldots, \lambda_n$ and a level ℓ we can associate points of the fundamental alcove $\mu_i = \frac{\kappa(\lambda_i)}{\ell} \in \mathcal{A}$, where $\kappa : \mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$ is the isomorphism induced by the (normalized) Killing form. Then the space $\mathcal{V}^\dagger_{\mathfrak{g},\vec{\lambda},\ell}$ has positive dimension (possibly after scaling the weights and level) if and only if the tuple $(\mu_1, \ldots, \mu_n)$ lies in the multiplicative polytope. This is only true when the genus of X is 0; otherwise conformal blocks are always nonzero after sufficient scaling (see [53]). We say that the weight data $\vec{w} = (\lambda_1, \ldots, \lambda_n, \ell)$ associated to a conformal block is on a facet of the multiplicative polytope if the associated $(\mu_1, \ldots, \mu_n)$ is on the facet.

Now assume $\vec{w}$ is on the regular facet of the multiplicative polytope associated to the cohomology product $\sigma_{u_1} \circledast_0 \cdots \circledast_0 \sigma_{u_n} = [\mathrm{pt}] \in \mathrm{QH}^*(G/P)$. Note that since $d = 0$, this corresponds to a product in the usual cohomology ring of G/P. Let $L \subseteq P$ be the Levi factor containing the chosen maximal torus T of G, and let $L' = [L, L]$. Then L' is semisimple and simply connected, and therefore is isomorphic to a product of simple groups; for simplicity we assume that there are two factors: $L' \cong G_1 \times G_2$. Then our main theorem gives an isomorphism between $\mathcal{V}^\dagger_{\mathfrak{g},\vec{\lambda},\ell}$ and conformal blocks associated to G_1 and G_2, which together have one less rank than G.

Theorem 0.1.2. *For weight data $\vec{w} = (\lambda_1, \ldots, \lambda_n, \ell)$ in the multiplicative polytope, lying on the facet corresponding to $\sigma_{u_1} \circledast_0 \cdots \circledast_0 \sigma_{u_n} = [pt] \in \mathrm{QH}^*(G/P)$, we have a natural isomorphism of conformal blocks*

$$\mathcal{V}^\dagger_{\mathfrak{g},\vec{w}} \cong \mathcal{V}^\dagger_{\mathfrak{g}_1,\vec{w}_1} \otimes \mathcal{V}^\dagger_{\mathfrak{g}_2,\vec{w}_2}$$

where $\vec{w}_1$ and $\vec{w}_2$ are the restrictions of following weight data to $\mathfrak{g}_1$ and $\mathfrak{g}_2$:

1. *Weights $u_1^{-1}\lambda_1, \ldots, u_n^{-1}\lambda_n$.*
2. *Levels $m_1\ell$ and $m_2\ell$, where m_1 and m_2 are the Dynkin indices of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ in $\mathfrak{g}$, respectively.*

Remark 0.1.3. The Dynkin indices m_1 and m_2 for simply-laced groups are always equal to 1.

Remark 0.1.4. In fact we prove the above theorem for products $\sigma_{u_1} * \cdots * \sigma_{u_n} = [\mathrm{pt}]$ and weights satisfying the corresponding inequality given in Theorem 0.1.1. For $G = \mathrm{SL}_{r+1}$ this is no generalization since the degenerated quantum product is the same as the usual one. However in other Lie types this is a weaker assumption than the one stated above, and the weights will lie on a face that is either not regular or not a facet.

Remark 0.1.5. In chapter 5 we restate this theorem for the groups SL_{r+1} and Sp_{2r}, and give a number of examples for those groups.

A simple argument shows that we can extend this isomorphism to conformal blocks *bundles*, which are vector bundles $\mathbb{V}_{\vec{w}}$ over the moduli space $\overline{\mathrm{M}}_{0,n}$ of genus zero stable pointed curves with n marked points, such that the fiber over $(X, p_1, \ldots, p_n) \in \overline{\mathrm{M}}_{0,n}$ is the dual of the space of conformal blocks, denoted $\mathcal{V}_{\mathfrak{g},\vec{w}}$. The proof uses the fact that in genus 0 these bundles are globally generated, and Roth's reduction theorem for invariants [17, 44].

Corollary 0.1.6. *With the same assumptions as above, we have an isomorphism of conformal blocks bundles on $\overline{\mathrm{M}}_{0,n}$:*

$$\mathbb{V}_{\mathfrak{g},\vec{w}} \cong \mathbb{V}_{\mathfrak{g}_1,\vec{w}_1} \otimes \mathbb{V}_{\mathfrak{g}_2,\vec{w}_2}.$$

Therefore the divisors $\mathbb{D}_{\mathfrak{g},\vec{w}}$ given by the first Chern classes of these vector bundles satisfy the relation

$$\mathbb{D}_{\mathfrak{g},\vec{w}} = rk(\mathbb{V}_{\mathfrak{g}_2,\vec{w}_2}) \cdot \mathbb{D}_{\mathfrak{g}_1,\vec{w}_1} + rk(\mathbb{V}_{\mathfrak{g}_1,\vec{w}_1}) \cdot \mathbb{D}_{\mathfrak{g}_2,\vec{w}_2}$$

Finally, since the inequality associated to $\sigma_{u_1} \circledast_0 \cdots \circledast_0 \sigma_{u_n} = [\mathrm{pt}]$ does not depend on the level ℓ, we can increase ℓ and the weight data will still be on the corresponding regular facet of the multiplicative polytope. In fact these faces also define a cone called the *additive eigencone* (see section 1.1). It is well known that at high enough level conformal blocks become isomorphic to spaces of invariants $\mathbb{A}_{\mathfrak{g},\vec{\lambda}} := (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^{\mathfrak{g}}$. Therefore as a final corollary we get Roth's reduction theorem for tensor product invariants:

Corollary 0.1.7. *([44]) Given weights $\vec{\lambda}$ lying on a facet of the additive eigencone corresponding to $\sigma_{u_1} \circledast_0 \cdots \circledast_0 \sigma_{u_n} = [pt]$, we have a canonical isomorphism of invariants:*

$$\mathbb{A}_{\mathfrak{g},\vec{\lambda}} \cong \mathbb{A}_{\mathfrak{g}_1,\vec{\lambda}_1} \otimes \mathbb{A}_{\mathfrak{g}_2,\vec{\lambda}_2},$$

where $\vec{\lambda}_1$, $\vec{\lambda}_2$ are given by the restrictions of $u_i^{-1}\lambda_i$ to $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

0.1.3 Reductions on positive degree faces

We also obtain reductions on the positive degree regular facets of the multiplicative polytope, however the statement is more complicated. We continue with the notation from the previous section. To the Levi subgroup $L \subseteq P$ we assign a degree k_L, which is the size of the kernel of the isogeny $Z_0 \to L/L'$, where Z_0 is the connected component of the identity of L. In [10], Belkale and Kumar showed the existence of a cocharacter μ_P lying in the fundamental alcove of L, such that $|\omega_P(\mu_P)| = 1$, where ω_P is the fundamental weight associated to P. Finally, let d_0 be the smallest integer such that $d + d_0\omega_P(\mu_P) \equiv 0 \pmod{k_L}$. Then by adding d_0 points to our pointed curve and twisting using the cocharacter μ_P, we obtain a rank reduction theorem for weight data on positive degree facets (see Proposition 4.2.3 for more details).

Theorem 0.1.8. *For weight data $\vec{w} = (\lambda_1, \ldots, \lambda_n, \ell)$ in the multiplicative polytope, lying on the facet corresponding to $\sigma_{u_1} \circledast_0 \cdots \circledast_0 \sigma_{u_n} = q^d[pt] \in \mathrm{QH}^*(G/P)$, we have a natural isomorphism of* *conformal blocks*

$$\mathcal{V}^{\dagger}_{\mathfrak{g},\vec{w}} \cong \mathcal{V}^{\dagger}_{\mathfrak{g}_1,\vec{w}_1} \otimes \mathcal{V}^{\dagger}_{\mathfrak{g}_2,\vec{w}_2}$$

where $\vec{w}_1$ and $\vec{w}_2$ are the restrictions of following weight data to $\mathfrak{g}_1$ and $\mathfrak{g}_2$:

1. *The first n weights are $u_1^{-1}\lambda_1, \ldots, u_n^{-1}\lambda_n$.*

2. *The last d_0 weights are $\ell\mu_P^*$, where μ_P^* is the dual with respect to the Killing form.*

3. *The levels are $m_1\ell$ and $m_2\ell$, where m_1 and m_2 are the Dynkin indices of $\mathfrak{g}_1 \times \mathfrak{g}_2$ in $\mathfrak{g}$.*

Remark 0.1.9. This result of course has no classical analogue for spaces of invariants. Furthermore Example 5.1.9 shows that this isomorphism does not extend to conformal blocks vector bundles in general. It would be interesting to know if there is a relationship between these bundles.

0.1.4 Conformal blocks over stacks of parahoric bundles

To prove these reduction theorems, we use the fact that spaces of conformal blocks can be canonically identified with spaces of generalized theta functions. More precisely, there is a line bundle $\mathcal{L}_{\vec{w}}$ over the moduli stack of parabolic bundles Parbun_G such that $\mathrm{H}^0(\mathrm{Parbun}_G, \mathcal{L}_{\vec{w}}) \cong \mathcal{V}^{\dagger}_{\mathfrak{g},\vec{w}}$. Parabolic bundles are principal G-bundles together with extra data over each point p_i, see section 1.2 for a precise definition. For arbitrary weight data, we need to work over a generalization of the stack of parabolic bundles: the moduli stack of parahoric bundles.

Parahoric bundles are torsors over a smooth group scheme $\mathcal{G} \to X$ determined by the choice of weight data $\vec{w}$. Moduli stacks of parahoric bundles are the natural setting in which work with conformal blocks as generalized theta functions when the weight data is on the affine wall of the alcove $\mathcal{A}$. When $G = \mathrm{SL}_{r+1}$, parahoric bundles can be identified with parabolic vector bundles where the underlying vector bundle has nonzero degree. For a general group, parahoric bundles can be more exotic.

Nevertheless, using the identification of parahoric bundles with equivariant bundles over a Galois cover $Y \to X$ [3], and analyzing the closed fibers of $\mathcal{G} \to X$, we have the following theorem.

Theorem 0.1.10. *Let $\mathcal{G}$ be the parahoric group scheme associated to the weight data $\vec{w}$ over a smooth projective curve of arbitrary genus X. Then the line bundle $\mathcal{L}_{\vec{w}}$ descends to* $\mathrm{Bun}_{\mathcal{G}}$, *and* $\mathrm{H}^0(\mathrm{Bun}_{\mathcal{G}}, \mathcal{L}_{\vec{w}})$ *is naturally isomorphic to the space of conformal blocks $\mathcal{V}^{\dagger}_{\mathfrak{g},\vec{w}}(X, \vec{p})$.*

Remark 0.1.11. To be more precise, the parahoric group schemes in the above theorem are the parahoric group schemes associated to the split group $G(K)$, where $K = k((z))$ is the field of formal Laurent series, and G is a simply connected linear algebraic group over $\mathbb{C}$. We do not prove this theorem for non-simply connected groups or attempt to generalize to quasi-split groups.

0.2 Outline of proof of the main theorem

Now we outline the proof of the reduction theorem. A more precise outline is given at the beginning of chapter 3, after parahoric bundles have been introduced. The proof is simplest when the weight data $\vec{w}$ lies in the interior of the alcove, and the degree of the wall is zero. Assume that the weights lie on a degree zero facet of the multiplicative polytope corresponding to a cohomology product $\sigma_{u_1} \circledast_0 \cdots \circledast_0 \sigma_{u_n} = [pt] \in \mathrm{QH}^*(G/P)$.

Now let Parbun_G be the moduli stack of parabolic bundles with full flags over $(X, p_1, \dots, p_n)$. To our weight data $\vec{w}$ we can associate a line bundle $\mathcal{L}_{\vec{w}}$ over Parbun_G. Then the space of conformal blocks $\mathcal{V}^{\dagger}_{\mathfrak{g},\vec{w}}$ can be identified with the space of global sections of $\mathcal{L}_{\vec{w}}$. The first step is to show that the following morphism of stacks induces an isomorphism of spaces of global sections of $\mathcal{L}_{\vec{w}}$ via pullback by

$$\mathrm{Parbun}_G \xleftarrow{\iota} \mathrm{Parbun}_L(0),$$

where the parabolic L-bundles are degree 0, and ι is given by extension of structure group and by twisting the flags over each p_i by u_i. The twisting by the u_i's makes the pullback $\iota^*\mathcal{L}_{\vec{w}}$ isomorphic to $\mathcal{L}_{\vec{w}'}$ where $\vec{w}'$ is the weight data described in the reduction theorem.

Therefore we want to show that global sections of $\mathcal{L}_{\vec{w}}$ over $\mathrm{Parbun}_L(0)$ extend to global sections of Parbun_G uniquely. To show this we use a method originally due to Ressayre [43]: we use another stack $\mathcal{C}$, with morphisms $\pi : \mathcal{C} \to \mathrm{Parbun}_G$, and $\xi : \mathcal{C} \to \mathrm{Parbun}_L(0)$.

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\pi"] \arrow[dr, "\xi"] & \\ \mathrm{Parbun}_G & \mathrm{Parbun}_L(0) \arrow[l, "\iota"'] \end{tikzcd}$$

The fibers of $\mathcal{C}$ over $\widetilde{\mathcal{E}} \in \mathrm{Parbun}_G$ are the degree d P-reductions of $\mathcal{E}$ with *relative position* $u_1, \dots, u_n$ (relative to each flag). Since $\sigma_{u_1} \circledast_0 \cdots \circledast_0 \sigma_{u_n} = [pt]$, generically π is one-to-one, and in fact is birational (see [10]). The morphism ξ in terms of vector bundles $W \subset V$ and flags $F_i^\bullet$ in type A is given by $(V, W, F_i^\bullet) \mapsto W \oplus V/W$ together natural induced flags on $W \oplus V/W$; this morphism is surjective. While the above diagram is not 2-commutative, the pullbacks of $\mathcal{L}_{\vec{w}}$ via π and $\iota \circ \xi$ can be identified over $\mathcal{C}$. This identification depends on the weight data $\vec{w}$ being on the facet corresponding to $\sigma_{u_1} \circledast_0 \cdots \circledast_0 \sigma_{u_n} = [pt]$.

The next step of the proof is to show that π is proper over the semistable locus of Parbun_G with respect to $\mathcal{L}_{\vec{w}}$. Starting with a one parameter family of semistable parabolic bundles $\mathcal{E} \to X \times C$ and a family of P-reductions of parabolic degree 0 over the punctured curve C^*, we embed and complete the family of P-reductions inside a Hilbert scheme. Then properness follows from a no-ghosts theorem proved by Holla and Narasimhan in [27]. Finally, we use Zariski's main theorem to show the pullback of global sections of $\mathcal{L}$ via π induces an isomorphism, and finish the reduction $\mathrm{H}^0(\mathrm{Parbun}_G, \mathcal{L}_{\vec{w}}) \xrightarrow{\sim} \mathrm{H}^0(\mathrm{Parbun}_L(0), \mathcal{L}_{\vec{w}'})$ with a simple diagram chase. For more details about the stack $\mathcal{C}$ and the proof of the properness of π over the semistable locus, see chapter 3. For the details about ι and ξ, and the reduction of conformal blocks to $\mathrm{Parbun}_L(0)$, see the beginning of chapter 4.

Finally, we need to reduce to the derived subgroup $L' = [L, L]$ to finish the proof of the reduction theorem. Again, we use a morphism of stacks

$$\mathrm{Parbun}_L(0) \xleftarrow{\iota'} \mathrm{Parbun}_{L'}$$

where ι' is given by extension of structure group. Then by a straightforward argument in section 7 of [10], since $\vec{w}$ is on the facet corresponding to $\sigma_{u_1} \circledast_0 \cdots \circledast_0 \sigma_{u_n} = [pt]$, ι' induces an isomorphism $\mathrm{H}^0(\mathrm{Parbun}_L(0), \mathcal{L}_{\vec{w}'}) \xrightarrow{\sim} \mathrm{H}^0(\mathrm{Parbun}_{L'}, \mathcal{L}_{\vec{w}'})$, finishing the proof of the reduction theorem, since $\mathrm{H}^0(\mathrm{Parbun}_{L'}, \mathcal{L}_{\vec{w}'})$ can be identified with a product of conformal blocks. This step requires more care when $d > 0$; for details see the discussion in chapter 4.

0.2.1 Modifications for general weights off the alcove wall

Now we drop the assumption that the weights are in the interior of the alcove. We call the linear faces of $\mathcal{A}$ the *chamber walls*, and the affine face the *alcove wall*. Assume first that one of the weights is on a chamber wall, but continue to assume they are off the alcove wall. Then we can define a stack $\mathcal{C}$ as above, but even restricted to semistable bundles, the morphism π is not proper over Parbun_G. To overcome this issue, we need to enlarge $\mathcal{C}$ to a stack $\mathcal{Y}$ which contains P-reductions that are certain specializations of P-reductions in $\mathcal{C}$. More precisely, the relative Schubert positions $w_1, \ldots, w_n$ of a P-reduction in $\mathcal{Y}$ all satisfy $w_i \leq u_i$, where $\leq$ denotes the Bruhat ordering, and $u_1, \ldots u_n$ are as above. Then $\mathcal{Y}$ is defined as the stack containing specializations of parabolic degree 0 with respect to $\vec{w}$ (see section 2.1 for the definition of parabolic degree). Note that every P-reduction in $\mathcal{C}$ has parabolic degree zero. Then by the same methods as above, the morphism $\mathcal{Y} \to \mathrm{Parbun}_G$ is proper over the semistable locus.

While enlarging $\mathcal{C}$ allows us to prove the properness result we need, unfortunately there is no extension of $\xi : \mathcal{C} \to \mathrm{Parbun}_L(0)$ to $\mathcal{Y}$. However, we can define a morphism $\xi : \mathcal{Y} \to \mathrm{Parbun}_L(\vec{Q}_L, 0)$, where $\mathrm{Parbun}_L(\vec{Q}_L, 0)$ is the stack of parabolic L-bundles with partial flags, i.e. flags $f_i \in L/Q_L$, for a parabolic subgroup Q_L. These morphisms fit into the following diagram

$$\begin{tikzcd}
\mathcal{Y} \arrow[d, "\pi"] \arrow[rrd, "\xi"] & & \\
\mathrm{Parbun}_G \arrow[r, "p"] & \mathrm{Parbun}_G(\vec{Q}) & \mathrm{Parbun}_L(\vec{Q}_L, 0) \arrow[l, "\iota"']
\end{tikzcd}$$

$\mathrm{Parbun}_G(\vec{Q}_L)$ is the stack of parabolic G-bundles with partial flags, p is the natural projection morphism, and ι is as in the previous section. It follows from the definition that the morphism p has connected and projective geometric fibers, and from this it is easy to see that the pullback of global sections of any line bundle induces an isomorphism. The proof of the reduction to $\mathrm{Parbun}_L(\vec{Q}, 0)$ then follows essentially as above. Finally, using a morphism $\iota' : \mathrm{Parbun}_{L'}(\vec{Q}_L) \to \mathrm{Parbun}_L(\vec{Q}_l, 0)$ we finish the proof of the reduction theorem in this case.

0.2.2 Weights on the alcove wall and parahoric bundles

For weights on the alcove wall descending to partial flags is not enough. In type A this can be resolved using the shifting technique in [7]. Given a parabolic vector bundle $\widetilde{V} = (V, F_i^\bullet)$, for each point p_i one can define a shifting operation, yielding a new parabolic bundle $\widetilde{V}'$ whose underlying vector bundle has degree one greater than V. This induces an isomorphism of stacks of parabolic bundles $\mathrm{Sh}_i : \mathrm{Parbun}(d) \xrightarrow{\sim} \mathrm{Parbun}(d+1)$.

Line bundles on both stacks are parametrized by weights $\lambda_1, \ldots, \lambda_n$, and a level; the effect of pulling back $\mathcal{L}_{\vec{\lambda},\ell}$ over $\mathrm{Parbun}(d+1)$ to $\mathrm{Parbun}(d)$ is to act on λ_i by an n^{th} root of unity corresponding to an element of the center of G. (Roots of unity act by automorphisms on the fundamental alcove of G.) Then if λ_i is on the alcove wall, by shifting enough times it is possible to move the weight off the alcove wall. If λ_i was on k chamber walls and the alcove wall, the effect of shifting is that λ_i is now on $k+1$ chamber walls, and off the alcove wall. Shifting over each p_i as needed, by working with positive degree vector bundles we can prove the reduction result in general using the above method for weights off the alcove wall.

In general type, the center of G is not large enough for shifting as described above to work. The solution is to descend to stacks of parahoric bundles, which are torsors over Bruhat-Tits group schemes. Let $A = k[[z]]$ and $K = k((z))$. Then to any weight data $\vec{w}$ we can associate parahoric subgoups $\mathcal{P}_1, \ldots, \mathcal{P}_n \subseteq G(k((z)))$, which determine a Bruhat-Tits group scheme $\mathcal{G} \to X$. Let $\mathrm{Bun}_{\mathcal{G}}$ be the moduli stack of $\mathcal{G}$-torsors. There is a natural morphism $\mathrm{Parbun}_G \to \mathrm{Bun}_{\mathcal{G}}$, and conformal blocks descend to this stack (cf. Theorem 0.1.10). When the weights are off the alcove wall this is just descent to partial flags since in this case $\mathrm{Bun}_{\mathcal{G}} \cong \mathrm{Parbun}_G(\vec{Q})$. On the alcove wall, $\mathcal{G}$-bundles are more exotic. Then the main diagram used to reduce down to the Levi L is the following.

$$\begin{tikzcd}
\mathcal{Y} \arrow[d, "\pi"] \arrow[drr, "\xi"] & & \\
\mathrm{Parbun}_G \arrow[r, "p"] & \mathrm{Bun}_{\mathcal{G}} & \mathrm{Bun}_{\mathcal{G}_L}(0) \arrow[l, "\iota"]
\end{tikzcd}$$

The stack $\mathcal{Y}$ again is an enlargement of $\mathcal{C}$, and the morphisms in this diagram are defined in a way analogous to the case for partial flags. Parahoric bundles thus allow us to prove Theorem 0.1.2 for any weights $\vec{w}$, following the strategy when the weights were assumed to be off the alcove wall. For more details on the construction of $\mathcal{Y}$ in the parahoric case, and proof of the properties of π and ξ, see chapter 3; we carry out the above strategy to prove the reduction theorem for arbitrary weight data and degree in chapter 4.

0.3 Outline of the dissertation

The dissertation is organized as follows. In chapter 1, we review some background material on the multiplicative polytope, quantum cohomology, conformal blocks, and parabolic bundles.

In chapter 2 we review the work of Balaji and Seshadri on parahoric bundles, semistability of parahoric bundles, and their moduli spaces, and prove Theorem 0.1.10. To show this we study the closed fiber of the Bruhat-Tits group schemes, and identify the fibers of the morphism $\mathrm{Parbun}_G \to \mathrm{Bun}_{\mathcal{G}}$ as certain flag varieties determined by the weight data.

In chapter 3 we start with an outline of the proof of the reduction theorem in terms of equivariant bundles. We then construct $\mathcal{Y}$, and show that $\pi : \mathcal{Y} \to \mathrm{Bun}_{\mathcal{G}}$ is proper over the semistable locus. Key to the properness proof is the lifting of families P-reductions of parabolic bundles to equivariant G-bundles over a ramified cover $Y \to X$. This process in general is discontinuous, but it is sufficient to assume that the family of P-reductions is of constant relative position.

In chapter 4 we use the results of the previous chapters to prove our main result as outlined above. We also prove that when $d = 0$, we get an isomorphism of conformal blocks bundles.

Finally, in chapter 5, we restate our main theorem for $G = \mathrm{SL}_{r+1}$ and Sp_{2r} and give some examples.

0.4 Notation and conventions

Let X be a smooth, connected and projective curve over $k = \mathbb{C}$ of genus g, and $p_1, \ldots, p_n$ be distinct points in X. Starting in chapter 3, we will assume $X \cong \mathbb{P}^1$. Assume we have sufficiently points to make the automorphism group of this pointed curve finite. All schemes and algebraic stacks are defined over k. For a morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$, we say f is representable if it is representable by schemes.

Let G be a simply connected simple algebraic group over k. Fix a Borel subgroup B, and a maximal torus T. Let $W = N(T)/T$ be the Weyl group. For a standard parabolic $P \supseteq B$ let $U = U_P$ be its unipotent radical, and let $L = L_P$ be the Levi subgroup of P containing T, so that P is a semi-direct product of L and U. Then $B_L = B \cap L$ is a Borel subgroup of L. We denote the Lie algebras of the groups G, B, P, U, L, B_L by $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{p}$, $\mathfrak{u}$, $\mathfrak{l}$, and $\mathfrak{b}_L$, respectively, and we denote by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}$ corresponding to T.

Let $R \subseteq \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$ with respect to the Cartan algebra $\mathfrak{h}$, and let R^+ be the set of positive roots (fixed by $\mathfrak{b}$). Let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be the set of simple roots, where r is the rank of G. Define the elements $x_i \in \mathfrak{h}$ by the equations $\alpha_i(x_j) = \delta_{ij}$ We denote the Killing form on $\mathfrak{h}$ and $\mathfrak{h}^*$

using angle brackets $\langle , \rangle$; assume it is normalized so that $\langle \theta, \theta \rangle = 2$, where θ is the highest root. The isomorphisms $\mathfrak{h} \to \mathfrak{h}^*$ $\mathfrak{h} \to \mathfrak{h}^*$ induced by the Killing form we will denote by κ. Define the coroots $\alpha_1^\vee, \ldots, \alpha_r^\vee \in \mathfrak{h}^*$ as $\alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$, and the fundamental weights $\omega_i \in \mathfrak{h}^*$ by $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$. Let $\mathfrak{h}_+$ and $\mathfrak{h}_+^*$ be the dominant Weyl chambers, and $\Lambda_+ \subseteq \mathfrak{h}_+^*$ be the semigroup of dominant integral weights. For a parabolic P, let Δ_P be the associated set of simple roots, and let R_L be the set of roots of $\mathfrak{l}$ with respect to $\mathfrak{h}$. If P is maximal, we will usually denote the excluded root and fundamental weight as α_P and ω_P, respectively. Let U_α denote the root group in G associated to α.

For any positive root β we will denote the associated reflection by s_β. Let $\{s_1, \ldots, s_r\} \in W$ be the generating set of simple reflections. For a parabolic subgroup P let W_P be the associated Weyl group, which is also the Weyl group of L. For every coset in W/W_P there is a unique minimal length representative. Let W^P be the set of minimal representatives. For any $w \in W$, we denote by $l(w)$ its length. The symbols $<, >, \leq, \geq$ will denote the Bruhat ordering in W.

For any $u \in W^P$ let $C_u = BuP \subseteq G/P$ be the Schubert cell associated to u and $X_u = \overline{BuP}$ the associated Schubert variety. Let $Z_u \subseteq X_u$ be the nonsingular locus. We denote by $\sigma_u \in \mathrm{H}^0(G/P)$ the Poincare dual of the homology class associated to X_u.

Let $\lambda_1, \ldots, \lambda_n$ be dominant integral weights and ℓ be a positive integer. We call the tuple $\vec{w} = (\lambda_1, \ldots, \lambda_n, \ell)$ the *Kac-Moody weight data* associated to $\lambda_1, \ldots, \lambda_n$ and ℓ. The integer ℓ is will be called the *level*. Given a weight λ and a level ℓ such that $\langle \lambda, \theta \rangle \leq \ell$, we can associate a point of the fundamental alcove

$$\mu = \frac{\kappa(\lambda)}{\ell}$$

and a dominant weight of the associated affine Kac-Moody algebra

$$\hat{\lambda} = (\ell - \langle \lambda, \theta \rangle)\omega_0 + \lambda.$$

Note that the second correspondence is one-to-one, but the first is not. We say that $\vec{w}$ is *in the multiplicative polytope* if $\vec{\mu} = (\frac{\kappa(\lambda_1)}{\ell}, \ldots, \frac{\kappa(\lambda_n)}{\ell})$ is in the multiplicative polytope.

CHAPTER 1: PRELIMINARIES ON THE MULTIPLICATIVE POLYTOPE AND CONFORMAL BLOCKS

In this expository chapter we introduce the main objects of interest in this dissertation. First we review the definition of the multiplicative polytope, and recall how the quantum cohomology of generalized Grassmannians G/P parametrize the faces of the polytope. We also relate the multiplicative polytope to the additive eigencone, and describe how integral points of this cone are related to invariants of tensor representations of G. In section 1.2, we give two definitions of conformal blocks: first as spaces of coinvariants, then as spaces of generalized theta functions. Conformal blocks are parametrized by the multiplicative polytope in the same way spaces of invariants are parametrized by the additive eigencone.

1.1 The multiplicative polytope and quantum cohomology

In this section we explain in more detail the definition of the multiplicative polytope, the inequalities defining its faces, and its classical analogues. As part of describing the faces we will also discuss the small quantum cohomology ring of the flag varieties G/P, and the degeneration of this ring that Belkale and Kumar used to describe the irredundant inequalities of the polytope.

1.1.1 The additive eigencone and the multiplicative polytope

First we describe the additive eigencone associated to G, which is a kind of classical analogue of the multiplicative polytope. The study of this problem goes back at least to Horn [18]. Let us first consider the algebra $\mathfrak{k}$ of Hermitian matrices. Every Hermitian matrix A has a unique set of real eigenvalues, which we denote by $\epsilon(A)$. Then the group K of unitary matrices acts on $\mathfrak{k}$ by conjugation, and ϵ is constant on each conjugacy class. A set of eigenvalues can be identified with a point in the dominant Weyl chamber of the Cartan algebra $\mathfrak{h}_+$ of $\mathfrak{k}$, and therefore we get a surjective map $\epsilon : \mathfrak{k}/K \to \mathfrak{h}_+$. The additive eigenvalue problem is as follows: for which sets of eigenvalues $\mu_1, \ldots, \mu_n \in \mathfrak{h}_+$ do there exist Hermitian matrices $A_1, \ldots, A_n$ such that $\epsilon(A_i) = \mu_i$ and $A_1 + \cdots + A_n = 0$?

We can generalize this to any Lie type as follows. Let $B \subseteq G$ be a Borel subgroup containing a maximal torus T, and let $K \subseteq G$ be a maximal compact subgroup such that $i\mathfrak{h}_{\mathbb{R}}$ is the Lie algebra of a maximal torus of K, where $\mathfrak{h}_{\mathbb{R}}$ is a real form of the Lie algebra $\mathfrak{h}$ of T. Then as for Hermitian matrices one can define an eigenvalue map $\epsilon : \mathfrak{k}/K \to \mathfrak{h}_+$, where K acts on its Lie algebra $\mathfrak{k}$ by the adjoint action. The eigenvalue problem is then: for which $\mu_1, \ldots, \mu_n \in \mathfrak{h}_+$ do there exist $A_1, \ldots, A_n \in \mathfrak{k}$ such that $\epsilon(A_i) = \mu_i$ and $A_1 + \cdots + A_n = 0$? Fixing n, it is well known that the set of tuples $(\mu_1, \ldots, \mu_n)$ satisfying this statement forms a full-dimensional convex polyhedral cone $\Gamma_n(G) \subseteq \mathfrak{h}_+^n$, called the *additive eigencone*. For more details and references, see Kumar's survey on this problem [33].

The multiplicative polytope is defined in a similar way. In type A, Hermitian matrices are replaced with unitary matrices, which have eigenvalues with complex norm 1. One can identify the eigenvalues of a unitary matrix A with a unique point in the *fundamental alcove* $\mathcal{A}$ of G, defined as

$$\mathcal{A} = \{\mu \in \mathfrak{h}_+ \mid \theta(\mu) \leq 1\}.$$

So as above, we have a well-defined eigenvalue map $\epsilon : K/\mathrm{Ad}K \to \mathcal{A}$. The multiplicative eigenvalue problem is then: for which $\mu_1, \ldots, \mu_n \in \mathcal{A}$ do there exist $A_1, \ldots, A_n \in K$ such that $\epsilon(A_i) = \mu_i$ and $A_1 \cdots A_n = \mathrm{Id}$? In general type one can also define an eigenvalue map ϵ, and the multiplicative eigenvalue problem is defined in the same way. One can show that the set of tuples $(\mu_1, \ldots, \mu_n)$ satisfying this statement forms a full-dimensional convex polyhedron $\Delta_n(G) \subseteq \mathcal{A}^n$ called the *multiplicative polytope* (see [39]).

There is a close relationship between the additive eigencone and the multiplicative polytope. Firstly, we have $\Delta_n(G) \subseteq \Gamma_n(G)$. Conversely, for any $\vec{\mu} \in \Gamma_n(G)$, there is a positive integer ℓ such that $\frac{1}{\ell}\vec{\mu}$ is contained in the multiplicative polytope. Therefore the *linear* faces (faces containing the origin) of the multiplicative polytope correspond exactly to the faces of the additive eigencone. This relationship can be proven using the exponential map and some basic differential geometry.

1.1.2 Quantum cohomology of G/P and the multiplicative polytope

Since the multiplicative polytope is a convex polyhedron, it is defined by a unique set of irredundant linear inequalities. The inequalities are parametrized by products in the small quantum cohomology ring of the flag varieties G/P, where P is a maximal parabolic. In type A these are the complex Grassmannians. Let us begin by reviewing the general type combinatorics of the cohomology of G/P.

For any flag variety G/P there is a canonical cell decomposition into *Schubert cells*. The Schubert cells are parametrized by cosets in W/W_P, where W is the Weyl group of G, and W_P is the Weyl group of P. These cosets each have a unique (minimal length) representative, the set of which is denoted W^P. We denote by C_w the Schubert cell corresponding to $w \in W^P$, and by $\sigma_w \in \mathrm{H}^*(G/P)$ the Poincaré dual of the homology class of C_w. It is well known that the cohomology ring $\mathrm{H}^*(G/P)$ is generated by the Schubert classes σ_w, and therefore the cohomology ring is determined by the set of positive numbers $c_{u,v}^w$ such that

$$\sigma_u \cdot \sigma_v = \sum_{w \in W^P} c_{u,v}^w \cdot \sigma_w^*.$$

The small quantum cohomology ring of G/P is defined as follows. Let $X = \mathbb{P}^1$ and fix 3 distinct points in X: say $p_1, p_2, p_3 \in X$. Then for any $d \geq 0$ and $w_1, w_2, w_3 \in W^P$, the *Gromov-Witten invariant* $\langle \sigma_{w_1}, \sigma_{w_2}, \sigma_{w_3} \rangle_d$ is defined as the number of degree d maps $f : X \to G/P$ such that $f(p_i) \in g_i C_{w_i}$ for generic $g_i \in G$, where the invariant is zero in the case that there are an infinite number of such maps. Additively the quantum cohomology ring is just $\mathrm{QH}^*(G/P) = \mathrm{H}^*(G/P)[q]$, where q is an indeterminant. The quantum product is then defined in terms of the Gromov-Witten invariants:

$$\sigma_{w_1} * \sigma_{w_2} = \sum_{\substack{d \geq 0 \\ w_3 \in W^P}} q^d \langle \sigma_{w_1}, \sigma_{w_2}, \sigma_{w_3} \rangle_d \cdot \sigma_{w_3}^*$$

where σ_w^* is the unique class such that $\sigma_w \cdot \sigma_w^* = [\mathrm{pt}]$. Since the image of a degree zero map $f : X \to G/P$ is just a point, by the uniqueness of σ_w^* it follows that $\langle \sigma_{w_1}, \sigma_{w_2}, \sigma_{w_3} \rangle_0 = c_{w_1,w_2}^{w_3}$.

Remarkably, the multiplicative polytope is determined by inequalities parametrized by products in $\mathrm{QH}^*(G/P)$. In [53], Teleman and Woodward proved the following.

Theorem 1.1.1. *([53]) For any maximal parabolic $P \subseteq G$, and any product $\sigma_{w_1} * \cdots * \sigma_{w_n} = q^d$[pt], a point $\vec{\mu} \in \Delta_n(G)$ must satisfy*

$$\sum_{i=1}^{n} \omega_P(w_i^{-1}\mu_i) \leq d,$$

where ω_P is the fundamental weight associated to P. The multiplicative polytope $\Delta_n(G)$ in the n-fold alcove $\mathcal{A}^n$ is determined by the above inequalities.

Corollary 1.1.2. *For any maximal parabolic $P \subseteq G$, and any product $\sigma_{w_1} \cdots \sigma_{w_n} =$ [pt], a point $\vec{\mu} \in \Gamma_n(G)$ must satisfy*

$$\sum_{i=1}^{n} \omega_P(w_i^{-1}\mu_i) \leq 0.$$

The additive eigencone $\Gamma_n(G)$ in $\mathfrak{h}_+^n$ is determined by the above inequalities.

While the above inequalities indeed determine the multiplicative polytope, they are not irredundant. The facets (codimension-one faces) intersecting the interior of $\mathcal{A}^n$ correspond to a subset of the products $\sigma_{w_1} * \cdots * \sigma_{w_n} = q^d[\mathrm{pt}]$. In the next section we will explain how the inequalities are reduced to an irredundant set. To facilitate discussion about the geometry of the multiplicative polytope, we make the following definitions.

Definition 1.1.3. The linear faces of $\mathcal{A}$ are called the *(Weyl) chamber walls*, and the affine face is called the *alcove wall.* A point in $\mathcal{A}^n$ is on a chamber or alcove wall if at least one μ_i in $(\mu_1, \ldots, \mu_n)$ is on a chamber or alcove wall. A *regular face* of $\Delta_n(G)$ is a face that intersects the interior of $\mathcal{A}^n$. If a regular facet of $\Delta_n(G)$ corresponds to a product $\sigma_{w_1} * \cdots * \sigma_{w_n} = q^d[\mathrm{pt}]$ in $\mathrm{QH}^*(G/P)$, we say that the facet is *degree d* and *of type P*.

1.1.3 Degeneration of cohomology products and irredundant inequalities

It remains to describe the irredundant set of inequalities corresponding to the regular facets of $\Delta_n(G)$. The solution is to *degenerate* the cohomology product described above so that the products corresponding to redundant inequalities become zero. Belkale and Kumar first defined this degeneration of the ordinary cohomology product in [9], and proved that the reduced set of inequalities are sufficient to define the multiplicative polytope. Ressayre then proved that these inequalities are irredundant in [43]. Finally, Belkale and Kumar built upon their work and Ressayre's to define a degenerate quantum product that gives rise to an irredundant set of inequalities for the multiplicative polytope [10].

The degeneration is based on the idea of *Levi-movable intersections.* Let $\Lambda_w = w^{-1}C_w$. Then if $\sigma_{w_1}\cdots\sigma_{w_n} = k[\mathrm{pt}]$, there exists, by Kleiman's transversality theorem, generic $p_1,\dots p_n \in P$ such that $\bigcap_i p_i\Lambda_i$ is transverse at the identity $\overline{e}\in G/P$. A product is Levi-movable if we can find $l_1,\dots,l_n$ in the Levi subgroup L of P such that $\bigcap_i l_i\Lambda_i$ is transverse at $\overline{e}$. The restriction to Levi-movable products reduces the above set of inequalities for the additive eigencone to an irredundant set in all types.

Belkale and Kumar show that the condition of being Levi-movable is an algebraic condition and can be expressed completely in terms of weights of G. For any $w\in W^P$, define χ_w as

$$\chi_w = \sum_{\beta\in(R^+\setminus R^+_L)\cap w^{-1}R^+}\beta.$$

Alternatively, χ_w can be shown (see [32, 1.3.22.3]) to be equal to $\rho - 2\rho^L + w^{-1}\rho$, where ρ and ρ^L are one-half the sums of the positive roots of G and L, respectively. Also, let $x_1,\dots,x_r\in\mathfrak{h}$ be the dual basis of $\alpha_1,\dots,\alpha_r$. Then Belkale and Kumar proved the following theorem.

Theorem 1.1.4. ([9]) *Suppose that $\sigma_{w_1}\cdots\sigma_{w_n} = k[\mathrm{pt}]$. Then the following inequality holds*

$$((\sum_{i=1}^n \chi_{w_i}) - \chi_1)(x_P) \le 0$$

and the product is Levi-movable if and only if this inequality is satisfied with equality.

For $u,v,w\in W^P$, let $A_P(u,v,w) = (\chi_1-\chi_u-\chi_v-\chi_w)(x_P)$. Then we can define a deformation of the normal cohomology product as follows. Additively we extend the cohomology ring to $\mathrm{H}^*(G/P)[\tau]$, where τ is an indeterminant. The product is given by

$$\sigma_u\odot\sigma_v = \sum_{w\in W^P}\tau^{A_P(u,v,w)}\cdot c_{u,v}^{w^*}\cdot\sigma_w$$

where $\sigma_w^* = \sigma_{w^*}$. This product gives $\mathrm{H}^*(G/P)[\tau]$ the structure of a commutative ring. Finally by setting τ to zero, we get the degeneration $\odot_0$ of the usual cohomology product. By definition this product is the result of throwing the summands of $\sigma_u\cdot\sigma_v$ such that $c_{u,v}^{w^*}>0$ and $A_P(u,v,w)>0$.

Remark 1.1.5. Belkale and Kumar prove that the product $\odot_0$ coincides with the usual cohomology product when G/P is *cominiscule*, which includes all Grassmannians, and Lagrangian Grassmannians, among other flag varieties. In general, a flag variety G/P is cominiscule if α_P appears with coefficient 1 in the highest root θ of G.

A similar approach gives a deformation and degeneration of the quantum cohomology product. For $u, v, w \in W^P$ and $d \geq 0$, let

$$A_P(u, v, w, d) = (\chi_1 - \chi_u - \chi_v - \chi_w)(x_P) + \frac{2g^* d}{\langle \alpha_P, \alpha_P \rangle}.$$

Then the deformed quantum product is given by

$$\sigma_{w_1} \circledast \sigma_{w_2} = \sum_{\substack{d \geq 0 \\ w_3 \in W^P}} \tau^{A_P(u,v,w,d)} q^d \langle \sigma_{w_1}, \sigma_{w_2}, \sigma_{w_3} \rangle_d \cdot \sigma_{w_3}^*.$$

By setting τ equal to zero, we get the product $\circledast_0$. In [10], Belkale and Kumar show that this product coincides with the usual quantum product when G/P is cominiscule. Their main theorem is that Levi-movable quantum products $\sigma_{u_1} \circledast_0 \cdots \circledast_0 \sigma_{u_n} = q^d[pt]$ exactly parametrize the irredundant regular facets of the multiplicative polytope, which we stated as Theorem 0.1.1 in the introduction.

1.2 Conformal blocks

In this section we define the main objects of interest in this dissertation: spaces and bundles of conformal blocks. We start with a review of the connection between the additive eigencone and invariants of tensor products of representations of G. Then we describe the construction of spaces of conformal blocks as certain invariant spaces of representations of infinite dimensional algebras related to the untwisted affine Kac-Moody algebra associated to G. In the last section, we introduce principal and parabolic G-bundles, and give an alternative definition of spaces of conformal blocks as spaces of generalized theta functions.

1.2.1 The additive eigencone and tensor representation invariants

Recall that for any dominant integral weight $\lambda \in \Lambda_+$ of G, there is a unique associated irreducible finite dimensional representation V_λ. Consider the following problem: for a tuple of such weights $(\lambda_1, \ldots, \lambda_n)$, when does the tensor representation $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$ have a non-trivial invariant subspace $\mathbb{A}_{\vec{\lambda}} = (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^G$? Remarkably, there is a very close connection between this problem and the additive eigenvalue problem.

Consider the following modification of the tensor invariants problem. For any dominant integral weight λ and integer $N > 0$, we can scale λ by N to get another dominant integral weight $N \cdot \lambda$. Then the *saturated tensor problem* is: for which tuple of weights $(\lambda_1, \ldots, \lambda_n)$ does there exist a positive integer N such that $\mathbb{A}_{N\vec{\lambda}} \neq \{0\}$? Let $\overline{\Gamma}_n(G)$ be the set of all such tuples of weights. This question is solved by the additive eigencone in the following way. The Killing form defines an isomorphism $\kappa : \mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$. Then $(\lambda_1, \ldots, \lambda_n) \in \overline{\Gamma}_n(G)$ if and only if $(\kappa(\lambda_1), \ldots, \kappa(\lambda_n))$ is in the additive eigencone. For more details and references, see [33].

Finally, we want to associate the above representation theory with some geometry, which is analogous to the theta function description of conformal blocks discussed below. Let $B \subseteq G$ be a Borel subgroup. Then G/B is a projective space called a generalized (full) flag variety. For any weight λ, the Borel-Weil theorem states that there is an equivariant line bundle L_λ on G/B such that the space global sections $\mathrm{H}^0(G/B, L_\lambda)$ is isomorphic to the irreducible representation V_λ^*, where we let G act on the global sections in the obvious way, and $*$ indicates the dual. More generally, if we have a tuple of weights $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$, then the tensor representation $(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^*$ can be realized as global sections of a line bundle $L_{\vec{\lambda}}$ over $X = \prod^n G/B$. Letting G act diagonally on X, the line bundle $L_{\vec{\lambda}}$ descends to the quotient stack $[X/G]$, and the space of global sections of $L_{\vec{\lambda}}$ over $[X/G]$ is exactly the dual of the space of invariants $\mathbb{A}^*_{\vec{\lambda}}$.

1.2.2 Conformal blocks as spaces of coinvariants

Let X be a smooth projective and connected algebraic curve over $k = \mathbb{C}$. We now give the definition of the space of conformal blocks over X in terms of representations of an infinite dimensional lie algebra $\hat{\mathfrak{g}}$. We will not use this definition in the rest of the dissertation. For more details on this definition see [4]; for a more comprehensive treatment of conformal blocks see [57];

for background on Kac-Moody algebras, see [29].

Let $K = \mathbb{C}((z))$ be the field of formal Laurent series with complex coefficients. Let $\hat{\mathfrak{g}} = (\mathfrak{g} \otimes K) \oplus \mathbb{C} \cdot c$. The bracket for $\hat{\mathfrak{g}}$ is given by

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + c \cdot \langle X, Y \rangle \mathrm{Res}(g \cdot \mathrm{d}f)$$

where $X, Y \in \mathfrak{g}$, $f, g \in K$, the product $\langle , \rangle$ is the normalized Killing form, and $\mathrm{Res}(g \cdot \mathrm{d}f)$ is the residue of $g \cdot \mathrm{d}f$. The vector c is central. This Lie algebra is a subalgebra of the completion of the untwisted affine Kac-Moody Lie algebra associated to G. Given a dominant integral weight λ, and an integer ℓ such that $\ell(\lambda) \leq \ell$, there is a unique associated simple $\hat{\mathfrak{g}}$-module $\mathcal{H}_{\lambda,\ell}$. Given a tuple of weights $(\lambda_1, \ldots, \lambda_n)$ and a level ℓ, we write $\mathcal{H}_{\vec{\lambda},\ell} = \mathcal{H}_{\lambda_1,\ell} \otimes \cdots \otimes \mathcal{H}_{\lambda_n,\ell}$.

The curve X, along with a collection of distinct points $p_1, \ldots, p_n \in X$ determines an action on $\mathcal{H}_{\vec{\lambda}}$ as follows. Let $X^* = X \setminus \{p_1, \ldots, p_n\}$. Then any function $f \in \mathcal{O}(X^*)$ determines $f_1, \ldots, f_n \in K$ by taking its Laurent series at each point. Thus given $X \otimes f \in \mathfrak{g}(X^*)$, we get an element $X \otimes f_i$ of the loop algebra $\mathfrak{g} \otimes K \subseteq \hat{\mathfrak{g}}$ for each i. The action of $\mathfrak{g}(X^*)$ on $\mathcal{H}_{\vec{\lambda}}$ is given in the obvious way:

$$(X \otimes f)(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} v_1 \otimes \cdots \otimes (X \otimes f_i) v_i \otimes \cdots \otimes v_n.$$

Then the space of conformal blocks is defined as follows.

Definition 1.2.1. For any tuple of weights $(\lambda_1, \ldots, \lambda_n)$ and level ℓ, the *space of conformal blocks* $\mathcal{V}^{\dagger}_{\vec{\lambda},\ell}(X, \vec{p})$ is defined as

$$\mathcal{V}^{\dagger}_{\vec{\lambda},\ell}(X, \vec{p}) = \mathrm{Hom}_{\mathfrak{g}(X^*)}(\mathcal{H}_{\vec{\lambda}}, \mathbb{C})$$

where $\mathbb{C}$ has the trivial $\mathfrak{g}(X^*)$-action. We denote the dual of this space as $\mathcal{V}_{\vec{\lambda},\ell}(X, \vec{p})$.

It is not obvious from the above definition, but the space of conformal blocks is in fact *finite dimensional.* Furthermore, the space of conformal blocks depends on the choice of points $p_1, \ldots, p_n \in X$, but remarkably its dimension does not. In fact the above definition works for families of pointed curves, and even allows degeneration to *stable pointed curves*, leading to the following definition.

Definition 1.2.2. For any tuple of weights $(\lambda_1, \ldots, \lambda_n)$, level ℓ, and genus g, the *bundle of conformal blocks* $\mathbb{V}_{\vec{\lambda},\ell}$ is the vector bundle over the moduli stack $\overline{\mathcal{M}}_{g,n}$ of genus g stable pointed curves with n

marked points, such that the fiber over $(X,\vec{p}) \in \overline{\mathcal{M}}_{g,n}$ is $\mathcal{V}_{\vec{\lambda},\ell}(X,\vec{p})$.

For the construction of the sheaf of conformal blocks for families of stable pointed curves, and for the proof that this sheaf is of finite rank and locally free over $\overline{\mathcal{M}}_{g,n}$, see [57].

Conformal blocks in genus zero are connected to the multiplicative polytope in the same way that tensor invariants are connected to the additive eigencone. The following theorem follows from the description of conformal blocks as generalized theta functions discussed below, and Theorem 5.2 in [10].

Theorem 1.2.3. *For any tuple of weights $(\lambda_1,\ldots,\lambda_n)$ and level ℓ, there exists an integer $N > 0$ such that $\dim(\mathcal{V}^{\dagger}_{N\vec{\lambda},N\ell}(X,\vec{p})) > 0$ if and only if $(\frac{\kappa(\lambda_1)}{\ell},\ldots,\frac{\kappa(\lambda_n)}{\ell}) \in \mathcal{A}^n$ lies in the multiplicative polytope.*

1.2.3 Conformal blocks as generalized theta functions

Let X be a scheme over k. Then a *principal G-bundle over X* is a G-scheme (with a right G-action) $\mathcal{E}$ together with a projection morphism $\pi : \mathcal{E} \to X$ such that $\mathcal{E}$ is locally trivial in the étale topology. In other words, there is a surjective étale cover $S \to X$ such that the pullback $\mathcal{E}_{|S}$ over S is isomorphic as a G-scheme to the trivial G-scheme $S \times G$.

Now fix a smooth, projective, and connected curve X over k. A *family of principal G-bundles over X* is simply a principal G-bundle over $X \times S$, for any k-scheme S. The category Bun_G of families of principal G-bundles over X forms a smooth algebraic stack over k. For an introduction to the moduli stack of G-bundles see Sorger's notes [48]; for detailed proofs of some of its basic geometric properties see Wang's senior thesis [58].

Now let $\vec{p} = (p_1,\ldots,p_n)$ be distinct closed points of X, and assume that G is semisimple, with the associated notations and conventions described in the introduction.

Definition 1.2.4. A *quasi-parabolic G-bundle over X (with full flags)* is a principal G-bundle $\mathcal{E} \to X$ together with choices of flags $\overline{g}_i \in (\mathcal{E}/B)_{p_i}$. A *family* of quasi-parabolic bundles parametrized by S is a principal G-bundle $\mathcal{E} \to X \times S$ together with sections $\overline{g}_i$ of $\mathcal{E}_{|p_i}/B \to S$ for each i. We denote the moduli stack of quasi-parabolic bundles $\mathrm{Parbun}_G(X,\vec{p})$, or more concisely as Parbun_G when X and $\vec{p}$ are understood.

We will often abuse terminology and call quasi-parabolic bundles simply parabolic bundles. The phrase “full flags” indicates that the flags are elements of a fiber of $\mathcal{E}$ modulo the Borel B, as opposed to “partial flags”, which would be elements of a fiber modulo a parabolic subgroup $Q \supset B$. The stack Parbun_G is again a smooth Artin stack, since the forgetful morphism $\mathrm{Parbun}_G \to \mathrm{Bun}_G$ is smooth.

Our methods rely on the fact that conformal blocks can be identified with spaces of *generalized theta functions*. Let $X(B)$ denote the character group of B. Then we have the following:

Theorem 1.2.5 ([37, 46])**.** *For any simple, simply-connected algebraic group G, there is a line bundle $\mathcal{L}$ on Parbun_G such that*

$$\mathrm{Pic}(\mathrm{Parbun}_G) \cong \mathbb{Z}\mathcal{L} \times \prod_{i=1}^{n} X(B).$$

Remark 1.2.6. In types A and C, $\mathcal{L}$ is a determinant of cohomology line bundle; in types B, D, and for $G = G_2$, $\mathcal{L}$ is the Pffafian line bundle, a canonical square root of a given determinant line bundle.

By identifying weights with characters, Kac-Moody weight data $\vec{w} = (\lambda_1, \dots, \lambda_n, \ell)$ corresponds to a line bundle $\mathcal{L}_{\vec{w}} = \mathcal{L}^{\ell} \otimes \mathcal{L}_{\lambda_1} \otimes \cdots \otimes \mathcal{L}_{\lambda_n}$ over Parbun_G. Suppose that we have a parabolic bundle $\mathcal{E}$ with full flags $\overline{g}_1, \dots, \overline{g}_n$, corresponding to a point in Parbun_G. Then the fiber of $\mathcal{L}_{\lambda_i}$ is the fiber over $\overline{g}_i$ of $\mathcal{E} \times^B \chi_i \to \mathcal{E}/B$, where χ_i is the character of B corresponding to λ_i. The line bundle $\mathcal{L}$ is a canonical root of a determinant of cohomology line bundle, discussed in more detail below.

Our main interest in parabolic bundles is that the global section of the line bundles $\mathcal{L}_{\vec{w}}$ can be identified with spaces of conformal block. The following theorem was proven in the form we need by Laszlo and Sorger in [37].

Theorem 1.2.7. *[37] Given Kac-Moody weight data $\vec{w}$, the space of global sections* $\mathrm{H}^0(\mathrm{Parbun}_G, \mathcal{L}_{\vec{w}})$ *is naturally isomorphic to the space of conformal blocks $\mathcal{V}^{\dagger}_{\mathfrak{g},\vec{w}}(X, \vec{p})$.*

We finish this chapter with a brief discussion of determinant of cohomology bundles mentioned above. Let V be a finite dimensional irreducible representation of G. Let $\mathcal{E}$ be a principal G-bundle over X, and $\mathcal{E}(V)$ the associated vector bundle. Then we make the following definition.

Definition 1.2.8. The *determinant of cohomology* line bundle $D(V)$ over Parbun_G associated to

Lie Type	$d(\mathfrak{g})$
A_r	1
B_r	2
C_r	1
D_r	2
G_2	2
F_4	6
E_6	6
E_7	12
E_8	60

Table 1.1: Minimal Dynkin indices

the representation V is the line bundle whose fiber over $(\mathcal{E}, \overline{g}_1, \ldots, \overline{g}_n)$ is

$$\left(\bigwedge^{\max} \mathrm{H}^0(X, \mathcal{E}(V))\right)^* \otimes \bigwedge^{\max} \mathrm{H}^1(X, \mathcal{E}(V)).$$

The bundle $D(V)$ can be identified in $\mathrm{Pic}(\mathrm{Parbun}_G)$ as the line bundle corresponding to trivial characters and a level ℓ equal to the *Dynkin index* of V: let $f : \mathfrak{g}_1 \to \mathfrak{g}$ be an embedding of simple Lie algebras, and assume that the Killing forms $\langle , \rangle_1$ and $\langle , \rangle$ of the algebras are normalized so that $\langle \theta_1, \theta_1 \rangle_1 = \langle \theta, \theta \rangle = 2$, where θ_1, θ are the highest roots of $\mathfrak{g}_1$, $\mathfrak{g}$, respectively. Then there is a unique integer m_f (the Dykin index) such that for any $x, y \in \mathfrak{g}$,

$$\langle f(x), f(y) \rangle = m_f \langle x, y \rangle_1.$$

For a faithful representation V of $\mathfrak{g}$, the Dynkin index is defined as the Dynkin index of $\mathfrak{g} \to \mathfrak{sl}(V)$. See Theorem 5.4 in [36] and section 6 of [37] for a proof of the identification of the level of $D(V)$.

Kumar and Narasimhan prove in [35, Prop 4.7] that for any simple Lie algebra $\mathfrak{g}$, there is a number $d(\mathfrak{g})$ such that for any representation V of $\mathfrak{g}$, $d(\mathfrak{g})$ divides the Dynkin index m_V. For the reader's convenience Table 1.1 reproduces the table of values of $d(\mathfrak{g})$. So we see that determinant of cohomology line bundles always have level divisible by $d(\mathfrak{g})$, which is greater than 1 when G is not of type A_r or C_r. Kumar and Narasimhan also show that there exists a determinant of cohomology line bundle of level $d(\mathfrak{g})$ in any Lie type. In [47], Sorger constructed canonical $d(\mathfrak{g})^{th}$ roots of these line bundles, explicitly giving the generators of the Picard group of the stack of G-bundles.

CHAPTER 2: PARAHORIC BUNDLES

The goal of this chapter is to prove some basic results about line bundles and their global sections on stacks of parahoric bundles. The main fact that we need is that conformal blocks descend to moduli stacks of parahoric bundles. We need to work with parahoric bundles to do the properness calculation in chapter 3, and we need conformal blocks to descend to complete the proof of the reduction theorem in chapter 4.

In section 2.1 we begin with a brief discussion of moduli spaces of parabolic G-bundles, which will serve to motivate the introduction of parahoric bundles. In the section 2.2 we introduce parahoric bundles and their moduli stacks, and review the main results of Balaji and Seshadri in [3], where they show that these stacks can be identified with stacks of equivariant G-bundles over a Galois cover of our curve $Y \to X$. In section 2.3 we study the special fibers of parahoric group schemes and the relative flag structure of stacks of parahoric bundles. Finally, in section 2.4, we prove that conformal blocks descend to stacks of parahoric bundles.

Throughout this chapter, G is a semisimple, connected and simply-connected algebraic group over $k = \mathbb{C}$, and X is a smooth, projective and connected curve over k of arbitrary genus.

2.1 Moduli spaces of parabolic bundles

The stack of parabolic bundles Parbun_G, while algebraic, smooth and irreducible, is not proper, or even separated. However, it is possible to construct projective moduli spaces of parabolic bundles. Moduli spaces for parabolic bundles depend on weight data $\vec{w} = (\lambda_1, \ldots, \lambda_n, \ell)$. In [52], Teleman defined the moduli space $\mathbb{M}_{\vec{w}}$ associated to $\vec{w}$ in terms of the graded algebra of conformal blocks:

$$\mathbb{M}_{\vec{w}} = \mathrm{Proj} \bigoplus_{N \geq 0} \mathrm{H}^0(\mathrm{Parbun}_G, \mathcal{L}_{\vec{w}}^N).$$

If the weight data corresponds to an interior point of $\mathcal{A}^n$, then the k-points of this moduli space correspond to *grade equivalence classes* of *semistable* parabolic G-bundles with flags in G/B. The locus of semistable parabolic bundles with respect to $\vec{w}$ is defined as the set of bundles $\widetilde{\mathcal{E}} \in \mathrm{Parbun}_G$ such that there exists a section $s \in \mathrm{H}^0(\mathrm{Parbun}_G, \mathcal{L}_{\vec{w}}^N)$ for some N such that $s(\widetilde{\mathcal{E}}) \neq 0$. Grade equivalence identifies bundles that must be identified in any separated moduli space of G-bundles; for the definition of grade equivalence see [53].

Alternatively, semistability can be defined in terms of P-reductions. Let $P \supseteq B$ be a maximal parabolic. Consider a parabolic G-bundle $\widetilde{\mathcal{E}} = (\mathcal{E}, \overline{g}_1, \ldots, \overline{g}_n) \in \mathrm{Parbun}_G$ and a P-reduction $\phi : X \to \mathcal{E}/P$. We can trivialize $\mathcal{E}$ over an open set U containing the points $p_1, \ldots, p_n$, and then clearly there are unique Weyl group elements $w_1, \ldots, w_n \in W^P$ such that $\phi(p_i) \in \tilde{g}_i C_{w_i}$. The w_i's do not depend on the trivialization and together are called the *relative position* of the P-reduction in $\widetilde{\mathcal{E}}$. Then semistability of $\widetilde{\mathcal{E}}$ is defined in [53] as follows. We say that $(\mathcal{E}, \overline{g}_1, \ldots, \overline{g}_n)$ is *semistable* if for every maximal parabolic P and every P-reduction the following inequality is satisfied

$$\sum_{i=1}^{n} \langle \omega_P, w_i^{-1} \lambda_i \rangle \leq \ell d,$$

where $w_1, \ldots, w_n \in W^P$ give the relative position of the reduction in $\widetilde{\mathcal{E}}$, and d is the degree of the reduction. The bundle is *stable* if strict inequality is satisfied for every P-reduction. We say that $\sum_{i=1}^{n} \langle \omega_P, w_i^{-1} \lambda_i \rangle - \ell d$ is the *parabolic degree* of the P-reduction. For a proof that these two definitions of semistability are equivalent, see Proposition 2.4.1.

2.1.1 Moduli spaces on the boundary of $\mathcal{A}$

When one or more of the weights are on a chamber wall, we can still construct a moduli space $\mathbb{M}_{\vec{w}}$, however it is too small to be a moduli space of parabolic bundles with full flags. Instead $\mathbb{M}_{\vec{w}}$ is a moduli space of parabolic bundles with partial flags in G/Q, for some $Q \supseteq B$.

A weight λ corresponds to a standard parabolic $Q \subseteq G$ in the following way: if Δ_Q is the set of simple roots α such that $\langle \lambda, \alpha \rangle = 0$, then Q is the parabolic corresponding to Δ_Q. So given weights $\lambda_1, \ldots, \lambda_n$ we get parabolics $Q_1, \ldots, Q_n$. Then we define $\mathrm{Parbun}_G(\vec{Q})$ to be the moduli stack of principal G-bundles over X along with choices of flags $\overline{g}_i \in \mathcal{E}_{|p_i}/Q_i$ for each i.

Suppose that a parabolic bundle $\mathcal{E}$ with full flags $\overline{g}_1, \ldots, \overline{g}_n$ is semistable with respect to weight data $\vec{w}$, and that one of the weights, say λ_1, is not regular, so that the corresponding parabolic Q is not equal to the Borel. Recall that the line bundle $\mathcal{L}_{\vec{w}}$ is defined as a product $\mathcal{L}^\ell \otimes \mathcal{L}_{\lambda_1} \otimes \cdots \otimes \mathcal{L}_{\lambda_n}$

where the fiber of $\mathcal{L}_{\lambda_i}$ is the fiber over $\overline{g}_i$ of $\mathcal{E} \times^B \chi_i$, where χ_i is the character of B corresponding to λ_i. But then χ_1 extends to Q, and therefore $\mathcal{L}_{\vec{w}}$ is constant over the fibers of $\mathcal{E}_{|p_1}/B \to \mathcal{E}_{|p_1}/Q$, and therefore $\mathcal{L}_{\vec{w}}$ identifies in $\mathbb{M}_{\vec{w}}$ a non-trivial family of parabolic bundles with fixed underlying G-bundle.

When one or more weight is on the alcove wall, i.e. if $\langle \lambda_i, \theta \rangle = \ell$, then $\mathbb{M}_{\vec{w}}$ identifies parabolic bundles in a similar way, but will identify bundles with different underlying principal G-bundles. In this case $\mathbb{M}_{\vec{w}}$ is naturally a moduli space of *parahoric bundles*. Parahoric bundles are by definition torsors over a smooth group scheme $\mathcal{G}$ over X associated to *parahoric subgroups* $\mathcal{P}_1, \ldots, \mathcal{P}_n$, which in turn are determined by our choice of weight data $\vec{w}$.

There is a natural morphism $\mathrm{Parbun}_G \to \mathrm{Bun}_{\mathcal{G}}$, where $\mathrm{Bun}_{\mathcal{G}}$ is the stack of parahoric bundles. The rest of the chapter is devoted to introducing the basic theory of parahoric bundles, and studying this morphism. Specifically, we will show that it is exactly analogous to the quotient morphism $\mathrm{Parbun}_G \to \mathrm{Parbun}_G(\vec{Q})$, and use this description to show that conformal blocks descend to $\mathrm{Bun}_{\mathcal{G}}$.

2.2 Parahoric bundles

Parahoric bundles are torsors over parahoric group schemes over X which are generically the trivial group scheme $X^* \times G$, but near each p_i are smooth group schemes originally arising in Bruhat-Tits theory [15]. The parabolic bundles above can be identified with parahoric bundles, but there are parahoric bundles that do not have an underlying principal G-bundle and therefore cannot be described as parabolic bundles.

Our primary technique for working with parahoric bundles is to work with associated equivariant bundles over a ramified extension $Y \to X$, following the work of Balaji and Seshadri in [3]; in this way we can view parahoric bundles as "orbifold bundles", with equivariant bundles acting as orbifold charts. We use this description of parahoric group schemes and bundles to describe the closed fibers of the group schemes over each p_i. The Levi factor of each closed fiber is isomorphic to the centralizer of a finite order element τ_i of T associated to each pair (λ_i, ℓ). One can therefore identify parabolic bundles $\mathcal{E}$ with an underlying parahoric bundle $\mathcal{E}_{\mathcal{G}}$ and a choice of flags in the closed fibers of $\mathcal{E}_{\mathcal{G}}$ modulo parabolic subgroups, mimicking the standard presentation of parabolic bundles. We develop this point of view in section 2.3.

We then use this description of parahoric bundles to prove results about the morphisms between stacks of parahoric bundles, line bundles over these stacks, and their global sections. The main result of the chapter is Theorem 0.1.10: the descent of conformal blocks to a stack of parahoric bundles $\mathrm{Bun}_{\mathcal{G}}$. This depends on the geometric fibers of the canonical morphism

$$\mathrm{Parbun}_G \to \mathrm{Bun}_{\mathcal{G}}$$

being connected and projective, which follows from the description of the closed fibers of parahoric group schemes.

2.2.1 Basic definitions

We mostly follow the notation and conventions of Tits in [56]. Let $K = k((z))$ and $A = k[[z]]$. Then K is naturally a local field with valuation ν. Assume that G is a connected, simply connected, semi-simple group over k. We will denote the associated split group over K as $G(K)$. Choose a maximal torus T of G, and let $X^*(T(K)) = \mathrm{Hom}_K(T(K), \mathbb{G}_m^K)$ denote the group of K-valued characters of T, and $X_*(T(K)) = \mathrm{Hom}_K(\mathbb{G}_m^K, T(K))$ the group of cocharacters. Let $V = X_*(T(K)) \otimes \mathbb{R}$. Then an *affine root* $\alpha + k$ is an affine function on V given by a root $\alpha \in R$ and an integer k (it will be clear whether k is an integer or a field in context). The vector space V acts on the *apartment* $A(T(K))$ associated to $T(K)$, making $A(T(K))$ an affine space. A choice of origin in $A(T(K))$ allows us to identify $A(T(K))$ with V, which we fix from now on. For every affine root $\alpha+k$, there is an associated *half-apartment* $A_{\alpha+k}$ defined as $A_{\alpha+k} = (\alpha + k)^{-1}([0, \infty))$, with its boundary denoted $\delta A_{\alpha+k}$. The *chambers* of $A(T(K))$ are the connected components of the complement of all the walls $\delta A_{\alpha+k}$. When G is simple, the chambers are simplices and the fundamental alcove $\mathcal{A}$ is identified with the chamber bounded by the walls corresponding to the simple roots $\alpha_1, \dots, \alpha_r$, and the affine root $\theta - 1$. When G is semisimple the chambers are polysimplices, and when G is not semisimple, the chambers are products of polysimplices and real affine spaces.

The *Bruhat-Tits building* $\mathcal{B}(G(K))$ is a space constructed by gluing together the apartments associated to each torus. Associated to each affine root $\alpha + k$ is a subgroup $X_{\alpha+k}$ of $U_\alpha(K)$: the choice of origin of $A(T(K))$ determines an isomorphism $U_\alpha(K) \cong \mathbb{G}_{a,K}$, and $X_{\alpha+k}$ is defined as $\nu^{-1}([k, \infty))$ in $U_\alpha(K)$ with respect to this isomorphism. Then the building $\mathcal{B}(G(K))$ has a

$G(K)$-action such that $X_{\alpha+k}$ fixes the half-apartment $A_{\alpha+k}$ pointwise. Furthermore $\mathcal{B}(G(K))$ is the union of $gA(T(K))$ for $g \in G(K)$, and the normalizer $N(K)$ of $T(K)$ fixes $A(T(K))$.

For simplicity assume G is simple. Just like we associate a parabolic subgroup to a weight λ, we can associate a *parahoric subgroup* $\mathcal{P}$ of $G(K)$ to each pair (λ, ℓ) such that $\ell(\lambda) = \langle \lambda, \theta \rangle \leq \ell$.

Definition 2.2.1. Let λ be a dominant integral weight, and ℓ be a level, corresponding to a point $\mu = \frac{1}{\ell}\kappa(\lambda)$ of the fundamental alcove. The point μ lies in the interior of a unique face F of the building $\mathcal{B}(G(K))$. Then the *parahoric subgroup associated to* (λ, ℓ) is defined as the stabilizer $\mathcal{P}$ in $G(K)$ of F. Alternatively, $\mathcal{P}$ is generated as a subgroup as follows:

$$\mathcal{P} = \langle T(A), X_{\alpha+k} \mid \mu \in A_{\alpha+k} \rangle.$$

Remark 2.2.2. Letting $G(A') = G(k[[z^{1/m}]])$ for some integer m, Balaji and Sesahdri showed that $\mathcal{P}$ can be identified non-canonically with an invariant subgroup of $G(A')$ under an action by a finite cyclic group Γ [3]. We will make this explicit in section 2.3.1.

Remark 2.2.3. Parahoric subgroups also correspond to subsets of the vertices $\{v_0, \ldots, v_r\}$ of the affine Dynkin diagram of G, with the empty set corresponding to an Iwahori subgroup $\mathcal{I}$. The Iwahori subgroup corresponding to B is defined as the inverse image of B with respect to the evaluation map $ev_0 : G(A) \to G$. Each standard parabolic P corresponds in the same way to a parahoric subgroup contained in $G(A)$, with vertex set the same as P. The vertex set of $G(A)$ is the set of all vertices $v_1, \ldots, v_r$ of the finite Dynkin diagram. The bijection between vertex sets and parahoric subgroups is inclusion preserving, and so parahoric subgroups corresponding to vertex sets containing the vertex v_0 are not contained in $G(A)$. The vertex set corresponding to (λ, ℓ) is the subset of $\{v_1, \ldots, v_r\}$ corresponding to simple roots α_i such that $\langle \lambda, \alpha_i \rangle = 0$, adding v_0 if in addition $\langle \lambda, \theta \rangle = \ell$.

Remark 2.2.4. When G is semisimple, the definition of a parahoric subgroup is exactly the same. However in this case there is a highest weight θ_i and level ℓ_i for each factor of the Dynkin diagram of G, making the identification of weight data $\vec{w}$ and a point in the alcove more complicated.

2.2.2 Parahoric group schemes, bundles, and associated loop groups

The main result of [15] is the existence of a group scheme $\mathcal{G}$, smooth over $\mathrm{Spec}(A)$, such that $\mathcal{G}(K) \cong G(K)$ and $\mathcal{G}(A) \cong \mathcal{P}$, for any parahoric $\mathcal{P}$. These group schemes are *éttofé*, which means the following: given any A-scheme $\mathcal{N}$ and K-morphism $u_K : \mathcal{G}_K \to \mathcal{N}_K$ such that $u(\mathcal{G}(A)) \subseteq \mathcal{N}(A)$, there is a unique extension to an A-morphism $u : \mathcal{G} \to \mathcal{N}$. This implies the uniqueness of $\mathcal{G}$ up to unique isomorphism.

Then we have the following definitions.

Definition 2.2.5. To weight data $\vec{w} = (\lambda_1, \ldots, \lambda_n, \ell)$ we associate a smooth group scheme $\mathcal{G}$ over X, which is the trivial group scheme $X^* \times G$ over $X^* = X \setminus \{p_1, \ldots, p_n\}$, and in a formal neighborhood of each p_i is isomorphic to the parahoric group scheme associated to each (λ_i, ℓ). A *parahoric $\mathcal{G}$-bundle* is simply a $\mathcal{G}$-torsor; that is, a scheme over X with a right $\mathcal{G}$-action that is étale-locally isomorphic to $\mathcal{G}$. We denote the moduli stack of $\mathcal{G}$-bundles by $\mathrm{Bun}_{\mathcal{G}}$.

We will also use the *loop groups* associated to the parahoric group schemes. For a k-algebra R, let $R[[z]]$ and $R((z))$ denote the ring of formal power series and formal Laurent series with coefficients in R, respectively. Note that $R((z))$ is a K-algebra, and $R[[z]]$ is an A-algebra. Then the loop groups associated to $G(K)$ and $\mathcal{P}$ are defined as follows.

Definition 2.2.6. For $K = k((z))$, the *loop group* LG associated to $G(K)$ is defined as the ind-scheme given by the functor

$$R \mapsto G(R((z))).$$

for any k-algebra R. The loop group $L^+\mathcal{P}$ associated to the parahoric subgroup $\mathcal{P}$ is the (infinite dimensional) affine group scheme associated to the functor

$$R \mapsto \mathcal{G}(R[[z]]),$$

where $\mathcal{G}$ is the group scheme associated to $\mathcal{P}$.

The *affine flag variety* associated to $\mathcal{P}$ is defined as the quotient sheaf

$$\mathrm{Gr}_{\mathcal{P}} = {}^{LG}/_{L^+\mathcal{P}}.$$

The stack $\mathrm{Bun}_{\mathcal{G}}$ can be seen as a quotient of a product of these affine flag varieties, and some of the properties of $\mathrm{Bun}_{\mathcal{G}}$ we are interested in can be proven by passing to these infinite dimensional ind-schemes. However for our purposes the primary interest is in quotients

$$L^{+}\mathcal{P} / L^{+}\mathcal{Q}.$$

where $\mathcal{Q} \subseteq \mathcal{P}$. These spaces are isomorphic to finite dimensional flag varieties given by quotients of the closed fiber of the corresponding parahoric group scheme. We choose to avoid the use of affine flag varieties to emphasize this finite dimensionality. Loop groups and their affine flag varieties are studied in much greater generality by Pappas and Rapoport in [41].

2.2.3 Parahoric bundles as quotients of equivariant bundles

Fixing weight data $\vec{w}$ we can understand parahoric bundles as quotients of bundles on a Galois cover $p : Y \to X$ that are equivariant with respect to the action of the Galois group Γ.

Let E be a Γ-equivariant principal G-bundle over a Galois cover $p : Y \to X$, with a right G-action and left Γ-action. If $y \in \mathcal{R}$ is a ramification point of p then by the work done in [53] we can find a formal neighborhood N_y of y such that E is isomorphic over N_y to the trivial bundle $N_y \times G$, with the action of Γ_y given by $\gamma \cdot (\omega, g) = (\gamma\omega, \tau(\gamma)g)$, where $\tau : \Gamma_y \to G$ does not depend on the formal parameter ω. We say that the *local type* of E at y is the conjugacy class of τ. The local type does not depend on the trivialization and is the same for every ramification point over p_i. The local type of E is the collection of local types $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)$ over each p_i.

Definition 2.2.7. The stack of (Γ, G)-bundles of local type $\boldsymbol{\tau}$ will be denoted $\mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})$. This stack is a smooth and connected Artin stack.

The weight data $\vec{w}$ determines ramification indices and local type representations. Let m_i be a positive integer such that $\frac{m_i}{\ell} \cdot \lambda_i$ is integral. We will usually take m_i to be the smallest such integer. Then it is well known that if $n \geq 3$ or $g \geq 2$ there exists a Galois covering $p : Y \to X$, ramified over each p_i with ramification index m_i. Let Γ be the Galois group of Y over X. To each weight λ_i we can associate a coweight $\frac{m_i}{\ell} \cdot \kappa(\lambda_i)$ and the associated cocharacter $\chi_i : \mathbb{G}_m \to T \subseteq G$. Let ζ_i be a primitive m_i-th root of unity, and let $\tau_i : \mathbb{Z}_{m_i} \to T$ be defined as $\tau_i(\gamma) = \chi_i(\zeta_i^{\gamma})$. If $y \in Y$ is a ramification point, and Γ_y the isotropy subgroup of Γ at y, then $\Gamma_y \cong \mathbb{Z}_{m_i}$ and we can therefore think of τ_i as a representation of Γ_y.

Let $\mathcal{G}$ be the parahoric group scheme associated to $\vec{w}$ as above. Then one of the main theorems in [3] is the following.

Theorem 2.2.8. *[3] Given weight data $\vec{w} = (\lambda_1, \ldots, \lambda_n, \ell)$, we have a natural isomorphism of stacks*

$$\mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau}) \xrightarrow{\sim} \mathrm{Bun}_{\mathcal{G}}.$$

where the ramification indices of $Y \to X$, local type $\boldsymbol{\tau}$, and $\mathcal{G}$ are determined by $\vec{w}$ as above.

A quick sketch of the proof will be useful for what follows. Balaji and Seshadri identify $\mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})$ with a stack of torsors over a group scheme $\mathcal{G}_0'$ over Y equivariant with respect to the Galois action. They then show that this stack is isomorphic to the stack of torsors over the invariant pushforward group scheme $\mathcal{G}' = p_*^\Gamma \mathcal{G}_0'$ over X. Finally they identify $\mathcal{G}'$ with the parahoric group scheme $\mathcal{G}$.

This theorem allows one to define the semistability of parahoric bundles in terms of semistability of equivariant bundles. It is easy to see that this definition does not depend on the choice of Galois cover Y.

Definition 2.2.9. We say that a (Γ, G)-bundle is Γ-*semistable* if for every maximal parabolic $P \subseteq G$ and every Γ-equivariant P-reduction $\sigma : Y \to \mathcal{F}/P$ we have $\sigma^*\mathcal{F}(\mathfrak{g}/\mathfrak{p}) \geq 0$. Stability is defined in the same way, replacing inequalities with strict inequalities. A $\mathcal{G}$-bundle is said to be *semistable* (resp. *stable*) with respect to some weight data $\vec{w}$ if the corresponding (Γ, G)-bundle is semistable (resp. stable).

2.3 Relative flag structures for parahoric bundles

In this section we show that parahoric bundles can be described as maximal parahoric bundles together with flag data. This can be seen as a generalization of the well-known fact that $\mathrm{Bun}_{\mathcal{G}_{\mathcal{I}}} \cong \mathrm{Parbun}_G$, where $\mathcal{G}_{\mathcal{I}}$ is the parahoric group scheme associated to the Iwahori subgroup $\mathcal{I}$ as each point $p_1, \ldots, p_n$. By definition there is a morphism $\mathrm{Parbun}_G \to \mathrm{Bun}_G$ with projective and connected geometric fibers.

The idea of flag structures for parahoric bundles is to replace Bun_G with $\mathrm{Bun}_{\mathcal{G}_{\mathcal{M}}}$ for some maximal parahorics $\mathcal{M}_1, \ldots, \mathcal{M}_n$, so that there exists a morphism $\mathrm{Bun}_{\mathcal{G}} \to \mathrm{Bun}_{\mathcal{G}_{\mathcal{M}}}$. The fibers can then be identified with connected flag varieties, which in general will be homogeneous spaces of groups of different Dynkin types than G.

In this section we study the more general situation of the morphism $\mathrm{Bun}_{\mathcal{G}_{\mathcal{Q}}} \to \mathrm{Bun}_{\mathcal{G}}$, where $\mathcal{G}_{\mathcal{Q}}$ is the group scheme associated to subgroups $\mathcal{Q}_i$ of the parahoric groups $\mathcal{P}_i$ defining $\mathcal{G}$. The main results of this section are the construction of an isomorphism of $\mathrm{Bun}_{\mathcal{G}_{\mathcal{Q}}}$ with the stack $\mathrm{Parbun}_{\mathcal{G}}(\vec{\mathcal{Q}})$ of $\mathcal{G}$-bundles with flags (Proposition 2.3.7), and the identification of the fibers of $\mathrm{Bun}_{\mathcal{G}_{\mathcal{Q}}} \to \mathrm{Bun}_{\mathcal{G}}$ (Corollary 2.3.8). We will use these results in section 2.4 to show that conformal blocks descend to stacks of parahoric bundles.

We note that most of the local results in this section follow from the general theory of Bruhat and Tits on reductive groups over local fields [14, 15]. We include proofs here for the sake of being explicit and because the general theory is not really necessary for a split simple group over K.

2.3.1 Special fibers of parahoric group schemes

Let $\mathcal{G}$ be a parahoric group scheme over $\mathrm{Spec}(A)$ corresponding to a parahoric subgroup $\mathcal{P}$. We want to describe the special fiber $\mathcal{G}(k)$. Choose a rational cocharacter μ in the interior of the face of the fundamental alcove corresponding to $\mathcal{P}$. Let m be an integer such that $m\mu$ is integral, and let $K' = k((z^{1/m})) = k((\omega))$ and $A' = k[[z^{1/m}]] = k[[\omega]]$. Write Δ for the restriction of $m\mu$ to $\mathrm{Spec}(K')$, and let τ be the representation of $\Gamma \cong \mathbb{Z}_m$ given by $\tau(\gamma) = m\mu(\zeta^\gamma)$, where ζ is a primitive m^{th} root of unity. Then the isomorphism of $\mathcal{G}$ with the invariant pushforward group scheme $\mathcal{G}'$ is the morphism induced by conjugation by Δ. In particular

$$\Delta \mathcal{P} \Delta^{-1} = \Delta \mathcal{G}(A) \Delta^{-1} = \mathcal{G}'(A) = G(A')^\Gamma,$$

where the action of Γ on $G(A')$ is the one induced by τ: for $f \in G(A')$, $(\gamma f)(\omega) = \tau(\gamma) f(\gamma^{-1}\omega) \tau(\gamma)^{-1}$. This identification induces an isomorphism of the group schemes $\mathcal{G} \cong \mathcal{G}'$ because both group schemes are éttofé.

Now for considering k as an A-module via the isomorphism $k \cong A/(z)$, we have $k \otimes_A A' \cong k[[\omega]]/(\omega^m)$. We will write this ring as $k[\epsilon]$. Then $\mathcal{G}'(k) = G(k[\epsilon])^\Gamma$. Furthermore the homomorphisms

$k \to k[\epsilon] \to k$ induce homomorphisms $G \to G(k[\epsilon]) \to G$ that commutes with the action of Γ, which is just conjugation by τ on G. Then we have homomorphisms $C_G(\tau) \to \mathcal{G}'(k) \to C_G(\tau)$, with the composition being the identity. Then we have the following description of the special fiber of $\mathcal{G}$.

Proposition 2.3.1. *Fixing μ, there is a canonical pair of homomorphisms $C_G(\tau) \xrightarrow{\iota} \mathcal{G}(k) \xrightarrow{\pi} C_G(\tau)$ with $\pi \circ \iota = Id$. Furthermore, the kernel of π is isomorphic to the group of m^{th} order Γ-invariant deformations of the identity of G, and is the unipotent radical of $\mathcal{G}(k)$. Finally, for any scheme S the natural map $L^+\mathcal{P}(S) \to \mathcal{G}(k)(S)$ is surjective. Its composition with π is given by conjugation by Δ and setting ω equal to zero.*

Proof. The kernel of π contains the unipotent radical of $\mathcal{G}(k)$ since $C_G(\tau)$ is reductive. Let $f \in \ker(\pi)$, and let $G \to \mathrm{GL}(V)$ be any faithful representation. Then clearly, we can identify f with a unipotent element of $\mathrm{GL}(V \otimes_k k[\epsilon])$, and therefore $\ker(\pi)$ is unipotent. Furthermore, shifting the parameter of f by $a \in k$ gives an m^{th}-order deformation $f(a\omega)$ that is still Γ-invariant and in $\ker(\pi)$. Taking $a \to 0$ connects f with the identity of $\mathcal{G}(k)$, showing that $\ker(\pi)$ is connected. Therefore $\ker(\pi)$ is the unipotent radical of $\mathcal{G}(k)$.

It remains to show that $L^+\mathcal{P}(S) \to \mathcal{G}(k)(S)$ is surjective for any scheme S. This argument is essentially identical to part of the proof of Lemma 2.5 in [53].

We use non-abelian cohomology, letting Γ act on $G(S[[\omega]])$ and $G(S[\epsilon])$ as above, following the notation and conventions in Serre's *Cohomologie galoisienne* [45]. Let $G_n = G(S[[\omega]]/\omega^n)$. We want to show that the natural map $\mathcal{P} \cong \mathrm{H}^0(\Gamma, G(S[[\omega]])) \to \mathrm{H}^0(\Gamma, G(S[\epsilon])) \cong \mathcal{G}(k)$ is surjective. It is sufficient to show that each $\mathrm{H}^0(\Gamma, G_{n+1}) \to \mathrm{H}^0(\Gamma, G_n)$ is surjective, since

$$\lim_{\leftarrow} \mathrm{H}^0(\Gamma, G_n) \subseteq \mathrm{H}^0(\Gamma, G(S[[\omega]])).$$

Since G is smooth, it follows that the morphism $G_{n+1} \to G_n$ is surjective for all n. Then consider the short exact sequence

$$1 \to K_n \to G_{n+1} \to G_n \to 1,$$

which induces an exact sequence of pointed sets

$$\mathrm{H}^0(\Gamma, G_{n+1}) \to \mathrm{H}^0(\Gamma, G_n) \to \mathrm{H}^1(\Gamma, K_n).$$

It is easy to see then that the kernel K_n is a $\mathbb{C}$-vector space: consider for example the case $n = 1$, where K_n is isomorphic to the Lie algebra of G. Therefore, since Γ is finite, we see that $\mathrm{H}^1(\Gamma, K_n)$ is trivial by [25, Proposition 6]. This proves the desired surjectivity. $\square$

2.3.2 The image of parahoric subgroups in $C_G(\tau)$

We want to describe the image of parahoric subgroups $\mathcal{Q} \subseteq \mathcal{P}$ in $C_G(\tau)$. For a root α, let (μ, α) denote the pairing of characters and cocharacters, and square brackets $[x]$ denote the smallest integer less than or equal to x. Then the parahoric subgroup $\mathcal{P}$ associated to $\mu \in \mathcal{A}$ is defined as the group

$$\mathcal{P} = \langle T(A), U_\alpha(z^{-[(\mu,\alpha)]}A), \alpha \in R \rangle,$$

where U_α denotes the root group associated to α and $U_\alpha(z^{-[(\mu,\alpha)]}A)$ is the group $X_{\alpha+[(\mu,\alpha)]}$ fixing the affine half-apartment $A_{\alpha+[(\mu,\alpha)]}$. We have the following proposition.

Lemma 2.3.2. *Let $\mathcal{Q} \subseteq \mathcal{P}$ be a parahoric subgroup corresponding to μ'. Then the image of $\mathcal{Q}$ in $C_G(\tau)$ is exactly the group generated by T and the root groups U_α such that $(\mu, \alpha) = [(\mu', \alpha)]$.*

Proof. For any $\alpha \in R$, $\Delta U_\alpha(z^{-[(\mu',\alpha)]}A)\Delta^{-1} = U_\alpha(z^{(\mu,\alpha)-[(\mu',\alpha)]}A)$. Therefore the result follows. $\square$

Corollary 2.3.3. *$C_G(\tau)$ is the group generated by T and the root groups U_α such that (μ, α) is an integer, and the image of the Iwahori subgroup $\mathcal{I}$ in $C_G(\tau)$ is the group generated by T and root groups U_α such that (μ, α) is a nonpositive integer.*

Proof. For the description of $C_G(\tau)$, simply take $\mathcal{Q} = \mathcal{P}$ in the proposition above.

The Iwahori subgroup corresponds to a cocharacter μ' in the interior of the alcove, and therefore if $(\mu, \alpha) = [(\mu', \alpha)]$, then (μ, α) is either 0 or -1. $\square$

The images of sub-parahoric subgroups of $\mathcal{P}$ are in fact parabolic subgroups of $C_G(\tau)$. This will allow us to identify the flags for parahoric bundles defined below with points in a (connected) flag variety.

Proposition 2.3.4. *The group $C_G(\tau)$ is a connected reductive subgroup of G, and the image of $\mathcal{I}$ in $C_G(\tau)$ is a Borel subgroup B'. Furthermore B' is the intersection of a Borel subgroup B_μ of G*

with $C_G(\tau)$, and $B_\mu = w_\mu B w_\mu$, where $w_\mu \in W$ is of order 2. In particular, if $\mathcal{P}$ is maximal, then $C_G(\tau)$ is a subgroup of G of maximal rank.

Proof. A proof that the centralizer of an element of a simply connected group is connected can be found in the lecture notes on conjugacy classes by Springer and Steinberg [50]. In the case that $\mathcal{P}$ is maximal, then $C_G(\tau)$ is a subgroup of maximal rank, as studied by Borel and de Siebenthal [13]. The Dynkin diagram of $C_G(\tau)$ is given by removing the vertices of the affine Dynkin diagram associated to $\mathcal{P}$.

Let B_μ be the subgroup of G generated by T and U_α such that either $(\mu, \alpha) < 0$ or $\alpha \in R^+$ and $(\mu, \alpha) = 0$. Then we have that the image of $\mathcal{I}$ in $C_G(\tau)$ is just $C_G(\tau) \cap B_\mu$. Let P be the parabolic subgroup associated to $\mathcal{P}$, and let w_μ be the product of the longest words in W and W_P. This element w_μ switches positive and negative roots for any root α such that $(\mu, \alpha) \neq 0$, and fixes all other roots. Clearly then w_μ is order 2, and $B_\mu = w_\mu B w_\mu$. □

Recall that the parahoric subgroups contained in $G(A)$ can also be defined as inverse images of parabolic subgroups of G. The following proposition generalizes this fact to other parahorics, and will be of crucial importance for what follows.

Proposition 2.3.5. *If Q is the image of $\mathcal{Q}$ in $C_G(\tau)$, then the inverse image of Q in $\mathcal{P}$ is exactly $\mathcal{Q}$.*

Proof. The inverse image of Q in $G(A')^\Gamma$ can be described as the group

$$\langle T(A), U_\alpha(\omega^{k_\alpha} A), \alpha \in R \rangle$$

for some non-negative integers k_α. There are two cases: either $(\mu, \alpha) = [(\mu', \alpha)]$ and $k_\alpha = 0$, or $k_\alpha > 0$. Note that α satisfies $(\mu, \alpha) = [(\mu', \alpha)]$ if and only if $-\alpha$ does. Then in the first case, we see that

$$\Delta^{-1} U_\alpha(\omega^{k_\alpha} A) \Delta = U_\alpha(\omega^{-(\mu,\alpha)} A) = U_\alpha(\omega^{-[(\mu',\alpha)]} A).$$

Now in the second case, we know that $(\mu, \alpha) > [(\mu', \alpha)]$. In the case that $\alpha \in R^+$, we therefore know that $[(\mu', \alpha)] = 0$, since $0 \leq (\mu', \alpha), (\mu, \alpha) \leq 1$. This implies that $0 \leq (\mu', \alpha) < 1$, and $0 < (\mu, \alpha) \leq 1$. We also know that $[(\mu', -\alpha)] < (\mu, -\alpha) < 0$, so that in fact $0 < (\mu', \alpha), (\mu, \alpha) < 1$. Therefore in this case we have

$$\Delta^{-1} U_\alpha(\omega^{k_\alpha} A)\Delta = U_\alpha(\omega^{-[(\mu,\alpha)]} A) = U_\alpha(\omega^{-[(\mu',\alpha)]} A),$$

finishing the proof. $\square$

2.3.3 Relative flag structures

Now we return to the global situation over X, and define relative flag structures for parahoric bundles.

Let $\mathcal{G}$ be the group scheme over X corresponding to parahoric subgroups $\mathcal{P}_1, \ldots, \mathcal{P}_n$, and let $\mathcal{Q}_i \subseteq \mathcal{P}_i$ be subgroups. As above, let μ_i be the cocharacter associated to each $\mathcal{P}_i$, with associated $\Delta_i \in G(K')$, and representation $\tau_i : \Gamma_{p_i} \to T$. Let Q_i be the image of each $\mathcal{Q}_i$ in $C_G(\tau_i)$, and let $\widetilde{Q}_i$ be the inverse image of Q_i in the closed fiber $\mathcal{G}(p_i)$. Then we have the following definition.

Definition 2.3.6. Let $\mathrm{Parbun}_{\mathcal{G}}(\vec{\mathcal{Q}})$ be the moduli stack of $\mathcal{G}$-bundles together with flags $\overline{g}_i \in \mathcal{E}(p_i)/\widetilde{Q}_i \cong C_G(\tau_i)/Q_i$.

Then we claim the following.

Proposition 2.3.7. *We have an isomorphism of stacks:*

$$\mathrm{Bun}_{\mathcal{G}_{\mathcal{Q}}} \cong \mathrm{Parbun}_{\mathcal{G}}(\vec{\mathcal{Q}}).$$

Proof. For simplicity, we describe the morphisms pointwise; the same constructions work with families.

Then there is a natural morphism $\mathcal{G}_{\mathcal{Q}} \to \mathcal{G}$ which induces a morphism of stacks $\mathrm{Bun}_{\mathcal{G}_{\mathcal{Q}}} \to \mathrm{Bun}_{\mathcal{G}}$. If $\mathcal{E}_{\mathcal{Q}}$ is a $\mathcal{G}_{\mathcal{Q}}$-bundle, via this projection we get a $\mathcal{G}$-bundle $\mathcal{E}$. We also have a natural morphism $\mathcal{E}_{\mathcal{Q}} \to \mathcal{E}$ and its restriction to each p_i: $\mathcal{E}_{\mathcal{Q}}(p_i) \to \mathcal{E}(p_i)$. This morphism gives a canonical point in $\mathcal{E}(p_i)/\widetilde{Q}_i$, since $\mathcal{E}_{\mathcal{Q}}(p_i)/\widetilde{Q}_i$ is simply a point. This defines a morphism $\mathrm{Bun}_{\mathcal{G}_{\mathcal{Q}}} \to \mathrm{Parbun}_{\mathcal{G}}(\vec{\mathcal{Q}})$.

We can also define this morphism in terms of a trivialization. Given a $\mathcal{G}_{\mathcal{Q}}$-bundle, we can describe it in terms of transition functions as

$$\begin{aligned} (\mathcal{E}_0)_{|U^*_{p_i}} &\xrightarrow{\sim} (\mathcal{E}_i)_{|U^*_{p_i}} \\ (z,g) &\mapsto (z, \Theta_i(z) g), \end{aligned}$$

where $U_{p_i}^* \cong \mathrm{Spec}(K)$ is a formal neighborhood around p_i. Then the associated $\mathcal{G}$ bundle is given by the same transition functions, and we take the flag $\overline{e} \in C_G(\tau_i)/Q_i \cong \mathcal{E}(p_i)/\widetilde{Q}_i$, with respect to this trivialization. Clearly if we change the trivialization, we get the same $\mathcal{G}$-bundle and flag, since the new transition function Θ_i' is given by $\Theta_i' = f\Theta_i g$, where $f \in \mathcal{Q}_i$, and g is the restriction of a morphism $g : X^* \to G$ to U_i^*. Multiplication on the right by g can be accounted for by changing the trivialization of $\mathcal{E}$ over X^*, while multiplication on the left by f fixes the flag $\overline{e}$.

Going the other direction, we simply choose a trivialization of the underlying $\mathcal{G}_{\mathcal{M}}$-bundle such that the flag is $\overline{e}$, then use these transition functions to construct the $\mathcal{G}$-bundle. We can always do this because the morphism $L^+\mathcal{M} \to \mathcal{G}(k)$ is surjective by Prop 2.3.1. By Prop 2.3.5, the subgroup of $\mathcal{P}_i$ fixing the flag $\overline{e}$ is exactly the parahoric subgroup $\mathcal{Q}_i$, so this definition does not depend on the choice of trivialization. □

Corollary 2.3.8. *For any parahoric subgroups $\mathcal{Q}_i \subseteq \mathcal{P}_i$ the morphism $\mathrm{Bun}_{\mathcal{G}_{\mathcal{Q}}} \to \mathrm{Bun}_{\mathcal{G}}$ is a smooth, proper and surjective representable morphism with connected and projective geometric fibers.*

Proof. By the above proposition, the morphism is clearly representable, and the geometric fibers are isomorphic to the product

$$\prod_{i=1}^{n} C_G(\tau_i)/Q_i.$$

Then by Proposition 2.3.4, the geometric fibers are connected and projective. □

Example 2.3.9. Let $G = \mathrm{Sp}_{2r}$, and $\mathcal{M}_1, \ldots, \mathcal{M}_n$ be maximal parahorics. Then $C_G(\tau_i)$ is a (semisimple) subgroup of maximal rank in G, and is isomorphic to $\mathrm{Sp}_{2s} \times \mathrm{Sp}_{2(r-s)}$ for some $0 \leq s \leq r$. Then letting $\mathcal{I}$ be the Iwahori subgroup associated to $B \subseteq G$, $\mathrm{Bun}_{\mathcal{G}_{\mathcal{I}}} \cong \mathrm{Parbun}_{\mathcal{G}_{\mathcal{M}}}(\vec{\mathcal{I}})$ is the stack of $\mathcal{G}_{\mathcal{M}}$-bundles $\mathcal{E}$ together with flags in $\mathcal{E}(p_i)/\widetilde{B}_i \cong C_G(\tau_i)/B_i$ for Borel subgroups $B_i \subseteq C_G(\tau_i)$. Each B_i is the intersection of a Borel subgroup of G with $C_G(\tau_i)$ (see Proposition 2.3.4).

Let $\mathcal{E}_{\mathcal{M}}$ be a $\mathcal{G}_{\mathcal{M}}$-bundle together with flags $\overline{g}_i$. Choose a trivialization of $\mathcal{E}_{\mathcal{M}}$ near p_i so that the flag $\overline{g}_i$ is trivial. Then acting by $C_G(\tau_i)$ on the flag we get a family of bundles corresponding to a morphism $C_G(\tau_i)/B_i \to \mathrm{Parbun}_{\mathcal{G}_{\mathcal{M}}}(\vec{\mathcal{I}}) \cong \mathrm{Bun}_{\mathcal{G}_{\mathcal{I}}} \cong \mathrm{Parbun}_G$. The composition of this morphism with $\mathrm{Parbun}_G \to \mathrm{Bun}_G$ factors through $C_G(\tau_i)/P_i$, where P_i is the image of $\mathcal{M}_i \cap G(A)$ in $C_G(\tau_i)$, and corresponds to a non-trivial family of G-bundles when $\mathcal{M}_i \neq G(A)$.

2.4 Line bundles on stacks of parahoric bundles

In this section we use the above description of $\mathrm{Bun}_{\mathcal{G}}$ to prove that conformal blocks descend to this stack, finishing the proof of Theorem 0.1.10. First we show that the locus of semistable bundles can be defined like semistability is defined in GIT, which will be needed later.

Proposition 2.4.1. *An equivariant bundle $E \in \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})$ is Γ-semistable if and only if there exists $s \in \mathrm{H}^0(\mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau}), D(V)^N)$ for some integer $N > 0$ and a faithful representation V of G such that $s(E) \neq 0$.*

Proof. Suppose E is Γ-semistable. Then E is also a semistable G-bundle. This follows from the uniqueness of the canonical reduction of an unstable G-bundle (see section 2.4 in [53]). Now the bundle E corresponds to a point $x \in \mathbb{M}$ of the moduli space $\mathbb{M}$ of G-bundles over Y. Then since a determinant of cohomology line bundle descends to an ample bundle $\mathcal{L}$ over $\mathbb{M}$ [36], there is a section $s \in \mathrm{H}^0(\mathbb{M}, \mathcal{L}^N)$ for some $N > 0$ such that $s(x) \neq 0$. In [37] Laszlo and Sorger showed that $\mathrm{H}^0(\mathbb{M}, \mathcal{L}^N) = \mathrm{H}^0(\mathrm{Bun}_G, \mathcal{L}^N)$, so pulling back and extending s over $\mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})$ gives a section $s \in \mathrm{H}^0(\mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau}), D(V)^N)$ such that $s(E) \neq 0$.

Now suppose there is an $s \in \mathrm{H}^0(\mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau}), D(V)^N)$ such that $s(E) \neq 0$. For the sake of contradiction, suppose that E is not Γ-semistable. Then there is a (unique) canonical P-reduction $\phi_E : Y \to E/P$ that is a maximum violator of semistability. This P-reduction gives a one-parameter family of equivariant bundles $\tau : \mathbb{A}^1 \to \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})$. But by Mumford's numerical criterion for semistability, since $s(E) \neq 0$, the index $\mu(E, \tau)$ is non-negative, which contradicts the assumption that ϕ is a maximal violator of semistability for E, since the index is a positive multiple of the degree of ϕ_E. (For more details on the construction of τ and calculation of its index, see the proof of our Proposition 4.1.3 and Lemma 3.16 in [10].) Therefore E is Γ-semistable. □

Note that the same result holds for $\mathrm{Bun}_{\mathcal{G}}$ and $\mathcal{L}_{\vec{w}}$, assuming this line bundle descends, which we prove below. The following proposition contains the basic geometric argument behind the proof of Theorem 0.1.10, assuming we know that the line bundle itself descends.

Proposition 2.4.2. *Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a representable morphism of Artin stacks, where $\mathcal{Y}$ is smooth over k and f is smooth, proper and surjective with connected geometric fibers. Then*

$f_*(\mathcal{O}_\mathcal{X}) = \mathcal{O}_\mathcal{Y}$*, and for any line bundle* $\mathcal{L}$ *over* $\mathcal{Y}$*, the pullback via* f *induces an isomorphism of global sections:* $\mathrm{H}^0(\mathcal{Y}, \mathcal{L}) \xrightarrow{\sim} \mathrm{H}^0(\mathcal{X}, \mathcal{L})$*.*

Proof. By Stein factorization of Artin stacks (see [40]) f factors as $\mathcal{X} \xrightarrow{f'} \mathcal{Y}' \xrightarrow{e} \mathcal{Y}$, where f' is proper with connected fibers, $f'_*(\mathcal{O}_V) \cong \mathcal{O}_{U'}$, and e is finite. But since f is surjective and has connected fibers, e must have connected fibers. But a finite morphism with connected fibers is an isomorphism since we are working over an algebraically closed field of characteristic 0, and $\mathcal{Y}$ is normal. Therefore $f_*(\mathcal{O}_\mathcal{X}) = \mathcal{O}_\mathcal{Y}$, and by the projection formula, $\mathrm{H}^0(\mathcal{Y}, \mathcal{L}) \to \mathrm{H}^0(\mathcal{X}, \mathcal{L})$ is an isomorphism. $\square$

We can now finish the proof of Theorem 0.1.10.

Proof of Theorem 0.1.10. By Proposition 2.4.2 the pullback of global sections of any line bundle on $\mathrm{Bun}_\mathcal{G}$ to Parbun_G is an isomorphism. It remains to show that $\mathcal{L}_{\vec{w}}$ descends to $\mathrm{Bun}_\mathcal{G}$, assuming $\mathcal{G}$ is the parahoric group scheme associated to $\vec{w}$. First note that by the work in section 6 of [10], a power of the line bundle $\mathcal{L}_{\vec{w}}$ can be identified with the pullback to Parbun_G of a determinant of cohomology bundle on $\mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})$. Therefore by Balaji and Seshadri's identification of the stacks of parahoric bundles and equivariant bundles, a power of $\mathcal{L}_{\vec{w}}$ descends to $\mathrm{Bun}_\mathcal{G}$. In particular, a power of $\mathcal{L}_{\vec{w}}$ is trivial over the fibers of $f : \mathrm{Parbun}_G \to \mathrm{Bun}_\mathcal{G}$. Now since the fibers of f are isomorphic to a product of connected flag varieties, the Picard groups of the fibers are torsion free, and therefore $\mathcal{L}_{\vec{w}}$ itself is trivial over the fibers of f.

We want to show that $f_*(\mathcal{L}_{\vec{w}})$ is a line bundle, and that its pullback to Parbun_G is $\mathcal{L}_{\vec{w}}$. (The following argument is essentially a solution of exercise III.12.4 in [23].) Let $U \to \mathrm{Bun}_\mathcal{G}$ be a smooth morphism and let V be the fiber product of U and $\mathrm{Parbun}_\mathcal{G}$. By definition, the pullback of $f_*(\mathcal{L}_{\vec{w}})$ to U is the pushforward of the pullback of $\mathcal{L}_{\vec{w}}$ to V. Since $\mathcal{L}_{\vec{w}}$ is trivial on the fibers, and the fibers are projective and connected, we have $\mathrm{H}^0(V_y, \mathcal{L}_{\vec{w}}) = k$ for any $y \in U$. Therefore by Grauert's Theorem, $f_*(\mathcal{L}_{\vec{w}})$ is locally free of rank 1 over U, and therefore over $\mathrm{Bun}_\mathcal{G}$ [23, Corollary III.12.9]. Now by the adjoint property of pullbacks, there is a natural morphism of sheaves $f^*f_*(\mathcal{L}_{\vec{w}}) \to \mathcal{L}_{\vec{w}}$. To show this is an isomorphism, it is sufficient to check it on fibers. Let $x \in \mathrm{Parbun}_G$ be a k-valued point, and y its image in $\mathrm{Bun}_\mathcal{G}$. Then the fiber of $f^*f_*(\mathcal{L}_{\vec{w}})$ over x is $\mathrm{H}^0((\mathrm{Parbun}_G)_y, \mathcal{L}_{\vec{w}})$, and the morphism to the fiber of $\mathcal{L}_{\vec{w}}$ is simply the evaluation map. But since $\mathrm{H}^0((\mathrm{Parbun}_G)_y, \mathcal{L}_{\vec{w}})$ is the space of constant functions on $(\mathrm{Parbun}_G)_y$, this map is nonzero, and therefore $f^*f_*(\mathcal{L}_{\vec{w}}) \cong \mathcal{L}_{\vec{w}}$. $\square$

Corollary 2.4.3. *Let $X \cong \mathbb{P}^1$. Then for any weight data $\vec{w}$, the space* $\mathrm{H}^0(\mathrm{Bun}_{\mathcal{G}}, \mathcal{L}_{\vec{w}}^N)$ *is nonzero for some N if and only if $\vec{w}$ is in the multiplicative polytope.*

Proof. This follows from Theorem 5.2 in [10] and Theorem 0.1.10. $\square$

CHAPTER 3: BEGINNING OF THE PROOF OF THE REDUCTION THEOREM: STACKS OF P-REDUCTIONS

We are now ready to begin the proof of the reduction theorem for conformal blocks. First we want to outline the strategy of the proof in the language of parahoric bundles and equivariant bundles. For the remainder of the proof of the reduction theorem we will fix the following data. Assume $X \cong \mathbb{P}^1$ and fix distinct points $p_1, \ldots, p_n \in X$. Let $\vec{w}$ be weight data in the multiplicative polytope. Assume that $\vec{w}$ lies on a face of the polytope corresponding to the quantum product $\sigma_{u_1} * \cdots * \sigma_{u_n} = q^d[\mathrm{pt}]$ in $\mathrm{QH}^*(G/P)$. Let $\mathcal{G}$ be the parahoric group scheme over X corresponding to $\vec{w}$, and let $\mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})$ be a stack of equivariant bundles over a curve Y such that $\mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau}) \cong \mathrm{Bun}_{\mathcal{G}}$. Let $L \subseteq P$ be the Levi subgroup containing T, and $L' = [L, L]$.

Consider the following diagram.

$$\begin{tikzcd}
\mathrm{Parbun}_G \arrow[d, "p"] & \mathrm{Parbun}_L \arrow[l] \arrow[d] & \mathrm{Parbun}_{L'} \arrow[l] \arrow[d, "p'"] \\
\mathrm{Bun}_{\mathcal{G}} \cong \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau}) & \mathrm{Bun}_Y^{\Gamma,L}(\boldsymbol{\tau}_u, 0) \arrow[l, "\iota"'] & \mathrm{Bun}_Y^{\Gamma,L'}(\boldsymbol{\tau}'_u) \arrow[l, "\iota'"']
\end{tikzcd}$$

The morphisms p and p' are the projection morphisms for parahoric bundles discussed above. The morphism ι is simply induced by extension of structure group. The morphism ι' is induced by extension of structure group and some additional non-canonical twisting if $d > 0$, and will be described in more detail in chapter 4. The other morphisms are the natural ones making the diagram commutative.

The basic strategy of the proof is to use these morphisms to prove that there is a natural isomorphism of global sections of the line bundle $\mathcal{L}_{\vec{w}}$ over Parbun_G and the associated line bundle $\mathcal{L}_{\vec{w}'}$ over $\mathrm{Parbun}_{L'}$. By the results in chapter 2, both p and p' induce an isomorphism of global sections. The proof that ι' induces an isomorphism of global sections is essentially the same as the argument in section 7 of [10], and we prove it in chapter 4.

In order to show that ι induces an isomorphism of sections, we use a method originally due to

Ressayre [43]. We start with a stack $\mathcal{C} \to \mathrm{Parbun}_G$, the fibers of which correspond to P-reductions of parabolic bundles of degree d and relative position $(u_1, \ldots, u_n)$. Our cohomology assumption guarantees that this morphism is *birational*, by which we simply mean there is an open subset of $\mathcal{C}$ mapping isomorphically to its image in Parbun_G. We then embed this stack into a larger stack $\mathcal{Y}$, containing $\mathcal{C}$ as a dense substack. This stack fits into the following diagram.

$$\begin{tikzcd}
\mathcal{Y} \arrow[d, "\pi"] \arrow[rrd, "\xi"] & & \\
\mathrm{Parbun}_G \arrow[r, "p"] & \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau}) & \mathrm{Bun}_Y^{\Gamma,L}(\boldsymbol{\tau}_u, 0) \arrow[l, "\iota"']
\end{tikzcd}$$

This diagram is *not* 2-commutative. However, it does induce a commutative diagram of global sections via pullback (see Proposition 4.1.3). The main theorem of this chapter is that π is proper over the semistable locus of Parbun_G.

Theorem 3.0.4. *The reduction stack $\mathcal{C}$ is embedded in a stack $\mathcal{Y}$ over $Parbun_G$ such that the restriction to semistable bundles $\pi : \mathcal{Y}^{ss} \to Parbun_G^{ss}$ is proper.*

The proof then goes as follows. By a version of Zariski's main theorem, pullback via the birational, proper morphism $\pi : \mathcal{Y}^{ss} \to \mathrm{Parbun}_G^{ss}$ induces an isomorphism of global sections for any line bundle. We will show that ξ is surjective (see Proposition 4.1.3 and Theorem 3.2.1) so pullback via ξ is injective. Then by a simple diagram chase, pullback via ι is an isomorphism.

3.1 Universal reduction stacks

We begin the construction of $\mathcal{Y}$ by reviewing the analogous space in the "classical" case, replacing conformal blocks with spaces of invariants, and parabolic bundles with tuples of flags. In [11] Belkale, Kumar and Ressayre use these varieties to prove a generalization of Fulton's conjecture.

Let $u_1, \ldots, u_n \in W^P$. Then B fixes the Schubert cell C_{u_i}, and we can form the fiber bundle $\mathfrak{C}_{u_i} = G \times^B C_{u_i}$ over G/B. Note that we have a natural projection map $\mathfrak{C}_{u_i} \to G/B$ and a map $\mathfrak{C}_{u_i} \to G/P$ defined by $[g, x] \mapsto gx$. Let $\mathfrak{C} = \mathfrak{C}_{u_1} \times \cdots \times \mathfrak{C}_{u_n}$. Then the *universal intersection scheme* $\mathcal{C}$ is defined as the fiber product of the map $\mathcal{C} \to (G/P)^n$ with the diagonal $\delta : G/P \to (G/P)^n$. In other words, the following diagram is Cartesian:

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d] & \mathfrak{C} \arrow[d] \\ G/P \arrow[r, hook] & (G/P)^n \end{tikzcd}$$

with the horizontal arrows being closed embeddings. Finally, we have natural maps $g : \mathcal{C} \to (G/B)^n$ and $h : \mathcal{C} \to G/P$, with the former being defined as the composition of the embedding $\mathcal{C} \hookrightarrow \mathfrak{C}$ and the projection $\mathfrak{C} \to (G/B)^n$. The points of $\mathcal{C}$ correspond to tuples $(\overline{g}_1, \ldots, \overline{g}_n, x)$ with each $\overline{g}_i \in G/B$, and $x \in G/P$ in the intersection of $g_1 C_{u_1}, \ldots, g_n C_{u_n}$. In a similar way we define the universal intersection schemes for the closed Schubert variety X_{u_i} and its smooth locus Z_{u_i}, denoting them $\mathcal{X}$ and $\mathcal{Z}$ respectively.

If we further assume that $\sigma_{u_1} \odot_0 \cdots \odot_0 \sigma_{u_n} = [pt]$, then $\mathcal{C} \to (G/B)^n$ is birational (see [11]). Let $\lambda_1, \ldots, \lambda_n$ be weights, and let $Q_1, \ldots, Q_n$ be the parabolic subgroups associated to each weight. Then replacing each C_{u_i} with $Q_i \cdot C_{u_i}$ we can construct a $\mathcal{Y}_0$ over $\prod_i G/Q_i$, and we can take the closure of $\mathcal{C}$ in the pullback of $\mathcal{Y}_0$ to get an integral scheme $\mathcal{Y}$ that surjects onto $\mathcal{Y}_0$. It fits into the following diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[dr] & \mathcal{Y} \arrow[r] \arrow[d] & \mathcal{Y}_0 \arrow[d] \\ & (G/B)^n \arrow[r] & \prod_i G/Q_i \end{tikzcd}$$

The following lemma gives us a different description of $\mathcal{Y}$.

Lemma 3.1.1. *If for $w, u \in W^P$ we have $w \leq u$, and for some weight λ we have*

$$\langle \omega_P, w^{-1}\lambda \rangle = \langle \omega_P, u^{-1}\lambda \rangle,$$

if and only if $uw^{-1} \in Q$, where Q is the parabolic subgroup associated to λ.

Proof. Now it is sufficient to prove the result when $w \xrightarrow{\beta} u$, or in other words when $u = s_\beta w$ and $l(u) = l(w) + 1$ for some positive root β. This is because for any $w \leq u$ we can find $\beta_1, \ldots, \beta_k$ such that $w = w_1 \xrightarrow{\beta_1} w_2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_k} w_{k+1} = u$ (cf. Prop 5.11 in [28]). Note that we are not assuming that β is a simple root. Then if $\lambda = \sum_i a_i \omega_i$ we have

$$\begin{aligned}
\langle \omega_P, u^{-1}\lambda \rangle = \langle u\omega_P, \lambda \rangle &= \sum_{i=1}^{n} a_i \langle u\omega_P, \omega_i \rangle \\
&= \sum_{i=1}^{n} a_i \langle w\omega_P, s_\beta \omega_i \rangle \\
&= \sum_{i=1}^{n} a_i \langle w\omega_P, \omega_i - \frac{2\langle \omega_i, \beta \rangle}{\langle \beta, \beta \rangle} \beta \rangle.
\end{aligned}$$

Now if $\beta = \sum_i b_i \alpha_i$ then by the definition of the fundamental weights ω_i we have $\langle \omega_i, \beta \rangle = b_i \langle \omega_i, \alpha_i \rangle = b_i \frac{\langle \alpha_i, \alpha_i \rangle}{2}$. Therefore we see that

$$\begin{aligned}
\langle \omega_P, u^{-1}\lambda \rangle &= \sum_{i=1}^{n} a_i \langle w\omega_P, \omega_i \rangle - \sum_{i=1}^{n} a_i \langle w\omega_P, \frac{2\langle \omega_i, \beta \rangle}{\langle \beta, \beta \rangle} \beta \rangle \\
&= \langle \omega_P, w^{-1}\lambda \rangle - \langle w\omega_P, \beta \rangle \sum_{i=1}^{n} \frac{a_i b_i \langle \alpha_i, \alpha_i \rangle}{2}.
\end{aligned}$$

Then by assumption we have

$$\langle w\omega_P, \beta \rangle \sum_{i=1}^{n} \frac{a_i b_i \langle \alpha_i, \alpha_i \rangle}{2} = 0.$$

We claim that $\langle w\omega_P, \beta \rangle > 0$. Clearly this is equivalent to showing that $(w^{-1}\beta)(x_P) > 0$. We know that $w^{-1}\beta \in R^+$, since $l(s_\beta w) = l(w) + 1$ (cf. Prop 5.7 in [28]). Therefore assume $(w^{-1}\beta)(x_P) = 0$. Then $s_{w^{-1}\beta}\omega_P = \omega_P$, and since $s_{w^{-1}\beta} = w^{-1}s_\beta w = w^{-1}u$ we have for any $\alpha \in R_P$

$$\langle u^{-1}w\alpha, \omega_P \rangle = \langle \alpha, w^{-1}u\omega_P \rangle = \langle \alpha, \omega_P \rangle = 0.$$

Therefore $u^{-1}wR_P = R_P$ and so $u^{-1}w$ is in W_P. But then u and w are in the same coset in W/W_P, and since they are different lengths and we assumed $u, w \in W^P$, we have reached a contradiction. Therefore, $(w^{-1}\beta)(x_P) > 0$.

Then we conclude that in fact $a_i b_i = 0$ for all i. In particular b_i may only be nonzero when $a_i = 0$. Therefore β is a sum of simple roots contained in the set of simple roots corresponding to Q. But $s_\beta \in N(T)/T$ is represented by $\exp(r(X_\beta + X_{-\beta}))$, for some real number r, where X_β and $X_{-\beta}$ are the root vectors corresponding to β and $-\beta$. But X_β and $X_{-\beta}$ are in the Lie algebra of Q, so we have $s_\beta = \exp(r(X_\beta + X_{-\beta})) \in Q$. $\square$

Now assuming that $\lambda_1, \ldots, \lambda_n$ are on the face of the eigencone corresponding to $\sigma_{u_1} \odot_0 \cdots \odot_0 \sigma_{u_n} = [pt]$, we can show that restricted to semistable flags, $\mathcal{Y}$ is proper over $(G/B)^n$. Let $f : C \to (G/B)^n$ be a family of semistable flags over a curve C, and $C^* \to \mathcal{Y}$ a compatible family over the punctured curve C^*. Since $\mathcal{Y} \subseteq \mathcal{X}$ and $\mathcal{X}$ is projective, we can take the limit and get a point $(\vec{g}, \overline{x}) \in \mathcal{X}$, where $\vec{g}$ is a semistable tuple of flags, and $\overline{x} \in G/P$. Any point in $\mathcal{X}$ corresponds to a Schubert position $w_1, \ldots, w_n$ such that $w_i \leq u_i$ for all i. By the above lemma, $\mathcal{Y}$ is the largest subvariety of $\mathcal{X}$ such that $\langle \omega_P, w_i^{-1}\lambda_i \rangle = \langle \omega_P, u_i^{-1}\lambda_i \rangle$ for any point in $\mathcal{Y}$ and any i. By the assumption that $\vec{\lambda}$ is on the given face of the eigencone, we know that

$$\sum_i \langle \omega_P, u_i^{-1}\lambda_i \rangle = 0.$$

Then since $\langle \omega_P, w_i^{-1}\lambda_i \rangle \geq \langle \omega_P, u_i^{-1}\lambda_i \rangle$ if $w_i \leq u_i$, we see that the limiting point $(\vec{g}, \overline{x})$ must be in $\mathcal{Y}$, since otherwise semistability would be violated.

The construction of $\mathcal{Y}$ and the properness proof for parabolic bundles will follow essentially the same strategy. The two main differences are that the space of partial flags $\prod_i G/Q_i$ needs to be replaced with the stack of parahoric bundles, and the space $\mathcal{X}$ needs to be replaced with a Hilbert scheme over C.

3.1.1 Stacks of P-reductions

We return to the notation and assumptions given at the beginning of the chapter. Then we have the following definitions.

Definition 3.1.2. The *universal reduction stack* $\mathcal{C}$ *relative to* $u_1, \ldots, u_n$ *and* d is the stack of pairs of parabolic bundles $\widetilde{\mathcal{E}} \in \mathrm{Parbun}_G$ and P-reductions $\mathcal{P} \times^P G \cong \mathcal{E}$ of degree d and relative Schubert position $u_1, \ldots, u_n$ (see section 2.1). The *smooth reduction stack* $\mathcal{Z}$ also includes P-reductions with relative positions $w_1, \ldots, w_n$ such that $C_{w_i} \subseteq Z_{u_i}$, where Z_{u_i} is the smooth locus of the closed Schubert variety X_{u_i}.

More formally, we can describe $\mathcal{C}$ as a locally closed substack of $\mathrm{Parbun}_{G,P}$, the moduli stack of parabolic bundles paired with P-reductions. Let $\widetilde{\mathcal{E}}$ be a family of parabolic bundles over S, and write $\mathcal{E}_i$ for the restriction of $\mathcal{E}$ to $\{p_i\} \times S$. Let $\mathcal{P}$ be the P-bundle corresponding to a

P-reduction $X \times S \to \mathcal{E}/P$, and similarly write the restrictions of $\mathcal{P}$ to each p_i as $\mathcal{P}_i$. Now we can consider $\mathfrak{C}_{u_i}$ as a locally closed subvariety of $G/B \times G/P$ via the morphism $\mathfrak{C}_{u_i} \to G/B \times G/P$ sending $(g, x) \mapsto (\overline{g}, gx)$. In a similar way we can consider the fiber bundle $\mathcal{E}_i \times^B \mathfrak{C}_{u_i}$ as a locally closed subscheme of $\mathcal{E}_i/B \times \mathcal{E}_i/P$. Now the B- and P-reductions of $\mathcal{E}_i$ correspond to a section $s_i : S \to \mathcal{E}_i/B \times \mathcal{E}_i/P$. The Schubert condition is that s_i must factor through $\mathcal{E}_i \times^B \mathfrak{C}_{u_i}$ for each i.

Clearly then $\mathcal{C}$ is a locally closed substack of $\mathrm{Parbun}_{G,P}$. Furthermore, since the morphism $\mathrm{Parbun}_{G,P} \to \mathrm{Parbun}_G$ is representable (see Proposition 3.3.1), so is $\mathcal{C} \to \mathrm{Parbun}_G$, and therefore $\mathcal{C}$ is an algebraic stack. Similarly we have stacks $\mathcal{Z}$ and $\mathcal{X}$ which are representable over Parbun_G. We also have natural open embeddings $\mathcal{C} \subseteq \mathcal{Z} \subseteq \mathcal{X}$.

Our next task will be to prove some properties of these intersection stacks.

3.1.2 Smoothness of $\mathcal{Z}$

The smoothness of $\mathcal{Z}$ simply comes down to the smoothness of the nonsingular loci of the Schubert varieties Z_{u_i}. Note that this implies that $\mathcal{C}$ is smooth as well.

Proposition 3.1.3. *The stack $\mathcal{Z}$ is smooth over Spec(k).*

Proof. We prove that $\mathcal{Z}$ is smooth over $\mathrm{Spec}(k)$ by showing it is formally smooth and locally of finite type. Note first that $\mathrm{Bun}_{G,P} \cong \mathrm{Bun}_P$ is smooth over $\mathrm{Spec}(k)$, and therefore locally of finite type. Furthermore, it is easy to see that the fibers of the projection $\mathcal{Z} \to \mathrm{Bun}_{G,P}$ are locally closed subschemes of Hilbert schemes, and therefore locally of finite type over the base.

Now we claim the projection morphism $\mathcal{Z} \to \mathrm{Bun}_P$ is formally smooth. This will finish the proof of the smoothness of $\mathcal{Z}$, since Bun_P is smooth. So let T be an affine scheme and $T \subseteq T'$ be an infinitesimal extension. Furthermore suppose we have a diagram

$$\begin{array}{ccc} T & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ T' & \longrightarrow & \mathrm{Bun}_P \end{array} \quad .$$

The top horizontal arrow corresponds to a G-bundle $\mathcal{E}$ over $X \times T$, with B-reductions over each p_i, and a P-reduction $\mathcal{F} \times^P G \cong \mathcal{E}$, satisfying the Schubert conditions as above. The bottom horizontal arrow corresponds to a P-bundle $\mathcal{F}'$ over $X \times T'$, and commutativity of the diagram corresponds to an isomorphism $\iota^*\mathcal{F}' \xrightarrow{\sim} \mathcal{F}$, where the bundle $\iota^*\mathcal{F}$ is a choice of pullback over the morphism $\iota : X \times T \to X \times T'$. Now letting $\mathcal{E}' = \mathcal{F} \times^P G$, we have a section $X \times T' \to \mathcal{E}'/P$, which we can furthermore restrict to each $\{p_i\} \times T'$ giving sections $s_i : T' \to p_i^*(\mathcal{E}'/P)$. Then since $\iota^*\mathcal{E}' \cong (\iota^*\mathcal{F}') \times^P G \cong \mathcal{E}$, we have a morphism $p_i^*\mathcal{E} \to p_i^*\mathcal{E}'$, and therefore a morphism of the associated bundles $(p_i^*\mathcal{E})(\mathfrak{Z}_{u_i}) \to (p_i^*\mathcal{E}')(\mathfrak{Z}_{u_i})$. The morphism $T \to \mathcal{Z}$ gives sections $T \to (p_i^*\mathcal{E})(\mathfrak{Z}_{u_i})$, and so we get morphisms $T \to (p_i^*\mathcal{E}')(\mathfrak{Z}_{u_i})$. These morphisms form a commutative diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d] & (p_i^*\mathcal{E}')(\mathfrak{Z}_{u_i}) \arrow[d] \\ T' \arrow[r] & (p_i^*\mathcal{E}')(G/P) \end{tikzcd} .$$

The right vertical arrow is smooth since the projection $\mathfrak{Z}_{u_i} \to G/P$ is smooth (see [11] section 5), and therefore we have a lift $T' \dashrightarrow (p_i^*\mathcal{E}')(\mathfrak{Z}_{u_i})$, which composed with $(p_i^*\mathcal{E}')(\mathfrak{Z}_{u_i}) \to (p_i^*\mathcal{E}')(G/B)$ gives the lift $T' \dashrightarrow \mathcal{Z}$. ∎

3.1.3 Irreducibility of $\mathcal{Z}$

Consider the natural projection $\mathcal{Z} \to \mathrm{Bun}_{G,P}(d)$, where $\mathrm{Bun}_{G,P}(d)$ is the stack of principal G-bundles over X paired with degree d P-reductions. We showed above that this projection is smooth; it is easy to see therefore that the induced morphism of topological spaces $|\mathcal{Z}| \to |\mathrm{Bun}_{G,P}(d)|$ is open (cf. [23, Exercise III.9.1]).

Now clearly $\mathrm{Bun}_{G,P}(d) \cong \mathrm{Bun}_P(d)$. Then $\mathrm{Bun}_P(d)$ is irreducible, since $\pi_0(\mathrm{Bun}_P) \cong \pi_1(P) \cong \mathbb{Z}$ (cf. [26] for a proof). Then we have the following.

Proposition 3.1.4. *The stack $\mathcal{Z}$ is irreducible.*

Proof. First we claim that the fibers of $\mathcal{Z} \to \mathrm{Bun}_{G,P}(d)$ over k-rational points are irreducible. The fiber of the projection over the point $\mathrm{Spec}(k) \to \mathrm{Bun}_{G,P}(d)$ is simply the independent choices of flags in G/B satisfying the Schubert conditions with respect to each point $x_i \in G/P$ given by the P-reduction $\mathcal{P}$ and our chosen Weyl group elements u_i. It is easy to see that each space of possible choices in G/B is an irreducible variety. Therefore the fiber over the given point is a product of irreducible varieties, which is irreducible.

Let U_1, U_2 be nonempty open subsets of $|\mathcal{Z}|$. Then since the projection $|\mathcal{Z}| \to |\mathrm{Bun}_{G,P}(d)|$ is open, and the stack $\mathrm{Bun}_{G,P}(d)$ is irreducible, the images of U_1 and U_2 in $|\mathrm{Bun}_{G,P}(d)|$ are open and intersect non-trivially; say the intersection is V, a nonempty open subset of $|\mathrm{Bun}_{G,P}(d)|$. Then choosing a presentation $Y \to \mathrm{Bun}_{G,P}(d)$, the pullback W of V to Y is a nonempty open subset of Y. But Y is a smooth variety over $\mathrm{Spec}(k)$, and therefore W contains a k-rational point. Then we have that the fiber over a k-rational point of $\mathrm{Bun}_{G,P}(d)$ intersects both U_1 and U_2 non-trivially. But we know that such a fiber is irreducible, and therefore that U_1 and U_2 must intersect non-trivially. $\square$

3.2 Lifting families of P-reductions

We need to embed $\mathcal{C}$ in a larger stack $\mathcal{Y}_0$ in order to construct $\mathcal{Y}$ by taking the closure. Our approach is to first lift the P-reductions in $\mathcal{C}$ to P-reductions of equivariant bundles.

Say $\widetilde{\mathcal{E}}$ is a parabolic G-bundle on X, and E is a (Γ, G)-bundle on Y, of local type $\boldsymbol{\tau}$ such that E is the image of $\widetilde{\mathcal{E}}$ in $\mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})$. Then P-reductions of $\mathcal{E}$ can clearly be lifted individually to E, since generically E is just the pullback of $\mathcal{E}$ to Y with the trivial Γ structure, and the closure of a generic P-reduction exists and is unique. However in families this process is *discontinuous*. This can be seen by considering a connected family of P-reductions of $\mathcal{E}$ that jumps in parabolic degree: once lifted to E this becomes a change in plain degree.

Let $\mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau}; \boldsymbol{\tau}_u, 0)$ be the stack of (Γ, G)-bundles of local type $\boldsymbol{\tau}$ together with invariant P-reductions of local type $\boldsymbol{\tau}_u = (u_1^{-1}\tau_1 u_1, \ldots, u_n^{-1}\tau_n u_n)$ and degree 0. The goal of this section is to prove the following theorem.

Theorem 3.2.1. *There exists a representable, surjective morphism $\mathcal{C} \to \mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau}; \boldsymbol{\tau}_u, 0)$.*

3.2.1 $\mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau}) \xrightarrow{\sim} \mathrm{Bun}_{\mathcal{G}}$ in terms of transition functions

We need to make the identification of parahoric bundles with equivariant bundles more explicit for what follows. For more details see [3].

By [5] and [53] we can describe a (Γ, G)-bundle E as follows. Let $\mathcal{R}$ be the ramification locus of $p : Y \to X$. Let E_0 be the trivial G-bundle over $Y^* = Y \setminus \mathcal{R}$ with trivial Γ action, and for each $y \in \mathcal{R}$ such that $p(y) = p_i$, let E_y be the trivial (Γ, G)-bundle over N_y with local type τ_i. Then E is isomorphic to the (Γ, G)-bundle corresponding to a choice of $\Theta_i \in G(K)$, giving transition functions

$$\begin{aligned}(E_0)_{|N_y^*} &\xrightarrow{\sim} (E_y)_{|N_y^*}\\ (\omega, g) &\mapsto (\omega, \Delta_i(\omega)\Theta_i(\omega)g),\end{aligned}$$

where $\Delta_i \in G(K')$ is associated to μ_i as in chapter 2. Note that the choice of Θ_i is not unique. Changing the trivialization of E_0 multiplies Θ_i on the right by an element of $G(K)$, and changing the trivialization of E_y multiplies Θ_i on the left by an element of the parahoric subgroup $\mathcal{P}_i$ corresponding to μ_i.

Now $\mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau}) \xrightarrow{\sim} \mathrm{Bun}_{\mathcal{G}}$ is given as follows. Let $F \in \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})$ be the bundle where each $\Theta_y = e$. Let $\mathcal{G}_F$ be the adjoint bundle $F \times^G G$, with G acting on itself by conjugation, and let $\mathcal{G}' = p_*^{\Gamma}(\mathcal{G}_F)$ be the invariant push-forward of this group scheme. The group scheme $\mathcal{G}_F$ can be identified with the sheaf of automorphisms of F, and $\mathcal{G}'$ is a representable by a smooth group scheme over X isomorphic to the parahoric group scheme $\mathcal{G}$. Let $\mathrm{Isom}(E, F)$ be the sheaf of local isomorphisms of E and F. This sheaf is a right $\mathcal{G}_F$-torsor. Then $p_*^{\Gamma}(\mathrm{Isom}(E, F))$ is representable by a smooth variety over X and is naturally a right $\mathcal{G}$-torsor.

Let E be an equivariant bundle with transition functions given by $\Delta_i(\omega)\Theta_i$ as above. Then the $\mathcal{G}$-bundle $\mathcal{E}$ corresponding to E can be described as follows. If $\mathcal{E}_0 = X^* \times G$ is the trivial G-bundle and $\mathcal{E}_i = \mathcal{G}_i$ is the parahoric group scheme corresponding to each μ_i, then $\mathcal{E}$ is isomorphic to the $\mathcal{G}$-bundle given by $\Theta_i \in G(A)$, giving transition morphisms

$$\begin{aligned}(\mathcal{E}_0)_{|U_{p_i}^*} &\xrightarrow{\sim} (\mathcal{E}_i)_{|U_{p_i}^*}\\ (z, g) &\mapsto (z, \Theta_i(z)g).\end{aligned}$$

3.2.2 Construction of $\mathcal{C} \to \mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau}; \boldsymbol{\tau}_u, 0)$

Consider a morphism $S \to \mathcal{C}$, where S is an arbitrary scheme. This corresponds to a family of parabolic bundles over S and a family of P-reductions of the underlying family of G-bundles $\mathcal{E} \to X \times S$. This gives a family (which we also denote $\mathcal{E}$) of $\mathcal{G}$-bundles and a family of *generic* P-reductions. Now by the uniformization theorem in [24] there is an étale cover $\widetilde{S} \to S$ such that $\mathcal{E}$ is trivialized over $X^* \times \widetilde{S}$ and $U_x \times \widetilde{S}$ for each $x \in \{p_1, \ldots, p_n\}$. Let $\mathcal{E}_0$ be the restriction of $\mathcal{E}$

to $X^* \times \widetilde{S}$, and let $\mathcal{E}_x$ be the restriction of $\mathcal{E}$ to each $U_x \times \widetilde{S}$. Suppose $\Theta_x : U_x^* \times \widetilde{S} \to G$ gives the transition map over $U_x^* \times \widetilde{S}$ with respect to some trivialization of $\mathcal{E}_0$ and $\mathcal{E}_x$. Then by the above work the corresponding Γ-equivariant bundle E is given by the transition functions $\Delta_y \Theta_x$ for each $p(y) = x$.

Let $\overline{g}_x : \widetilde{S} \to G/B$ be the family of flags at x. Then taking a further refinement of $\widetilde{S}$ (which we continue to denote $\widetilde{S}$) we can lift this morphism to $g_x : \widetilde{S} \to G$. So clearly we can choose a trivialization of $\mathcal{E}$ near x such that the family of flags is identically trivial. Now locally near x, the P-reduction of $\mathcal{E}$ corresponds to a morphism $\psi : U_x \times \widetilde{S} \to G/P$. Then since the flags are trivial, the corresponding generic P-reduction of $\mathcal{G}$ near x is just the restriction of ψ to $U_x^* \times \widetilde{S}$, and the generic P-reduction of E is given by $\Delta_y \psi : N_y^* \times \widetilde{S} \to G/P$. Our first task is to show that this morphism extends to all of $N_y \times \widetilde{S}$, and that therefore the generic P-reduction of E extends to all of $Y \times S$.

Lemma 3.2.2. *For a scheme S and a morphism $\psi : U \times S \to G$, if the restriction ψ_0 to* $\mathrm{Spec}(k) \times S$ *factors through B, then $\Delta \psi \Delta^{-1}$ is defined on all of $N \times S$.*

Proof. Since ψ_0 lands in B, the morphism ψ corresponds to a morphism $S \to L^+\mathcal{I} \to L^+\mathcal{P}$, where $\mathcal{P}$ is the parahoric subgroup corresponding to Δ. Conjugation by Δ induces an isomorphism of group schemes $\mathcal{G} \cong \mathcal{G}'$, where $\mathcal{G}$ is the parahoric group scheme corresponding to $\mathcal{P}$, and $\mathcal{G}'$ is the group scheme obtained by invariant pushforward. Therefore $L^+\mathcal{P} = L^+\mathcal{G} \cong L^+\mathcal{G}'$, and $L^+\mathcal{G}' \subseteq L^+G(A')$. □

Proposition 3.2.3. *For any scheme S and morphism $\psi : N \times S \to G$ such that $\psi_0 = \psi(0, s)$ factors through the Schubert cell C_w^P, the morphism $\Delta\psi : N^* \times S \to G/P$ can be uniquely extended to $N \times S$.*

Proof. By assumption ψ_0 factors through $BwP \subseteq G$. Now as shown in section 8.3 of [49], $U_{w^{-1}} \times P \cong BwP$, where $U_{w^{-1}}$ is a subgroup of the unipotent radical of B, and the isomorphism is given by $(u, p) \mapsto uwp$. Let f_0 and g_0 be the compositions of ψ_0 and the projections to P and $U_{w^{-1}}$, and let $\psi' = wf_0$, extended to $N \times S$. Clearly then, since Δ is a one-parameter subgroup of T, $(\psi')^{-1}\Delta^{-1}\psi'$ maps to P. Therefore $\Delta\psi \cdot (\psi')^{-1}\Delta^{-1}\psi'$ composed with $G \to G/P$ is equal to $\Delta\psi$. But $\psi \cdot (\psi')^{-1}$ is just $g_0 : S \to B$ at $\omega = 0$, and therefore by Lemma 3.2.2 $\Delta\psi \cdot (\psi')^{-1}\Delta^{-1}\psi'$ is defined for all of $N \times S$. Since G/P is projective, the extension is clearly unique. □

Corollary 3.2.4. *There exists a morphism of stacks $\mathcal{C} \to \mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau})$.*

Proof. By the above proposition, we have constructed a P-reduction of E over $Y \times \widetilde{S}$. The descent data for $\mathcal{E}$ gives descent data for E, and it is easy to see that the P-reduction we constructed descends to E over $Y \times S$. Let $\mathcal{T}$ be the stack over $\mathcal{C}$ adding the data of a trivialization near each ramification point making the flag trivial. Then we have constructed a morphism $\mathcal{T} \to \mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau})$. It is well known that $\mathcal{T} \to \mathcal{C}$ is a torsor with respect to the action on $\mathcal{T}$ by

$$\prod^{n} L^+\mathcal{I},$$

and therefore it suffices to show our construction does not depend on the choice of trivialization (see for example [37]). A change in trivialization of $\mathcal{E}$ multiplies ψ on the left by some $f \in L^+\mathcal{I}(\widetilde{S})$. But then $\Delta f\psi = \Delta f \Delta^{-1}\Delta\psi$ and $\Delta f \Delta^{-1}$ corresponds to a change of trivialization of E, since $\Delta\mathcal{I}\Delta^{-1} \subseteq G(A')^{\Gamma}$. Therefore the P-reduction does not depend on the choice of trivialization, finishing the proof. $\square$

We also want to identify the degree and local type of the reductions in the image of this morphism.

Proposition 3.2.5. *The morphism $\mathcal{C} \to \mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau})$ factors through $\mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau};\boldsymbol{\tau}_u,0)$, the substack of degree 0 P-reductions of local type $\boldsymbol{\tau}_u$.*

Proof. Let $\widetilde{\mathcal{E}}$ be a parabolic bundle and $\mathcal{F}$ be a P-reduction, together corresponding to a point in $\mathcal{C}$. Let E be the corresponding (Γ, G)-bundle, and F be the corresponding Γ-invariant P-reduction of E. Then Teleman and Woodward showed that the degree of F is a positive scalar multiple of the parabolic degree of the original P-reduction $\mathcal{F}$ [53]. Since by assumption the parabolic degree is 0, the degree of F is also 0.

Following the proof of Proposition 3.2.3, if the P-reduction $\mathcal{F}$ is given near x by $\psi : U_x \to G/P$, then the P reduction of E is given by $\Delta_y\psi$. That is, locally the P-reduction corresponds to a map:

$$\begin{aligned}(F_y)_{|N_y^*} &\to (E_y)_{|N_y^*}\\ (\omega, p) &\mapsto (\omega, \Delta_y(\omega)\psi(z)p).\end{aligned}$$

Now say $\psi(0) = bu_ip$. Then the completion of $\Delta_y\psi$ is $\Delta_y\psi p^{-1}u_i^{-1}\Delta_y^{-1}u_ip$. Note that $f(\omega) = \Delta_y\psi p^{-1}u_i^{-1}\Delta_y^{-1}$ is in $G(A')^{\Gamma_y}$, since $\psi p^{-1}u_i^{-1}$ is in the Iwahori subgroup $\mathcal{I}$, and therefore $f(0)$ is in the centralizer $C_G(\tau_i)$.

Now the P-reduction given by $\phi = fu_ip$ is Γ_y-invariant, which means that for every $\gamma \in \Gamma_y$ there is a $p(\gamma,\omega) \in P(A')$ such that

$$\tau_i(\gamma)\phi(\omega) = \phi(\gamma\omega)p(\gamma,\omega).$$

Then the induced Γ_y action on F_y is given by $p(\gamma,\omega) = \phi(\gamma\omega)^{-1}\tau_i(\gamma)\phi(\omega)$, and therefore changing trivializations of F_y as in [53], we see that the local type of F is $\phi(0)^{-1}\tau_i(\gamma)\phi(0) = p^{-1}u_i^{-1}f(0)^{-1}\tau_if(0)u_ip = p^{-1}u_i^{-1}\tau_iu_ip$, finishing the proof. □

3.2.3 Proof of Theorem 3.2.1

We've proven that we have a morphism of stacks $\mathcal{C} \to \mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau};\boldsymbol{\tau}_u,0)$. It remains to show this morphism is representable and surjective.

Proof of Theorem 3.2.1. Consider the following diagram, where $\mathcal{Y}_0$ is the pullback of $\mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau};\boldsymbol{\tau}_u,0)$.

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr] & \mathcal{Y}_0 \arrow[r] \arrow[d] & \mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau};\boldsymbol{\tau}_u,0) \arrow[d] \\ & \mathrm{Parbun}_G \arrow[r] & \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau}) \end{tikzcd}$$

Then since $\mathrm{Parbun}_G \to \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})$ is representable, $\mathcal{Y}_0 \to \mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau};\boldsymbol{\tau}_u,0)$ is representable. Therefore we just need to show $\mathcal{C} \to \mathcal{Y}_0$ is representable. We claim that this morphism is a monomorphism and locally of finite type, and therefore representable [38, Cor 8.1.3] [55, Tag 0B89]. The morphism is locally of finite type since $\mathcal{C} \to \mathrm{Parbun}_G$ is locally of finite type. Suppose we have two parabolic bundles $\widetilde{\mathcal{E}}_1$, $\widetilde{\mathcal{E}}_2$, with P-reductions $\phi_i : X \times S \to \mathcal{E}_i$ giving morphisms $f_1, f_2 : S \to \mathcal{C}$. Let E_1, E_2 be the corresponding equivariant bundles, and ϕ_i^Y the corresponding invariant P-reductions. Then to show that $\mathcal{C} \to \mathcal{Y}_0$ is a monomorphism, it is sufficient (by definition) to show that an isomorphism $\widetilde{\mathcal{E}}_1 \cong \widetilde{\mathcal{E}}_2$ identifies ϕ_1 and ϕ_2 if and only if it identifies ϕ_1^Y and ϕ_2^Y. But since E_i is just the pullback of $\mathcal{E}_i$ away from $p_1,\ldots,p_n$, clearly this is true generically. Then since $\mathcal{E}_i/P$ and E_i/P are separated over $X \times S$, if the P-reductions are identified away from $p_1,\ldots,p_n$, they are the same over all of $X \times S$. Therefore $\mathcal{C} \to \mathcal{Y}_0$ is a monomorphism and representable.

Second proof: We sketch a second proof that provides a local description of $\mathcal{C}$ in $\mathcal{Y}_0$, and additionally shows that $\mathcal{C}$ is immersed in $\mathcal{Y}_0$. Suppose we have a smooth morphism $S \to \mathcal{Y}_0$. We need to show that the stack theoretic pullback of this morphism is representable by a scheme. This morphism corresponds to the following data: a family of parabolic bundles $\widetilde{\mathcal{E}}$ over S, a corresponding Γ-equivariant family of G-bundles $E \to Y \times S$ and a Γ-invariant P-reduction $Y \times S \to E/P$, with the given local types. Then passing to an étale cover of S we can trivialize $\mathcal{E}$ over $X^* = X \setminus \{p_1, \ldots, p_n\}$ and formal neighborhoods $x \in U_x$ for each branch point $x \in \{p_1, \ldots, p_n\}$ so that the flags are trivial. This induces a trivialization of E over Y^* and formal neighborhoods N_y for each ramification point y. Say the P-reduction near y is given by $\phi : N_y \times S \to G$, so that letting F be the P-bundle corresponding to the P-reduction of E the P-reduction then corresponds to

$$\begin{aligned} (F)_{|N_y} &\to (E)_{|N_y} \\ (\omega, s, p) &\mapsto (\omega, s, \phi(\omega, s)p), \end{aligned}$$

where the Γ-action on E and F are constant with respect to ω and s and given by τ and τ_u, respectively, passing to another étale cover if necessary.

We see that $\phi(\omega, s)u_i^{-1}$ gives a morphism $S \to L^+G(A')^\Gamma$, since

$$\begin{aligned} \tau(\gamma)\phi(\gamma^{-1}\omega, s)u_i^{-1}\tau(\gamma)^{-1} &= \tau(\gamma)\phi(\gamma^{-1}\omega, s)u_i^{-1}\tau(\gamma)^{-1}u_i u_i^{-1} && (3.1) \\ &= \tau(\gamma)\phi(\gamma^{-1}\omega, s)\tau_u(\gamma)^{-1}u_i^{-1} && (3.2) \\ &= \phi(\omega, s)u_i^{-1}. && (3.3) \end{aligned}$$

Let Δ_y be the rational OPS as above, and let B' be the image of $\Delta_y \mathcal{I} \Delta_y^{-1}$ in G. Then B' is contained in a Borel subgroup B_μ of G, where $B_\mu = w_\mu B w_\mu$ (see Proposition 2.3.4). So letting $P_\mu = w_\mu P w_\mu$ we see that $\phi(\omega, s)w_\mu$ gives a well-defined morphism to $C_G(\tau_i)u_i w_\mu P_\mu/P_\mu$, and that $B' u_i w_\mu P_\mu/P_\mu$ is contained in $C_G(\tau_i)u_i w_\mu P_\mu/P_\mu$. Let $S_\mathcal{C}$ be the pullback of S to $B' u_i w_\mu P_\mu/P_\mu$ over each ramification point. Then it is not hard to see using Proposition 2.3.5 that ϕu_i^{-1} restricted to $N_y \times S_\mathcal{C}$ corresponds to a morphism to $L^+\mathcal{I}$. Following the construction of the morphism $\mathcal{C} \to \mathrm{Bun}_Y^{\Gamma, G; P}(\boldsymbol{\tau}; \boldsymbol{\tau}_u, 0)$ in reverse, we see therefore that $\Delta^{-1}\phi$ restricted to $S_\mathcal{C}$ gives a local P-

reduction of Schubert position u_i of $\widetilde{\mathcal{E}}$. Therefore we get a morphism $S_{\mathcal{C}} \to \mathcal{C}$. It is easy to see the any restriction $T \to S \to \mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau};\boldsymbol{\tau}_u,0)$ that factors through $\mathcal{C}$ must locally factor through $B'u_iw_\mu P_\mu/P_\mu$, and therefore through $S_{\mathcal{C}}$.

Surjectivity: Now to show the morphism is surjective, suppose we have an equivariant bundle E over $Y \times \mathrm{Spec}(k')$ with an invariant P-reduction. Then since $G(A')^\Gamma$ surjects onto $C_G(\tau)$, we can choose trivializations of E such that the P-reduction gives elements $x \in B'u_iw_\mu P_\mu/P_\mu$ over the ramification points of Y. Then taking the transition functions of E with respect to this trivialization, and modifying them by Δ_y^{-1}, we get transition functions for a G-bundle over X. Taking the trivial flags we get a parabolic bundle $\widetilde{\mathcal{E}}$ which maps to E. By the above work the P-reduction of E corresponds a P-reduction of $\mathcal{E}$ giving a point in $\mathcal{C}$. □

3.3 Properness over the semistable locus

Now we can set up the properness calculation. Let $\mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau};\boldsymbol{\tau}_u,0)$ be as above. The letters ss will mean we are working with semistable objects with respect to the given weight data $\vec{w}$; when it appears on Parbun_G we mean the inverse image of the semistable locus of $\mathrm{Bun}_{\mathcal{G}}$. Then by the above work there exists an embedding $\mathcal{C} \hookrightarrow \mathcal{Y}_0 = \mathrm{Parbun}_G \times_{\mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})} \mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau};\boldsymbol{\tau}_u,0)$ making the following diagram commute:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook] \arrow[dr] & \mathcal{Y}_0 \arrow[r, two heads] \arrow[d] & \mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau};\boldsymbol{\tau}_u,0) \arrow[d] \\
& \mathrm{Parbun}_G \arrow[r, two heads] & \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})
\end{tikzcd}.$$

Let $\mathcal{Y}$ be the closure of $\mathcal{C}$ in $\mathcal{Y}_0$. Like $\mathcal{C}$, $\mathcal{Y}$ is an integral algebraic stack over Parbun_G. Then from this diagram it is clear that if the projection $\mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau},ss;\boldsymbol{\tau}_u,0) \to \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau},ss)$ is proper, then $\mathcal{Y}^{ss} \to \mathrm{Parbun}_G^{ss}$ is proper.

Let $\mathcal{R}_0$ be the stack of all degree 0 P-reductions over $\mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})$. The following propositions are a standard application of the theory of Quot schemes.

Proposition 3.3.1. *The morphism $\mathcal{R}_0 \to \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})$ is representable, separated and of finite type.*

Proof. Let S be noetherian scheme, and consider the fiber $\mathcal{W}$ of $\mathcal{R}_0 \to \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})$ over a morphism $S \to \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})$. Then it is easy to see that $\mathcal{W}$ is isomorphic to the stack of P-reductions of the bundle

$\mathcal{E} \to Y \times S$ corresponding to $S \to \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})$, which is isomorphic to the stack $\mathfrak{Mor}_{Y\times S}(Y \times S, \mathcal{E}/P)$ of sections of $\mathcal{E}/P \to Y \times S$. But since $Y \times S$ and $\mathcal{E}/P$ are strongly projective (in the sense of Altman and Kleiman) over $Y \times S$, $\mathfrak{Mor}_{Y\times S}(Y \times S, \mathcal{E}/P)$ is representable by a quasi-projective scheme over $Y \times S$ (see [2]). Therefore $\mathcal{R}_0 \to \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})$ is representable, separated and of finite type. $\square$

Proposition 3.3.2. *The stack* $\mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau};\boldsymbol{\tau}_u,0)$ *is a closed substack of the stack* $\mathcal{R}_0$.

Proof. First we claim that the substack of Γ-invariant P-reductions is closed. The finite group Γ acts by isomorphisms on the scheme $\mathfrak{Mor}_{Y\times S}(Y \times S, \mathcal{E}/P)$ of sections of $\mathcal{E}/P \to Y \times S$ in the obvious way, using the fact that Γ acts on Y and $\mathcal{E}$ by isomorphisms. Then since by definition Γ acts trivially on S, for each $\gamma \in \Gamma$ we get a morphism $f_\gamma : \mathfrak{Mor}_{Y\times S}(Y \times S, \mathcal{E}/P) \to \mathfrak{Mor}_{Y\times S}(Y \times S, \mathcal{E}/P) \times_S \mathfrak{Mor}_{Y\times S}(Y \times S, \mathcal{E}/P)$ given by the identity and the isomorphism induced by γ. But since $\mathfrak{Mor}_{Y\times S}(Y \times S, \mathcal{E}/P)$ is separated over S, its diagonal morphism Δ is closed, and so the space of γ-invariant P-reductions is the closed subscheme given by the fiber product of f_γ and Δ. Then the space of Γ invariant P-reductions of $\mathcal{E}$ is given by the scheme-theoretic intersection of these closed subschemes; we denote this space $\mathfrak{Mor}^{\Gamma}_{Y\times S}(Y \times S, \mathcal{E}/P)$. Finally, if we have $S \to \mathcal{R}_0$, we get a morphism $S \to \mathfrak{Mor}_{Y\times S}(Y \times S, \mathcal{E}/P)$; intersecting with $\mathfrak{Mor}^{\Gamma}_{Y\times S}(Y \times S, \mathcal{E}/P)$ we get a closed subscheme S^Γ of S and a morphism $S^\Gamma \to \mathrm{Bun}^{\tau}_{Y,P}$. Thus $\mathrm{Bun}^{\tau}_{Y,P} \to \mathcal{R}_0$ is a representable closed embedding of stacks.

The fact that Γ-invariant P-reductions of fixed local type form a closed substack of $\mathcal{R}_0$ follows from the observation that the conjugacy class of $u_i^{-1}\tau_i u_i$ is closed in P, since $u_i^{-1}\tau_i u_i$ is a semisimple element. If $E \to Y \times S$ is a family of bundles in $\mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})$, and F is the (Γ, P)-bundle associated to an invariant P-reduction of E, then choosing trivializations of E near each ramification point y, the action of Γ_y gives a morphism $f_y : S \to P$. Then by taking the fiber product of each f_y with the conjugacy classes $C_P(u_i^{-1}\tau_i u_i)$, we get a closed subscheme of S, which is corresponds to the restriction of the family of P-reductions to those of local type $\boldsymbol{\tau}_u$. Using the same method as in Corollary 3.2.4, one can show this construction defines a closed substack of the Γ-invariant substack of $\mathcal{R}_0$. $\square$

Therefore $\mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau}, ss;\boldsymbol{\tau}_u, 0) \to \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau}, ss)$ is proper if $\mathcal{R}_0^{ss} \to \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau}, ss)$ is proper, and it is sufficient to show that $\mathcal{R}_0^{ss} \to \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau}, ss)$ satisfies the existence part of the valuative criterion for properness to complete the proof of Theorem 3.0.4.

The following proposition is an easy consequence of the main lemma in [27], which is used to prove a no-ghosts theorem similar to the one we need, the main difference being that our G-bundle is not fixed.

Proposition 3.3.3. *Suppose C is a DVR with an algebraically closed residue field, and that X and Y are integral schemes, flat and projective of relative dimension 1 over C, with Y furthermore smooth over C. Suppose we have a flat C-morphism $f : X \to Y$ that is an isomorphism over C^*, and ξ is a relatively ample line bundle on X.*

Then if the restriction $f_0 : X_0 \to Y_0$ of f to the closed point of C is not an isomorphism, there is a unique component D of X_0 such that $f_0 : D_{red} \to Y_0$ is an isomorphism and $deg(D_{red}, \xi) < deg(X_{|C^*}, \xi)$.

Now we are ready to complete the properness proof.

Proof of Theorem 3.0.4. Now suppose we have the following diagram, where C is the spectrum of a complete discrete valuation ring with an algebraically closed residue field k', and C^* is the spectrum of its quotient field.

$$\begin{array}{ccc} C^* & \longrightarrow & \mathcal{R}_0^{ss} \\ \downarrow & & \downarrow \\ C & \longrightarrow & \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau}, ss) \end{array}$$

In order to prove the right vertical arrow is universally closed, it is sufficient (see [21] II.7.3.8 and [38] Theorem 7.3 and 7.5) to show that we can find a lift $C \to \mathcal{R}_0^{ss}$ making the above diagram 2-commutative. This diagram corresponds to a family of semistable (Γ, G)-bundles $\mathcal{E} \to Y \times C$ and a family of degree 0 P-reductions $\phi : Y \times C^* \to \mathcal{E}/P$. Now since $\mathcal{E}/P$ is projective over C, we can complete the subscheme $\phi(Y \times C^*)$ to a closed subscheme $Z \subseteq \mathcal{E}/P$, with Z flat over C. Our goal is to show that this subscheme corresponds to a section of $\mathcal{E}/P \to Y \times C$.

We claim that $f : Z_0 \to Y_0 \cong Y \times \mathrm{Spec}(k')$ is an isomorphism. Suppose not. Then by the above proposition there is a unique component D of Z_0 such that $f : D_{\mathrm{red}} \to Y_0$ is an isomorphism. Now let T_π be the tangent bundle along the fibers of $\mathcal{E}/P \to Y \times C$, and let ξ be the restriction of the determinant of this bundle to Z_0. Then ξ is ample and therefore by the above proposition we have $\deg(D_{red}, \xi) < \deg(Z_{|C^*}, \xi)$. But by assumption $\deg(Z_{|C^*}, \xi) = 0$ and therefore $\deg(D_{red}, \xi) < 0$, which violates the Γ-semistability of $\mathcal{E}_{|p_0}$. (It is well-known that a bundle is Γ-semistable if and only if it is semistable; see the proof of Proposition 2.4.1.) Therefore $f : Z_0 \to Y_0$ is an isomorphism.

Finally we need to prove that the map $f : Z \to Y \times C$ is an isomorphism. Now we know that Z is integral, since it is the closure of a subscheme of $\mathcal{E}/P$ isomorphic to $Y \times C^*$. Furthermore f is birational by assumption. Let V be the open subset of points x in $Y \times C$ such that their fibers $f^{-1}(x)$ are zero-dimensional. Then the restriction $f : f^{-1}(V) \to V$ is projective and quasi-finite, and therefore finite. Then since it is also birational and V is normal, it is an isomorphism by Zariski's main theorem: see Lemme 8.12.10.1 in [22]. Therefore V is contained in the largest open set U such that there exists a morphism $U \to Z$ representing the birational inverse of f. But then by the above work, $Y_0 \subseteq V \subseteq U$. Therefore $f : Z \to Y \times C$ is an isomorphism, proving that we have a lift $C \to \mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau}, ss; 0)$, finishing the proof of Theorem 3.0.4. $\square$

CHAPTER 4: CONCLUSION OF THE PROOF OF THE REDUCTION THEOREM

We now have all the necessary geometric properties of the P-reduction stacks needed to prove the reduction theorem. We continue to assume that $X \cong \mathbb{P}^1$, and $\vec{w}$ is weight data in the multiplicative polytope lying on a regular facet of the polytope corresponding to Weyl group elements $u_1, \ldots, u_n \in W^P$ and a degree d.

We prove the reduction in two steps: first, we reduce to the Levi subgroup L of P using the properties of $\mathcal{Y}$ proven in chapter 3. Then we reduce to the derived subgroup $L' = [L, L]$ using an argument similar to the one in section 7 of [10]. For an outline of the strategy of the proof, see the discussion at the beginning of chapter 3.

4.1 Reduction to the Levi subgroup $L \subseteq P$

The first step is to lift conformal blocks to $\mathcal{Y}$.

Proposition 4.1.1. *Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a representable morphism of Artin stacks, where $\mathcal{Y}$ is smooth over k, $\mathcal{X}$ is integral, and f is birational and proper. (By birational we simply mean that there is a non-empty open substack $\mathcal{U} \subseteq \mathcal{X}$ such that f restricted to $\mathcal{U}$ is an isomorphism onto its image.) Then for any line bundle $\mathcal{L}$ over $\mathcal{Y}$, the pullback via f induces an isomorphism of global sections:* $\mathrm{H}^0(\mathcal{Y}, \mathcal{L}) \xrightarrow{\sim} \mathrm{H}^0(\mathcal{X}, \mathcal{L})$.

Proof. Let $U \to \mathcal{Y}$ be a smooth morphism from a connected (and therefore irreducible) scheme U. First we show that, using Zariski's main theorem, the pullback of global sections is an isomorphism $\mathrm{H}^0(U, \mathcal{L}) \xrightarrow{\sim} \mathrm{H}^0(V, \mathcal{L})$, where V is the pullback of U to $\mathcal{X}$.

Assume that f is birational and proper, and that $\mathcal{X}$ is integral. Let $V \to \mathcal{X}$ be the pullback of U. Our first goal is to show that V is irreducible. Let $X = |\mathcal{X}|$ and $Y = |\mathcal{Y}|$ be the sets of points of these stacks with the Zariski topology. Now by definition $|V| \to X$ and $|U| \to Y$ are continuous, open, and surjective maps. Furthermore by assumption, the map $X \to Y$ is an isomorphism over some open set $T \subseteq Y$, and therefore $|V| \to |U|$ is an isomorphism over some $S \subseteq |U|$, since $|V|$ is the fiber product of X and $|U|$ over Y. Now suppose we have two non-empty open sets $V_1, V_2 \subseteq |V|$. Then their images in X must intersect in T, since X is irreducible. But then V_1 and V_2 must both intersect S, which is irreducible since $|U|$ is irreducible. Therefore they must intersect, so V is irreducible.

So $V \to U$ is a birational and proper morphism of integral schemes over k, with U smooth over k. Then by the projection formula, $f_* f^* \mathcal{L} \cong \mathcal{L} \otimes f_* \mathcal{O}_V$. Now since f is proper, $f_* \mathcal{O}_V$ is a coherent $\mathcal{O}_U$-module, and in fact we have $\mathcal{O}_U \subseteq f_* \mathcal{O}_V \subseteq K$ where K is the function field of U and V. But since U is nonsingular, it is in particular normal, and so the structure sheaf $\mathcal{O}_U$ is locally integrally closed, and since $f_* \mathcal{O}_V$ is coherent, we have $f_* \mathcal{O}_V = \mathcal{O}_U$. But then $\mathrm{H}^0(V, f^* \mathcal{L}) = \mathrm{H}^0(U, f_* f^* \mathcal{L}) = \mathrm{H}^0(U, \mathcal{L})$.

Now let a collection of smooth morphisms $U_i \to \mathcal{Y}$ as above be jointly surjective, and let $\mathbf{U} = \bigsqcup_i U_i$. Now let $R = \mathbf{U} \times_{\mathcal{Y}} \mathbf{U}$. Let $\mathbf{V}, S$ be the pullbacks of $\mathbf{U}, R$ to $\mathcal{X}$. The morphisms $R \rightrightarrows \mathbf{U}$ and $S \rightrightarrows \mathbf{V}$ are the natural projections, and $S \to R$ is the natural map induced by $\mathbf{V} \to \mathbf{U}$. Note that $S \cong \mathbf{V} \times_{\mathcal{X}} \mathbf{V}$. Now f is surjective since it is birational and proper. Therefore $S \to R$ is surjective. Note that both R and S are reduced schemes since they are locally smooth over the reduced schemes $\mathbf{U}$ and $\mathbf{V}$, respectively. Therefore the pullback of sections of line bundles via $S \to R$ is injective. Since global sections of $\mathcal{L}$ on $\mathbf{U}$ descend to global sections on $\mathcal{Y}$ if and only if they pull back to equal sections via the projections $R \rightrightarrows \mathbf{U}$ (and via some isomorphism of the pullbacks of $\mathcal{L}$ to R), it is easy to see then that the isomorphism $\mathrm{H}^0(\mathbf{U}, \mathcal{L}) \xrightarrow{\sim} \mathrm{H}^0(\mathbf{V}, f^* \mathcal{L})$ descends to $\mathrm{H}^0(\mathcal{Y}, \mathcal{L}) \xrightarrow{\sim} \mathrm{H}^0(\mathcal{X}, \mathcal{L})$. □

Proposition 4.1.2. *We have via the natural pullback map, an isomorphism:*

$$\mathrm{H}^0(\mathrm{Parbun}_G, \mathcal{L}_{\vec{w}}) \xrightarrow{\sim} \mathrm{H}^0(\mathcal{Y}, \mathcal{L}_{\vec{w}}).$$

Proof. It was shown in [10] that $\mathcal{C} \to \mathrm{Parbun}_G$ is birational. Since by Theorem 3.0.4 the morphism $\mathcal{Y}^{ss} \to \mathrm{Parbun}_G^{ss}$ is proper, by Proposition 4.1.1 we have an isomorphism $\mathrm{H}^0(\mathrm{Parbun}_G^{ss}, \mathcal{L}_{\vec{w}}) \xrightarrow{\sim} \mathrm{H}^0(\mathcal{Y}^{ss}, \mathcal{L}_{\vec{w}})$. By Theorem 9.10 in [52] and Proposition 2.4.1 the left vertical arrow in the following diagram is an isomorphism:

$$\begin{tikzcd} \mathrm{H}^0(\mathrm{Parbun}_G, \mathcal{L}_{\vec{w}}) \arrow[r] \arrow[d, "\sim"] & \mathrm{H}^0(\mathcal{Y}, \mathcal{L}_{\vec{w}}) \arrow[d] \\ \mathrm{H}^0(\mathrm{Parbun}_G^{ss}, \mathcal{L}_{\vec{w}}) \arrow[r, "\sim"] & \mathrm{H}^0(\mathcal{Y}^{ss}, \mathcal{L}_{\vec{w}}) \end{tikzcd} .$$

By commutativity of this diagram, the right vertical arrow is surjective. Furthermore, the right vertical arrow is injective, since $\mathcal{Y}$ is integral and $\mathcal{Y}^{ss}$ is a nonempty open substack. Note we're assuming $\vec{w}$ is in the eigen-polytope, so that some power of $\mathcal{L}_{\vec{w}}$ has global sections. Then the right vertical arrow is an isomorphism, and therefore we have $\mathrm{H}^0(\mathrm{Parbun}_G, \mathcal{L}_{\vec{w}}) \xrightarrow{\sim} \mathrm{H}^0(\mathcal{Y}, \mathcal{L}_{\vec{w}})$.

□

Now let P be the maximal parabolic associated to the product $\sigma_{u_1} * \cdots * \sigma_{u_n} = q^d[pt]$, and $L \subseteq P$ the Levi factor containing the maximal torus of G. Let $\mathrm{Bun}_Y^{\Gamma,L}(\boldsymbol{\tau}_u, d)$ be the moduli stack of Γ-equivariant L-bundles of local type $\boldsymbol{\tau}_u = (u_1^{-1}\tau_1 u_1, \ldots, u_n^{-1}\tau_n u_n)$ and degree d. Let $\xi : \mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau}; \boldsymbol{\tau}_u, 0) \to \mathrm{Bun}_Y^{\Gamma,L}(\boldsymbol{\tau}_u, d)$ be the natural projection given by $P \to L$, and $\iota : \mathrm{Bun}_Y^{\Gamma,L}(\boldsymbol{\tau}_u, d) \to \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})$ be the morphism given by extending the structure group to G.

Then we have the following proposition.

Proposition 4.1.3. *The following diagram*

$$\begin{tikzcd} \mathcal{Y} \arrow[r] \arrow[d] & \mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau}; \boldsymbol{\tau}_u, 0) \arrow[d, "\pi"] \arrow[dr, "\xi"] & \\ \mathrm{Parbun}_G \arrow[r] & \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau}) & \mathrm{Bun}_Y^{\Gamma,L}(\boldsymbol{\tau}_u, 0) \arrow[l, "\iota"] \end{tikzcd}$$

induces a commutative diagram of global sections of $\mathcal{L}_{\vec{w}}$. Furthermore ξ is surjective.

Proof. The morphism ξ is surjective because there is a section $j : \mathrm{Bun}_Y^{\Gamma,L}(\boldsymbol{\tau}_u, 0) \to \mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau}; \boldsymbol{\tau}_u, 0)$ given by extension of structure group.

The left square is 2-commutative, so we just need to show the triangle on the right induces a commutative diagram of global sections. We use the methods of [10], translated to equivariant bundles. Let N_P be the smallest positive integer such that $N_P x_P$ is in the coroot lattice, and let $\bar{x}_P = N_P x_P$. Let $t^{\bar{x}_P}$ be the associated cocharacter, and $\phi_t : P \to P$ be the homomorphism sending

$p \mapsto t^{\bar{x}_P} p t^{-\bar{x}_P}$. Then the family of homomorphisms $\phi : P \times \mathbb{G}_m \to P$ extends to a family over $\mathbb{A}^1$, and $\phi_0 : P \to P$ factors through $L \subseteq P$.

Now assume we have a morphism $S \to \mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau}; \boldsymbol{\tau}_u, 0)$, corresponding to an equivariant G-bundle E and a P-reduction $\sigma : Y \times S \to E/P$, which in turn corresponds to an equivariant P-bundle F with local type $\boldsymbol{\tau}_u$. Then let $F_t = F \times^{\phi_t} P$, and $E_t = F_t \times^P G$. Clearly then (E_1, F_1) is isomorphic to the pair (E, F) (in fact the same is true for (E_t, F_t) for any $t \neq 0$), and E_0 has a reduction to an L-bundle, which we will also denote F_0. This is what Belkale and Kumar call the *Levification process.*

So we get a canonical morphism $f : S \times \mathbb{A}^1 \to \mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau}; \boldsymbol{\tau}_u, 0)$, and F_0 is (isomorphic to) the image of $S \to \mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau}; \boldsymbol{\tau}_u, 0)$ via ξ. Let $\mathcal{L}_1$ be the pullback of $\mathcal{L}_{\vec{w}}$ via π, and $\mathcal{L}_2$ the pullback via $\iota \circ \xi$. Clearly then there is a canonical isomorphism $f_1^* \mathcal{L}_2 \cong f_0^* \mathcal{L}_1$. Our goal is to show that this is a canonical isomorphism $f_1^* \mathcal{L}_1 \cong f_0^* \mathcal{L}_1$, which will complete the proposition. We will do this using a $\mathbb{G}_m$-action on $f^* \mathcal{L}_1$, which we will show is trivial over $t = 0$.

The equivariant $\mathbb{G}_m$ action is defined as follows. There is a natural $\mathbb{G}_m$-action on the G-bundle corresponding to f, defined over $\mathbb{A}^1$ by right multiplication by $t^{\bar{x}_P}$. This induces a $\mathbb{G}_m$-action on $f^* \mathcal{L}_1$. Now Belkale and Kumar show in section 6 of [10] that there is some $N > 0$ such that $\mathcal{L}_{\vec{w}}^N$ is isomorphic to the determinant bundle $\mathcal{L} = D(V)$, where V is the adjoint representation of G. Then $f_0^* \mathcal{L}_1^N \cong D(F_0 \times^L \mathfrak{g})$.

Let $s \in S$ be a point and $(F_0)_s$ be the fiber over s. Now we can filter $\mathfrak{g}$ so that the action of $\mathrm{Ad}(t^{\bar{x}_P})$ on the associated graded pieces is $t^{-\gamma}$, for some integer γ. Denote the associated graded piece with an action of $t^{-\gamma}$ as $\mathfrak{g}_\gamma$, and $(F_0)_s \times^L \mathfrak{g}_\gamma$ as V_γ (note that $\mathfrak{g}_\gamma$ is fixed by L, since $t^{\bar{x}_P}$ is central). Then $\mathbb{G}_m$ acts by $t^{-\gamma}$ on V_γ, and therefore by $t^{\chi(Y, V_\gamma)\gamma}$ on $D(V_\gamma)$. Then the exponent of the action on $D((F_0)_s \times^L \mathfrak{g})$ is

$$\sum_\gamma \chi(Y, V_\gamma)\gamma = \sum_\gamma (\deg(V_\gamma) + (1 - g)\mathrm{rk}(V_\gamma))\gamma \tag{4.1}$$

$$= \sum_\gamma \deg(V_\gamma)\gamma \tag{4.2}$$

where the first equality is Riemann-Roch, and the second follows from the fact that $\dim \mathfrak{g}_\gamma = \dim \mathfrak{g}_{-\gamma}$.

Now let $R_\gamma \subseteq R$ be the set of roots in $\mathfrak{g}_\gamma$. Then clearly

$$\deg(V_\gamma) = \sum_{\alpha \in R_\gamma} \deg((F_0)_s \times^L \mathbb{C}_{-\omega_P}) \cdot \alpha(\alpha_P^\vee),$$

noting that $\alpha(\alpha_P^\vee)$ is the coefficient of ω_P in α. But $d = \deg((F_0)_s \times^L \mathbb{C}_{-\omega_P})$ is zero. Therefore the $\mathbb{G}_m$-action on $D(F_0 \times^L \mathfrak{g})$ is trivial, which implies the action on $f_0^*\mathcal{L}_1$ is trivial. Then the $\mathbb{G}_m$-action gives a canonical identification $f_1^*\mathcal{L}_1 \cong f_0^*\mathcal{L}_1$ by taking the limit $t \to 0$. $\square$

Now we are ready to reduce to the Levi.

Corollary 4.1.4. *We have* $\mathrm{H}^0(\mathrm{Bun}_Y^{\Gamma,L}(\boldsymbol{\tau}_u, 0), \iota^*\mathcal{L}_{\vec{w}}) \xleftarrow{\sim} \mathrm{H}^0(\mathrm{Bun}_{\mathcal{G}}, \mathcal{L}_{\vec{w}}) \cong \mathrm{H}^0(\mathrm{Parbun}_G, \mathcal{L}_{\vec{w}})$.

Proof. Since ξ and $\mathcal{Y} \to \mathrm{Bun}_Y^{\Gamma,G;P}(\boldsymbol{\tau}; \boldsymbol{\tau}_u, 0)$ are surjective by the above proposition and Theorem 3.2.1, $\mathrm{H}^0(\mathrm{Bun}_Y^{\Gamma,L}(\boldsymbol{\tau}_u, 0), \mathcal{L}_{\vec{w}}) \to \mathrm{H}^0(\mathcal{Y}, \mathcal{L}_{\vec{w}})$ is injective. Since by Proposition 4.1.2 pullback via $\mathcal{Y} \to \mathrm{Parbun}_G$ is surjective, by Proposition 4.1.3 and the fact that conformal blocks descend to $\mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau}) \cong \mathrm{Bun}_{\mathcal{G}}$ (Theorem 0.1.10,) the proof is finished by a simple diagram chase. $\square$

4.2 Reductions to L' and completion of the proof of the main theorem

To complete the proof of the reduction theorem, we need to identify $\mathrm{H}^0(\mathrm{Bun}_Y^{\Gamma,L}(\boldsymbol{\tau}_u, 0), \iota^*\mathcal{L}_{\vec{w}})$ with a space of global sections over $\mathrm{Parbun}_{L'}$. In order to accomplish this for an arbitrary degree d, we need to add weights to our weight data. Having done this, we conclude the chapter with a proof of the identification of conformal blocks bundles when $d = 0$.

We write our weight data for L bundles as $\vec{w}^L = (\lambda_1^L, \ldots, \lambda_n^L, \ell)$, where $\lambda_i^L = u_i^{-1}\lambda_i$. Note that these weights satisfy the equation $\sum_{i=1}^n \langle \omega_P, \lambda_i^L \rangle = \ell \cdot d$.

It is easy to see that L' is simply connected, since G is simply connected. Furthermore we know the Dynkin type of L': it is given by removing the vertex of the Dynkin diagram of G corresponding to P. Therefore $L' \cong G_1 \times G_2 \times G_3$, where G_1, G_2 and G_3 are simple, simply connected groups. Note that one or more of the groups may be trivial, and most commonly there are exactly two non-trivial factors. The following discussion follows closely section 7 of [10].

Let Z_0 be the connected component of the identity of L, and let $L' \times Z_0 \to L$ be the natural homomorphism. This homomorphism is in fact an isogeny, with kernel say of size k_L. Alternatively,

k_L is the size of the kernel of the isogeny $Z_0 \to L/L'$. Let N_P be the smallest positive integer such that $N_P x_P$ is in the coroot lattice, and let $\bar{x}_P = N_P x_P$. Then it is easy to see that

$$k_L = \omega_P(\bar{x}_P) = 2N_P \frac{\langle \omega_P, \omega_P \rangle}{\langle \alpha_P, \alpha_P \rangle}.$$

The basic result we use to reduce conformal blocks to L' is the following proposition.

Proposition 4.2.1. *[10] Suppose weight data $\vec{w}^L = (\lambda_1^L, \dots, \lambda_n^L, \ell)$ for L satisfies the equation $\sum_{i=1}^n \langle \omega_P, \lambda_i^L \rangle = \ell \cdot d$ and that $d = d' k_L$. Then there is a surjective morphism $\iota' : \mathrm{Parbun}_{L'} \to \mathrm{Parbun}_L(d)$ such that the induced pullback of global sections of $\mathcal{L}_{\vec{w}}$ is an isomorphism.*

To reduce down to L' for a general degree, one needs to change the degree of the L bundles. For each parabolic P Belkale and Kumar show the existence of an element of the coroot lattice μ_P lying in the fundamental alcove of L such that $|\omega_P(\mu_P)| = 1$. They use μ_P to shift the degree of the stack of parabolic L-bundles, since for the reduction to L', it is necessary that k_L divides the degree. Let d_0 be the smallest positive integer such that $d + d_0 \omega_p(\mu_P) \equiv 0 \pmod{k_L}$. Let $\mathrm{Parbun}_L^{[d_0]}(d)$ be the stack of parabolic degree d L-bundles with full flags over $n + d_0$ points in $X \cong \mathbb{P}^1$. Let $\mathcal{L}_{\vec{w}^L}$ be the pullback of $\mathcal{L}_{\vec{w}}$ to $\mathrm{Parbun}_L^{[d_0]}(d)$ via ι and the forgetful functor. This is the line bundle associated to a level ℓ, weights $\lambda_1^L, \dots, \lambda_n^L$, and the zero weight on the remaining d_0 points. Then Corollary 7.6 in [10] says the following.

Proposition 4.2.2. *[10] Associated to μ_P is a natural isomorphism $\tau_\mu : \mathrm{Parbun}_L^{[d_0]}(d + d_0\omega_p(\mu_P)) \to \mathrm{Parbun}_L^{[d_0]}(d)$. The weights of $\tau_\mu^* \mathcal{L}_{\vec{w}^L}$ are $\lambda_1^L, \dots, \lambda_n^L$, and d_0 copies of $\ell \cdot \kappa(\mu_P)$, and the level remains the same.*

Note that the forgetful morphism $\mathrm{Parbun}_L^{[d_0]}(d) \to \mathrm{Parbun}_L(d)$ induces an isomorphism of global sections for any line bundle for the same basic reason that conformal blocks descend to stacks of parahoric bundles. Combining this fact and the above propositions, we can identify global sections of $\mathcal{L}_{\vec{w}^L}$ over $\mathrm{Parbun}_L(d)$ with its pullback to $\mathrm{Parbun}_{L'}^{[d_0]}$ via τ_μ and ι'.

There is a morphism $\mathrm{Parbun}_L(d) \to \mathrm{Bun}_Y^{\Gamma,L}(\boldsymbol{\tau}_u, 0)$, defined in the same way as $\mathrm{Parbun}_G \to \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})$, so that the pullback of $\iota^* \mathcal{L}_{\vec{w}}$ to $\mathrm{Parbun}_L(d)$ is the line bundle associated to the weight data $\vec{w}^L$. One way to finish the proof of the reduction theorem would be to show that the pullback of global sections of any line bundle with respect to this morphism is an isomorphism. This could be proven in the same way that we showed that conformal blocks descend to stacks of parahoric bundles: the geometric fibers of this morphism should be products of quotients of centralizers in L by Borel subgroups. Any centralizer of a torus element of L will be reductive and connected, since L is connected and L' is simply connected. Unfortunately, we do not have the references in the reductive case to feel confident in this approach.

Instead, we simply replicate the above propositions for equivariant bundles. More precisely, we want to construct a morphism $\iota' : \mathrm{Bun}_Y^{\Gamma,L'}(\boldsymbol{\tau}'_u) \to \mathrm{Bun}_Y^{\Gamma,L}(\boldsymbol{\tau}_u, 0)$ so that it fits into the following diagram.

$$\begin{tikzcd}
\mathrm{Parbun}_{L'}^{[d_0]} \arrow[r, "\iota'"] \arrow[d] & \mathrm{Parbun}_{L}^{[d_0]}(d + d_0\omega_p(\mu_P)) \arrow[r, "\tau_\mu"] \arrow[dr] & \mathrm{Parbun}_L(d) \arrow[d] \\
\mathrm{Bun}_Y^{\Gamma,L'}(\boldsymbol{\tau}'_u) \arrow[rr, "\iota'"] & & \mathrm{Bun}_Y^{\Gamma,L}(\boldsymbol{\tau}_u, 0)
\end{tikzcd}$$

First let's review the definition of ι'. Suppose $d + d_0\omega_p(\mu_P) = d'k_L$, and let $\mathcal{F}$ be an L'-bundle. Then $F \times \mathcal{O}_X(d')$ is an $L' \times Z_0$ bundle, and therefore extending the structure group via $L' \times Z_0 \to L$ we get an L-bundle $\mathcal{F}_L$ of degree $d + d_0\omega_p(\mu_P)$. Parabolic structures are transferred in the obvious way. The idea of the construction of ι' for equivariant bundles is to use an equivariant version of $\mathcal{O}_X(d')$ over Y.

There is a canonical identification of the rational coweight of L, and the rational coweight of $L' \times Z_0$. Therefore, given a rational coweight μ of L, we can factor it uniquely as $\mu' \cdot \mu''$, where μ' is a rational coweight of L', and μ'' is a rational coweight of Z_0. Note that a coweight of L may factor into rational coweight of L' and Z_0.

Assume we have chosen Y so that all its ramification indices are divisible by k_L, and such that there are d_0 extra ramified orbits of Γ, with the isotropy subgroup acting trivially over these points. This is already necessary for $\mathrm{Bun}_Y^{\Gamma,L'}(\boldsymbol{\tau}'_u)$ to be defined. Let $\mu_1, \ldots, \mu_{n+d_0}$ be the rational coweights associated to $\vec{w}^L$. In other words, $\mu_i = \frac{1}{\ell}\kappa(\lambda_i^L)$ for $1 \leq i \leq n$, and $\mu_i = \mu_P$ for $n+1 \leq i \leq n+d_0$. Then given a parabolic L'-bundle $\mathcal{F}$, the coweights $\mu'_1, \ldots, \mu'_{n+d_0}$ allow one to construct the associated equivariant bundle F. Similarly, the Z_0-coweights $\mu''_1, \ldots, \mu''_{n+d_0}$ allow one to construct an equivariant line bundle $\mathcal{O}_Y(d', \vec{\mu})$. Then $\iota' : \mathrm{Bun}_Y^{\Gamma,L'}(\boldsymbol{\tau}'_u) \to \mathrm{Bun}_Y^{\Gamma,L}(\boldsymbol{\tau}_u, 0)$ is defined as follows: for any $F \in \mathrm{Bun}_Y^{\Gamma,L'}(\boldsymbol{\tau}'_u)$, extend the structure group of $F \times \mathcal{O}_Y(d', \vec{\mu})$ to L via $L' \times Z_0 \to L$. It is easy to check that this morphism is well defined and fits into the above diagram.

Let $\mathcal{L}_{\vec{w}^L}$ be the pullback of $\mathcal{L}_{\vec{w}}$ via $\iota : \mathrm{Bun}_Y^{\Gamma,L}(\boldsymbol{\tau}_u, 0) \to \mathrm{Bun}_Y^{\Gamma,G}(\boldsymbol{\tau})$; note that this line bundle pulls back to $\mathcal{L}_{\vec{w}^L}$ over $\mathrm{Parbun}_L(d)$. Then we have the following proposition, where we let $\lambda_{n+i} = \ell\kappa(\mu_P)$.

Proposition 4.2.3. *Suppose weight data $\vec{w}^L = (\lambda_1^L, \dots, \lambda_{n+d_0}^L, \ell)$ for L satisfies the equation $\sum_{i=1}^{n+d_0} \langle \omega_P, \lambda_i^L \rangle = \ell \cdot d$. Then the morphism $\iota' : \mathrm{Bun}_Y^{\Gamma,L'}(\boldsymbol{\tau}'_u) \to \mathrm{Bun}_Y^{\Gamma,L}(\boldsymbol{\tau}_u, 0)$ induces an isomorphism of global sections of $\mathcal{L}_{\vec{w}^L}$.*

Proof. Firstly, we note that ι' is surjective. It is easy to see that $\mathrm{Parbun}_{L'}^{[d_0]} \to \mathrm{Bun}_Y^{\Gamma,L'}(\boldsymbol{\tau}'_u)$ and $\mathrm{Parbun}_L^{[d_0]}(d + d_0\omega_p(\mu_P)) \to \mathrm{Bun}_Y^{\Gamma,L}(\boldsymbol{\tau}_u, 0)$ are surjective: the first case is well known since L' is semi-simple. In the other case, given a (Γ, L)-bundle F over Y, one constructs a parabolic L-bundle over X by simply taking the quotient over Y^*, and using étale-local trivializations of F over the ramification points to construct a parabolic L-bundle over X, following the above work for (Γ, G)-bundles. Note that we do not need a generic trivialization of F or an understanding of the effect of the choice of trivialization to show the morphism is surjective; we defer such analysis to future work. Therefore by the above diagram ι' is surjective, and therefore the pullback of global sections of any line bundle is injective.

To show the pullback of global sections is surjective we follow the proof of Lemma 7.1 in [10]. Assume we have two (Γ, L)-bundles F_1 and F_2, and choose lifts to (Γ, L')-bundles F'_1, F'_2. Suppose further we have an isomorphism $\phi : F_1 \xrightarrow{\sim} F_2$. We want to show we can modify this isomorphism by multiplication by an element of Z_0 such that it lifts to an isomorphism of F'_1 and F'_2. This will give a canonical identification of the fibers of $\mathcal{L}_{\vec{w}}$ and its pullback, since Z_0 acts trivially on $\mathcal{L}_{\vec{w}}$ (see proof of Prop 4.1.3), and therefore show that the pullback of global sections is surjective. But ϕ gives an isomorphism of the associated L/L'-bundles, and since L/L' is a torus, the isomorphism therefore corresponds canonically to some $zL' \in L/L'$. Some more care could be taken here: the L/L'-bundles associated to F'_1, F'_2 can be canonically identified with $\mathcal{O}_Y(d', \vec{\mu})$ extended to an L/L'-bundle; ϕ then induces an automorphism of this bundle giving z. But $Z_0 \to L/L'$ is surjective, so we can lift zL' to $z \in Z_0$. It can be easily checked that composing ϕ with the automorphism of F_2 induced by z^{-1} gives an automorphism that lifts to $\phi' : F'_1 \to F'_2$. □

By the results in chapter 2, the morphism $\mathrm{Parbun}_{L'} \to \mathrm{Bun}_Y^{\Gamma,L'}(\boldsymbol{\tau}'_u)$ induces an isomorphism of global sections of $\mathcal{L}_{\vec{w}'}$. Note that the weights in $\vec{w}'$ are the restrictions of $u_i^{-1}\lambda_i$ to L'. All that remains is the identification of the levels.

The level(s) of the reduced conformal blocks depends on the Dynkin indices (see the section 1.2.3) of L' in G. Let m_1, m_2, and m_3 be the Dynkin indices of each subalgebra $\mathfrak{g}_1$, $\mathfrak{g}_2$, and $\mathfrak{g}_3$ in $\mathfrak{g}$. Let V be a faithful representation of G, and $D(V)$ be the associated determinant bundle over Parbun_G. The level of $D(V)$ is the Dynkin index of V. Then the pullback of this line bundle to Parbun_{G_i} is just $D(V_{|G_i})$. But by the results in section 5 of [36] and section 7 of [10], the level of this bundle is the Dynkin index of $V_{|G_i}$, which is equal to the index of V times the index of G_i in G. Therefore by linearity pulling back a line bundle $\mathcal{L}$ of level ℓ gives a bundle of level $m_i\ell$ over Parbun_{G_i}.

This completes the proof of our main theorem.

Theorem 4.2.4. *(Theorem 0.1.2) For weight data $\vec{w} = (\lambda_1, \dots, \lambda_n, \ell)$ in the multiplicative polytope, lying on the face corresponding to $\sigma_{u_1} * \cdots * \sigma_{u_n} = q^d[pt] \in \mathrm{QH}^*(G/P)$ such that $k_L|d$, we have a natural isomorphism of vector spaces*

$$\mathrm{H}^0(\mathrm{Parbun}_G, \mathcal{L}_{\vec{w}}) \xrightarrow{\sim} \mathrm{H}^0(Parbun_{L'}, \mathcal{L}_{\vec{w}'})$$

where the weight data $\vec{w}'$ is as described above. Therefore, we have a natural isomorphism of conformal blocks $\mathcal{V}^\dagger_{\mathfrak{g},\vec{w}} \cong \mathcal{V}^\dagger_{\mathfrak{g}_1,\vec{w}_1} \otimes \mathcal{V}^\dagger_{\mathfrak{g}_2,\vec{w}_2} \otimes \mathcal{V}^\dagger_{\mathfrak{g}_3,\vec{w}_3}$.

Finally, we prove that when $d = 0$ this isomorphism can be extended to an isomorphism of vector bundles.

Corollary 4.2.5 (Corollary 0.1.6)**.** *When $d = 0$ we in fact have an isomorphism of conformal blocks bundles on $\overline{M}_{0,n}$:*

$$\mathbb{V}_{\mathfrak{g},\vec{w}} \cong \mathbb{V}_{\mathfrak{g}_1,\vec{w}_1} \otimes \mathbb{V}_{\mathfrak{g}_2,\vec{w}_2} \otimes \mathbb{V}_{\mathfrak{g}_3,\vec{w}_3}.$$

Proof. Letting $\mathbb{A}_{\mathfrak{g},\vec{w}}$ be the trivial bundle of invariants over $\overline{\mathrm{M}}_{0,n}$, we have the following diagram of vector bundles:

$$\begin{tikzcd} \mathbb{A}_{\mathfrak{g},\vec{w}} \arrow[d, two heads] & \mathbb{A}_{\mathfrak{g}_1,\vec{w}_1} \otimes \mathbb{A}_{\mathfrak{g}_2,\vec{w}_2} \otimes \mathbb{A}_{\mathfrak{g}_3,\vec{w}_3} \arrow[l, "\sim"'] \arrow[d, two heads] \\ \mathbb{V}_{\mathfrak{g},\vec{w}} & \mathbb{V}_{\mathfrak{g}_1,\vec{w}_1} \otimes \mathbb{V}_{\mathfrak{g}_2,\vec{w}_2} \otimes \mathbb{V}_{\mathfrak{g}_3,\vec{w}_3} \end{tikzcd}$$

where the horizontal isomorphism follows from the factorization result for invariants in [44], or alternatively, the above theorem, choosing a high enough level. So we just need to show that the composition $\mathbb{A}_{\mathfrak{g}_1,\vec{w}_1} \otimes \mathbb{A}_{\mathfrak{g}_2,\vec{w}_2} \otimes \mathbb{A}_{\mathfrak{g}_3,\vec{w}_3} \to \mathbb{V}_{\mathfrak{g},\vec{w}}$ descends to $\mathbb{V}_{\mathfrak{g}_1,\vec{w}_1} \otimes \mathbb{V}_{\mathfrak{g}_2,\vec{w}_2} \otimes \mathbb{V}_{\mathfrak{g}_3,\vec{w}_3}$, since we've already shown the conformal blocks bundles are the same rank. Furthermore, it is sufficient to check this on $\mathrm{M}_{0,n}$, which is dense in $\overline{\mathrm{M}}_{0,n}$, and since these are vector bundle morphisms, we can check it fiber by fiber. The necessary diagram of fibers is induced by the following diagram:

$$\begin{tikzcd}
U_{\mathrm{triv}} \arrow[d, hook] & V_{\mathrm{triv}} \arrow[l] \arrow[d, hook] \\
\mathrm{Parbun}_G & \mathrm{Parbun}_{L'} \arrow[l]
\end{tikzcd}$$

where U_{triv} and V_{triv} are the substacks of trivial bundles, and the diagram of fibers is obtained by taking global sections of $\mathcal{L}_{\vec{w}}$, then taking the duals of each map. □

The above method fails when $d > 0$, because in this case trivial bundles in $\mathrm{Parbun}_{L'}$ do not map to trivial bundles in Parbun_G. Furthermore we have an example showing the bundles are not isomorphic (see Example 5.1.9). It would be interesting to know if there is nevertheless a relationship between these vector bundles.

CHAPTER 5: EXAMPLES

5.1 Rank reduction in type A

We want to work out the combinatorics of the above result in more detail when $G \cong \mathrm{SL}_{r+1}$, and give some examples. Of course all the examples in [30] and [44] are valid for conformal blocks with the same weights and high enough level, but we want to focus here on applications to conformal blocks bundles and their divisors.

Firstly let's restate our main theorem in terms of the combinatorics specific to SL_{r+1}. Weights are represented by partitions with $r+1$ parts: $\lambda = (\lambda^1 \geq \cdots \geq \lambda^{r+1} \geq 0)$. Adding 1 to each part gives a new partition $(\lambda^1 + 1 \geq \cdots \geq \lambda^{r+1} + 1 \geq 0)$ corresponding to the same weight; we will always assume the partitions are normalized so that $\lambda^{r+1} = 0$. To go from a dominant weight $\lambda = \sum_{i=1}^{r} a_i \omega_i$ to a (normalized) partition, one simply adds the coefficients: $\lambda^i = \sum_{j=i}^{r} a_i$. The spaces G/P are Grassmannians, so we will write P_k for $G/P_k \cong \mathrm{Gr}(k, r+1)$. The Schubert varieties in $\mathrm{Gr}(k, r+1)$ are parametrized by subsets of $I \subseteq \{1, \ldots, r+1\}$ with k elements. Since all Grassmannians are cominiscule, all products are Levi-movable, so we can work with standard quantum cohomology. Then given $\sigma_{I_1} * \cdots * \sigma_{I_n} = q^d[pt]$, the inequality the weights $\lambda_1, \ldots, \lambda_n$ and level ℓ must satisfy to be in the multiplicative polytope is

$$\sum_{j=1}^{n} \sum_{i \in I_j} \lambda_j^i - \frac{k}{r+1} \sum_{j=1}^{n} |\lambda_j| \leq \ell d,$$

and the weight data lies on the corresponding facet of the multiplicative polytope if it satisfies this inequality with equality. These inequalities can be derived from the inequalities stated in the introduction, or alternatively the inequalities in [7].

The group $L' = [L, L]$ is isomorphic to $\mathrm{SL}_k \times \mathrm{SL}_{r-k+1}$. The splitting into λ_1, λ_2 of a weight λ with respect to $I = (i_1 < \cdots < i_k)$ is given by $\lambda_1 = (\lambda^{i_1} \geq \cdots \geq \lambda^{i_k})$, and λ_2 the same with respect to the complement I^c; note that these partitions may need to be normalized. The level ℓ induces the bilevel (ℓ, ℓ) on $\mathrm{Parbun}_{L'}$, since the roots of SL_{r+1} are all the same length, and therefore the

Dykin indices are both 1. From the list in Section 7 of [10], we know that $\mu_{P_k} = -\alpha_k^\vee$, and so $\omega_k(\mu_{P_k}) = -1$. Therefore $d_0 = d \pmod{k_L}$. Furthermore, it is easily seen that $\kappa(\mu_{P_k})$ restricted to L' corresponds to the partition $(1 \geq 1 \geq \cdots \geq 1 \geq 0)$ for the SL_k factor and $(1 \geq 0 \geq \cdots \geq 0 \geq 0)$ for the SL_{r-k+1} factor.

It remains to calculate k_L. Since all roots are the same length and we normalize the Killing form so that $\langle \theta, \theta \rangle = 2$, the formula becomes $k_L = N_k \langle \omega_k, \omega_k \rangle$. Direct calculation shows furthermore that

$$\langle \omega_k, \omega_k \rangle = \frac{k(r-k+1)}{r+1}.$$

The integer N_k is the smallest integer that makes the k-th column of the inverse Cartan matrix integral. In general N_k depends on the divisibility of $r+1$, which is the determinant of the Cartan matrix of G. It can be easily seen then that if $r+1$ is prime, then $k_L = k(r-k+1)$, which is also the dimension of the Grassmannian $\mathrm{Gr}(k, r+1)$. Additionally, when $k = 1$ or r then k_L is always $r+1$. The existence of walls with positive degree divisible by k_L will therefore depend on n being sufficiently large.

So we can restate our main theorem for SL_{r+1} as follows.

Theorem 5.1.1. *Let the weight data $\vec{w} = (\lambda_1, \ldots, \lambda_n, \ell)$ lie in the multiplicative polytope. If it furthermore lies on the facet corresponding to $\sigma_{I_1} * \cdots * \sigma_{I_n} = q^d[pt] \in \mathrm{QH}^*(Gr(k, r+1))$, i.e., if*

$$\sum_{j=1}^{n} \sum_{i \in I_j} \lambda_j^i - \frac{k}{r+1} \sum_{j=1}^{n} |\lambda_j| = \ell d,$$

then we have a natural isomorphism of conformal blocks

$$\mathcal{V}^\dagger_{\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell} \cong \mathcal{V}^\dagger_{\mathfrak{sl}_k, \vec{\lambda}', \ell} \otimes \mathcal{V}^\dagger_{\mathfrak{sl}_{r-k+1}, \vec{\lambda}'', \ell}$$

where $\vec{\lambda}'$ and $\vec{\lambda}''$ are as follows:

1. *The weight λ_i' is given by the subpartition of λ_i given by I_i for $1 \leq i \leq n$, and the last $d_0 = d \pmod{k_L}$ weights are all $(\ell \geq \ell \geq \cdots \geq \ell \geq 0)$.*

2. *The weight λ_i'' is given by the subpartition of λ_i given by I_i^c for $1 \leq i \leq n$, and the last d_0 weights are all $(\ell \geq 0 \geq \cdots \geq 0 \geq 0)$.*

When $d = 0$ this isomorphism extends to an isomorphism of conformal blocks bundles over $\overline{\mathrm{M}}_{0,n}$.

5.1.1 Level one conformal blocks

Level one conformal blocks are the simplest non-trivial case, but even here the bundles and associated divisors we obtain are quite interesting (see for example [19]). Non-trivial level one conformal blocks correspond to a subset of the vertices of the multiplicative polytope, and tuples of central elements of G multiplying to the identity. We will refer to these vertices of the multiplicative polytope as the *trivial vertices*.

The level one weights λ_i can be identified with integers. For simplicity, we will assume that $\mathbb{V}_{\mathfrak{sl}_{r+1},\vec{\lambda},1} \neq \{0\}$. Now in this case we must have $r+1 \mid \sum_{i=1}^{n} \lambda_i$. Let $\sum_{i=1}^{n} \lambda_i = s(r+1)$. It is a fact that $2 \leq s \leq n-1$. We will start with the simplest walls of the multiplicative polytope, where $k = 1$ and $d = 0$, so that $\mathrm{Gr}(k, r+1) \cong \mathbb{P}^r$. So in this case our subsets I_j can also be identified with integers, with $1 \leq I_j \leq r+1$. Note that I_j corresponds to a hyperplane of dimension $I_j - 1$ in $\mathbb{P}^r$. Then $\sigma_{I_1} \cdots \sigma_{I_n} = [pt]$ if and only if the codimensions $r - I_k + 1$ add up to r, which is equivalent to $\sum I_j = (n-1)(r+1)+1$. Let $[a \leq b]$ be one if $a \leq b$ and zero otherwise. Then the above inequalities become

$$\sum_{j=1}^{n} [I_j \leq \lambda_j] \leq s.$$

In order to apply our reduction result, we need to find weights and Schubert varieties making the above inequality an equality. In order for this to be possible, the largest s weights must sum up to at least $(s-1)(r+1)+1$, in order for the codimension equation to be satisfied. Therefore we have the following proposition.

Proposition 5.1.2. *Suppose $\lambda_1, \ldots, \lambda_n$ are level one weights such that $\mathbb{V}_{\mathfrak{sl}_{r+1},\vec{\lambda},1} \neq \{0\}$. Let N be a subset of $\{1, \ldots, n\}$ such that $|N| = s$. Then if $\sum_{j=1}^{n} \lambda_j = s(r+1)$ and $\sum_{j \in N} \lambda_j \geq (s-1)(r+1)+1$, we have*

$$\mathbb{V}_{\mathfrak{sl}_{r+1},\vec{\lambda},1} \cong \mathbb{V}_{\mathfrak{sl}_r,\vec{\lambda}',1}.$$

where $\lambda'_j = \lambda_j - 1$ for $j \in N$ and $\lambda'_j = \lambda_j$ otherwise.

Note that $\sum_{i=1}^{n} \lambda'_i = sr$, so that one can iterate this reduction to obtain weights such that any selection of s of them have a sum less than or equal to $(s-1)(r'+1)$, where r' is the reduced rank.

We are also interested in the intersection of the corresponding divisors with so-called F-curves. Modulo numerical equivalence, classes of the one-dimensional strata of $\overline{\mathrm{M}}_{0,n}$ correspond to partitions of $\{1,\ldots,n\}$ into four non-empty subsets $N_1,\ldots,N_4$. The importance of F-curves is that the numerical equivalence class of the divisors $\mathbb{D}_{\mathfrak{sl}_{r+1},\vec{\lambda},1}$ is determined by its intersection with the F-curves. In [17], Fakhruddin gave a formula for the intersection of level-one type A conformal blocks with F-curves in terms of these partitions. The hypothesis of the above proposition also implies that the conformal blocks divisor $\mathbb{D}_{\mathfrak{sl}_{r+1},\vec{\lambda},1}$ intersects a number of F-curves trivially:

Proposition 5.1.3. *Suppose $\lambda_1,\ldots,\lambda_n$ are level one weights such that $\mathbb{V}_{\mathfrak{sl}_{r+1},\vec{\lambda},1} \neq \{0\}$, and $\sum_{j=1}^{n}\lambda_j = s(r+1)$. Let N be a subset of $\{1,\ldots,n\}$. Then if $\sum_{j\in N}\lambda_j \geq (s-1)(r+1)$, $\mathbb{D}_{\mathfrak{sl}_{r+1},\vec{\lambda},1}$ intersects trivially all F-curves associated to partitions $\{1,\ldots,n\} = \sqcup_{i=1}^{4} N_i$ such that $N_i = N$ for some i.*

Proof. Let $\nu_k = \sum_{j\in N_k}\lambda_j \pmod{r+1}$. By the assumed inequality we see that $\sum_k \nu_k \leq r+1$. But by Fakhruddin's formula, for the intersection to be non-zero, we need $\sum_k \nu_k = 2(r+1)$. Therefore all given intersections are trivial. □

We would like to understand level-one conformal blocks on walls with $k > 1$. The first observation is that there is a *duality* among the walls containing a trivial vertex of the multiplicative polytope. This depends on the action of the center on the multiplicative polytope.

Consider the vertex of the multiplicative polytope given by the level-one weights $\lambda_1,\ldots,\lambda_n$. This corresponds to a tuple of elements of the center $\vec{c} = (c_1,\ldots,c_n)$ such that c_i is equal to $\zeta^{\lambda_i}\mathrm{Id}$, where ζ is a primitive root of unity, and $\prod_i c_i = \mathrm{Id}$. Then this tuple of central elements acts on the multiplicative polytope in the obvious way. For a subset I of $\{1,\ldots,r+1\}$, let $I-1$ be the set obtained from I by subtracting 1 from each element, and replacing any 0 with $r+1$. For any positive integer λ, $I-\lambda$ is obtained by iterating this process. Then Agnihotri and Woodward proved the following proposition.

Proposition 5.1.4. ([1] Prop 7.2) *If $\lambda_1,\ldots,\lambda_n$ are level-one weights corresponding to a trivial vertex of the multiplicative polytope, and if F is the facet of the polytope corresponding to $\sigma_{I_1} * \cdots * \sigma_{I_n} = q^d[\mathrm{pt}] \in \mathrm{QH}^*(\mathrm{Gr}(k,r+1))$, then $\vec{c}^{\,-1}\cdot F$ is the facet corresponding to*

$$\sigma_{I_1-\lambda_1} * \cdots * \sigma_{I_n-\lambda_n} = q^{d'}[\mathrm{pt}]$$

where $d' = d - \left(\sum_{j=1}^{n}[I_j \leq \lambda_j] - \frac{k}{r+1}\sum_{j=1}^{n}\lambda_j\right)$*. In particular, if* F *contains* $(\lambda_1, \dots, \lambda_n)$*, then* $d' = 0$*.*

The duality also arises from Grassmann duality. For each σ_I class in $\mathrm{H}^*(\mathrm{Gr}(k, r+1))$, there is a dual class σ_{I^t} given in the following way. To $I = (i_1 < \cdots < i_k)$ we can associate a partition $\lambda(I)$ defined as $\lambda(I)^j = r - k + 1 + j - i_j$. Then I^t is the $(r-k+1)$-subset of $\{1, \dots r+1\}$ corresponding to the transpose of $\lambda(I)$. Then we have the following lemma.

Lemma 5.1.5. *For any* $1 \leq \lambda \leq r$*, and subset* I *of* $\{1, \dots, n\}$*, we have* $(I - \lambda)^t = I^t - (r - \lambda + 1)$*.*

Proof. Clearly it is sufficient to consider the case $\lambda = 1$. There are then two cases. First, if $1 \in I$, then $r + 1 \notin I^t$. Then $\lambda(I - 1)$ is the partition obtained from λ by removing the largest part, and taking the transpose, we get $\lambda(I^t)$ with 1 subtracted from each part. On the other hand $I^t - r$ is just the set obtained by adding 1 to each element, or in other words subtracting 1 from each part.

Now suppose $1 \notin I$. Note that $r + 1 \in I^t$. Then $\lambda(I - 1)$ is the partition obtained by adding 1 to each part of $\lambda(I)$, and $\lambda(I - 1)^t$ is the partition obtained from $\lambda(I^t)$ by adding k as its highest part. The subset $I^t - r$ is obtained from I^t by replacing $r + 1$ with 1 and adding 1 to all other elements, finishing the proof. □

Proposition 5.1.6. *There is a natural bijection between the* $d = 0$ *regular faces of type* k *and type* $r - k + 1$ *containing a trivial vertex of the multiplicative polytope. Given a facet* F *containing the vertex* $(\lambda_1, \dots, \lambda_n)$*, corresponding to the product* $\sigma_{I_1} \cdots \sigma_{I_n} = [\mathrm{pt}]$*, the dual facet corresponds to the product* $\sigma_{(I_1-\lambda_1)^t} \cdots \sigma_{(I_n-\lambda_n)^t} = [\mathrm{pt}]$*. In other words, the dual facet containing the given trivial vertex is* $\vec{c} \cdot F^t = (\vec{c}^{-1} \cdot F)^t$*, where* t *indicates the Grassmann dual.*

Proof. Let F be the given facet containing $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$, and denote by $\vec{c}$ the corresponding tuple of elements of the center. For any $d = 0$ facet denote by F^t its Grassmann dual. Since F contains the given vertex, by Proposition 5.1.4 $\vec{c}^{-1} \cdot F$ is degree 0, and so we can take the dual $(\vec{c}^{-1} \cdot F)^t$. On the other hand, F^t is degree 0, so it contains the origin, and therefore $\vec{c} \cdot F^t$ contains $\vec{\lambda}$. But by the above lemma, $\vec{c} \cdot F^t = (\vec{c}^{-1} \cdot F)^t$. □

5.1.2 Specific examples

We now give examples illustrating the above results. In each example, we compute the rank of a conformal blocks bundle $\mathbb{V}_{\mathfrak{sl}_{r+1},\vec{\lambda},\ell}$, and the symmetrized conformal blocks divisor $\mathbb{SD}_{\mathfrak{sl}_{r+1},\vec{\lambda},\ell}$, which is obtained by permuting the weights in all possible ways and summing each associated divisor. These symmetrized divisors lie in the *symmetrized nef cone*, which has the advantage of being significantly smaller than the full nef cone, making computations easier. The symmetrized nef cone has a natural basis $D_2, \ldots, D_{\lfloor \frac{n}{2} \rfloor}$. The computations were done on a home computer using Swinarski's conformal blocks package for Macaulay 2, the LiE software package, and Buch's Littlewood-Richardson calculator package in Sage, as well as the author's own code to compute the inequalities and find examples [51, 20, 16, 54]. For more details on the symmetrized nef cone and the formulas used to compute the symmetrized divisors, see [17].

Example 5.1.7. Let $G = \mathrm{SL}_7$, and let

$$\lambda_1 = \omega_6 = (1 \geq 1 \geq 1 \geq 1 \geq 1 \geq 1 \geq 0)$$
$$\lambda_2 = \omega_6 = (1 \geq 1 \geq 1 \geq 1 \geq 1 \geq 1 \geq 0)$$
$$\lambda_3 = \omega_3 = (1 \geq 1 \geq 1 \geq 0 \geq 0 \geq 0 \geq 0)$$
$$\lambda_4 = \omega_3 = (1 \geq 1 \geq 1 \geq 0 \geq 0 \geq 0 \geq 0)$$
$$\lambda_5 = \omega_2 = (1 \geq 1 \geq 0 \geq 0 \geq 0 \geq 0 \geq 0)$$
$$\lambda_6 = \omega_1 = (1 \geq 0 \geq 0 \geq 0 \geq 0 \geq 0 \geq 0)$$

Then the $d = 0$ walls these weights lie on are the following:

1. $k = 1$ and classes {6}, {6}, {7}, {3}, {7}, {7}.

2. $k = 1$ and classes {6}, {6}, {3}, {7}, {7}, {7}.

3. $k = 2$ and classes {5,6}, {5,6}, {3,7}, {3,7}, {6,7}, {6,7}.

4. $k = 5$ and classes {3, 4, 5, 6, 7}, {3, 4, 5, 6, 7}, {2, 3, 5, 6, 7}, {2, 3, 5, 6, 7}, {1, 2, 5, 6, 7}, {1, 4, 5, 6, 7}.

5. $k = 6$ and classes {2, 3, 4, 5, 6, 7}, {2, 3, 4, 5, 6, 7}, {2, 3, 4, 5, 6, 7}, {1, 2, 3, 5, 6, 7}, {1, 2, 4, 5, 6, 7}, {1, 3, 4, 5, 6, 7}.

6. $k = 6$ and classes {2, 3, 4, 5, 6, 7}, {2, 3, 4, 5, 6, 7}, {1, 2, 3, 5, 6, 7}, {2, 3, 4, 5, 6, 7}, {1, 2, 4, 5, 6, 7}, {1, 3, 4, 5, 6, 7}.

Focusing on the $k = 2$ wall, it is easy to see that the resulting $\mathfrak{sl}_2$ weights give a trivial rank 1 bundle. The $\mathfrak{sl}_5$ weights are $\lambda''_1 = \lambda''_2 = (1 \geq 1 \geq 1 \geq 1 \geq 0)$, $\lambda''_3 = \lambda''_4 = \lambda''_5 = (1 \geq 1 \geq 0 \geq 0 \geq 0)$, and $\lambda''_6 = (1 \geq 0 \geq 0 \geq 0 \geq 0)$. Since the degree of the wall is 0, the reduction result will work for any level. Then the rank of the bundles $\mathbb{V}_{\mathfrak{sl}_7,\vec{\lambda},\ell}$ and $\mathbb{V}_{\mathfrak{sl}_5,\vec{\lambda}'',\ell}$ at $\ell = 1$ is 1, and the symmetrized divisor is $\mathbb{SD}_{\mathfrak{sl}_7,\vec{\lambda},\ell} = \mathbb{SD}_{\mathfrak{sl}_5,\vec{\lambda}'',\ell} = 288D_2 + 360D_3$. At level $\ell = 2$, the bundles are both rank 7, and the symmetrized divisor is $144D_2 + 432D_3$. Finally, at $\ell = 3$ (and any higher level) the conformal blocks are isomorphic to the corresponding spaces of invariants (since 3 is greater than the critical level, see [8]) which are both dimension 8.

Example 5.1.8. Here we exhibit a higher level reduction, with the degree of the cohomology product still zero. Let $G = \mathrm{SL}_4$, and $k = 2$. Let $I_1 = I_2 = \{2, 3\}$ and $I_3 = \cdots = I_6 = \{3, 4\}$. Then $\sigma_{I_1} \cdots \sigma_{I_6} = [\mathrm{pt}]$, so this product defines a degree 0 facet of the multiplicative polytope. Let $\lambda_1 = \lambda_2 = (5 \geq 5 \geq 4 \geq 0)$, and $\lambda_3 = \cdots = \lambda_6 = (2 \geq 1 \geq 1 \geq 0)$, and $\ell = 5$. Then this weight data lies on the given facet, so letting $\lambda'_1 = \cdots = \lambda'_6 = (1 \geq 0)$, $\lambda''_1 = \lambda''_2 = (5 \geq 0)$, and $\lambda''_3 = \cdots = \lambda''_6 = (1 \geq 0)$, one can calculate that $\mathrm{rk}(\mathbb{V}_{\mathfrak{sl}_4,\vec{\lambda},5}) = 10$, $\mathrm{rk}(\mathbb{V}_{\mathfrak{sl}_2,\vec{\lambda}',5}) = 5$, and $\mathrm{rk}(\mathbb{V}_{\mathfrak{sl}_2,\vec{\lambda}'',5}) = 2$. Five is above the critical level for $\vec{\lambda}'$, so the corresponding vector bundle is trivial. One can calculate then that $\mathbb{SD}_{\mathfrak{sl}_4,\vec{\lambda},5} = 5 \cdot \mathbb{SD}_{\mathfrak{sl}_2,\vec{\lambda}'',5} = 1920D_2 + 2160D_3$.

Example 5.1.9. Here we give an example of a reduction on a positive degree wall. Our group is SL_4, and our Grassmannian $\mathrm{Gr}(1, 4)$. Now it is well-known that $\mathrm{QH}^*(\mathrm{Gr}(1, 4)) \cong \mathbb{Z}[T, q]/(T^4 - q)$. Let $I_1 = \{2\}$ and $I_2 = \cdots = I_6 = \{3\}$. Then $\sigma_{I_1} * \cdots * \sigma_{I_6} = q[\mathrm{pt}]$. Let $\lambda_1 = \cdots \lambda_4 = (2 \geq 2 \geq 2 \geq 0)$, and $\lambda_5 = \lambda_6 = (2 \geq 1 \geq 1 \geq 0)$ and $\ell = 2$. It is easy to see that these weights lie on the wall corresponding to the given product. Furthermore $k_L = 4$ in this case, so that $d_0 = 1$. Then $\lambda'_1 = \cdots \lambda'_4 = (2 \geq 2 \geq 0)$ and $\lambda'_5 = \lambda'_6 = (2 \geq 1 \geq 0)$, and we add seventh weight $\lambda'_7 = (2 \geq 0 \geq 0)$. Then one calculates that $\mathrm{rk}(\mathcal{V}^{\dagger}_{\mathfrak{sl}_4,\vec{\lambda},2}) = 1$ and $\mathrm{rk}(\mathcal{V}^{\dagger}_{\mathfrak{sl}_3,\vec{\lambda}'',2}) = 1$, satisfying the statement of the theorem.

Increasing the level moves the weight data off the given facet of the polytope, and so we would expect the ranks to be different in general. Indeed: $\mathrm{rk}(\mathcal{V}^{\dagger}_{\mathfrak{sl}_4,\vec{\lambda},3}) = 12$ and $\mathrm{rk}(\mathcal{V}^{\dagger}_{\mathfrak{sl}_3,\vec{\lambda}'',3}) = 24$. If we raise the level above the critical level (which is 7 in both cases) so that the spaces of conformal blocks become isomorphic to spaces of tensor invariants, we see that $\mathrm{rk}(\mathcal{V}^{\dagger}_{\mathfrak{sl}_4,\vec{\lambda},\ell}) = 21$ and $\mathrm{rk}(\mathcal{V}^{\dagger}_{\mathfrak{sl}_3,\vec{\lambda}'',\ell}) = 124$, showing that the spaces of invariants are not isomorphic.

Furthermore, one can check that the symmetrized conformal blocks divisors arising from $\vec{\lambda}$ and $\vec{\lambda}''$ at level $\ell = 2$ are non-zero but not equal, showing that the isomorphism does not extend to an isomorphism of conformal blocks bundles over all of $\overline{\mathrm{M}}_{0,n}$.

5.1.3 Symmetric weights

Suppose now that our weights are symmetric, that is, $\lambda_1 = \cdots = \lambda_n$. In this case weights and a level ℓ are in the multiplicative polytope exactly when they satisfy all *symmetric* inequalities, that is, inequalities coming from a product $\sigma_I^n = q^d[\mathrm{pt}]$. This follows from the uniqueness of the canonical reduction of an unstable parabolic bundle. The symmetric case has the advantage of vastly reducing the number of inequalities, making higher rank and level examples more accessible. In the following example, the inequalities are so few we can compute the extremal rays of the face of the eigencone, thus giving an infinite family of potential examples of reductions.

Example 5.1.10. Let $r = 11$ and $n = 6$. Then $n \nmid k(r-k+1)$ unless $k = 6$. Using Buch's Littlewood-Richardson calculator, one can check that there are exactly 2 non-zero symmetric products in this case, given by the sets $I = \{1, 8, 9, 10, 11, 12\}$ and $J = \{6, 7, 8, 9, 10, 11\}$. The corresponding inequalities a partition λ must satisfy to be in the symmetric eigencone are then

$$\lambda^1 + \lambda^8 + \lambda^9 + \lambda^{10} + \lambda^{11} \leq \frac{1}{2}|\lambda|$$
$$\lambda^6 + \lambda^7 + \lambda^8 + \lambda^9 + \lambda^{10} + \lambda^{11} \leq \frac{1}{2}|\lambda|.$$

Then assuming λ satisfies the first inequality with equality, it is sufficient for λ to satisfy $\lambda^6 + \lambda^7 \leq \lambda^1$ to be in the symmetric eigencone; similarly, if λ satisfies the second inequality with equality, it is sufficient that $\lambda^1 \leq \lambda^6 + \lambda^7$. It is then easy to compute the extremal rays for the symmetric eigencone in this case using Sage's Polyhedron package. We list in Table 5.1 (bases for) the extremal rays for the 10-dimensional subcone of weights in the symmetric eigencone satisfying the first inequality

$\lambda = \sum_i a_i\omega_i$	Partition	λ'	λ''	$\dim(\mathbb{A}_{\vec{\lambda}'})$	$\dim(\mathbb{A}_{\vec{\lambda}''})$
ω_2	$(1,1,0,0,0,0,0,0,0,0,0)$	ω_1	ω_1	1	1
$\omega_1+\omega_3$	$(2,1,1,0,0,0,0,0,0,0,0)$	$2\omega_1$	ω_2	1	15
$2\omega_1+\omega_4$	$(3,1,1,1,0,0,0,0,0,0,0)$	$3\omega_1$	ω_3	1	40
$3\omega_1+\omega_5$	$(4,1,1,1,1,0,0,0,0,0,0)$	$4\omega_1$	ω_4	1	15
$4\omega_1+\omega_6$	$(5,1,1,1,1,1,0,0,0,0,0)$	$5\omega_1$	ω_5	1	1
$5\omega_1+\omega_7$	$(6,1,1,1,1,1,1,0,0,0,0)$	$6\omega_1$	0	1	1
$4\omega_1+\omega_8$	$(5,1,1,1,1,1,1,1,0,0,0)$	$4\omega_1+\omega_2$	0	265	1
$3\omega_1+\omega_9$	$(4,1,1,1,1,1,1,1,1,0,0)$	$3\omega_1+\omega_3$	0	7570	1
$2\omega_1+\omega_{10}$	$(3,1,1,1,1,1,1,1,1,1,0)$	$2\omega_1+\omega_4$	0	7570	1
$\omega_1+\omega_{11}$	$(2,1,1,1,1,1,1,1,1,1,1)$	$\omega_1+\omega_5$	0	265	1

Table 5.1: Extremal rays of a symmetric wall

with equality. We also list the reductions for each extremal ray, and the dimension of the spaces of invariants for each factor. We were able to check that these dimensions multiply to the dimension of the associated SL_{12} space of invariants in some, but not all cases, showing the computational advantage of rank reduction.

Therefore, a weight $\lambda = \sum_i a_i\omega_i$ is in the above cone exactly when

$$a_1 = \sum_{i=2}^{7}(i-2)a_i + \sum_{i=8}^{11}(12-i)a_i.$$

Furthermore, we see that the image of the reduction map is not surjective on the first factor, since the multiplicative polytope is trivial when $r+1=n$ (that is, the multiplicative polytope is simply $\mathcal{A}^n$). This follows from the fact that the order of the center of SL_{r+1} is $r+1$, and therefore the symmetric multiplicative polytope contains all the symmetric vertices of $\mathcal{A}^n$. Finally, note that the divisors arising from weights on this wall contain the divisors arising from 6 symmetric SL_6 weights. It is not known if divisors arising from weights on a wall will always contain all divisors arising from the images of the reduction map.

5.2 Examples in type C

We want to give some examples of the reduction theorem when $G = \mathrm{Sp}_{2r}$ and $d = 0$. Weights again can be represented by partitions $\lambda^1 \geq \lambda^2 \geq \cdots \geq \lambda^r \geq 0$, with $\ell(\lambda) \geq \lambda^1$. The correspondence between weights in terms of the basis of fundamental weights and partitions is the same as in type

A, however in this case there is a one-to-one correspondence between weights of G and partitions with r parts. The spaces G/P are isomorphic to the symplectic isotropic Grassmannians $\mathrm{IG}(k, 2r)$, which are the moduli spaces of k-dimensional isotropic subspaces of $\mathbb{C}^{2r}$, where $1 \leq k \leq r$. The classes in $H^*(\mathrm{IG}(k, 2r))$ of Schubert varieties again correspond to subsets $I \subseteq \{1, \dots, 2r\}$, with the following condition: if $i \in I$, then $2r - i + 1 \notin I$. For each such subset I, let $I' = I \cap \{1, \dots, r\}$, and $I'' = \{2r - i + 1 \mid i \in I, i \geq r + 1\}$; note that $I' \cap I'' = \emptyset$. Then for a product $\sigma_{I_1} \odot_0 \cdots \odot_0 \sigma_{I_n} = [\mathrm{pt}]$, the corresponding inequality is

$$\sum_{j=1}^{n} \left(\sum_{i \in I'_j} \lambda_j^i - \sum_{i \in I''_j} \lambda_j^i \right) \leq 0.$$

Given such a product in $\mathrm{IG}(k, 2r)$, and weights in the multiplicative polytope on this wall, the reduction will be to the group $L' = \mathrm{SL}_k \times \mathrm{Sp}_{2(r-k)}$. For a subset I, let $I^c = I' \sqcup I''$. Then given a weight λ, the reduced weight associated to the $\mathrm{Sp}_{2(r-k)}$ factor is simply the subpartition given by I^c. The reduction to the SL_k factor is more complicated, and we will illustrate it with an example.

Our examples will be for the group $G = \mathrm{Sp}_6$, and we use the irredundant list of inequalities calculated by Kumar, Leeb, and Milson in [34]. Again we use the LiE software package, and Swinarski's conformal blocks package to compute the ranks and divisors.

Example 5.2.1. Let $k = 1$ and $n = 4$. Then $\mathrm{IG}(1, 6) \cong \mathbb{P}^5$, so the products in the standard cohomology ring are easy to calculate. Note however that not every non-zero product will be Levi-movable, since G/P is not cominiscule. Let $I_1 = \{3\}$, $I_2 = I_3 = \{5\}$, and $I_6 = \{6\}$. Then this product is Levi-movable and equal to a point by the calculations in [34], and the corresponding inequality is then

$$\lambda_1^3 \leq \lambda_2^2 + \lambda_3^2 + \lambda_4^1.$$

Let $\lambda_1 = \omega_1 + \omega_3 = (2 \geq 1 \geq 1)$, $\lambda_2 = \omega_1 = (1 \geq 0 \geq 0)$, $\lambda_3 = 2\omega_1 = (2 \geq 0 \geq 0)$, and $\lambda_4 = \omega_3 = (1 \geq 1 \geq 1)$. Clearly these weights lie on the given wall, and therefore the reduction theorem applies to these weights, assuming they lie in the multiplicative polytope. Indeed, letting $\lambda'_1 = \omega_1 + \omega_2 = (2 \geq 1)$, $\lambda'_2 = \omega_1 = (1 \geq 0)$, $\lambda'_3 = 2\omega_1 = (2 \geq 0)$, and $\lambda'_4 = \omega_2 = (1 \geq 1)$, one can calculate that $\mathrm{rk}(\mathbb{V}_{\mathfrak{sp}_6, \vec{\lambda}, 2}) = \mathrm{rk}(\mathbb{V}_{\mathfrak{sp}_4, \vec{\lambda}', 2}) = 2$, and that $\mathbb{SD}_{\mathfrak{sp}_6, \vec{\lambda}, 2} = \mathbb{SD}_{\mathfrak{sp}_4, \vec{\lambda}', 2} = 8D_2$.

Example 5.2.2. Finally, we give an example when G/P is a Lagrangian Grassmannian. Let

$G = \mathrm{Sp}_6$, $n = 3$, and $k = 3$. Let $I_1 = \{1, 4, 5\}$, $I_2 = \{2, 4, 6\}$, and $I_3 = \{3, 4, 5\}$. Then again by [34], this product corresponds to an irredundant inequality of the multiplicative polytope, which is

$$\lambda_1^1 + \lambda_2^2 + \lambda_3^3 \leq \lambda_1^2 + \lambda_1^3 + \lambda_2^1 + \lambda_2^3 + \lambda_3^1 + \lambda_3^2.$$

Let $\lambda_1 = 2\omega_1 + \omega_2$, $\lambda_2 = \omega_1 + \omega_2$, and $\lambda_3 = \omega_3$. These weights lie in the additive eigencone, and on the given wall. For any weight $\lambda = a\omega_1 + b\omega_2 + c\omega_3$, the reduction to SL_3 is given by

$$I_1 : \lambda \mapsto (a + b + 2c)\omega_1 + b\omega_2$$

$$I_2 : \lambda \mapsto (b + 2c)\omega_1 + (a + b)\omega_2$$

$$I_3 : \lambda \mapsto (b + 2c)\omega_1 + a\omega_2.$$

Therefore for the given weights we get $\lambda_1' = 3\omega_1 + \omega_2$, $\lambda_2' = \omega_1 + 2\omega_2$, and $\lambda_3' = 2\omega_1$. It is easily checked that the dimensions of the spaces of invariants are both 1. Note that the level of the weights has increased in this case, and if working with conformal blocks, the level ℓ is doubled after reducing to SL_3, since the Dynkin index is 2. This example also shows that weights on the alcove wall do not necessarily reduce to weights on the alcove wall(s) of the smaller group.

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