

Long Time Stability and Control Problems for Stochastic Networks in Heavy Traffic

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ABSTRACT

CHIHOOON LEE: Long Time Stability and Control Problems for Stochastic
Networks in Heavy Traffic

(Under the direction of Dr. Amarjit Budhiraja)

Stochastic processing networks arise commonly from applications in computers, telecommunications, and large manufacturing systems. Study of stability and control for such networks is an active and important area of research. In general the networks are too complex for direct analysis and therefore one seeks tractable approximate models. Heavy traffic limit theory yields one of the most useful collection of such approximate models. Typical results in the theory say that, when the network processing resources are roughly balanced with the system load, one can approximate such systems by suitable diffusion processes that are constrained to live within certain polyhedral domains (e.g., positive orthants). Stability and control problems for such diffusion models are easier to analyze and, once these are resolved, one can then infer stability properties and construct good control policies for the original physical networks. In my dissertation we consider three related problems concerning stability and long time control for such networks and their diffusion approximations.

In the first part of the dissertation we establish results on long time asymptotic properties, in particular geometric ergodicity, for limit diffusion models obtained from heavy traffic analysis of stochastic networks. The results provide the rate of convergence to steady state, moment estimates for steady state, uniform in time moment estimates for the process and central limit type results for time averages of such processes. In the second part of the dissertation we consider invariant distributions of an

important subclass of stochastic networks, namely the generalized Jackson networks (GJN). It is shown that, under natural stability and heavy traffic conditions, the invariant distributions of GJN converge to unique invariant probability distribution of the corresponding constrained diffusion model. The result leads to natural methodologies for approximation and simulation of steady state behavior of such networks. In the final part of the dissertation we consider a rate control problem for stochastic processing networks with an ergodic cost criterion. It is shown that value functions and near optimal controls for limit diffusion models serve as good approximations for the same quantities for the underlying physical queueing networks that are heavily loaded.

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CHAPTER 1

Introduction

The study of stochastic networks is an active area of research in applied probability with diverse applications arising from computer, telecommunications, and complex manufacturing systems. From engineering and performance perspective, stability and control issues for such network models are of central concern. Excepting simple cases, the networks of interest are too complex to be analyzed directly and thus one seeks tractable approximate models. When the network is in “heavy traffic,” that is, processing resources are roughly balanced with the system load, one can, using tools from theory of diffusion processes and weak convergence, approximate such systems by suitable constrained diffusion processes. A typical approximating model is a reflecting diffusion in some polyhedral cone with (possibly) oblique directions of constraint (at the boundary), which change discontinuously from one face of the cone to another. The main goals of this work are

- (a) to study long time asymptotics for such constrained diffusions;
- (b) to develop rigorous mathematical results that relate time asymptotic properties of constrained diffusion processes with the steady state behavior of the underlying stochastic networks;
- (c) to study a rate control problem for stochastic networks with an ergodic cost criterion and to show that value functions and near optimal controls for limit diffusion models serve as good approximations for the same quantities for the underlying physical queueing networks which are heavily loaded.

A common theme in all three works is that the analysis is based on a detailed understanding of time asymptotics of both stochastic networks and their approximating diffusion models. Additionally, topics in (b) and (c) require a careful treatment of the interchange of the two limits, namely the heavy traffic limit and the long time limit.

In Chapter 2 we study geometric ergodicity and related issues for certain families of constrained diffusions that arise in the heavy traffic analysis of multiclass queueing networks. We first consider the classical diffusion model with constant coefficients, namely a semimartingale reflecting Brownian motion (SRBM) in a positive orthant. Under a natural stability condition on a related deterministic dynamical system, the existence and uniqueness of a stationary distribution for this model have been studied in [25]. We strengthen this result considerably by establishing *geometric ergodicity* for the process, under exactly the conditions of [25]. Namely, we show that the statistical distribution of the process at time t converges to the unique stationary distribution, as $t \rightarrow \infty$, at an exponential rate. The result that we establish is in fact much stronger in that we identify an exponentially growing Lyapunov function V and show that the process is V -uniformly ergodic. Such a result says, for example, that expected values of unbounded (possibly exponentially growing) functions of the state converge to those under the unique invariant distribution, at an exponential rate. This result is then used to prove that the unique invariant measure of the SRBM admits a finite moment generating function in a neighborhood of zero. As other consequences we establish uniform (in time and initial condition in a compact set) estimates on exponential moments of an SRBM. Growth estimates on polynomial moments of the process as a function of the initial condition are obtained. Finally we establish a functional central limit theorem for time averages of functionals of an SRBM and characterize the asymptotic variance in this limit result via the solution of the related Poisson equation. In Section 2.3 of the chapter, we also consider a family of diffusion models with state dependent coefficients, constrained to take values in

some convex polyhedral cone in \mathbb{R}^d . Such diffusions arise as approximating models for stochastic networks with state dependent arrival and processing rates. Positive recurrence for such constrained diffusions under suitable conditions on the drift vector field was established in [2]. As in the case of an SRBM, we strengthen this result by establishing V -uniform ergodicity with a function V that grows exponentially. As consequences of this result we establish, similar to the constant coefficients case, exponential moment bounds, moment stability results and functional central limit theorems.

Chapter 3 is concerned with the study of convergence of invariant measures for a certain class of queueing networks. Since the original queueing networks can be quite complex, practitioners often use the steady state behavior of limit diffusion models to approximate the steady state of the queueing system. However, a rigorous mathematical justification for such approximation procedures has received little attention. In recent work [28], a family of queueing networks (the so-called generalized Jackson networks) was considered and it was shown that the stationary distributions of the queue length processes for the network, under suitable scaling, converge to the unique stationary distribution of the approximating constrained diffusion process as the traffic intensity approaches its limit. One of the key assumptions made in [28] is that the inter-arrival and service times have finite moment generating functions (m.g.f.) in the neighborhood of origin. Finiteness of the m.g.f. of the primitive processes is a critical ingredient in the strong approximation techniques underlying their analysis. In Chapter 3 of this work we present a proof of the above result, based on some elementary stability properties of a related deterministic dynamical system, *without* imposing any exponential integrability conditions on the primitives of the network. We make the usual i.i.d. and second moment assumptions on inter-arrival and service times that are typically used in heavy traffic analysis for invoking a functional central limit theorem. Our result significantly goes beyond [28] and gives mathematical

validity to the interchange of limits heuristic for many queueing models with Pareto type primitives (which commonly arise from communication and internet networks) as well.

In Chapter 4 we consider a rate control problem for stochastic processing networks with an ergodic cost criterion. One of the primary means of control of flow in a network is by adjusting arrival or processing rates. One is particularly interested in network performance under such control actions over long periods of time. Goal is to construct control policies that give “near optimal” performance that does not degrade over time. A natural mathematical formulation for addressing these issues is through an optimal stochastic control problem with an ergodic cost criterion. In general a direct analysis of the control problem for networks is quite intractable and therefore one considers related diffusion control problems that arise in a formal heavy traffic limit. In the current setting such a formal limit analysis leads to a drift control problem for constrained diffusion processes with an ergodic cost criterion. We show that the value function (i.e., optimum value of the cost) of the rate control problem for the network converges under a suitable heavy traffic scaling to that of the limit diffusion model. Since there are well studied numerical schemes for computing near optimal controls for controlled diffusions, our results suggest natural approaches for obtaining near (asymptotically) optimal rate control algorithms for such a family of processing networks.

CHAPTER 2

Long time asymptotics for constrained diffusions

Study of stability of stochastic networks is of central importance. (cf. [42, 36, 21, 17, 16, 43, 15, 14, 41, 18]) Excepting special cases, the networks of interest are too complex to be analyzed directly and thus one seeks tractable approximate models. In this respect, constrained diffusion processes which arise as appropriate scaling limits of critically loaded queueing networks are key.

In this chapter we will consider a variety of stability properties of such diffusion processes. We will begin our study with semimartingale reflecting Brownian motions (SRBMs) in a positive orthant \mathbb{R}_+^d , $d \in \mathbb{N}$. Roughly speaking, an SRBM is a stochastic process with continuous sample paths which in the interior of the orthant $S \doteq \mathbb{R}_+^d$, behaves like a Brownian motion with a constant drift and when it hits the boundary of S , an instantaneous reflection occurs so as to constrain the process in the orthant S . Such Markov processes commonly arise in the heavy traffic analysis of *multiclass open queueing networks* and have been extensively studied [31, 51, 52, 57, 25, 19, 58]. In particular, the study of stability properties of an SRBM plays a critical role in the stability analysis of stochastic networks.

The main paper on the long term properties of an SRBM is [25], where it is established that under suitable stability conditions on a closely related deterministic dynamical system, an SRBM is positive recurrent and admits a unique invariant probability measure. The goal of this chapter is to study the rate of convergence of

the transition probability kernel to the invariant distribution and other refined long term asymptotics for an SRBM. This study is undertaken in Section 2.2.

We begin, in Section 2.1, by reviewing some standard concepts regarding Markov processes with a general state space. In particular we recall some basic notions, such as irreducibility, recurrence, geometric and uniform ergodicity, for general state space Markov processes. Also discussed are, sampled and resolvent chains, generators and drift criterion for stability.

Section 2.2 is devoted to the study of an SRBM. We give basic definitions and review classical results [57] on existence and uniqueness of an SRBM. We also present the key result of [25] which gives sufficient conditions for ergodicity of an SRBM. We then proceed to new results obtained in this chapter. Our first main result (Theorem 2.2.18) shows that an SRBM is strong Feller. Although such a result is ‘folk lore’ in the literature, our work provides the first rigorous proof of the statement. The strong Feller property is central in establishing the key irreducibility property of an SRBM (Theorem 2.2.19). Next we study several stability properties of an SRBM under the main stability condition of [25] (see Condition 2.2.8 in Section 2.2.1). In Theorem 2.2.31, by identifying a suitable Lyapunov function, we show that the invariant measure, existence and uniqueness of which follows from [25], has a finite moment generating function in a neighborhood of zero. This result is then used to establish uniform (in time and initial condition in a compact set) estimates on exponential moments of an SRBM. Growth of polynomial moments of the process, as a function of the initial condition, is investigated in Corollary 2.2.34 and Theorem 2.2.35. Finally in Theorem 2.2.38 we establish a functional central limit theorem for functionals of an SRBM and characterize the asymptotic variance in this limit result via the solution of the related Poisson equation.

We next consider a family of diffusion models with state dependent coefficients, constrained to take values in some convex polyhedral cone in \mathbb{R}^d with the vertex at the

origin. Positive recurrence for such constrained diffusions under suitable conditions on the drift coefficient was established in [2]. In this chapter we strengthen this result by establishing V -uniform ergodicity with a function V that grows exponentially. As consequences of this result we establish, as in the constant coefficients case, exponential moment bounds, moment stability results and functional central limit theorems. These results are given in Corollary 2.3.11.

Our proofs make critical use of Lyapunov function methods developed in [45, 22]. At the heart of the proofs for the SRBM is Theorem 2.2.27 which obtains suitable bounds on exponential moments of hitting times of compact sets. Once these estimates are available, the results of [22] (cf. Theorem 4.4) yield a Lyapunov function V for which the inequality (2.39) holds and as a consequence the process is V -uniformly ergodic. Lemma 2.2.28 establishes that V has exponential upper and lower bounds. From these estimates one immediately obtains finiteness of exponential moments of invariant measure (Theorem 2.2.31) and convergence of expected value of unbounded (exponentially growing) functionals of the state process to the expectation under the invariant measure, at an exponential rate. Furthermore, these estimates are key in proving stability results (Corollary 2.2.34 and Theorem 2.2.35) for polynomial moments of the process. Finally we obtain, as a consequence of results in [30], functional central limit theorems for processes $\xi_n(t) \doteq \frac{1}{\sqrt{n}} \left(\int_0^{nt} [F(Z_s) - \pi(F)] ds \right)$, where Z is the underlying Markov process, π the unique invariant measure and F is allowed to have exponential growth. In the state dependent case (see Section 2.3) although one can prove similar bounds on exponential moments of hitting times as in the constant coefficients case, we are unable to establish an exponential lower bound (2.34) as in Lemma 2.2.28. The main obstacle to such a result is that, in Section 2.3, the drift vector field of the underlying constrained diffusion is allowed to have linear growth, and as a result estimate (2.37) which critically uses the boundedness of the drift coefficients fails. In view of this difficulty we proceed by making a different choice of a

Lyapunov function V (cf. Lemma 2.3.9) that, from results in [2], is known to have exponential upper and lower bounds. We show that this Lyapunov function satisfies the multiplicative drift condition (2.54) for a sampled Markov chain. The results of [45] can then be brought to bear to establish V -uniform ergodicity and as a consequence one obtains similar exponential moment estimates, moment stability results and functional central limit theorems as in the constant coefficients case.

Some of the notation used in this chapter is as follows. For a metric space X , let $\mathcal{B}(X)$ be the Borel σ -field on X . The Dirac measure at the point x is denoted by δ_x . The set of positive integers is denoted by \mathbb{N} and $\mathbb{N}_0 \doteq \mathbb{N} \cup \{0\}$. The set of real numbers by \mathbb{R} and nonnegative real numbers by \mathbb{R}_+ . Let \mathbb{R}^d be the d -dimensional Euclidean space. For sets $A, B \subseteq \mathbb{R}^d$, $dist(A, B)$ will denote the distance between two sets, i.e., $\inf\{|x - y| : x \in A, y \in B\}$ and 1_A , A° , \bar{A} denote an indicator function of A , set of interior points of A , closure of A , respectively. For a given matrix M denote by M' its transpose and by M^i the i^{th} row of M . For $a, b \in \mathbb{R}$, let $a \vee b \doteq \max\{a, b\}$, $a \wedge b \doteq \min\{a, b\}$ and $a^+ \doteq \max\{0, a\}$. The class of continuous functions $f : X \rightarrow Y$ is denoted by $C(X, Y)$, real continuous bounded functions on X by $C_b(X)$. Let $D(X, Y)$, when X is a subset of \mathbb{R} , denote the class of right continuous functions having left limits, defined from X to Y , equipped with the usual Skorohod topology. Finally, let $C(X)$, $C[0, 1]$, $C[0, \infty)$ denote $C(X, \mathbb{R})$, $C([0, 1], \mathbb{R})$, $C([0, \infty), \mathbb{R})$, respectively and $D[0, 1]$, $D[0, \infty)$ denote $D([0, 1], \mathbb{R})$, $D([0, \infty), \mathbb{R})$, respectively. Inequalities for vectors are interpreted componentwise.

2.1 Preliminaries

2.1.1 Modes of stability

The term ‘stability’ in stochastic processes literature does not have a single definition, but rather could have different meanings depending on the context of use. In this section, we will introduce a series of increasingly stronger concepts of ‘stability’ for a

continuous time Markov process Φ .

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ be a filtered measurable space on which is given an X -valued stochastic process $\Phi = \{\Phi_t : t \in \mathbb{R}_+\}$ and a collection of probability measures $\{P_x, x \in X\}$ such that $(\Phi, \{P_x\})$ form a time homogeneous Markov family. More precisely, under each of the measures P_x , (Φ_t) is a Markov process with initial distribution δ_x and transition probability kernel

$$P^t(y, A) \doteq P_y(\Phi_t \in A), \quad y \in X, \quad A \in \mathcal{B}(X).$$

Frequently, when the family $\{P_x\}$ is clear, we will suppress it from the notation and refer to Φ as the Markov process. Here X is a locally compact, complete and separable metric space. We assume that Φ is a strong Markov process with RCLL paths and $\{P^t\}$ maps Borel functions to Borel functions. I.e. for all bounded Borel maps $f : X \rightarrow \mathbb{R}$, the map $x \mapsto \int_X f(y)P^t(x, dy)$ is a Borel measurable map. Let $\mathbb{E}_x[\cdot]$ denote the expectation with respect to the measure P_x .

We will occasionally need to refer to discrete time Markov processes. An X -valued discrete time stochastic process $\check{\Phi} = \{\check{\Phi}_n : n \in \mathbb{N}_0\}$ along with a family of probability measure $\{\check{P}_x\}_{x \in X}$ defined on some filtered measurable space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n \in \mathbb{N}_0})$ is called a homogeneous Markov family if under each measure \check{P}_x , $(\check{\Phi}_n)$ is a Markov chain with initial distribution δ_x and one step transition kernel

$$\check{P}(y, A) \doteq \check{P}_y[\check{\Phi}_1 \in A], \quad y \in X, \quad A \in \mathcal{B}(X).$$

Once more reference to $\{\check{P}_x\}$ will be omitted when clear from the context.

For a real bounded measurable function f and a σ -finite measure μ on X , define

$$P^t f(x) \doteq \int P^t(x, dy) f(y), \quad \mu P^t(A) \doteq \int \mu(dx) P^t(x, A).$$

Note that $P^t f(x)$ equals the conditional expectation of $f(\Phi_t)$ conditioned on $\Phi_0 = x$. For a measurable set $A \in \mathcal{B}(X)$ let $\tau_A \doteq \inf\{t \geq 0 : \Phi_t \in A\}$ and $\eta_A \doteq \int_0^\infty 1_{\{\Phi_t \in A\}} dt$, the first hitting time of A and a sojourn time in A , respectively.

A set $A \subseteq X$ is said to be *precompact* if the closure of A is a compact subset of X . Consider a family of open precompact sets $\{O_n : n \in \mathbb{N}_0\}$ such that $O_n \uparrow X$ as $n \rightarrow \infty$. Define the exit time of Φ from O_m as $T^m \doteq \tau_{O_m^c}$ and let $\xi \doteq \lim_{m \rightarrow \infty} T^m$. The random variable ξ is referred to as the *explosion time* of the process. Our first notion of ‘stability’ is that of *nonexplosivity*, defined as below.

Definition 2.1.1. (Nonexplosivity. [cf. p. 520 [47]]) *We say the Markov process Φ is nonexplosive if $P_x\{\xi = \infty\} = 1$ for all $x \in X$.*

Note that nonexplosive processes remain bounded for bounded time intervals (almost surely). A somewhat stronger notion of ‘stability’ is that of *nonevanescence*. Define the set $\{\Phi \rightarrow \infty\}$ as $\{\omega \in \Omega : \forall \text{ compact sets } C \subseteq X, \Phi_t(\omega) \in C^c \text{ for all sufficiently large } t\}$.

Definition 2.1.2. (Nonevanescence. [cf. Section 3.1 [47]]) *We say the process Φ is nonevanescient if $P_x\{\Phi \rightarrow \infty\} = 0$ for each $x \in X$.*

It is easy to check that if Φ is nonevanescient then it is nonexplosive [cf. Theorem 3.1 [47]]. In countable state Markov process theory an important concept is that of irreducibility. In continuous state theory, although there is not a single agreed upon definition of irreducibility, the following notion of *φ -irreducibility* plays a very important role.

By convention, all measures μ on $(X, \mathcal{B}(X))$ in this chapter will be nontrivial (i.e. $\mu(X) \neq 0$).

Definition 2.1.3. (φ -irreducibility. [cf. p. 490 [46]]) *Let φ be a σ -finite measure on $(X, \mathcal{B}(X))$. The Markov process Φ is called φ -irreducible if whenever $A \in \mathcal{B}(X)$ is such that $\varphi(A) > 0$, we have $\mathbb{E}_x[\eta_A] > 0$, $\forall x \in X$. The measure φ is called*

an irreducibility measure for the process Φ . For a discrete time Markov process $\check{\Phi}$, φ -irreducibility is defined in a similar way on setting $\eta_A \doteq \sum_{n=0}^{\infty} 1_{[\check{\Phi}_n \in A]}$.

Taking φ as the counting measure one sees that the notion of φ -irreducibility coincides with the usual concept of irreducibility for countable state space Markov processes. An irreducibility measure ψ for the Markov process Φ is called *maximal* if every other irreducibility measure for Φ is absolutely continuous with respect to ψ . For proof of existence of a maximal irreducibility measure of a φ -irreducible process Φ , we refer the reader to Section 3 of [22]. All through this chapter the symbol ψ will exclusively be used for a maximal irreducibility measure. Let $\mathcal{B}^+(X) \doteq \{A \in \mathcal{B}(X) : \psi(A) > 0\}$. A σ -finite measure π on $(X, \mathcal{B}(X))$ is called an *invariant* measure for Φ if and only if for all $A \in \mathcal{B}(X)$, $x \in X$ and $t \geq 0$, $\pi(A) = \pi P^t(x, A)$. If there exists an invariant probability measure π then it can be shown that π and ψ are mutually absolutely continuous [cf. Section 3 [22]], and so we can write $\mathcal{B}^+(X) = \{A \in \mathcal{B}(X) : \pi(A) > 0\}$.

We now introduce the next stronger notion of stability, namely, Harris recurrence. Once more, this reduces to the usual notion of recurrence for countable state Markov processes on taking φ in the definition below as the counting measure.

Definition 2.1.4. (Harris recurrence. [cf. Section 2.2 in [46]]) *The process Φ is called Harris recurrent if for some σ -finite measure φ on $(X, \mathcal{B}(X))$,*
 $P_x(\{\eta_A = \infty\}) = 1$ *whenever $\varphi(A) > 0$.*

Sometimes in order to emphasize the special choice of φ , we will say Φ is φ -Harris recurrence. It can be shown (see Theorem 2.4 in [44]), that Φ is φ -Harris recurrent if and only if there exists a σ -finite measure μ on $(X, \mathcal{B}(X))$ such that $P_x(\{\tau_A < \infty\}) = 1$ whenever $\mu(A) > 0$. We remark that φ and μ do not coincide in general [cf. p. 493 in [46]]. Clearly φ -Harris recurrence implies φ -irreducibility. Furthermore, it can be shown that if Φ is Harris recurrent then it has a unique (up to a scalar multiplier) invariant measure π [cf. p. 491 [46]].

The next stronger notion of stability is the finiteness of the invariant measure π . If π is finite then it can be normalized to a probability measure and with an abuse of notation we denote the normalized probability measure once more by π . In this case we say that Φ is positive Harris recurrent. This unique invariant probability measure π plays a central role in the study of asymptotic properties of Φ .

Definition 2.1.5. (Positive Harris recurrence. [cf. Section 2.2 in [46]])

Suppose that Φ is Harris recurrent with a finite invariant measure π . Then Φ is called positive Harris recurrent.

Positive Harris recurrence is the natural generalization to general state space models, of the usual concept of positive recurrence in Markov chain theory for countable state space. Once positive Harris recurrence is assured, then the next important and practical question of interest is the convergence and speed of convergence to steady state. For example, does the transition function P^t converge to invariant measure π as $t \rightarrow \infty$, if so, how fast is the convergence? For a signed measure μ on $\mathcal{B}(X)$ define its *total variation norm* $\|\mu\|$ as $\|\mu\| \doteq \sup_{f:|f|\leq 1} |\mu(f)|$.

Definition 2.1.6. (Ergodicity. [cf. p. 533 on [47]]) *A Markov process Φ will be called ergodic if it has a unique invariant probability measure π and*

$$\lim_{t \rightarrow \infty} \|P^t(x, \cdot) - \pi\| = 0, \quad \forall x \in X, \quad (2.1)$$

where $\|\cdot\|$ is the total variation norm.

Remark 2.1.7. If Φ is positive Harris recurrent and the skeleton chain

$\check{\Phi}_n \doteq \{\Phi_{n\Delta} : n \in \mathbb{N}_0\}$ for some $\Delta > 0$ is φ -irreducible, then Φ is ergodic [cf. Theorem 6.1 [46]].

Ergodicity ensures the convergence of the expectation $\mathbb{E}_x[f(\Phi_t)]$ to the steady state value $\pi(f)$, for all bounded measurable functions f , as $t \rightarrow \infty$. To investigate

such convergence for an unbounded function f , we need the concept of the so-called f -norm. For any signed measure μ on $\mathcal{B}(X)$ and $f \geq 1$, define its f -norm as

$$\|\mu\|_f \doteq \sup_{|g| \leq f} |\mu(g)| = \sup_{|g| \leq f} \left| \int \mu(dy) g(y) \right|.$$

Note that f -norm is same as the total variation norm if $f \equiv 1$.

Definition 2.1.8. (*f -Ergodicity.* [cf. p. 511 on [46]]) *For a measurable function $f \geq 1$, a Markov process Φ will be called f -ergodic if it is positive Harris recurrent with invariant probability measure π , such that $\pi(f) < \infty$ and*

$$\lim_{t \rightarrow \infty} \|P^t(x, \cdot) - \pi\|_f = 0, \quad \forall x \in X.$$

For an f -ergodic Markov process, although the convergence of $\mathbb{E}_x[f(\Phi(t))]$ to the steady state $\pi(f)$ is guaranteed, the ‘practical convergence’ needed for numerical purposes may take a very long time. Thus the rate of convergence to steady state is particularly important. Exponential rate of convergence is perhaps one of the most sought after ergodic properties of a Markov process. Now we introduce the notion of f -exponential ergodicity from [22].

Definition 2.1.9. (*f -Exponential ergodicity.* [cf. Section 3 [22]]) *Let a Markov process Φ be positive Harris recurrent with invariant probability measure π . For a measurable function $f \geq 1$, Φ is called f -exponentially ergodic if there exist a constant $\beta \in (0, 1)$ and a function $B : X \rightarrow \mathbb{R}_+$ such that for all $t \in \mathbb{R}_+$ and $x \in X$,*

$$\|P^t(x, \cdot) - \pi\|_f \leq B(x)\beta^t. \tag{2.2}$$

When $f \equiv 1$ we simply say that Φ is exponentially ergodic.

A somewhat stronger property than the above exponential bound is the following.

Definition 2.1.10. (*f*-Uniform ergodicity. [cf. Section 3 [22]]) *Let a Markov process Φ be positive Harris recurrent with invariant probability measure π . For a measurable function $f : X \rightarrow [1, \infty)$, Φ is called *f*-uniformly ergodic if there exist constants $D \in (0, \infty)$, $\rho \in (0, 1)$ such that for all $t \in \mathbb{R}_+$ and $x \in X$,*

$$\|P^t(x, \cdot) - \pi\|_f \leq f(x)D\rho^t. \quad (2.3)$$

Sometimes when looking for estimates uniform in initial condition, we will find it more useful to work with a norm which is defined for operators (i.e. transition kernels) rather than for measures. For measurable functions $h : X \rightarrow \mathbb{R}$ and f as above define $\|h\|_f \doteq \sup_{x \in X} \frac{|h(x)|}{f(x)} < \infty$, and L_∞^f be the vector space of such functions. Also for kernel $P \doteq P(x, dy)$, define the *f*-norm $\|P\|_f$ by

$$\|P\|_f \doteq \sup_{h \in L_\infty^f, \|h\|_f \neq 0} \frac{\|Ph\|_f}{\|h\|_f}.$$

With an abuse of notation define the transition kernel $\pi(x, A) \doteq \pi(A)$, $A \in \mathcal{B}(X)$, $x \in X$. It is easy to check that *f*-uniform ergodicity is equivalent to the *f*-norm exponential convergence of the transition kernel P^t to π . In the special case when $f \equiv 1$, we will refer to *f*-uniform ergodicity merely as uniform ergodicity.

2.1.2 Sampled chains

In our analysis of continuous time Markov process $\Phi = \{\Phi_t : t \in \mathbb{R}_+\}$ with transition kernel (P^t) , we will consider some Markov chains derived from Φ . In this section, we will describe sampled Markov chains such as Δ -skeleton chain, \mathfrak{R} -chain, and K_a -chain corresponding to different sampling schemes. These sampled chains play a central role in proving results such as φ -irreducibility and other ergodicity properties [cf. Section 2.3 [46]].

Roughly speaking a K_a -chain is the Markov chain obtained by sampling the con-

tinuous time Markov process Φ using a sampling distribution a , a probability measure on $[0, \infty)$. Resolvent chain \mathfrak{R}_β corresponds to the special case when a is an exponential distribution with parameter β , and Δ -skeleton chain corresponds to the case when a is a point mass at $\Delta \in (0, \infty)$. Precise definitions are as follows.

Definition 2.1.11. [cf. pp. 491-2 [46]] *For a probability measure a on \mathbb{R}_+ , define the Markov transition function $K_a : X \times \mathcal{B}(X) \rightarrow [0, 1]$ as*

$$K_a(x, A) \doteq \int_0^\infty P^t(x, A) a(dt).$$

We will call the discrete time Markov chain with one step transition kernel $K_a(x, A)$ as the K_a -chain for the Markov process Φ . If a is an exponential distribution with parameter β we will denote K_a by \mathfrak{R}_β , i.e.

$$\mathfrak{R}_\beta(x, A) \doteq \int_0^\infty P^t(x, A) \beta \exp(-\beta t) dt$$

and call the corresponding sampled chain as the \mathfrak{R}_β -chain; if $\beta = 1$, we write \mathfrak{R}_β as merely \mathfrak{R} . Finally, if a is degenerate at $\Delta \in (0, \infty)$, we will call the associated Markov chain $\{\Phi(j\Delta) : j \in \mathbb{N}_0\}$ as the Δ -skeleton chain.

In other words, K_a is the transition kernel for the discrete time Markov chain $\{\Phi_{t_k} : k \in \mathbb{N}_0\}$ where $\{t_{k+1} - t_k : k \in \mathbb{N}_0\}$ are independent of Φ and *i.i.d.* with common law a . The following theorems illustrate how the analysis of sampled chains play important roles in the study of stability properties of Φ .

In all through this chapter, $(\check{\Phi}, \{\check{P}_x\}_{x \in X})$ will denote a discrete time Markov chain with one step transition kernel $\check{P}(x, A) \doteq \check{P}_x[\check{\Phi}_1 \in A]$.

Theorem 2.1.12. [cf. Proposition 2.2 (ii) [46]] *The Markov process Φ is φ -irreducible if and only if the \mathfrak{R} -chain is φ -irreducible.*

Theorem 2.1.13. [cf. Theorem 3.1 [44]] *Suppose that a is a general probability*

measure on \mathbb{R}_+ . If the K_a -chain is Harris recurrent, then so is the process Φ ; and then the K_a -chain is positive Harris recurrent if and only if the process Φ is positive Harris recurrent.

In many cases, showing φ -irreducibility for a continuous time Markov chain Φ in a direct way is not straightforward. By Theorem 2.1.12, it suffices to show φ -irreducibility of its \mathfrak{R} -chain, denoted by $\check{\Phi}_{\mathfrak{R}}$. We first consider the notion of *open set irreducibility* of a discrete time chain such as $\check{\Phi}_{\mathfrak{R}}$.

Definition 2.1.14. (Reachable points, open set irreducibility. [cf. Section 6.1.2 [45]]) *A point $x \in X$ is called reachable for a chain $\check{\Phi}$ if for every open set $O \in \mathcal{B}(X)$ containing x we have*

$$\sum_n \check{P}^n(y, O) > 0, \quad \forall y \in X,$$

where \check{P}^n is the n -step transition kernel of the chain. If every point $x \in X$ is reachable, then the Markov chain $\check{\Phi}$ is called *open set irreducible*.

Note that if the state space X is countable and is equipped with the discrete topology then the open set irreducibility is equivalent to the usual irreducibility. Next, we will define and use the following concepts in proving φ -irreducibility of Φ . For the transition probability kernel \check{P} , $x \in X$ and bounded function f on X consider the mapping $\check{P} : f(x) \mapsto \check{P}f(x) \doteq \int \check{P}(x, dy)f(y)$. Let C_b denote the class of bounded continuous functions from X to \mathbb{R} . If \check{P} maps C_b to C_b then we say \check{P} is (*weak*) *Feller*.

Definition 2.1.15. (Strong Feller) *We say \check{P} is strong Feller if \check{P} maps all bounded measurable functions to C_b . For an X -valued continuous time Markov process Φ with transition kernel P^t , we call P^t (or, Φ) is strong Feller if for all bounded measurable functions f on X , $t > 0$ we have $P^t f \in C_b$.*

Theorem 2.1.16. [Proposition 6.2.1 [45]] *If $\check{\Phi}$ is strong Feller, and X contains one reachable point x^* , then $\check{\Phi}$ is φ -irreducible.*

Remark 2.1.17. More can be said about the irreducibility measure. In fact, one maximal irreducibility measure of $\check{\Phi}$ is $\psi = P(x^*, \cdot)$.

2.1.3 Generators of Markov processes

Many properties of a Markov process are often more conveniently characterized in terms of *generator* rather than through the probability transition semigroup. Some basic definitions and properties of a generator are summarized in this section. We begin by introducing the *drift operator* Δ which is the analogue of a generator for discrete time Markov chains.

Definition 2.1.18. (Drift operator. [cf. p. 174 [45]]) *For a discrete time Markov chain $\check{\Phi}$ with a transition function \check{P} and a measurable function $V : X \rightarrow \mathbb{R}_+$, let*

$$\Delta V(x) \doteq \int \check{P}(x, dy)V(y) - V(x), \quad x \in X. \quad (2.4)$$

We refer to Δ as the drift operator for the Markov chain $\check{\Phi}$.

We refer the reader to [45] (Parts II and III) for a whole spectrum of stability results for discrete time Markov chains that can be obtained by studying ‘Lyapunov functions’ V that satisfy suitable drift inequalities. For example, the existence of a suitable set $C \subseteq X$, constants $b \in \mathbb{R}$, $\beta \in (0, \infty)$, and a function $V : X \rightarrow [1, \infty)$ satisfying $\Delta V(x) \leq -\beta V(x) + b1_C(x)$, $x \in X$ ensures the *geometric ergodicity* of the chain $\check{\Phi}$: $\|\check{P}^n(x, \cdot) - \pi\|_V = O(\rho^n)$ for some $\rho \in (0, 1)$ [cf. Theorem 15.0.1 [45]].

For continuous time processes the notion analogous to the drift operator Δ is that of the *generator* of a Markov process. We now present the two most commonly used definitions of a generator of a Markov process. We first define the *bounded-pointwise generator*. Let \mathfrak{L} be the space of bounded measurable, real-valued functions from X

to \mathbb{R} . For $\{f_k\} \subseteq \mathfrak{L}$, if $\sup_k \sup_{x \in X} |f_k(x)| < \infty$ and $\lim_{k \rightarrow \infty} f_k(x) = f(x)$ then we say $\{f_k\}$ converges boundedly and pointwisely to f , and write, $bp \lim_{k \rightarrow \infty} f_k = f$.

Definition 2.1.19. (Bounded-pointwise generator. [cf. [27]]) Let $\bar{\mathcal{A}}$ be a linear operator defined on $\mathcal{D}(\bar{\mathcal{A}}) \subseteq \mathfrak{L}$,

$$\mathcal{D}(\bar{\mathcal{A}}) \doteq \left\{ f \in \mathfrak{L} : bp \lim_{t \downarrow 0} \frac{P^t f - f}{t} \text{ exists} \right\},$$

given as $\bar{\mathcal{A}}f = bp \lim_{t \downarrow 0} \frac{P^t f - f}{t}$ for $f \in \mathcal{D}(\bar{\mathcal{A}})$. We refer to $\bar{\mathcal{A}}$ as the bounded-pointwise generator for the Markov process Φ and $\mathcal{D}(\bar{\mathcal{A}})$ the domain of the generator.

We observe that $\bar{\mathcal{A}}f(x)$ is the derivative of the expectation $\mathbb{E}_x f(\Phi(t))$ with respect to t at $t = 0$. This is the natural analogue in the continuous time theory of the one step increment (as in (2.4)), for a discrete time Markov chain.

In practice bounded-pointwise generator is too restrictive since many functions of interest in stability theory fail to be in the domain of bounded-pointwise operator. To include a bigger class of functions we introduce the *extended generator* of a Markov process.

Definition 2.1.20. (Extended generator. [cf. Section 1.3 [47] or Section 3 [22]]) Denote by $\mathcal{D}(\tilde{\mathcal{A}})$ the set of all functions $V : X \rightarrow \mathbb{R}$ for which $\mathbb{E}_x |V(\Phi_t)| < \infty$ and there exists a measurable function $W : X \rightarrow \mathbb{R}$ satisfying

$$\int_0^t \mathbb{E}_x |W(\Phi_s)| ds < \infty,$$

$$\mathbb{E}_x [V(\Phi_t)] = V(x) + \mathbb{E}_x \left[\int_0^t W(\Phi_s) ds \right],$$

for each $x \in X, t > 0$. For $V \in \mathcal{D}(\tilde{\mathcal{A}})$, and a corresponding W we write $(V, W) \in \tilde{\mathcal{A}}$ (or sometimes with an abuse of notation $W = \tilde{\mathcal{A}}V$). We refer to the (multivalued) function $\tilde{\mathcal{A}}$ as the extended generator of Φ and $\mathcal{D}(\tilde{\mathcal{A}})$ the domain of the extended generator.

Remark 2.1.21. Under suitable conditions on $V \in \mathcal{D}(\tilde{\mathcal{A}})$ (and Φ) one has that if

$$\hat{\mathcal{A}}V(x) = \lim_{h \downarrow 0} \frac{\int P^h(x, dy)V(y) - V(x)}{h},$$

where the limit on the right side above is taken pointwise, then $(V, \hat{\mathcal{A}}V) \in \tilde{\mathcal{A}}$. Also if $V \in \mathcal{D}(\bar{\mathcal{A}})$ then one can easily check that $V \in \mathcal{D}(\tilde{\mathcal{A}})$.

2.1.4 Lyapunov criteria for stochastic stability

Lyapunov's method is one of the basic techniques for studying stability properties of deterministic and stochastic dynamical systems. For discrete time Markov chains on a general state space, [45] gives explicit Lyapunov criteria for various notions of stochastic stability, described in Section 2.1.1. For analogous criteria for continuous time Markov processes we refer the reader to [46], [47] and [48]. We summarize the results below.

For this summary we will assume that Φ is strong Feller and φ -irreducible. The general theory does not require this assumption, however, restricting to strongly Feller Markov process greatly simplifies the presentation.

(C1) *Condition for Harris recurrence:* There exist some compact set $C \in \mathcal{B}(X)$, some constant $b < \infty$ and some measurable function $V : X \rightarrow [0, \infty)$, $V \in \mathcal{D}(\tilde{\mathcal{A}})$ such that $\tilde{\mathcal{A}}V(x) \leq b1_C(x)$.

(C2) *Condition for Positive Harris recurrence:* There exist some compact set $C \in \mathcal{B}(X)$, some constant $b < \infty$ and some measurable function $V : X \rightarrow [0, \infty)$, $V \in \mathcal{D}(\tilde{\mathcal{A}})$ such that $\tilde{\mathcal{A}}V(x) \leq -1 + b1_C(x)$.

(C3) *Condition for V-exponential ergodicity:* There exist some compact set $C \in \mathcal{B}(X)$, some constants $b, c > 0$, and a function $V : X \rightarrow [1, \infty)$, $V \in \mathcal{D}(\tilde{\mathcal{A}})$ such that $\tilde{\mathcal{A}}V(x) \leq -cV(x) + b1_C(x)$.

2.2 Semimartingale reflecting Brownian motion

2.2.1 Definitions and formulation

A semimartingale reflecting Brownian motion (SRBM) is a stochastic process with values in a d -dimensional positive orthant (\mathbb{R}_+^d). It commonly arises as a diffusion approximation to open queueing networks that are in ‘heavy traffic’ (cf. [51]). In this section we will give some basic definitions and summarize the (weak) existence & uniqueness result of [57], for an SRBM. We will also present a key oscillation inequality (Theorem 2.2.10) that will be used in several estimates in this chapter.

Let $d \in \mathbb{N}$, $S \doteq \{x = (x_1, \dots, x_d)' \in \mathbb{R}^d : x_i \geq 0, i = 1, \dots, d\}$ and consider column vectors $r^0, r^1, \dots, r^d \in \mathbb{R}^d$. Let $R \doteq [r^1, \dots, r^d]_{d \times d}$ and Σ be a $d \times d$ strictly positive definite matrix. For $x \in \partial S$, define the set of directions of reflection as

$$r(x) \doteq \left\{ \sum_{i=1}^d q_i r^i : \sum_{i=1}^d q_i = 1, q_i \geq 0, \text{ and } q_i > 0 \text{ only if } x_i = 0 \right\}. \quad (2.5)$$

We call the quadruple (S, r^0, Σ, R) as the *data* for an SRBM.

Definition 2.2.1. (SRBM) *For $x \in S$, an SRBM associated with the data (S, r^0, Σ, R) that starts from x is a continuous, $\{\mathcal{F}_t\}$ -adapted d -dimensional process Z , defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ such that:*

- (i) $Z(t) \doteq B(t) + r^0 t + RY(t) \in S$ for all $t \geq 0$, P -a.s.
- (ii) $B(\cdot)$ is a d -dimensional $\{\mathcal{F}_t\}$ Brownian motion with covariance matrix Σ such that $B(0) = x$, P -a.s.
- (iii) Y is an $\{\mathcal{F}_t\}$ -adapted d -dimensional process such that $Y_i(0) = 0$ for $i = 1, \dots, d$, P -a.s. For each $i = 1, \dots, d$, Y_i is continuous, nondecreasing, and Y_i can in-

crease only when $Z(\cdot)$ is on the face $F^i \doteq \{x \in S : x_i = 0\}$, i.e.,

$$\int_0^t 1_{\{Z_i(s) \neq 0\}} dY_i(s) = 0, \quad \forall t \geq 0.$$

Roughly speaking, an SRBM behaves like a Brownian motion with a drift component $r^0 t$ in the interior of S ; when $Z(\cdot)$ hits the boundary of S , an instantaneous reflection occurs, that is, for some $1 \leq i \leq d$ the process $Y_i(\cdot)$ increases by the minimal amount needed to keep Z in the orthant S . Allowed directions of reflection (constraint) at $x \in \partial S$ are given by the set $r(x)$. In particular, on the relative interior of the i^{th} face F^i the direction of reflection is given by the i^{th} column of the reflection matrix R .

Next we introduce one of the key conditions required for the existence of an SRBM. Throughout this work all vector inequalities will be interpreted componentwise.

Definition 2.2.2. (completely- \mathcal{S}) A $d \times d$ matrix R is said to be an \mathcal{S} -matrix if there exists a d -dimensional column vector $u \geq 0$ such that $Ru > 0$. A principal sub-matrix of R is any square matrix obtained from R after deleting columns and rows of R which have indices in any subset (possibly empty) of $\{1, \dots, d\}$. The matrix R is called completely- \mathcal{S} if every principal sub-matrix \tilde{R} of R is an \mathcal{S} -matrix.

In Theorem 2 of [52], it was shown that a *necessary* condition for the existence of an SRBM is that the reflection matrix R is *completely- \mathcal{S}* . The following result from [57] shows that the condition is *sufficient* as well.

Define $C \doteq \{(z, y) : [0, \infty) \rightarrow S \times S, z \text{ and } y \text{ are continuous functions}\}$, $\mathcal{M} \doteq \sigma\{(z, y)(s) : 0 \leq s < \infty, (z, y) \in C\}$, $\mathcal{M}_t \doteq \sigma\{(z, y)(s) : 0 \leq s \leq t, (z, y) \in C\}$, where $t \geq 0$.

Theorem 2.2.3. [57] Assume that R is completely- \mathcal{S} . Fix $x \in S$. There exists an SRBM associated with the data (S, r^0, Σ, R) that starts from x . Let Z , defined on

some filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, be such an SRBM and let Y denote its ‘pushing’ process as described in Definition 2.2.1 (iii). Let Q_x denote the probability measure induced on (C, \mathcal{M}) by (Z, Y) : $Q_x(A) = P((Z, Y) \in A)$ for all $A \in \mathcal{M}$. Then Q_x is unique and hence the law of any SRBM, together with its associated pushing process, for the data (S, r^0, Σ, R) and starting point x is unique.

The canonical process $z(\cdot)$ under the measure Q_x defines an SRBM starting from x on $(C, \mathcal{M}, \{\mathcal{M}_t\})$, where for the semimartingale decomposition (i) in Definition 2.2.1 one can take $Y = y$ and $B = z - Ry - r^0$. The family $(z, \{Q_x\}_{x \in S})$ is a Feller continuous strong Markov process.

Remark 2.2.4. The completely- \mathcal{S} condition on R will be a standing hypothesis for this chapter. Henceforth by an SRBM, associated with the data (S, r^0, Σ, R) , $(\{Z_t\}_{t \geq 0}, (P_x)_{x \in S})$ we will mean a Markov family given on some filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ with transition kernel $P^t(x, A) = P_x Z_t^{-1}(A) = Q_x(z(t) \in A)$, where Q_x and z are as in Theorem 2.2.3. Also it will be implied that there are processes Y and B given on (Ω, \mathcal{F}) such that (i), (ii), (iii) in Definition 2.2.1 hold for P there replaced by P_x , for all $x \in S$.

Remark 2.2.5. The completely- \mathcal{S} condition can be geometrically interpreted as follows. Recall the set of reflection direction given in (2.5). The completely- \mathcal{S} condition ensures that for every $x \in \partial S$, there exists a convex combination of vectors in $r(x)$ which points into S° from x . In the operations research literature, a completely- \mathcal{S} matrix is also referred to as completely- \mathcal{Q} or strictly semi-monotone matrix [cf. [12]].

An SRBM is said to be *positive recurrent* if for each closed set $A \subseteq S$ having positive Lebesgue measure, we have $\mathbb{E}_x[\tau_A] < \infty$ for all $x \in S$, where $\tau_A \doteq \inf\{t \geq 0 : Z(t) \in A\}$. We now formulate the key condition for positive recurrence of an SRBM in terms of the associated ‘fluid limit’ trajectories. Let $C([0, \infty), \mathbb{R}^d)$ be the class of continuous functions $f : [0, \infty) \rightarrow \mathbb{R}^d$.

Definition 2.2.6. (Skorohod Problem) Let $\psi \in C([0, \infty), \mathbb{R}^d)$ with $\psi(0) \in S$. Then $(\phi, \eta) \in C([0, \infty), \mathbb{R}^d) \times C([0, \infty), \mathbb{R}^d)$ solves the Skorohod problem (SP) for ψ (with respect to S and R) if the following hold:

- (i) $\phi(t) = \psi(t) + R\eta(t) \in S$, for all $t \geq 0$;
- (ii) η is such that, for $i = 1, \dots, d$, (a) $\eta_i(0) = 0$, (b) η_i is nondecreasing, and (c) η_i can increase only when ϕ is on F^i , that is, $\int_0^t 1_{\{\phi_i(s) \neq 0\}} d\eta_i(s) = 0$, for all $t \geq 0$.

Definition 2.2.7. We say that a path $\phi \in C([0, \infty), \mathbb{R}^d)$ is attracted to the origin if for any $\epsilon > 0$ there exists $T < \infty$ such that $t \geq T$ implies $|\phi(t)| \leq \epsilon$.

Condition 2.2.8. The ϕ component of all solutions of the SP for $\psi(\cdot)$ of the form $\psi(t) = x + r^0 t$, $t \geq 0$, $x \in S$, is attracted to the origin.

Theorem 2.2.9. [25] Suppose that Condition 2.2.8 holds. Then the SRBM is positive recurrent and it has a unique stationary distribution.

The above result will be our starting point for obtaining sharper asymptotic properties, such as geometric ergodicity, later in this chapter. The following *oscillation inequality* plays an important role in the course of proving several results in this chapter. For any $0 \leq t_1 < t_2 < \infty$, let $D([t_1, t_2], \mathbb{R}^d)$ denote the set of functions $w : [t_1, t_2] \rightarrow \mathbb{R}^d$ that are right continuous on $[t_1, t_2)$ and have finite left limits on $(t_1, t_2]$. Define $Osc(f, [t_1, t_2]) \doteq \sup\{|f(t) - f(s)| : t_1 \leq s < t \leq t_2\}$ for $f \in D([t_1, t_2], \mathbb{R}^d)$ and $|a| \doteq \max_{1 \leq i \leq d} |a_i|$ for $a \in \mathbb{R}^d$.

Theorem 2.2.10. (Oscillation Inequality. [cf. Theorem 5.1 [58]]) Suppose that $\delta \geq 0$, $0 \leq t_1 < t_2 < \infty$ and $w, x, y \in D([t_1, t_2], \mathbb{R}^d)$ are such that

- (i) $w(t) = x(t) + Ry(t) \in S$, for all $t \in [t_1, t_2]$,
- (ii) For each $i = 1, \dots, d$, $y_i(t_1) \geq 0$, y_i is nondecreasing and y_i cannot increase when $w_i > \delta$, i.e., $\int_{[t_1, t_2]} 1_{\{w_i(t) > \delta\}} dy_i(t) = 0$.

Then there is a constant $C > 0$ depending only on R , such that

$$\text{Osc}(y, [t_1, t_2]) + \text{Osc}(w, [t_1, t_2]) \leq C(\text{Osc}(x, [t_1, t_2]) + \delta).$$

2.2.2 φ -irreducibility for SRBM

As seen in Chapter 2 (Section 2.1.4), irreducibility plays a central role in the study of stability properties of Markov processes. The main result of this section will show that an SRBM associated with the data (S, r^0, Σ, R) is φ -irreducible. Since this data will be fixed henceforth, we will omit any reference to it. The central idea in proving φ -irreducibility for an SRBM is to consider its discrete time \mathfrak{R} -chain. By Theorem 2.1.12, SRBM is φ -irreducible *if and only if* its \mathfrak{R} -chain is φ -irreducible. Also, from Theorem 2.1.16, in order to prove the φ -irreducibility of the \mathfrak{R} -chain it suffices to show that it is strong Feller and S contains one reachable point x^* for the chain.

Many of the ideas for proving the strong Feller property and existence of a reachable point are adapted from [3]. We present arguments in full detail below because their setup does not cover the full generality of an SRBM and also because their proofs require minor corrections at a few places. We begin with the following lemma.

Lemma 2.2.11. *Let $(\{Z_t\}_{t \geq 0}, \{P_x\}_{x \in S})$ be an SRBM. Let $U \subseteq S$ be a bounded open (relative to S) set and let $\tau_U \doteq \inf\{t \geq 0 : Z(t) \notin U\}$. Then we have*

$$\sup_{x \in U} \mathbb{E}_x(\tau_U) < \infty.$$

Proof. For $f \in C^2(S)$ define

$$\mathcal{L}f(x) \doteq \frac{1}{2} \sum_{i,j=1}^d \Sigma_{ij} \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^d r_i^0 \frac{\partial f(x)}{\partial x_i} \quad \text{for } x \in S,$$

$$J_i f(x) \doteq \langle r^i(x), \nabla f(x) \rangle = \sum_{j=1}^d r_j^i \frac{\partial f(x)}{\partial x_j} \quad \text{for } x \in F^i, \quad i = 1, \dots, d.$$

From the *completely-S* property of R , there exists $\theta = (\theta_1, \dots, \theta_d)'$ satisfying $\theta \geq 0$ and $R\theta > 0$. Define $h(x) \doteq \exp(k_0 \sum_{i=1}^d \theta_i x_i)$, where $x \in S$ and $k_0 \in (0, \infty)$ is a constant which will be suitably chosen later in this proof. Clearly, $h \in C_b^2(\bar{U})$. Note that for $x \in U$

$$\mathcal{L}h(x) = \left(k_0^2 \frac{1}{2} \sum_{i,j=1}^d \Sigma_{ij} \theta_i \theta_j + k_0 \sum_{i=1}^d \theta_i r_i^0 \right) h(x).$$

Choose $k_0 > 0$ such that $\left(k_0^2 \frac{1}{2} \sum_{i,j=1}^d \Sigma_{ij} \theta_i \theta_j + k_0 \sum_{i=1}^d \theta_i r_i^0 \right) > 1$. For such a k_0 we have $\mathcal{L}h(x) \geq 1$, since $h(x) \geq 1$. Applying Itô's formula, we have

$$\begin{aligned} & h(Z(t \wedge \tau_U)) - h(Z(0)) \\ &= \int_0^{t \wedge \tau_U} \mathcal{L}h(Z(s)) ds + \sum_{i=1}^d \int_0^{t \wedge \tau_U} J_i h(Z(s)) dY_i(s) + \int_0^{t \wedge \tau_U} \nabla h(Z(s)) dB(s). \end{aligned}$$

Note that the last term $\int_0^{t \wedge \tau_U} \nabla h(Z(s)) dB(s)$ is a P_x -martingale. Since $\mathcal{L}h(x) \geq 1$ for $x \in S$ and for $x \in F^i$

$$J_i h(x) = \sum_{j=1}^d r_j^i \frac{\partial}{\partial x_j} h(x) = k_0 h(x) r^i \cdot \theta \geq 0$$

we have that

$$\mathbb{E}_x [h(Z(t \wedge \tau_U)) - h(Z(0))] \geq \mathbb{E}_x(t \wedge \tau_U).$$

Note that $h(x)$ is a bounded function on \bar{U} and thus taking $t \rightarrow \infty$, we have that $\sup_{x \in U} \mathbb{E}_x(\tau_U) < \infty$. ■

Lemma 2.2.12. *Let $(\{Z_t\}_{t \geq 0}, \{P_x\}_{x \in S})$ be an SRBM. Then for a bounded open set $O \subseteq S$, $P^t(x, O) > 0$ for all $t \in (0, \infty)$ and $x \in S$. In particular, the \mathfrak{R} -chain $\{\check{\Phi}_n\}_{n \geq 1}$, associated with the Markov process $\check{\Phi} \equiv Z$, is open set irreducible.*

Proof. Note that the second part of the lemma is immediate from the first, since the transition kernel of the \mathfrak{R} -chain is $\mathfrak{R}(y, A) = \int_0^\infty e^{-t} P^t(y, A) dt$, $y \in S$, $A \in \mathcal{B}(S)$. We will now show that $P^t(x, O) > 0$ for each *fixed* $t > 0$ and $x \in S$. The proof is adapted from Lemma 3.1 (a) in [3]. We first consider $x \in S^\circ$. Let $\Omega' \doteq C([0, \infty), \mathbb{R}^d)$ and let $X_t(w) \doteq w(t)$ be the coordinate projection. Also let $\mathcal{B}_t \doteq \sigma\{X(s) : s \leq t\}$ and $\hat{P}_x \doteq P_x \circ Z^{-1}$ be the measure induced by Z on Ω' .

Let $\hat{Q}_x = P_x \circ U^{-1}$ be the measure on Ω' induced by $U(t) \doteq B(t) + r^0 t$, $t \geq 0$. Note that $X(\cdot)$ under \hat{P}_x ‘behaves like’ the process $X(\cdot)$ under \hat{Q}_x up to the first hitting time of ∂S . Let $\delta_0 \doteq d(x, \partial S)$. Let A_0 be a nonempty bounded open set and $\delta \in (0, \delta_0 \wedge 1)$ be such that $\bar{A}_0 \subseteq (O \cap S^\circ)$ and $d(A_0, \partial O) > \delta$.

Fix $t > 0$ and let $w_0 \in \Omega'$ be such that $w_0(0) = x$, $w_0(t) \in A_0$ and w_0 is linear on $[0, t]$. Now define the neighborhood of the sample path of w_0 by

$$N(w_0) \doteq \{w \in \Omega' : |w(s) - w_0(s)| < \delta/2, \forall 0 \leq s \leq t\}$$

and note that any $w \in N(w_0)$ stays in S° until t and $w(t) \in O$. For a Borel set $A \subseteq S$ let $\sigma_A \doteq \inf\{t \geq 0 : X(t) \in A\}$. Then

$$P^t(x, O) = P_x(Z(t) \in O) \geq \hat{P}_x(X(t) \in O, \sigma_{\partial S} > t) \geq \hat{Q}_x(N(w_0)) > 0,$$

where the next to last inequality follows from the observation that the distribution of $X(\cdot \wedge \sigma_{\partial S})$ under \hat{P}_x and \hat{Q}_x is the same and the last inequality follows from Schilder’s Theorem (cf. Lemma 5.2.1 of [20]). This proves the result for $x \in S^\circ$.

Finally, consider $x \in \partial S$. From Lemma 2.1 in [57], $P_x(Z(t_0) \in \partial S) = 0$ for almost all $t_0 > 0$ (see the equation (2.1) therein) and $x \in S$. From Markov property of Z , we have that for t_0 as above, and all $t \geq t_0$

$$\begin{aligned}
P_x(Z(t) \in O) &= \int_S P_y(Z(t - t_0) \in O) P_x \circ Z_{t_0}^{-1}(dy) \\
&= \int_{S^c} P_y(Z(t - t_0) \in O) P_x \circ Z_{t_0}^{-1}(dy) > 0,
\end{aligned}$$

where the second equality follows from the fact $P_x(Z(t_0) \in \partial S) = 0$ and the last inequality comes from the result in the previous paragraph. Thus $P^t(x, O) > 0$ for all $t > 0$ and $x \in S$. This proves the lemma. ■

We will now argue that the \mathfrak{A} -chain for an SRBM is strong Feller. In order to prove this result, we begin by showing that the SRBM itself is strong Feller. The key step in establishing the strong Feller property of an SRBM is proving that its transition probability function admits a measurable density. In order to prove this result, we begin with the following lemma.

Lemma 2.2.13. *Let $G \subseteq S$ be a bounded open set. Let $(\{Z_t\}_{t \geq 0}, \{P_x\}_{x \in S})$ be an SRBM. Define $\tau_G \doteq \inf\{r \geq 0 : Z(r) \notin G\}$. Then there exists a constant $K > 0$ such that*

$$\sup_{x \in \bar{G}} \mathbb{E}_x \left[\sum_{i=1}^d Y_i(\tau_G) \right] \leq K.$$

Proof. Let $g \in C_b^2(S)$ be such that

$$D_i g \geq 1 \text{ on } F^i \text{ for all } i \in \{1, \dots, d\}, \tag{2.6}$$

where $D_i g \doteq \langle \nabla g, r^i \rangle = \sum_{j=1}^d r_j^i \frac{\partial g}{\partial x_j}$. The existence of such a g has been established in Theorem 3.2 of [19].

An application of Itô's formula gives

$$\begin{aligned}
g(Z_{t \wedge \tau_G}) &= g(Z_0) + \int_0^{t \wedge \tau_G} \langle \nabla g(Z_s), dB_s \rangle \\
&\quad + \int_0^{t \wedge \tau_G} \langle \nabla g(Z_s), r^0 \rangle ds + \int_0^{t \wedge \tau_G} \left(\frac{1}{2} \text{tr}[D^2 g(Z(s))\Sigma] \right) ds \\
&\quad + \sum_{i=1}^d \int_0^{t \wedge \tau_G} \langle \nabla g(Z_s), r^i \rangle dY_i(s),
\end{aligned}$$

where D^2 is the Hessian matrix. Taking expectations and recalling that $g \in C_b^2(S)$ we see that there exists $C \in \mathbb{R}_+$ such that

$$C [1 + \mathbb{E}_x(t \wedge \tau_G)] \geq \sum_{i=1}^d \mathbb{E}_x \int_0^{t \wedge \tau_G} \langle \nabla g(Z_s), r^i \rangle dY_i(s) \geq \mathbb{E}_x \left[\sum_{i=1}^d Y_i(t \wedge \tau_G) \right],$$

where the last inequality is a consequence of (2.6). The result now follows on combining the above estimate with Lemma 2.2.11. ■

We now proceed to the proof of existence of a measurable transition probability density function for an SRBM. Let $\Gamma(t, x, z)$ be the transition density function of a Brownian motion starting at x with drift r^0 and covariance Σ , i.e.,

$$\Gamma(t, x, z) = |2\pi\Sigma t|^{-1/2} \exp \left[\frac{(x - r^0 t - z)' \Sigma^{-1} (x - r^0 t - z)}{2t} \right].$$

Let Y be as in Definition 2.2.1 and set $K(t) \doteq RY(t)$. For a bounded open set $G \subseteq S, t > 0$, define

$$\begin{aligned}
p_G(t, x, z) &\doteq \Gamma(t, x, z) - \mathbb{E}_x \left[1_{[0, t)}(\tau_G) \Gamma(t - \tau_G, Z(\tau_G), z) \right] \\
&\quad + \mathbb{E}_x \left[\int_0^{t \wedge \tau_G} \langle \nabla \Gamma(t - r, Z(r), z), dK(r) \rangle \right] \tag{2.7}
\end{aligned}$$

where $x \in \bar{G}, z \in G \cap S^\circ$ and $\nabla \Gamma(t, \zeta, z)$ denotes the gradient of Γ in the variable ζ . Henceforth we will abbreviate τ_G as τ .

Remark 2.2.14. Note that there exist $K_1, K_2 \in (0, \infty)$ such that for all $T_0 > 0$ and for all $x, z \in \mathbb{R}^d, 0 < t < T_0$, $|D_x^\alpha \Gamma(t, x, z)| \leq K_1 t^{-\left(\frac{d+|\alpha|}{2}\right)} \exp\left[\frac{-K_2|z-x|^2}{t}\right]$ where $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multi-index with $0 \leq |\alpha| \leq 2$ where $|\alpha| = \alpha_1 + \dots + \alpha_d$ and D_x^α denotes differentiation with respect to x . Observing that $\sup_{0 < s \leq t} \frac{e^{-c/s}}{s^m} < \infty$ for all $m \in \mathbb{R}, c \in (0, \infty)$, and since $z \in G \cap S^\circ$, $d(z, \partial G) > 0$ and $d(z, \partial S) > 0$, we have together with Lemma 2.2.13, that $p_G(t, x, z)$ is well-defined. Similarly one can check that the map $z \mapsto p_G(\cdot, \cdot, z)$ is continuous for all $z \in G \cap S^\circ$.

Lemma 2.2.15. *For any continuous function f with compact support $K \subseteq G \cap S^\circ$, $x \in \bar{G}$ and $t > 0$*

$$\int_G f(z) p_G(t, x, z) dz = \mathbb{E}_x \left[\mathbf{1}_{[t, \infty)}(\tau) f(Z(t)) \right]. \quad (2.8)$$

Proof. Let us denote the LHS of (2.8) by $u(t, x)$. We begin by showing that

$$\left\{ u(t - (\tau \wedge r), Z(\tau \wedge r)) : 0 \leq r < t \right\} \text{ is a } P_x\text{-martingale w.r.t. } \{\mathcal{F}_{\tau \wedge r} : 0 \leq r < t\}. \quad (2.9)$$

For $0 \leq r_1 \leq r_2 < t$, let $\mathcal{F}_1 \doteq \mathcal{F}_{\tau \wedge r_1}$, $\mathcal{F}_2 \doteq \mathcal{F}_{\tau \wedge r_2}$ and $\xi_1 \doteq Z(\tau \wedge r_1)$, $\xi_2 \doteq Z(\tau \wedge r_2)$.

We abbreviate $\tau \wedge r_i$ by τ_i , $i = 1, 2$. We first show that

$$\begin{aligned} & \mathbb{E}_x \left[\int_{S^\circ} f(z) \Gamma(t - \tau_2, \xi_2, z) dz \Big| \mathcal{F}_1 \right] \\ &= \int_{S^\circ} f(z) \Gamma(t - \tau_1, \xi_1, z) dz + \int_{S^\circ} f(z) \mathbb{E} \left[\int_{\tau_1}^{\tau_2} \langle \nabla \Gamma(t - r, Z(r), z), dK(r) \rangle \Big| \mathcal{F}_1 \right] dz. \end{aligned} \quad (2.10)$$

By applying Itô's formula to $\Gamma(t - (\tau \wedge r), Z(\tau \wedge r), z)$ over $[r_1, r_2]$, we have

$$\begin{aligned} & \Gamma(t - \tau_2, \xi_2, z) - \Gamma(t - \tau_1, \xi_1, z) \\ &= \int_{\tau_1}^{\tau_2} \left(-\frac{\partial}{\partial t} \Gamma(t - r, Z(r), z) + \mathcal{L} \Gamma(t - r, Z(r), z) \right) dr \\ &+ \int_{\tau_1}^{\tau_2} \langle \nabla \Gamma(t - r, Z(r), z), dK(r) \rangle + \int_{\tau_1}^{\tau_2} \langle \nabla \Gamma(t - r, Z(r), z), dB(r) \rangle, \end{aligned}$$

where $\mathcal{L}\Gamma(t, x, z) = \left(\frac{1}{2} \sum_{i,j=1}^d \Sigma_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d r_i^0 \frac{\partial}{\partial x_i} \right) \Gamma(t, x, z)$. Conditioning on \mathcal{F}_1 and taking expectation on both sides, we have on using Kolmogorov's backward equation, $\frac{\partial}{\partial t} \Gamma(t, x, z) = \mathcal{L}\Gamma(t, x, z)$, that

$$\mathbb{E} [\Gamma(t - \tau_2, \xi_2, z) | \mathcal{F}_1] = \Gamma(t - \tau_1, \xi_1, z) + \mathbb{E} \left[\int_{\tau_1}^{\tau_2} \langle \nabla \Gamma(t - r, Z(r), z), dK(r) \rangle \Big| \mathcal{F}_1 \right].$$

Equality (2.10) now follows from the above equality via an application of Fubini's Theorem.

Next we show that

$$\{U_{\tau \wedge r}, \mathcal{F}_{\tau \wedge r} : 0 \leq r < t\} \text{ is a martingale,} \quad (2.11)$$

where $U_r \doteq \int p_1(t - r, Z(r), z) f(z) dz$, $0 \leq r < t$ and

$$p_1(t - r, x, z) \doteq \mathbb{E}_x \left[1_{[0, t-r)}(\tau) \Gamma(t - r - \tau, Z(\tau), z) \right].$$

From strong Markov property of Z we have that

$$\mathbb{E}_x \left[1_{[0, t)}(\tau) \Gamma(t - \tau, Z(\tau), z) \Big| \mathcal{F}_{\tau \wedge r} \right] = p_1(t - \tau \wedge r, Z(\tau \wedge r), z). \quad (2.12)$$

This is a consequence of the fact that from strong Markov property, for a measurable function $f : [0, \infty) \times \bar{G} \rightarrow [0, \infty)$, $\mathbb{E}[f(\tau, Z(\tau)) | \mathcal{F}_{\tau \wedge r}] = g(\tau \wedge r, Z(\tau \wedge r))$ where $g(r, z) \doteq \mathbb{E}_z[f(\tau + r, Z(\tau))]$. Thus from (2.12)

$$U_{\tau \wedge r} = \mathbb{E} \left[\int_{S^o} f(z) 1_{[0, t)}(\tau) \Gamma(t - \tau, Z(\tau), z) dz \Big| \mathcal{F}_{\tau \wedge r} \right]$$

which proves (2.11).

Next let $p_3(t, x, z) \doteq \mathbb{E}_x \left[\int_0^{t \wedge \tau} \langle \nabla \Gamma(t-r, Z(r), z), dK(r) \rangle \right]$.

For $r \leq t$,

$$p_3(t - \tau \wedge r, Z(\tau \wedge r), z) = \mathbb{E} \left[\int_{\tau \wedge r}^{\tau \wedge t} \langle \nabla \Gamma(t-u, Z(u), z), dK(u) \rangle \middle| \mathcal{F}_{\tau \wedge r} \right].$$

This is a consequence of the fact that from strong Markov property, for nonnegative measurable function $\psi : [0, \infty) \times G \rightarrow \mathbb{R}^d$

$$\mathbb{E}_x \left[\int_{\tau \wedge r}^{\tau \wedge t} \langle \psi(u, Z(u)), dK(u) \rangle \middle| \mathcal{F}_{\tau \wedge r} \right] = \tilde{g}(\tau \wedge r, Z(\tau \wedge r)),$$

where for $r \leq t$, $\tilde{g}(r, z) \doteq \mathbb{E}_z \left[\int_0^{(t-r) \wedge \tau} \langle \psi(u+r, Z(u)), dK(u) \rangle \right]$. Thus

$$\begin{aligned} & \mathbb{E} [p_3(t - \tau_2, \xi_2, z) | \mathcal{F}_1] - p_3(t - \tau_1, \xi_1, z) \\ &= -\mathbb{E} \left[\int_{\tau_1}^{\tau_2} \langle \nabla \Gamma(t-u, Z(u), z), dK(u) \rangle \middle| \mathcal{F}_1 \right]. \end{aligned}$$

Integrating on S° , we have

$$\begin{aligned} & \mathbb{E}_x \left[\int_{S^\circ} f(z) p_3(t - \tau_2, \xi_2, z) dz \middle| \mathcal{F}_1 \right] - \mathbb{E}_x \left[\int_{S^\circ} f(z) p_3(t - \tau_1, \xi_1, z) dz \right] \\ &= - \int_{S^\circ} \mathbb{E}_x \left[\int_{\tau_1}^{\tau_2} \langle \nabla \Gamma(t-u, Z(u), z), dK(u) \rangle \middle| \mathcal{F}_1 \right] f(z) dz. \end{aligned} \quad (2.13)$$

Proof of (2.9) now follows on combining (2.7), (2.10), (2.11), and (2.13).

Next observe that (2.8) holds trivially for $x \in \partial G$. Finally, for any $x \in G$ and continuous f with compact support $K \subseteq G \cap S^\circ$

$$\begin{aligned}
\int_{S^\circ} f(z)p_G(t, x, z)dz &= \lim_{r \uparrow t} \mathbb{E}_x [u(t - \tau \wedge r, Z(\tau \wedge r))] \\
&= \lim_{r \uparrow t} \mathbb{E}_x \left[\int_K f(z)p_G(t - \tau \wedge r, Z(\tau \wedge r), z)dz \right] \\
&= \lim_{r \uparrow t} \mathbb{E}_x \left[\int_K f(z)1_{[t, \infty)}(\tau)p_G(t - \tau \wedge r, Z(\tau \wedge r), z)dz \right] \\
&\quad + \lim_{r \uparrow t} \mathbb{E}_x \left[\int_K f(z)1_{[0, t)}(\tau)p_G(t - \tau \wedge r, Z(\tau \wedge r), z)dz \right] \\
&\equiv I_1 + I_2, \tag{2.14}
\end{aligned}$$

where the first equality follows on using the martingale property (2.9). Next note that for all compact set $K \subseteq G \cap S^\circ$,

$$\sup_{z \in K} \sup_{\tilde{z} \in \partial G} \sup_{0 \leq s \leq t} \Gamma(s, \tilde{z}, z) < \infty, \tag{2.15}$$

and

$$\sup_{z \in K} \sup_{\tilde{z} \in \partial S} \sup_{0 \leq s \leq t} |\nabla \Gamma(s, \tilde{z}, z)| < \infty. \tag{2.16}$$

Combining the uniform estimates (2.15), (2.16) with the Feller property of Z , we have by the dominated convergence theorem: If $x \in G, y(s) \in G$ are such that $y(s) \rightarrow x$ as $s \rightarrow t$, then

$$\lim_{s \uparrow t} \int_{S^\circ \cap G} f(z)p_G(t - s, y(s), z)dz = f(x). \tag{2.17}$$

Next noting that $\sup_{0 \leq s \leq t} \sup_{x \in G, z \in K} |p_G(s, x, z) - \Gamma(s, x, z)| < \infty$, we have via another application of dominated convergence theorem that

$$\begin{aligned}
I_1 &= \mathbb{E}_x \left[1_{[t, \infty)}(\tau) \lim_{r \uparrow t} \int_{S^\circ} f(z)p_G(t - \tau \wedge r, Z(\tau \wedge r), z)dz \right] \\
&= \mathbb{E}_x \left[1_{[t, \infty)}(\tau) f(Z(t)) \right], \tag{2.18}
\end{aligned}$$

$$\begin{aligned}
I_2 &= \mathbb{E}_x \left[1_{[0,t)}(\tau) \lim_{r \uparrow t} \int_{S^\circ} f(z) p_G(t - \tau \wedge r, Z(\tau \wedge r), z) dz \right] \\
&= \mathbb{E}_x \left[1_{[0,t)}(\tau) \int_{S^\circ} f(z) p_G(t - \tau, Z(\tau), z) dz \right] = 0,
\end{aligned} \tag{2.19}$$

where the second equality in (2.18) follows from (2.17) and the last inequality is a consequence of the observation that $p_G(t, x, z) = 0$ for all $t > 0$, $x \in \partial G$, $z \in K$. The result now follows on combining (2.14), (2.18), and (2.19). ■

As an immediate corollary of Lemma 2.2.15 we have the following.

Corollary 2.2.16. *For all $t > 0$, $x \in \bar{G}$ and $z \in G \cap S^\circ$, $p_G(t, x, z) \geq 0$. For any Borel set $E \subseteq G \cap S^\circ$, $t > 0$, and $x \in \bar{G}$,*

$$P_x(Z(t) \in E, \tau \geq t) = \int_G 1_E(z) p_G(t, x, z) dz. \tag{2.20}$$

From Feller property of Z it follows that for all $t > 0$, $x \mapsto p_G(t, x, z)$ is continuous on \bar{G} . Combining this with Remark 2.2.14 we see that for each $t > 0$, $(x, z) \mapsto p_G(t, x, z)$ is continuous on $\bar{G} \times (G \cap S^\circ)$. Defining $p_G(t, \cdot, \cdot)$ to be zero outside $\bar{G} \times (G \cap S^\circ)$, we have that $p_G(t, \cdot, \cdot)$ is jointly measurable on $S \times S$ for all $t > 0$.

For $n \in \mathbb{N}$, let $B(0; n)$ be the open ball centered at 0 with radius n . Let $G_n \doteq B(0; n) \cap S$ and define $p(t, x, z) \doteq \lim_{n \rightarrow \infty} p_{G_n}(t, x, z)$. Note that for fixed t, x, z $p_{G_n}(t, x, z)$, is nondecreasing in n , so the above limit is well defined. By definition $(x, z) \mapsto p(t, x, z)$ is a jointly measurable nonnegative function on $S \times S$. Furthermore, observe that for $A \in \mathcal{B}(S)$

$$\begin{aligned}
P_x[Z(t) \in A] &= P_x[Z(t) \in A \cap S^\circ] \\
&= \lim_{n \rightarrow \infty} P_x[Z(t) \in A \cap S^\circ \cap G_n, \tau_{G_n} \geq t] \\
&= \lim_{n \rightarrow \infty} \int_S 1_A(z) p_n(t, x, z) dz = \int_S 1_A(z) p(t, x, z) dz.
\end{aligned} \tag{2.21}$$

Thus we have shown that an SRBM admits a transition probability density $p(t, x, z)$

which is jointly measurable in (x, z) . The strong Feller property of an SRBM now follows from the following well known result.

Lemma 2.2.17. [cf. Lemma 11 [55] pp. 60-61] *Let X be a locally compact metric space and C_X the space of real continuous functions defined on X . Let $p(x, y)$ be a measurable function on $X \times X$, and $\lambda(dy)$ a finite measure on X , with*

$$C1. \int p(x, y)\lambda(dy) = 1;$$

$$C2. \int p(x, y)\phi(y)\lambda(dy) \in C_X \text{ if } \phi \in C_X.$$

Then $\int \psi(y)p(x, y)\lambda(dy) \in C_X$ for all real bounded measurable functions ψ .

Theorem 2.2.18. *The SRBM $(\{Z_t\}_{t \geq 0}, \{P_x\}_{x \in S})$ is strong Feller.*

Proof. Fix $t > 0$. Let ψ be a positive smooth function on S such that $\int_S \psi(z)dz = 1$. Now let $\lambda(dz) \doteq \psi(z)dz$, $q(t, x, z) \doteq \frac{1}{\psi(z)}p(t, x, z)$. Then $q(t, \cdot, \cdot)$ is jointly measurable on $S \times S$ and for $x \in S$ we have $\int_S q(t, x, z)\lambda(dz) = 1$. For any $f \in C_S$, we have $x \mapsto \int_S f(z)q(t, x, z)\lambda(dz)$ is bounded continuous by Feller continuity of SRBM. Since conditions $C1, C2$ of the previous lemma are satisfied, for any bounded measurable function h on S , $\int_S h(z)p(t, x, z)dz = \int_S h(z)q(t, x, z)\lambda(dz)$ is bounded continuous in x . ■

We are now ready to establish the φ -irreducibility of SRBM.

Theorem 2.2.19. *The SRBM $(\{Z_t\}_{t \geq 0}, \{P_x\}_{x \in S})$ is φ -irreducible.*

Proof. In view of Theorem 2.1.12, it suffices to show that the \mathfrak{R} -chain of an SRBM, *i.e.*, the Markov chain with transition kernel $\mathfrak{R}(y, A) \doteq \int_0^\infty P^t(y, A)e^{-t}dt$, is φ -irreducible. To prove the last statement, in turn, it suffices to show in view of Lemma 2.2.12 that the Markov chain $\check{\Phi}$ with transition kernel $\mathfrak{R}(y, A)$ is strong Feller (cf. Proposition 6.1.5 of [45]). Finally, the strong Feller property of $\check{\Phi}$ is an immediate consequence of Theorem 2.2.18 via an application of dominated convergence theorem. This completes the proof. ■

2.2.3 Geometric ergodicity, V -uniform ergodicity

The main result of this section is the V -uniform ergodicity (see Definition 2.1.10) of an SRBM under Condition 2.2.8 for a function V , which is exponentially growing. We begin by introducing the following *Lyapunov* function $W(\cdot)$, which was constructed in [25].

Theorem 2.2.20. [cf. [25]] *Suppose that Condition 2.2.8 holds. Then there exists a continuous map $W : \mathbb{R}^d \rightarrow \mathbb{R}$ such that the following hold.*

(P1) $W(\cdot) \in C^2(\mathbb{R}^d \setminus \{0\})$.

(P2) *Given $N < \infty$, there is an $M < \infty$ such that $x \in S$ and $|x| \geq M$ imply $W(x) \geq N$.*

(P3) *Given $\epsilon > 0$, there is an $M < \infty$ such that $x \in S$ and $|x| \geq M$ imply $|D^2W(x)| \leq \epsilon$.*

(P4) *There exists a $c > 0$ such that*

$$\langle DW(x), r^0 \rangle \leq -c, \text{ for all } x \in S \setminus \{0\},$$

$$\langle DW(x), r \rangle \leq -c, \text{ for all } r \in r(x), x \in \partial S \setminus \{0\}.$$

(P5) *$W(\cdot)$ is radial homogeneous: $W(\alpha x) = \alpha W(x)$ for $\alpha \geq 0, x \in S$.*

Some consequences of properties (P1) – (P5) are the following.

(P6) For every $M \in (0, \infty)$ there exists a $\gamma \equiv \gamma(M) \in (0, \infty)$ such that

$$\sup_{|x| \leq M} |W(x)| \leq \gamma.$$

(P7) $\Gamma \doteq \sup_{x \in S \setminus \{0\}} |DW(x)| < \infty$.

(P8) There exist $c_1, c_2 \in (0, \infty)$ such that $c_1|x| \leq W(x) \leq c_2|x|$, for all $x \in S$.

For a proof of the following elementary lemma we refer the reader to Lemma 4.2 in [2].

Lemma 2.2.21. *Let $x \in S$. Suppose that $\{\alpha(t) : t \geq 0\}$ is an \mathbb{R}^d -valued \mathcal{F}_t -progressively measurable process such that there exists $\bar{\alpha} \in (0, \infty)$ for which $|\alpha(t)| \leq \bar{\alpha}$, for all $t \in (0, \infty)$, P_x -a.s. Then for $\lambda \in (0, \infty)$,*

$$\mathbb{E}_x \left(\exp \left\{ \lambda \left| \int_0^t \langle \alpha(s), dB(s) \rangle \right| \right\} \right) \leq 2 \exp \left\{ \frac{\lambda^2 \bar{\alpha}^2 \gamma t}{2} \right\},$$

where $\gamma \in (0, \infty)$ depends only on the norm of the covariance matrix Σ^{-1} .

Although W may not be differentiable at 0, with an abuse of notation, we set $DW(0) = 0$ and $D^2W(0) = 0$.

Lemma 2.2.22. *Let $x \in S$ and $\Delta > 0$ be fixed. For $m \in \mathbb{N}$ let ν_m be defined as follows:*

$$\nu_m \doteq \sup_{(m-1)\Delta \leq t \leq m\Delta} \left| \int_{(m-1)\Delta}^t \langle DW(Z(s)), dB(s) \rangle \right|.$$

Then for any $\kappa \in (0, \infty)$ and $m, n \in \mathbb{N}; m \leq n$,

$$\mathbb{E}(e^{\kappa \sum_{i=m}^n \nu_i}) \leq [2\sqrt{2}e^{\kappa^2 \Gamma^2 \gamma \Delta}]^{(n-m+1)}, \quad (2.22)$$

where $\gamma \in (0, \infty)$ is as in Lemma 2.2.21 and Γ as in (P7).

Proof. Let $\{\mathcal{F}_t\}$ be as in Definition 2.2.1. Then using Lemma 2.2.21,

$$\begin{aligned} & \mathbb{E}_x(\exp\{\kappa \nu_n\} | \mathcal{F}_{(n-1)\Delta}) \\ &= \mathbb{E}_{Z((n-1)\Delta)} \left(\sup_{0 \leq t \leq \Delta} \exp \left(\kappa \left| \int_0^t \langle DW(Z(s)), dB(s) \rangle \right| \right) \right) \\ &\leq 2 \left(\mathbb{E}_{Z((n-1)\Delta)} \left(\exp \left\{ 2\kappa \left| \int_0^\Delta \langle DW(Z(s)), dB(s) \rangle \right| \right\} \right) \right)^{\frac{1}{2}} \\ &\leq 2\sqrt{2} \exp(\kappa^2 \Gamma^2 \gamma \Delta), \end{aligned} \quad (2.23)$$

where the first equality follows by the Markov property of $Z(\cdot)$ and the next to last inequality from an application of Doob's maximal inequality for submartingales. The last inequality is a consequence of (P7) and Lemma 2.2.21. Iterating the above $(n - m + 1)$ times yields the estimates in (2.22). \blacksquare

By (P3), there exists $M_0 > 1$ such that $x \in S$ and $|x| \geq M_0$ imply

$$\text{tr} [D^2W(x)\Sigma] < c,$$

where c is as in (P4). Fix $M \in (M_0, \infty)$ and define $B_1 \doteq \{x : |x| \leq M\}$. Choose $L \in (0, \infty)$ large enough so that $B_2 \doteq \{x : |W(x)| \leq L\} \supseteq B_1$. Let $\sigma_1 \doteq \inf\{t \geq 0 : Z(t) \in B_2\}$. The proof of the following theorem is similar to that of Theorem 4.1 of [2] except that we consider an exponential moment bound rather than a polynomial moment bound for the hitting time σ_1 .

Theorem 2.2.23. *There exist $\beta \in (0, \infty)$ and $\alpha_1, \alpha_2 \in (0, \infty)$ such that for all $x \in S$ $\mathbb{E}_x[e^{\beta\sigma_1}] < \alpha_1 e^{\alpha_2|x|}$. In particular, for any compact set $K \subseteq S$ $\sup_{x \in K} \mathbb{E}_x[e^{\beta\sigma_1}] < \infty$.*

Proof. Fix $\Delta \in (0, \infty)$ and $m \in \mathbb{N}$. By applying Itô's formula to $W(\cdot)$, we have

$$\begin{aligned} W(Z(m\Delta) \wedge \sigma_1) &= W(Z(m-1)\Delta \wedge \sigma_1) + \int_{(m-1)\Delta \wedge \sigma_1}^{m\Delta \wedge \sigma_1} \left(\frac{1}{2} \text{tr} [D^2W(Z(s))\Sigma] \right) ds \\ &\quad + \int_{(m-1)\Delta \wedge \sigma_1}^{m\Delta \wedge \sigma_1} \langle DW(Z(s)), r^0 \rangle ds \\ &\quad + \int_{(m-1)\Delta \wedge \sigma_1}^{m\Delta \wedge \sigma_1} \langle DW(Z(s)), dB(s) \rangle \\ &\quad + \sum_{i=1}^d \int_{(m-1)\Delta \wedge \sigma_1}^{m\Delta \wedge \sigma_1} \langle DW(Z(s)), r^i \rangle dY_i(s). \end{aligned} \tag{2.24}$$

For $n \in \mathbb{N}$, define $A_n \doteq \{\omega \in \Omega : \inf_{s \in [0, n\Delta]} |W(Z(s))| > L\}$. Note that when $\omega \in A_n$

and $m \leq n$ we have from (2.24) that

$$\begin{aligned}
W(Z(m\Delta)) &= W(Z(m-1)\Delta) + \int_{(m-1)\Delta}^{m\Delta} \left(\frac{1}{2} \text{tr} [D^2W(Z(s))\Sigma] \right) ds \\
&\quad + \int_{(m-1)\Delta}^{m\Delta} \langle DW(Z(s)), r^0 \rangle ds + \int_{(m-1)\Delta}^{m\Delta} \langle DW(Z(s)), dB(s) \rangle \\
&\quad + \sum_{i=1}^d \int_{(m-1)\Delta}^{m\Delta} \langle DW(Z(s)), r^i \rangle dY_i(s) \\
&\equiv T_1 + T_2 + T_3 + T_4 + T_5.
\end{aligned}$$

On A_n , $T_2 \leq \frac{c}{2}\Delta$ and by (P4) we have $T_3 \leq -c\Delta$ and $T_5 < 0$. As a result, on the set A_n and for $m \leq n$

$$\begin{aligned}
W(Z(m\Delta)) &\leq W(Z(m-1)\Delta) - \frac{c}{2}\Delta + \int_{(m-1)\Delta}^{m\Delta} \langle DW(Z(s)), dB(s) \rangle \\
&\leq W(Z(m-1)\Delta) - \frac{c}{2}\Delta + \sup_{(m-1)\Delta \leq t \leq m\Delta} \left| \int_{(m-1)\Delta}^t \langle DW, dB \rangle \right|,
\end{aligned}$$

where in the above display, $DW(Z(s))$, $dB(s)$ are abbreviated as DW and dB , respectively.

Thus for $1 \leq m \leq n$ and on the set A_n ,

$$L < W(Z(m\Delta)) \leq W(Z(m-1)\Delta) - \frac{c}{2}\Delta + \sup_{(m-1)\Delta \leq t \leq m\Delta} \left| \int_{(m-1)\Delta}^t \langle DW, dB \rangle \right|. \quad (2.25)$$

Define ν_m as in Lemma 2.2.22. For $m = 1, \dots, n$ iterating inequality (2.25) we have that, on A_n , $L < W(Z(n\Delta)) \leq W(x) + \sum_{j=1}^n \nu_j - \frac{c}{2}n\Delta$.

Define $D_n \doteq \{\omega \in \Omega : L < W(Z(n\Delta)) \leq W(x) + \sum_{j=1}^n \nu_j - \frac{c}{2}n\Delta\}$, then

$$\begin{aligned}
P(A_n) &\leq P(D_n) \leq P\left(L + \frac{c}{2}n\Delta - W(x) \leq \sum_{j=1}^n \nu_j\right) \\
&\leq [\mathbb{E}(\exp(\alpha \sum_{j=1}^n \nu_j))]/[\exp(\alpha(L + \frac{c}{2}n\Delta - W(x)))] \\
&\leq [2\sqrt{2} \exp(\alpha^2 \Gamma^2 \gamma \Delta)]^n / [\exp(\alpha(L + \frac{c}{2}n\Delta - W(x)))] \\
&= \exp(\alpha W(x) - \alpha L) \exp\left(n\Delta\left(\frac{\log 8}{2\Delta} + \alpha^2 \Gamma^2 \gamma - \frac{\alpha c}{2}\right)\right),
\end{aligned}$$

where $\alpha > 0$, the third inequality follows from Chebyshev's inequality and the fourth inequality follows from (2.22). Let $-\eta \doteq (\frac{\log 8}{2\Delta} + \alpha^2 \Gamma^2 \gamma - \frac{\alpha c}{2})$ and choose sufficiently large $\Delta > 0$ and sufficiently small $\alpha > 0$ so that $\eta > 0$. Let $t \in (0, \infty)$ be arbitrary and pick $n_0 \in \mathbb{N}$ such that $t \in [n_0\Delta, (n_0 + 1)\Delta]$. Then

$$\begin{aligned}
P(\sigma_1 > t) &= P(Z(s) \notin B_2 : 0 \leq s \leq t) \\
&\leq P(Z(s) \notin B_2 : 0 \leq s \leq n_0\Delta) \\
&\leq \exp(\alpha W(x) - \alpha L) \cdot \exp(-\eta(n_0 + 1)\Delta) \\
&\leq C \cdot e^{\alpha W(x)} e^{-\eta t},
\end{aligned}$$

where the last inequality follows on defining $C \doteq \exp(-\alpha L)$ and noting that $t \leq (n_0 + 1)\Delta$.

Finally, for $\beta \in (0, \eta)$

$$\begin{aligned}
\mathbb{E}_x[e^{\beta\sigma_1}] &= \int_0^\infty \beta e^{\beta t} P[\sigma_1 > t] dt \\
&\leq C\beta e^{\alpha W(x)} \int_0^\infty e^{(\beta-\eta)t} dt \\
&= C\beta e^{\alpha W(x)} \left[\frac{e^{(\beta-\eta)t}}{\beta - \eta} \right]_0^\infty = \frac{C\beta}{\eta - \beta} e^{\alpha W(x)}.
\end{aligned}$$

The result now follows from the above estimates on recalling (P8). ■

We will now proceed to the construction of a Lyapunov function V that will enable us to prove geometric ergodicity of Z . The starting point of our analysis is the following result of [22] (cf. see Theorems 6.2 and 5.1 therein). For $\delta \in (0, \infty)$ and a compact set $C \subseteq S$, let $\tau_C(\delta) \doteq \inf\{t \geq \delta : Z(t) \in C\}$.

Theorem 2.2.24. *Suppose that for some compact set $C \subseteq S$ and $\eta, \delta \in (0, \infty)$ we have $\mathbb{E}_x e^{\eta\tau_C(\delta)} < \infty$ for all $x \in S$. Let*

$$V_0(x) \doteq \frac{1}{\eta} [\mathbb{E}_x e^{\eta\tau_C(\delta)} - 1] + 1 \quad (2.26)$$

and suppose that $\sup_{x \in C} V_0(x) < \infty$. Then for all $\beta > 0$ $(V_\beta, W_\beta) \in \tilde{\mathcal{A}}$, where $V_\beta \doteq \mathfrak{R}_\beta V_0$, $W_\beta \doteq \beta V_\beta - V_0$ and $\tilde{\mathcal{A}}$ is the extended generator of Z (see Definition 2.1.20). Furthermore, there exist $b, c \in (0, \infty)$ such that

$$\tilde{\mathcal{A}}V_\beta(x) \leq -cV_\beta(x) + b1_C(x) \quad \text{for all } x \in S. \quad (2.27)$$

Proof. The result follows on taking $f \equiv 1$ in Theorem 6.2 of [22] and using the (b) part of the cited theorem along with (a) part of Theorem 5.1 in the same paper. ■

Let $\tilde{L} \geq L$ be large enough so that $\text{dist}(B_2, \partial B_3) > 1$, where $B_3 \doteq \{x \in S : |W(x)| \leq \tilde{L}\}$. Note that $B_1 \subseteq B_2 \subseteq B_3$. Let $\sigma_0 \doteq \inf\{t \geq 0 : Z(t) \in \partial B_3\}$.

Lemma 2.2.25. *For each fixed $\delta \in (0, \infty)$ there exists $\epsilon_0 \equiv \epsilon_0(\delta) \in (0, 1)$ such that*

$$\sup_{x \in B_3} P_x(\sigma_0 > \delta) < \epsilon_0, \quad (2.28)$$

$$\sup_{x \in B_3} P_x(\sigma_0 < \delta) < \epsilon_0. \quad (2.29)$$

Proof. We will only prove (2.28), the proof of (2.29) is similar and is omitted. We will prove this by the method of contradiction. Fix $\delta \in (0, \infty)$ and suppose that (2.28) does not hold for any $\epsilon_0 \in (0, 1)$. Then there exist sequences $\{x_n\}, \{\epsilon_n\}$ such

that $P_{x_n}[\sigma_0 > \delta] \geq \epsilon_n$, where $\{x_n\} \subseteq B_3$, $\epsilon_n \in (0, 1)$ and $\epsilon_n \uparrow 1$ as $n \rightarrow \infty$. From the Feller property of SRBM it follows that if $\{x_{n_k}\}_{k \geq 1}$ is a subsequence of $\{x_n\}$ such that $x_{n_k} \rightarrow x$ as $k \rightarrow \infty$ then $\limsup P_{x_{n_k}}[\sigma_0 > \delta] \leq P_x[\sigma_0 > \delta]$. Hence $P_x[\sigma_0 > \delta] = 1$. However this contradicts Lemma 2.2.12 and hence the lemma follows. \blacksquare

Let $\tau_1 \doteq \inf\{t \geq \sigma_0 : Z(t) \in \partial B_2\}$. The following lemma is the key step in the proof of Theorem 2.2.27.

Lemma 2.2.26. *Under Condition 2.2.8, there exists $\beta_1 \in (0, \infty)$ and $A \in (0, \infty)$ such that*

$$\sup_{x \in B_2} \mathbb{E}_x[e^{\beta_1 \tau_1}] < A. \quad (2.30)$$

Proof. In view of Theorem 2.2.23 and strong Markov property of SRBM it suffices to show that there exists $r \in (0, \infty)$ such that $\sup_{x \in B_2} \mathbb{E}_x[e^{r\sigma_0}] < \infty$. This will follow if we show that there exists $\theta_0 \in (0, \infty)$ such that for all $k \in \mathbb{N}$ and $x \in B_2$,

$$P_x[\sigma_0 > k] < e^{-\theta_0 k}. \quad (2.31)$$

Next note that $P_x[\sigma_0 > k] = \mathbb{E}_x(\mathbb{E}_x[1_{\{\sigma_0 > k\}} | \mathcal{F}_{k-1}] 1_{\{\sigma_0 > k-1\}})$. Furthermore,

$$\mathbb{E}_x[1_{\{\sigma_0 > k\}} | \mathcal{F}_{k-1}] 1_{\{\sigma_0 > k-1\}} \leq \sup_{x \in B_3} P_x[\sigma_0 > 1] 1_{\{\sigma_0 > k-1\}} < \tilde{\epsilon}_0 1_{\{\sigma_0 > k-1\}},$$

where $\tilde{\epsilon}_0 = \epsilon_0(1) \in (0, 1)$ is as in Lemma 2.2.25. Inequality (2.31) now follows on iterating the above conditioning argument k times. This completes the proof of the lemma. \blacksquare

The following is the key estimate in the proof of geometric ergodicity.

Theorem 2.2.27. *Suppose Condition 2.2.8 holds. Fix $\delta \in (0, \infty)$ and let $\beta_1 \in (0, \infty)$ be as in Lemma 2.2.26. Then $\sup_{x \in B_2} \mathbb{E}_x[e^{\beta_1 \tau_{B_2}(\delta)}] < \infty$.*

Proof. Define a sequence of stopping times $\{\hat{\tau}_n\}, \{\hat{\sigma}_n\}, n \in \mathbb{N}$ as follows:

$$\hat{\sigma}_n \doteq \inf\{t \geq \hat{\tau}_{n-1} : Z(t) \in \partial B_3\}, \quad \hat{\tau}_n \doteq \inf\{t \geq \hat{\sigma}_n : Z(t) \in B_2\}, \quad n = 1, 2, \dots,$$

where $\hat{\tau}_0 \doteq 0$. Let $\theta \doteq \epsilon_0(\delta)$, where $\epsilon_0(\cdot)$ is as in Lemma 2.2.25. Also for $n \in \mathbb{N}$ set $m_n \doteq \hat{\tau}_n - \hat{\tau}_{n-1}$ and let $m_0 \doteq 0$. From the second inequality in Lemma 2.2.25, $P_n[\hat{\tau}_n - \hat{\tau}_{n-1} > \delta | \mathcal{F}_{\hat{\tau}_{n-1}}] \geq 1 - \theta$ for all $n \geq 1$. Thus by the second Borel Cantelli lemma (see [26], p. 240) $\hat{\tau}_n \rightarrow \infty$ a.s. as $n \rightarrow \infty$. Therefore, for $x \in B_2$,

$$\mathbb{E}_x[e^{\beta_1 \tau_{B_2}(\delta)}] = e^{\beta_1 \delta} \mathbb{E}_x \sum_{n=1}^{\infty} 1_{\delta \in (\hat{\tau}_{n-1}, \hat{\tau}_n]} e^{\beta_1 (\tau_{B_2}(\delta) - \delta)}. \quad (2.32)$$

Observing that $1_{\delta \in (\hat{\tau}_{n-1}, \hat{\tau}_n]} e^{\beta_1 (\tau_{B_2}(\delta) - \delta)} \leq 1_{\delta \in (\hat{\tau}_{n-1}, \hat{\tau}_n]} e^{\beta_1 (\hat{\tau}_n - \hat{\tau}_{n-1})}$ we have for $n \geq 1$,

$$\begin{aligned} \mathbb{E}_x 1_{\delta \in (\hat{\tau}_{n-1}, \hat{\tau}_n]} e^{\beta_1 (\tau_{B_2}(\delta) - \delta)} &\leq \mathbb{E}_x 1_{m_1 < \delta} \cdots 1_{m_{n-1} < \delta} e^{\beta_1 (\hat{\tau}_n - \hat{\tau}_{n-1})} \\ &= \mathbb{E}_x 1_{m_1 < \delta} \cdots 1_{m_{n-1} < \delta} \mathbb{E}_x [e^{\beta_1 (\hat{\tau}_n - \hat{\tau}_{n-1})} | \mathcal{F}_{\hat{\tau}_{n-1}}] \\ &\leq A \mathbb{E}_x 1_{m_1 < \delta} \cdots 1_{m_{n-1} < \delta}, \end{aligned} \quad (2.33)$$

where the last inequality is a consequence of the strong Markov property of SRBM and Lemma 2.2.26. Next note that from Lemma 2.2.25, for $n \geq 2$, $\mathbb{E}_x [1_{m_{n-1} < \delta} | \mathcal{F}_{\hat{\tau}_{n-2}}] < \theta$. By a successive conditioning argument we now have that the right side of (2.33) is bounded by $A\theta^{n-1}$. The result now follows on substituting this bound in (2.32):

$$\sup_{x \in B_2} \mathbb{E}_x [e^{\beta_1 \tau_{B_2}(\delta)}] \leq A e^{\beta_1 \delta} \sum_{n=1}^{\infty} \theta^{n-1} = \frac{A e^{\beta_1 \delta}}{1 - \theta}. \quad \blacksquare$$

Lemma 2.2.28. *Let $(\{Z_t\}_{t \geq 0}, \{P_x\}_{x \in S})$ be an SRBM. Under Condition 2.2.8 there exist $a_1, a_2, A_1, A_2 \in (0, \infty)$, such that*

$$a_1 e^{a_2 |x|} \leq V_0(x) \leq A_1 e^{A_2 |x|} \quad \text{for each } x \in S. \quad (2.34)$$

Furthermore, there exist $\beta \in (0, \infty)$ and $\tilde{a}_1, \tilde{a}_2, \tilde{A}_1, \tilde{A}_2 \in (0, \infty)$, such that

$$\tilde{a}_1 e^{\tilde{a}_2 |x|} \leq V_\beta(x) \leq \tilde{A}_1 e^{\tilde{A}_2 |x|} \quad \text{for each } x \in S. \quad (2.35)$$

Proof. We begin by showing the first inequality of (2.34). Note that $V_0(x) \geq 1$, so in order to prove the inequality it suffices to show that there exist $M, a_1, a_2 \in (0, \infty)$ such that for all $|x| \geq M$, $V_0(x) \geq a_1 e^{a_2 |x|}$. By Jensen's inequality $\mathbb{E}_x[e^{\beta_1 \tau_{B_2}(\delta)}] \geq e^{\beta_1 \mathbb{E}_x \tau_{B_2}(\delta)}$. For the lower bound on $\mathbb{E}_x \tau_{B_2}(\delta)$, let M be large enough so that for all $|x| \geq M$, $d(x, B_2) \geq \frac{1}{2}|x|$. Define for $\vartheta \in (0, 1)$, $E_\vartheta \doteq \{\sup_{0 \leq s \leq \vartheta |x|} |Z_s - x| \leq \frac{1}{2}|x|\}$. Then we have

$$\begin{aligned} \mathbb{E}_x \tau_{B_2}(\delta) &\geq \mathbb{E}_x \tau_{B_2}(\delta) 1_{E_\vartheta} \\ &\geq \vartheta |x| P_x \left[\sup_{0 \leq s \leq \vartheta |x|} |Z_s - x| \leq \frac{1}{2}|x| \right]. \end{aligned} \quad (2.36)$$

By Markov inequality and Theorem 2.2.10, we have that

$$\begin{aligned} P_x \left[\sup_{0 \leq s \leq \vartheta |x|} |Z_s - x| \geq \frac{1}{2}|x| \right] &\leq \frac{2 \mathbb{E}_x \sup_{0 \leq s \leq \vartheta |x|} |Z_s - x|}{|x|} \\ &\leq \frac{4C \mathbb{E}_x \sup_{0 \leq s \leq \vartheta |x|} |B_s + r^0 s| - x}{|x|} \\ &\leq \frac{C^* \vartheta |x|}{|x|} = C^* \vartheta, \end{aligned} \quad (2.37)$$

where C as in Theorem 2.2.10 and C^* is a global constant. Choosing $\vartheta < \frac{1}{2C^*}$, we now have from (2.36) that for all $|x| \geq M$, $\mathbb{E}_x \tau_{B_2}(\delta) \geq \frac{\vartheta}{2}|x|$. The desired lower bound in (2.34) now follows. Next recalling that $V_\beta(x) \doteq \mathfrak{R}_\beta V_0 = \int_0^\infty \mathbb{E}_x V_0(Z_t) \beta e^{-\beta t} dt$, we have

$$\begin{aligned} V_\beta(x) &\geq \mathbb{E}_x \int_0^\infty a_1 e^{a_2 |Z_t|} \beta e^{-\beta t} dt \geq a_1 e^{a_2 |x|} \mathbb{E}_x \int_0^\infty e^{-a_2 |Z_t - x|} \beta e^{-\beta t} dt \\ &\geq a_1 e^{a_2 |x|} \mathbb{E}_x \int_0^\infty e^{-a_2 \sup_{0 \leq s \leq t} |Z_s - x|} \beta e^{-\beta t} dt. \end{aligned}$$

Again by Theorem 2.2.10 and a similar argument as used in (2.37),

$$V_\beta(x) \geq a_1 e^{a_2|x|} \int_1^\infty \beta e^{-bt} e^{-\beta t} dt,$$

for some global constant $b \in (0, \infty)$. This proves the lower bound in (2.35).

For the second inequality of (2.34) recall the stopping time σ_1 introduced above Theorem 2.2.23. By a conditioning argument and the strong Markov property,

$$\begin{aligned} \mathbb{E}_x e^{\beta_1 \tau_{B_2}(\delta)} &\leq \mathbb{E}_x e^{\beta_1 \sigma_1} + \mathbb{E}_x [e^{\beta_1 \tau_{B_2}(\delta)} 1_{\sigma_1 \leq \delta}] \\ &= \mathbb{E}_x e^{\beta_1 \sigma_1} + \mathbb{E}_x [1_{\sigma_1 \leq \delta} \mathbb{E}_x (e^{\beta_1 \tau_{B_2}(\delta)} | \mathcal{F}_{\sigma_1})] \\ &\leq \mathbb{E}_x e^{\beta_1 \sigma_1} + \sup_{x \in B_2} \mathbb{E}_x [e^{\beta_1 \tau_{B_2}(\delta)}]. \end{aligned}$$

The desired upper bound in (2.34) now follows on using Theorem 2.2.23 and 2.2.27 in the above display.

Finally,

$$\begin{aligned} V_\beta(x) &\leq \mathbb{E}_x \int_0^\infty A_1 e^{A_2|Z_t|} \beta e^{-\beta t} dt \leq A_1 e^{A_2|x|} \mathbb{E}_x \int_0^\infty \beta e^{A_2|Z_t-x|} e^{-\beta t} dt \\ &\leq A_1 e^{A_2|x|} \mathbb{E}_x \int_0^\infty \beta e^{A_2 \sup_{0 \leq s \leq t} |Z_s-x|} e^{-\beta t} dt. \end{aligned}$$

Once more from Theorem 2.2.10, we have

$$V_\beta(x) \leq A_1 e^{A_2|x|} \int_0^\infty \beta e^{A_3 t} e^{-\beta t} dt,$$

for some global constant A_3 . The upper bound in (2.35) now follows on choosing $\beta \in (A_3, \infty)$. ■

Henceforth we will fix a $\beta > A_3$ and denote the corresponding V_β by V . (2.38)

Corollary 2.2.29. *Under Condition 2.2.8, an SRBM $(\{Z_t\}_{t \geq 0}, \{P_x\}_{x \in S})$ satisfies the following drift criteria: There exist $b, c \in (0, \infty)$, some compact set $C \subseteq S$ such that*

$$\tilde{A}V(x) \leq -cV(x) + b1_C(x), \quad \text{for all } x \in S. \quad (2.39)$$

Proof. From Lemma 2.2.28,

$$\sup_{x \in B_2} V_0(x) \doteq \frac{1}{\beta_1} [\mathbb{E}_x e^{\beta_1 \tau_{B_2}(\delta)} - 1] + 1 < \infty, \quad \mathbb{E}_x e^{\beta_1 \tau_{B_2}(\delta)} < \infty, \quad \forall x \in S.$$

Result follows on combining these two facts and applying Theorem 2.2.24. ■

Corollary 2.2.30. *Suppose that Condition 2.2.8 holds. Let π be the unique invariant probability distribution of the SRBM $(\{Z_t\}_{t \geq 0}, \{P_x\}_{x \in S})$. Then $\pi(V) < \infty$.*

Proof. Theorem 5.1 (c) of [22] along with equation (2.39) implies the existence of $\lambda \in (0, 1)$ and $\tilde{b} \in (0, \infty)$ such that $\mathfrak{R}V(x) \leq V(x) - (1 - \lambda)V(x) + \tilde{b}1_C(x)$ for all $x \in S$. Combining the above drift condition with Theorem 2.2.3 and Theorem 14.3.7 of [45], we get $\pi(V) \leq b\pi(C)/(1 - \lambda) < \infty$. ■

Theorem 2.2.31. *Suppose that Condition 2.2.8 holds for the SRBM*

$(\{Z_t\}_{t \geq 0}, \{P_x\}_{x \in S})$ and let π be the unique invariant distribution for $\{Z_t\}_{t \geq 0}$. Let

$\tilde{a}_2 \in (0, \infty)$ be as in Lemma 2.2.28. Then for all $c \in \mathbb{R}^d$ with $|c| \leq \tilde{a}_2$ we have

$$\int_S e^{c \cdot x} \pi(dx) < \infty.$$

Proof. Let $\tilde{a}_1 \in (0, \infty)$ be as in Lemma 2.2.28. Then

$$\tilde{a}_1 \int_S e^{c \cdot x} \pi(dx) \leq \tilde{a}_1 \int_S e^{\tilde{a}_2 |x|} \pi(dx) \leq \int_S V(x) \pi(dx) < \infty,$$

where the last inequality follows from Corollary 2.2.30. In particular, from the proof of Corollary 2.2.30 we see that $\int_S e^{c \cdot x} \pi(dx) < \frac{b\pi(C)}{\tilde{a}_1(1-\lambda)}$. ■

Now we present the main result of this section.

Theorem 2.2.32. *Under Condition 2.2.8, an SRBM $(\{Z_t\}_{t \geq 0}, \{P_x\}_{x \in S})$ is V -uniformly ergodic; i.e., there exist constants $D \in (0, \infty)$, $\rho \in (0, 1)$ such that for all $t \in \mathbb{R}_+$ and $x \in S$,*

$$\|P^t(x, \cdot) - \pi\|_V \leq V(x)D\rho^t.$$

Proof. Result is immediate consequence of Corollary 2.2.29 along with Theorem 5.2 (c) of [22]. ■

2.2.4 Long time stability

From V -uniform ergodicity established in Theorem 2.2.32, we can now establish several results on moment stability of an SRBM.

Theorem 2.2.33. *Suppose that Condition 2.2.8 holds. Then there exists $\tilde{D} \in (0, \infty)$ such that for all $g \in L_\infty^V$, $x \in S$, and $t \geq 0$,*

$$\mathbb{E}_x g(Z_t) \leq \tilde{D}[1 + V(x)\rho^t],$$

where $\rho \in (0, \infty)$ is as in Theorem 2.2.32. In particular

$$\mathbb{E}_x e^{\tilde{a}_2 |Z_t|} \leq \tilde{D}[1 + V(x)\rho^t],$$

where $\tilde{a}_2 \in (0, \infty)$ is as in Lemma 2.2.28, and for every compact set $K \subseteq S$ we have $\sup_{t \geq 0} \sup_{x \in K} \mathbb{E}_x e^{\tilde{a}_2 |Z_t|} < \infty$.

Proof. For $g \in L_\infty^V$ let $\tilde{g} \doteq \frac{g}{\|g\|_V}$. Then $\tilde{g} \leq V$ and by Theorem 2.2.32, we have that for all $t \in \mathbb{R}_+$ and $x \in S$,

$$\sup_{\tilde{g}: \|\tilde{g}\| \leq V} \left| \mathbb{E}_x \left\{ \tilde{g}(Z_t) - \int_S \tilde{g}(y)\pi(dy) \right\} \right| \leq DV(x)\rho^t,$$

where $D \in (0, \infty)$ is as in Theorem 2.2.32. So

$$\mathbb{E}_x \tilde{g}(Z_t) \leq \int_S \tilde{g}(y) \pi(dy) + DV(x) \rho^t.$$

Since $\int_S \tilde{g}(y) \pi(dy) < \infty$ and $\|g\|_V < \infty$ there is a $\tilde{D} \in (0, \infty)$ such that $\mathbb{E}_x g(Z_t) \leq \tilde{D}[1 + V(x) \rho^t]$. In particular choosing $g(x)$ to be $\tilde{a}_1 e^{\tilde{a}_2 |x|}$ as in Lemma 2.2.28 so that $|g| \leq V$ yields that $\mathbb{E}_x e^{\tilde{a}_2 |Z_t|} \leq \tilde{D}[1 + V(x) \rho^t]$, and for every compact set $K \subseteq S$ we have $\sup_{t \geq 0} \sup_{x \in K} \mathbb{E}_x e^{\tilde{a}_2 |Z_t|} < \infty$ from Lemma 2.2.28 and Corollary 2.2.30. ■

Corollary 2.2.34. *Suppose that Condition 2.2.8 holds for the SRBM $(\{Z_t\}_{t \geq 0}, \{P_x\}_{x \in S})$.*

Then there exists a $t_0 > 0$ such that for all $p > 0$,

$$\lim_{|x| \rightarrow \infty} \sup_{t \geq t_0} \frac{1}{|x|^{p+1}} \mathbb{E}_x (|Z(t|x)|^{p+1}) = 0.$$

Proof. Fix $p > 0$ and choose $\theta_{p+1} \in (0, \infty)$ small enough so that $\theta_{p+1} x^{p+1} \leq e^{\tilde{a}_2 x}$ for $x \in \mathbb{R}_+$, where \tilde{a}_2 is as in Theorem 2.2.33. Then from Theorem 2.2.33

$$\begin{aligned} \frac{1}{|x|^{p+1}} \mathbb{E}_x (|Z(t_0|x)|^{p+1}) &\leq \frac{1}{|x|^{p+1} \theta_{p+1}} \mathbb{E}_x e^{\tilde{a}_2 |Z(t_0|x)|} \\ &\leq \frac{\tilde{D}}{|x|^{p+1} \theta_{p+1}} [1 + V(x) \rho^{t_0|x|}] \\ &\leq \frac{\tilde{D}}{|x|^{p+1} \theta_{p+1}} [1 + A_1 e^{\tilde{A}_2 |x|} \rho^{t_0|x|}], \end{aligned}$$

where the last inequality follows from Lemma 2.2.28. The result now follows on taking t_0 large enough so that $\rho^{t_0} \leq e^{-\tilde{A}_2}$. ■

Theorem 2.2.35. *Suppose that Condition 2.2.8 holds for the SRBM $(\{Z_t\}_{t \geq 0}, \{P_x\}_{x \in S})$.*

Then for each $p > 0$ there exists a constant $\kappa_p \in (0, \infty)$ such that

$$\frac{1}{t} \int_0^t \mathbb{E}_x [|Z(s)|^p] ds \leq \kappa_p \left\{ \frac{1}{t} |x|^{p+1} + 1 \right\}, \quad t > 0, \quad x \in S.$$

Proof. By Corollary 2.2.34, there exists an $L \in (0, \infty)$ such that with $D \doteq \{x \in S : |x| < L\}$

$$\mathbb{E}_x[Z(t_0|x|)^{p+1}] \leq \frac{1}{2}|x|^{p+1}, \quad \forall x \in D^c, \quad (2.40)$$

where t_0 is as in Corollary 2.2.34. Let $\delta \doteq t_0L$ and set $\tau(\delta) = \inf\{t \geq \delta : |Z(t)| \leq L\}$. Define $\hat{V}(x) \doteq \mathbb{E}_x \left[\int_0^{\tau(\delta)} (|Z(t)|^p + 1) dt \right]$, $x \in S$. We will next show that there exists a $d \in (0, \infty)$ such that

$$\hat{V}(x) \leq d(|x|^{p+1} + 1), \quad \forall x \in S. \quad (2.41)$$

The result will then follow as an immediate consequence of Proposition 5.4 of [18]. Define a sequence of stopping times σ_n as $\sigma_0 \doteq 0$, $\sigma_n = \sigma_{n-1} + t_0[|Z(\sigma_{n-1})| \vee L]$, $n \in \mathbb{N}$. Also, let $n_0 \doteq \min\{n \geq 1 : |Z(\sigma_n)| \leq L\}$. Then

$$\begin{aligned} \hat{V}(x) &\leq \mathbb{E}_x \left[\int_0^{\sigma_{n_0}} (|Z(t)|^p + 1) dt \right] \\ &= \sum_{k=0}^{\infty} \mathbb{E}_x \left[\int_{\sigma_k}^{\sigma_{k+1}} (|Z(t)|^p + 1) dt \mathbf{1}_{k < n_0} \right]. \end{aligned} \quad (2.42)$$

An application of strong Markov property and Theorem 2.2.10 shows that there exists a $d_1 \in (0, \infty)$ such that

$$\mathbb{E}_x \left[\int_{\sigma_k}^{\sigma_{k+1}} (|Z(t)|^p + 1) dt \middle| \mathcal{F}_{\sigma_k} \right] \mathbf{1}_{k < n_0} \leq d_1 (|Z(\sigma_k)|^{p+1} + 1) \mathbf{1}_{k < n_0}. \quad (2.43)$$

Using this estimate in (2.42) we get by suitable conditioning

$$\hat{V}(x) \leq d_1 \mathbb{E}_x \left[\sum_{k=0}^{n_0-1} (|Z(\sigma_k)|^{p+1} + 1) \right]. \quad (2.44)$$

Next note that $\{Z(\sigma_k)\}_{k \geq 1}$ is a Markov chain with transition kernel

$$\check{P}(x, A) \doteq \int_A p(t_0(|x| \vee L), x, y) dy, \quad x \in S, A \in \mathcal{B}(S),$$

where $p(t, x, y)$ is the density of SRBM introduced in (2.21). Using Theorem 2.2.10 once more and (2.40), one sees that there exists a $\tilde{b} \in (0, \infty)$ such that

$$\int_S \check{P}(x, dy) |y|^{p+1} \leq |x|^{p+1} - \frac{1}{2} |x|^{p+1} + \tilde{b} 1_{[0, L]}(|x|). \quad (2.45)$$

Using Theorem 14.2.2 of [45] we have now that

$$\mathbb{E}_x \sum_{k=0}^{n_0-1} [|Z(\sigma_k)|^{p+1} + 1] \leq 2 \left\{ |x|^{p+1} + \tilde{b} 1_{[0, L]}(|x|) \right\}.$$

The inequality (2.41) now follows on using the above estimate in (2.44). ■

Finally, in this section we use the results of [30] and geometric ergodicity results obtained above to derive central limit theorems (CLT) for

$$S_t \doteq \int_{[0, t)} F(Z(s)) ds, \quad \text{as } t \rightarrow \infty,$$

for a broad family of measurable functions $F : S \rightarrow \mathbb{R}$. We begin by considering the Poisson equation, the solution of which characterizes the asymptotic variance in the CLT for S_t . Recall that for a function $V \geq 1$, L_∞^V is the vector space of functions $h : S \rightarrow \mathbb{R}$ such that $\|h\|_V \doteq \sup_{x \in S} \frac{|h(x)|}{V(x)} < \infty$. Also we recall from Section 2.1.1 that the V -norm for a kernel $P(\doteq P(x, dy))$ is defined as $\|P\|_V \doteq \sup_{h \in L_\infty^V, \|h\|_V \neq 0} \frac{\|Ph\|_V}{\|h\|_V}$.

Theorem 2.2.36. *Suppose that Condition 2.2.8 holds for the SRBM*

($\{Z_t\}_{t \geq 0}, \{P_x\}_{x \in S}$). Let V be as in (2.38). Then the following hold.

(i) *For all $F \in L_\infty^V$ and $x \in S$, the limit, as $t \rightarrow \infty$, of $\mathbb{E}_x[S_t - t\pi(F)]$ exists.*

Denoting the limit by $\hat{F}(x)$, we have that $\hat{F}(x) \in L_\infty^V$.

(ii) \hat{F} solves the Poisson equation for F , i.e., $\hat{F}(x) \in \mathcal{D}(\tilde{\mathcal{A}})$, where $\tilde{\mathcal{A}}$ is the extended generator of Z (see Definition 2.1.20) and

$$\tilde{\mathcal{A}}\hat{F}(x) = \pi(F) - F(x), \quad x \in S. \quad (2.46)$$

(iii) The convergence in (i) is exponentially fast, i.e., denoting $\mathbb{E}_x[S_t - t\pi(F)]$ by $F_t^c(x)$, we have that for some $b_0, B_0 \in (0, \infty)$, $\|F_t^c - \hat{F}\|_V \leq B_0 e^{-b_0 t}$ for all $t \in (0, \infty)$.

Remark 2.2.37. Note that from Corollary 2.2.30 $\pi(|F|) < \infty$ for all $F \in L_\infty^V$, thus the statements in the above theorem are meaningful. Also for any $F \in L_\infty^V$, Poisson equation $\tilde{\mathcal{A}}g = \pi(F) - F$ admits at most one solution g , up to a constant factor, with the property $\pi(|g|) < \infty$. I.e. if g, \check{g} are two solutions and $\pi(|g| + |\check{g}|) < \infty$, then $g - \check{g} = c$ a.s. $[\pi]$ for some $c \in \mathbb{R}$. Proof of this statement follows along the lines of Proposition 17.4.1 of [45].

Proof. (i) Fix $F \in L_\infty^V$. From Theorem 2.2.32,

$$\int_0^\infty |P^t(x, F) - \pi(F)| dt \leq V(x) \int_0^\infty D\rho^t dt \leq D_0 V(x),$$

where $\rho \in (0, 1)$ is as in Theorem 2.2.32 and $D_0 \in (0, \infty)$. Thus

$$\lim_{t \rightarrow \infty} [\mathbb{E}_x S_t - t\pi(F)] = \lim_{t \rightarrow \infty} \int_0^t [P^s(x, F) - \pi(F)] ds$$

exists and denoting the limit by $\hat{F}(x)$, we have $\hat{F}(x) \leq D_0 V(x)$ for all $x \in S$. Thus $\hat{F} \in L_\infty^V$.

(ii) Using the exponential bounds on V obtained in Lemma 2.2.28 one can check that $\mathbb{E} \int_0^t |\hat{F}(Z_s)| ds < \infty$. Next for $t > 0$,

$$\begin{aligned}
\mathbb{E}_x \hat{F}(X_t) &= \int_0^\infty \mathbb{E}_x [P^s(X_t, F) - \pi(F)] ds = \int_0^\infty [P_x(X_{s+t}, F) - \pi(F)] ds \\
&= \int_t^\infty [P^s(x, F) - \pi(F)] ds = \hat{F}(x) - \int_0^t [P^s(x, F) - \pi(F)] ds \\
&= \hat{F}(x) + \int_0^t \mathbb{E}_x [\pi(F) - F(X_s)] ds.
\end{aligned}$$

This establishes that $\hat{F} \in \mathcal{D}(\tilde{\mathcal{A}})$ and $\tilde{\mathcal{A}}\hat{F} = \pi(F) - F$.

(iii) For $0 \leq t < T < \infty$,

$$\begin{aligned}
|\mathbb{E}_x[S_t - t\pi(F)] - \mathbb{E}_x[S_T - T\pi(F)]| &\leq \int_t^T |P^s(x, F) - \pi(F)| ds \\
&\leq V(x) \int_t^T D\rho^s ds \leq V(x)D_1\rho^t,
\end{aligned}$$

where the next to last inequality is a consequence of Theorem 2.2.32 and $D_1 \in (0, \infty)$.

The result now follows on sending $T \rightarrow \infty$. ■

We now present the main central limit result of this section, which is an immediate consequence of Theorem 4.4 of [30] and Corollary 2.2.29. Define for a $F \in L_\infty^V$

$$\xi_n(t) \doteq \frac{1}{\sqrt{n}} \left(\int_0^{nt} \bar{F}(Z_s) ds \right), \quad t \in [0, 1],$$

where $\bar{F} \doteq F - \pi(F)$. Let $C[0, 1]$ denote the set of continuous functions defined from $[0, 1]$ to \mathbb{R} .

Theorem 2.2.38. *Let the SRBM $(\{Z_t\}_{t \geq 0}, \{P_x\}_{x \in S})$ satisfy Condition 2.2.8. Let V be as in (2.38). Let $F : S \rightarrow \mathbb{R}$ be a measurable function such that $F^2(x) \leq V(x)$ for all $x \in S$. Define $\gamma_F^2 \doteq 2 \int \hat{F}(x) \bar{F}(x) \pi(dx)$, where \hat{F} is the solution of Poisson equation (2.46). Then, as $n \rightarrow \infty$, ξ_n converges in distribution to $\gamma_F B$ in $C[0, 1]$, where B is a one dimensional standard Brownian motion.*

Remark 2.2.39. Note that since any two solutions of the Poisson equation (2.46) differ by a constant and $\int \bar{F}(x)\pi(dx) = 0$, the choice of the solution in the definition of $\gamma_{\bar{F}}^2$ is immaterial. The nonnegativity and finiteness of the expression defining $\gamma_{\bar{F}}^2$ under the drift condition (2.39) is established in [30].

2.3 Constrained diffusion processes in polyhedral domains

In Sections 2.2.3 and 2.2.4 we studied geometric ergodicity properties for a constrained diffusion in \mathbb{R}_+^d with constant drift and diffusion coefficients. In the current section we will address stability properties for a class of diffusion processes, with general state dependent coefficients, constrained to take values in a convex polyhedral cone S in \mathbb{R}^d with the vertex at the origin. Our assumption on the reflection vector field $r(x)$ here will be somewhat more restrictive than the completely- \mathcal{S} assumption made in Section 2.2.1. In particular we assume that the Skorohod map associated with the reflection data is well defined for all RCLL trajectories and it satisfies a Lipschitz property (details are given below). Study of such diffusions is motivated by queueing networks with state dependent arrival and service rates. It is well known (see eg. [59]) that under suitable heavy traffic conditions, appropriately scaled state descriptors of such networks converge weakly to reflected diffusions of the form considered in this section. Under natural stability conditions on the drift vector field, existence of a unique invariant probability distribution for this class of diffusions was established in [2]. In this section, we investigate the rate of convergence to steady state; in particular we establish geometric ergodicity. Since many arguments are quite similar to the constant coefficients case studied in Sections 2.2.3 and 2.2.4, only sketches of proofs will be provided.

We now describe the precise model that will be studied in this section. We assume that S is given as the intersection of half spaces S_i , $i = 1, \dots, N$, $N \geq d$. Let n_i be the

unit vector associated with S_i via the relation $S_i \doteq \{x \in \mathbb{R}^d : \langle x, n_i \rangle \geq 0\}$. Define F^i to be the face of S corresponding to n_i , i.e., $F^i \doteq \{x \in \partial S : \langle x, n_i \rangle = 0\}$. With each face F^i we associate the *direction of constraint* unit vector r_i satisfying $\langle r_i, n_i \rangle > 0$. Denote the $d \times N$ matrix $[r_1, \dots, r_N]$ by R . At points on the boundary ∂S where more than one faces meet, there are more than one allowed directions of constraint. In general, for $x \in \partial S$ define

$$r(x) \doteq \left\{ r \in \mathbb{R}^d : r = \sum_{i \in In(x)} \alpha_i r_i; \alpha_i \geq 0; |r| = 1 \right\},$$

where $In(x) \doteq \{i \in \{1, 2, \dots, N\} : \langle x, n_i \rangle = 0\}$. The set $r(x)$ represents the directions of constraint allowed at the point x . Let $D_S([0, \infty), \mathbb{R}^d) \doteq \{\psi \in D([0, \infty), \mathbb{R}^d) : \psi(0) \in S\}$. For $\eta \in D_S([0, \infty), \mathbb{R}^d)$, $T \in (0, \infty)$ let $|\eta|(T)$ denote the total variation of η on $[0, T]$ with respect to the Euclidean norm on \mathbb{R}^d .

Definition 2.3.1. (Skorohod Map) *Let $\psi \in D_S([0, \infty), \mathbb{R}^d)$ be given. Then $(\phi, \eta) \in D([0, \infty), \mathbb{R}^d) \times D([0, \infty), \mathbb{R}^d)$ solves the Skorohod problem (SP) for ψ with data (S, R) , if and only if $\phi(0) = \psi(0)$ and for all $t \in [0, \infty)$: (i) $\phi(t) = \psi(t) + \eta(t)$; (ii) $\phi(t) \in S$; (iii) $|\eta|(t) < \infty$; (iv) $|\eta|(t) = \int_{[0,t]} I_{\{\phi(s) \in \partial S\}} d|\eta|(s)$; (v) There exists a Borel measurable function $\gamma : [0, \infty) \rightarrow \mathbb{R}^d$ such that $\gamma(t) \in r(\phi(t))$, $d|\eta|$ - almost everywhere and $\eta(t) = \int_{[0,t]} \gamma(s) d|\eta|(s)$.*

On the domain $D \subseteq D_S([0, \infty), \mathbb{R}^d)$ on which there is a unique solution to the Skorohod problem we define the Skorohod map (SM) Γ as $\Gamma(\psi) \doteq \phi$ if $(\phi, \psi - \phi)$ is the unique solution of the Skorohod problem posed by ψ . We will make the following assumption on the regularity of the Skorohod map defined by the data $\{(r_i, n_i) : i = 1, 2, \dots, N\}$.

Condition 2.3.2. *The Skorohod map is well defined on all of $D_S([0, \infty), \mathbb{R}^d)$, that is, $D = D_S([0, \infty), \mathbb{R}^d)$, and the SM is Lipschitz continuous in the following sense.*

There exists a constant $K \in (0, \infty)$ such that for all $\psi_1, \psi_2 \in D_S([0, \infty), \mathbb{R}^d)$:

$$\sup_{0 \leq t < \infty} |\Gamma(\psi_1)(t) - \Gamma(\psi_2)(t)| < K \sup_{0 \leq t < \infty} |\psi_1(t) - \psi_2(t)|. \quad (2.47)$$

We will assume without loss of generality that $K \geq 1$. We refer the reader to [23, 24] for sufficient conditions under which Condition 2.3.2 holds.

We now introduce the constrained diffusion process that will be studied in this section. Let (Ω, \mathcal{F}, P) be a complete probability space on which is given a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual hypotheses. Let $(B(t), \{\mathcal{F}_t\})$ be a d -dimensional standard Wiener process on the above probability space. We will study the constrained diffusion process given as a solution to equation

$$Z(t) = \Gamma \left(x + \int_0^t \sigma(Z(s)) dB(s) + \int_0^t b(Z(s)) ds \right) (t), \quad (2.48)$$

where $\sigma : S \rightarrow \mathbb{R}^{d \times d}$ and $b : S \rightarrow \mathbb{R}^d$ are maps satisfying the following condition:

Condition 2.3.3. *There exists a $\gamma \in (0, \infty)$ such that*

$$|\sigma(x) - \sigma(y)| + |b(x) - b(y)| \leq \gamma |x - y|, \quad \forall x, y \in S, \quad (2.49)$$

$$|\sigma(x)| \leq \gamma, \quad \forall x \in S. \quad (2.50)$$

Under Condition 2.3.3 equation (2.48) admits a unique strong solution and as a consequence there exists a filtered measurable space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0})$ on which are given a family of probability measures $\{P_x\}_{x \in S}$ and continuous stochastic processes Z and B such that for all $x \in S$, under P_x , $\{B(t), \{\mathcal{F}_t\}_{t \geq 0}\}$ is a d -dimensional standard Wiener process and (Z, B) satisfy (2.48) P_x -a.s. Furthermore, $(Z, \{P_x\}_{x \in S})$ is a strong Markov family. Henceforth, we will refer to this family merely as Z .

We will make the following uniform nondegeneracy assumption on the diffusion coefficient.

Condition 2.3.4. *There exists a $c \in (0, \infty)$ such that for all $x \in S$ and $\alpha \in \mathbb{R}^d$,*

$$\alpha'(\sigma(x)\sigma'(x))\alpha \geq c\alpha'\alpha.$$

We now introduce the main condition on the drift field b for the process Z to be positive recurrent. Define

$$\mathcal{C} \doteq \left\{ -\sum_{i=1}^N \alpha_i r_i : \alpha_i \geq 0, i \in \{1, \dots, N\} \right\}. \quad (2.51)$$

The cone \mathcal{C} was used to characterize stability of certain semimartingale reflecting Brownian motions in [8]. For $\delta \in (0, \infty)$, define

$$\mathcal{C}_\delta \doteq \{v \in \mathcal{C} : \text{dist}(v, \partial\mathcal{C}) \geq \delta\}. \quad (2.52)$$

Our key stability assumption on the diffusion model is the following.

Condition 2.3.5. *There exists a $\delta \in (0, \infty)$ and a bounded set $A \subseteq S$ such that for all $x \in S \setminus A$, $b(x) \in \mathcal{C}_\delta$.*

The following is the main result of [2] (Theorem 2.2 therein).

Theorem 2.3.6. *Assume that Conditions 2.3.2-2.3.5 hold. Then Z has a unique invariant probability measure π .*

For the rest of this section Conditions 2.3.2-2.3.5 will be assumed to hold. We will now study geometric ergodicity of Z . We begin with the following result on φ -irreducibility of Z . Let λ denote the Lebesgue measure on $(S, \mathcal{B}(S))$.

Lemma 2.3.7. *For every $A \in \mathcal{B}(S)$ with $\lambda(A) > 0$, $P^t(x, A) > 0$ for all $t > 0, x \in S$. In particular, Z is λ -irreducible.*

The proof is provided in the appendix. The following result from [2] (cf. Lemmas 3.1 and 4.1 therein) will be key in our analysis.

Lemma 2.3.8. *There is a function $T : S \rightarrow [0, \infty)$ such that the following properties hold.*

(1) *For some $c_1 \in (0, \infty)$, $|T(x) - T(y)| \leq c_1|x - y|$ for all $x, y \in S$.*

(2) *For some $c_2, c_3 \in (0, \infty)$, $c_2|x| \leq T(x) \leq c_3|x|$ for all $x \in S$.*

(3) *For some $c_4 \in (0, \infty)$,*

$$T(Z(t \wedge \sigma_A)) \leq [T(z) - (t \wedge \sigma_A)]^+ + c_4 \eta_t^*, \quad (2.53)$$

for all $t \geq 0$, P_z -a.s., for all $z \in S$, where A is as in Condition 2.3.5,

$$\sigma_A \doteq \inf\{t \geq 0 : Z_t \in A\},$$

$$\text{and } \eta_t^* \doteq \sup_{0 \leq s \leq t} \left| \int_0^s \sigma(Z(r)) dB_r \right|.$$

We now have the following result.

Lemma 2.3.9. *The 1-skeleton chain $\{\check{Z}_n \doteq Z(n)\}_{n \in \mathbb{N}_0}$ satisfies the following drift inequality: There are $\delta, \beta, b_2 \in (0, \infty)$ and a compact set $C_2 \subseteq S$ such that*

$$\mathbb{E}_x V(\check{Z}_1) \leq (1 - \beta)V(x) + b_2 1_{C_2}(x), \quad x \in S, \quad (2.54)$$

with $V(x) \doteq e^{\delta T(x)}$.

Proof. From Lemma 2.3.8 (3), for $\delta \in (0, \infty)$

$$V(x)^{-1} \mathbb{E}_x V(\check{Z}_1) 1_{\sigma_A > 1} \leq \mathbb{E}_x e^{\delta [T(x) - 1]^+ + c_4 \eta_1^* - T(x)} 1_{\sigma_A > 1}. \quad (2.55)$$

Thus for $x \in S_1 \doteq \{x : T(x) > 1\}$,

$$V(x)^{-1} [\mathbb{E}_x e^{\delta T(\check{Z}_1)} 1_{\sigma_A > 1}] \leq \mathbb{E}_x e^{\delta c_4 \eta_1^* - \delta} 1_{\sigma_A > 1} \leq e^{-\delta} e^{\delta^2 c_5}, \quad (2.56)$$

where $c_5 \in (0, \infty)$ is an appropriate constant independent of δ and x . Now fix δ small enough so that $e^{-\delta}e^{\delta^2 c_5} \doteq (1 - 2\beta) < 1$. Then for $x \in S_1$,

$$\mathbb{E}_x V(\check{Z}_1)1_{\sigma_A > 1} \leq (1 - 2\beta)V(x). \quad (2.57)$$

From the strong Markov property of Z we see that for all $x \in S$,

$$\mathbb{E}_x V(\check{Z}_1)1_{\sigma_A \leq 1} = \mathbb{E}_x [\mathbb{E}_{Z(\sigma_A)} V(Z_{1-\sigma_A})1_{\sigma_A \leq 1}].$$

Thus

$$\mathbb{E}_x V(\check{Z}_1)1_{\sigma_A \leq 1} \leq \sup_{y: y \in A} \mathbb{E}_y \sup_{0 \leq t \leq 1} V(Z_t) \leq \sup_{y: y \in A} \mathbb{E}_y \sup_{0 \leq t \leq 1} e^{\delta c_3 |Z_t|},$$

where the last inequality follows from Lemma 2.3.8 (2). Using the above inequality; the Lipschitz property (2.47); and Condition 2.3.3 we now see that, for some $\tilde{K} \in (0, \infty)$

$$\mathbb{E}_x V(\check{Z}_1)1_{\sigma_A \leq 1} \leq \tilde{K}, \quad \forall x \in S. \quad (2.58)$$

Choose $M \in (1, \infty)$ such that $\beta V(x) \geq \tilde{K}$ for all $x \in S_M \doteq \{x : T(x) \geq M\}$. Then

$$\mathbb{E}_x V(\check{Z}_1)1_{\sigma_A \leq 1} \leq \beta V(x), \quad \forall x \in S_M. \quad (2.59)$$

Combining (2.57) and (2.59) we have

$$\mathbb{E}_x V(\check{Z}_1) \leq (1 - \beta)V(x), \quad \forall x \in S_M. \quad (2.60)$$

Also for $x \in C_2 \doteq S_M^c$, we have from (2.55) and (2.58) that

$$\begin{aligned} \mathbb{E}_x V(\check{Z}_1) &= \mathbb{E}_x V(\check{Z}_1)1_{\sigma_A \leq 1} + \mathbb{E}_x V(\check{Z}_1)1_{\sigma_A > 1} \\ &\leq \tilde{K} + e^{\delta T(x)} \mathbb{E}_x e^{\delta_4 \eta_1^*} \leq \tilde{K} + e^{\delta M} e^{\delta^2 c_5} \doteq b_2. \end{aligned} \quad (2.61)$$

Combining (2.60) and (2.61) we have the result. ■

Corollary 2.3.10. *The invariant measure π satisfies $\pi(V) < \infty$. Furthermore, the 1-skeleton chain $(\{\check{Z}_n\}, P_x)$ is V -uniformly ergodic, i.e., there exist $\rho \in (0, 1)$ and $D \in (0, \infty)$ such that for all $x \in S$,*

$$\|P^n(x, \cdot) - \pi\|_V \leq DV(x)\rho^n.$$

Proof. The first part of the corollary is an immediate consequence of Theorem 14.0.1 of [45], while the second follows from Theorem 16.0.1 of the same reference. ■

We now summarize the stability results that follow as a corollary to the above result.

Corollary 2.3.11. *Let π be the unique invariant distribution for Z . Then the following hold.*

1. *Let $\delta \in (0, \infty)$ be as in Lemma 2.3.9 and c_2 as in Lemma 2.3.8. Then for all $c \in \mathbb{R}^d$ with $|c| \leq c_2\delta$, $\int_S e^{c \cdot x} \pi(dx) < \infty$.*
2. *Let V be as in Lemma 2.3.9. Then, Z is V -uniformly ergodic; i.e., there exist constants $D \in (0, \infty)$, $\rho \in (0, 1)$ such that for all $t \in \mathbb{R}_+$ and $x \in S$,*

$$\|P^t(x, \cdot) - \pi\|_V \leq V(x)D\rho^t.$$

3. *Let $g \in L_\infty^V$, where L_∞^V is as defined below Theorem 2.2.32. Then there exists a $\tilde{D} \in (0, \infty)$ such that for all $g \in L_\infty^V$, $x \in S$, and $t \geq 0$,*

$$E_x g(Z_t) \leq \tilde{D}[1 + V(x)\rho^t],$$

where $\rho \in (0, \infty)$ is as in Corollary 2.3.10. In particular

$$\mathbb{E}_x e^{c_2|Z_t|} \leq \tilde{D}[1 + V(x)\rho^t],$$

where c_2 is as in Lemma 2.3.8, and for every compact set $K \subseteq S$ we have

$$\sup_{t \geq 0} \sup_{x \in K} \mathbb{E}_x e^{c_2|Z_t|} < \infty.$$

4. There exists a $t_0 > 0$ such that for all $p > 0$,

$$\lim_{|x| \rightarrow \infty} \sup_{t \geq t_0} \frac{1}{|x|^{p+1}} \mathbb{E}_x (|Z(t|x)|^{p+1}) = 0.$$

5. For each $p > 0$ there exists a constant $\kappa_p \in (0, \infty)$ such that

$$\frac{1}{t} \int_0^t \mathbb{E}_x [|Z(s)|^p] ds \leq \kappa_p \left\{ \frac{1}{t} |x|^{p+1} + 1 \right\}, \quad t > 0, \quad x \in S.$$

6. Conclusions (i), (ii), (iii) of Theorem 2.2.36 hold.

7. Let $F : S \rightarrow \mathbb{R}$ be a measurable function such that $F^2(x) \leq V(x)$ for all $x \in S$.

Define $\gamma_F^2 \doteq 2 \int \hat{F}(x) \bar{F}(x) \pi(dx)$, where \hat{F} is the solution of Poisson equation (2.46). Then, as $n \rightarrow \infty$, ξ_n converges weakly to $\gamma_F B$ in $C[0, 1]$, where B is a one dimensional standard Brownian motion.

Proof. 1. This is immediate from Corollary 2.3.10 and Lemma 2.3.8 (2).

2. Let $\|P^t - \pi\|_V \doteq \sup_{x \in S} \frac{\|P^t(x, \cdot) - \pi\|_V}{V(x)}$. From Corollary 2.3.10 we have that

$$\|P^n - \pi\|_V \leq D\rho^n,$$

for all $n \in \mathbb{N}$. It is easy to check (for example, cf. Proposition 16.1.3 of [45])

that for $t \in (0, \infty)$

$$\begin{aligned} \|P^t - \pi\|_V &\leq \|P^{\lfloor t \rfloor} - \pi\|_V \sup_{0 \leq r \leq 1} \|P^r - \pi\|_V \\ &\leq D\rho^{\lfloor t \rfloor} \sup_{0 \leq r \leq 1} \sup_{x \in S} \frac{\mathbf{E}_x[V(Z(r))] + \pi(V)}{V(x)}, \end{aligned} \quad (2.62)$$

where $\lfloor t \rfloor$ denotes the greatest integer less or equal to t . Using arguments similar to those in Lemma 2.3.9 we see (cf. (2.61)) that for $r \in [0, 1]$ and $x \in S$,

$$\mathbf{E}_x[V(Z(r))] \leq \tilde{K} + V(x)e^{\delta^2 c_5},$$

where c_5 is as in (2.56) and \tilde{K} is as in (2.58). Substituting this estimate in (2.62) we now have that

$$\|P^t - \pi\|_V \leq \tilde{D}\rho^t,$$

where $\tilde{D} = \frac{D}{\rho}[\pi(V) + \tilde{K} + e^{\delta^2 c_5}]$. This proves 2.

3 – 7. The proofs of 3 – 7 are now carried out exactly as in the case of deterministic coefficients studied in Section 2.2.4 on noting that V introduced in Corollary 2.3.10 satisfies

$$\tilde{a}_1 e^{\tilde{a}_2 |x|} \leq V(x) \leq \tilde{A}_1 e^{\tilde{A}_2 |x|} \quad \text{for each } x \in S,$$

for suitable $\tilde{a}_1, \tilde{a}_2, \tilde{A}_1, \tilde{A}_2 \in (0, \infty)$. The proof of 5 requires minor modifications to the proof of Theorem 2.2.35. In particular, the analog of (2.43) is obtained from (2.42) by applying (2.53) instead of Theorem 2.2.10) and in obtaining (2.45) one uses (in addition to 4) the Lipschitz property (2.47), linear growth condition (2.49), boundedness of σ (2.50), and Gronwall's lemma. Details are omitted. ■

2.4 Appendix

Proof of Lemma 2.3.7. The proof is adapted from arguments in [3], [33] and [52].

We begin by observing that for all $x \in S$,

$$\mathbb{E}_x \int_0^\infty 1_{\{Z(s) \in \partial S\}} ds = 0. \quad (2.63)$$

Although the result is proved in analogous way as Lemma 2.1 of [57], we provide a quick proof for the sake of completeness. For $i = 1, \dots, N$, let $\xi_i(t) \doteq Z(t) \cdot n_i$, where n_i is the inward unit normal to the face F^i . In order to prove (2.63) it suffices to show that for each i ,

$$\int_0^t 1_{\{\xi_i(s)=0\}} ds = 0, P_x\text{-a.s.}, \quad (2.64)$$

for all $t > 0$ and $x \in S$. Note that ξ_i is a continuous $\{\mathcal{F}_t\}$ -semimartingale with quadratic variation $\langle \xi_i \rangle_t = \int_0^t n_i' a(Z_s) n_i ds$, where $a = \sigma \sigma'$. From Condition 2.3.4, we have that

$$\int_0^t 1_{\{\xi_i(s)=0\}} d\langle \xi_i \rangle_s \geq c \int_0^t 1_{\{\xi_i(s)=0\}} ds. \quad (2.65)$$

From Corollary 1, p. 216 of [50], the left side of (2.65) equals $\int_0^\infty L_t^a 1_{\{0\}}(a) da = 0$, where $\{L_t^a\}_{t \geq 0}$ is the local time process (at level a) of the continuous semimartingale ξ_i . (See page 211 [50] for definition of local times.) This proves (2.64) and hence (2.63) follows.

We next show that

$$P_x[Z(t) \in \partial S] = 0, \quad \forall x \in S, \quad t > 0. \quad (2.66)$$

Suppose first that $x \in S^\circ$. Let $\eta \doteq \inf\{t > 0 : Z(t) \in \partial S\}$. Without loss of generality we can assume that on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0})$ we have probability measures $\{Q_x\}_{x \in S}$ such that under Q_x , Z has the same law as the

unconstrained diffusion:

$$X(t) = x + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dB(s).$$

From uniform nondegeneracy of σ (Condition 2.3.4) we have that the measure $m_x \doteq Q_x \circ (\eta, Z(\eta))^{-1}$ on $[0, \infty) \times \partial S$ is absolutely continuous with respect to the Lebesgue measure. Using the strong Markov property of Z ,

$$\begin{aligned} P_x[Z(t) \in \partial S] &= P_x[\eta \leq t, Z(t) \in \partial S] \\ &= \int_{[0, t] \times \partial S} P_y[Z(t-u) \in \partial S] m_x(du, dy) = 0, \end{aligned}$$

where the last equality is an immediate consequence of (2.63).

Finally consider $x \in \partial S$. Then since (2.66) holds for $x \in S^\circ$ we have from Markov property that for all $s < t$, $P_x[Z(t) \in \partial S] = P_x[Z(t) \in \partial S, Z(s) \in \partial S]$. Since $s \in (0, t)$ is arbitrary, we get $P_x[Z(t) \in \partial S] = P_x[Z(t) \in \partial S, Z(q) \in \partial S, \forall q \in \mathbb{Q} \cap [0, t]]$, where \mathbb{Q} is the set of rational numbers. Sample path continuity of Z now gives

$$P_x[Z(t) \in \partial S] = P_x[Z(s) \in \partial S, \forall 0 \leq s \leq t].$$

However the last expression is 0 from (2.63). This proves (2.66).

To prove the result it suffices to show that for every $x \in S$ and $t > 0$, $m_{t,x} \doteq P_x \circ Z(t)^{-1}$ is mutually absolutely continuous with respect to the Lebesgue measure λ on $(S, \mathcal{B}(S))$. From nondegeneracy of σ (Condition 2.3.4) it follows that for all $y \in S$, under Q_y , (Z_t, η) has a nowhere vanishing joint density on $\mathbb{R}^d \times (0, \infty)$, for every $t > 0$. In particular for all $y \in S$ and $t > 0$,

$$Q_y(Z_t \in A, \eta > t) = 0 \Leftrightarrow \lambda(A) = 0. \quad (2.67)$$

We now show that $m_{t,x} \ll \lambda$. Let $A \in \mathcal{B}(S)$ be such that $\lambda(A) = 0$. In proving $m_{t,x}(A) = 0$ we can assume in view of (2.66) that A is contained in a compact subset of S° . Introduce sequences of stopping times as follows. Let $\sigma_0 \doteq \inf\{t : Z_t \in A\}$, and for $n \geq 1$, $\tau_n \doteq \inf\{t \geq \sigma_{n-1} : Z_t \in \partial S\}$ and $\sigma_n \doteq \inf\{t \geq \tau_n : Z_t \in A\}$. Then

$$\begin{aligned} m_{t,x}(A) &= P_x[Z_t \in A] = \sum_{n=0}^{\infty} P_x[Z_t \in A, t \in [\sigma_n, \tau_{n+1})] \\ &= \sum_{n=0}^{\infty} \int_{[0,t) \times A} Q_y[Z_{t-r} \in A, \eta \geq t-r] dm^n(r, y), \end{aligned} \quad (2.68)$$

where m^n is the joint law of (σ_n, Z_{σ_n}) and last equality is a consequence of the strong Markov property of $(Z, \{P_x\})$ and the observation that $P_x \circ (Z_{\cdot \wedge \eta}, \eta)^{-1} = Q_x \circ (Z_{\cdot \wedge \eta}, \eta)^{-1}$ for all $x \in S^\circ$. Now since $\lambda(A) = 0$, we have from (2.67) that the right side of (2.68) is zero and thus we have shown that $m_{t,x} \ll \lambda$. Finally, we show that $\lambda \ll m_{t,x}$. Let $A \in \mathcal{B}(S)$ and $(t, x) \in (0, \infty) \times S$ be such that $\lambda(A) > 0$. Once more, since $\lambda(\partial S) = 0$, for purposes of establishing $m_{t,x}(A) > 0$, we can assume without loss of generality that A is contained in a compact subset of S° . The desired inequality is now an immediate consequence of (2.67) and (2.68). ■

CHAPTER 3

Stationary distribution convergence for GJN in heavy traffic

Jackson network is one of the most commonly studied stochastic networks in queueing theory. For the setting where inter-arrival and service times are exponential, such a network was first considered in [35]. Subsequently, there have been many works that treat general arrival and service distributions. Such generalized Jackson networks (GJN) are the subject of this chapter. In a recent paper [28], the authors have shown that under appropriate conditions the stationary distributions of suitably scaled queue length processes for GJN converge to the stationary distribution of the associated reflected Brownian motion (RBM) in the heavy traffic limit. One of the key assumptions made in the analysis is that the inter-arrival and service times have finite moment generating functions (m.g.f.) in the neighborhood of origin (the precise assumption is a bit stronger and formulated in terms of residual service times and arrival times at time zero). The proof is based on strong approximation techniques and detailed uniform (in the scaling parameter) estimates on certain exponential moments of the state process. Finiteness of the m.g.f. of the primitive processes is a critical ingredient in these estimates. Indeed the authors in [28] suggest that for primitives with certain Pareto like distributions, steady state of RBM may be a poor approximation for steady state of the underlying physical network.

In this chapter we provide an elementary proof of the main result of [28] without imposing any exponential integrability conditions on the primitives of the network.

We make standard i.i.d. and second moment assumptions on inter-arrival and service times; see Conditions 3.1.1-3.1.2 in Section 3.1.1. These assumptions are typically used in heavy traffic analysis for invoking a functional central limit theorem [31]. In addition, similar to [28], we assume the heavy traffic condition (Condition 3.1.5) and a natural stability condition (Condition 3.1.8). Our proof is based on uniform stability estimates (see proof of Theorem 3.2.3, in particular (3.23)) on a family of certain deterministic dynamical systems obtained from a fluid limit analysis of the underlying queueing networks. We also make critical use of Lyapunov function methods developed in [45] and [18].

This chapter is organized as follows. In Section 3.1.1 we recall the formulation of a GJN and introduce basic assumptions on the inter-arrival and service time distributions. A description of the dynamics of the queue length process in terms of a Skorohod map is presented in Section 3.1.1. Diffusion scaling and appropriate heavy traffic conditions are introduced in Section 3.1.2. We also recall in this section the basic heavy traffic limit result of [31]. Next, our main stability assumption (Condition 3.1.8) is introduced and the well known results of [18] and [33] on existence of steady state distributions for GJN and RBM, respectively, are recalled. Section 3.2 presents our main result (Theorem 3.2.1) that establishes weak convergence of the stationary queue length distributions for the diffusion scaled GJN networks to the unique stationary distribution of the RBM.

Some of the notation used in this chapter is as follows. For $x \in \mathbb{R}^d$ the L_1 norm of x , i.e., $\sum_{i=1}^d |x_i|$, will be denoted by $|x|$. Let $\mathbb{I} = \mathbb{I}_{K \times K}$ denote the identity matrix for given K . For a metric space X , let $\mathcal{P}(X)$ denote the collection of all probability measures on X . The convergence in distribution of random variables (with values in some Polish space) Φ_n to Φ will be denoted as $\Phi_n \Rightarrow \Phi$. With an abuse of notation weak convergence of probability measures (on some Polish space) μ_n to μ will also be denoted as $\mu_n \Rightarrow \mu$.

3.1 Problem formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Unless specified otherwise, all the random variables considered in this chapter are assumed to be defined on this probability space.

3.1.1 Generalized Jackson network

Network structure. We start by describing a network with K service stations, where the i^{th} station is denoted by P_i , $i \in \mathbb{K} \doteq \{1, \dots, K\}$. We assume that each station has an infinite capacity buffer. We consider a single class network, that is, all customers at a station are homogeneous in terms of service requirement and routing decision. Arrivals of jobs can be from outside the system and/or from internal routing. Upon completion of service at station P_i a customer is routed to other service stations (or exits the system) according to a probabilistic routing matrix \mathbb{P} . At every station the jobs are assumed to be processed by First-In-First-Out discipline. We assume that the network is open, that is, any customer entering the network eventually leaves it. (See below (3.2) for a precise formulation.) This network was considered by Jackson in [35] with exponential inter-arrival/service time distributions and is referred to as Jackson network in this Markovian setting. We allow general inter-arrival and service time distributions. Hereafter, this single class network will be referred to as a generalized Jackson network (GJN).

Stochastic primitives and assumptions

For $k \in \mathbb{N}$, let $\eta_i(k)$, $\Delta_i(k)$ denote the k^{th} inter-arrival time and k^{th} service time, respectively, at station P_i , since time 0. (We only consider exogenous arrivals here.)

We assume:

Condition 3.1.1. For $\ell, i \in \mathbb{K}$, $\{\eta_\ell(k) : k \geq 1\}$, $\{\Delta_i(k) : k \geq 1\}$ are i.i.d. sequences with values in $[0, \infty]$.

A station P_j is said to have nonnull exogenous arrivals if $\mathbb{P}[\eta_j(1) < \infty] > 0$. Let \mathbb{K}_e denote the set of indices of stations with nonnull exogenous arrivals. Whenever external arrivals are under discussion, we consider only the nonnull exogenous arrivals. Our second assumption on the network is the following:

Condition 3.1.2. For $\ell \in \mathbb{K}_e$ and $i \in \mathbb{K}$, $\mathbb{E}[\eta_\ell(1)^2] < \infty$ and $\mathbb{E}[\Delta_i(1)^2] < \infty$.

Denote for $i \in \mathbb{K}_e$, $\alpha_i \doteq 1/\mathbb{E}[\eta_i(1)]$ the external arrival rate into station P_i and for $i \notin \mathbb{K}_e$, we set $\alpha_i = 0$. Then $\alpha = (\alpha_1, \dots, \alpha_K)'$ is the vector of external arrival rates. Let $m_i \doteq \mathbb{E}[\Delta_i(1)]$ denote the mean service time at station P_i and $\mu_i \doteq 1/m_i$ the service rate at P_i . We set $\mu = (\mu_1, \dots, \mu_K)'$ and M to be the diagonal matrix with m_1, \dots, m_K as diagonal entries. We assume μ_i and α_ℓ are finite for $i \in \mathbb{K}$ and $\ell \in \mathbb{K}_e$.

For $i \in \mathbb{K}$, let $U_{i,0}$ and $V_{i,0}$ be random variables representing the residual inter-arrival time and service time at time 0, at P_i . Here we adopt the convention $U_{i,0} = \infty$ for $i \notin \mathbb{K}_e$. Define for $t \geq 0$,

$$A_i(t) \doteq \max\{r \geq 1 : \eta_i(0) + \eta_i(1) + \dots + \eta_i(r-1) \leq t\}, \quad i \in \mathbb{K}, \quad (3.1)$$

where $\eta_i(0) \doteq U_{i,0}$ and we follow the convention that max over an empty set is 0. Thus $A_i(t)$ represents the total number of arrivals at P_i by time t . Denote by $D_j(t)$ the total number of service completions at P_j by time t and let $D_{ji}(t)$ be the number of those jobs that are routed to P_i immediately upon completion at station P_j . Denote by $Q_i(t)$ the queue length at time t , i.e., number of customers that are in queue or currently in service at P_i . Then, for $i \in \mathbb{K}$,

$$Q_i(t) = Q_i(0) + A_i(t) - D_i(t) + \sum_{j=1}^K D_{ji}(t). \quad (3.2)$$

The routing decisions at each station are to be of Bernoulli type. More precisely,

we consider a $K \times K$ sub-stochastic matrix $\mathbb{P} = (p_{ji})_{j,i \in \mathbb{K}}$, where the entry p_{ji} is the probability of the event that upon completion at P_j the job is routed to station P_i . The spectral radius of the transition matrix \mathbb{P} is assumed to be strictly less than unity, which ensures that all customers eventually leave the network. For $i \in \mathbb{K}$ and each $k \in \mathbb{N}$, let $\phi^i(k)$ be the routing (column) vector for the k^{th} customer at station P_i upon finishing service. Then $\phi^i(k)$ is a K -dimensional ‘‘Bernoulli random vector’’ with parameter $(\mathbb{P}^i)'$, where \mathbb{P}^i denotes the i^{th} row of \mathbb{P} . More precisely, $\phi^i(k) = e_j$ with probability p_{ij} and $\phi^i(k) = \mathbf{0}$ with probability $1 - \sum_{j=1}^K p_{ij}$. Here e_j is the K -dimensional j^{th} coordinate vector. We assume

Condition 3.1.3. *The random variables $\{\eta_i(k), \Delta_i(k), \phi^i(k) : i \in \mathbb{K}, k \geq 1\}$ are independent. Also, this collection of random variables is independent of $\{U_{i,0}, V_{i,0}, Q_i(0) : i \in \mathbb{K}\}$.*

Define

$$R^i(k) \doteq \sum_{l=1}^k \phi^i(l),$$

which measures aggregated routing decisions up to k^{th} service completion at station P_i . In particular, $R_j^i(k)$ will denote the j^{th} component of $R^i(k)$, representing total number of routings from P_i to P_j among the first k services completed.

Let $E_i(t)$ be the total number of service completions at the station P_i in t units of service time since time 0. Note that E_i in general will be different from D_i due to service idleness. I.e.,

$$E_i(t) \doteq \max\{r \geq 1 : \Delta_i(0) + \Delta_i(1) + \cdots + \Delta_i(r-1) \leq t\}, \quad (3.3)$$

where $\Delta_i(0) \doteq V_{i,0}$ and as before, max over an empty set is 0. Also let $T_i(t)$ be the cumulative amount of service time that the station P_i has spent on customers by time t . Let $I_i(t) \doteq t - T_i(t)$ denote the amount of time that the station P_i has been idle by time t . We assume the network is nonidling, that is, a service station is idle only

when there are no customers at that station requiring service. Then

$$D_j(t) = E_j(T_j(t)), \quad D_{ji}(t) = R_i^j(E_j(T_j(t))).$$

Henceforth, we will refer to α, μ and \mathbb{P} as the parameters of the GJN. We define traffic intensity vector $\rho = (\rho_1, \dots, \rho_K)'$ of this GJN as $\rho_i \doteq ([\mathbb{I} - \mathbb{P}']^{-1}\alpha)_i / \mu_i, i \in \mathbb{K}$. Initial condition of the network is specified by random variables $(Q_i(0), U_{i,0}, V_{i,0}; i \in \mathbb{K})$.

System dynamics and Skorohod mapping

The evolution of the state of the system satisfies the following equations: For $i \in \mathbb{K}$,

$$Q_i(t) = Q_i(0) + A_i(t) - E_i(T_i(t)) + \sum_{j=1}^K R_i^j(E_j(T_j(t))), \quad (3.4)$$

$$\int_0^\infty Q_i(t) dI_i(t) = 0. \quad (3.5)$$

We note that these processes satisfy,

$$Q_i(t) \geq 0, \quad T_i \text{ and } I_i \text{ are nondecreasing and } T_i(0) = I_i(0) = 0, \quad i \in \mathbb{K}. \quad (3.6)$$

Equations (3.4), (3.5) and (3.6) describe the system dynamics. Next consider the ‘‘centered’’ process $\tilde{Q} \doteq (\tilde{Q}_i(t) : t \geq 0, i \in \mathbb{K})$, where

$$\begin{aligned} \tilde{Q}_i(t) &\doteq Q_i(0) + (A_i(t) - \alpha_i t) - (E_i(T_i(t)) - \mu_i T_i(t)) \\ &\quad + \sum_{j \in \mathbb{K}} (R_i^j(E_j(T_j(t))) - p_{ji} E_j(T_j(t))) \\ &\quad + \sum_{j \in \mathbb{K}} p_{ji} (E_j(T_j(t)) - \mu_j T_j(t)) + \left(\alpha_i + \sum_{j \in \mathbb{K}} \mu_j p_{ji} - \mu_i \right) t. \end{aligned} \quad (3.7)$$

Denote $Y_i \doteq \mu_i I_i$. Set $Q \doteq (Q_1, \dots, Q_K)'$ and analogously define T, I, E , and Y . The dynamics in (3.4) - (3.6) can now be represented in the following succinct vector

forms:

$$Q(t) = \tilde{Q}(t) + [\mathbb{I} - \mathbb{P}']Y(t), \quad t \in \mathbb{R}_+, \quad (3.8)$$

$$\int_0^\infty Q_i(t) dY_i(t) = 0, \quad i \in \mathbb{K}, \quad (3.9)$$

$$Q(t) \geq 0, \quad Y \text{ is nondecreasing.} \quad (3.10)$$

The above dynamics can equivalently be stated in terms of a Skorohod map as we describe below. Recall the definition of Skorohod map given in Chapter 2 (Definition 2.3.1). We consider here the case where $N = d = K$; $S = \mathbb{R}_+^K$; $S_i = \{x \in \mathbb{R}_+^K : x_i \geq 0\}$, $i \in \mathbb{K}$ and $R = [\mathbb{I} - \mathbb{P}']$.

Define the map Γ_1 by $\Gamma_1(\psi) \doteq R^{-1}[\phi - \psi]$. The following result (see [31, 23]) gives the regularity of the Skorohod map defined by the data $(S, [\mathbb{I} - \mathbb{P}'])$.

Proposition 3.1.4. *The Skorohod map is well defined on all of $D_S([0, \infty), \mathbb{R}^K)$, that is, $D = D_S([0, \infty), \mathbb{R}^K)$, and the SM is Lipschitz continuous in the following sense. There exists a constant $L \in (1, \infty)$ such that for all $\psi_1, \psi_2 \in D_S([0, \infty), \mathbb{R}^K)$:*

$$\sup_{0 \leq t < \infty} \{|\Gamma(\psi_1)(t) - \Gamma(\psi_2)(t)| + |\Gamma_1(\psi_1)(t) - \Gamma_1(\psi_2)(t)|\} < L \sup_{0 \leq t < \infty} |\psi_1(t) - \psi_2(t)|. \quad (3.11)$$

Equivalent form of dynamics (3.8) - (3.10) in terms of the SM can now be written as follows:

$$Q = \Gamma(\tilde{Q}), \quad Q - \tilde{Q} = [\mathbb{I} - \mathbb{P}']Y. \quad (3.12)$$

3.1.2 GJN in heavy traffic

Under appropriate heavy traffic conditions, the queue lengths of suitably scaled GJN can be approximated by a diffusion with constant coefficients, constrained to take values in S . More precisely, consider a sequence of GJN networks indexed by $n \in \mathbb{N}$ with parameters $(\alpha^n, \mu^n, \mathbb{P})$. Conditions 3.1.1 through 3.1.3 are assumed to hold for

each fixed n . Processes $Q^n, \tilde{Q}^n, Y^n, M^n, T^n$ are defined in a manner analogous to Section 3.1.1. In particular, (3.8) - (3.10) and (3.12) hold with (Q, \tilde{Q}, Y) replaced by (Q^n, \tilde{Q}^n, Y^n) . The heavy traffic assumption we make is as follows:

Condition 3.1.5.

$$\alpha^n = \alpha - \frac{\tilde{v}^n}{\sqrt{n}}, \quad \tilde{v}^n \rightarrow \tilde{v}, \quad \mu^n = \mu - \frac{\tilde{\beta}^n}{\sqrt{n}}, \quad \tilde{\beta}^n \rightarrow \tilde{\beta}, \quad [\mathbb{I} - \mathbb{P}']^{-1}\alpha = \mu.$$

Here $\alpha, \mu, \tilde{v}^n, \tilde{\beta}^n, \tilde{v}, \tilde{\beta} \in \mathbb{R}^K$. Also μ^n, μ are strictly positive and $\alpha^n, \alpha \geq 0$ (α_i, α_i^n are strictly positive when $i \in \mathbb{K}_e$ and 0 when $i \notin \mathbb{K}_e$). Note that the traffic intensity vector ρ^n of the n^{th} GJN can be written as

$$\rho_i^n = 1 - \frac{([\mathbb{I} - \mathbb{P}']^{-1}\tilde{v}^n)_i - \tilde{\beta}_i^n}{\sqrt{n}\mu_i^n},$$

and as $n \rightarrow \infty, \rho^n \rightarrow 1$.

To state the precise heavy traffic limit result, we begin with the following Markovian state description for GJN. Recall the initial residual times, $U_{i,0}, V_{i,0}$ introduced in Section 3.1.1. We denote the analogous quantities for the n^{th} GJN by $U_{i,0}^n$ and $V_{i,0}^n$, respectively. For $t \geq 0$, and $i \in \mathbb{K}$, let $U_i^n(t)$ and $V_i^n(t)$ denote the remaining time, at instant t , before the next exogenous arrival and next service completion, respectively, at station P_i , for the n^{th} GJN. They can be explicitly written as follows:

$$U_i^n(t) = U_i^n(0) + \sum_{k=1}^{A_i^n(t)} \eta_i^n(k) - t, \quad V_i^n(t) = V_i^n(0) + \sum_{k=1}^{D_i^n(t)} \Delta_i^n(k) - T_i^n(t).$$

Define the *state* of the system at time t at P_i by

$$X^n(t) \doteq (Q_i^n(t), U_\ell^n(t), V_i^n(t) : i \in \mathbb{K}, \ell \in \mathbb{K}_e).$$

Let $X^n = (X^n(t) : t \geq 0)$. Note that the process $(Q^n(t) : t \geq 0)$ alone is not Markovian due to the residual inter-arrival/service times, but one can check that the augmented process $(X^n(t) : t \geq 0)$ is indeed a strong Markov process with state space

$\mathbb{X} \doteq \mathbb{R}_+^K \times \mathbb{R}_+^{|\mathcal{K}_e|} \times \mathbb{R}_+^K$. See [13] for details. Define the diffusion scaled processes:

$$\widehat{X}^n(t) \doteq \frac{X^n(nt)}{\sqrt{n}}, \quad \widehat{W}^n(t) \doteq \frac{\widetilde{Q}^n(nt)}{\sqrt{n}}, \quad \widehat{I}^n(t) \doteq \frac{I^n(nt)}{\sqrt{n}}.$$

We next recall the definition of a semimartingale reflecting Brownian motion (SRBM) from Section 2.2. We will consider an SRBM with more general initial distributions, than Dirac probability measures, as introduced below.

Definition 3.1.6. *Let $\{\mathcal{F}_t\}$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$. For $\mu_0 \in \mathcal{P}(\mathbb{R}_+^K)$, $b \in \mathbb{R}^K$ and a positive definite $K \times K$ matrix Σ , an SRBM associated with the data $(S, b, \Sigma, [\mathbb{I} - \mathbb{P}'])$ with initial distribution μ_0 , is a continuous $\{\mathcal{F}_t\}$ -adapted K -dimensional process Z such that*

- (i) $Z = \Gamma(Z(0) + B + b\iota)$, \mathbb{P} -a.s. Here $\iota : [0, \infty) \rightarrow [0, \infty)$ is the identity map.
- (ii) B is a K -dimensional $\{\mathcal{F}_t\}$ -Brownian motion with covariance Σ , mean $\mathbf{0}$ and $B(0) = \mathbf{0}$.
- (iii) $Z(0)$ is distributed according to μ_0 .

We will write $\xi(t) \doteq Z(t) - Z(0) - B(t) - bt$, $t \geq 0$ and for brevity refer to Z as a (b, Σ) -SRBM.

Recall from Chapter 2 that Z is a strong Markov process. The following heavy traffic limit theorem follows upon slightly modifying the arguments in [51]. The proof is omitted. Recall that $\widehat{Q}^n(t) \doteq \frac{Q^n(nt)}{\sqrt{n}}$.

Theorem 3.1.7. *Assume that the sequence of measures $\mathbb{P} \circ [\widehat{X}^n(0)]^{-1}$ converges weakly to some $\nu \in \mathcal{P}(\mathbb{X})$. Then the process $(\widehat{Q}^n, \widehat{W}^n, M_n^{-1}\widehat{I}^n)$ converges in distribution in $D([0, \infty), \mathbb{R}^K)^{\otimes 3}$ to $(Z, B + b, \xi)$ as in Definition 3.1.6 with μ_0 given as $\mu_0(A) \doteq \nu(A \times \mathbb{R}_+^{|\mathcal{K}_e|} \times \mathbb{R}_+^K)$, for $A \in \mathcal{B}(\mathbb{R}_+^K)$; $b = [\mathbb{I} - \mathbb{P}']\tilde{\beta} - \tilde{v}$; and some positive definite matrix Σ .*

We now introduce our main stability condition on the sequence of GJN that will be used in this chapter.

Condition 3.1.8. *There exists a $\theta \in \mathbb{R}^K$, $\theta > 0$, such that, for all $n \in \mathbb{N}$,*

$$M_n[\tilde{\beta}^n - [\mathbb{I} - \mathbb{P}']^{-1}\tilde{v}^n] < -\theta.$$

Note that Condition 3.1.8 is equivalent to the requirement that for some $\theta_0 \in (0, \infty)$,

$$\sup_n \max_{i \in \mathbb{K}} \sqrt{n}(\rho_i^n - 1) \leq -\theta_0 < 0. \quad (3.13)$$

In particular we have that for each fixed n , $\max_i \rho_i^n < 1$. This traffic intensity property implies the stability of GJN for each fixed n as summarized in the following result from [54]. (See also [43] and [18].)

Theorem 3.1.9. *There exists a stationary probability distribution for the Markov process \widehat{X}^n .*

We remark that, in general uniqueness of invariant measures for \widehat{X}^n may not hold unless additional conditions are imposed. In what follows, if the initial condition of the Markov process \widehat{X}^n is x for some $x \in \mathbb{X}$, i.e., $(\widehat{Q}^n(0), \widehat{U}^n(0), \widehat{V}^n(0)) \equiv (q, u, v) \doteq x$, we will write the process as \widehat{X}_x^n .

Next we consider stability of the limiting diffusion model, i.e. an SRBM. The following theorem from [33] gives a necessary and sufficient condition for positive recurrence of a (b, Σ) -SRBM.

Theorem 3.1.10. *The SRBM with data (b, Σ) has a unique stationary probability distribution π if and only if $[\mathbb{I} - \mathbb{P}']^{-1}b < 0$.*

Note that Condition 3.1.8 in particular implies that $\tilde{\beta} - [\mathbb{I} - \mathbb{P}']^{-1}\tilde{v} < 0$ and so from Theorem 3.1.10, under Condition 3.1.8, a (b, Σ) -SRBM has a unique stationary probability distribution, where b and Σ are as in Theorem 3.1.7.

3.2 Convergence of invariant measures

Conditions 3.1.1 through 3.1.8 will hold throughout this section and explicit reference to them in statement of our results will be omitted.

Recall that we denote the unique invariant probability measure of the (b, Σ) -SRBM on \mathbb{R}_+^K by π . Also, recall that from Theorem 3.1.9, for each $n \in \mathbb{N}$, the Markov process \widehat{X}^n admits a stationary probability measure π_n . Denote by π_n^0 the marginal distribution of π_n on the first coordinate of \mathbb{X} , i.e., $\pi_n^0(A) = \pi_n(A \times \mathbb{R}_+^{|\mathcal{K}_e|} \times \mathbb{R}_+^K)$, $A \in \mathcal{B}(\mathbb{R}_+^K)$. The following is the main result of this chapter.

Theorem 3.2.1. *The sequence $\{\pi_n^0\}$ of probability measures on $(\mathbb{R}_+^K, \mathcal{B}(\mathbb{R}_+^K))$ converges weakly to the unique invariant probability measure, π , of the (b, Σ) -SRBM, where b and Σ are as in Theorem 3.1.7.*

The main step in proving Theorem 3.2.1 is the following tightness result.

Theorem 3.2.2. *The sequence $\{\pi_n\}_{n \in \mathbb{N}}$ is a tight family of probability measures on \mathbb{X} .*

Theorem 3.2.1 follows from Theorem 3.2.2 using standard arguments. Indeed, from Theorem 3.2.2 we have that every subsequence of $\{\pi_n\}$ admits a convergent subsequence. Denoting a typical limit point by $\tilde{\pi}$ we see from Theorem 3.1.7 that the process $(\widehat{Q}^n, \widehat{W}^n, M_n^{-1}\widehat{I}^n)$, with $\widehat{X}^n(0)$ distributed as π_n , converges in distribution to (Z, B, ξ) as in Definition 3.1.6 with $\mu_0 = \tilde{\pi}^0$, and b, Σ as in Theorem 3.1.7, where $\tilde{\pi}^0(A) = \tilde{\pi}(A \times \mathbb{R}_+^{|\mathcal{K}_e|} \times \mathbb{R}_+^K)$, $A \in \mathcal{B}(\mathbb{R}_+^K)$. Stationarity of \widehat{Q}^n implies that $\tilde{\pi}^0$ is an invariant measure for the (b, Σ) -SRBM. Theorem 3.1.10 then gives that $\tilde{\pi}^0 = \pi$. Thus Theorem 3.2.1 follows.

Rest of the section is devoted to the proof of Theorem 3.2.2. We begin with the following moment stability estimate on \widehat{X}_x^n that is uniform in n .

Theorem 3.2.3. *There exists a $t_0 \in (0, \infty)$ such that for all $t \geq t_0$,*

$$\lim_{|x| \rightarrow \infty} \sup_n \frac{1}{|x|^2} \mathbb{E} \left(|\widehat{X}_x^n(t|x)|^2 \right) = 0.$$

Proof. We first show that for some $t_0 \in (0, \infty)$ and all $t \geq t_0$,

$$\lim_{|x| \rightarrow \infty} \sup_n \frac{1}{|x|^2} \mathbb{E} \left(|\widehat{Q}_x^n(t|x)|^2 \right) = 0. \quad (3.14)$$

Fix $x = (q, u, v) \in \mathbb{X}$. Recall that Q^n is given by the representation (3.7) and (3.12) with all processes there written with a superscript n . We will now write a slightly modified dynamical description for \widetilde{Q}^n that makes explicit the dependence on initial residual times (u, v) . We set $u_i = \infty$ for $i \notin \mathbb{K}_e$. We suppress x from the notation unless needed and rewrite \widetilde{Q}_i^n as

$$\begin{aligned} \widetilde{Q}_i^n(t) &= q + \left(A_i(t) - \alpha_i(t - \sqrt{n}u_i)^+ \right) - \left(E_i(T_i(t)) - \mu_i(T_i(t) - \sqrt{n}v_i)^+ \right) \\ &\quad + \sum_{j \in \mathbb{K}} p_{ji} \left(E_j(T_j(t)) - \mu_j(T_j(t) - \sqrt{n}v_j)^+ \right) \\ &\quad + \sum_{j \in \mathbb{K}} \left(R_i^j(E_j(T_j(t))) - p_{ji} E_j(T_j(t)) \right) \\ &\quad + \left[\alpha_i(t - \sqrt{n}u_i)^+ + \sum_{j \in \mathbb{K}} p_{ji} \mu_j(t - \sqrt{n}v_j)^+ - \mu_i(t - \sqrt{n}v_i)^+ \right] \\ &\quad + \sum_{j \in \mathbb{K}} p_{ji} \mu_j \left((T_j(t) - \sqrt{n}v_j)^+ - T_j(t) \right) + \sum_{j \in \mathbb{K}} p_{ji} \mu_j \left(t - (t - \sqrt{n}v_j)^+ \right) \\ &\quad - \mu_i \left[t - (t - \sqrt{n}v_i)^+ - (T_i(t) - (T_i(t) - \sqrt{n}v_i)^+) \right]. \end{aligned}$$

Thus $\widehat{Q}_x^n(t) = \Gamma(q + N^n + b^n)(t)$, where $N^n(t) = N_1^n(t) + N_2^n(t) + N_3^n(t)$ and for $i \in \mathbb{K}$,

$$N_{1,i}^n(t) \doteq \frac{1}{\sqrt{n}} \left(A_i(nt) - \alpha_i^n(nt - \sqrt{n}u_i)^+ \right), \quad (3.15)$$

$$\begin{aligned}
N_{2,i}^n(t) &\doteq \frac{1}{\sqrt{n}} \left[\sum_{j \in \mathbb{K}} p_{ji} \left(E_j(T_j(nt)) - \mu_j^n(T_j(nt) - \sqrt{n}v_j)^+ \right) \right. \\
&\quad \left. - \left(E_i(T_i(nt)) - \mu_i^n(T_i(nt) - \sqrt{n}v_i)^+ \right) \right], \\
N_{3,i}^n(t) &\doteq \frac{1}{\sqrt{n}} \sum_{j \in \mathbb{K}} (R_i^j(E_j(T_j(nt))) - p_{ji}E_j(T_j(nt))), \\
b_i^n(t) &\doteq \sqrt{n} \left[\alpha_i^n(t - u_i/\sqrt{n})^+ + \sum_{j \in \mathbb{K}} p_{ji} \mu_j^n(t - v_j/\sqrt{n})^+ - \mu_i^n(t - v_i/\sqrt{n})^+ \right] \\
&\quad + \frac{1}{\sqrt{n}} \left[\sum_{j \in \mathbb{K}} p_{ji} \mu_j^n \left((T_j(nt) - \sqrt{n}v_j)^+ - T_j(nt) \right) \right. \\
&\quad + \sum_{j \in \mathbb{K}} p_{ji} \mu_j^n \left(nt - (nt - \sqrt{n}v_j)^+ \right) \\
&\quad \left. - \mu_i^n \left[(nt - (nt - \sqrt{n}v_i)^+) - (T_i(nt) - (T_i(nt) - \sqrt{n}v_i)^+) \right] \right].
\end{aligned}$$

Define

$$Z_x^n(t) \doteq \Gamma(q + b^n \iota)(t), \quad t \geq 0, \quad (3.16)$$

where $\iota : [0, \infty) \rightarrow [0, \infty)$ is the identity map. Using the Lipschitz property of the Skorohod map (see Proposition 3.1.4), we have

$$|\widehat{Q}_x^n(t) - Z_x^n(t)| \leq L \sup_{0 \leq s \leq t} |N^n(s)|. \quad (3.17)$$

Let $\bar{t} \doteq |x|$ and set $\bar{z}^n \doteq Z_x^n(\bar{t})$. Observing that $\bar{t} \geq \max_{i \in \mathbb{K}} \{ \frac{u_i}{\sqrt{n}}, \frac{v_i}{\sqrt{n}} \}$ and $T_i(t) = t$ for $t \in [0, \sqrt{n}v_i]$, $i \in \mathbb{K}$, $n \geq 1$, we get

$$Z_x^n(t) = \Gamma(\bar{z}^n + \bar{b}^n \iota)(t - \bar{t}), \quad t \geq \bar{t}, \quad (3.18)$$

and

$$\bar{b}^n \doteq \sqrt{n}[\alpha^n - (\mathbb{I} - \mathbb{P}')\mu^n] = \sqrt{n}[(\mathbb{I} - \mathbb{P}')\tilde{\beta}^n - \tilde{v}^n].$$

Next note that

$$|\bar{z}^n| = |\Gamma(q + b^n \iota)(|x|)| \leq L \left[|q| + \sup_{0 \leq t \leq |x|} |b^n(t)| \right]. \quad (3.19)$$

Define $c_0 \doteq \sup_{i,n} \{ \alpha_i^n + 3(\sum_{j \in \mathbb{K}} p_{ji} \mu_j^n + \mu_i^n) \}$. If $t < \frac{|x|}{\sqrt{n}}$ we see that $|b_i^n(t)| \leq c_0 |x|$ for all $n \geq 1$ and $i \in \mathbb{K}$. On the other hand, if $t \geq \frac{|x|}{\sqrt{n}}$

$$|b^n(t)| \leq \left| \sqrt{n}[\alpha^n - (\mathbb{I} - \mathbb{P}')\mu^n] \right| t + K c_0 |x|.$$

Combining the above observation with (3.19) and the heavy traffic condition (i.e., Condition 3.1.5), we see that one can find $L_0 \in (0, \infty)$ such that

$$|\bar{z}^n| \leq L_0 |x|. \quad (3.20)$$

Next recall the cone \mathcal{C} introduced in Chapter 2 (see (2.51)). In notation of this chapter, $\mathcal{C} = \{v \in \mathbb{R}^K : [\mathbb{I} - \mathbb{P}']^{-1}v \leq 0\}$, we see that from Conditions 3.1.5-3.1.8 that there exists a $\delta > 0$ such that

$$\inf_n \text{dist}(\bar{b}_n, \partial \mathcal{C}) \geq \delta. \quad (3.21)$$

Recall the definition of \mathcal{C}_δ in (2.52). For $q_0 \in \mathbb{R}_+^K$, let $\mathcal{A}(q_0)$ be the collection of all trajectories $\psi : [0, \infty) \rightarrow \mathbb{R}_+^K$ of the form

$$\psi(t) \doteq \Gamma(q_0 + v \iota)(t), \quad t \geq 0, \quad (3.22)$$

where v ranges over all of \mathcal{C}_δ . Define the ‘‘hitting time to the origin’’ function

$$T(q_0) \doteq \sup_{\psi \in \mathcal{A}(q_0)} \inf \{t \in [0, \infty) : \psi(t) = 0\}.$$

Lemma 3.1 of [2] shows that

$$T(q_0) \leq \frac{4L^2}{\delta} |q_0|, \text{ and for all } \psi \in \mathcal{A}(q_0), \psi(t) = 0 \text{ for all } t \geq T(q_0). \quad (3.23)$$

Combining this with (3.18), (3.21) and (3.20) we now have that $Z_x^n(t) = 0$, for all $t \geq L|x|$, where $L \doteq [1 + \frac{4L^2}{\delta} L_0]$. Using this in (3.17) we now see that

$$|\widehat{Q}_x^n(t|x)| \leq L \sup_{0 \leq s \leq t|x|} |N^n(s)|, \quad (3.24)$$

for all $t \geq L$ and for all initial conditions x . Next we obtain an estimate on second moment of the right side of (3.24). Define $A_{i,0}^n$ and $E_{i,0}^n$ by (3.1), (3.3) with $U_{i,0}^n$ and $V_{i,0}^n$ there replaced by 0. Note that

$$A_i^n(t) = A_{i,0}^n((t - \sqrt{nu_i})^+) + 1_{[\sqrt{nu_i}, \infty)}(t), \quad E_i^n(t) = E_{i,0}^n((t - \sqrt{nv_i})^+) + 1_{[\sqrt{nv_i}, \infty)}(t). \quad (3.25)$$

From standard estimates for renewal processes (see e.g., [9] Lemma 3.5), we have that for some $\kappa_0 \in (0, \infty)$, and all $n \geq 1$

$$\begin{aligned} & \frac{1}{n} \mathbb{E} \sup_{0 \leq s \leq t} \left([A_{i,0}^n(ns) - n\alpha_i^n s]^2 + [E_{i,0}^n(ns) - n\mu_i^n s]^2 \right. \\ & \left. + [R_i^j(E_{j,0}^n(ns)) - p_{ji} E_{j,0}^n(ns)]^2 \right) \leq \kappa_0(1+t), \end{aligned}$$

for all $i \in \mathbb{K}_e, j \in \mathbb{K}$. Using these estimates in (3.15), recalling (3.25), and noting that for all $t \geq 0$, $[T_i(nt) - \sqrt{nv_i}]^+ \leq nt$, we obtain for some $\kappa_1 \in (0, \infty)$,

$$\mathbb{E} \sup_{0 \leq s \leq t} [|N_{1,i}^n(s)|^2 + |N_{2,i}^n(s)|^2 + |N_{3,i}^n(s)|^2] \leq \kappa_1(1+t). \quad (3.26)$$

Applying this estimate in (3.24) we now have that for all $t \geq L, x \in \mathbb{X}$ and for some

$\kappa_2 \in (0, \infty)$,

$$\mathbb{E}|\widehat{Q}_x^n(t|x)|^2 \leq \kappa_2 L^2(1+t|x|). \quad (3.27)$$

Choosing $t_0 = L$ the result (3.14) now follows.

It remains to show that for all $t \geq t_0$

$$\lim_{|x| \rightarrow \infty} \sup_n \frac{1}{|x|^2} \mathbb{E} \left(|\widehat{U}_x^n(t|x)|^2 \right) = 0, \quad \lim_{|x| \rightarrow \infty} \sup_n \frac{1}{|x|^2} \mathbb{E} \left(|\widehat{V}_x^n(t|x)|^2 \right) = 0. \quad (3.28)$$

Recall $\widehat{V}_i^n(t|x) = \frac{1}{\sqrt{n}} V_i^n(nt|x)$ and thus

$$|\widehat{V}_i^n(t|x)|^2 \leq \frac{1}{n} |\Delta_i^n(E_i^n(T_i^n(nt|x)))|^2 \leq \frac{1}{n} \sum_{k=1}^{E_i^n(T_i^n(nt|x))} |\Delta_i^n(k)|^2 \leq \frac{1}{n} \sum_{k=1}^{E_{i,0}^n(nt|x)} |\Delta_i^n(k)|^2,$$

where the second inequality above uses the fact that $E_i^n(T_i^n(nt|x)) \geq 1$ since

$V_{i,0}^n = \sqrt{nv_i} \leq nt_0|x|$ and the last inequality follows on $T_i^n(t) \leq t$ for all $t \geq 0$. Using

Wald's identity we have for some $\kappa_3 \in (0, \infty)$, and all $x \in \mathbb{X}$, $t \geq t_0$

$$\begin{aligned} \mathbb{E} \left(|\widehat{V}_i^n(t|x)|^2 \right) &\leq \frac{1}{n} \mathbb{E} \left(E_{i,0}^n(nt|x) + 1 \right) \mathbb{E}[\Delta_i^n(1)]^2 \\ &\leq \frac{\kappa_3(1+nt|x|)}{n} \leq \kappa_3(1+t|x|). \end{aligned} \quad (3.29)$$

This proves the second statement in (3.28). The first statement is shown similarly. ■

For $\bar{\delta} \in (0, \infty)$, define the return time to a compact set $C \subseteq \mathbb{X}$ by $\tau_C^n(\bar{\delta}) \doteq \inf\{t \geq \bar{\delta} : \widehat{X}^n(t) \in C\}$. The proof of the following result is adapted from that of Proposition 5.3 of [18].

Theorem 3.2.4. *For some constants $c, \bar{\delta} \in (0, \infty)$, and a compact set $C \subseteq \mathbb{X}$,*

$$\sup_n \mathbb{E} \left[\int_0^{\tau_C^n(\bar{\delta})} (1 + |\widehat{X}_x^n(t)|) dt \right] \leq c(1 + |x|^2), \quad x \in \mathbb{X}. \quad (3.30)$$

Proof of this Theorem 3.2.4 follows as a special case of Proposition 4.3.3. Proof of the latter result is given in the Appendix of Chapter 4.

Theorem 3.2.5. *Let $f : \mathbb{X} \rightarrow \mathbb{R}_+$, and define for $\bar{\delta} \in (0, \infty)$, and a compact set $C \subseteq \mathbb{X}$*

$$V_n(x) \doteq \mathbb{E} \left[\int_0^{\tau_C^n(\bar{\delta})} f(\widehat{X}_x^n(t)) dt \right], \quad x \in \mathbb{X}.$$

If $\sup_n V_n$ is everywhere finite and uniformly bounded on C , then there exists $\bar{\kappa} \in (0, \infty)$ such that for all $n \in \mathbb{N}$, $t > 0$, $x \in \mathbb{X}$

$$\frac{1}{t} \mathbb{E}[V_n(\widehat{X}_x^n(t))] + \frac{1}{t} \int_0^t \mathbb{E}[f(\widehat{X}_x^n(s))] ds \leq \frac{1}{t} V_n(x) + \bar{\kappa}. \quad (3.31)$$

Theorem 3.2.5 is a special case of a more general result in Chapter 4, namely Theorem 4.3.2: Proof of Theorem 4.3.2 is given in the Appendix of Chapter 4.

Proof of Theorem 3.2.2. We apply Theorem 3.2.5 with $f(x) \doteq 1 + |x|$ for $x \in \mathbb{X}$ and $\bar{\delta}$, C as in Theorem 3.2.4. To prove the result it suffices to show that for all $n \in \mathbb{N}$, $\langle \pi_n, f \rangle \doteq \int_{\mathbb{X}} f(x) \pi_n(dx) \leq \bar{\kappa}$. Since π_n is an invariant measure, for any nonnegative, real measurable function Φ on \mathbb{X} ,

$$\int_{\mathbb{X}} \mathbb{E}[\Phi(\widehat{X}_x^n(t))] \pi_n(dx) = \langle \pi_n, \Phi \rangle. \quad (3.32)$$

Fix $k \in \mathbb{N}$ and let $V_n^k(x) \doteq V_n(x) \wedge k$. Let

$$\Psi_n^k(x) \doteq \frac{1}{t} V_n^k(x) - \frac{1}{t} \mathbb{E}[V_n^k(\widehat{X}_x^n(t))].$$

From (3.32), we have that $\int_{\mathbb{X}} \Psi_n^k(x) \pi_n(dx) = 0$. Let $\Psi_n(x) \doteq \frac{1}{t} V_n(x) - \frac{1}{t} \mathbb{E}[V_n(\widehat{X}_x^n(t))]$. Monotone convergence theorem yields that $\Psi_n^k(x) \rightarrow \Psi_n(x)$ as $k \rightarrow \infty$. Next we will

show that $\Psi_n^k(x)$ is bounded from below for all $x \in \mathbb{X}$. If $V_n(x) \leq k$,

$$\Psi_n^k(x) = \frac{1}{t}V_n^k(x) - \frac{1}{t}\mathbb{E}[V_n^k(\widehat{X}_x^n(t))] \geq \frac{1}{t}V_n(x) - \frac{1}{t}\mathbb{E}[V_n(\widehat{X}_x^n(t))] \geq -\bar{\kappa},$$

where the last inequality follows from (3.31). On the other hand, if $V_n(x) \geq k$

$$\Psi_n^k(x) = \frac{1}{t}k - \frac{1}{t}\mathbb{E}[V_n^k(\widehat{X}_x^n(t))] \geq 0, \quad (3.33)$$

where the second inequality follows on noting that $V_n^k \leq k$. Hence $\Psi_n^k(x) \geq -\bar{\kappa}$ for all $x \in \mathbb{X}$. By an application of Fatou's Lemma we conclude that

$$\int_{\mathbb{X}} \Psi_n(x) \pi_n(dx) \leq \liminf_{k \rightarrow \infty} \int_{\mathbb{X}} \Psi_n^k(x) \pi_n(dx) = 0. \quad (3.34)$$

From (3.31), $\Psi_n(x) \geq \frac{1}{t} \int_0^t \mathbb{E}[f(\widehat{X}_x^n(s))] ds - \bar{\kappa}$. Combining this with (3.34) we have

$$0 \geq \int_{\mathbb{X}} \Psi_n(x) \pi_n(dx) \geq \frac{1}{t} \int_0^t \int_{\mathbb{X}} \mathbb{E}[f(\widehat{X}_x^n(s))] \pi_n(dx) ds - \bar{\kappa}.$$

Using (3.32) once more we now have that $\langle \pi_n, f \rangle \leq \bar{\kappa}$. This completes the proof. ■

CHAPTER 4

Ergodic rate control problems for single class queueing networks

The study of control of stochastic processing queueing systems is of great current interest (cf. [37, 4, 1, 38, 29, 49, 32, 11, 40]). Barring special cases, control problems for stochastic networks are too complex to be analyzed directly and thus one is interested in approximations, that are more amenable to analysis. In this respect, controlled constrained diffusion processes which arise as scaling limits of critically loaded queueing networks are extremely useful. Since there are well studied numerical schemes for computing near optimal controls for controlled diffusions, it is of great interest to establish that under appropriate conditions, value functions and near optimal control for limiting diffusion control problem are good approximations for analogous quantities for physical networks that are in heavy traffic. The goal of this chapter is to study such questions for an important class of rate control problems for queueing networks.

Indeed, one of the primary means of control of flow in a network is by adjusting arrival or processing rates. Network performance under such control actions, particularly over long periods of time, is of interest. A natural mathematical formulation of such control issues is given in terms of an optimal stochastic control problem for stochastic networks with an ergodic cost criterion. A formal heavy traffic approximation of such a control problem leads to a class of drift control problems for reflected diffusions in positive orthants with an ergodic cost criterion.

The main result of the chapter is Theorem 4.1.5 which shows that the value

function (i.e., optimum value of the cost) of the rate control problem for the network converges, under a suitable heavy traffic scaling, to that of the limit ergodic control problem for reflected diffusions. Furthermore, we show in this theorem that near optimal policies for the network control problem can be well approximated by those in the limit diffusion control problem when the network is close to heavy traffic. Thus our results provide rigorous mathematical justification for the formal (controlled) diffusion approximation and suggest natural approaches for using the limit model in obtaining near (asymptotically) optimal rate control algorithms for a family of stochastic processing networks in heavy traffic.

Estimates that are developed for analyzing the controlled model can be used as well for studying the setting where the arrival and service rates are uncontrolled but depend on the current value of the state process. For this uncontrolled state dependent rate setting we show in Theorem 4.1.6 a result analogous to one established in Chapter 3 for generalized Jackson network with constant rates. More precisely, from result of [59] it is known that under appropriate heavy traffic conditions suitably scaled queue length processes, obtained from such network with state dependent rates, converge weakly to a reflecting diffusion with drift and diffusion coefficients that are state dependent. Also, sufficient conditions for existence of unique invariant distributions for such reflecting diffusions have been obtained in [2]. In Theorem 4.1.6, we show that under natural stability and heavy traffic conditions, the steady states of the sequence of suitably scaled queue length processes converge to the unique invariant distribution of the limit reflecting diffusion.

This chapter is organized as follows. We begin, in Section 4.1, with problem formulation and assumptions. We consider (generalized) Jackson type networks under fixed routing in which arrival and service rates are state (i.e., queue length) dependent. Additionally, we allow dynamic service rate control in the system (see (4.1)). We introduce the set of admissible service controls, the ergodic cost criterion of in-

terest and definition of value function for the queueing system. We then describe the formal controlled diffusion approximation to this queueing system that arises in the heavy traffic limit. We introduce an analogous cost criterion and value function for this limit model. Main results of the chapter are as follows. In Theorem 4.1.2 we show that the value function of the diffusion control problem is a lower bound for the asymptotic value function for the physical network control problem, in the heavy traffic limit. For the upper bound, we restrict to a setting where the diffusion coefficient in the limit model is constant. The main reason for this restrictive condition is that a key ingredient to our proof of the upper bound is Theorem 4.1.4, which we are currently only able to establish for the setting of constant diffusion coefficients. This theorem says that for every $\epsilon > 0$, there exists a continuous ϵ -optimal Markov control. The proof of this result relies on certain transition probability density estimates for reflected diffusions which are only known to us for the setting of a constant diffusion coefficient. We use Theorem 4.1.4 to show in Theorem 4.1.5 that an ϵ -optimal continuous Markov control for the limit model can be used to construct a sequence of admissible rate controls for physical networks and the costs associated with this sequence of controls converge to that for the ϵ -optimal Markov control in the limit control problem. As an immediate consequence of this result we get the main result of the chapter, namely Theorem 4.1.5, showing the convergence of value functions of physical network to that of the limit diffusion control problem. Proofs of Theorems 4.1.2 and 4.1.3 are given in Section 4.4 while the proof of Theorem 4.1.4 is in Section 4.5. The related proofs of convergence of invariant distributions in the uncontrolled setting is treated in Section 4.3. Section 4.2 presents some key stability and tightness results that are needed in other proofs.

Some of the notation used in this chapter is as follows. For a metric space X , let $\mathcal{P}(X)$ denote the collection of all probability measures on X . When $\sup_{0 \leq s \leq t} |f_n(s) - f(s)| \rightarrow 0$ as $n \rightarrow \infty$, for all $t \geq 0$, we say that $f_n \rightarrow f$ uniformly on compact sets.

For real continuous bounded function f on X , define $\|f\|_\infty \doteq \sup_{x \in X} |f(x)|$. We will denote generic constants by c_1, c_2, \dots , and their values may change from one proof to another.

4.1 Problem formulation and main results

Network structure. The network description is very similar to that in Chapter 3, however, for the sake of keeping the presentation self contained, we repeat this description below. Consider a sequence of networks indexed by n . Each network has a similar structure, in particular there are K service stations, with the i^{th} station denoted by P_i , $i \in \mathcal{K} \doteq \{1, \dots, K\}$. We assume that each station has an infinite capacity buffer. Arrivals of jobs can be from outside the system and/or from internal routing. Upon completion of service at station P_i a customer is routed to other service stations (or exits the system) according to a probabilistic routing matrix $\mathbb{P} = (p_{ij})_{i,j \in \mathcal{K}}$. At every station the jobs are assumed to be processed by First-In-First-Out discipline. We assume that the network is open, that is, any customer entering the network eventually leaves it. More precisely, we require that spectral radius of \mathbb{P} is strictly less than 1 and $p_{ii} = 0$ for all $i \in \mathcal{K}$. We allow arrival and service rates to be time varying random processes. They may be described as deterministic functions of the state processes or, more generally, given as nonanticipative control processes.

A more precise description of the model is as follows. Let λ_i^n, μ_i^n , $n \geq 1, i \in \mathcal{K}$ be measurable functions from $\mathbb{R}_+^K \rightarrow \mathbb{R}_+$. These functions will correspond to state dependent arrival and service rates. External arrivals are assumed to occur any for $i \in \mathcal{K}_e$, where \mathcal{K}_e is a subset of \mathcal{K} . Thus $\lambda_i^n = 0$ for all $n \geq 1, x \in S \doteq \mathbb{R}_+^K$ and $i \in \mathcal{K} \setminus \mathcal{K}_e$. Let $R \doteq [\mathbb{I} - \mathbb{P}']$ and set $a^n(x) \doteq \lambda^n(x) - R\mu^n(x)$. We define $\hat{a}^n \doteq \frac{1}{\sqrt{n}}a^n$. The following are our standing assumptions.

Assumption 4.1.1.

- (i) For each $n \in \mathbb{N}$, $\lambda^n, \mu^n \in C(S)$.
- (ii) For some $\kappa_1 \in (0, \infty)$, $|\lambda^n(x)| \leq n\kappa_1$, $|\mu^n(x)| \leq n\kappa_1$ for all $n \geq 1$ and $x \in S$.
- (iii) There exists a constant $\kappa_2 \in (0, \infty)$ such that $\sup_x |\hat{a}^n(x)| \leq \kappa_2$.
- (iv) There exists $a \in C_b(S)$ such that $\frac{a^n(\sqrt{n}x)}{\sqrt{n}} \rightarrow a(x)$ uniformly on compact sets as $n \rightarrow \infty$.
- (v) There exist Lipschitz functions $\lambda, \mu \in C_b(\mathbb{R}^K)$, such that $\frac{\lambda^n(\sqrt{n}x)}{n} \rightarrow \lambda(x)$, $\frac{\mu^n(\sqrt{n}x)}{n} \rightarrow \mu(x)$ uniformly on compact sets as $n \rightarrow \infty$. Furthermore, $\lambda = R\mu$.
- (vi) For each $i \in \mathbb{K}_e^c$, there exists $j \in \mathbb{K}_e$ such that $p_{ji} > 0$.
- (vii) $\inf_{i \in \mathbb{K}_e} \inf_{x \in S} \inf_n \frac{\lambda_i^n(\sqrt{n}x)}{n} > 0$, $\inf_{i \in \mathbb{K}} \inf_{x \in S} \inf_n \frac{\mu_i^n(\sqrt{n}x)}{n} > 0$.

We now introduce the set of controls: Fix $\delta_0, M \in (0, \infty)$. Let

$$\bar{\alpha} \doteq \sup_{x \in S, i \in \mathbb{K}} [R^{-1}a(x)]_i.$$

Define $\Lambda \doteq \{u \in \mathbb{R}^K : u \geq [\delta_0 + \bar{\alpha}]\mathbf{1}, |u| \leq M\}$. Let $\Lambda_n \doteq \{u \in \mathbb{R}^K : \frac{u}{\sqrt{n}} \in \Lambda\}$. A control for the n^{th} network will be a stochastic process with values in Λ_n as we now describe. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ be a filtered probability space on which are given unit rate independent Poisson processes, N_i, N_{ij} , $i \in \mathbb{K}, j \in \mathbb{K} \cup \{0\}$, which are $\{\mathcal{F}_t\}$ adapted and are such that $(N_i(t) - N_i(s), N_{ij}(t) - N_{ij}(s); i, j)$ is independent of $\{\mathcal{F}_s\}$ for $0 \leq s < t < \infty$. For $i \in \mathbb{K}$, N_i will be used to define the stream of jobs entering the i^{th} buffer and for $i, j \in \mathbb{K}$, N_{ij} will be used to represent the flow of jobs to buffer j from buffer i . For $i \in \mathbb{K}$ and $j = 0$, N_{ij} will be associated with jobs that leave the system after service at station P_i . Precise state evolution is described below.

Let $U^n = (U_1^n, U_2^n, \dots, U_K^n)'$ be a Λ_n valued $\{\mathcal{F}_t\}$ -progressively measurable process. U^n will represent the service rate control in the system. We will refer to such a process

as an admissible control and denote by \mathcal{A}_n the collection of all admissible controls. Under the control U^n , the state of the system $Q^n = (Q_1^n, Q_2^n, \dots, Q_K^n)'$ is given by the following equation.

$$\begin{aligned} Q_i^n(t) &= Q_i^n(0) + N_i \left(\int_0^t \lambda_i^n(Q^n(s)) ds \right) \\ &\quad + \sum_{j=1}^K N_{ji} \left(p_{ji} \int_0^t 1_{\{Q_j^n(s) > 0\}} [\mu_j^n(Q^n(s)) + U_j^n(s)] ds \right) \\ &\quad - \sum_{j=0}^K N_{ij} \left(p_{ij} \int_0^t 1_{\{Q_i^n(s) > 0\}} [\mu_i^n(Q^n(s)) + U_i^n(s)] ds \right), \quad i \in \mathbb{K}, \end{aligned} \quad (4.1)$$

where Q^n represents the queue length vector process obtained under the (rate) control process U^n . Next define martingales

$$\begin{aligned} M_{i0}^n(t) &\doteq N_i \left(\int_0^t \lambda_i^n(Q^n(s)) ds \right) - \int_0^t \lambda_i^n(Q^n(s)) ds, \\ M_{ij}^n(t) &\doteq N_{ij} \left(p_{ij} \int_0^t [\mu_i^n(Q^n(s)) + U_i^n(s)] ds \right) - p_{ij} \int_0^t [\mu_i^n(Q^n(s)) + U_i^n(s)] ds. \end{aligned}$$

Letting $M_i^n \doteq M_{i0}^n + \sum_{j=1}^K M_{ji}^n - \sum_{j=0}^K M_{ij}^n$, we can rewrite the evolution (4.1) as

$$\begin{aligned} Q_i^n(t) &= Q_i^n(0) + \int_0^t \left[\lambda_i^n(Q^n(s)) + \sum_{j=1}^K p_{ji} \mu_j^n(Q^n(s)) - \mu_i^n(Q^n(s)) \right] ds \\ &\quad + \int_0^t \left[\sum_{j=1}^K p_{ji} U_j(s) - U_i(s) \right] ds + M_i^n(t) + RY_i^n(t), \end{aligned} \quad (4.2)$$

where

$$Y_i^n(t) = \sum_{j=0}^K N_{ij} \left(p_{ij} \int_0^t 1_{\{Q_i^n(s) = 0\}} [\mu_i^n(Q^n(s)) + U_i^n(s)] ds \right), \quad i \in \mathbb{K}.$$

Note that Y_i^n is an RCLL nondecreasing $\{\mathcal{F}_t\}$ adapted process and Y_i^n increases only when $Q_i^n(t) = 0$, i.e., $\int_0^\infty 1_{\{Q_i^n(t) \neq 0\}} dY_i^n(t) = 0$ a.s.

Thus the state evolution can be described succinctly by the following equation:

$$Q^n(t) = Q^n(0) + \int_0^t a^n(Q^n(s))ds - \int_0^t RU^n(s)ds + M^n(t) + RY^n(t). \quad (4.3)$$

The above dynamics can equivalently be described in terms of a Skorohod map as described below. Recall the definition of Skorohod problem and Skorohod map given in Chapter 2 (Definition 2.3.1). In the current chapter $N = d = K$, $S = \mathbb{R}_+^K$, $S_i = \{x \in \mathbb{R}_+^K : x_i \geq 0\}$. Also recall Proposition 3.1.4 from Chapter 3 that gives the well-posedness of the Skorohod problem and regularity of the Skorohod map for the setting where $R = [\mathbb{I} - \mathbb{P}']$ with \mathbb{P} as in the current chapter.

The dynamics in (4.3) can be equivalently described in terms of the SM as follows:

$$Q^n = \Gamma(\tilde{Q}^n), \quad Q^n - \tilde{Q}^n = RY^n, \quad (4.4)$$

where $\tilde{Q}^n(t) = \tilde{Q}^n(0) + \int_0^t a^n(Q^n(s))ds - \int_0^t RU^n(s)ds + M^n(t)$.

For asymptotic analysis we will consider processes under the diffusion scaling. Given a stochastic process X^n , we will denote by \hat{X}^n the process defined as $\hat{X}^n(t) \doteq X^n(t)/\sqrt{n}$, $t \geq 0$ and refer to it as the diffusion scaled form of X^n . With this notation, we have from (4.4) that

$$\hat{Q}^n(t) = \Gamma \left(\hat{Q}^n(0) + \int_0^t \tilde{a}^n(\sqrt{n}\hat{Q}^n(s))ds - \int_0^t R\hat{U}^n(s)ds + \hat{M}^n(\cdot) \right) (t). \quad (4.5)$$

When $\hat{Q}^n(0) \equiv x$, we will sometimes write the corresponding scaled state process as \hat{Q}_x^n .

In this chapter we are concerned with an ergodic cost problem associated with the sequence of controlled queueing systems $\{Q^n\}_{n \geq 1}$. Let $k(\cdot)$ be a continuous and bounded function from S to \mathbb{R} and $c \in \mathbb{R}^K$ be fixed. Define for each $n \in \mathbb{N}$, $x \in S$,

$T \in [0, \infty)$, the ‘average cost’ on time interval $[0, T]$ for the control $U^n \in \mathcal{A}_n$ by

$$J_T^n(U^n, x) \doteq \frac{1}{T} \mathbb{E} \int_0^T \left[k(\widehat{Q}_x^n(s)) + c \cdot \widehat{U}^n(s) \right] ds.$$

Asymptotic average cost is defined as $J^n(U^n, x) \doteq \limsup_{T \rightarrow \infty} J_T^n(U^n, x)$. Define the value function V^n for the n^{th} queueing system as

$$V^n(x) \doteq \inf_{U^n \in \mathcal{A}_n} J^n(U^n, x). \quad (4.6)$$

The main goal of this chapter is to show that the value function and near optimal control policies in the n^{th} network, for large n , can be approximated by the same quantities in an associated diffusion control problem. Towards this goal, we now introduce the associated diffusion control problem.

Define for $x \in S$, a $[K \times [K + K(K + 1)]]$ -dimensional matrix $\Sigma(x)$ as

$$\Sigma(x) \doteq (A(x), B_1(x), \dots, B_K(x)), \quad (4.7)$$

where A and $B_i, i \in \mathbb{K}$ are $K \times K$ and $K \times (K + 1)$ matrices given as follows. For $x = (x_1, \dots, x_K)'$,

$$A(x) = \text{diag}(\sqrt{\lambda_1(x)}, \dots, \sqrt{\lambda_K(x)}), \quad B_i(x) = (B_i^0(x), B_i^1(x), \dots, B_i^K(x))',$$

where $B_i^0(x) \doteq -\sqrt{\mu_i(x)}1_i$ and for $j \in \mathbb{K}, j \neq i, B_i^j(x) \doteq 1_{ij}\sqrt{p_{ij}\mu_i(x)}$. Here 1_i is a K -dimensional vector with 1 at the i^{th} coordinate and 0 elsewhere; 1_{ij} is a K -dimensional vector with -1 at the i^{th} ; 1 at the j^{th} coordinates; and zeroes elsewhere. Finally, $B_i^i(x) \doteq 0$. It is easy to see (cf. proof of Proposition 1 in [59]) that due to Assumption 4.1.1 (vi) and (vii), $\Sigma(x)\Sigma(x)'$ is uniformly nondegenerate. I.e. there exists a $\kappa \in (0, \infty)$ such that for all $v \in \mathbb{R}^K, v'(\Sigma(x)\Sigma(x)')v \geq \kappa v'v$ for all $x \in$

S . One can then find a Lipschitz $K \times K$ matrix valued function $\sigma(x)$ such that $\Sigma(x)\Sigma(x)' = \sigma(x)\sigma(x)'$. (See Theorem 5.2.2 in [56].) Note that $\sigma\sigma'$ is also uniformly nondegenerate.

Let B be a K -dimensional standard Brownian motion given on some filtered probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}}, \{\widehat{\mathcal{F}}_t\})$. We will denote the set $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}}, \{\widehat{\mathcal{F}}_t\}, B)$ by Ξ and refer to it as a system. Recall $\Lambda \doteq \{u \in \mathbb{R}^K : u \geq [\delta_0 + \bar{\alpha}]\mathbf{1}, |u| \leq M\}$ and denote by $\mathcal{A}(\Xi)$ the collection of all $\{\widehat{\mathcal{F}}_t\}$ adapted Λ -valued processes. We will refer to such processes as admissible controls (for the diffusion control problem). For $U \in \mathcal{A}(\Xi)$ and $x \in S$, let $Z \equiv Z^{x,U}$ be the unique solution of

$$Z(t) = \Gamma \left(x + \int_0^\cdot a(Z_s)ds - \int_0^\cdot RU(s)ds + \int_0^\cdot \sigma(Z_s)dB_s \right) (t). \quad (4.8)$$

Ergodic cost for this limit diffusion model is given as follows.

$$J(U, x) \doteq \limsup_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \int_0^T [k(Z(s)) + c \cdot U(s)] ds.$$

The value is defined as

$$V \doteq \inf_{\Xi} \inf_{U \in \mathcal{A}(\Xi)} J(U, x). \quad (4.9)$$

From Theorem 3.4 of [7] it follows that the infimum on the right side of (4.9) does not depend on $x \in S$.

Our main results can now be stated as follows.

Theorem 4.1.2. *For every bounded sequence $\{x_n\} \subseteq S$, $\liminf_n V_n(x_n) \geq V$.*

For $b : S \rightarrow \Lambda$, let U_b be an admissible control for the limit diffusion model, given on some system Ξ as:

$$U_b(t) = b(Z_t), \quad Z_t = \Gamma \left(x - \int_0^\cdot [Rb(Z_s) - a(Z_s)]ds + \int_0^\cdot \sigma(Z_s)dB_s \right) (t), \quad t \geq 0.$$

Define $U_b^n \in \mathcal{A}_n$ as $U_b^n(s) \doteq \sqrt{nb}(\widehat{Q}_x^n(s))$, where \widehat{Q}_x^n is given by (4.5) with U^n on the right side of (4.5) replaced by U_b^n . Note that we have suppressed the dependence of U_b^n and U_b on x in the notation.

Theorem 4.1.3. *Let $\tilde{b} : S \rightarrow \Lambda$ be continuous. Let $\{x_n\} \subseteq S$ be such that $x_n \rightarrow x$ as $n \rightarrow \infty$. Then $J^n(U_{\tilde{b}}^n, x_n) \rightarrow J(U_{\tilde{b}}, x)$ as $n \rightarrow \infty$.*

Theorem 4.1.4. *Suppose that for all $x \in S$, $\sigma(x) \equiv \sigma$. For each $\epsilon > 0$, there exists a continuous $b : S \rightarrow \Lambda$ such that for all $x \in S$, $J(U_b, x) \leq V + \epsilon$.*

As an immediate consequence of Theorems 4.1.2, 4.1.3 and 4.1.4 we get the following main result of the chapter.

Theorem 4.1.5. *Suppose that for all $x \in S$, $\sigma(x) \equiv \sigma$. Let $\{x_n\} \subseteq S$ be a bounded sequence. Then as $n \rightarrow \infty$, $V_n(x_n) \rightarrow V$. Also, for every $\epsilon > 0$, there exists a continuous $b : S \rightarrow \Lambda$ such that*

$$0 \leq \limsup_{n \rightarrow \infty} [J^n(U_b^n, x_n) - V_n(x_n)] \leq \epsilon.$$

Proofs of Theorems 4.1.2 and 4.1.3 are given in Section 4.4, while Theorem 4.1.4 is proved in Section 4.5. As another consequence of estimates used in the proofs of Theorems 4.1.2-4.1.4, we obtain the following result on convergence of steady states of the Markov process \widehat{Q}^n in the uncontrolled setting (i.e., when \widehat{U}^n on right side of (4.5) equals zero).

Theorem 4.1.6. *Suppose that $0 \in \Lambda_n$. Let \widehat{Q}^n be given by (4.5) with $\widehat{U}^n \equiv 0$. Then the Markov process \widehat{Q}^n admits a stationary probability distribution π_n . Furthermore, Markov process Z given by (4.8) with $U \equiv 0$ admits a unique stationary probability distribution π . Finally, if π_n is an arbitrary stationary law for \widehat{Q}^n , $n \geq 1$ then $\pi_n \Rightarrow \pi$ as $n \rightarrow \infty$.*

Proof of Theorem 4.1.6 is given in Section 4.3.

4.2 Some stability results

The following stability results are key ingredients in the proofs.

Proposition 4.2.1. *Let \widehat{Q}_x^n be defined by (4.5) with $\widehat{Q}_x^n(0) \equiv x \in S$ and $U^n \in \mathcal{A}_n$.*

Then there exists a $t_0 \in (0, \infty)$ such that for all $t \geq t_0$,

$$\lim_{|x| \rightarrow \infty} \sup_n \sup_{U^n \in \mathcal{A}_n} \frac{1}{|x|^2} \mathbb{E} \left(|\widehat{Q}_x^n(t|x)|^2 \right) = 0.$$

Proof. Fix $x \in S$ and $U^n \in \mathcal{A}_n$. Write (4.5) as, $\widehat{Q}_x^n(t) \equiv \Gamma \left(x + r^n(\cdot) + \widehat{M}^n(\cdot) \right) (t)$,

where

$$r^n(t) \doteq \int_0^t \widehat{a}^n(\sqrt{n}\widehat{Q}^n(s)) ds - \int_0^t R\widehat{U}^n(s) ds \equiv \int_0^t \bar{b}^n(s) ds.$$

Define $Z_x^n(t) \doteq \Gamma(x + r^n(\cdot))(t)$, $t \geq 0$. Using the Lipschitz property of the Skorohod map (Proposition 3.1.4), we have

$$|\widehat{Q}_x^n(t) - Z_x^n(t)| \leq L \sup_{0 \leq s \leq t} |\widehat{M}^n(s)|. \quad (4.10)$$

Recall the cone $\mathcal{C} \doteq \{v \in \mathbb{R}^K : R^{-1}v \leq 0\}$, introduced in Section 2.3. From below Assumption 4.1.1, we see that there exists a $\delta \in (0, \infty)$ satisfying

$$\inf_n \inf_s \text{dist}(\bar{b}^n(s), \partial\mathcal{C}) \geq \delta. \quad (4.11)$$

Thus for all $n \geq 1$ and $s \geq 0$, $\bar{b}^n(s) \in \mathcal{C}_\delta \doteq \{v \in \mathcal{C} : \text{dist}(v, \partial\mathcal{C}) \geq \delta\}$.

Combining this observation with (3.23) we now have that $Z_x^n(t) = 0$, for all $t \geq \bar{D}|x|$, where $\bar{D} \doteq \frac{4L^2}{\delta}$. Using this in (4.10) we now see that

$$|\widehat{Q}_x^n(t|x)| \leq L \sup_{0 \leq s \leq t|x|} |\widehat{M}^n(s)|, \quad (4.12)$$

for all $t \geq \bar{D}$ and for all initial conditions x .

Next we obtain an estimate on the second moment of the right side of (4.12). Noting that \widehat{M}_i^n is an $\{\mathcal{F}_t\}_{t \geq 0}$ square integrable martingale, one gets using Doob's inequality that

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq s \leq t} |\widehat{M}_i^n(s)|^2 &\leq 4\mathbb{E}|\widehat{M}_i^n(t)|^2 \\
&\leq \frac{c_1}{n} \mathbb{E} \left[\int_0^t \left[\lambda_i^n(Q^n(s)) + \sum_{j=1}^K \mu_j^n(Q^n(s)) + U_i^n(s) \right] ds \right] \\
&\leq c_2(1+t),
\end{aligned} \tag{4.13}$$

where the last inequality follows from Assumption 4.1.1 (ii). (Here $c_1, c_2 \in (0, \infty)$ are independent of n .) Applying this estimate in (4.12) we now have that for all $t \geq \bar{D}$ and $x \in S$,

$$\mathbb{E}|\widehat{Q}_x^n(t|x)|^2 \leq c_3(1+t|x|). \tag{4.14}$$

The result now follows on choosing $t_0 = \bar{D}$. ■

The proof of the following result is analogous to that of Lemma 4.4 of [2]. For completeness, we give a sketch in the Appendix. For $M \in (0, \infty)$, let $S_M \doteq \{x \in S : |x| \leq M\}$.

Proposition 4.2.2. *Let \widehat{Q}_x^n be defined by (4.5) with $\widehat{Q}^n(0) \equiv x \in S$ and $U^n \in \mathcal{A}_n$. Then for every $M \in (0, \infty)$, the collection $\{\widehat{Q}_x^n(t) : n \geq 1, t > 0, x \in S_M\}$ is a tight family of random variables.*

4.3 Uncontrolled case: Convergence of invariant distributions

This section will be concerned with the uncontrolled setting, namely, the case where $U^n \equiv 0$. Throughout this section we will make the assumption that $0 \in \Lambda$ and that \widehat{Q}^n is given by (4.5) with $U^n = 0$. It is well known that \widehat{Q}^n is a strong Markov process

with state space S . Furthermore, the estimate in Proposition 4.2.2 shows that for each $n \geq 1$ and $x \in S$, $\{\widehat{Q}_x^n(t) : t \geq 0\}$ forms a tight family of random variables. In particular, Markov process \widehat{Q}^n admits an invariant probability distribution. We denote such a distribution by π_n . Note that in general additional conditions are needed in order to assert uniqueness of π_n . Next we present a heavy traffic limit theorem from [59] for the sequence of stochastic processes $\{\widehat{Q}^n\}_{n \geq 1}$ with paths in $D([0, \infty), S)$.

Given $\mu \in \mathcal{P}(S)$, let Z_μ be a continuous $\{\overline{\mathcal{F}}_t\}$ adapted process given on some filtered probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \{\overline{\mathcal{F}}_t\}, \overline{\mathbb{P}})$, supporting a K -dimensional standard Brownian motion \overline{B} , such that Z_μ solves

$$Z_\mu(t) = \Gamma \left(Z_\mu(0) + \int_0^t a(Z_\mu(s)) ds + \int_0^t \sigma(Z_\mu(s)) d\overline{B}_s \right) (t), \quad \overline{\mathbb{P}} \circ Z_\mu^{-1}(0) = \mu.$$

From the uniform nondegeneracy and Lipschitz continuity of σ , we have that such a process is unique in law. The following Theorem follows from Theorem 1 of [59].

Theorem 4.3.1. *Assume that $\widehat{Q}^n(0)$ converges in distribution to some $\mu \in \mathcal{P}(S)$. Then \widehat{Q}^n converges weakly in $D([0, \infty), S)$ to Z_μ .*

Remark on the Proof. The proof largely follows that of Theorem 1 of [59]. We note that Assumption 4.1.1 (iv) made here is stronger than assumption (A2) of [59]. The latter weaker condition requires an additional local time argument in the proof which can be avoided in our setting. Also, the limit process in [59] is expressed in terms of a $[K + K(K + 1)]$ dimensional Brownian motion and the diffusion coefficient matrix Σ . However, noting that $\Sigma\Sigma' = \sigma\sigma'$, standard martingale representation arguments show that by suitably augmenting the probability space one can describe the limit using a K dimensional Brownian motion and with the different coefficient matrix σ .

The collection $\{Z_\mu : \mu \in \mathcal{P}(S)\}$ describes a strong Markov process, stability properties of which have been studied in [2]. In particular, note that since $0 \in \Lambda_n$, we

have from Assumption 4.1.1 (iv) that $R^{-1}a(x) \leq -\delta$ for all $x \in S$ and $\delta \in (0, \infty)$ as in (4.11). Therefore, using Theorem 2.2 of [2] one has that the above Markov process has a unique invariant probability measure. Henceforth, this probability measure will be denoted by π . We remark here that Theorem 2.2 of [2] was stated under an additional Lipschitz assumption on the drift vector a . However, an examination of the proof shows that this assumption is not needed for the proof. In fact, due to the uniform nondegeneracy of σ , the result continues to hold if a is merely bounded and measurable.

We now present the proof of Theorem 4.1.6 in Section 4.1. Recall that π_n is a stationary distribution for \widehat{Q}^n (not necessarily unique). The proof is based on the following propositions, the proofs of which, being very similar to the proof of Theorem 3.5 of [10], are given in the Appendix. For $\bar{\delta} \in (0, \infty)$, define the return time to a compact set $C \subseteq S$ by $\tau_C^n(\bar{\delta}) \doteq \inf\{t \geq \bar{\delta} : \widehat{Q}^n(t) \in C\}$.

Proposition 4.3.2. *For $f : S \rightarrow \mathbb{R}_+$, $\bar{\delta} \in (0, \infty)$, and a compact set $C \subseteq S$, define*

$$G_n(x) \doteq \mathbb{E} \left[\int_0^{\tau_C^n(\bar{\delta})} f(\widehat{Q}_x^n(t)) dt \right], \quad x \in S.$$

If $\sup_n G_n$ is everywhere finite and uniformly bounded on C , then there exists a $\bar{\kappa} \in (0, \infty)$ such that for all $n \in \mathbb{N}$, $t > 0$, $x \in S$

$$\frac{1}{t} \mathbb{E}[G_n(\widehat{Q}_x^n(t))] + \frac{1}{t} \int_0^t \mathbb{E}[f(\widehat{Q}_x^n(s))] ds \leq \frac{1}{t} G_n(x) + \bar{\kappa}. \quad (4.15)$$

Proposition 4.3.3. *For some constants $c, \bar{\delta} \in (0, \infty)$, and a compact set $C \subseteq S$,*

$$\sup_n \mathbb{E} \left[\int_0^{\tau_C^n(\bar{\delta})} (1 + |\widehat{Q}_x^n(t)|) dt \right] \leq c(1 + |x|^2), \quad x \in S. \quad (4.16)$$

Proof of Theorem 4.1.6. The proof is similar to that of Theorems 3.2.1 and 3.2.2. To keep the presentation self contained we provide the details. Since π is the unique

invariant probability measure for $\{Z_\mu : \mu \in \mathcal{P}(S)\}$, we have from Theorem 4.3.1 that it suffices to establish the tightness of the family $\{\pi_n\}$. We apply Proposition 4.3.2 with $f(x) \doteq 1 + |x|$ for $x \in S$ and $\bar{\delta}$, C as in Proposition 4.3.3. To prove the desired tightness it suffices to show that for all $n \in \mathbb{N}$, $\langle \pi_n, f \rangle \doteq \int_S f(x) \pi_n(dx) \leq \bar{\kappa}$. Note that for any nonnegative, real measurable function ξ on S ,

$$\int_S \mathbb{E}[\xi(\widehat{Q}_x^n(t))] \pi_n(dx) = \langle \pi_n, \xi \rangle. \quad (4.17)$$

Fix $k \in \mathbb{N}$ and let $G_n^k(x) \doteq G_n(x) \wedge k$. Let

$$\Psi_n^k(x) \doteq \frac{1}{t} G_n^k(x) - \frac{1}{t} \mathbb{E}[G_n^k(\widehat{Q}_x^n(t))].$$

From (4.17), we have that $\int_S \Psi_n^k(x) \pi_n(dx) = 0$. Let $\Psi_n(x) \doteq \frac{1}{t} G_n(x) - \frac{1}{t} \mathbb{E}[G_n(\widehat{Q}_x^n(t))]$. Monotone convergence theorem yields that $\Psi_n^k(x) \rightarrow \Psi_n(x)$ as $k \rightarrow \infty$. Next we will show that Ψ_n^k is bounded from below uniformly in n and k . If $G_n(x) \leq k$,

$$\Psi_n^k(x) = \frac{1}{t} G_n^k(x) - \frac{1}{t} \mathbb{E}[G_n^k(\widehat{Q}_x^n(t))] \geq \frac{1}{t} G_n(x) - \frac{1}{t} \mathbb{E}[G_n(\widehat{Q}_x^n(t))] \geq -\bar{\kappa},$$

where the last inequality follows from (4.15). On the other hand, if $V_n(x) \geq k$

$$\Psi_n^k(x) = \frac{1}{t} k - \frac{1}{t} \mathbb{E}[G_n^k(\widehat{Q}_x^n(t))] \geq 0, \quad (4.18)$$

where the second inequality follows on noting that $G_n^k \leq k$. Hence $\Psi_n^k(x) \geq -\bar{\kappa}$ for all $x \in S$. By an application of Fatou's Lemma we conclude that

$$\int_S \Psi_n(x) \pi_n(dx) \leq \liminf_{k \rightarrow \infty} \int_S \Psi_n^k(x) \pi_n(dx) = 0. \quad (4.19)$$

From (4.15), $\Psi_n(x) \geq \frac{1}{t} \int_0^t \mathbb{E}[f(\widehat{Q}_x^n(s))] ds - \bar{\kappa}$. Combining this with (4.19) and inte-

grability with respect to π_n , we have

$$0 \geq \int_S \Psi_n(x) \pi_n(dx) \geq \frac{1}{t} \int_0^t \int_S \mathbb{E}[f(\widehat{Q}_x^n(s))] \pi_n(dx) ds - \bar{\kappa}.$$

Using (4.17) once more we now have that $\langle \pi_n, f \rangle \leq \bar{\kappa}$. This completes the proof. \blacksquare

4.4 Controlled case: Convergence of value functions

In this section we return to the control problem introduced in Section 4.1 and present the proofs of Theorems 4.1.2 and 4.1.3. The proofs rely on the functional occupation measure approach developed in [39]. We begin with some notation and definitions.

For a Polish space E , let E_{path} be the space of $D([0, \infty), E)$ valued stochastic processes. For an E -valued stochastic process $\{z(t) : t \geq 0\}$ with paths in $D([0, \infty), E)$, define, for $t \geq 0$, stochastic processes $\{z_p(t)(s) : s \geq 0\}$, $\{\Delta z_p(t)(s) : s \geq 0\}$ with paths in $D([0, \infty), E_{\text{path}})$ as

$$z_p(t, \omega)(s) \doteq z(t + s, \omega), \quad \Delta z_p(t, \omega)(s) \doteq z(t + s, \omega) - z(t, \omega), \quad s, t \geq 0.$$

We rewrite (4.5) as follows

$$\begin{aligned} \widehat{Q}^n(t) &= \widehat{Q}^n(0) + \int_0^t \widehat{a}^n(\sqrt{n}\widehat{Q}^n(s)) ds - \int_0^t R\widehat{U}^n(s) ds + \widehat{M}^n(t) + R\widehat{Y}^n(t) \\ &\equiv \widehat{Q}^n(0) + \widehat{A}^n(t) - R\widehat{C}^n(t) + \widehat{M}^n(t) + R\widehat{Y}^n(t). \end{aligned} \quad (4.20)$$

Suppose that $\widehat{Q}^n(0) = x_n$ for $n \geq 1$ and $\sup_n |x_n| \leq M$. For $t \geq 0$, let $\{\widehat{H}_p^n(t)\}$ be a stochastic process with paths in $D([0, \infty), \mathbb{S}_{\text{path}})$ defined as

$$\widehat{H}_p^n(t) \doteq (\widehat{Q}_p^n(t), \Delta \widehat{A}_p^n(t), \Delta \widehat{C}_p^n(t), \Delta \widehat{M}_p^n(t), \Delta \widehat{Y}_p^n(t)),$$

where $\mathbb{S} \doteq (S \times \mathbb{R}^K \times \mathbb{R}^K \times \mathbb{R}^K \times \mathbb{R}^K)$. Note that we have suppressed the dependence of the processes on x_n in our notation. Then for any $t, s \geq 0$, (4.20) can be rewritten as

$$\widehat{Q}_p^n(t)(s) = \widehat{Q}^n(t+s) = \Gamma \left(\widehat{Q}_p^n(t) + \Delta \widehat{A}_p^n(t) - R \Delta \widehat{C}_p^n(t) + \Delta \widehat{M}_p^n(t) \right) (s). \quad (4.21)$$

Recall, we say a family of random variables is tight, if the corresponding collection of laws (i.e., induced measures) is tight.

Proposition 4.4.1. *Let $(n_\ell, T_\ell)_{\ell=1}^\infty$ be a sequence such that $n_\ell \rightarrow \infty$, $T_\ell \rightarrow \infty$ as $\ell \rightarrow \infty$. Let $\{\mathcal{Q}_\ell : \ell \geq 1\}$ be a sequence of $\mathcal{P}(\mathbb{S}_{\text{path}})$ valued random variables defined as*

$$\mathcal{Q}_\ell(F) \doteq \frac{1}{T_\ell} \int_0^{T_\ell} 1_F(\widehat{H}_p^{n_\ell}(s)) ds,$$

where $F \in \mathcal{B}(\mathbb{S}_{\text{path}})$. Then $\{\mathcal{Q}_\ell : \ell \geq 1\}$ is a tight family of random variables.

Proof of the proposition is based on the following well known result (see, e.g., Theorem 5.4 in [39]).

Lemma 4.4.2. *Suppose that the sequence $\{\mathbb{E} \mathcal{Q}_n\}_{n \geq 1}$ is tight family of probability measures on \mathbb{S}_{path} . Then $\{\mathcal{Q}_n : n \geq 1\}$ is a tight family of random variables.*

Proof of Proposition 4.4.1. From Lemma 4.4.2, it suffices to show that the collection

$$\{\widehat{H}_p^{n_\ell}(t) : \ell \geq 1, t \geq 0\} \quad (4.22)$$

is a tight family of \mathbb{S}_{path} valued random variables. Tightness of $\{(\Delta \widehat{A}_p^{n_\ell}(t), \Delta \widehat{C}_p^{n_\ell}(t)) : \ell \geq 1, t \geq 0\}$ is immediate on recalling the uniform boundedness of \widehat{a}^n and the compactness of Λ . Next we argue the tightness of

$$\{\Delta \widehat{M}_p^{n_\ell}(t) : \ell \geq 1, t \geq 0\}. \quad (4.23)$$

Note that for $\ell \geq 1, t \geq 0$, $\Delta \widehat{M}_p^{n\ell}(t)$ is a martingale (with respect to its own filtration). Furthermore, if τ is a bounded stopping time (with respect to this filtration), we have that for $\beta > 0$,

$$\mathbb{E}|\Delta \widehat{M}_p^{n\ell}(t)(\tau + \beta) - \Delta \widehat{M}_p^{n\ell}(t)(\tau)|^2 \leq c_1 \beta, \quad (4.24)$$

for some $c_1 \in (0, \infty)$. The constant c_1 can be chosen to be uniform in ℓ and t . (It may depend on the upper bound on τ .) The estimate in (4.24) follows exactly as the one in (4.13) and so the proof is omitted. Using Aldous's criterion (see, e.g., Theorem 16.10 of [5]) we now have the tightness of the family in (4.23). Next from Proposition 4.2.2 we have that

$$\{\widehat{Q}^{n\ell}(t) : \ell \geq 1, t \geq 0\} \text{ is tight.} \quad (4.25)$$

The tightness of $\{\widehat{Q}_p^{n\ell}(t) : \ell \geq 1, t \geq 0\}$ now follows on combining the above with (4.21) and recalling Lipschitz property of Γ given in Proposition 3.1.4. Similarly, using the Lipschitz property of Γ_1 , one has the tightness of $\{\widehat{Y}_p^{n\ell}(t) : \ell \geq 1, t \geq 0\}$. This proves the tightness of (4.22) and the result follows. ■

Suppose $\{\mathcal{Q}_\ell : \ell \geq 1\}$ along some subsequence $\{\ell_m\}_{m \geq 1}$ converges in distribution to $\widetilde{\mathcal{Q}}$ defined on some probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$. We will denote expectation under \mathbb{P}_0 by \mathbb{E}_0 , a generic element of Ω_0 by ω_0 and write $\widetilde{\mathcal{Q}}(\omega_0)$ as $\widetilde{\mathcal{Q}}^{\omega_0}$. Denote the canonical coordinate process on \mathbb{S}_{path} as $H^* = (Q^*, A^*, C^*, M^*, Y^*)$. With an abuse of notation, we will also denote a typical element of \mathbb{S}_{path} by the same symbol. The following theorem describes the law of H^* under $\widetilde{\mathcal{Q}}^{\omega_0}$ for \mathbb{P}_0 a.e. ω_0 .

Theorem 4.4.3.

(i) For \mathbb{P}_0 -almost every ω_0 , we have the following properties:

(1) For all $t \geq 0$,

$$Q^*(t) = \Gamma(Q^*(0) + A^*(\cdot) - RC^*(\cdot) + M^*(\cdot))(t), \quad \widetilde{\mathcal{Q}}^{\omega_0} \text{ a.s.} \quad (4.26)$$

(2) H^* is stationary under \tilde{Q}^{ω_0} in the following sense: The probability measure

$$(\tilde{Q}^{\omega_0})^{-1}(Q_p^*(t), \Delta A_p^*(t), \Delta C_p^*(t), \Delta M_p^*(t), \Delta Y_p^*(t))$$

is the same for every $t \geq 0$.

(3) Under \tilde{Q}^{ω_0} , M^* is a square integrable \mathcal{G}_t -martingale, where $\mathcal{G}_t \doteq \sigma\{H^*(s) : s \leq t\}$.

(4) $\langle M_t^*, M_t^* \rangle = \int_0^t [\sigma(Q^*(s))\sigma(Q^*(s))'] ds$, $t \geq 0$, a.s. \tilde{Q}^{ω_0} .

(5) $A^*(t) = \int_0^t a(Q^*(s)) ds$, for all $t \geq 0$, a.s. \tilde{Q}^{ω_0} .

(6) There is a \mathcal{G}_t progressively measurable process U^* with values in Λ such that $C^*(t) = \int_0^t U^*(s) ds$, for all $t \geq 0$, a.s. \tilde{Q}^{ω_0} .

(ii) Suppose that $0 \in \Lambda_n$ for every n and \widehat{Q}^n is defined by (4.20) with $U^n(s) = 0$ for all $n \geq 1$ and $s \geq 0$. Then conclusions of part (i) continue to hold with C^* in (4.26) replaced by 0.

The proof of the above result is similar to that of Theorem 6.3 of [39]. For completeness we give a sketch in the Appendix.

Proof of Theorem 4.1.2. In order to prove the result, it suffices to show that for every sequence (T_ℓ, n_ℓ) such that $T_\ell \rightarrow \infty, n_\ell \rightarrow \infty$ as $\ell \rightarrow \infty$ and arbitrary $U^{n_\ell} \in \mathcal{A}_{n_\ell}$,

$$\liminf_{\ell \rightarrow \infty} \frac{1}{T_\ell} \mathbb{E} \int_0^{T_\ell} [k(\widehat{Q}^{n_\ell}(s)) + c \cdot \widehat{U}^{n_\ell}(s)] ds \geq V. \quad (4.27)$$

Define $K : \mathbb{S}_{\text{path}} \rightarrow \mathbb{R}$ as $K(H^*) \doteq k(Q^*(0)) + c \cdot C^*(1)$. Then for $\ell \geq 1$,

$$\frac{1}{T_\ell} \int_0^{T_\ell} [k(\widehat{Q}_x^{n_\ell}(s)) + c \cdot \widehat{U}^{n_\ell}(s)] ds = \int_{\mathbb{S}_{\text{path}}} K(H^*) d\mathcal{Q}_\ell(H^*) + \delta_\ell,$$

where

$$\begin{aligned}\delta_\ell &= \frac{1}{T_\ell} \left[\int_0^1 c \cdot \widehat{U}^{n_\ell}(s)(1-s)ds - \int_{T_\ell}^{T_\ell+1} c \cdot \widehat{U}^{n_\ell}(s)(T_\ell - s + 1)ds \right] \\ &\leq \frac{c_1}{T_\ell} \rightarrow 0, \text{ as } \ell \rightarrow \infty.\end{aligned}$$

By a usual subsequential argument, we can assume without loss of generality that \mathcal{Q}_ℓ converges in distribution to $\widetilde{\mathcal{Q}}$ as given in Theorem 4.4.3. Thus

$$\mathbb{E} \liminf_{\ell \rightarrow \infty} \frac{1}{T_\ell} \int_0^{T_\ell} [k(\widehat{Q}^{n_\ell}(s)) + c \cdot \widehat{U}^{n_\ell}(s)]ds = \mathbb{E}_0 \int_{\mathbb{S}_{\text{path}}} K(H^*) d\widetilde{\mathcal{Q}}(H^*). \quad (4.28)$$

Using stationarity (Theorem 4.4.3 (i)(2)) and noting that $C^*(0) = 0$, the right side of (4.28) can be written as

$$\mathbb{E}_0 \left[\lim_{T \rightarrow \infty} \int \left(\frac{1}{T} \int_0^T \{k(Q^*(s)) + c \cdot [C^*(s+1) - C^*(s)]\} ds \right) d\widetilde{\mathcal{Q}}(H^*) \right].$$

Using Theorem 4.4.3 (i)(5) and recalling the boundedness of U^* , the last expression is the same as

$$\mathbb{E}_0 \left[\lim_{T \rightarrow \infty} \int \left(\frac{1}{T} \int_0^T \{k(Q^*(s)) + c \cdot U^*(s)\} ds \right) d\widetilde{\mathcal{Q}}(H^*) \right]. \quad (4.29)$$

By appealing to Martingale Representation Theorem (see e.g. [34], Proposition 6.2) one has, for a.e. ω_0 , by suitably augmenting the filtered probability space $(\mathbb{S}_{\text{path}}, \mathcal{B}(\mathbb{S}_{\text{path}}), \{\mathcal{G}_t\}, \widetilde{\mathcal{Q}}^{\omega_0})$, that $M_t^* = \int_0^t \sigma(Q^*(s))dW(s)$, $t \geq 0$, a.e. $\widetilde{\mathcal{Q}}^{\omega_0}$, where W is a standard K -dimensional Brownian motion on the augmented filtered probability space. Thus the expression inside the expectation operator in (4.29) represents the cost for some admissible control (and some initial condition) for the diffusion control problem of Section 4.1. Hence the expression in (4.29) can be bounded below by V . This proves (4.27) and Theorem 4.1.2 follows. ■

We now proceed to the proof of Theorem 4.1.3.

Proof of Theorem 4.1.3. We begin by noting that when U is replaced with $U_{\tilde{b}}$ in (4.8), the corresponding state process is a strong Markov process which admits a unique stationary probability distribution $\eta_{\tilde{b}}$. Furthermore, for all $x \in S$

$$J(U_{\tilde{b}}, x) = \int_S [k(y) + c \cdot \tilde{b}(y)] \eta_{\tilde{b}}(dy). \quad (4.30)$$

For proofs of these statements see Theorem 3.2 and Lemma 4.1 of [7]. Henceforth, we will denote $k + c \cdot \tilde{b}$ by $k_{\tilde{b}}$ and write the right side of (4.30) as $\langle \eta_{\tilde{b}}, k_{\tilde{b}} \rangle$. Let $(T_\ell, n_\ell)_{\ell \geq 1}$ be, as before, a sequence such that $T_\ell \rightarrow \infty$ and $n_\ell \rightarrow \infty$ as $\ell \rightarrow \infty$. In order to prove the theorem, it suffices to show that

$$\mathbb{E} \frac{1}{T_\ell} \int_0^{T_\ell} [k(\widehat{Q}^{n_\ell}(s)) + c \cdot \widehat{U}_{\tilde{b}}^{n_\ell}(s)] ds \rightarrow \langle \eta_{\tilde{b}}, k_{\tilde{b}} \rangle, \quad (4.31)$$

as $\ell \rightarrow \infty$, where \widehat{Q}^{n_ℓ} is defined as in (4.20) with \widehat{U}^{n_ℓ} there replaced by $\widehat{U}_{\tilde{b}}^{n_\ell}$. Note that the left side of (4.31) can be rewritten as

$$\mathbb{E} \int_{\mathbb{S}_{\text{path}}} k_{\tilde{b}}(Q^*(0)) d\mathcal{Q}_\ell(H^*). \quad (4.32)$$

From Proposition 4.4.1, we have that $\{\mathcal{Q}_\ell\}_{\ell \geq 1}$ is a tight family of random variables. Once more one can assume by a subsequential argument that \mathcal{Q}_ℓ converges to some $\tilde{\mathcal{Q}}$, defined on some probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$, that satisfies all the properties in Theorem 4.4.3. So the expression in (4.32) converges to

$$\mathbb{E}_0 \int_{\mathbb{S}_{\text{path}}} k_{\tilde{b}}(Q^*(0)) d\tilde{\mathcal{Q}}(H^*). \quad (4.33)$$

We will now apply part (ii) of Theorem 4.4.3 with (a^n, a) replaced by (a_1^n, a_1) , where $a_1^n(x) \doteq a^n(x) - \sqrt{n}R\tilde{b}(\frac{x}{\sqrt{n}})$, $a_1(x) \doteq a(x) - R\tilde{b}(x)$, $x \in S$ and Λ^n replaced by Λ_1^n ,

where Λ_1^n is defined analogously to Λ^n with $\bar{\alpha}$ replaced by $\bar{\alpha}_1 \doteq \sup_{x \in S, i \in \mathbb{K}} [R^{-1}a_1(x)]_i$. Since $\tilde{b}(x) \in \Lambda$ for all x , we have that $0 \in \Lambda_1^n$. Thus from part (ii) of Theorem 4.4.3, for \mathbb{P}_0 -almost every ω_0 , (1) to (6) in that theorem hold with a in (5) replaced by a_1 and U^* in (6) replaced by 0. Once more by suitably augmenting the filtered probability space $(\mathbb{S}_{\text{path}}, \mathcal{B}(\mathbb{S}_{\text{path}}), \{\mathcal{G}_t\}, \tilde{\mathcal{Q}}^{\omega_0})$ one has the representation

$$Q^*(t) = \Gamma \left(Q^*(0) + \int_0^t a_1(Q^*(s)) ds + \int_0^t \sigma(Q^*(s)) dW(s) \right) (t), \quad a.s. \tilde{\mathcal{Q}}^{\omega_0},$$

for some K -dimensional standard Brownian motion W . Recalling the stationarity property from part (2) of the Theorem 4.4.3 and the uniqueness property of the invariant measure $\eta_{\tilde{b}}$, we see that $Q^*(0)$ has law $\eta_{\tilde{b}}$ under $\tilde{\mathcal{Q}}^{\omega_0}$, for \mathbb{P}_0 -a.e. ω . Thus $\int_{\mathbb{S}_{\text{path}}} k_{\tilde{b}}(Q^*(0)) d\tilde{\mathcal{Q}}(H^*) = \langle \eta_{\tilde{b}}, k_{\tilde{b}} \rangle$ a.e. ω_0 . Using this observation in (4.33) we have (4.31) and thus the result follows. ■

4.5 Proof of Theorem 4.1.4

This section is devoted to the proof of the Theorem 4.1.4. Thus throughout this section we assume $\sigma(x) \equiv \sigma$. From Theorem 3.4 of [7] we have that there exists a measurable map $b_* : S \rightarrow \Lambda$ such that $V = \int_S [k(x) + c \cdot b_*(x)] \eta_{b_*}(dx)$. Recall that from Theorem 4.1 of [7], if $b : S \rightarrow \Lambda$ is a measurable map and U is replaced with U_b in (4.8), the corresponding state process is a strong Markov process that admits a unique stationary probability distribution which we denote as η_b . Furthermore, $J(U_b, x) = \langle \eta_b, k_b \rangle$ for all $x \in S$. Thus in order to prove the theorem it suffices to show that there is a sequence of continuous maps $b_n : S \rightarrow \Lambda$ such that $\eta_{b_n} \Rightarrow \eta_{b_*}$. We begin with the following lemma. Let $\gamma_g \in \mathcal{P}(S)$ be defined as $\gamma_g(A) \doteq c \int_A e^{-|x|^2/2} dx$, where $A \in \mathcal{B}(S)$ and c is the normalization constant.

Lemma 4.5.1. *For each $n \in \mathbb{N}$, there exist $b_n \in \Lambda$ and compact sets $A_n \subseteq S$ such*

that b_n is continuous,

$$\{x \in S : b_\star(x) \neq b_n(x)\} \subseteq A_n^c \quad \text{and} \quad \gamma_g(A_n^c) \leq \frac{1}{2^{n+1}}. \quad (4.34)$$

Proof. From Lusin's Theorem (see, e.g., Theorem 2.24 [53]) one can find a continuous function $\hat{b}_n : S \rightarrow \mathbb{R}^K$ such that (4.34) is satisfied with b_n replaced by \hat{b}_n . Note that Λ is a convex closed subset of \mathbb{R}^K . Let $\Pi_\Lambda : \mathbb{R}^K \rightarrow \Lambda$ be the projection map. Then Π_Λ is a Lipschitz function. The result follows on setting $b_n(x) \doteq \Pi_\Lambda(\hat{b}_n(x))$, $x \in S$. ■

Let $\{\mu_n\} \subseteq \mathcal{P}(S)$ be such that $\mu_n \Rightarrow \mu$, for some $\mu \in \mathcal{P}(S)$. Given on some filtered probability space $\Upsilon_n \doteq (\Omega^n, \mathcal{F}^n, \{\mathcal{F}_t^n\}, \mathbb{P}^n)$, let X^n be the unique weak solution of

$$X^n(t) = \Gamma \left(X^n(0) + \int_0^\cdot (a - Rb_n)(X^n(s))ds + \sigma W^n(\cdot) \right) (t), \quad X^n(0) \sim \mu_n, \quad (4.35)$$

where b_n is as in Lemma 4.5.1 and W^n is a K -dimensional $\{\mathcal{F}_t^n\}$ standard Brownian motion. Let

$$Y^n(t) = \Gamma_1 \left(X^n(0) + \int_0^\cdot (a - Rb_n)(X^n(s))ds + \sigma W^n(\cdot) \right) (t), \quad (4.36)$$

where $\Gamma_1(\cdot)$ is defined below Definition 2.2.7.

Theorem 4.5.2. *Let μ_n, μ, b_n and (X^n, Y^n) be as above. Then $(X^n, Y^n) \Rightarrow (X, Y)$ as $C([0, \infty), \mathbb{R}^K \times \mathbb{R}^K)$ -valued random variables, where X is a continuous $\{\overline{\mathcal{F}}_t\}$ adapted process, on some filtered probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \{\overline{\mathcal{F}}_t\}_{t \geq 0}, \overline{\mathbb{P}})$, such that $\overline{\mathbb{P}} \circ X_0^{-1} = \mu$ and*

$$\begin{aligned} X(t) &= \Gamma \left(X(0) + \int_0^\cdot (a - Rb_\star)(X(s))ds + \sigma W(\cdot) \right) (t), \quad t \geq 0 \\ Y(t) &= \Gamma_1 \left(X(0) + \int_0^\cdot (a - Rb_\star)(X(s))ds + \sigma W(\cdot) \right) (t), \quad t \geq 0. \end{aligned} \quad (4.37)$$

Proof of the above Theorem is given immediately after the proof of Theorem 4.1.4.

Proof of Theorem 4.1.4. It suffices to show that, as $n \rightarrow \infty$,

$$\langle \eta_{b_n}, k_{b_n} \rangle \rightarrow \langle \eta_{b_\star}, k_{b_\star} \rangle. \quad (4.38)$$

From Lemma 7.1 of [7], the family $\{\eta_{b_n} : n \geq 1\}$ is tight. Let η^0 be a limit point of the sequence $\{\eta_{b_n}\}$. Denote the subsequence, along which η_{b_n} converges to η^0 , again by $\{\eta_{b_n}\}$.

Let X^n, Y^n be as in (4.35), (4.36) with μ_n there replaced by η_{b_n} . From Theorem 4.5.2 we get that $X^n \Rightarrow X$, where X is given by (4.37) with μ replaced by η^0 . Since X^n is stationary, the same is true for X . Thus from Theorem 3.4 of [7] we get that $\eta^0 = \eta_{b_\star}$. This proves $\eta_{b_n} \Rightarrow \eta_{b_\star}$ as $n \rightarrow \infty$. Next the uniform boundedness of $a - Rb_n$ and (4.35) imply that

$$\sup_n \mathbb{E} \left(\sup_{0 \leq t \leq 1} |Y^n(t)|^2 \right) < \infty.$$

In particular, for all $t \in [0, 1]$ we have $|\mathbb{E}Y^n(t) - \mathbb{E}Y(t)| \rightarrow 0$ as $n \rightarrow \infty$. Also,

$$\left| \mathbb{E} \int_0^t a(X^n(s)) ds - \mathbb{E} \int_0^t a(X(s)) ds \right| \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Thus taking expectation in (4.35), (4.37), and using the stationarity of X^n and X , we get $\int_S b_n(x) \eta_{b_n}(dx) \rightarrow \int_S b_\star(x) \eta_{b_\star}(dx)$. This proves (4.38) and the result follows. \blacksquare

For a Polish space \mathcal{T} , denote by $\mathcal{M}(\mathcal{T})$ the space of subprobability measures on $(\mathcal{T}, \mathcal{B}(\mathcal{T}))$ with the usual topology of weak convergence. Let $G \doteq [0, 1] \times S \times \Lambda$, where Λ is as introduced in Section 4.1.

Proof of Theorem 4.5.2. It suffices to prove the result with $[0, \infty)$ replaced by $[0, T]$ where $T > 0$ is arbitrary. Without loss of generality we can assume $T = 1$. We first consider the case $\mu_n = \delta_{x_n}$ and $\mu = \delta_x$, where $x_n, x \in S$ and $x_n \rightarrow x$ as $n \rightarrow \infty$.

For $t \in [0, 1]$, define $m_t^n \in \mathcal{M}(G)$ as

$$m_t^n(A \times B \times C) \doteq \int_0^t 1_A(s) 1_B(X^n(s)) 1_C(b_n(X^n(s))) ds,$$

where $A \in \mathcal{B}[0, 1]$, $B \in \mathcal{B}(S)$, $C \in \mathcal{B}(\Lambda)$. Note that $\{m_t^n\}_{0 \leq t \leq 1}$ is a continuous stochastic process, with values in $\mathcal{M}([0, 1] \times S \times \Lambda)$, defined on the filtered probability space Υ_n . Furthermore, $\int_0^t b_n(X^n(s)) ds = \int_G um_t^n(ds \, dx \, du)$ and thus

$$X^n(t) = X^n(0) + \int_G um_t^n(ds \, dx \, du) + \int_0^t a(X^n(s)) ds + \sigma W^n(t) + RY^n(t).$$

Since Λ is compact and a is bounded, we can find $c_1 \in (0, \infty)$ such that for all $n \geq 1$ and $0 \leq s \leq t < \infty$,

$$|X^n(t) - X^n(s)| + |Y^n(t) - Y^n(s)| \leq c_1 (|t - s| + w_{W^n}(|t - s|)),$$

where for $g \in C([0, \infty), \mathbb{R}^K)$ and $\delta > 0$, $w_g(\delta) \doteq \sup_{0 \leq s \leq t < \infty, |t-s| \leq \delta} |g(t) - g(s)|$. This along with the tightness of $\{X^n(0)\}_{n \geq 1}$ gives that (X^n, Y^n) are tight in $C([0, 1], \mathbb{R}^K \times \mathbb{R}^K)$. Let $f \in C_b(G)$. Then for $0 \leq s < t < 1$ we have $|m_t^n(f) - m_s^n(f)| \leq \|f\|_\infty |t - s|$. This along with the observation $m_0^n = 0$ shows that $\{m^n\}_{n \geq 1}$ is a tight family of $C([0, 1], \mathcal{M}(G))$ valued random variables. Thus $\{(X^n, Y^n, W^n, m^n)\}_{n \geq 1}$ is a tight family of $C([0, 1], \mathcal{E}_0)$ valued random variables, where $\mathcal{E}_0 \doteq \mathbb{R}^{3K} \times \mathcal{M}(G)$.

Consider a weak limit point (X, Y, W, m) of the sequence $\{(X^n, Y^n, W^n, m^n)\}_{n \geq 1}$. Abusing notation, we will denote the subsequence once more with the superscript n . Denote by $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ the probability space on which all the limit processes are given. Then

$$\begin{aligned} \text{(i)} \quad X(t) &= X(0) + \int_G um_t^n(ds \, dx \, du) + \int_0^t a(X(s)) ds \\ &\quad + \sigma W(t) + RY(t), \quad \mathbb{P}^* \text{ a.s.}, \end{aligned} \tag{4.39}$$

- (ii) $Y(t) = \Gamma_1 \left(X(0) + \int_G um_t(ds dx du) + \int_0^t a(X(s))ds + \sigma W(\cdot) \right) (t)$, \mathbb{P}^* a.s.,
(iii) W is a Wiener process.

Also $\mathbb{P}^* \circ X_0^{-1} = \mu$.

Define $\mathcal{F}_t^* = \sigma\{(X(s), Y(s), W(s), m(s)) : 0 \leq s \leq t\}$. We now show that W is an $\{\mathcal{F}_t^*\}$ martingale. It suffices to show for all $p \geq 1$, $0 \leq t_1 \leq t_2 \leq \dots \leq t_p \leq s \leq t \leq 1$,

$$\mathbb{E}^* (\psi(Z_{t_i}, i \leq p)[W_t - W_s]) = 0, \quad (4.40)$$

where $\psi \in C_b(\mathcal{E}_0^p, \mathbb{R})$ is arbitrary and $Z_t = (X_t, Y_t, W_t, m_t)$. Left side of (4.40) can be expressed as

$$\lim_{n \rightarrow \infty} \mathbb{E}_n (\psi(Z_{t_i}^n, i \leq p)[W_t^n - W_s^n]),$$

where $Z^n \doteq (X^n, Y^n, W^n, m^n)$. However, the last expression is clearly 0 since W^n is an $\{\mathcal{F}_t^n\}$ martingale and Z^n is $\{\mathcal{F}_t^n\}$ adapted.

We will now argue that for all $t \in [0, 1]$,

$$\int_G um_t(ds dx du) = \int_0^t b_*(X(s))ds, \quad \text{a.s. } \mathbb{P}^*. \quad (4.41)$$

This along with weak uniqueness of solutions to (4.37) will prove the result. Since $m_t^n([0, s] \times S \times \Lambda) = s \wedge t$ for all $s, t \in [0, 1]$ a.s. \mathbb{P}^n , we have that $m_t([0, s] \times S \times \Lambda) = s \wedge t$ for all $s, t \in [0, 1]$ a.s. \mathbb{P}^* . Thus for $f \in C_b[0, 1]$, $\int_0^1 f(s) \hat{m}_t(ds) = \int_0^t f(s) ds$ where $\hat{m}_t \in \mathcal{M}([0, 1])$ is defined as $\hat{m}_t(A) \doteq m_t(A \times S \times \Lambda)$ for $A \in \mathcal{B}[0, 1]$.

Next, for $f \in C_b(S)$ we have $\int_G f(x) m_t^n(ds dx du) = \int_0^t f(X^n(s)) ds$. Thus

$$\int_G f(x) m_t(ds dx du) = \int_0^t f(X(s)) ds, \quad \text{a.s. } \mathbb{P}^*, \quad (4.42)$$

and

$$\int_0^t \left(\int_S f(x) k_t(s, dx) \right) ds = \int_0^t f(X(s)) ds,$$

where $k_t : \Omega^* \times [0, t] \rightarrow \mathcal{P}(S)$ is a measurable map satisfying

$$\int_A k_t(\omega, s, B) ds = m_t(\omega, A \times B \times \Lambda), \quad \forall A \times B \subseteq \mathcal{B}([0, 1] \times S), \text{ a.s. } \mathbb{P}^*.$$

Thus

$$k_t(\omega, s, dx) = \delta_{X_s(\omega)}(dx) \text{ a.e. } (s, \omega) \text{ } [\hat{m}_t \otimes \mathbb{P}^*]. \quad (4.43)$$

Recall the definition of A_n given in (4.34). Define $B_n \doteq \bigcup_{m=n}^{\infty} A_m^c$ and $E_n = B_n^c$.

Then

$$\gamma_g(E_n) \geq 1 - \frac{1}{2^n} \quad \text{for all } n \geq 1 \quad (4.44)$$

and $b_*(x) = b_n(x) = b_{n+1}(x) = \dots$ for all $x \in E_n$. Let $F \subseteq S$ be a compact set such that $\{x_n\} \subseteq F$. Fix $\epsilon > 0$. Let $F_1 \subseteq S$ be a compact set such that

$$\sup_n \sup_{0 \leq t \leq 1} \mathbb{P}^n[X^n(t) \in F_1^c] < \frac{\epsilon}{2}. \quad (4.45)$$

For $\delta > 0$, define $F^\delta \doteq F_1 \cap \{x \in S : \text{dist}(x, \partial S) \geq \delta\}$. For $t > 0$, let $p(t, x, y)$ be the transition probability density function of $X_0^x(t) \doteq \Gamma(x + \sigma W(\cdot))(t)$. Then from the representation of $p_G(t, x, y)$ in (2.7) and its relation with the transition density $p(t, x, y)$, described below Corollary 2.2.16, along with Remark 2.2.14, we have for each $\delta > 0$, there is a function $\Psi_\delta : [0, 1] \rightarrow \mathbb{R}_+$ and $\alpha > 0$ such that

$$\begin{aligned} \sup_{x \in F, z \in F^\delta} p(t, x, z) &\leq \Psi_\delta(t), \quad t \in (0, 1), \\ \int_0^1 e^{-\alpha/t} \Psi_\delta(t) dt &< \infty. \end{aligned} \quad (4.46)$$

Henceforth, fix $\delta > 0$. Let $Q_n^{x_n}, Q_0^{x_n}$ be probability measures induced by $X^n, X_0^{x_n}$ on $C([0, 1], S)$. Then by Girsanov's theorem, and uniform boundedness of b_n and σ ,

there is a $\theta \in (0, \infty)$ such that

$$Q_n^{x_n} \circ \pi_t^{-1}(A) \leq \theta \sqrt{Q_0^{x_n} \circ \pi_t^{-1}(A)}, \quad \forall n \geq 1 \text{ and } A \in \mathcal{B}(S), \quad (4.47)$$

where π_t is the usual coordinate map from $C([0, 1], S)$ to S . Since the Lebesgue measure on S is absolutely continuous with respect to γ_g , we have from (4.46), (4.44) that there is $n_0 \in \mathbb{N}$ such that

$$\lambda(E_{n_0}^c \cap F_1) \int_0^1 e^{-\alpha/t} \Psi_\delta(t) dt < \frac{\epsilon^2}{4\theta^2}, \quad (4.48)$$

where λ is the Lebesgue measure on S .

Note that (4.48) implies that for all $x \in F$, $t \in [0, 1]$

$$\mathbb{E} \int_{[0, t] \times (E_{n_0}^c \cap F^\delta)} e^{-\alpha/s} \delta_{X_0^\delta(s)} ds \leq \frac{\epsilon^2}{4\theta^2}. \quad (4.49)$$

Let $S_\delta \doteq \{x \in S : \text{dist}(x, \partial S) < \delta\}$. From (4.47) and (4.45) we now have that

$$\begin{aligned} \int_0^1 e^{-\alpha/2s} Q_n^{x_n} \circ \pi_s^{-1}(E_{n_0}^c) ds &= \int_0^1 e^{-\alpha/2s} Q_n^{x_n} \circ \pi_s^{-1}(E_{n_0}^c \cap F_1) ds + \frac{\epsilon}{2} \\ &\leq \theta \int_0^1 e^{-\alpha/2s} \sqrt{Q_0^{x_n} \circ \pi_s^{-1}(E_{n_0}^c \cap F_1)} ds + \frac{\epsilon}{2} \\ &\leq \theta \left(\int_0^1 e^{-\alpha/s} Q_0^{x_n} \circ \pi_s^{-1}(E_{n_0}^c \cap F_1) ds \right)^{1/2} + \frac{\epsilon}{2} \\ &\leq \theta \left(\int_0^1 e^{-\alpha/s} Q_0^{x_n} \circ \pi_s^{-1}(E_{n_0}^c \cap F^\delta) ds \right)^{1/2} \\ &\quad + \theta \left(\int_0^1 e^{-\alpha/s} Q_0^{x_n} \circ \pi_s^{-1}(S_\delta) ds \right)^{1/2} + \frac{\epsilon}{2} \\ &\leq \epsilon + \ell(x_n, \delta), \end{aligned} \quad (4.50)$$

where $\ell(x_n, \delta) = \theta \left(\int_0^1 e^{-\alpha/s} Q_0^{x_n} \circ \pi_s^{-1}(S_\delta) ds \right)^{1/2}$ and the last step follows from (4.49).

From Feller property of $\{X_0^x\}$ we have that

$$\ell(x_n, \delta) \rightarrow \ell(x, \delta) \text{ as } n \rightarrow \infty. \quad (4.51)$$

Let $\bar{m}_t^n(A \times B) \doteq m_t^n(A \times B \times \Lambda)$, $A \in \mathcal{B}([0, t])$, $B \in \mathcal{B}(S)$. For $n, n_0 \in \mathbb{N}$, $n \geq n_0$ and $f \in C(\Lambda)$, $h \in C[0, 1]$,

$$\begin{aligned} & \int_G e^{-\alpha/2s} h(s) f(u) m_t^n(ds dx du) \\ &= \int_{[0,1] \times S} e^{-\alpha/2s} h(s) f(b_n(x)) \bar{m}_t^n(ds dx) \\ &= \int_{[0,1] \times E_{n_0}} e^{-\alpha/2s} h(s) f(b_{n_0}(x)) \bar{m}_t^n(ds dx) + \int_{[0,1] \times E_{n_0}^c} e^{-\alpha/2s} h(s) f(b_n(x)) \bar{m}_t^n(ds dx). \end{aligned}$$

Thus we have

$$\begin{aligned} & \left| \int_G e^{-\alpha/2s} h(s) f(u) m_t^n(ds dx du) - \int_{[0,1] \times S} e^{-\alpha/2s} h(s) f(b_{n_0}(x)) \bar{m}_t^n(ds dx) \right| \\ & \leq 2 \|f\|_\infty \|h\|_\infty \int_{[0,1] \times E_{n_0}^c} e^{-\alpha/2s} \bar{m}_t^n(ds dx). \end{aligned} \quad (4.52)$$

From (4.50), the left side of (4.52) is bounded above by $2 \|f\|_\infty \|h\|_\infty [\epsilon + \ell(x_n, \delta)]$.

Now, letting $n \rightarrow \infty$ in (4.52) we obtain

$$\begin{aligned} & \mathbb{E} \left| \int_G e^{-\alpha/2s} h(s) f(u) m_t(ds dx du) - \int_{[0,1] \times S} e^{-\alpha/2s} h(s) f(b_{n_0}(x)) \bar{m}_t(ds dx) \right| \\ & \leq 2 \|f\|_\infty \|h\|_\infty [\epsilon + \ell(x, \delta)], \end{aligned}$$

where \bar{m}_t is defined analogously to \bar{m}_t^n . Thus noting that $b_{n_0}(x) = b_\star(x)$ on E_{n_0} we get

$$\begin{aligned} & \mathbb{E} \left| \int_G e^{-\alpha/2s} h(s) f(u) m_t(ds dx du) - \int_{[0,1] \times S} e^{-\alpha/2s} h(s) f(b_\star(x)) \bar{m}_t(ds dx) \right| \\ & \leq 2 \|f\|_\infty \|h\|_\infty \left[\epsilon + \ell(x, \delta) + \mathbb{E} \int_{[0,1] \times E_{n_0}^c} e^{-\alpha/2s} \bar{m}_t(ds dx) \right]. \end{aligned} \quad (4.53)$$

Since $E_{n_0}^c$ is an open set, we have from (4.50) and (4.51)

$$\mathbb{E} \int_{[0,1] \times E_{n_0}^c} e^{-\alpha/2s} \bar{m}_t(ds dx) \leq \liminf_{n \rightarrow \infty} \mathbb{E} \int_{[0,1] \times E_{n_0}^c} e^{-\alpha/2s} \bar{m}_t^n(ds dx) < \epsilon + \ell(x, \delta).$$

Noting that $\ell(x, \delta) \rightarrow 0$ as $\delta \rightarrow 0$, we have, letting $\epsilon \rightarrow 0$, $\delta \rightarrow 0$ in (4.53), for all $t \in (0, 1)$, $h \in C[0, 1]$ and $f \in C(\Lambda)$,

$$\int_G e^{-\alpha/2s} h(s) f(u) m_t(ds dx du) = \int_{[0,1] \times S} e^{-\alpha/2s} h(s) f(b_\star(x)) \bar{m}_t(ds dx) \quad \text{a.e. } \mathbb{P}^\star.$$

Combining this with (4.42) we now obtain for all $t \in [0, 1]$,

$$\mathbb{P}^\star \otimes m_t\{(\omega, s, X_s(\omega), b_\star(X_s(\omega))) : s \in [0, t]\} = 1.$$

As an immediate consequence, we obtain (4.41). This proves the result for the case $\mu_n = \delta_{x_n}$ and $\mu = \delta_x$.

Next let μ_n, μ be arbitrary such that $\mu_n \Rightarrow \mu$. Let $\widehat{Q}_n^{\mu_n}$ (resp. \widehat{Q}^μ) be the measure induced by (X^n, Y^n) (resp. (X, Y)) on $C([0, 1], S \times \mathbb{R}^K)$, where (X^n, Y^n) is as in (4.35), (4.36) and (X, Y) as in (4.37). We will write $\widehat{Q}_n^{\mu_n}$ as $\widehat{Q}_n^{x_n}$ when $\mu_n = \delta_{x_n}$. Similarly we will write \widehat{Q}^x for \widehat{Q}^μ when $\mu = \delta_x$. We then have from the first part of the proof that for all $f \in C_b(C([0, 1], S \times \mathbb{R}^K))$

$$\sup_{x \in F} |\langle f, \widehat{Q}_n^{x_n} \rangle - \langle f, \widehat{Q}^x \rangle| \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

for all compact subset $F \subseteq S$. Using this along with the continuity of the map $x \mapsto \langle f, \widehat{Q}^x \rangle$, for $f \in C_b(C([0, 1], S \times \mathbb{R}^K))$, and the weak convergence of measures μ_n to μ , we have, as $n \rightarrow \infty$,

$$\begin{aligned}
|\langle f, \widehat{Q}_n^{\mu_n} \rangle - \langle f, \widehat{Q}^\mu \rangle| &\leq \int_S |\langle f, \widehat{Q}_n^x \rangle - \langle f, \widehat{Q}^x \rangle| \mu_n(dx) \\
&\quad + \left| \int_S \langle f, \widehat{Q}^x \rangle \mu_n(dx) - \int_S \langle f, \widehat{Q}^x \rangle \mu(dx) \right| \\
&\rightarrow 0.
\end{aligned}$$

The result follows. ■

4.6 Appendix

Proof of Proposition 4.3.3. By Proposition 4.2.1, there exists $\bar{L} \in (0, \infty)$ such that with $C \doteq \{x \in S : |x| \leq \bar{L}\}$,

$$\sup_n \mathbb{E}[\widehat{Q}_x^n(t_0|x)^2] \leq \frac{1}{2}|x|^2, \quad \forall x \in C^c, \quad (4.1)$$

where t_0 is as in Proposition 4.2.1. Let $\bar{\delta} \doteq t_0 \bar{L}$ and set $\tau_C^n(\bar{\delta}) \equiv \tau^n \doteq \inf\{t \geq \bar{\delta} : |\widehat{Q}_x^n(t)| \leq \bar{L}\}$. Define a sequence of stopping times σ_m as

$$\sigma_0 \doteq 0, \quad \sigma_m = \sigma_{m-1} + t_0[|\widehat{Q}_x^n(\sigma_{m-1})| \vee \bar{L}], \quad m \in \mathbb{N}.$$

Note that the dependence of these stopping times on n and x has been suppressed in notation. Also, let $m_0^n \doteq \min\{m \geq 1 : |\widehat{Q}_x^n(\sigma_m)| \leq \bar{L}\}$. Define

$$\widehat{G}_n(x) \doteq \mathbb{E} \left[\int_0^{\tau^n} (1 + |\widehat{Q}_x^n(t)|) dt \right], \quad x \in S.$$

Then

$$\widehat{G}_n(x) \leq \mathbb{E} \left[\int_0^{\sigma_{m_0^n}} (1 + |\widehat{Q}_x^n(t)|) dt \right] = \sum_{k=0}^{\infty} \mathbb{E} \left[\int_{\sigma_k}^{\sigma_{k+1}} (1 + |\widehat{Q}_x^n(t)|) dt 1_{k < m_0^n} \right]. \quad (4.2)$$

Let $\mathcal{F}_t \doteq \sigma\{\widehat{Q}_x^n(s) : 0 \leq s \leq t\}$ (we suppress n and x in the notation). We claim that for some constant $c_1 \in (0, \infty)$, and for all $n, k \in \mathbb{N}$, $x \in S$,

$$\mathbb{E} \left[\int_{\sigma_k}^{\sigma_{k+1}} (1 + |\widehat{Q}_x^n(t)|) dt \middle| \mathcal{F}_{\sigma_k} \right] 1_{k < m_0^n} \leq c_1 \left(1 + |\widehat{Q}_x^n(\sigma_k)|^2 \right) 1_{k < m_0^n}. \quad (4.3)$$

The claim is proved below (4.6). Assuming that the claim holds and using this estimate in (4.2) we get by suitable conditioning

$$\sup_n \widehat{G}_n(x) \leq c_1 \sup_n \mathbb{E} \left[\sum_{k=0}^{m_0^n-1} \left(1 + |\widehat{Q}_x^n(\sigma_k)|^2 \right) \right]. \quad (4.4)$$

Next note that $\{\widehat{Q}_x^n(\sigma_k)\}_{k \geq 1}$ is a Markov chain with the one step transition kernel

$$\check{P}_n(x, A) \doteq P_n^{t_0(|x| \vee \bar{L})}(x, A), \quad x \in S, \quad A \in \mathcal{B}(S),$$

where P_n^t is the transition probability kernel for the Markov process \widehat{Q}_n . Using (4.14) and (4.1) one has for some constant $c_2 \in (1, \infty)$,

$$\sup_n \int_S \check{P}_n(x, dy) |y|^2 \leq |x|^2 - \frac{1}{2}|x|^2 + c_2 1_{[0, \bar{L}]}(|x|). \quad (4.5)$$

Using Theorem 14.2.2 of [45] we have that

$$\begin{aligned} \sup_n \mathbb{E} \sum_{k=0}^{m_0^n-1} [1 + |\widehat{Q}_x^n(\sigma_k)|^2] &\leq 3 \left[|x|^2 + 2 \sup_n \mathbb{E} \sum_{k=0}^{m_0^n-1} c_2 1_C(\widehat{Q}_x^n(\sigma_k)) \right] \\ &= 3 \left[|x|^2 + 2c_2 1_{[0, \bar{L}]}(|x|) \right], \end{aligned} \quad (4.6)$$

where the second equality follows from the fact that whenever $1 \leq k \leq m_0^n - 1$, $|\widehat{Q}_x^n(\sigma_k)| > \bar{L}$ (we assume without loss of generality $\bar{L} > 2$). The inequality (4.16) now follows on using the above estimate in (4.4).

Thus it only remains to prove the claim in (4.3). By an application of strong

Markov property this is equivalent to showing for some $c_3 \in (0, \infty)$ and all n, x

$$\mathbb{E} \left[\int_0^{\sigma_1} (1 + |\widehat{Q}_x^n(t)|) dt \right] \leq c_3 (1 + |x|^2). \quad (4.7)$$

From definition of σ_1 , we see that

$$\sigma_1 \leq c_4(1 + |x|) \quad (4.8)$$

for some constant $c_4 \in (0, \infty)$. With notation introduced in proof of Proposition 4.2.1, we have

$$\begin{aligned} \sup_n \mathbb{E} \sup_{0 \leq t \leq \sigma_1} |\widehat{Q}_x^n(t)| &= \sup_n \mathbb{E} \sup_{0 \leq t \leq \sigma_1} \left| \Gamma \left(x + \widehat{M}^n(\cdot) + r^n(\cdot) \right) (t) - Z_x^n(t) + Z_x^n(t) \right| \\ &\leq \sup_n L \mathbb{E} \sup_{0 \leq t \leq \sigma_1} |\widehat{M}^n(t)| + \sup_n \mathbb{E} \sup_{0 \leq t \leq \sigma_1} |Z_x^n(t)|. \end{aligned} \quad (4.9)$$

Using the boundedness assumption (iii) in Assumption 4.1.1 we see that for some $c_5 \in (0, \infty)$

$$\mathbb{E} \sup_{0 \leq t \leq \sigma_1} |Z_x^n(t)| \leq c_5(1 + |x|), \quad \forall x \in S.$$

Using this estimate along with (4.13) and (4.8) in (4.9) we now have for some $c_6 \in (0, \infty)$ and $x \in S$,

$$\mathbb{E} \left[\int_0^{\sigma_1} (1 + |\widehat{Q}_x^n(t)|) dt \right] \leq c_6(1 + |x|^2). \quad (4.10)$$

This proves (4.3) and the result follows. ■

Proof of Proposition 4.3.2. The proof is adapted from Proposition 5.4 of [18]. We begin by showing that:

$$\text{For all } m \in \mathbb{N}, \quad \mathbb{E} \left[\int_0^{\tau_C^{n(m\bar{\delta})}} f(\widehat{Q}_x^n(t)) dt \right] \leq G_n(x) + b_1 m \bar{\delta}, \quad x \in S, \quad (4.11)$$

where $b_1 \doteq \sup_n \sup_{x \in C} G_n(x) / \bar{\delta}$. The proof is by induction. For $m = 1$, the inequality

in (4.11) holds trivially. Suppose now that (4.11) holds for $m = k \in \mathbb{N}$. In what follows, instead of indicating the dependence on the initial condition as a subscript to \widehat{Q}^n , we will indicate it in the expectation operation. For example, $\mathbb{E}[f(\widehat{Q}_x^n(t))]$ will be written as $\mathbb{E}_x[f(\widehat{Q}^n(t))]$, etc. Using the strong Markov property of \widehat{Q}^n we have that

$$\begin{aligned}
\mathbb{E}_x \left[\int_0^{\tau_C^n((k+1)\bar{\delta})} f(\widehat{Q}^n(t)) dt \right] &\leq \mathbb{E}_x \left[\int_0^{\tau_C^n(\bar{\delta})} f(\widehat{Q}^n(t)) dt \right] \\
&\quad + \mathbb{E}_x \left[\mathbb{E}_{\widehat{Q}^n(\tau_C^n(\bar{\delta}))} \left[\int_0^{\tau_C^n(k\bar{\delta})} f(\widehat{Q}^n(t)) dt \right] \right] \\
&\leq G_n(x) + \sup_{x \in C} \mathbb{E}_x \left[\int_0^{\tau_C^n(k\bar{\delta})} f(\widehat{Q}^n(t)) dt \right] \\
&\leq G_n(x) + \sup_n \sup_{x \in C} G_n(x) + b_1 k \bar{\delta}, \tag{4.12}
\end{aligned}$$

where the last inequality follows from the induction hypothesis. Proof of (4.11) now follows on noting that the right side of (4.12) coincides with $G_n(x) + b_1(k+1)\bar{\delta}$. Using the monotonicity in m of the expression on the left side of (4.11) we now have that for all $t \geq \bar{\delta}$,

$$\mathbb{E}_x \left[\int_0^{\tau_C^n(t)} f(\widehat{Q}^n(s)) ds \right] \leq G_n(x) + 2b_1 t. \tag{4.13}$$

Note that (4.13) is trivially satisfied for all $t < \bar{\delta}$. Thus (4.13) holds for all $t \geq 0$. Using the strong Markov property once again, we have (see proof of Proposition 5.4 of [18] for analogous arguments)

$$\begin{aligned}
\mathbb{E}_x[G_n(\widehat{Q}^n(t))] &\leq G_n(x) - \int_0^t \mathbb{E}_x \left[f(\widehat{Q}^n(s)) \right] ds \\
&\quad + \mathbb{E}_x \left[\mathbb{E}_{\widehat{Q}^n(\tau_C^n(\bar{\delta}))} \left[\int_0^{\tau_C^n(t)} f(\widehat{Q}^n(s)) ds \right] \right] \\
&\leq G_n(x) - \int_0^t \mathbb{E}_x \left[f(\widehat{Q}^n(s)) \right] ds + \sup_n \sup_{x \in C} \mathbb{E}_x \left[\int_0^{\tau_C^n(t)} f(\widehat{Q}^n(s)) ds \right] \\
&\leq G_n(x) - \int_0^t \mathbb{E}_x \left[f(\widehat{Q}^n(s)) \right] ds + \sup_n \sup_{x \in C} G_n(x) + 2b_1 t, \tag{4.14}
\end{aligned}$$

where the last inequality follows from (4.13). We obtain (4.15) by dividing both sides in (4.14) by t and putting $\bar{\kappa} \doteq \left(\frac{2}{\delta} + 1\right) \sup_n \sup_{x \in C} G_n(x)$. ■

Proof of Theorem 4.2.2. The proof follows along the lines of Lemma 4.4 of [2] and thus only a sketch will be provided. Let for $\Delta > 0$, $\nu_{ij}^n \doteq \sup_{0 \leq s \leq \Delta} |\widehat{M}_{ij}^n(s)|$. We will show that for each (i, j) , given $c_1, c_2 \in (0, \infty)$, there are $\gamma_0, \Delta, \eta \in (0, \infty)$ such that

$$\limsup_n \sup_{U^n \in \mathcal{A}_n} e^{-\gamma_0 \Delta} \mathbb{E} e^{\gamma_0 c_1 \nu_{ij}^n} \leq c_2 e^{-\eta \Delta}. \quad (4.15)$$

The proof of the Proposition will then follow as in [2]. (See displays above (4.9) therein.) We will only consider the case $j \neq 0$. Proof for $j = 0$ follows along similar lines. Without loss of generality, we assume that $p_{ij} \neq 0$. Let $c_3, c_4 \in (0, \infty)$, $n_0 \geq 1$ be such that for all $x \in S$, $u \in \Lambda_n$, $n \geq n_0$;

$$nc_3 \leq p_{ij}(\mu_i^n(x) + u_i) \leq nc_4. \quad (4.16)$$

Henceforth we will only consider $n \geq n_0$. For $\delta > 0$, let $\tau(\delta)$ be an $\{\mathcal{F}_t^n\}_{t \geq 0}$ stopping time such that

$$p_{ij} \int_0^{\tau(s)} [\mu_i^n(Q^n(t)) + U_i^n(t)] dt = s. \quad (4.17)$$

Let

$$\widetilde{M}_{ij}^n(s) \doteq M_{ij}^n(\tau(s)), \quad s \geq 0. \quad (4.18)$$

Then $\widetilde{M}_{ij}^n \stackrel{\mathcal{L}}{=} N_0$, where N_0 is a unit rate compensated Poisson process. (See Theorem T16, Chapter II in [6].)

Note that (4.16)-(4.18) yield $\sqrt{n} \nu_{ij}^n \leq \sup_{0 \leq s \leq nc_3 \Delta} |\widetilde{M}_{ij}^n(s)|$. Therefore, for $\gamma, c_1 \in (0, \infty)$ we have $\mathbb{E} e^{\gamma c_1 \sqrt{n} \nu_{ij}^n} \leq \mathbb{E} e^{\gamma c_1 \nu_0^n}$, where $\nu_0^n \doteq \sup_{0 \leq s \leq nc_3 \Delta} |N_0(s)|$. Applying Doob's maximal inequality for submartingales, we get

$$\mathbb{E} e^{\gamma c_1 \nu_0^n} \leq 4 \mathbb{E} e^{\gamma c_1 |N_0(nc_3 \Delta)|}.$$

Assume that γ is small enough so that $\gamma c_1 < 1$. Then a straightforward calculation shows that with $c_5 \doteq \frac{\epsilon}{2} c_1^2 c_3$, $\mathbb{E} e^{\gamma c_1 \nu_0^n} \leq 8 e^{c_5 n \Delta \gamma^2}$. Thus for all $\gamma \in (0, \infty)$ such that $\gamma c_1 < 1$,

$$e^{-\sqrt{n}\gamma\Delta} \mathbb{E} e^{\gamma c_1 \sqrt{n} \nu_{ij}^n} \leq 8 e^{c_5 n \Delta \gamma^2} e^{-\sqrt{n}\gamma\Delta}.$$

Now choose $\gamma = \frac{\gamma_0}{\sqrt{n}}$, $\gamma_0 \in (0, \frac{1}{c_1})$. Then

$$e^{-\gamma_0\Delta} \mathbb{E} e^{\gamma_0 c_1 \sqrt{n} \nu_{ij}^n} \leq 8 e^{c_5 n \Delta \gamma_0^2} e^{-\gamma_0\Delta}.$$

Now choose γ_0 sufficiently small and Δ sufficiently large so that for some $\eta \in (0, 1)$,

$$\frac{\log 8 - \log c_2}{\Delta} + c_5 \gamma_0^2 - \gamma_0 \leq -\eta.$$

Thus (4.15) holds with such a choice of γ_0, Δ, η . The result follows. ■

Proof of Theorem 4.4.3. We will only prove (i) since (ii) follows in an analogous fashion. For $x \in \mathbb{R}^K$, let $|x|^* \doteq |x| \wedge 1$. In order to prove (1), it suffices to show that for all $t \geq 0$

$$\int_{\mathbb{S}_{\text{path}}} |\psi(t)|^* d\tilde{\mathcal{Q}}(H^*) = 0, \quad \text{a.s.}, \quad (4.19)$$

where $\psi(t) \doteq Q^*(t) - \Gamma(Q^*(0) + A^*(\cdot) - RC^*(\cdot) + M^*(\cdot))(t)$. Note that

$$\int_{\mathbb{S}_{\text{path}}} [j(Q^*) + j(M^*)] d\mathcal{Q}^\ell(H^*) \leq \frac{1}{\sqrt{n_\ell}},$$

where for $z \in D([0, \infty), \mathbb{R}^K)$, $j(z) \doteq \sup_{t>0} |z(t) - z(t-)|$. Thus

$$\tilde{\mathcal{Q}}(\mathbb{S}_{\text{path}}^0) = 1, \quad \text{a.s.}, \quad (4.20)$$

where $\mathbb{S}_{\text{path}}^0 \doteq C([0, \infty), \mathbb{R}^{4K})$. In particular,

$$\mathbb{E}_0 \int_{\mathbb{S}_{\text{path}}} |\psi(t)|^* d\tilde{\mathcal{Q}}(H^*) = \lim_{\ell \rightarrow \infty} \mathbb{E} \int_{\mathbb{S}_{\text{path}}} |\psi(t)|^* d\mathcal{Q}^\ell(H^*) = 0. \quad (4.21)$$

Next from (4.21) $\int_{\mathbb{S}_{\text{path}}} |\psi(t)|^* d\mathcal{Q}^\ell(H^*)$ is equal to

$$\frac{1}{T_\ell} \int_0^{T_\ell} \left| \widehat{Q}_p^{n_\ell}(s)(t) - \Gamma \left(\widehat{Q}_p^{n_\ell}(s)(0) + \Delta \widehat{A}_p^{n_\ell}(s)(\cdot) - R \Delta \widehat{C}_p^{n_\ell}(s)(\cdot) + \Delta \widehat{M}_p^{n_\ell}(s)(\cdot) \right) (t) \right|^* ds,$$

which is 0 a.s. The equality in (4.19) now follows on combining (4.21) with the above display. This proves (1).

For $t \geq 0$, define $H_p^* \in \mathbb{S}_{\text{path}}$ as $H_p^*(t) \doteq (Q_p^*(t), \Delta A_p^*(t), \Delta C_p^*(t), \Delta M_p^*(t), \Delta Y_p^*(t))$.

In order to prove (2), it suffices to show that for all $f \in C_b(\mathbb{S}_{\text{path}})$,

$$\left| \int_{\mathbb{S}_{\text{path}}} [f(H_p^*(t)) - f(H_p^*(0))] d\tilde{\mathcal{Q}}(H^*) \right| = 0. \quad (4.22)$$

Note that left side above is the limit, as $\ell \rightarrow \infty$, of

$$\left| \int_{\mathbb{S}_{\text{path}}} [f(H_p^*(t)) - f(H_p^*(0))] d\mathcal{Q}^\ell(H^*) \right|. \quad (4.23)$$

The expression in (4.23) can be rewritten as

$$\left| \frac{1}{T_\ell} \int_0^{T_\ell} [f(\widehat{H}_p^{n_\ell}(t+s)) - f(\widehat{H}_p^{n_\ell}(s))] ds \right| \leq \frac{2\|f\|_\infty t}{T_\ell} \rightarrow 0, \text{ as } \ell \rightarrow \infty.$$

This proves (4.22) and thus (2) follows.

Next note that for any $t \geq 0$ and $p \geq 1$,

$$\sup_{\ell \geq 1} \sup_{s \geq 0} \mathbb{E} |\widehat{M}^{n_\ell}(t+s) - \widehat{M}^{n_\ell}(s)|^p \doteq \mathcal{N}(p, t) < \infty. \quad (4.24)$$

This in particular shows that for all $t \geq 0$, $\mathbb{E}_0 \int_{\mathbb{S}_{\text{path}}} |M^*(t)|^2 d\tilde{\mathcal{Q}}(H^*) < \infty$. In order to

prove (3) it suffices now to show that for all $k \geq 1$, $0 \leq t_1 < t_2 < \dots < t_k \leq s \leq t$, $\psi \in C_b(\mathbb{S}^k)$,

$$\mathbb{E}_0 \left| \int_{\mathbb{S}_{\text{path}}} \psi(H^*(t_1), \dots, H^*(t_k)) [M^*(t) - M^*(s)] d\tilde{Q}(H^*) \right|^2 = 0. \quad (4.25)$$

In view of (4.24) and weak convergence of \tilde{Q}^{n_ℓ} to \tilde{Q} , the expression on the left side of (4.25) is

$$\lim_{\ell \rightarrow \infty} \mathbb{E} \left| \frac{1}{T_\ell} \int_0^{T_\ell} \psi(\hat{H}^{n_\ell}(u)(t_1), \dots, \hat{H}^{n_\ell}(u)(t_k)) [\hat{M}^{n_\ell}(u+t) - \hat{M}^{n_\ell}(u+s)] du \right|^2. \quad (4.26)$$

Denote the expression inside the time integral by $\Lambda(u)$. Then the above can be rewritten as

$$\lim_{\ell \rightarrow \infty} \mathbb{E} \left[\frac{2}{T_\ell^2} \int_0^{T_\ell} \int_0^v \Lambda(u) \cdot \Lambda(v) dudv \right]. \quad (4.27)$$

Define the sets

$$L_0 \doteq \{(u, v) \in [0, T_\ell]^2 : 0 \leq u - v < t - s\}, \quad L_1 \doteq \{(u, v) \in [0, T_\ell]^2 : u - v > t - s\}.$$

Using the fact that $\hat{M}^{n_\ell}(t)$ is $\mathcal{F}_t^n \doteq \sigma\{\hat{H}^{n_\ell}(s) : s \leq t\}$ martingale, we see that $\mathbb{E}[\Lambda(u) \cdot \Lambda(v)] = 0$ for all $(u, v) \in L_1$. Thus the expression in (4.27) is the same as

$$\frac{2}{T_\ell^2} \int_{L_0} \mathbb{E}[\Lambda(u) \cdot \Lambda(v)] dudv. \quad (4.28)$$

Using (4.24) the expression in (4.28) can be bounded by

$$\frac{2\|\psi\|_\infty^2 \mathcal{N}(2, t-s) T_\ell(t-s)}{T_\ell^2}, \quad (4.29)$$

which approaches 0 as $\ell \rightarrow \infty$. This proves the expression in (4.26) is zero. Thus (4.25) holds and (3) follows.

Proof of (4) is very similar to that of (3) and so we only give a sketch. One needs to establish (4.25) with M^* replaced by the $\mathbb{R}^{K \times K}$ valued stochastic process N^* defined as

$$N^*(t) \doteq M^*(t)[M^*(t)]' - \int_0^t \sigma(Q^*(s))\sigma(Q^*(s))' ds.$$

In order for this it suffices to show that the expression in (4.27) approaches 0 as $\ell \rightarrow \infty$, when $\Lambda(u)$ is defined as before but with \widehat{M}^{n_ℓ} replaced by \widehat{N}^{n_ℓ} , which is defined as

$$\begin{aligned} \widehat{N}^{n_\ell}(t) &\doteq \widehat{M}^{n_\ell}(t)[\widehat{M}^{n_\ell}(t)]' - \int_0^t \sigma(\widehat{Q}^{n_\ell}(s))\sigma(\widehat{Q}^{n_\ell}(s))' ds \\ &= \widehat{M}^{n_\ell}(t)[\widehat{M}^{n_\ell}(t)]' - \int_0^t \Sigma(\widehat{Q}^{n_\ell}(s))\Sigma(\widehat{Q}^{n_\ell}(s))' ds. \end{aligned}$$

Let Σ^n be as in (4.7) with λ, μ there replaced by λ^n, μ^n . Let $\widetilde{\Sigma}^n(x) \doteq \frac{1}{\sqrt{n}}\Sigma^n(\sqrt{n}x)$.

Then

$$\widetilde{N}^n(t) \doteq \widehat{N}^n(t) - \int_0^t [(\Sigma\Sigma^*) - (\widetilde{\Sigma}^n(\widetilde{\Sigma}^n)^*)]\widehat{Q}^n(s) ds$$

is an $\{\widehat{\mathcal{F}}^n(t)\}$ martingale. We will now show that

$$\mathbb{E} \frac{1}{T_\ell} \int_0^{T_\ell} \left(\int_{u+s}^{u+t} \left| (\Sigma\Sigma^*)(\widehat{Q}^{n_\ell}(r)) - (\widetilde{\Sigma}^{n_\ell}(\widetilde{\Sigma}^{n_\ell})^*)(\widehat{Q}^{n_\ell}(r)) \right| dr \right) du \rightarrow 0, \text{ as } \ell \rightarrow \infty. \quad (4.30)$$

Once (4.30) is established, the term (analogous to) (4.27) is shown to converge to 0, as $\ell \rightarrow \infty$, upon following steps similar to those leading to (4.29). Let $f_\ell : \mathbb{S}_{\text{path}} \rightarrow \mathbb{R}$ be defined as

$$f_\ell(H^*) \doteq \sup_{s \leq r \leq t} \left| (\Sigma\Sigma^*)(Q^*(r)) - (\widetilde{\Sigma}^{n_\ell}(\widetilde{\Sigma}^{n_\ell})^*)(Q^*(r)) \right|.$$

Then $f_\ell(H^*) \rightarrow 0$ as $\ell \rightarrow \infty$ uniformly on compact subsets of \mathbb{S}_{path} . Since \mathcal{Q}^ℓ converges

to $\tilde{\mathcal{Q}}$, we have

$$\mathbb{E} \left| \int_{\mathbb{S}_{\text{path}}} f_\ell(H^\star) d\mathcal{Q}^\ell(H^\star) \right| \rightarrow 0. \quad (4.31)$$

Finally, the expression on left side of (4.30) is bounded above by

$$\mathbb{E} \frac{(t-s)}{T_\ell} \int_0^{T_\ell} f_\ell(\hat{H}^{n_\ell}(u)) du = (t-s) \mathbb{E} \left| \int_{\mathbb{S}_{\text{path}}} f_\ell(H^\star) d\mathcal{Q}^\ell(H^\star) \right|.$$

Combining this with (4.31), we have (4.30). This proves (4).

We now consider (5). It suffices to show that for all $t \geq 0$,

$$\mathbb{E}_0 \int \left| A^\star(t) - \int_0^t a(Q^\star(s)) ds \right| d\tilde{\mathcal{Q}}^\omega(H^\star) = 0. \quad (4.32)$$

Let $\tilde{a}^n(x) \doteq \frac{a(\sqrt{nx})}{\sqrt{n}}$, $x \in S$. Then $\tilde{a}^n \rightarrow a$ uniformly on compact sets. Let $g_n : \mathbb{S}_{\text{path}} \rightarrow \mathbb{R}$ be defined as $g_n(H^\star) \doteq \sup_{0 \leq s \leq t} |(\tilde{a}^n - a)(Q^\star(s))|$. Then

$$\mathbb{E} \int_{\mathbb{S}_{\text{path}}} g_{n_\ell}(H^\star) d\mathcal{Q}^\ell(H^\star) \rightarrow 0, \text{ as } \ell \rightarrow \infty. \quad (4.33)$$

Now the left side of (4.32) can be written as (4.33)

$$\begin{aligned} & \limsup_{\ell \rightarrow \infty} \mathbb{E} \frac{1}{T_\ell} \int_0^{T_\ell} \left| \hat{A}^{n_\ell}(t+u) - \hat{A}^{n_\ell}(u) - \int_u^{t+u} a(\hat{Q}^{n_\ell}(s)) ds \right| du \\ & \leq t \limsup_{\ell \rightarrow \infty} \mathbb{E} \left| \int_{\mathbb{S}_{\text{path}}} g_{n_\ell}(H^\star) d\mathcal{Q}^\ell(H^\star) \right|, \end{aligned}$$

where the inequality follows on recalling the representation (4.20). The last expression is 0 in view of (4.33) and thus (4.32) follows. This proves (5).

To prove (6), it suffices to show that for all $0 < s < t < \infty$,

$$\mathbb{E}_0 \int \text{dist} \left(\frac{C^\star(t) - C^\star(s)}{t-s}, \Lambda \right) d\tilde{\mathcal{Q}}(H^\star) = 0, \quad (4.34)$$

where for $x \in \mathbb{R}^K$, $\text{dist}(x, \Lambda) \doteq \inf_{y \in \Lambda} |x - y|$. However, (4.34) is immediate on using

weak convergence of \mathcal{Q}^ℓ to $\tilde{\mathcal{Q}}$ and recalling that

$$\frac{\hat{C}^{n_\ell}(t+u) - \hat{C}^{n_\ell}(s+u)}{t-s} \in \Lambda$$

a.s. for all $\ell \geq 1$ and $u \geq 0$. ■

LIST OF NOTATION AND SYMBOLS

General mathematical notation

\mathbb{N}	Set of natural numbers
\mathbb{N}_0	$\mathbb{N} \cup \{0\}$
\mathbb{R}	Set of real numbers
\mathbb{R}_+	Non-negative real numbers
\mathbb{R}^d	d -dimensional Euclidean space
$\mathbb{R}^{d \times m}$	The space of real $d \times m$ -matrices
S	\mathbb{R}_+^d , d -dimensional positive orthant
1_A	Indicator function of set A
A°	Set of interior points of set A
\bar{A}	Closure of set A
M'	Transpose of matrix M
$C(X, Y)$	Class of continuous functions $f : X \rightarrow Y$
$C_b(X)$	Continuous real bounded functions on X
$C_b^2(X)$	Class of continuous, real bounded, and twice differentiable functions defined on X
$C(X)$	$C(X, \mathbb{R})$
$C[0, 1]$	$C([0, 1], \mathbb{R})$
$C[0, \infty)$	$C([0, \infty), \mathbb{R})$
$D(X, Y)$	Class of right continuous functions having left limits defined from X to Y
$D[0, 1]$	$D([0, 1], \mathbb{R})$
$D[0, \infty)$	$D([0, \infty), \mathbb{R})$
δ_x	Dirac measure concentrated on x
∇	Gradient operator: $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d})$
∇^2	The Hessian Matrix
$\langle \cdot, \cdot \rangle$	Inner product operator
$\mathcal{P}(X)$	The collection of all probability measures on X
$\mathcal{M}(X)$	The collection of all subprobability measures on X
λ	Lebesgue measure on \mathbb{R}
$\text{dist}(A, B)$	The distance between two sets $A, B \subseteq \mathbb{R}^d$: $\inf\{ x - y : x \in A, y \in B\}$
$\mathbb{I} = \mathbb{I}_{K \times K}$	The identity matrix for some $K \in \mathbb{N}$
$ x $	The L_1 norm of $x \in \mathbb{R}^d$, i.e., $\sum_{i=1}^d x_i $
$a \wedge b$	$\min\{a, b\}$
$a \vee b$	$\max\{a, b\}$
a^+	$\max\{0, a\}$

Markov process notation

Φ	A continuous time Markov process
$\check{\Phi}$	A discrete time sampled Markov chain from Φ
$\mathcal{B}(X)$	Borel σ -field on metric space X
$P_x(\cdot)$	Probability measure conditional on $\Phi_0 = x$
$\mathbb{E}_x[\cdot]$	Expectation operator with respect to the law P_x
φ	Irreducibility measure
ψ	Maximal irreducibility measure
$\mathcal{B}^+(X)$	Sets in $\mathcal{B}(X)$ with positive ψ measure
τ_A	The first hitting time on A : $\inf\{t \geq 0 : \Phi_t \in A\}$
η_A	Sojourn time on A : $\int_0^\infty 1_{\{\Phi_t \in A\}} dt$
$P^t(x, A)$	Transition kernel of a continuous time Markov process Φ : $P(\Phi_t \in A \Phi_0 = x)$
$\check{P}^n(x, A)$	Transition kernel of a discrete time Markov process $\check{\Phi}$: $P(\check{\Phi}_n \in A \check{\Phi}_0 = x)$
\mathfrak{R}_β	Resolvent kernel: $\int P^t(x, A) \beta \exp(-\beta t) dt$
$K_a(x, A)$	K_a -chain transition kernel: $\int P^t(x, A) a(dt)$
$\mu(f)$	$\int f d\mu$
π	Invariant measure
$X \stackrel{\mathcal{L}}{=} Y$	Random variables X and Y are the same in law.

Drift and generator notation

$\Delta V(x)$	Drift operator: $\int P(x, dy) V(y) - V(x)$
$\bar{\mathcal{A}}$	The bounded-pointwise generator
$\mathcal{D}(\bar{\mathcal{A}})$	The domain of the generator $\bar{\mathcal{A}}$
$\tilde{\mathcal{A}}$	The extended generator
$\mathcal{D}(\tilde{\mathcal{A}})$	The domain of the generator $\tilde{\mathcal{A}}$

Norms

$\ \mu\ $	Total variation norm of a signed measure μ : $\sup_{f: f \leq 1} \mu(f) $
$\ \mu\ _f$	f -norm of μ : $f \geq 1$, $\sup_{ g \leq f} \mu(g) $
$\ h\ _V$	V -norm of measurable function h : $\sup_{x \in X} \frac{ h(x) }{V(x)}$
$\ P\ _V$	V -norm of kernel $P = P(x, dy)$: $\sup_{h: \ h\ _V \neq 0} \frac{\ Ph\ _V}{\ h\ _V}$
$\ f\ _\infty$	$\sup_{x \in X} f(x) $ for $f \in C_b(X)$

Abbreviations and symbols

SRBM	Semimartingale Reflecting Brownian Motion
RCLL	Right Continuous and having Left Limit
CLT	Central Limit Theorem
LLN	Law of Large Numbers

FCLT	Functional Central Limit Theorem
LHS, RHS	Left Hand Side, Right Hand Side
\mathbb{P} -a.s.	almost surely with respect to probability measure \mathbb{P}
$M \stackrel{\mathcal{L}}{=} N$	Random variables M and N are equal in law
$\mu_n \Rightarrow \mu$	μ_n converges weakly to μ
$f_n \rightarrow f$ u.o.c.	$\sup_{0 \leq s \leq t} f_n(s) - f(s) $ for all $t \geq 0$: $f_n \rightarrow f$ uniformly on compact sets
$A \doteq B$	A is defined by B
$A \equiv B$	A and B are identically equal

BIBLIOGRAPHY

- [1] B. Ata, J. M. Harrison, and L. A. Shepp. Drift rate control of a Brownian processing system. *Ann. Appl. Probab.*, 15(2):1145–1160, 2005.
- [2] R. Atar, A. Budhiraja, and P. Dupuis. On positive recurrence of constrained diffusion processes. *Ann. Probab.*, 29(2):979–1000, 2001.
- [3] S. Balaji and S. Ramasubramanian. Asymptotics of reflecting diffusions in an orthant. *Proc. Intl. Conf. Held at Cochin, India*, 1997.
- [4] A. Bassamboo, J. M. Harrison, and A. Zeevi. Dynamic routing and admission control in high-volume service systems: asymptotic analysis via multi-scale fluid limits. *Queueing Systems Theory Appl.*, 51(3-4):249–285, 2005.
- [5] P. Billingsley. *Convergence of probability measures, Second Edition*. Wiley Series in Probability and Statistics. John Wiley and Sons, Inc., New York, 1999.
- [6] P. Brémaud. *Point processes and queues*. Springer-Verlag, New York, 1981. Martingale dynamics, Springer Series in Statistics.
- [7] A. Budhiraja. An ergodic control problem for constrained diffusion processes: Existence of optimal Markov control. *SIAM J. Control Optim.*, 42(2):532–558, 2003.
- [8] A. Budhiraja and P. Dupuis. Simple necessary and sufficient conditions for the stability of constrained processes. *SIAM J. Appl. Math.*, 59(5):1686–1700 (electronic), 1999.
- [9] A. Budhiraja and A. Ghosh. Diffusion approximations for controlled stochastic networks: An asymptotic bound for the value function. *Ann. Appl. Probab.*, 16(4):1962–2006, 2006.
- [10] A. Budhiraja and C. Lee. Stationary distribution convergence for generalized Jackson networks in heavy traffic. *Submitted for Publication*, 2007.
- [11] H. Chen, P. Yang, and D. Yao. Control and scheduling in a two-station queueing network: optimal policies and heuristics. *Queueing Systems Theory Appl.*, 18(3-4):301–332, 1994.
- [12] R. W. Cottle. A note on completely-q matrices. *Math. Program.*, 19:347–351, 1980.
- [13] J. G. Dai. On positive Harris recurrence of queueing networks: A unified approach via fluid limit models. *Ann. Appl. Probab.*, 5:49–77, 1995.

- [14] J. G. Dai. Stability of open multiclass queueing networks via fluid models. In *Stochastic networks*, volume 71 of *IMA Vol. Math. Appl.*, pages 71–90. Springer, New York, 1995.
- [15] J. G. Dai. A fluid limit model criterion for instability of multiclass queueing networks. *Ann. Appl. Probab.*, 6(3):751–757, 1996.
- [16] J. G. Dai and O. B. Jennings. Stability of general processing networks. In *Stochastic modeling and optimization*, pages 193–243. Springer, New York, 2003.
- [17] J. G. Dai and C. Li. Stabilizing batch-processing networks. *Oper. Res.*, 51(1):123–136, 2003.
- [18] J. G. Dai and S. P. Meyn. Stability and convergence of moments for multiclass queueing networks via fluid limit models. *IEEE Transactions on Automatic Control*, 40:1889–1904, 1995.
- [19] J. G. Dai and R. J. Williams. Existence and uniqueness of semimartingale reflecting Brownian motions in convex polyhedrons. *Theory of Probability and Its Applications*, 40:3–53, 1995.
- [20] A. Dembo and O. Zeitouni. *Large deviations techniques and applications, Second edition*. Application of Mathematics, Vol 38. Springer, New York, 1998.
- [21] D. Down and S. P. Meyn. Piecewise linear test functions for stability and instability of queueing networks. *Queueing Systems Theory Appl.*, 27(3-4):205–226 (1998), 1997.
- [22] D. Down, S. P. Meyn, and R. L. Tweedie. Exponential and uniform ergodicity of Markov processes. *Ann. Probab.*, 23:1671–1691, 1996.
- [23] P. Dupuis and H. Ishii. On Lipschitz continuity of the solution mapping to the Skorohod problem, with applications. *Stochastics*, 35:31–62, 1991.
- [24] P. Dupuis and K. Ramanan. Convex duality and the Skorokhod problem. I, II. *Probab. Theory Related Fields*, 115(2):153–195, 197–236, 1999.
- [25] P. Dupuis and R. J. Williams. Lyapunov functions for semimartingale reflecting Brownian motions. *Ann. Probab.*, 22(2):680–702, 1994.
- [26] R. Durrett. *Probability: Theory and Examples*. Third Edition. Duxbury Press, 2005.
- [27] S. N. Ethier and T. G. Kurtz. *Markov processes: Characterization and convergence*. Wiley, New York, 1986.
- [28] D. Gamarnik and A. Zeevi. Validity of heavy traffic steady-state approximations in open queueing networks. *Ann. Appl. Probab.*, 16(1):56–90, 2006.

- [29] J. M. George and J. M. Harrison. Dynamic control of a queue with adjustable service rate. *Oper. Res.*, 49(5):720–731, 2001.
- [30] P. W. Glynn and S. P. Meyn. A Lyapunov bound for solutions of the Poisson equation. *Ann. Appl. Probab.*, 24(2):916–931, 1996.
- [31] J. M. Harrison and M. I. Reiman. Reflected Brownian motion on an orthant. *Ann. Probab.*, 9(2):302–308, 1981.
- [32] J. M. Harrison and J. A. Van Mieghem. Dynamic control of Brownian networks: state space collapse and equivalent workload formulations. *Ann. Appl. Probab.*, 7(3):747–771, 1997.
- [33] J. M. Harrison and R. J. Williams. Brownian models of open queueing networks with homogeneous customer populations. *Stochastics*, 22(2):77–115, 1987.
- [34] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes*, volume 24 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1989.
- [35] J. R. Jackson. Networks of waiting lines. *Operations Res.*, 5:518–521, 1957.
- [36] P. R. Kumar. Scheduling queueing networks: stability, performance analysis and design. In *Stochastic networks*, volume 71 of *IMA Vol. Math. Appl.*, pages 21–70. Springer, New York, 1995.
- [37] H. J. Kushner. *Heavy traffic analysis of controlled queueing and communication networks*, volume 47 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 2001. Stochastic Modelling and Applied Probability.
- [38] H. J. Kushner and P. Dupuis. *Numerical methods for stochastic control problems in continuous time*, volume 24 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 2001. Stochastic Modelling and Applied Probability.
- [39] H. J. Kushner and L. F. Martins. Limit theorems for pathwise average cost per unit time problems for controlled queues in heavy traffic. *Stochastics Stochastics Rep.*, 42(1):25–51, 1993.
- [40] H. J. Kushner and K. M. Ramachandran. Optimal and approximately optimal control policies for queues in heavy traffic. *SIAM J. Control Optim.*, 27(6):1293–1318, 1989.
- [41] S. P. Meyn. Stability and optimization of queueing networks and their fluid models. In *Mathematics of stochastic manufacturing systems (Williamsburg, VA, 1996)*, volume 33 of *Lectures in Appl. Math.*, pages 175–199. Amer. Math. Soc., Providence, RI, 1997.

- [42] S. P. Meyn. Stability, performance evaluation, and optimization. In *Handbook of Markov decision processes*, volume 40 of *Internat. Ser. Oper. Res. Management Sci.*, pages 305–346. Kluwer Acad. Publ., Boston, MA, 2002.
- [43] S. P. Meyn and D. Down. Stability of generalized Jackson networks. *Ann. Appl. Probab.*, 4(1):124–148, 1994.
- [44] S. P. Meyn and R. L. Tweedie. Generalized resolvents and Harris recurrence of Markov processes. *Proceedings of the Doeblin Conference held November 2-7, at the University of Tübingen's Heinrich Fabri Institut*, 1991.
- [45] S. P. Meyn and R. L. Tweedie. *Markov chains and stochastic stability*. Springer-Verlag, London, 1993.
- [46] S. P. Meyn and R. L. Tweedie. Stability of Markovian processes II: Continuous-time processes and sampled chains. *Advanced Applied Probability*, 25:497–517, 1993.
- [47] S. P. Meyn and R. L. Tweedie. Stability of Markovian processes III: Foster-Lyapunov criteria for continuous-time processes. *Advanced Applied Probability*, 25:518–548, 1993.
- [48] S. P. Meyn and R. L. Tweedie. *A survey of Foster-Lyapunov conditions for general state space Markov processes*. Proceedings of the Workshop on Stochastic Stability and Stochastic Stabilization. Springer-Verlag, 1994.
- [49] E. Plambeck, S. Kumar, and J. M. Harrison. A multiclass queue in heavy traffic with throughput time constraints: asymptotically optimal dynamic controls. *Queueing Syst.*, 39(1):23–54, 2001.
- [50] P. E. Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, second edition, 2004.
- [51] M. I. Reiman. Open queueing networks in heavy traffic. *Math. Oper. Res.*, 9(3):441–458, 1984.
- [52] M. I. Reiman and R. J. Williams. A boundary property of semimartingale reflecting Brownian motions. *Probability Theory and Related Fields*, 77:87–97, 1988.
- [53] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.
- [54] K. Sigman. The stability of open queueing networks. *Stochastic Process. Appl.*, 35(1):11–25, 1990.
- [55] A. V. Skorohod. *Asymptotic methods in the theory of stochastic differential equations*. Translations of Mathematical Monographs Vol. 78. American Mathematical Society, Providence, Rhode Island, 1989.

- [56] D. W. Stroock and S. R. S. Varadhan. *Multidimensional diffusion processes*. Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1997 edition.
- [57] L. M. Taylor and R. J. Williams. Existence and uniqueness of semimartingale reflecting Brownian motions in an orthant. *Probability Theory and Related Fields*, 96:283–317, 1993.
- [58] R. J. Williams. Diffusion approximations for open multiclass queueing networks: Sufficient conditions involving state space collapse. *Queueing Systems Theory Appl.*, 30(1-2):27–88, 1998.
- [59] K. Yamada. Diffusion approximation for open state-dependent queueing networks in the heavy traffic situation. *Ann. Appl. Probab.*, 5(4):958–982, 1995.