DESCENDING G-EQUIVARIANT LINE BUNDLES TO GIT QUOTIENTS

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ABSTRACT

Nathaniel Bushek: Descending G-equivariant line bundles to GIT quotients
(Under the direction of Shrawan Kumar)

In part one, we consider descent of line bundles to GIT quotients of products of flag varieties. Let G be a simple, connected, algebraic group over $\mathbb{C}$, B a Borel subgroup, and $T \subset B$ a maximal torus. Consider the diagonal action of G on the projective variety $(G/B)^3 = G/B \times G/B \times G/B$. For any triple (χ_1, χ_2, χ_3) of regular characters there is a G-equivariant line bundle $\mathcal{L}$ on $(G/B)^3$. Then, $\mathcal{L}$ is said to descend to the GIT quotient $\pi : [(G/B)^3(\mathcal{L})]^{ss} \to (G/B)^3(\mathcal{L})//G$ if there exists a line bundle $\hat{\mathcal{L}}$ on $(G/B)^3(\mathcal{L})//G$ such that $\mathcal{L}\mid_{[(G/B)^3(\mathcal{L})]^{ss}} \cong \pi^*\hat{\mathcal{L}}$.

Let Q be the root lattice, Λ the weight lattice, and d the least common multiple of the coefficients of the highest root θ of $\mathfrak{g}$, the Lie algebra of G, written in terms of simple roots. We show that $\mathcal{L}$ descends if $\chi_1, \chi_2, \chi_3 \in d\Lambda$ and $\chi_1 + \chi_2 + \chi_3 \in \Gamma$, where Γ is the intersection over root lattices $Q_{\mathfrak{s}}$ of all semisimple Lie subalgebras $\mathfrak{s} \subset \mathfrak{g}$ of maximal rank. Moreover, we show that $\mathcal{L}$ never descends if $\chi_1 + \chi_2 + \chi_3 \notin Q$.

In part two, we discuss joint work with Shrawan Kumar. Let $\mathfrak{g}$ be any simple Lie algebra over $\mathbb{C}$. Recall that there exists a principal TDS embedding of $\mathfrak{sl}_2$ into $\mathfrak{g}$ passing through a principal nilpotent element of $\mathfrak{g}$. Moreover, $\wedge(\mathfrak{g}^*)^{\mathfrak{g}}$ is generated by primitive elements $\omega_1, \ldots, \omega_\ell$, where ℓ is the rank of $\mathfrak{g}$. N. Hitchin conjectured that for any primitive element $\omega \in \wedge^d(\mathfrak{g}^*)^{\mathfrak{g}}$, there exists an irreducible $\mathfrak{sl}_2$-submodule $V_\omega \subset \mathfrak{g}$ of dimension d such that ω is non-zero on the line $\wedge^d(V_\omega)$. We prove that the validity of this conjecture for simple simply-laced Lie algebras implies its validity for any simple Lie algebra.

Let G be a connected, simply-connected, simple, simply-laced algebraic group and let σ be a diagram automorphism of G with fixed subgroup K. Then, we show that the restriction map $R(G) \to R(K)$ is surjective, where R denotes the representation ring over $\mathbb{Z}$. As a corollary, we show that the restriction map in the singular cohomology $H^*(G) \to H^*(K)$ is surjective. Our proof of the reduction of Hitchin's conjecture to the simply-laced case relies on this cohomological surjectivity.

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INTRODUCTION

In this thesis we consider two separate questions that, in common, utilize the relationship between Geometric Invariant Theory (GIT) of a simple, complex algebraic group G acting on a variety and the corresponding representation theory of G. The first problem considers under what conditions a line bundle $\mathcal{L}$ on $(G/B \times G/B \times G/B)$ descends to the GIT quotient $(G/B \times G/B \times G/B)//G$. The question of descent in this context has implications to the tensor product decomposition problem. In part two, we consider a conjecture of N. Hitchin that remains an open question for all simple, complex Lie algebras $\mathfrak{g}$. In joint work with S. Kumar, we prove that if this conjecture holds for all simple $\mathfrak{g}$ of simply-laced type, then it holds for all simple $\mathfrak{g}$.

0.1 Descending line bundles to $(G/B \times G/B \times G/B)//G$

Let G be a simple, connected, complex linear algebraic group. Let B be a fixed Borel subgroup and $T \subset B$ a fixed maximal torus. Let $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{t}$, $\mathfrak{b}$, the Lie algebras of T and B. Let Λ be the weight lattices of $\mathfrak{g}$ and $X(T)$ the character group of T. We let Λ^{++} denote the regular weights and $X(T)^{++} = X(T) \cap \Lambda^{++}$ the regular characters.

Then $(G/B)^3 = (G/B \times G/B \times G/B)$ is a projective variety with a natural action of G given by the diagonal of left multiplication. Let $\mathcal{L}$ be an ample line bundle on $(G/B)^3$. By taking the external tensor product of three ample G-equivariant line bundles associated to a regular characters on G/B, ample line bundles on $(G/B)^3$ correspond to triples of regular characters on T. Let $\mathcal{L}(\chi_1, \chi_2, \chi_3)$ be the line bundle associated to the triple of regular characters (χ_1, χ_2, χ_3). Then, we consider the following question.

Question 0.1.1. *What conditions can be placed on a triple of regular characters (χ_1, χ_2, χ_3) to know that the corresponding line bundle $\mathcal{L}(\chi_1, \chi_2, \chi_3)$ on $(G/B)^3$ will or will not descend to the GIT quotient $(G/B)^3//G$?*

Although the question of descending line bundles has been considered in many contexts (e.g., [24]), the primary model for descent in this context comes from the work of Shrawan Kumar. In [17], Kumar considered the descent question for the GIT quotient $(G/P)//T$, where $B \subset P$ is a parabolic and the torus action is by left multiplication. Line bundles over G/P correspond to a single dominant weight χ (Kumar considered the simply-connected group) such that χ vanishes on the subspace of $\mathfrak{t}$ spanned by the roots of the Levi subgroup of P.

Now, for a simple $\mathfrak{g}$, let Γ denote the intersection over all root lattices $Q_{\mathfrak{s}}$ of all semisimple Lie subalgebras $\mathfrak{s}$ of $\mathfrak{g}$ of maximal rank. Using Borel-de Siebenthal theory (cf. [33]) of maximal subalgebras, Kumar gives an explicit description of the lattice Γ for each simple $\mathfrak{g}$, which is listed in Table 2.1. Finally, Kumar proves the following ([17], Theorem 3.10).

Theorem 0.1.1. *The line bundle $\mathcal{L}(\chi)$ descends to $(G/P)//T$ if and only if $\chi \in \Gamma$.*

When considering descent of a line bundle $\mathcal{L}(\chi_1, \chi_2, \chi_3)$ to $(G/B)^3//G$, the lattice Γ continues to play an important role. Let d be the least common multiple of the coefficients of the longest root θ of $\mathfrak{g}$ when expressed in terms of the simple roots (cf. Table 2.2). The following theorem is the main result of part one.

Theorem 0.1.2. *Given $\chi_1, \chi_2, \chi_3 \in X(T)^{++}$, if $\chi_1, \chi_2, \chi_3 \in d\Lambda$ and $\chi_1 + \chi_2 + \chi_3 \in \Gamma$, then $\mathcal{L}(\chi_1, \chi_2, \chi_3)$ descends to $(G/B)^3//G$.*

The proof of this theorem follows methods similar to those used by Kumar in Theorem 0.1.1. We utilize Kempf's 'descent' lemma (cf. [9], Theorem 2.3), which states that a line bundle descends if for every point $x \in (G/B)^3//G$ such that the orbit $G \cdot x$ is closed in $((G/B)^3)^{ss}$, the isotropy subgroup G_x acts trivially on the fiber $\mathcal{L}(\chi_1, \chi_2, \chi_3)_x$. However, the case of $(G/B)^3//G$ becomes complicated and additional assumptions are required that keep Theorem 0.1.2 from being optimal. Removing these assumptions requires a better understanding of $((G/B)^3)^{ss}$ than is currently available. Also, the isotropy subgroups G_x are, in general, unwieldy objects, so an important reduction is to prove that it suffices to consider only such points x with G_x a reductive group. This reduction is done in Section 2.2.

We also have the following necessary condition.

Proposition 0.1.1. *If $\chi_1 + \chi_2 + \chi_3 \notin Q$, then $\mathcal{L}(\chi_1, \chi_2, \chi_3)$ does not descend.*

Moreover, we show that when $(\chi_1, \chi_2, \chi_3) = (2\rho, \rho, \rho)$, where $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$, the corresponding line bundle always descends to $(G/B)^3//G$. Yet, the triple $(2\rho, \rho, \rho)$ violates one or both of the conditions in Theorem 0.1.2 depending on the type of $\mathfrak{g}$ chosen. Thus, we know that, outside of type A, Theorem 0.1.2 is not optimal.

The major motivation for the descent question is that when $\mathcal{L}(\chi_1, \chi_2, \chi_3)$ descends to a line bundle $\hat{\mathcal{L}}$ on $(G/B)^3//G$, we have the following isomorphism.

$$H^0((G/B)^3//G, \hat{\mathcal{L}}) \cong H^0((G/B)^3, \mathcal{L}(\chi_1, \chi_2, \chi_3))^G. \tag{1}$$

Now, using the Borel-Weil theorem, the dimension (over $\mathbb{C}$) of right hand side of equation (1) is

$$\dim[V(\chi_1) \otimes V(\chi_2) \otimes V(\chi_3)]^G,$$

which is exactly the multiplicity of the irreducible representation $V(\chi_1)^*$ inside $V(\chi_2) \otimes V(\chi_3)$. On the other hand, the left hand side of equation (1) is, due to the vanishing of higher cohomology, the Euler-Poincaré characteristic of $\hat{\mathcal{L}}$ over $(G/B)^3//G$. Then, by the Riemann-Roch Theorem for singular varieties, this value varies polynomially on open convex subsets of $X(T)^{++}$.

In this sense, we say that the tensor product multiplicity function is *piecewise polynomial*. This work is still in progress. The method for proving piecewise polynomiality is understood and follows as in [19], yet the sectors of polynomiality are not yet known. Having a full description of the sectors of polynomiality depends on knowing precisely which line bundles have a semistable locus differing from the stable locus. This has proven a difficult problem since a useful characterization of semistability and stability is still lacking. Moreover, proving a complete piecewise polynomiality result using our methods requires optimal conditions for descent.

Piecewise polynomiality of the tensor product multiplicity function is already known separately by the works of Berenstein-Zelevinsky in [2] and Meinrenken-Sjamaar in [22]. However, the proof of Berenstein-Zelevinsky is non-constructive and the proof of Meinrenken-Sjamaar uses sympletic geometry. Our proof method aims to give an explicit construction of the polynomial using algebraic-geometry, and an explicit description of the sectors of polynomiality. Therefore, we believe this proof of piecewise polynomiality will be a worthwhile contribution to the literature.

In chapter one, we develop the necessary notation and preliminary theory used in part one of this thesis. In chapter two, we give the proof of Theorem 0.1.2, prove a necessary condition for descent, providing the counter example “sufficient is necessary”, and answer a few questions on semistability.

In the appendix, we develop some characterizations of semistability and stability.

0.2 Diagram Automorphisms, GIT, and Hitchin's Conjecture

Let $\mathfrak{g}$ be a simple, complex Lie algebra and G the corresponding simply-connected, connected, linear algebraic group. In [16], Kostant proved the existence of a unique (up to the action of $\mathrm{Ad}(G)$) embedding of $\mathfrak{sl}_2$ into $\mathfrak{g}$, called a *principal TDS*, such that the image passes through a principal nilpotent element of $\mathfrak{g}$ (i.e., the image meets the open orbit of the nilpotent cone.) Under the adjoint action of a principal TDS, the Lie algebra $\mathfrak{g}$ decomposes as a direct sum of exactly ℓ irreducible $\mathfrak{sl}_2$-submodules

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_\ell,$$

such that

$$\dim(V_i) = 2m_i + 1,$$

where ℓ is the rank of $\mathfrak{g}$ and $m_1, \ldots, m_\ell$ are the exponents of $\mathfrak{g}$.

On the other hand, the singular cohomology $H^*(G) = H^*(G, \mathbb{C})$ with complex coefficients is a Hopf algebra with co-multiplication induced by the multiplication map of G. Let $P(\mathfrak{g}) \subset H^*(G)$ be the graded subspace of primitive elements. Then, $P(\mathfrak{g})$ has a basis in degrees $2m_1 + 1, \ldots, 2m_\ell + 1$, where again m_i are the exponents of $\mathfrak{g}$. We naturally identify $H^*(G)$ with $\wedge(\mathfrak{g}^*)^{\mathfrak{g}}$ and hence consider $P(\mathfrak{g})$ as a subspace of $\wedge(\mathfrak{g}^*)^{\mathfrak{g}}$.

Now, N. Hitchin made the following conjecture [13].

Conjecture 0.2.1. *Let $\mathfrak{g}$ be any simple Lie algebra. For any primitive element $\omega \in P_d \subset \wedge^d(\mathfrak{g}^*)^{\mathfrak{g}}$, there exists an irreducible sub-module $V_\omega \subset \mathfrak{g}$ of dimension d with respect to the principal TDS action such that*

$$\omega|_{\wedge^d(V_\omega)} \neq 0.$$

The main motivation for Hitchin behind the above conjecture lies in its connection with the study of polyvector fields on the moduli space $M_G(\Sigma)$ of semistable principal G-bundles on a smooth projective curve Σ of any genus $g > 2$. Specifically, observe that the cotangent space at a smooth point E of $M_G(\Sigma)$ is isomorphic with $H^0(\Sigma, \mathfrak{g}(E) \otimes \Omega)$, where $\mathfrak{g}(E)$ denotes the associated adjoint bundle and Ω is the canonical bundle of the curve Σ. Given a bi-invariant differential form ω of degree

k on G, i.e., $\omega \in \wedge^k(\mathfrak{g}^*)^{\mathfrak{g}}$, and elements $\Phi_j \in H^0(\Sigma, \mathfrak{g}(E) \otimes \Omega), 1 \leq j \leq k,\ \ \omega(\Phi_1, \ldots, \Phi_k)$ defines a skew form with values in the line bundle Ω^k. Dually, it defines a homomorphism

$$\Theta_\omega : H^1(\Sigma, \Omega^{1-k}) \to H^0(M_G(\Sigma), \wedge^k T),$$

where T is the tangent bundle of $M_G(\Sigma)$. As shown by Hitchin, the validity of the above conjecture would imply that the map Θ_ω is injective for any invariant form $\omega \in \wedge^k(\mathfrak{g}^*)^{\mathfrak{g}}$ (cf. [13]).

The following reduction theorem is the main result of part two ([4] Theorem 2.5).

Theorem 0.2.1. *If Hitchin's conjecture is valid for any simply-laced simple Lie algebra $\mathfrak{g}$, then it is valid for any simple Lie algebra.*

More precisely, if Hitchin's conjecture is valid for $\mathfrak{g}$ of type $(A_{2\ell-1}; A_{2\ell}; D_4; E_6)$, then it is valid for $\mathfrak{g}$ of type $(C_\ell; B_\ell; G_2; F_4)$ respectively.

Thus, one needs to verify the conjecture only for the simple Lie algebras of types A, D and E. The reduction of the conjecture from the simply-laced, simple Lie algebras to all simple Lie algebra relies on the realization of any simple Lie algebra $\mathfrak{k}$ as the fixed point subalgebra of a diagram automorphism of an appropriate simple simply-laced Lie algebra $\mathfrak{g}$ (cf. [28]).

Let K be the algebraic subgroup of G with Lie algebra $\mathfrak{k}$, where $\mathfrak{k}$ is the fixed-point subalgebra under a diagram automorphism of a simple simply-laced Lie algebra $\mathfrak{g}$. We utilized the description of the root systems given by Springer in [28] to prove the following theorem ([4] Theorem 3.1).

Theorem 0.2.2. *The canonical map $\phi : R(G) \to R(K)$ is surjective, where $R(G)$ denotes the representation ring of G (over $\mathbb{Z}$).*

In particular, the canonical map $K//\mathrm{Ad}\,K \to G//\mathrm{Ad}\,G$, between the GIT quotients, is a closed embedding.

Let $S^\bullet(V)$ be the symmetric algebra on V. We have the following Lie algebraic analogue ([4] Theorem 3.4) of the previous theorem.

Theorem 0.2.3. *The canonical restriction map $\psi : S^\bullet(\mathfrak{g}^*)^{\mathfrak{g}} \to S^\bullet(\mathfrak{k}^*)^{\mathfrak{k}}$ is surjective.*

Finally, we use H. Cartan's transgression map (cf. [5], [21])

$$\tau : (S^+\mathfrak{g}^*)^{\mathfrak{g}} \to (\wedge \mathfrak{g}*)^{\mathfrak{g}},$$

and similarly for $\mathfrak{k}$, and the surjectivity of ψ, to obtain the surjectivity of $\gamma_o : P(\mathfrak{g}) \to P(\mathfrak{k})$, and thereby the surjectivity of $\gamma : H^*(G) \to H^*(K)$. In our view, the surjectivity of ϕ, γ and γ_o is of independent interest. Then, the proof of Theorem 0.2.1 relies on constructing a principal TDS in $\mathfrak{k}$ which remains a principal TDS in $\mathfrak{g}$. We use the surjectivity of γ_o to lift primitive elements $\omega_d \in \wedge^d(\mathfrak{k}^*)^{\mathfrak{k}}$ to primitive elements $\widetilde{\omega}_d \in \wedge^d(\mathfrak{g}^*)^{\mathfrak{g}}$. In this way, we are able to use non-vanishing assumptions on $\widetilde{\omega}_d$ to imply non-vanishing results of ω_d .

In chapter six, we develop the notation and preliminary theory needed for part two of this thesis, including a full construction of the root datum of K as the fixed point subgroup of G, a discussion of principal TDS embeddings, and a description of the transgression map. In chapter seven, we give a full proof of Theorem 0.2.2. Last, in chapter eight, we give a full proof of Theorems 0.2.1 and 0.2.3.

CHAPTER 1: NOTATION AND PRELIMINARIES I

In this chapter we give a brief overview of the theory and notation needed for part one of the thesis. In this first section, we discuss the necessary structure of linear algebraic groups. In section two, we introduce the basics of Geometric Invariant Theory (GIT) and the ideas fundamental to variation of GIT quotients. In section three, we discuss the notation and background of the representation theory of G, as well as the Borel-Weil theorem, which provides the connection between geometry and representation theory. In section four, we give a quick overview of the intersection theory later used.

1.1 Group Theory

Let G be a *linear algebraic group* over $\mathbb{C}$, that is, a group that is also a complex affine variety. For any G, there is a unique maximal, closed, connected, normal, solvable subgroup $R(G)$ of G, called the *radical* of G. There is also a unique maximal closed, connected, unipotent subgroup $R_u(G)$ of G, called the *unipotent radical* of G. Note that $R_u(G) \subset R(G)$. Then, we say that G is *reductive* if $R_u(G) = \{e\}$ and *semisimple* if $R(G) = \{e\}$.

Any maximal, solvable, closed subgroup of G is called a *Borel* subgroup, denoted by B. Borel subgroups have the property of being minimal subgroups of G such that the quotient G/B is a projective variety. All Borel subgroups of G are conjugate to each other. A subgroup of G is called a *torus* if it is isomorphic to $(\mathbb{C}^*)^k$ for some k. The maximal tori of G are all conjugate, and for any Borel subgroup B, the maximal tori of G contained in B are conjugate by B. We will fix a maximal torus T contained in a fixed Borel subgroup B. When G is reductive, the *Weyl group* of G is $W := N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G, and acts on T by conjugation.

Given any algebraic group G, the tangent space at the identity is a Lie algebra $\mathfrak{g}$. Therefore, we also have Lie algebras $\mathfrak{b}$ and $\mathfrak{t}$ corresponding to B and T. The dimension of $\mathfrak{t}$ is called the *rank* of $\mathfrak{g}$, denoted $\mathrm{rank}(\mathfrak{g})$. A Lie algebra $\mathfrak{g}$ is called *simple* if it has no non-zero, proper ideals. We say that the algebraic group G is *simple* if its Lie algebra $\mathfrak{g}$ is simple. Any simple group is semisimple.

There are natural connections between G and $\mathfrak{g}$. First, there is the exponential map: $\exp : \mathfrak{g} \to G$.

In general, exp is neither a homomorphism nor a morphism of varieties. Second, G acts naturally on its Lie algebra $\mathfrak{g}$ by the Adjoint action. For $g \in G$ and $X \in \mathfrak{g}$,

$$\mathrm{Ad}(g) \cdot X = \frac{d}{dt}_{t=0}(g \exp(tX) g^{-1}).$$

Two important properties of the Adjoint action relevant to our use are the following. For any $g \in G$ and $X, Y \in \mathfrak{g}$,

$$\mathrm{Ad}(g)^{-1} = \mathrm{Ad}(g^{-1}), \quad \& \quad \mathrm{Ad}(g) \cdot [X, Y] = [\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y]. \tag{1.1}$$

Through the Adjoint action, W acts naturally on $\mathfrak{t}$ and $\mathfrak{t}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})$. Moreover, by differentiation of $\mathrm{Ad} : G \to GL(\mathfrak{g})$, we get the adjoint action $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ given by

$$\mathrm{ad}(X) \cdot Y = [X, Y].$$

Hence, $\mathfrak{t}$ acts naturally on $\mathfrak{g}$ by the adjoint action, and since $\mathfrak{t}$ is diagonalizable, this affords an eigenspace decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha. \tag{1.2}$$

The linear functions $\alpha : \mathfrak{t} \to \mathbb{C}$ arising as eigenvalues in this decomposition are called the *roots* of $\mathfrak{g}$. The set of roots is denoted R and is W-invariant.

There is a basis of R, denoted by Δ. We call the roots in Δ the simple roots. Note that $|\Delta| = \mathrm{rank}(\mathfrak{g})$. Then, every root $\alpha \in R$ is an integral linear combination of simple roots with either all non-negative coefficients or all non-positive coefficients. The non-negative (non-positive) linear combinations are called the *positive (negative) roots*, and the set is denoted by R^+ (R^-). Note that $R^- = -R^+$. Then, fixing a Borel subgroup amounts to fixing R^+ via

$$\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha. \tag{1.3}$$

Given any $w \in W$, define the *set of inversions* of w, denoted $R(w)$, as follows.

$$R(w) := \{\beta \in R^+ \mid \ w \cdot \beta \in -R^+\}. \tag{1.4}$$

Assume that G is semisimple and $n = \mathrm{rank}(\mathfrak{g})$. There is a natural bilinear form $(\cdot,\cdot) : \mathfrak{t}^* \times \mathfrak{t}^* \to \mathbb{C}$ induced by the Killing form on $\mathfrak{g}$. If $\Delta = \{\alpha_1, \cdots, \alpha_n\}$, then $\alpha_i^\vee := \dfrac{2\alpha_i}{(\alpha_i,\alpha_i)}$ are called the *co-roots* of $\mathfrak{g}$. Let $\varpi_1, \cdots, \varpi_n \in \mathfrak{t}^*$ be such that $(\varpi_i, \alpha_j^\vee) = \delta_{i,j}$; these ϖ_i are called the *fundamental weights*.

The *character group* of T, denoted $X(T)$, is the group of all homomorphisms $T \to \mathbb{C}^*$. $X(T)$ is a rank n lattice. Moreover, the $\mathbb{Z}$-span of the fundamental weights inside $\mathfrak{t}^*$ forms a lattice Λ, called the *weight lattice*, of rank n; and the $\mathbb{Z}$ span of Δ forms a rank n lattice Q called the *root lattice*. Upon differentiation of characters, we have the following containment.

$$Q \subset X(T) \subset \Lambda. \tag{1.5}$$

Note that Q and Λ only depend on $\mathfrak{g}$ whereas $X(T)$ depends on G. Several different groups G will all have the same Lie algebra $\mathfrak{g}$. When $X(T) = \Lambda$, G is called the *simply-connected* group, and when $X(T) = Q$, G is called the *adjoint* group.

An element $\lambda \in \Lambda$ is called a *weight*. If $\lambda \in \Lambda$ is a non-negative (resp. positive) linear combination of fundamental weights, λ is a *dominant* (resp.*regular*) weight. We denote the set of dominant (resp. regular) weights as Λ^+ (resp. Λ^{++}). Moreover, we define the set of *dominant characters* (resp. *regular characters*) to be $X(T)^+ := X(T) \cap \Lambda^+$ (resp. $X(T)^{++} := X(T) \cap \Lambda^{++}$). When we are considering a weight $\lambda \in \Lambda$ as a character, we will write e^λ to emphasize e^λ as a homomorphsim $T \to \mathbb{C}^*$.

Now, for each $\beta \in R$, the restriction of the exponential map to $\mathfrak{g}_\beta$ is an isomorphism of varieties. The image, $U_\beta := \exp(\mathfrak{g}_\beta)$ is a one-dimensional, closed subgroup of G. In particular, for each $\beta \in R$ there is an isomorphism of varieties $u_\beta : \mathbb{C} \to U_\beta$ chosen to satisfy the following:

(i) $\dot{w} u_\beta(s) \dot{w}^{-1} = u_{w\cdot\beta}(s)$ for $w \in W$ and $\dot{w} \in N_G(T)$ any representative,

(ii) and $t u_\beta(s) t^{-1} = u_\beta(e^\beta(t)s)$ for all $t \in T$.

Let $U^+ = R_u(B)$. Then it is well known that $B = TU^+$. We shall refer to this as the *TU decomposition* of B. Moreover, given any ordering $(\beta_1, \beta_2, \ldots, \beta_m)$ of R^+, where $m = |R^+|$, the product map

$$\mathbb{C}^m \to U^+ , \quad (s_1, \ldots, s_m) \mapsto u_{\beta_1}(s_1) u_{\beta_2}(s_2) \cdots u_{\beta_m}(s_m), \tag{1.6}$$

is an isomorphism of varieties. In particular $U^+ \cong \prod_{\beta\in R^+} U_\beta$. Similarly, if we let $U^- = \prod_{\beta\in -R^+} U_\beta$, then we define the *opposite Borel subgroup* of G to be $B^- := TU^-$ (cf. [29] Proposition 8.2.1). An isomorphsim $\mathbb{C}^m \to U^-$ also exists similar to in equation (1.6) for any ordering of $-R^+$.

Lastly, we have the Bruhat decomposition.

$$G = \coprod_{w \in W} B\dot{w}B = \coprod_{w \in W} U_{w^{-1}}\dot{w}B = \coprod_{w \in W} \dot{w}U^-_{w^{-1}}B. \tag{1.7}$$

Here $U_{w^{-1}} = \prod_{\beta \in R(w^{-1})} U_\beta \subset U$ and $U^-_{w^{-1}} = \dot{w}^{-1}U_{w^{-1}}\dot{w} = \prod_{\beta \in R(w^{-1})} U_{w^{-1}\beta} \subset U^-$, and $\dot{w} \in N_G(T)$ denotes any lift of $w \in W$. Then,

$$G/B \cong \coprod_{w \in W} C(w),$$

where $C(w) := U_{w^{-1}}\dot{w}B/B \cong \mathbb{C}^{\ell(w)}$, where $\ell(w)$ is the *length* of $w \in W$ (cf. [29] §8.3.1).

1.2 Geometric Invariant Theory

Here we consider the theory of quotients by a reductive group G acting on a projective variety X. Given a line bundle $\mathcal{L}$ over X, we say that $\mathcal{L}$ is a G-*equivariant* line bundle, or G-*linearized*, if there is a G action on $\mathcal{L}$ that is linear on fibers and such that the bundle map $\mathcal{L} \to X$ is G-equivariant. Given a G-equivariant ample line bundle $\mathcal{L}$ on X, the subsets of *semistable* points, denoted X^{ss}, and *stable* points, denoted X^s, are described as follows.

$$\begin{aligned} X^{ss} &:= \{x \in X \mid \exists \sigma \in H^0(X, \mathcal{L}^{\otimes N})^G \text{ such that } \sigma(x) \neq 0\} \\ X^s &:= \{x \in X^{ss} \mid G \cdot x \text{ closed in } X^{ss} \ \& \ \dim G_x = 0\}. \end{aligned} \tag{1.8}$$

Here, G_x denotes the isotropy subgroup of x.

Mumford (cf. [23]) devised a numerical criterion for determining semistability and stability. Given any one-parameter subgroup δ of G, i.e., an algebraic group homomorphism $\delta : \mathbb{C}^* \to G$, and any $x \in X$, the projectivity of X implies that the limit $x_0 := \lim_{s \to 0} \delta(s) \cdot x$ exists. Clearly, x_0 is a δ-fixed point. Thus, $\delta(\mathbb{C}^*)$ acts on the fiber $\mathcal{L}_{x_0}$ by a $\mathbb{C}^*$-character, i.e., $\delta(s) \cdot v_{x_0} = s^r v_{x_0}$ for some $r \in \mathbb{Z}$. Then, define the Mumford index $\mu^{\mathcal{L}}(x, \delta) := -r$. Semistability and stability then have the following characterization

$$\begin{aligned} x \in X^{ss} &\iff \mu^{\mathcal{L}}(x, \delta) \geq 0 \quad \text{for all non-trivial one-parameter subgroups } \delta \text{ of } G \\ x \in X^s &\iff \mu^{\mathcal{L}}(x, \delta) > 0 \quad \text{for all non-trivial one-parameter subgroups } \delta \text{ of } G. \end{aligned} \tag{1.9}$$

For any one parameter subgroup $\delta : \mathbb{C}^* \to G$, the Mumford index satisfies the following properties (cf. [23]).

Proposition 1.2.1. *(i)* $\mu^{\mathcal{L}}(g \cdot x, g\delta g^{-1}) = \mu^{\mathcal{L}}(x, \delta)$ *for all* $g \in G$ *and* $x \in X$*,*

(ii) $\mu^{\mathcal{L}}(\lim_{s\to 0} \delta(s) \cdot x, \delta) = \mu^{\mathcal{L}}(x, \delta)$ *for all* $x \in X$*,*

(iii) if X_1*,* X_2*, and* X_3 *respectively have* G*-equivariant line bundles* $\mathcal{L}_1$*,* $\mathcal{L}_2$*, and* $\mathcal{L}_3$*, and* $p_i : X_1 \times X_2 \times X_3 \to X_i$ *is the* i*th projection; then, for* $\underline{\mathcal{L}} := p_1^*\mathcal{L}_1 \otimes p_2^*\mathcal{L}_2 \otimes p_3^*\mathcal{L}_3$*,*

$$\mu^{\underline{\mathcal{L}}}((x_1, x_2, x_3), \delta) = \mu^{\mathcal{L}_1}(x_1, \delta) + \mu^{\mathcal{L}_2}(x_2, \delta) + \mu^{\mathcal{L}_3}(x_3, \delta),$$

when $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$*.*

A fundamental result of geometric invariant theory (GIT) provides the existence of a projective variety $X//G$, called the *GIT quotient*, and a surjective morphism $\pi : X^{ss} \to X//G$ that is a *good quotient*(cf. [23] Theorem 1.10). Now we come to a central definition.

Definition 1.2.1. *We say that a line bundle* $\mathcal{L}$ *on* X *descends to a line bundle on* $X//G$ *if there exists a line bundle* $\hat{\mathcal{L}}$ *on* $X//G$ *such that* $\pi^*(\hat{\mathcal{L}}) \cong \mathcal{L}|_{X^{ss}}$*, where the isomorphism is* G*-equivariant.*

Following ([31], section 3), the *invariant direct image*, $\pi_*^G(\mathcal{L})$ of a line bundle $\mathcal{L}$ over X is the coherent sheaf on $X//G$ whose local sections are the G-invariant sections of $\pi^*\mathcal{L}$. Then, $\pi_*^G \circ \pi^* = \mathrm{Id}$. Further, if we suppose that there are two line bundles $\hat{\mathcal{L}}_1$ and $\hat{\mathcal{L}}_2$ on $X//G$ such that $\pi^*(\hat{\mathcal{L}}_1) \cong \mathcal{L}_{X^{ss}} \cong \pi^*(\hat{\mathcal{L}}_2)$, by applying π_*^G to both sides we get the following known lemma.

Lemma 1.2.1. *If* $\mathcal{L}$ *descends to* $\hat{\mathcal{L}}$ *on* $X//G$*, then* $\hat{\mathcal{L}}$ *is unique up to isomorphism.*

Now, recall the following 'descent' lemma of Kempf ([9], Theorem 2.3) adapted to our setting.

Lemma 1.2.2. $\mathcal{L}$ *descends to* $X//G$ *if and only if for any* $x \in X^{ss}$*, the isotropy subgroup* G_x *acts trivially on the fiber* $\mathcal{L}_x$*. In fact, for the 'if' part, it suffices to assume that* G_x *acts trivially for only those* $x \in X^{ss}$ *such that the orbit* $G \cdot x$ *is closed in* X^{ss}*.*

The question of how GIT quotients vary depending on the line bundle $\mathcal{L}$ was considered by Dolgachev-Hu in [8]. Define the *G-ample cone*, denoted $C^G(X)$, to be the cone generated by ample line bundles $\mathcal{L}$ such that $X^{ss}(\mathcal{L}) \neq \emptyset$, where $X^{ss}(\mathcal{L})$ is the semistable locus determined by $\mathcal{L}$. Then,

$C^G(X)$ is a convex cone inside $\mathrm{NS}^G(X) \otimes_{\mathbb{Z}} \mathbb{R}$, where $\mathrm{NS}^G(X)$ is the Neron-Severi group which is a quotient of the Abelian group $\mathrm{Pic}^G(X)$ of G-equivariant line bundles over X. When $X = (G/B)^3$, $NS^G(X) = Pic^C(X)$ and $C^G(X) \subset (X(T)^{++})^3$, since χ_1, χ_2, and χ_3 must all be regular in order for $\mathcal{L}(\chi_1, \chi_2, \chi_3)$ to be ample.

Within $C^G(X)$ define an equivalence relation $\mathcal{L} \sim \mathcal{L}'$ if and only if $X^{ss}(\mathcal{L}) = X^{ss}(\mathcal{L}')$. Then, the equivalence classes of $\mathcal{L}$ such that $X^s(\mathcal{L}) = X^{ss}(\mathcal{L})$ are called *chambers*. If $\mathcal{L} \in C^G(X)$ does not belong to a chamber, i.e., $X^{ss}(\mathcal{L}) \neq X^s(\mathcal{L})$, then the equivalence class of $\mathcal{L}$ is called a *wall*. In this way, we partition $C^G(X)$ into chambers and walls. Chambers are open convex cones and walls are closed convex cones. There are finitely many walls, and since the chambers are the connected components of the complements of the union of walls, there are also finitely many chambers ([8], Theorem 3.3.3). In the case of $(G/B)^3$, the boundary of $C^G(X) = (X(T)^+)^3$ consists of precisely those ample line bundles with $X^{ss} \neq \emptyset$ and $X^s = \emptyset$ ([8], Proposition 3.2.8, Proposition 3.3.5, and Corollary 4.1.9). Further, in our setting, walls are always of positive co-dimension in $C^G(X)$.

1.3 Representation Theory and the Borel-Weil Theorem

Ample line bundles on G/B correspond to regular characters of T. Because of $B = TU$ decomposition, any character $e^\chi \in X(T)$ extends to a character on B by setting $e^\chi|_U \equiv 1$. Let B act on $\mathbb{C}$ by the character $e^{-\chi}$ and denote this one dimensional B representation by $\mathbb{C}_{-\chi}$. Then, define the quotient $\sim$ on $G \times \mathbb{C}_{-\chi}$ by $(gb^{-1}, bz) \sim (g, z)$ for all $b \in B$, $g \in G$ and $z \in \mathbb{C}_{-\chi}$. We denote the class of (g, z) by $[g, z]$. Then, define $G \times_B \mathbb{C}_{-\chi} := G \times \mathbb{C}_{-\chi}/\sim$. This is a G-equivariant line bundle over G/B, denoted $\mathcal{L}(\chi)$, with the bundle map $[g, z] \mapsto gB$ and G action given by the left multiplication. In fact, every G-equivariant line bundles over G/B is formed in this way.

All representations of G considered here will be finite dimensional, and admit an eigenspace decomposition with respect to the action of T. We call such a decomposition a *weight space decomposition*; the T-eigenvalues are called the *weights* of the representation and the T-eigenspaces are called the *weight spaces*. There is a partial order on the set of weights, where $\lambda > \mu$ if $\mu = \lambda - \sum_{\alpha \in \Delta} k_\alpha \alpha$ and $k_\alpha \in \mathbb{Z}_{\geq 0}$. Moreover, in each irreducible representation there is a unique line that is fixed by B and the corresponding weight is highest with respect to the partial order. We call any vector spanning the unique B-fixed line the *highest weight vector* and the corresponding weight, χ, the *highest weight*. Each irreducible representation is determined by its unique highest weight.

Conversely, given any dominant weight χ, there is a unique irreducible representation with χ as its highest weight. We denote the representation of highest weight χ by $V(\chi)$. Given a representation $V(\chi)$, the dual representation $V(\chi)^*$ has highest weight $-w_0\chi$, where $w_0 \in W$ is the longest element in the Weyl group. Then, we have the following well known Borel-Weil theorem (cf. [26]).

Theorem 1.3.1.

$$H^0(G/B, \mathcal{L}(\chi)) \cong V(\chi)^*. \tag{1.10}$$

Similarly, triples of regular characters (χ_1, χ_2, χ_3) correspond to the ample line bundles on $(G/B)^3$ as follows. Let $p_i : (G/B)^3 \to G/B$ be the projection onto the i-th coordinate. Define

$$\mathcal{L}(\chi_1) \boxtimes \mathcal{L}(\chi_2) \boxtimes \mathcal{L}(\chi_3) := p_1^*\mathcal{L}(\chi_1) \otimes p_2^*\mathcal{L}(\chi_2) \otimes p_3^*\mathcal{L}(\chi_3),$$

which is an ample line bundle on $(G/B)^3$. Let us simply denote $\mathcal{L}(\chi_1)\boxtimes\mathcal{L}(\chi_2)\boxtimes\mathcal{L}(\chi_3)$ by $\mathcal{L}(\chi_1, \chi_2, \chi_3)$, or even by $\mathcal{L}$ when no confusion is likely. By applying the Borel-Weil theorem we have the following.

$$H^0((G/B)^3, \mathcal{L}(\chi_1, \chi_2, \chi_3))^G \cong [V(\chi_1)^* \otimes V(\chi_2)^* \otimes V(\chi_3)^*]^G. \tag{1.11}$$

Since

$$\dim[V(\chi_1)^* \otimes V(\chi_2)^* \otimes V(\chi_3)^*]^G = \dim[V(\chi_1) \otimes V(\chi_2) \otimes V(\chi_3)]^G,$$

it follows that there is a semistable point of $(G/B)^3$ relative to $\mathcal{L}(\chi_1, \chi_2, \chi_3)$ if and only if, for some $N > 0$,

$$\dim[V(N\chi_1) \otimes V(N\chi_2) \otimes V(N\chi_3)]^G \neq 0.$$

In particular, $C^G((G/B)^3)$ is exactly the saturated tensor semigroup $\Gamma_3(G)$ of G (cf. [1]).

1.4 Intersection Theory

In this section we cover some of the basics of intersection theory that are relevant later in this work. We follow the treatment in [10]. Here, we let X be any projective variety. Then, the group of *k-cycles*, denoted $Z_k(X)$, is the free abelian group generated by the k-dimensional irreducible subvarieties of X; i.e., a k-cycle has the form $\sum n_i[Y_i]$, where Y_i are k-dimensional irreducible subvarieties of X and

$n_i \in \mathbb{Z}$. We say that two k-cycles are *rationally equivalent* if their difference is an integral sum

$$\sum m_i[\mathrm{div}(f_i)],$$

where f_i are non-zero rational functions on some $k+1$ dimensional subvarieties Z_i and $\mathrm{div}(f_i)$ is the principal divisor of f_i. We denote the group of k-cycles modulo rational equivalence by $A_k(X)$. Then, the *cycle class group* of X is the free abelian group

$$A_*(X) := \bigoplus A_k(X). \tag{1.12}$$

A proper morphism between varieties $f : X \to Y$, induces a homomorphism $f_* : A_k(X) \to A_k(Y)$, called the *proper push-forward*. A particularly important push-forward is when $p : X \to \mathrm{Spec}(\mathbb{C})$ is the projection to a point for a projective variety X. Then, for $\alpha \in A_0(X)$, we set

$$\int_X \alpha := p_*(\alpha). \tag{1.13}$$

This takes integral values by the obvious identification $A_0(\mathrm{Spec}(\mathbb{C})) \cong \mathbb{Z}$. This makes sense as well for any $\alpha \in A_*(X)$ since $p_*(\beta) = 0$ for any $\beta \in A_k(X)$ with $k > 0$.

Given a line bundle $\mathcal{L}$ on X, we think of the first *Chern class* of $\mathcal{L}$ is an operator (cf. [10] §2.5)

$$c_1(\mathcal{L}) : A_k(X) \to A_{k-1}(X).$$

The most relevant property of the first Chern class to our use is the following additive property. For line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X,

$$c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2). \tag{1.14}$$

Then, the *Chern character* is given by

$$\mathrm{Ch}(\mathcal{L}) := e^{c_1(\mathcal{L})} = \sum_{j=0}^{\infty} \frac{c_1(\mathcal{L})^j}{j!}, \tag{1.15}$$

where the power $c_1(\mathcal{L})^i : A_k(X) \to A_{k-i}(X)$ is taken as composition of maps. Clearly, this is a polynomial in $c_1(\mathcal{L})$ of degree $\leq \dim(X)$. The most relevant property of the Chern character to our use is the following multiplicative property. Given line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X,

$$\mathrm{Ch}(\mathcal{L}_1 \otimes \mathcal{L}_2) = \mathrm{Ch}(\mathcal{L}_1) \cdot \mathrm{Ch}(\mathcal{L}_2), \tag{1.16}$$

where the latter is viewed as a composition of operators on $A_*(X)$. Then, we have the following fundamental result known as the Riemann-Roch Theorem for singular varieties. ([10] Corollary 18.3.1)

Theorem 1.4.1. *For any projective variety X and line bundle $\mathcal{L}$ on X we have*

$$\mathcal{X}(X, \mathcal{L}) = \int_X Ch(X) \cap Td(X),$$

where $\mathrm{Td}(X) \in A_*(X)_{\mathbb{Q}}$ is an element of the rational extension of the cycle class group independent of $\mathcal{L}$ and for any projective variety Y and any coherent sheaf $\mathcal{F}$,

$$\mathcal{X}(Y, \mathcal{F}) = \sum_{i \geq 0} (-1)^i \dim H^i(Y, \mathcal{F}) \tag{1.17}$$

is the Euler-Poincaré characteristic.

CHAPTER 2: CONDITIONS FOR DESCENT

In this chapter we prove conditions for descent of line bundles to $(G/B)^3//G$. We denote $X := (G/B)^3$ in this chapter. After some basic results in the first section, in the second section, we reduce the problem of descent to considering only points $x \in X$ with stabilizer G_x that is a reductive group. This is a useful simplification. In section three, we prove a theorem giving sufficient conditions for descent. In section four, we give a necessary condition for descent, although, except in type A, this necessary condition does not match the sufficient conditions. In section five, we give an example to show that the sufficient conditions are not necessary.

2.1 Beginning Statements

We begin with $(\chi_1, \chi_2, \chi_3) \in (X(T)^{++})^3$ such that the semistable locus of the line bundle $\mathcal{L}(\chi_1, \chi_2, \chi_3)$ is non-empty, i.e., $(\chi_1, \chi_2, \chi_3) \in \Gamma_3(G)$ where $\Gamma_3(G)$ denotes the saturated tensor semigroup (cf. Section 1.3). Here, we produce some sufficient conditions when $\mathcal{L}(\chi_1, \chi_2, \chi_3)$ descends to $X//G$ under the diagonal action of G.

The relation $g \cdot (g_1B, g_2B, g_3B) = (gg_1B, gg_2B, gg_3B) = (g_1B, g_2B, g_3B)$ is equivalent to $g_1^{-1}gg_1, g_2^{-1}gg_2, g_3^{-1}gg_3 \in B$. Therefore, for $x = (g_1B, g_2B, g_3B)$, the isotropy subgroup is

$$G_x = g_1Bg_1^{-1} \cap g_2Bg_2^{-1} \cap g_3Bg_3^{-1}. \tag{2.1}$$

This is, *a priori*, much too complex to deal with and we want to simplify this.

Lemma 2.1.1. *If X is any G-variety with a G-linearized line bundle $\mathcal{L}$, then G_x acts trivially on $\mathcal{L}_x$ if and only if G_{gx} acts trivially on $\mathcal{L}_{gx}$ for any $g \in G$.*

Proof: The result is symmetric so we only need to show one direction. Suppose $v_x \in \mathcal{L}_x$ and $h \cdot v_x = v_x$ for all $h \in G_x$. If $h' \in G_{gx}$ and $v_{hx} \in \mathcal{L}_{gx}$, then $h' = ghg^{-1}$ for $h \in G_x$ and $v_{gx} = g \cdot v_x$ for some $v_x \in \mathcal{L}_x$, so $h' \cdot v_{gx} = ghg^{-1} \cdot (g \cdot v_x) = gh \cdot v_x = v_{gx}$. ■

Recall the Bruhat decomposition from section 1.1

$$G = \coprod_{w \in W} U_{w^{-1}} \dot{w} B = \coprod_{w \in W} \dot{w} U^-_{w^{-1}} B,$$

where $U_{w^{-1}} = \prod_{\alpha \in R(w^{-1})} U_\alpha \subset U$ and $U^-_{w^{-1}} = \dot{w}^{-1} U_{w^{-1}} \dot{w} = \prod_{\alpha \in R(w^{-1})} U_{w^{-1}\alpha} \subset U^-$, and $\dot{w} \in N_G(T)$ denotes any lift of $w \in W$. Here, $R(w^{-1})$ is the set of inversions of w^{-1} as defined in equation (1.4). In light of Lemma 2.1.1, we can consider points of the form $x = (B, \dot{w}_1 u_1 B, \dot{w}_2 u_2 B)$ with $w_1, w_2 \in W$ and $u_i \in U^-_{w_i^{-1}}$ for $i = 1, 2$. Let $tu \in G_x \subset B$, then by equation (2.1) there exist $t_1, t_2 \in T$ and $v_1, v_2 \in U^+$ such that

$$tu = \dot{w}_1 u_1 t_1 v_1 u_1^{-1} \dot{w}_1^{-1} = \dot{w}_2 u_2 t_2 v_2 u_2^{-1} \dot{w}_2^{-1}. \tag{2.2}$$

Lemma 2.1.2. *If $tu \in B$ and $t_1, t_2 \in T$ satisfy equation (2.2), then $t = \dot{w}_1 t_1 \dot{w}_1^{-1} = \dot{w}_2 t_2 \dot{w}_2^{-1}$.*

Proof:

We show $t = \dot{w}_1 t_1 \dot{w}_1^{-1}$, the proof for the second identity follows similarly.

$$tu = \dot{w}_1 (t_1 \dot{w}_1^{-1} \dot{w}_1 t_1^{-1}) u_1 t_1 v_1 u_1^{-1} \dot{w}_1^{-1} = (\dot{w}_1 t_1 \dot{w}_1^{-1}) \dot{w}_1 (t_1^{-1} u_1 t_1) v_1 u_1^{-1} \dot{w}_1^{-1}. \tag{2.3}$$

Since this lies in B and $(\dot{w}_1 t_1 \dot{w}_1^{-1}) \in T$, left multiplying by $(\dot{w}_1 t_1 \dot{w}_1^{-1})^{-1}$ gives

$$\dot{w}_1 (t_1^{-1} u_1 t_1) v_1 u_1^{-1} \dot{w}_1^{-1} = \dot{w}_1 (t_1^{-1} u_1 t_1)(\dot{w}_1^{-1} \dot{w}_1) v_1 (\dot{w}_1^{-1} \dot{w}_1) u_1^{-1} \dot{w}_1^{-1} \in B. \tag{2.4}$$

Since T normalizes all root subgroups, $t_1^{-1} u_1 t_1 \in U^-_{w_1^{-1}}$. Thus,

$$\dot{w}_1 (t_1^{-1} u_1 t_1) \dot{w}_1^{-1}, \;\; \dot{w}_1 u_1^{-1} \dot{w}_1^{-1} \in \dot{w}_1 U^-_{w_1^{-1}} \dot{w}_1^{-1} = U_{w_1^{-1}} \subset B.$$

By the appropriate left and right multiplication on equation (2.4), we have $\dot{w}_1 v_1 \dot{w}_1^{-1} \in B$. But since $v_1 \in U$, this implies that $\dot{w}_1 v_1 \dot{w}_1^{-1} \in U$. Hence, the expression in equation (2.4) is also in U. Finally, apply $TU = B$ decomposition to equation (2.3) to conclude the desired result. ■

Lemma 2.1.3. *Let $x = (B, \dot{w}_1 u_1 B, \dot{w}_2 u_2 B)$ where $w_1, w_2 \in W$ and $u_i \in U^-_{w_i^{-1}}$ for $i = 1, 2$. If $tu \in G_x$, then tu acts trivially on $\mathcal{L}_x$ if and only if $e^{\chi_1 + w_1 \chi_2 + w_2 \chi_3}(t) = 1$.*

Proof: For $tu \in G_x$ and $z_1, z_2, z_3 \in \mathbb{C}^*$,

$$tu \cdot [e, z_1] \otimes [\dot{w}_1 u_1, z_2] \otimes [\dot{w}_2 u_2, z_3]$$

$$= [tu, z_1] \otimes [tu\dot{w}_1 u_1, z_2] \otimes [tu\dot{w}_2 u_2, z_3].$$

Since $tu = \dot{w}_1 u_1 t_1 v_1 u_1^{-1} \dot{w}_1^{-1} = \dot{w}_2 u_2 t_2 v_2 u_2^{-1} \dot{w}_2^{-1}$ for some $t_1, t_2 \in T$ and $v_1, v_2 \in U^+$, as was seen in equation (2.2), this becomes

$$[tu, z_1] \otimes [\dot{w}_1 u_1 t_1 v_1 u_1^{-1} \dot{w}_1^{-1} \dot{w}_1 u_1, z_2] \otimes [\dot{w}_2 u_2 t_2 v_2 u_2^{-1} \dot{w}_2^{-1} \dot{w}_2 u_2, z_3]$$

$$= [tu, z_1] \otimes [\dot{w}_1 u_1 t_1 v_1, z_2] \otimes [\dot{w}_2 u_2 t_2 v_2, z_3]$$

$$= [e, tu \cdot z_1] \otimes [\dot{w}_1 u_1, t_1 v_1 \cdot z_2] \otimes [\dot{w}_2 u_2, t_2 v_2 \cdot z_3]$$

$$= e^{-\chi_1}(t) e^{-\chi_2}(t_1) e^{-\chi_3}(t_2) [e, z_1] \otimes [\dot{w}_1 u_1, z_2] \otimes [\dot{w}_2 u_2, z_3].$$

But, by Lemma 2.1.2 this coefficient is just

$$e^{-\chi_1}(t) e^{-\chi_2}(t_1) e^{-\chi_3}(t_2) = e^{-\chi_1 - w_1\chi_2 - w_2\chi_3}(t).$$

■

Proposition 2.1.1. *Let $p_T : B \twoheadrightarrow T$ be the projection with kernel U^+. Let $x \in X^{ss}$ be such that $G_x \subset B$ and let $H \subset p_T(G_x)$ be any subgroup. If H is a divisible group, then $e^{\chi_1 + w_1\chi_2 + w_2\chi_3} |_H \equiv 1$. In particular, if $p_T(G_x)$ is divisible, then G_x acts trivially on $\mathcal{L}_x$.*

Proof:

Let $x \in X^{ss}$ and $tu \in G_x$ be as above. Semistability of x implies existence of $\sigma \in H^0(X, \mathcal{L}^N)^G$, for some $N > 0$, such that $\sigma(x) \neq 0$. Recall that $\mathcal{L}^N = \mathcal{L}(N\chi_1) \boxtimes \mathcal{L}(N\chi_2) \boxtimes \mathcal{L}(N\chi_3)$ and let

$$\sigma(x) = [e, z_1] \otimes [\dot{w}_1 u_1, z_2] \otimes [\dot{w}_2 u_2, z_3].$$

G-invariance of σ implies $\sigma(x) = (tu \cdot \sigma)(x) = tu \cdot \sigma((tu)^{-1} \cdot x) = tu \cdot \sigma(x)$ when $tu \in G_x$. Just as in the proof of Lemma 2.1.3,

$$tu \cdot \sigma(x) = e^{-N\chi_1 - w_1 N\chi_2 - w_2 N\chi_3}(t)[e, z_1] \otimes [\dot{w}_1 u_1, z_2] \otimes [\dot{w}_2 u_2, z_3].$$

Hence

$$e^{-N\chi_1 - Nw_1\chi_2 - Nw_2\chi_3}|_{p_T(G_x)} \equiv 1.$$

Now given $t \in H$, divisibility implies there is some $s \in \pi(G_x)$ such that $s^N = t$, from which the result follows. The final statement follows immediately from Lemma 2.1.3. ■

2.2 Reductive Stabilizers

Recall from section 1.1 that given any $\beta \in R$, there is an isomorphism $u_\beta : \mathbb{C} \to U_\beta$ such that $du_\beta(\mathbb{C}) = \mathfrak{g}_\beta$, where $\mathfrak{g}_\beta$ is the β root subspace of $\mathfrak{g}$. Let $\phi : \mathbb{C}^m \to U^-$ be such an isomorphism given by $\phi(x_1, \ldots, x_m) = u_{\beta_1}(x_1) \cdots u_{\beta_m}(x_m)$ corresponding the ordering of negative roots $(\beta_1, \ldots, \beta_n)_\phi$. Here we use the subscript on the ordering to denote that the ordering corresponds to the isomorphism ϕ.

Given some $u \in U^-$ and some isomorphism $\phi : \mathbb{C}^m \to U^-$, let

$$\mathcal{R}_\phi(u) = \{\beta \in -R^+ \mid p_\beta(\phi^{-1}(u)) \neq 0\}$$

where p_β is the projection from $\mathbb{C}^m$ to the coordinate $x_\beta = \phi^{-1}(u_\beta(x_\beta))$. The following lemma proceeds similar to Lemma 3.5 of [17].

Lemma 2.2.1. *Given any isomorphism $\phi : \mathbb{C}^n \to U^-$ and $u \in U^-$,*

$$T \cap uBu^{-1} = \cap_{\alpha \in \mathcal{R}_\phi(u)} \ker(e^\alpha),$$

where $e^\alpha : T \to \mathbb{C}^*$ is the character corresponding to the root α. This subgroup is independent of ϕ.

Proof:

Let $ubu^{-1} = t$, then $u^{-1}tu = b$ and so $u^{-1}tut^{-1} \in B \cap U^- = \{e\}$. Hence, $tut^{-1} = u$, and expanding u we have

$$\begin{aligned} u_{\alpha_{i_1}}(e^{\alpha_{i_1}}(t)x_{i_1})\cdots u_{\alpha_{i_k}}(e^{\alpha_{i_k}}(t)x_{i_k}) &= tu_{\alpha_{i_1}}(x_{i_1})\cdots u_{\alpha_{i_k}}(x_{i_k})t^{-1} \\ &= u_{\alpha_{i_1}}(x_{i_1})\cdots u_{\alpha_{i_k}}(x_{i_k}). \end{aligned} \tag{2.5}$$

Hence,

$$\phi(\dots,x_{i_1},\dots,x_{i_2},\dots,x_{i_k},\dots) = \phi(\dots,e^{\alpha_{i_1}}(t)x_{i_1},\dots,e^{\alpha_{i_2}}(t)x_{i_2},\dots,e^{\alpha_{i_k}}(t)x_{i_k},\dots),$$

where similar entries are taken in the same coordinates and zero is in all other coordinates. Since ϕ is an isomorphism this implies $e^{\alpha_{i_j}}(t)x_{i_j} = x_{i_j}$ for all $1 \le i_j \le k$, but $x_{i_j} \neq 0$, so $e^{\alpha_{i_j}}(t) = 1$. The reverse inclusion follows immediately from equation (2.5). The final statement follows immediately. ■

Remark 2.2.1. Note that if $ubu^{-1} = t$, then $b = u^{-1}tu$, and as in the proof above $u^{-1}tut^{-1} = e$. So $b = t$. Thus, $T \cap uBu^{-1} = T \cap uTu^{-1}$.

Lemma 2.2.2. *Let, ϕ and ϕ' be two isomorphisms $\mathbb{C}^m \to U^-$ and $u \in U^-$. Suppose that $(\dots,\beta_1,\dots,\beta_2,\dots,\beta_k,\dots)_\phi$ for all $\beta_i \in \mathcal{R}_\phi(u)$, where $k = |\mathcal{R}_\phi(u)|$. If $(\dots,\beta_1,\dots,\beta_2,\dots,\beta_k,\dots)_{\phi'}$, then $\mathcal{R}_\phi(u) = \mathcal{R}_{\phi'}(u)$.*

Proof:

Just observe that $\phi(\phi^{-1}(u)) = u_{\beta_1}(x_{\beta_1})\cdots u_{\beta_k}(x_{\beta_k}) = \phi'(\phi'^{-1}(u))$ since the relative ordering of $\beta_1,\dots,\beta_k$ determined by ϕ also obeys the ordering determined by ϕ'. ■

Remark 2.2.2. (i) For each $x = (B, \dot{w}_1u_1B, \dot{w}_2u_2B) \in X$, we can always choose some $\phi : \mathbb{C}^n \to U^-$ such that for all $\beta_i \in \mathcal{R}_\phi(u_2)$, $(\beta_1,\beta_2,\dots,\beta_k,\widetilde{\beta}_1,\dots,\widetilde{\beta}_j)_\phi$ for all $\widetilde{\beta}_i \in -R^+ \setminus \mathcal{R}_\phi(u_2)$. For, assuming ϕ does not satisfy this property, if $(\dots,\beta_1,\dots,\beta_2,\dots,\beta_k,\dots)_\phi$ for all $\beta_i \in \mathcal{R}_\phi(u_2)$, then, there is some ϕ' such that $(\beta_1,\beta_2,\dots,\beta_k,\widetilde{\beta}_1,\dots,\widetilde{\beta}_j)_{\phi'}$ for all $\beta_i \in \mathcal{R}_\phi(u_2)$ and all $\widetilde{\beta}_i \in -R^+ \setminus \mathcal{R}_\phi(u_2)$. Then, by Lemma 2.2.2, $\mathcal{R}_{\phi'}(u_2) = \mathcal{R}_\phi(u_2)$.

(ii) For the remainder of this chapter, at each $x = (B, \dot{w}_1u_1B, \dot{w}_2u_2B) \in X$, we fix an isomorphism ϕ that satisfies the property in (i). We will drop notational dependence on ϕ and write only $\mathcal{R}(u) := \mathcal{R}_\phi(u)$ for $u \in U^-$ and take $(\beta_1,\beta_2,\dots,\beta_k,\widetilde{\beta}_1,\dots,\widetilde{\beta}_j)$ to mean $(\beta_1,\beta_2,\dots,\beta_k,\widetilde{\beta}_1,\dots,\widetilde{\beta}_j)_\phi$. This convention certainly depends on x, but this dependence will always be clear from the context.

For any subset $S \subset R$ and $w \in W$,

$$\dot{w}(\cap_{\alpha \in S} \ker(e^{\alpha}))\dot{w}^{-1} = \cap_{\beta \in wS} \ker(e^{\beta}).$$

Then, for $x = (B, \dot{w}_1 u_1 B, \dot{w}_2 u_2 B) \in X$ with $u_i \in U^{-}_{w_i^{-1}}$ and $w_i \in W$, define

$$T_x := G_x \cap T.$$

By the above and Lemma 2.2.1, we have

$$\begin{aligned} T_x &= T \cap \dot{w}_1 u_1 B u_1^{-1} \dot{w}_1^{-1} \cap \dot{w}_2 u_2 B u_2^{-1} \dot{w}_2^{-1} \\ &= \left[T \cap \dot{w}_1 u_1 B u_1^{-1} \dot{w}_1^{-1}\right] \cap \left[T \cap \dot{w}_2 u_2 B u_2^{-1} \dot{w}_2^{-1}\right] \\ &= \left[\cap_{\alpha \in w_1 \mathcal{R}(u_1)} \ker(e^{\alpha})\right] \cap \left[\cap_{\alpha \in w_2 \mathcal{R}(u_2)} \ker(e^{\alpha})\right] \\ &= \cap_{\alpha \in (w_1 \mathcal{R}(u_1) \cup w_2 \mathcal{R}(u_2))} \ker(e^{\alpha}). \end{aligned} \tag{2.6}$$

Remark 2.2.3. Note that T_x depends only on $\mathcal{R}(u_i)$ and w_i, $i = 1, 2$.

The following lemma appears in [17] but with the weight lattice Λ in place of $X(T)$. While one can simply observe from the proof given there that the lemma holds just as well for $X(T)$, we include a proof for completeness.

Lemma 2.2.3. *For $S \subset R$ any collection of roots, let $T_S := \cap_{\alpha \in S} \ker(e^{\alpha}) \subset T$. For any character $\mu \in X(T)$, $e^{\mu}|_{T_S} \equiv 1$ if and only if $\mu \in \mathbb{Z}S$.*

Proof: If $\mu \in \mathbb{Z}S$, then it is clear that $e^{\mu}|_{T_S} \equiv 1$. For the reverse inclusion consider the isomorphism $\xi : T \to \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{C}^*)$, $\xi(t)(\mu) = e^{\mu}(t)$. Then, the following is immediate upon considering the definition of T_S

$$\xi\big(T_S\big) = \{\phi \in \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{C}^*) \mid \phi|_{\mathbb{Z}S} \equiv 1\}.$$

Now, suppose that $\mu \in X(T) \setminus \mathbb{Z}S$, we claim there must be some $\phi \in \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{C}^*)$, with $\phi|_{\mathbb{Z}S} \equiv 1$, such that $\phi(\mu) \neq 1$. If we prove the claim, then by the above inequality there must be some $t \in T_S$ such that $e^{\mu}(t) \neq 1$. This a contradiction and hence $\mu \in \mathbb{Z}S$.

To see the claim, let $X = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n$ and $\mu = \sum_{i=1}^{n} a_i e_i$. Now we must have $a_i e_i \notin \mathbb{Z}S$ for some i, we may assume $i = 1$. Now, if $\mathbb{Z}e_1 \cap \mathbb{Z}S = \emptyset$, then define $\phi(e_1)$ to be any $z \neq 1 \in \mathbb{C}^*$, and define $\phi(e_j) = 1$. If there is some $n \in Z$ such that $ne_1 \in \mathbb{Z}S$, we can choose n of minimum absolute value. Then define $\phi(e_1) = \zeta_n$, where ζ_n is a primitive n-th root of unity, and $\phi(e_j) = 1$. ■

Let $w \in W$ and $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be the simple roots. Define

$$\epsilon_w(\alpha) = \begin{cases} 1 & w^{-1}\alpha < 0 \\ -1 & w^{-1}\alpha > 0 \end{cases}.$$

That is, $\epsilon_w(\alpha)$ is the appropriate sign such that $\epsilon(\alpha)w^{-1}\alpha \in -R^+$. Let the $\alpha_i \in \Delta$ be indexed so that $(\ldots, \epsilon(\alpha_1)w^{-1}\alpha_1, \ldots, \epsilon(\alpha_n)w^{-1}\alpha_n, \ldots)$. The function ϵ_w is certainly dependent on $w \in W$, and the ordering on Δ is dependent on $x \in X$, but these dependencies are always clear from context. For notational simplicity, we will write $\epsilon(\alpha) := \epsilon_w(\alpha)$. Given any subset P of the root lattice Q, we define $\mathbb{Z}P$ to be the sublattice of Q spanned by the elements of P.

Lemma 2.2.4. *Let $x = (B, \dot{w}_1 u_1 B, \dot{w}_2 u_2 B) \in X$ and define $S \subset \Delta$ to be any subset of simple roots satisfying $\epsilon(\alpha)w_2^{-1}\alpha \notin \mathcal{R}(u_2)$, for all $\alpha \in S$, then*

$$\mathbb{Z}[w_2\mathcal{R}(u_2 \prod_{\alpha \in S} u_{\epsilon(\alpha)w_2^{-1}\alpha}(x_\alpha))] = \mathbb{Z}(w_2\mathcal{R}(u_2)) + \mathbb{Z}S,$$

whenever $x_\alpha \neq 0$ for all $\alpha \in S$.

Proof: Since for $\alpha \in S$, $\epsilon(\alpha)w_2^{-1}\alpha \notin \mathcal{R}(u_2)$, $\beta < \epsilon(\alpha)w_2^{-1}\alpha$ via remark 2.2.2 for all $\beta \in \mathcal{R}(u_2)$. We have

$$\mathcal{R}(u_2 \prod_{\alpha \in S} u_{\epsilon(\alpha)w_2^{-1}\alpha}(x_\alpha)) = \mathcal{R}(u_2) \cup \{\epsilon(\alpha)w_2^{-1}\alpha\}_{\alpha \in S}. \tag{2.7}$$

So

$$\mathbb{Z}w_2\mathcal{R}(u_2 \prod_{\alpha \in S} u_{\epsilon(\alpha)w_2^{-1}\alpha}(x_\alpha)) = \mathbb{Z}(w_2\mathcal{R}(u_2)) + \mathbb{Z}S.$$

■

2.3 Sufficient Conditions for Descent

Lemma 2.3.1. *If G_x acts trivially on $\mathcal{L}_x$ for every $x \in X^{ss}$ such that G_x is reductive, then $\mathcal{L}$ descends.*

Proof:

Recall from Lemma 1.2.2 that it suffices to show that G_x acts trivially on $\mathcal{L}_x$ for all $x \in X^{ss}$ such that $G \cdot x$ is closed in X^{ss}. By ([8], Lemma 3.3.12), if $G \cdot x$ is closed in X^{ss}, then, G_x is reductive (this is just an application of Matsushima's Theorem). ∎

Proposition 2.3.1. *Any reductive subgroup H of B must be contained in some torus. In particular, $bHb^{-1} \subset T$ for some $b \in B$.*

Proof:

Recall the projection $p_T : B \to T$ sending tu to t, this is a homomorphism of algebraic groups. Compose p_T with the inclusion $H \subset B$ to get a homomorphism $\psi : H \to T$. We claim that $\ker(\psi) = H \cap U = \{e\}$. H is a closed subgroup of a solvable group, so H is solvable and hence $R(H) = H^\circ$, where $R(H)$ is the radical of H. Upon observing that $H_u = (H_u)^\circ = (H^\circ)_u = \{e\}$ the claim follows. The last equality follows from the reductivity of H. The first two equalities follow by using the fact that H_u and $(H^\circ)_u$ are closed subgroups of U and thus connected.

Then, $\psi : H \hookrightarrow T$ is an injective algebraic group homomorphism. To determine if elements of an algebraic group are semisimple, it suffices to see if their images are semisimple under any faithful representation. Any faithful representation of T gives a faithful representation of H, under which all the elements of H then are semisimple. Last, any subgroup of B which consists of all semi-simple elements must lie in a maximal torus $S \subset B$ ([29] §6.3.6). But all maximal tori in B are B-conjugates, so there is some $b \in B$ such that $bHb^{-1} \subset T$. ∎

Corollary 2.3.1. *$\mathcal{L}$ descends if G_x acts trivially on $\mathcal{L}_x$ for all $x \in X^{ss}$ such that $G_x = T_x$.*

Proof:

Assume G_x acts trivially on $\mathcal{L}_x$ for all $x \in X^{ss}$ such that $G_x = T_x$. Let $x \in X^{ss}$ be such that G_x is reductive. By Matsushima's Theorem, it follows that for any $b \in B$, $bG_xb^{-1} = G_{bx}$ is also reductive. By Proposition 2.3.1, we have $G_{bx} = T_{bx}$ for some $b \in B$. Thus, G_{bx} acts trivially on $\mathcal{L}_{bx}$ according to our assumption. But, by Lemma 2.1.1, this implies that G_x acts trivially on $\mathcal{L}_x$. Now use Lemma 2.3.1. ■

Recalling Lemma 2.1.1 and noting that T_x° is a divisible group (cf. [29] 3.2.7) we have the following.

Corollary 2.3.2. *Let $x = (B, \dot{w}_1 u_1 B, \dot{w}_2 u_2 B) \in X^{ss}$ such that G_x is reductive, then $e^{\chi_1 + w_1\chi_2 + w_2\chi_3}|_{T_x^\circ} \equiv 1$.*

For $\mathfrak{s}$ a semisimple subalgebra of $\mathfrak{g}$, let $Q_\mathfrak{s}$ be the root lattice of $\mathfrak{s}$.

Theorem 2.3.1. *For $x = (B, \dot{w}_1 u_1 B, \dot{w}_2 u_2 B) \in X^{ss}$, if $\chi_1 + w_1\chi_2 + w_2\chi_3 \in Q_\mathfrak{s}$ for all semisimple subalgebras $\mathfrak{s}$ containing $\mathfrak{t}$, then $e^{\chi_1 + w_1\chi_2 + w_2\chi_3} |_{T_x} \equiv 1$.*

Proof:

There is a correspondence between sublattices of Q of finite index and the Lie subalgebras of $\mathfrak{s}$ of $\mathfrak{g}$ of maximal rank (cf. [17], §3). Now, suppose $\mathbb{Z}(w_1\mathcal{R}(u_1) \cup w_2\mathcal{R}(u_2))$ is finite index in Q. Then there is a semisimple subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ containing $\mathfrak{t}$ such that

$$\mathbb{Z}(w_1\mathcal{R}(u_1) \cup w_2\mathcal{R}(u_2)) = Q_\mathfrak{s},$$

hence $\chi_1 + w_1\chi_2 + w_2\chi_3 \in \mathbb{Z}(w_1\mathcal{R}(u_1) \cup w_2\mathcal{R}(u_2))$. Now apply lemma 2.2.3.

Now, suppose that $\mathbb{Z}(w_1\mathcal{R}(u_1) \cup w_2\mathcal{R}(u_2))$ fails to be finite index in Q. Let $S \subset \Delta$ be a subset of simple roots such that

$$\mathbb{Q}S \cap \mathbb{Q}(w_1\mathcal{R}(u_1) \cup w_2\mathcal{R}(u_1)) = \emptyset \tag{2.8}$$

and

$$\mathbb{Z}(w_1\mathcal{R}(u_1) \cup w_2\mathcal{R}(u_1)) + \mathbb{Z}S$$

is finite index in Q.

For each $\alpha \in S$, the fact that $\{\pm\alpha\} \cap w_2\mathcal{R}(u_2) = \emptyset$ implies $\epsilon(\alpha)w_2^{-1}\alpha \notin \mathcal{R}(u_2)$. Let $\widetilde{u} = \prod_{\alpha \in S} u_{\epsilon(\alpha)w^{-1}\alpha}(x_\alpha)$ for $x_\alpha \in \mathbb{C}^*$ with the product taken in the order in the spirit of Remark 2.2.2. Then by Lemma 2.2.4 we have

$$\mathbb{Z}(w_2\mathcal{R}(u_2\widetilde{u})) = \mathbb{Z}(w_2\mathcal{R}(u_2)) + \mathbb{Z}S.$$

Hence,

$$\mathbb{Z}(w_1\mathcal{R}(u_1) \cup w_2\mathcal{R}(u_2\widetilde{u})) = \mathbb{Z}(w_1\mathcal{R}(u_1) \cup w_2\mathcal{R}(u_2)) + \mathbb{Z}S. \tag{2.9}$$

Consider $\widetilde{x} = (B, \dot{w}_1 u_1 B, \dot{w}_2 u_2 \widetilde{u} B) \in X$, where $\dot{w}_1 u_1$ and $\dot{w}_2 u_2$ are as before. Then by equations (2.9) and (2.7) of Lemma 2.2.4,

$$T_{\widetilde{x}} = \bigcap_{\alpha \in w_1\mathcal{R}(u_1) \cup w_2\mathcal{R}(u_2\widetilde{u})} \ker(e^{\alpha}) = \bigcap_{\alpha \in w_1\mathcal{R}(u_1) \cup w_2\mathcal{R}(u_2) \cup S} \ker(e^{\alpha}).$$

Moreover, $\chi_1 + w_1\chi_2 + w_2\chi_3 \in \mathbb{Z}(w_1\mathcal{R}(u_1) \cup w_2\mathcal{R}(u_2\widetilde{u}))$ since this lattice is finite index in Q. Thus, $e^{\chi_1 + w_1\chi_2 + w_2\chi_3} \mid_{T_{\widetilde{x}}} \equiv 1$ by Lemma 2.2.3.

Next, there is a short exact sequence

$$\mathbb{Z}S \hookrightarrow X(T)/\mathbb{Z}(w_1\mathcal{R}(u_1) \cup w_2\mathcal{R}(u_2)) \twoheadrightarrow X(T)/[\mathbb{Z}S + \mathbb{Z}(w_1\mathcal{R}(u_1) \cup w_2\mathcal{R}(u_2))].$$

Equation (2.8) forces $\mathbb{Z}S \cap \mathrm{Tor}(X(T)/\mathbb{Z}(w_1\mathcal{R}(u_1) \cup w_2\mathcal{R}(u_2))) = \emptyset$. Hence,

$$\mathrm{Tor}(X(T)/\mathbb{Z}(w_1\mathcal{R}(u_1) \cup w_2\mathcal{R}(u_2))) \hookrightarrow \mathrm{Tor}(X(T)/\mathbb{Z}S + \mathbb{Z}(w_1\mathcal{R}(u_1) \cup w_2\mathcal{R}(u_2)))). \tag{2.10}$$

Now, $T_{\widetilde{x}} \subset T_x$ since the former is an intersection over a larger set of roots. By Lemma 3.7 of [17], observing that the proof given there holds for $X(T)$ in place of Λ,

$$T_x/T_x^{\circ} \cong \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Tor}(X(T)/\mathbb{Z}(w_1\mathcal{R}(u_1) \cup w_2\mathcal{R}(u_2))), \mathbb{C}^*\big)$$

and

$$T_{\widetilde{x}}/T_{\widetilde{x}}^{\circ} \cong \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Tor}(X(T)/[\mathbb{Z}S + \mathbb{Z}(w_1\mathcal{R}(u_1) \cup w_2\mathcal{R}(u_2))]), \mathbb{C}^*\big).$$

Hence, we have a natural surjection using the injectivity of $\mathbb{C}^*$ on the map in equation (2.10),

$$T_{\widetilde{x}}/T_{\widetilde{x}}^{\circ} \twoheadrightarrow T_x/T_x^{\circ},$$

that commutes with the inclusion $T_{\widetilde{x}} \subset T_x$ and the quotient maps. In particular, the surjectivity implies that for each T_x° coset, we can choose some representative in T_x that is also in $T_{\widetilde{x}}$.

Since x is semistable, by Corollary 2.3.2, $e^{\chi_1 + w_1\chi_2 + w_2\chi_3} \mid_{T_x^{\circ}} \equiv 1$. On the other hand, $e^{\chi_1 + w_1\chi_2 + w_2\chi_3} \mid_{T_{\widetilde{x}}} \equiv$

Table 2.1: The Lattice Γ

$A_\ell \ (\ell \geq 1):$	$\Gamma = Q.$
$B_\ell \ (\ell \geq 3):$	$\Gamma = 2Q.$
$C_\ell \ (\ell \geq 2):$	$\Gamma = 2\Lambda.$
$D_4 \ :$	$\Gamma = \{n_1\alpha_1 + 2n_2\alpha_2 + n_3\alpha_3 + n_4\alpha_4 \mid n_i \in \mathbb{Z} \text{ and } n_1 + n_3 + n_4 \in 2\mathbb{Z}\}.$
$D_\ell \ (\ell \geq 5):$	$\Gamma = \{2n_1\alpha_1 + 2n_2\alpha_2 + \cdots + 2n_{\ell-2}\alpha_{\ell-2} + n_{\ell-1}\alpha_{\ell-1} + n_\ell\alpha_\ell \mid n_i \in \mathbb{Z} \text{ and } n_{\ell-1} + n_\ell \in 2\mathbb{Z}\}.$
$G_2 \ :$	$\Gamma = \mathbb{Z}6\alpha_1 + \mathbb{Z}2\alpha_2.$
$F_4 \ :$	$\Gamma = \mathbb{Z}6\alpha_1 + \mathbb{Z}6\alpha_2 + \mathbb{Z}12\alpha_3 + \mathbb{Z}12\alpha_4.$
$E_6 \ :$	$\Gamma = 6\Lambda.$
$E_7 \ :$	$\Gamma = 12\Lambda.$
$E_8 \ :$	$\Gamma = 60Q.$

1. Hence by the surjectivity of the above map, $e^{\chi_1 + w_1\chi_2 + w_2\chi_3} |_{T_x} \equiv 1$ as well. ■

Remark 2.3.1. (i) Although the proof doesn't require that $\widetilde{x}$ be semistable, it follows easily that it can be chosen to be so. The reason is that x is semistable, and x is obtained from $\widetilde{x}$ be setting certain coordinates to zero. Thus, there is an open subset of x of semistable points from which to choose $\widetilde{x}$.

(ii) This proof can be significantly simplified by just defining the subgroup $T_{\widetilde{x}}$ as the intersection of kernels of roots we want. Then, we proceed with final steps of the proof. The key here is that this subgroup $T_{\widetilde{x}}$ doesn't actually need to be an isotropy subgroup of any point. However, we consider it worthwhile to include the full construction here and it may prove worthwhile to know that the desired subgroup is an honest isotropy subgroup of a semistable point.

For any $\mathfrak{g}$, let Γ be the intersection of lattices $Q_\mathfrak{s}$ for all semisimple Lie subalgebras $\mathfrak{s}$ of $\mathfrak{g}$ containing $\mathfrak{t}$. The following description of Γ, proved by Kumar in [17], Theorem 3.10, relies on the Borel-de Siebenthal classification of semisimple subalgebras of $\mathfrak{g}$ of maximal rank (cf. [33]). In the proof, Γ is an intersection $\cap_{w\in W} wM$ for some fixed lattice M. In particular, this means that Γ is always W-invariant.

Theorem 2.3.2. *For each type of G, Γ is given in Table 2.1.*

Let us define the subset W^{ss}_{red} of $W \times W$ as follows

$$\{(w_1, w_2) \mid \exists\, u_1 \in U^-_{w_1^{-1}}\ ,\ u_2 \in U^-_{w_2^{-1}} \text{ with } x = (B, \dot{w}_1 u_1 B, \dot{w}_2 u_2 B) \in X^{ss} \text{ and } G_x = T_x\}.$$

The following is an obvious consequence of Theorem 2.3.2, Corollary 2.3.1, Theorem 2.3.1, and Lemma

2.1.3.

Corollary 2.3.3. *If $\chi_1 + w_1\chi_2 + w_2\chi_3 \in \Gamma$ for all $(w_1, w_2) \in W^{ss}_{red}$, then $\mathcal{L}$ descends to $X//G$.*

Lemma 2.3.2. *For $\mathfrak{g}$ of any type, if $\chi \in \Lambda$ and $w \in W$, then $\chi - w\chi \in Q$.*

Proof:

Let $\chi = \sum_{i=1}^{\ell} a_i \varpi_i$, for $a_i \in \mathbb{Z}$ and ϖ_i fundamental weights. Let $w = s_{i_r} \cdots s_{i_1}$ be a reduced decomposition for w in terms of simple reflections. We prove the claim by induction on r. If $r = 1$, we have

$$s_{i_1}\chi = \sum_{i=1}^{\ell} a_i s_{i_1} \varpi_i = \sum_{i=1}^{\ell} a_i \varpi_i - a_{i_1}\alpha_{i_1} = \chi - a_{i_1}\alpha_{i_1},$$

where α_{i_1} is the corresponding simple root. Then,

$$w\chi = s_{i_r}(\chi + Q) = \chi - a_{i_r}\alpha_{i_r} + Q = \chi + Q.$$

■

Let θ be the longest root of $\mathfrak{g}$ and d be the least common multiple of the coefficients of θ in terms of the simple roots. For every type of $\mathfrak{g}$, both θ and d is given in Table 2.2. It is obvious that $dQ \subset \Gamma$ in all cases.

Table 2.2: θ and d for each type of $\mathfrak{g}$

A_ℓ :	$\theta = \alpha_1 + \alpha_2 + \cdots + \alpha_\ell$	$d = 1$
B_ℓ :	$\theta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_\ell$	$d = 2$
C_ℓ :	$\theta = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$	$d = 2$
D_ℓ :	$\theta = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell$	$d = 2$
G_2 :	$\theta = 3\alpha_1 + 2\alpha_2$	$d = 6$
F_4 :	$\theta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$	$d = 12$
E_6 :	$\theta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	$d = 6$
E_7 :	$\theta = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	$d = 12$
E_8 :	$\theta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$	$d = 60$

The following is the main result of part one of this thesis and follows *a fortiori* from Corollary 2.3.3 and Lemma 2.3.2.

Theorem 2.3.3. *Given $\chi_1, \chi_2, \chi_3 \in X(T)^{++}$, if $\chi_1, \chi_2, \chi_3 \in d\Lambda$ and $\chi_1 + \chi_2 + \chi_3 \in \Gamma$, then $\mathcal{L}$ descends to $X//G$.*

Remark 2.3.2. Although the proof technically only requires that $\chi_2, \chi_3 \in d\Lambda$ and $\chi_1 + \chi_2 + \chi_3 \in \Gamma$. The following results show that this condition as stated is not at any loss of generality. Indeed, one would hope that the inherent symmetry of the problem would appear in the statement of the theorem.

Lemma 2.3.3. *In all cases except G_2 and F_4 we have $\Gamma \subset d\Lambda$. If G is of type G_2 or F_4, then, $d\Lambda \subset \Gamma$.*

Proof:

It is clear that $\Gamma \subset d\Lambda$ for all cases except of type D_ℓ, $\ell \geq 4$, G_2, and F_4. In the D_ℓ case. Since $2Q \subset 2\Lambda$, it suffices to show for $\ell \geq 5$, that $m\alpha_{\ell-1} + n\alpha_\ell \in 2\Lambda$ when $n + m$ is even, and for $\ell = 4$, that $a\alpha_1 + b\alpha_3 + c\alpha_4 \in 2\Lambda$ when $a + b + c$ is even.

For D_ℓ with $\ell \geq 4$, $2\varpi_{\ell-1} - 2\varpi_\ell = \alpha_{\ell-1} - \alpha_\ell$. Moreover, $2\alpha_{\ell-1} \in 2\Lambda$ since $2Q \subset 2\Lambda$. Hence,

$$m\alpha_{\ell-1} + n\alpha_\ell = (m+n)\alpha_{\ell-1} - n(\alpha_{\ell-1} - \alpha_\ell) \in 2\Lambda$$

whenever $m + n$ is even. For $\ell = 4$, in addition to the above, we have $2\varpi_1 - 2\varpi_3 = \alpha_1 - \alpha_3 \in 2\Lambda$. Hence

$$(a+b+c)\alpha_1 - (b+c)(\alpha_1 - \alpha_3) - c(\alpha_3 - \alpha_4) = a\alpha_1 + b\alpha_3 + c\alpha_4 \in 2\Lambda$$

whenever $a + b + c$ is even.

For G_2 and F_4, note that $\Lambda = Q$, and since $dQ \subset \Gamma$, we are done. ■

Then following slight strengthening of Theorem 2.3.3 for G_2 and F_4 follows immediately upon noting that Γ is W-invariant, and hence, $\chi - w\chi \in \Gamma$ for any $\chi \in \Gamma$ and any $w \in W$.

Theorem 2.3.4. *For G of type G_2 and F_4, if $\chi_1, \chi_2, \chi_3 \in \Gamma$ then $\mathcal{L}$ descends to $X//G$.*

Example 2.3.1. *Let's compare Theorem 2.3.3 with the few cases where explicit computations are possible, i.e., $G = SL(n)$ for $n = 2, 3$.*

For $SL(2)$, it is easy to see that the different possibilities for $\pi(G_x)$ are T and $\{\pm I\}$. Hence, the only case where $\pi(G_x)$ is not divisible is $\{\pm I\}$. If for $i = 1, 2, 3$,

$$\chi_i = b_i \varpi_1$$

we have for $w_1, w_2 \in W$ arbitrary

$$e^{-\chi_1 - w_1\chi_2 - w_2\chi_3}(-I) = (-1)^{b_1+b_2+b_3}.$$

Hence, we have descent if $2 \mid b_1 + b_2 + b_3$, which is equivalent to $\chi_1 + \chi_2 + \chi_3 \in Q$.

For the $SL(3)$ case, a more involved computation shows that the only possibilities for $\pi(G_x)$ are T, $\mathbb{C}^*$, and the three element group generated by

$$\zeta I = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta \end{pmatrix},$$

where ζ is a primitive cube root of unity. The first two cases are divisible. Then, if for $i = 1, 2, 3$,

$$\chi_i = (a_i - b_i)\varpi_1 + b_i\varpi_2,$$

we have for any $w_1, w_2 \in W$,

$$e^{-\chi_1 - w_1\chi_2 - w_2\chi_3}(\zeta I) = \zeta^{\sum_{i=1}^{3} a_i + b_i}.$$

In particular, this is trivial if $3 | \sum_{i=1}^{3} a_i + b_i$, which is again equivalent to $\chi_1 + \chi_2 + \chi_3 \in Q$.

2.4 A Necessary Condition

The proof of the following proposition was suggested by S. Kumar.

Theorem 2.4.1. *If $\chi_1 + \chi_2 + \chi_3 \notin Q$, then $\mathcal{L}(\chi_1, \chi_2, \chi_3)$ does not descend.*

Before we prove the theorem, we consider two lemmas.

Lemma 2.4.1. *If $\chi_1 + \chi_2 + \chi_3 \notin Q$ then* $\dim H^0(X, \mathcal{L}(\chi_1, \chi_2, \chi_3))^G = 0$.

Proof:

Suppose that $[V(-w_0\chi_1) \otimes V(-w_0\chi_2) \otimes V(-w_0\chi_3)]^G \neq 0$, then $V(\chi_1)$ is a component of $V(-w_0\chi_2) \otimes V(-w_0\chi_3)$. Hence, $\chi_1 = -w_0\chi_2 - w_0\chi_3 + Q$ (cf. [18], Proposition 3.2). By Lemma 2.3.2 we know that $\chi_i - w_0\chi_i \in Q$ for $i = 2, 3$, so we have $\chi_1 + \chi_2 + \chi_3 \in Q$. ■

Lemma 2.4.2. *Let* $\theta_{\chi_1,\chi_2,\chi_3}(N) := \dim\left(H^0(X, \mathcal{L}(\chi_1,\chi_2,\chi_3)^{\otimes N})^G\right)$ *for* $N > 0$. *If* $\mathcal{L}(\chi_1,\chi_2,\chi_3)$ *descends, then* $\theta_{\chi_1,\chi_2,\chi_3}(N)$ *is a polynomial in* N *with rational coefficients.*

Proof:

Note that by Borel-Weil-Bott, $H^p(X, \mathcal{L}) = 0$ if $p > 0$. Let $\hat{\mathcal{L}}$ be the descended line bundle on $X//G$. By [31], Theorem 3.2.a, we have for any $N > 0$, $H^p(X//G, \hat{\mathcal{L}}^N) \cong H^p(X, \mathcal{L}^N)^G$. In particular,

$$\theta_{\chi_1,\chi_2,\chi_3}(N) = \dim\left(H^0(X, \mathcal{L}^N)^G\right) \cong \mathcal{X}(X//G, \hat{\mathcal{L}}^N),$$

where $\mathcal{X}(X//G, \hat{\mathcal{L}})$ is the Euler-Poincarè characteristic of $\hat{\mathcal{L}}$ on $X//G$ (cf. §1.4).

Now, recall that the first Chern class as an operator on k-cycle classes satisfies

$$c_1(\hat{\mathcal{L}}^N) = Nc_1(\hat{\mathcal{L}}),$$

and so the Chern character satisfies the following

$$\mathrm{ch}(\hat{\mathcal{L}}^N) = \sum_{i\geq 0}(1/i!)c_1^i(\hat{\mathcal{L}}^N) = \sum_{i\geq 0}(1/i!)N^i c_1^i(\hat{\mathcal{L}}).$$

Then, by the Riemann-Roch theorem for singular varieties ([10] Corollary 18.3.1)

$$\mathcal{X}(X//G, \hat{\mathcal{L}}^N) = \int_{X//G} \mathrm{ch}(\hat{\mathcal{L}}^N) \cap \mathrm{Td}(X//G) = \sum_{i\geq 0}(1/i!)N^i \int_{X//G} c_1^i(\hat{\mathcal{L}}) \cap \mathrm{Td}(X//G).$$

Here, $\mathrm{Td}(X//G) \in A_*(X//G)_{\mathbb{Q}}$ is independent of N. To complete the proof, we only need to observe that the sum is finite. But, $\mathrm{Td}(X//G)$ is in the rational extension $A_*(X//G)_{\mathbb{Q}}$ of the cycle class group. Since c_1 only lowers the degree of a cycle class, it follows that $c_1^i(\hat{\mathcal{L}}) \cap \mathrm{Td}(X//G) = 0$ for $i > \dim(X//G)$. ■

Corollary 2.4.1. *If* $\mathcal{L}(\chi_1,\chi_1,\chi_3)$ *descends, then* $\chi_1 + \chi_2 + \chi_3 \in Q$.

Proof:

First note that if $\chi_1 + \chi_2 + \chi_3 \notin Q$ then $\theta_{\chi_1,\chi_2,\chi_3}(1) = 0$. For, if $\theta_{\chi_1,\chi_2,\chi_3}(1) \neq 0$, then, $V(\chi_1)$ occurs in $V(-w_0\chi_1) \otimes V(-w_0\chi_2)$, and so $\chi_1 + w_0\chi_2 + w_0\chi_3 \in Q$. By Lemma 2.3.2, this implies that $\chi_1 + \chi_2 + \chi_3 \in Q$ which is a contradiction.

Now, assume $\chi_1 + \chi_2 + \chi_3 \notin Q$ and $\mathcal{L}$ descends. By the last lemma, $\theta_{\chi_1,\chi_2,\chi_3}(N)$ is a polynomial in N. Since Q is finite index in Λ, there is some integer $k > 1$ such that $jk(\chi_1 + \chi_2 + \chi_3) \in Q$ for all integers $j > 0$. This implies $(jk+1)(\chi_1 + \chi_2 + \chi_3) \notin Q$ for all $j > 0$. Thus,

$$0 = \theta_{(jk+1)\chi_1,(jk+1)\chi_2,(jk+1)\chi_3}(1) = \theta_{\chi_1,\chi_2,\chi_3}(jk+1)$$

for all $j > 0$. So, $\theta_{\chi_1,\chi_2,\chi_3}(N)$ is a polynomial with infinitely many zeros, and so it must be that $\theta_{\chi_1,\chi_2,\chi_3}(N) = 0$ for all N. But, we assume that $(X)^{ss} \neq \emptyset$, which means exactly that $\theta_{\chi_1,\chi_2,\chi_3}(N) \neq 0$ for some $N > 0$. This is a contradiction. ■

2.5 A Counter Example to "Sufficient is Neccesary"

Let $\chi_1 = 2\rho$ and $\chi_2 = \chi_3 = \rho$. Note that $H^0(X, \mathcal{L}(2\rho,\rho,\rho)^N)^G \cong [V(2N\rho)^* \otimes V(N\rho)^* \otimes V(N\rho)^*]^G$ is one dimensional since $V(2N\rho)^*$ is the Cartan component of $V(N\rho)^* \otimes V(N\rho)^*$ (cf. [18] Lemma 3.1). Then, $V(N\rho)^*$ has a highest weight vector $\phi_{N\rho}$ of weight $N\rho$, i.e. $\phi_{N\rho}$ is dual to the lowest weight vector $v_{-N\rho}$ of $V(N\rho)$. Thus, we have an explicit G-isomorphism $V(N\rho)^* \cong V(N\rho)$ by extending G-linearly the map $\phi_{N\rho} \mapsto v^+_{N\rho}$.

Let ψ° be the equivariant embedding of $V(2N\rho) \hookrightarrow V(N\rho) \otimes V(N\rho)$ given by the Cartan component, i.e., $\psi^\circ(v^+_{2N\rho}) = v^+_{N\rho} \otimes v^+_{N\rho}$. Composing ψ° with the isomorphism above gives $\psi \in \mathrm{Hom}_G(V(2N\rho), V(N\rho)^* \otimes V(N\rho)^*)$, i.e., G-linearly extend $\psi(v^+_{2N\rho}) = \phi_{N\rho} \otimes \phi_{N\rho}$.

Let $\{w^i_\gamma\}$, $\{v^i_\mu\}$ be bases for $V(2N\rho)$ and $V(N\rho)$, respectively, where the basis vector v^i_μ is taken in weight space μ, with i indexing basis vectors within each weight space, and similarly for the w^i_γ. Moreover, let $\{(w^i_\gamma)^*\}$, $\{(v^i_\mu)^*\}$, be respective dual bases for $V(2N\rho)^*$ and $V(N\rho)^*$ where the weight space subscript indicates the weight space the vector is dual to, i.e. $(v_{-N\rho})^* = \phi_{N\rho}$. Then, applying the usual isomorphism $(V^* \otimes W)^G \cong \mathrm{Hom}_G(V, W)$ to the above, we have

$$\psi \leftrightarrow \sum_{\gamma,i} (w^i_\gamma)^* \otimes \psi(w^i_\gamma).$$

Thus, we have the unique (up to scaling) element of $[V(2N\rho)^* \otimes V(N\rho)^* \otimes V(N\rho)^*]^G$ given above.

Under the Borel-Weil isomorphism, this gives the G-invariant section σ^N. Write $\sigma^N = \sigma_1 + \sigma_2$, where σ_1 is the section corresponding to $(w^i_{2N\rho})^* \otimes \psi(w^i_{2N\rho}) = (w^i_{2N\rho})^* \otimes \phi_{N\rho} \otimes \phi_{N\rho}$ and σ_2 is the section corresponding to $\sum_{\gamma<2N\rho,i}(w_\gamma^i)^* \otimes \psi(w_\gamma^i)$.

Since semistability is constant on G-orbits, to determine the zero set of σ^N it suffices to consider points of the form (B, g_1B, g_2B). Since, $(w_\gamma^i)^*(v_{2N\rho}^+)$ is non-zero if and only if $\gamma = 2N\rho$, $\sigma_2(B, g_1B, g_2B) = 0$. Then

$$\sigma^N(B, g_1B, g_2B) = \sigma_1(B, g_1B, g_2B) = [e, 1] \otimes [g_1, \phi_{N\rho}(g_1 v_{N\rho}^+)] \otimes [g_2, \phi_{N\rho}(g_2 v_{N\rho}^+)].$$

Then, we have the following description:

$$X^{ss} = G \cdot \{(B, g_1B, g_2B) \mid [g_1 v_{N\rho}^+]_{-N\rho} \neq 0, \text{ and } [g_3 v_{N\rho}^+]_{-N\rho} \neq 0 \text{ for some } N > 0\},$$

where $[w]_\mu$ denotes the μ-weight space component of w. In fact, using Bruhat decomposition one can see that then $X^{ss} = G \cdot (B, Bw_0B, Bw_0B) = G \cdot (B, w_0U^-B, w_0U^-B)$.

By a past lemma, to prove that $\mathcal{L}$ descends it suffices to check trivial action at semistable points $x = (B, w_1u_1B, w_2u_2B)$ such that $G_x = T_x$ (here, g_1, g_2 are again expressed in the WU^-B Bruhat decomposition.) Moreover, recall that for such x, T_x acts trivially on $\mathcal{L}_x$ if and only if $e^{2\rho+w_1\rho+w_2\rho}|_{T_x} \equiv 1$. Since $w_1 = w_2 = w_0$ for any such point, this condition is always satisfied.

2.6 Application of Descent to Tensor Product Decomposition

In this short section we provide a corollary to Theorem 2.3.3. Assume that G is of type A. In particular, this is the case where the sufficient conditions for descent match the necessary conditions.

Recalling the lattice Γ for each type of $\mathfrak{g}$, we define the following set

$$\Sigma = \{(\chi_1, \chi_2, \chi_3) \in X(T)^3 \mid \chi_1 + \chi_2 + \chi_3 \in Q\}.$$

Since Q is a lattice, it follows that Σ is also a lattice under component-wise addition.

Let $\{C_1, \ldots, C_r\}$ be the GIT classes which are chambers, i.e. where $X(\mathcal{L})^{ss} = X(\mathcal{L})^s$ (cf. §1.2).

Corollary 2.6.1. *Let G be of type A. For $j = 1, \ldots, m$, function*

$$f_j(\chi_1, \chi_2, \chi_3) = \dim[V(\chi_1) \otimes V(\chi_2) \otimes V(\chi_3)]^G$$

is a non-zero polynomial with rational coefficients on

$$\big((X(T)^{++})^3 \cap \Sigma\big) \cap C_j \tag{2.11}$$

and 0 *on* $\Sigma^c \cap (X(T)^{++})^3$.

The proof follows exactly as in [17], Theorem 4.1. While the GIT chambers are not yet precisely known, we do have the following partial description of them. Note that the following proposition holds for any type of simple G.

Proposition 2.6.1. *If* $X^{ss} \neq X^s$ *when* $X^{ss} \neq \emptyset$*, then there exist* $w_1, w_2, w_3 \in W$ *and* $i = 1, \ldots, n$ *such that*

$$(w_1\lambda + w_2\mu + w_3\nu)(x_i) = 0, \tag{2.12}$$

where $x_i \in \mathfrak{t}$ *is dual to the simple root* $\alpha_i \in \Delta$*.*

Proof:

The proof here follows just as in [19], Proposition 3.5. If there is some $x \in X^{ss} \setminus X^s$, we must have $\mu^{\mathcal{L}}(x, \delta) = 0$ for some $\delta \in OPS(G)$. Since, $X^{ss} \setminus X^s$ is G-stable, and $\mu^{\mathcal{L}}(g \cdot x, \delta) = \mu^{\mathcal{L}}(x, g\delta g^{-1})$, we can assume $\delta \in OPS(T)$. Moreover, by the action of $N(T)$, we can assume that δ is G-dominant, i.e., the derivative $\dot{\delta} \in \mathfrak{t}^+$. Since $\mu^{\mathcal{L}}(x, \delta) = 0$, by Proposition 1.2.1 (ii), we know $\lim_{t\to 0} \delta(t) \cdot x =: x_0$ is also semistable.

Now, let G^δ be the fixed point subgroup of δ under the conjugation action. Then G^δ is a connected, Levi subgroup of G. Let Δ_δ be the simple roots of G^δ and W_δ the Weyl group. Let W^δ be the set of minimal length coset representative in the cosets W/W_δ. Then, similar to as in the proof of Proposition 3.5 in [19], the fixed point set of X under the action of G^δ is

$$X^\delta = \bigsqcup_{w_1, w_2, w_3 \in W^\delta} G^\delta w_1^{-1} B/B \times G^\delta w_2^{-1} B/B \times G^\delta w_3^{-1} B/B. \tag{2.13}$$

In particular, since $x_0 \in X^\delta$, there exist $w_1, w_2, w_3 \in W^\delta$ such that $x_0 \in G^\delta w_1^{-1} B/B \times G^\delta w_2^{-1} B/B \times G^\delta w_3^{-1} B/B$. Take $x_0 = (g_1 w_1^{-1} B, g_2 w_2^{-1} B, g_3 w_3^{-1} B)$, then by Lemma A.1.1 we know that for some $N > 0$

$$[g_1 v_{w_1^{-1} N\lambda} \otimes g_2 v_{w_2^{-1} N\mu} \otimes g_3 v_{w_3^{-1} N\nu}]^T \neq 0.$$

But, the weight spaces of $g_1 v_{w_1^{-1}N\lambda} \otimes g_2 v_{w_2^{-1}N\mu} \otimes g_3 v_{w_3^{-1}N\nu}$ are of the form

$$w_1^{-1}N\lambda + w_2^{-1}N\mu + w_3^{-1}N\nu + \sum_{\alpha\in\Delta_\delta} k_\alpha \alpha,$$

for $k_\alpha \in \mathbb{Z}$. So, we must have

$$w_1^{-1}N\lambda + w_2^{-1}N\mu + w_3^{-1}N\nu \in \oplus_{\alpha\in\Delta_\delta}\mathbb{Z}\alpha.$$

In particular, since some simple root $\alpha_i \in \Delta$ is not in Δ_δ, we must have

$$(w_1^{-1}N\lambda + w_2^{-1}N\mu + w_3^{-1}N\nu)(x_i) = 0.$$

■

Then,

$$\left(\cup_{j=1}^r C_j\right)^c \subset \bigcup_{\substack{w_1,w_2,w_3\in W\\ i=1,\dots,n}} H_{w_1,w_2,w_3,i}, \tag{2.14}$$

where $\{C_1,\dots,C_r\}$ are the GIT classes which are chambers and $H_{w_1,w_2,w_3,i}$ is the hyperplane in $X(T)^3$ defined by $(w_1\lambda + w_2\mu + w_3\nu)(x_i) = 0$, for $w_1,w_2,w_3 \in W$ and $x_i \in \mathfrak{t}$ is dual to the simple root α_i. Let us denote by $\{\mathcal{S}_1,\dots,\mathcal{S}_m\}$ the connected components of the complement of

$$\bigcup_{\substack{w_1,w_2,w_3\in W\\ i=1,\dots,n}} H_{w_1,w_2,w_3,i} \subset X(T)^3.$$

Then, we have the following corollary.

Corollary 2.6.2. *Let G be of type A. The function $f_j(\chi_1,\chi_2,\chi_3)$, as given above, is a non-zero polynomial with rational coefficients on*

$$\left((X(T)^{++})^3 \cap \Sigma\right) \cap \mathcal{S}_j \tag{2.15}$$

and 0 *on* $\Sigma^c \cap (X(T)^{++})^3$.

CHAPTER 3: NOTES ON SEMISTABILITY

In this short chapter we provide some results on semistability. In section one, we prove that certain subvarieties of $(G/B)^3$ can never meet the semistable locus. In section two, we provide a counter-example to a natural question in the study of semistability.

3.1 Embedding Flag Varieties of Levi Subgroups

It is natural to ask, if for a Levi subgroup $L \subset G$, the image of the natural embedding

$$(L/B_L)^3 \hookrightarrow (G/B)^3,$$

meets $((G/B)^3)^{ss}$, where $B_L = B \cap L$ is the Borel subgroup of L. We have the following proposition.

Proposition 3.1.1. *If L is a proper Levi subgroup of G and $l_i \in L$ for $i = 1, 2, 3$, then points of the form $(l_1B, l_2B, l_3B) \in (G/B)^3$ are never semistable.*

Proof:

Suppose that $(l_1B, l_2B, l_3B) \in ((G/B)^3)^{ss}$, then this implies that the image of the embedding $(L/B_L)^3 \hookrightarrow (G/B)^3$ meets $((G/B)^3)^{ss}$. Now, $(U_L w_0^L B_L/B_L)^3$ is open in $(L/B_L)^3$, where $B_L = B \cap L$ is the Borel of L, U_L is the unipotent radical of B_L, and w_0^L is the longest element of the Weyl group of L. So, there must be a semistable point of the form $(u_1 w_0^L B_L, u_2 w_0^L B_L, u_3 w_0^L B_L)$, with $u_i \in U_L$.

Then, by Lemma A.1.1, for some $N > 0$,

$$[u_1 v_{w_0^L N\chi_1} \otimes u_2 v_{w_0^L N\chi_2} \otimes u_3 v_{w_0^L N\chi_3}]^G \neq 0,$$

where $v_{w_0^L N\chi_i}$ is a non-zero vector in the corresponding extremal weight space. This implies that all weight spaces of a non-zero, G-invariant, component are also T-invariant, i.e. are of weight zero.

However, all weight spaces of this vector are of the form

$$w_0^L N\chi_1 + w_0^L N\chi_2 + w_0^L N\chi_3 + \sum_{\alpha \in \Delta(L)} k_\alpha \alpha,$$

where $\Delta(L)$ is the set of simple roots for L and $k_\alpha \geq 0$ for all α. So we must have some expression of this form equal to zero. By left multiplying such an equation by w_0^L and rearranging

$$N\chi_1 + N\chi_2 + N\chi_3 = \sum_{\alpha \in \Delta(L)} k_\alpha(-w_0^L \alpha).$$

Now, for each $\alpha \in \Delta(L)$, $(-w_0^L \alpha)$ is a positive root for L. In particular, the right hand side is a positive linear combination of roots only in $\Delta(L)$. Yet, there exists some $\beta \in \Delta \setminus \Delta(L)$. It is impossible that β is a linear combination of $\Delta(L)$. On the other hand, by dominance and by observing charts for fundamental weights in terms of simple roots [14], Table 1, one can see that $N\chi_1 + N\chi_2 + N\chi_3$ has a positive coefficient for every simple root. This is a contradiction.

■

3.2 Embedding Flag Varieties of Subgroups of Maximal Rank

First, let $S \subset G$ be a semisimple subgroup of maximal rank and B_S a Borel subgroup of S such that $B_S \subset B$. In our study, the following question naturally occurred.

Question 3.2.1. *Does the image of the natural embedding*

$$(S/B_S)^3 \hookrightarrow (G/B)^3 \tag{3.1}$$

meet the semistable locus?

The answer, in general, is no. Here we give a counter example. If the answer were yes, then we would have existence of semistable points of a very useful form. In particular, existence of semistable points in $(S/B_S)^3$ would aid in sharpening the descent theorem.

Consider a line bundle $\mathcal{L}(\chi_1, \chi_2, \chi_3)$ on $(G/B)^3$, then the pull-back is the restriction and we have an S-equivariant map of sections

$$H^0((G/B)^3, \mathcal{L}^N) \to H^0((S/B_S)^3, \mathcal{L}|^N_{(S/B_S)^3}),$$

which induces

$$H^0((G/B)^3, \mathcal{L}^N)^S \to H^0((S/B_S)^3, \mathcal{L}|^N_{(S/B_S)^3})^S.$$

Composing with the inclusion of G-invariants on the left-hand side gives the map

$$\rho : H^0((G/B)^3, \mathcal{L}^N)^G \to H^0((S/B_S)^3, \mathcal{L}|^N_{(S/B_S)^3})^S.$$

Then, to prove that the embedding (3.1), in general, does not meet the $((G/B)^3)^{ss}$, it suffices to find a G and an S such that $[V(N\chi_1) \otimes V(N\chi_2) \otimes V(N\chi_3)]^G \neq 0$ for some $N > 0$ yet $[V_S(N\chi_1) \otimes V_S(N\chi_2) \otimes V_S(N\chi_3)]^S = 0$ for all N.

The relationship between G and S is given by Borel-de Siebenthal theory (cf. [33], [17] Theorem 3.10). We take, as a counter example, the subalgebra of type A_3 of the Lie algebra B_3. This is obtained from B_3 by deleting the short root and adjoining $-\theta$, where $\theta = \alpha_1 + 2\alpha_2 + 2\alpha_3$ is the highest root and $\{\alpha_1, \alpha_2, \alpha_3\}$ are the simple roots for B_3.

Then, we have A_3 as the subalgebra with root lattice $\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\theta$. We consider $\mathfrak{s}$ to be the subalgebra of type A_3 with simple roots

$$\begin{aligned} \beta_1 &= \alpha_2 \\ \beta_2 &= \alpha_1 \\ \beta_3 &= \alpha_2 + 2\alpha_3. \end{aligned}$$

It is straightforward to check that $\langle \beta_1, \beta_2 \rangle \langle \beta_2, \beta_1 \rangle = 1$, $\langle \beta_2, \beta_3 \rangle \langle \beta_3, \beta_2 \rangle = 1$, and $\langle \beta_1, \beta_3 \rangle \langle \beta_3, \beta_1 \rangle = 0$. Also, $\langle \beta_3, \beta_3 \rangle = -2$, and the A_3 roots generated by this choice of simple roots are still B_3 roots. Thus, these are the simple roots of a maximal rank subalgebra of type A_3.

Expressing the simple roots of B_3 in terms of the simple roots of A_3 gives the restriction map on roots:

$$\alpha_1 \mapsto \beta_2$$
$$\alpha_2 \mapsto \beta_1$$
$$\alpha_3 \mapsto (1/2)(\beta_3 - \beta_1).$$

For reference, we record the fundamental weights of both types. First, for B_3:

$$\begin{aligned} \omega_1 &= \alpha_1 + \alpha_2 + \alpha_3 \\ \omega_2 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 \\ \omega_3 &= (1/2)(\alpha_1 + 2\alpha_2 + 3\alpha_3). \end{aligned}$$

Second, for A_3:

$$\begin{aligned} \nu_1 &= (1/4)(3\beta_1 + 2\beta_2 + \beta_3) \\ \nu_2 &= (1/4)(2\beta_1 + 4\beta_2 + 2\beta_3) \\ \nu_3 &= (1/4)(\beta_1 + 2\beta_2 + 3\beta_3). \end{aligned}$$

Applying the restriction map to the fundamental weights we get the following.

$$\omega_1 \mapsto \nu_2$$
$$\omega_2 \mapsto \nu_1 + \nu_3$$
$$\omega_3 \mapsto \nu_3.$$

If we express a highest weight for type B_3 as $(a, b, c) = a\omega_1 + b\omega_2 + c\omega_3$, then this gives the highest weight of the type A_3 subalgebra $(b, a, b+c) = b\nu_1 + a\nu_2 + (b+c)\nu_3$. Let $V(a, b, c)$ denote the obvious representation. We wish to construct an example with (a_i, b_i, c_i), $i = 1, 2, 3$, such that

$$\left[V(a_1, b_1, c_1) \otimes V(a_2, b_2, c_2) \otimes V(a_3, b_3, c_3)\right]^G \neq 0 \tag{3.2}$$

however

$$\left[V_S(b_1, a_1, b_1 + c_1) \otimes V_S(b_2, a_2, b_2 + c_2) \otimes V_S(b_3, a_3, b_3 + c_3)\right]^S = 0. \tag{3.3}$$

Now, recall that for B_n and $\lambda \in \Lambda(B_n)$, $-w_0\lambda = \lambda$ and for A_3,

$$-w_0(a\nu_1 + b\nu_2 + c\nu_3) = c\nu_1 + b\nu_2 + a\nu_3.$$

Hence, equation (3.2) holds if and only if $V(a_1, b_1, c_1)$ is a component in $V(a_2, b_2, c_2) \otimes V(a_3, b_3, c_3)$. While, equation (3.3) holds if and only if $V_S(b_1 + c_1, a_1, b_1)$ is a component of $V_S(b_2, a_2, b_2 + c_2) \otimes V_S(b_3, a_3, b_3 + c_3)$.

Using [20], we then check that $V(2, 1, 5)$ has multiplicity 4 inside $V(1, 1, 2) \otimes V(1, 2, 1)$. However, $V_S(6, 2, 1)$ does not occur inside $V_S(1, 1, 3) \otimes V_S(2, 1, 3)$.

Since the tensor cone forms a semi-group, we have for any $N > 0$

$$\big[V(N2, N1, N5) \otimes V(N1, N1, N2) \otimes V(N1, N2, N1)\big]^G \neq 0.$$

Yet, since the saturation property holds for $SL(n)$ (cf. [15]) and $4\nu_1 + 4\nu_2 + 12\nu_3 \in Q_{\mathfrak{s}}$, we have for any $N > 0$,

$$\big[V_S(N1, N2, N6) \otimes V_S(N1, N1, N3) \otimes V_S(N2, N1, N3)\big]^S = 0.$$

This gives the desired counter example

CHAPTER 4: NOTATION AND PRELIMINARIES II

In this chapter, we develop the needed notation and preliminaries to understand joint work with Shrawan Kumar found in [4], which we cover in chapters six and seven. In the first section, we discuss diagram automorphisms, showing all such automorphisms, and constructing the root datum of the fixed point subgroups following [28]. In sections two, we introduce the theory of principal TDS embeddings due to Kostant in [16]. In section three, we give a brief treatment of primitive elements in $H^*(G)$. Last, in section four, we define the transgression map and state related theorems.

4.1 Diagram Automorphisms

The theory discussed in this section can all be found in [28]. In this section let $\mathfrak{g}$ be a simple, simply-lace Lie algebra of type A_ℓ, D_ℓ, or E_6 with Cartan subalgebra $\mathfrak{t}$. For each of the corresponding Dynkin Diagrams, there is a non-trivial automorphism σ called a *diagram automorphism.* In a natural way, the diagram automorphism σ induces an automorphism on the simply-connected algebraic group G with Lie algebra $\mathfrak{g}$, and hence σ also induces an automorphism on $\mathfrak{g}$ (cf. [28] for how this is defined). By abuse of notation, we also use σ to denote the automorphism of G and $\mathfrak{g}$. Let $G^\sigma \subset G$ be the fixed point subgroup of σ on G. If T^σ is the fixed point subgroup of T, then $\mathfrak{t}^\sigma$ is both the Lie algebra of T^σ and the σ-fixed point subalgebra of $\mathfrak{t}$. Similarly, $\mathfrak{g}^\sigma$ is simultaneously the Lie algebra of G^σ and the σ-fixed point subalgebra of $\mathfrak{g}$.

Then, we construct the root datum of G^σ. Let $X(T)$ be the character group of T and Λ the weight lattice, so $\Lambda = X(T)$ as G is simply connected. Let R be the roots with basis Δ, Q be the root lattice. For each $\alpha \in R$, let $\alpha^\vee$ be the co-root such that $\langle \alpha, \alpha^\vee \rangle = 2$. Consider σ as a permutation on R. G^σ has maximal torus T^σ consisting of the σ fixed points of T. The roots of G^σ, denoted R_σ, are given by the restriction of R to $\mathfrak{t}^\sigma$. Alternatively, we can can think of R_σ as the image of R in the quotient $X(T)/(\sigma - 1)X(T)$. For $\alpha \in R$, we will denote by $\alpha_{\mathcal{O}} \in R_\sigma$ the image of α in the quotient $X(T)/(\sigma - 1)X(T)$. Observe that Δ_σ, the simple roots of R_σ, is the image of Δ in R_σ.

If $\mathcal{O}$ is a σ-orbit in R, write

$$\alpha_{\mathcal{O}}^{\vee} = \sum_{\alpha \in \mathcal{O}} \alpha^{\vee}.$$

Then the co-roots of G^{σ} corresponding to $\alpha_{\mathcal{O}} \in \Delta_{\sigma}$ is as follows:

(i) If G is not of type A_{2n}, then take $\alpha_{\mathcal{O}}^{\vee}$ when α has either $|\mathcal{O}| = 1$ or $|\mathcal{O}| > 1$ and α is orthogonal to every root in its orbit.

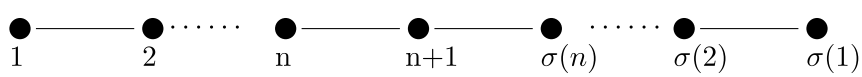
(ii) If G is of type A_{2n}, then take $\alpha_{\mathcal{O}}^{\vee}$ when α has $|\mathcal{O}| > 1$ and α is orthogonal to every root in its orbit. If $|\mathcal{O}| > 1$ and $\langle \sigma\alpha, \alpha^{\vee} \rangle \neq 0$, then take $2\alpha_{\mathcal{O}}^{\vee}$.

In what follows we let $\Delta_{\sigma}^{\vee}$ denote the co-roots corresponding to Δ_{σ} as described here. In the following we use these rules to compute the Dynkin diagrams for types of pairs (G^{σ}, G), what is called *diagram folding.*

4.1.1 (C_{n+1}, A_{2n+1})

We have σ as follows:

Figure 4.1: σ on A_{2n+1}



All orbits are size two except the $(n+1)$th node, Δ_{σ} has one root for every orbit. Since every root with a non-trivial orbit is orthogonal to every other root in its orbit,

$$\Delta_{\sigma}^{\vee} = \{\alpha_{n+1}^{\vee},\ \alpha_i^{\vee} + \sigma(\alpha_i)^{\vee} \mid i \neq n+1\}.$$

Then, for $i \neq n, n+1$, $i, j \leq n+1$,

$$\langle (\alpha_j)_{\mathcal{O}}, \alpha_i^{\vee} + \sigma(\alpha_i)^{\vee} \rangle = \begin{cases} 0 & |i-j| > 1 \\ -1 & |i-j| = 1 \end{cases}$$

and

$$\langle(\alpha_j)_{\mathcal{O}}, \alpha_{n+1}^\vee\rangle = \begin{cases} -1 & j = n \\ 0 & o.w. \end{cases}.$$

Also,

$$\langle(\alpha_j)_{\mathcal{O}}, \alpha_n^\vee + \sigma(\alpha_n)^\vee\rangle = \begin{cases} 0 & |i-j| > 1 \\ -2 & j = n+1 \\ -1 & j = n-1 \end{cases}.$$

So, we have single bonds between all nodes for $\beta_i := (\alpha_i)_{\mathcal{O}}$ for $i \leq n-1$, and the double bond between $\beta_n := (\alpha_n)_{\mathcal{O}}$ and $\beta_{n+1} := (\alpha_{n+1})_{\mathcal{O}}$, and β_{n+1} is the longer root. So, the diagram for (G^σ, T^σ) in this case is

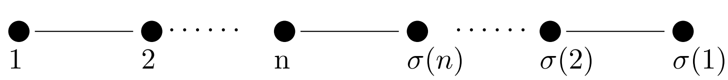
Figure 4.2: C_{n+1} as G^σ

Hence G^σ is of type C_{n+1}.

4.1.2 (B_n, A_{2n})

We have σ as follows:

Figure 4.3: σ on A_{2n}



All orbits are size two, so Δ_σ has one root for every orbit. Although it is *a priori* possible that multiple orbits are identified in Δ_σ, as we cannot use [28] Lemma 10.3.2 here, we know from example 4.1.6 below that G^σ has rank n. Since all roots except n and $n+1$ are orthogonal in their orbits, we have

$$\Delta_\sigma^\vee = \{2(\alpha_n^\vee + \alpha_{n+1}^\vee),\ \alpha_i^\vee + \sigma(\alpha_i)^\vee \mid i \neq n\}.$$

Then, for $i \neq n$, $i, j \leq n$,

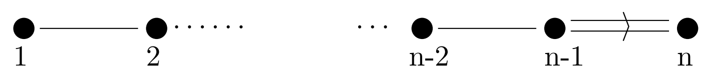
$$\langle (\alpha_j)_{\mathcal{O}}, \alpha_i^\vee + \sigma(\alpha_i)^\vee \rangle = \begin{cases} 0 & |i-j| > 1 \\ -1 & |i-j| = 1 \end{cases}$$

and

$$\langle (\alpha_j)_{\mathcal{O}}, 2(\alpha_n^\vee + \alpha_{n+1}^\vee) \rangle = \begin{cases} -2 & j = n-1 \\ 0 & j < n-1 \end{cases}.$$

Let $\beta_i = (\alpha_i)_{\mathcal{O}}$ for $1 \leq i \leq n$. So, we have single bonds between all nodes except the double bond between β_n and β_{n-1}, and β_n is the shorter root. So, the diagram for (G^σ, T^σ) in this case is

Figure 4.4: B_n as G^σ

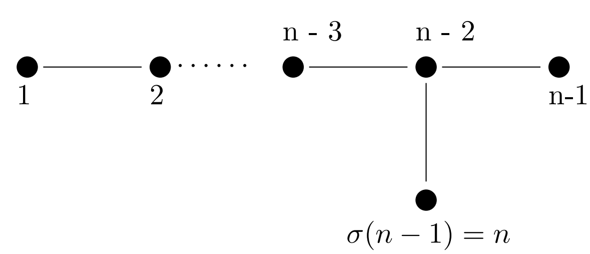


Hence G^σ is of type B_n.

4.1.3 (B_{n-1}, D_n)

We have σ as follows:

Figure 4.5: σ on D_n



All orbits are size one except the $n-1$ and n nodes of size two, Δ_σ has one roots for each root in the original diagram and one root for the $n-1$, n orbit, and

$$\Delta_\sigma^\vee = \{\alpha_n^\vee + \alpha_{n-1}^\vee,\ \alpha_i^\vee \mid i \leq n-1\}.$$

Then, for $i \leq n-1$, $j \leq n$,

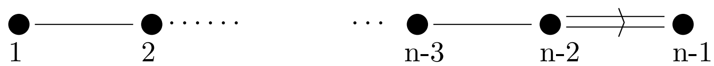
$$\langle (\alpha_j)_{\mathcal{O}}, \alpha_i^\vee \rangle = \begin{cases} 0 & |i-j| > 1 \\ -1 & |i-j| = 1 \end{cases}$$

and

$$\langle (\alpha_j)_{\mathcal{O}}, \alpha_n^\vee + \alpha_{n-1}^\vee \rangle = \begin{cases} -2 & j = n-2 \\ 0 & j < n-1 \end{cases}.$$

For $1 \leq i \leq n-1$ let $\beta_i := (\alpha_i)_{\mathcal{O}}$, then there are single bonds between β_i β_{i+1} except the double bond between β_{n-2} and β_{n-1}, and β_{n-1} is the shorter root. So, the diagram for (G^σ, T^σ) in this case is

Figure 4.6: B_{n-1} as G^σ

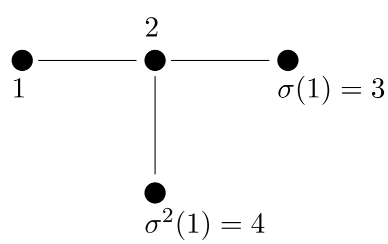


Hence G^σ is of type B_{n-1}.

4.1.4 (G_2, D_4)

We have the order three σ as follows:

Figure 4.7: Order three σ on D_4



There is one orbit of size three $\{\alpha_1, \alpha_3, \alpha_4\}$ and one fixed simple root $\{\alpha_2\}$, so Δ_σ has two roots, and

$$\Delta_\sigma^\vee = \{\alpha_1^\vee + \alpha_3^\vee + \alpha_4^\vee \ , \ \alpha_2^\vee\}.$$

Then,

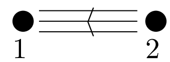
$$\langle (\alpha_1)_{\mathcal{O}}, \alpha_2^\vee \rangle = -1$$

and

$$\langle(\alpha_2)_{\mathcal{O}}, \alpha_1^\vee + \alpha_3^\vee + \alpha_4^\vee\rangle = -3.$$

So, we have triple bond between $\beta_1 := (\alpha_1)_{\mathcal{O}}$ and $\beta_2 := (\alpha_2)_{\mathcal{O}}$, and β_2 is the longer root. So, the diagram for (G^σ, T^σ) in this case is

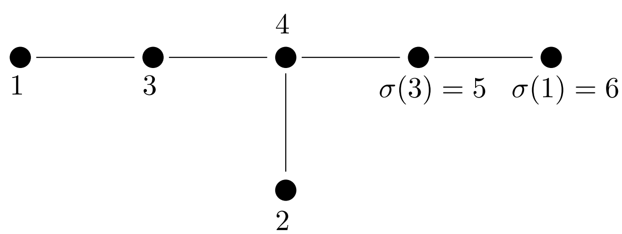
Figure 4.8: G_2 and G^σ



Hence G^σ is of type G_2.

4.1.5 (F_4, E_6)

Figure 4.9: σ on E_6



There are two orbits of size two $\{\alpha_1, \alpha_6\}$ and $\{\alpha_3, \alpha_5\}$, and the fixed roots α_2 and α_4. So, D_σ has four roots corresponding to $(\alpha_1)_{\mathcal{O}}, \ldots, (\alpha_4)_{\mathcal{O}}$. and

$$\Delta_\sigma^\vee = \{\alpha_1^\vee + \alpha_6^\vee, \alpha_3^\vee + \alpha_5^\vee\ \alpha_2^\vee, \alpha_4^\vee \mid i \leq n-1\}.$$

Then, the non-zero inner products are

$$\langle(\alpha_4)_{\mathcal{O}}, \alpha_2^\vee\rangle = \langle(\alpha_2)_{\mathcal{O}}, \alpha_4^\vee\rangle = -1,$$

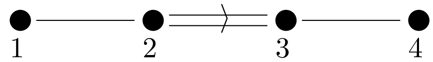
$$\langle(\alpha_3)_{\mathcal{O}}, \alpha_4^\vee\rangle = -1, \quad \langle(\alpha_4)_{\mathcal{O}}, \alpha_3^\vee + \alpha_5^\vee\rangle = -2,$$

and

$$\langle(\alpha_3)_{\mathcal{O}}, \alpha_1^\vee + \alpha_6^\vee\rangle = \langle(\alpha_1)_{\mathcal{O}}, \alpha_3^\vee + \alpha_5^\vee\rangle = -1.$$

So, we have single bonds between all nodes except the double bond between $(\alpha_3)_{\mathcal{O}}$ and $(\alpha_4)_{\mathcal{O}}$, and $(\alpha_4)_{\mathcal{O}}$ is the longer root. To label the roots of G^σ in a convenient manner, let $\beta_1 = (\alpha_2)_{\mathcal{O}}$, $\beta_2 = (\alpha_4)_{\mathcal{O}}$, $\beta_3 = (\alpha_5)_{\mathcal{O}}$ and $\beta_4 = (\alpha_6)_{\mathcal{O}}$. So, the diagram for (G^σ, T^σ) in this case is

Figure 4.10: F_4 as G^σ



Hence G^σ is of type F_4.

4.1.6 Example $G = SL(2n+1)$

Let us explicitly look at the case of $SL(2n+1)$. This is the group of determinant one automorphisms of $\mathbb{C}^{2n+1}$ in the standard basis $\{e_i\}_{i=1}^{2n+1}$. Then, the maximal torus is

$$T = \{\mathrm{diag}(a_1, \ldots, a_{2n+1}) \mid a_1 \cdots a_{2n+1} = 1\},$$

and the Borel subgroup B is upper triangular matrices. We consider the following involution of $SL(2n+1)$

$$\sigma(A) = E^{-1}(A^t)^{-1}E,$$

where

$$E = \begin{pmatrix} 0 & 0 & J \\ 0 & 2 & 0 \\ J & 0 & 0 \end{pmatrix}$$

and J is the $n \times n$ anti-identity. It is routine to check that σ is a diagram automorphism of $SL(2n+1)$. Moreover, since A fixes the bilinear form determined by E if and only if $A^tEA = E$, we see this condition is equivalent to $A = E^{-1}(A^t)^{-1}E = \sigma(A)$. In particular,$SL(2n+1)^\sigma$ is the subgroup $SO(2n+1)$ stabilizing the symmetric bilinear form associated to E.

4.2 Principal TDS Embeddings

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of rank ℓ and $\mathfrak{t}$ a fixed Cartan subalgebra. An element $X \in \mathfrak{g}$ is said to be *nilpotent* if $\mathrm{ad}(X)$ is a nilpotent linear transformation, where $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is the adjoint representation $\mathrm{ad}(X)(Y) = [X, Y]$ (alternatively one can use the image of X under any representation). An example of a nilpotent element is any $X_\alpha \in \mathfrak{g}_\alpha$ where $\alpha \in R^+$ and $\mathfrak{g}_\alpha$ is the root subspace of α.

Now if G is a group with $\mathfrak{g}$ as its Lie algebra, using the properties of $\mathrm{Ad}(g)$ listed in section 1.1, we have for any $g \in G$ and any $X, Y \in G$,

$$(\mathrm{ad}(X))^N Y = [\mathrm{Ad}(g)(X), \cdots, [\mathrm{Ad}(g)(X), Y] \cdots] = \mathrm{Ad}(g) \cdot \mathrm{ad}(X)^N (\mathrm{Ad}(g^{-1})(Y)).$$

In particular, $\mathrm{Ad}(g)(X)$ is nilpotent when X is nilpotent. If we let $\mathcal{N}$ be the collection of all nilpotent elements in $\mathfrak{g}$, then we see that $\mathcal{N}$ is $\mathrm{Ad}(G)$-stable. Moveover, it follows from [16], Lemma 5.4, that $\mathcal{N}$ is exactly the $\mathrm{Ad}(G)$ orbit of $\mathfrak{b}$. There is a unique open orbit in $\mathcal{N}$, and any element in this open orbit is called *principal nilpotent*. Then, the following is due to Kostant ([16], Theorem 5.3).

Theorem 4.2.1. *For $\alpha \in R^+$, let $X_\alpha \in \mathfrak{g}_\alpha$, i.e., such that $\mathfrak{b} = \bigoplus_{\alpha \in R^+} \mathbb{C} X_\alpha$. Then,*

$$X = \sum_{\alpha \in R^+} a_\alpha X_\alpha \in \mathfrak{b}$$

is a principal nilpotent if and only if $a_\alpha \neq 0$ for every simple root $\alpha \in \Delta$.

Now, let $\mathfrak{sl}_2$ be the Lie algebra of traceless 2×2 matrices over $\mathbb{C}$ with the standard basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this case, E is the principal nilpotent of $\mathfrak{sl}_2$. We have the following definition.

Definition 4.2.1. *A Lie algebra embedding $\varphi : \mathfrak{sl}_2 \to \mathfrak{g}$ (or its image) is called a principal TDS if $\varphi(E)$ is a principal nilpotent element of $\mathfrak{g}$.*

It follows from [16], Corollary 3.7, that all principal TDS are $\mathrm{Ad}(G)$-conjugates. That is, if $\varphi' : \mathfrak{sl}_2 \to \mathfrak{g}$ is another principal TDS, then, there exists a $g \in G$ such that

$$\varphi' = \mathrm{Ad} g \cdot \varphi. \tag{4.1}$$

Now, a principal TDS embedding defines an action of $\mathfrak{sl}_2$ on $\mathfrak{g}$ by the adjoint representation of $\varphi(\mathfrak{sl}_2)$, i.e., $\mathfrak{g}$ becomes a representation of $\mathfrak{sl}_2$. Recall that the irreducible representations of $\mathfrak{sl}_2$ correspond to nonnegative integers n such that n is the highest weight. Moreover, the irreducible $\mathfrak{sl}_2$ representation of highest weight n has dimension $n+1$ (cf. [14], §7.2). Then, we decompose $\mathfrak{g}$ into irreducible $\mathfrak{sl}_2$ components:

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_\ell,$$

labeling them so that

$$n_1 \leq \cdots \leq n_\ell, \quad \text{where} \quad n_i = \dim V_i. \tag{4.2}$$

By the identity (4.1), we see that the decomposition of $\mathfrak{g}$ with respect to another principal TDS φ' looks like

$$\mathfrak{g} = \big(\mathrm{Ad}g \cdot V_1\big) \oplus \big(\mathrm{Ad}g \cdot V_2\big) \oplus \cdots \oplus \big(\mathrm{Ad}g \cdot V_\ell\big).$$

Then, by [16] Corollaries 5.3 and 8.7 for *(i)* & *(ii)*, and by any table of exponents for *(iii)*, we have the following.

Proposition 4.2.1. *(i) $\ell =$ rank of $\mathfrak{g}$.*

(ii) Each n_i is an odd integer $2m_i + 1$. Moreover,

$$m_1 \leq m_2 \leq \cdots \leq m_\ell$$

are the exponents of $\mathfrak{g}$.

(iii) Except when $\mathfrak{g}$ is of type D_ℓ, with ℓ even, each V_i is an isotypical component (in particular, uniquely determined) for the principal TDS φ, i.e., $m_1 < m_2 < \cdots < m_\ell$.

When $\mathfrak{g}$ is of type D_ℓ, with ℓ even, the exponents are

$$1, 3, 5, \cdots, \ell-3, \ell-1, \ell-1, \ell+1, \cdots, 2\ell-3. \tag{4.3}$$

Hence, the isotypical component for the highest weight $2\ell-2$ is a direct sum of two copies of the irreducible module $V_{\mathfrak{sl}_2}(2\ell-2)$ with highest weight $2\ell-2$.

4.3 Primitive Elements

Let G be the simple, connected group with simple Lie algebra $\mathfrak{g}$. Let the singular cohomology of G be considered with complex coeffcicients, i.e., $H^*(G) := H^*(G, \mathbb{C})$. The multiplication map $G \times G \to G$ induces a graded homomorphism on cohomology $\Delta : H^*(G) \to H^*(G) \otimes H^*(G)$. In particular, this is a *co-mulitplication* map making $H^*(G)$ a *Hopf Algebra*.

Then, an element $x \in H^*(G)$ is called a *primitive element* if $\Delta(x) = 1 \otimes x + x \otimes 1$. Since Δ is a graded homomorphism, the collection of primitive elements in $H^*(G)$ forms a graded subspace which we denote by $P(\mathfrak{g})$. Note, e.g. by [7], Theorem 15.1, that $H^*(G)$ does not depend on the isogeny type of G, so the notation $P(\mathfrak{g})$ is justified.

Now, let G_0 be a maximal compact subgroup of G. We can identify $\wedge(\mathfrak{g}^*)$ with the left invariant, $\mathbb{C}$-valued forms on G_0, and hence identify $\wedge(\mathfrak{g}^*)^{\mathfrak{g}}$ with the bi-invariant forms on G_0. Then, $\wedge(\mathfrak{g}^*)^{\mathfrak{g}} \cong H^*_{dR}(G_0, \mathbb{C})$, where the latter denotes the de Rham cohomology of G_0, follows from the fact that each de Rham cohomology class on a compact, connected Lie group has exactly one bi-invariant representative (cf. [7], Theorem 12.1). Moreover, by the de Rham Theorem $H^*_{dR}(G_0, \mathbb{C}) \cong H^*(G_0, \mathbb{C})$. Lastly, since G_0 is a deformation retract of G, we have $H^*(G_0, \mathbb{C}) \cong H^*(G)$. In summary, we have a canonical isomorphism

$$\wedge (\mathfrak{g}^*)^{\mathfrak{g}} \cong H^*(G). \tag{4.4}$$

A Hopf algebra structure is naturally defined on $\wedge(\mathfrak{g}^*)^{\mathfrak{g}}$ (cf. [21] §10.3.1). It is a tedious but routine computation to check that isomorphsim (4.4) respects co-multiplication, and is thus an isomorphism of Hopf algebras. In particular, we will consider $P(\mathfrak{g}) \subset \wedge(\mathfrak{g}^*)^{\mathfrak{g}}$.

Then, we have the following useful facts on primitive elements (cf. [21] Proposition 10.12 and Threom 10.2); the latter is well known as the Hopf-Kozul-Samelson Theorem.

Proposition 4.3.1. *All primitive elements of $\wedge(\mathfrak{g}^*)^{\mathfrak{g}}$ occur in odd degrees.*

Theorem 4.3.1. *The space of primitive elements $P(\mathfrak{g}) \subset \wedge(\mathfrak{g}^*)^{\mathfrak{g}}$ generates the algebra $\wedge(\mathfrak{g}^*)^{\mathfrak{g}}$.*

In addition, for any $d \geq 1$, let P_d be the subspace of $P(\mathfrak{g})$ of (homogeneous) degree d elements. Then, it is well known that

$$\dim P_d = \#\{1 \leq i \leq \ell \mid n_i = d\}, \tag{4.5}$$

where n_i's are the dimensions of irreducible components of $\mathfrak{g}$ under the principal-$\mathfrak{sl}_2$ action.

In particular, if $\mathfrak{g}$ is not of type D_ℓ (with ℓ even), then

$$\dim P_d \leq 1 \tag{4.6}$$

and P_d is of dimension 1 if and only if d is equal to one of the $n_i' s$. If $\mathfrak{g}$ is of type D_ℓ (with ℓ even),

$$\dim P_d \leq 1 \ \text{ if } \ d \neq 2\ell - 1, \ \text{ and } \ \dim P_{2\ell-1} = 2. \tag{4.7}$$

4.4 The Transgression Map

Here we give a description of the transgression map. The transgression map is originally due to the work of H. Cartan in [5]. However, we follow the treatment given buy E. Meinrenken in [21].

Let $\mathfrak{g}^*[k]$ be the graded Lie algebra that has non-zero elements only in degree $-k$. Then, the *Weil Algebra* of $\mathfrak{g}$, denoted $W\mathfrak{g}$, is the Koszul algebra for $\mathfrak{g}^*[-1]$. In practice, we have

$$W\mathfrak{g} = S(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*), \tag{4.8}$$

where S denotes the symmetric algebra and $\wedge$ denotes the exterior algebra. Equation (4.8) requires some additional explanation. In particular, the Weil algebra is a graded algebra with a degree one differential d. To each $\mu \in \mathfrak{g}^*$ we have a degree one element $\mu \in \mathfrak{g}^*[-1] = S^1(\mathfrak{g}^*)$, and a degree two element $d\mu \in \mathfrak{g}^*[-2] = \wedge^1(\mathfrak{g}*)$ of the Weil algebra. Set $\hat{\mu} = d\mu - d_\wedge \mu \in W\mathfrak{g}$, where, $d_\wedge$ is the differential on $\wedge(\mathfrak{g}^*)$ used to define the Lie algebra cohomology of $\mathfrak{g}^*$. Then, the collection of μ and $\hat{\mu}$, for all $\mu \in \mathfrak{g}^*$, generate $W\mathfrak{g}$.

Now, we have two different naturally defined actions of $\mathfrak{g}$ on $W\mathfrak{g}$. Namely, for each $X \in \mathfrak{g}$, we have the *contraction operation*, denoted $i(X)$, which is a degree -1 operator, and the *Lie derivative*, denoted $L(X)$, which is a degree 0 operator. The Lie derivative is the tensor product of the extensions of the co-adjoint representation of $\mathfrak{g}$ to $S(\mathfrak{g}^*)$ and $\wedge(\mathfrak{g}^*)$. The contraction operator $i(X)$ is defined on degree one generators by $i(X)(\mu) = \mu(X)$ and on degree two generators by

$$i(X)d\mu = L(X)\mu.$$

Then, we define the *invariant subspace* of the Weil algebra, denoted $(W\mathfrak{g})^\mathfrak{g}$, to be the subspace of

$W\mathfrak{g}$ annihilated by $L(X)$ for all $X \in \mathfrak{g}$. Also, define the *basic subspace* of $W\mathfrak{g}$, denoted $(W\mathfrak{g})_{\text{bas}}$, to be the subspace of $(W\mathfrak{g})^{\mathfrak{g}}$ annihilated by $i(X)$ for all $X \in \mathfrak{g}$. It follows from basic properties (cf. [21] §6.6) of contraction and Lie derivatives that the differential d preserves both $(W\mathfrak{g})^{\mathfrak{g}}$ and $(W\mathfrak{g})_{\text{bas}}$. Then, we have the following proposition (cf. [21], Theorem 6.2 and Proposition 6.9).

Proposition 4.4.1. $(W\mathfrak{g})_{bas} = (S(\mathfrak{g}^*))^{\mathfrak{g}}$*, where* $S(\mathfrak{g}^*)$ *is the symmetric algebra in the generators* $\hat{\mu}$*. Moreover,* $(W_{\mathfrak{g}})^{\mathfrak{g}}$ *with the restriction of the Weil differential, is an acyclic differential algebra.*

Now, there is a canonically defined, $\mathfrak{g}$-equivariant (with respect to both contraction and Lie operators), morphism of graded differential algebras

$$\pi : W\mathfrak{g} \to \wedge(\mathfrak{g}^*).$$

Restricting to the invariant subspaces (i.e., just with respect to Lie differentiation) we have a morphism of differential algebras

$$\pi : (W\mathfrak{g})^{\mathfrak{g}} \to (\wedge(\mathfrak{g}^*))^{\mathfrak{g}}.$$

The differential $d_\wedge$ is trivial on $(\wedge(\mathfrak{g}^*))^{\mathfrak{g}}$ since $(\wedge(\mathfrak{g}^*))^{\mathfrak{g}} \cong H^*(\mathfrak{g})$, where the former is the Lie algebra cohomology of $\mathfrak{g}$ (cf. [21] § 6.13, 6.8 and Proposition 6.11). In particular, since π is a morphism of differential algebras, $\pi(dx) = 0$ for any $x \in W\mathfrak{g}$ such that $dx \in (W\mathfrak{g})^{\mathfrak{g}}$.

We have the following definition/proposition the the *transgression* map τ ([21] Proposition 6.17)

Proposition 4.4.2. *There is a well-defined linear map,*

$$\tau : (S^+\mathfrak{g}^*)^{\mathfrak{g}} \to (\wedge\mathfrak{g}*)^{\mathfrak{g}}$$

such that $\tau(p) = \pi(C)$*, where* $C \in (W\mathfrak{g})^{\mathfrak{g}}$ *is any odd element such that* $dC = p$*. If* p *has degree* r*, then* $\tau(p)$ *has degree* $2r - 1$*.*

Since we have stated all the necessary theory, let us prove the existence of such a C and that τ is well-defined. For existence, consider $p \in (S\mathfrak{g}^*)^{\mathfrak{g}} = (W\mathfrak{g})_{\text{bas}}$, since $(W\mathfrak{g})_{\text{bas}} \subset (W\mathfrak{g})^{\mathfrak{g}}$ and, by Proposition 4.4.1, $(W\mathfrak{g})^{\mathfrak{g}}$ is acyclic, there must be some $C \in (W\mathfrak{g})^{\mathfrak{g}}$ such that $dC = p$. To see that τ is well-defined, it suffices to show that if $d(C_1 - C_2) = 0$, then $\pi(C_1 - C_2) = 0$. Again, since $(W\mathfrak{g})^{\mathfrak{g}}$ is acyclic, $d(C_1 - C_2) = 0$ implies that $C_1 - C_2 = dC'$ for some $C' \in (W\mathfrak{g})^{\mathfrak{g}}$. Then,

$\pi(C_1 - C_2) = \pi(dC') = 0$.

Also, note that in the proof of Theorem 2.2.1 we give a definition of τ in terms of a basis for $\mathfrak{g}$. Finally, we have the following important theorem originially due to H. Cartan (cf. [5], [21]).

Theorem 4.4.1. *The kernel of τ is $(S^+(\mathfrak{g}^*)^{\mathfrak{g}}) \cdot (S^+(\mathfrak{g}^*)^{\mathfrak{g}})$ and the image of τ is $P(\mathfrak{g})$.*

CHAPTER 5: DIAGRAM AUTOMORPHISMS & GIT

Let $\mathfrak{g}$ be a simple, simply-laced Lie algebra over $\mathbb{C}$ and let G be the connected, simply-connected complex algebraic group with Lie algebra $\mathfrak{g}$. Let σ be a diagram automorphism of $\mathfrak{g}$ and let $\mathfrak{k} = \mathfrak{g}^\sigma$ be the fixed subalgebra. The, $\mathfrak{k}$ is a simple Lie algebra again of type discussed in the previous chapter. Let $K = G^\sigma$ be the fixed-point subgroup of G with Lie algebra $\mathfrak{k}$. Recall the construction of root datum for $\mathfrak{g}^\sigma$ as given in the previous chapter.

We have the following main result of this section.

Theorem 5.0.2. *The canonical map $\phi : R(G) \to R(K)$ is surjective, where $R(G)$ denotes the representation ring of G (over $\mathbb{Z}$).*

In particular, the canonical map $K//Ad\,K \to G//Ad\,G$, between the GIT quotients, is a closed embedding.

Before proving this theorem, we must develop some useful lemmas. Let $\Lambda^+(\mathfrak{g}) \subset \mathfrak{t}^*$ (resp. $\Lambda^+(\mathfrak{k}) \subset \mathfrak{t}_\mathfrak{k}^*$) be the set of dominant integral weights for the root system of $\mathfrak{g}$ (resp. $\mathfrak{k}$) and let $X(T_K)^+ \subset \Lambda^+(\mathfrak{k})$ be the submonoid of dominant characters for the group K, i.e., $X(T_K)^+$ is the set of characters of the maximal torus $T_K = T^\sigma$ (with Lie algebra $\mathfrak{t}_\mathfrak{k}$) of K which are dominant with respect to the group K. Let $\{\nu_1, \ldots, \nu_{\ell_\mathfrak{k}}\}$ be the fundamental weights of $\mathfrak{k}$, where $\ell_\mathfrak{k} = rank(\mathfrak{k})$, and recall $\{\varpi_1, \ldots, \varpi_\ell\}$ are the fundamental weights of $\mathfrak{g}$. Observe that since G is simply-connected, $X(T)^+ = \Lambda^+(\mathfrak{g})$. Moreover, under the restriction map $\rho : \mathfrak{t}^* \to \mathfrak{t}_\mathfrak{k}^*$,

$$\rho(\Lambda^+(\mathfrak{g})) = X(T_K)^+. \tag{5.1}$$

To see this, let $X(T_K)$ be the character lattice of K (similarly for $X(T) = \Lambda(\mathfrak{g})$). Then, by Springer's original construction of $X(T_K)$ [28], the restriction $\rho : \Lambda(\mathfrak{g}) \to X(T_K)$ is surjective. Further, from the description of the coroots of $\mathfrak{k}$ as in [28], $\rho(\Lambda^+(\mathfrak{g})) \subset \Lambda^+(\mathfrak{k})$. Thus, we have

$$\rho(\Lambda^+(\mathfrak{g})) \subset \Lambda^+(\mathfrak{k}) \cap X(T_K) = X(T_K)^+.$$

Conversely, in all cases except for $\mathfrak{g}$ of type A_{2n}, by Lemma 5.0.2, $\rho(\Lambda^+(\mathfrak{g})) = \Lambda^+(\mathfrak{k}) \supset X(T_K)^+$, so equation (5.1) holds in these cases. When $\mathfrak{g}$ is of type A_{2n}, again by Lemma 5.0.2,

$$\rho(\Lambda^+(\mathfrak{g})) = \left(\oplus_{i=1}^{n-1} \mathbb{Z}_+\nu_i\right) \oplus 2\mathbb{Z}_+\nu_n,$$

and

$$X(T_K) = \rho(\Lambda(\mathfrak{g})) = \left(\oplus_{i=1}^{n-1} \mathbb{Z}\nu_i\right) \oplus 2\mathbb{Z}\nu_n.$$

From this again, we see that (5.1) is satisfied. This proves (5.1) in all cases.

For any $\lambda \in \Lambda^+(\mathfrak{g})$, let $V(\lambda)$ be the irreducible G-module with highest weight λ. Similarly, for $\mu \in X(T_K)^+$, let $W(\mu)$ be the irreducible K-module with highest weight μ. We denote the fundamental representations $V(\varpi_i)$ of $\mathfrak{g}$ by V_i and $W(\nu_j)$ of $\mathfrak{k}$ by W_j.

Lemma 5.0.1. *For any $\lambda \in \Lambda^+(\mathfrak{g})$, $W(\rho(\lambda))$ has multiplicity one in $V(\lambda)$ as a $\mathfrak{k}$-module. (Observe that by 5.1, $\rho(\lambda) \in X(T_K)^+$.)*

Proof: Note that the Borel subalgebra $\mathfrak{b}_{\mathfrak{k}}$ of $\mathfrak{k}$ is contained in the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. So, if v_λ is the highest weight vector of $V(\lambda)$ (of weight λ), then v_λ remains a highest weight vector of weight $\rho(\lambda)$ in $V(\lambda)$ for the action of $\mathfrak{k}$. Hence, $W(\rho(\lambda)) \subset V(\lambda)$.

Multiplicity one is clear from the weight consideration. ■

We next prove two facts unique to our context. For any simple root α, we denote the corresponding coroot by $\alpha^\vee$. We follow the indexing convention as in section 4.1.

Lemma 5.0.2. *(a) If G is not of type A_{2n} or E_6, then $\rho(\varpi_i) = \nu_i$ for $1 \le i \le \ell_{\mathfrak{k}} :=$ rank$(\mathfrak{k})$.*

(b) If G is of type A_{2n}, then $\rho(\varpi_i) = \rho(\varpi_{2n-i+1}) = \nu_i$ for $1 \le i \le n-1$, and $\rho(\varpi_n) = \rho(\varpi_{n+1}) = 2\nu_n$.

(c) If G is of type E_6, $\rho(\varpi_1) = \rho(\varpi_6) = \nu_4$; $\rho(\varpi_2) = \nu_1$; $\rho(\varpi_3) = \rho(\varpi_5) = \nu_3$; $\rho(\varpi_4) = \nu_2$.

Proof: (a) It suffices to show

$$\langle \rho(\varpi_i), \beta_j^\vee \rangle = \delta_{i,j}, \quad \text{for } 1 \le i, j \le \ell_{\mathfrak{k}}. \tag{5.2}$$

In this case, we have (cf. 4.1)

$$\beta_j^\vee = \sum \alpha_k^\vee,$$

where the summation runs over the orbit of α_j under σ. For $1 \leq j \leq \ell_{\mathfrak{k}}$, no α_k is in the σ-orbit of α_j for any $1 \leq k \leq \ell_{\mathfrak{k}}$. Thus, the equation (5.2) follows.

(b) When G is of type A_{2n}, by section 4.1,

$$\beta_j^\vee = \begin{cases} \alpha_j^\vee + \alpha_{2n-j+1}^\vee, & \text{for } j \leq n-1, \\ \\ 2\alpha_n^\vee + 2\alpha_{n+1}^\vee, & \text{for } j = n. \end{cases}$$

So, for $1 \leq i \leq 2n$,

$$\begin{aligned} \langle \rho(\varpi_i), \beta_j^\vee \rangle &= \begin{cases} \langle \varpi_i, \alpha_j^\vee \rangle + \langle \varpi_i, \alpha_{2n-j+1}^\vee \rangle, & \text{for } j \leq n-1, \\ \\ 2\langle \varpi_i, \alpha_n^\vee \rangle + 2\langle \varpi_i, \alpha_{n+1}^\vee \rangle, & \text{for } j = n. \end{cases} \\ &= \begin{cases} \delta_{i,j} + \delta_{i,2n-j+1}, & \text{for } j \leq n-1, \\ \\ 2\delta_{i,n} + 2\delta_{i,n+1}, & \text{for } j = n. \end{cases} \end{aligned}$$

From this (b) follows.

(c) Following the indexing convention as in section 4.1, we get that

$$\beta_1^\vee = \alpha_2^\vee,\ \beta_2^\vee = \alpha_4^\vee,\ \beta_3^\vee = \alpha_3^\vee + \alpha_5^\vee,\ \beta_4^\vee = \alpha_1^\vee + \alpha_6^\vee.$$

Thus,

$$\begin{aligned} &\rho(\varpi_1) = \rho(\varpi_6) = \nu_4, \\ &\rho(\varpi_2) = \nu_1, \\ &\rho(\varpi_3) = \rho(\varpi_5) = \nu_3, \\ &\rho(\varpi_4) = \nu_2. \end{aligned}$$

■

5.1 Proof of Theorem 5.0.2

Let $\{\mu_1, \dots, \mu_N\} \subset X(T_K)^+$ be a set of semigroup generators of $X(T_K)^+$. Then, the classes $\{[W(\mu_j)]\}_{1 \leq j \leq N}$ generate the $\mathbb{Z}$-algebra $R(K)$, where $[W(\mu_j)] \in R(K)$ denotes the class of the irreducible K-module $W(\mu_j)$ (cf. [25], Theorem 3.12).

We proceed separately for each of the five cases depending on the type of $(\mathfrak{g}, \mathfrak{k})$.

Case I (A_{2n+1}, C_{n+1}):

By Lemmas 5.0.2 and 5.0.1, for $1 \leq j \leq n+1$, $W_j \subset V_j$ (as $\mathfrak{k}$-modules). Recall that $V_1 \cong W_1 \cong \mathbb{C}^{2n+2}$ (so $W_1 = V_1$) and $V_j = \wedge^j V_1$ for all $1 \leq j \leq 2n+1$. Also, for $2 \leq j \leq n+1$, W_j is given as the kernel of the surjective $\mathfrak{k}$-equivariant contraction map $\wedge^j W_1 \to \wedge^{j-2} W_1$. Hence, for $2 \leq j \leq n+1$, in $R(\mathfrak{k})$ (where $R(\mathfrak{k})$ is the representation ring of $\mathfrak{k}$), by [11], Theorem 17.5,

$$[W_j] + [\wedge^{j-2} W_1] = [\wedge^j W_1].$$

Thus,

$$\phi([V_1]) = [W_1], \text{ and } \phi([V_j]) - \phi([V_{j-2}]) = [W_j], \text{ for } 2 \leq j \leq n+1,$$

where V_0 is interpreted as the trivial one dimensional module $\mathbb{C}$. Thus, the class $[W_j]$ of each fundamental representation lies in the image of ϕ, and hence ϕ is surjective.

Case II. (A_{2n}, B_n):

By Lemmas 5.0.2 and 5.0.1, for $1 \leq j \leq n-1$, $W_j \subset V_j$ and $W(2\nu_n) \subset V_n$ (as $\mathfrak{k}$-modules). Recall that $V_1 \cong W_1 \cong \mathbb{C}^{2n+1}$ (so $W_1 = V_1$), and $V_j = \wedge^j V_1$ for all $1 \leq j \leq 2n$. Also, $W_j = \wedge^j W_1$ for $1 \leq j \leq n-1$ and $W(2\nu_n) = \wedge^n W_1$ (see, e.g., [11], Theorem 19.14). Thus, as $\mathfrak{k}$-modules,

$$W_j = V_j, \ \ j \leq n-1; \qquad W(2\nu_n) = V_n.$$

Thus,

$$[W_1], \dots, [W_{n-1}], [W(2\nu_n)] \in \text{ Image } \phi.$$

By Lemma 5.0.2 (b) and the identity (5.1), $X(T_K)^+$ is generated (as a semigroup) by $\{\nu_1, \ldots, \nu_{n-1}, 2\nu_n\}$. Hence, ϕ is surjective in this case.

Case III. (D_n, B_{n-1})**:**

Recall that $V_1 \cong \mathbb{C}^{2n}$ and $W_1 \cong \mathbb{C}^{2n-1}$. By Lemmas 5.0.2 and 5.0.1, for $1 \leq j \leq n-1$, $W_j \subset V_j$ (as $\mathfrak{k}$-modules). Since $W_1 \subset V_1$ (as $\mathfrak{k}$-modules), we get (as $\mathfrak{k}$-modules):

$$V_1 = W_1 \oplus \mathbb{C}.$$

Thus, for $1 \leq k \leq n-2$, as $\mathfrak{k}$-modules,

$$V_k = \wedge^k V_1 = \wedge^k (W_1 \oplus \mathbb{C}) \cong (\wedge^k W_1) \oplus (\wedge^{k-1} W_1) = W_k \oplus W_{k-1},$$

where the first equality is by [11], Theorem 19.2; W_0 is interpreted as the one dimensional trivial module and the last equality is from the proof of Case II.

Since $W_{n-1} \subset V_{n-1}$ as $\mathfrak{k}$-modules, and both being spin representations have the same dimension 2^{n-1} (see, e.g., [12], Section 6.2.2), we get $V_{n-1} = W_{n-1}$. Therefore,

$$\phi([V_k]) = [W_k] + [W_{k-1}] \text{ for } 1 \leq k \leq n-2, \text{ and } \phi([V_{n-1}]) = [W_{n-1}].$$

In particular, each of $[W_1], \ldots, [W_{n-1}]$ lies in the image of ϕ, proving the surjectivity of ϕ in this case.

Case IV. (D_4, G_2)**:**

The two fundamental representations W_1 and W_2 have respective dimensions 7 and 14 ([11], Section 22.3). On the other hand, V_1 is eight dimensional and $V_2 = \wedge^2 V_1$. Since $\rho(\varpi_1) = \nu_1$ (by Lemma 5.0.2), by Lemma 5.0.1 we get $W_1 \subset V_1$ (as $\mathfrak{k}$-modules). So, we have the decomposition (as $\mathfrak{k}$-modules):

$$V_1 = W_1 \oplus \mathbb{C}.$$

Thus, as $\mathfrak{k}$-modules,

$$V_2 = \wedge^2 V_1 = \wedge^2 (W_1 \oplus \mathbb{C}) \cong \big(\wedge^2 W_1\big) \oplus W_1.$$

But, $\wedge^2 W_1 \cong W_2 \oplus W_1$ ([11], Section 22.3). Hence, as $\mathfrak{k}$-modules,

$$V_2 = W_2 \oplus W_1^{\oplus 2}.$$

This gives

$$\phi([V_1]) = [W_1] + 1 \text{ and } \phi([V_2]) = [W_2] + 2[W_1],$$

which proves the surjectivity of ϕ in this case.

Case V. (E_6, F_4):

By Lemma 5.0.2(c), we see that ρ is surjective with kernel given by $\{a\varpi_1 + b\varpi_3 - b\varpi_5 - a\varpi_6 \mid a, b \in \mathbb{Z}\}$. Considering the images of ϖ_i under ρ, we have as $\mathfrak{k}$-modules (by Lemmas 5.0.2(c) and 5.0.1),

$$\begin{aligned} &W_1 \subset V_2, \\ &W_2 \subset V_4, \\ &W_3 \subset V_3, V_5, \\ &W_4 \subset V_1, V_6. \end{aligned}$$

Using [27], Tables 44 and 47 or [20], we obtain

$$\begin{aligned} &\dim(W_1) = 52, && \dim(V_2) = 78, \\ &\dim(W_2) = 1274, && \dim(V_4) = 2925, \\ &\dim(W_3) = 273, && \dim(V_3) = \dim(V_5) = 351, \\ &\dim(W_4) = 26, && \dim(V_1) = \dim(V_6) = 27. \end{aligned}$$

Along with the fundamental $\mathfrak{k}$-modules, there are only three other irreducible $\mathfrak{k}$-modules of dimensions at most 1651 ([27], Table 44, or [20]). These are $\dim(W(2\nu_4)) = 324$, $\dim(W(\nu_1 + \nu_4)) = 1053$, and $\dim(W(2\nu_1)) = 1053$.

Let U^k denote an arbitrary $\mathfrak{k}$-module of dimension k. Considering the dimensions, we get (as

$\mathfrak{k}$-modules):

$$\begin{aligned}
V_1 &= V_6 = W_4 \oplus \mathbb{C}, \\
V_2 &= W_1 \oplus U^{26}, \\
V_3 &= V_5 = W_3 \oplus U^{78}, \\
V_4 &= W_2 \oplus U^{1651}.
\end{aligned}$$

Now, U^{26} must be either W_4 or the trivial module $\mathbb{C}^{26}$, and U^{78} must be some combination of W_4, W_1 and $\mathbb{C}$. Since $\phi([V_1]) - 1 = [W_4]$, this implies that $[W_4]$, $[W_1]$ and $[W_3]$ are in the image of ϕ. (We remark that [27] gives $F_4 \subset E_6$ branching, but we continue without these results for clarity and completeness.)

Using appropriate tensor product decompositions in [20], we get

$$[W(2\nu_4)] = [W_4]^2 - [W_3] - [W_1] - [W_4] - 1, \tag{5.3}$$

$$[W(\nu_1 + \nu_4)] = [W_1][W_4] - [W_3] - [W_4], \tag{5.4}$$

$$[W(2\nu_1)] = [W_1]^2 - [W_2] - [W(2\nu_4)] - [W_1] - 1. \tag{5.5}$$

Since W_2 appears in V_4 as a $\mathfrak{k}$-submodule exactly once by Lemma 5.0.1, from the above identities, we get that $[W_2]$ lies in the image of ϕ if $W(2\nu_1)$ is not a component of V_4. In fact, we prove below that $2\nu_1$ is not a $\mathfrak{k}$-weight of V_4 at all.

In order that $2\nu_1$ be a $\mathfrak{k}$-weight of V_4, we should have $2\nu_1 = \mu|_{\mathfrak{t}_{\mathfrak{k}}}$, where μ is a weight of V_4. This is only possible if there exists a weight of V_4 of the form $\mu = a\varpi_1 + 2\varpi_2 + b\varpi_3 - b\varpi_5 - a\varpi_6$, for some $a, b \in \mathbb{Z}$. We claim this is impossible. Indeed, all weights of V_4 are of the form $\varpi_4 - \sum_{i=1}^{6} d_i\alpha_i$, where $d_i \in \mathbb{Z}^+$. If such μ existed, then by [3], Planche V,

$$\begin{aligned}
&\textstyle\sum_{i=1}^{6} d_i\alpha_i = \varpi_4 - \mu \\
&= \varpi_4 + a(\varpi_6 - \varpi_1) - 2\varpi_2 + b(\varpi_5 - \varpi_3) \\
&= (2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6) + (a/3)(-2\alpha_1 - \alpha_3 + \alpha_5 + 2\alpha_6) \\
&-2(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6) + (b/3)(-\alpha_1 - 2\alpha_3 + 2\alpha_5 + \alpha_6),
\end{aligned}$$

from which we immediately see a contradiction since the α_2 coefficient is -1.

This completes the proof in this last case and hence the proof of the first part of Theorem 2.2.2 is completed.

To prove that $\eta : K//\mathrm{Ad}\ K \to G//\mathrm{Ad}\ G$ is a closed embedding, it suffices to show that the induced map between the affine coordinate rings $\eta^* : \mathbb{C}[G//\mathrm{Ad}\ G] \to \mathbb{C}[K//\mathrm{Ad}\ K]$ is surjective. But, by [25], Theorem 3.5, there is a functorial isomorphism

$$\mathbb{C} \otimes_{\mathbb{Z}} R(G) \to \mathbb{C}[G//\mathrm{Ad}\ G],$$

and similarly we have an isomorphism

$$\mathbb{C} \otimes_{\mathbb{Z}} R(K) \to \mathbb{C}[K//\mathrm{Ad}\ K].$$

From this the surjectivity of η^* follows from the surjectivity of $R(G) \to R(K)$. This proves the theorem.

$\square$

CHAPTER 6: REDUCTION OF HITCHIN'S CONJECTURE

In this section, building upon Theorem 5.0.2, we give the proof for the reduction of the validity of Hitchin's conjecture to the simply-laced, simple Lie algebras. Recall the notation and defintion of chapters 4 and 5. Fix a principal TDS. Then, N. Hitchin made the following conjecture.

Conjecture 6.0.1. *Let $\mathfrak{g}$ be any simple Lie algebra. For any primitive element $\omega \in P_d \subset \wedge^d(\mathfrak{g}^*)^{\mathfrak{g}}$, there exists an irreducible sub-module $V_\omega \subset \mathfrak{g}$ of dimension d with respect to the principal TDS action such that*

$$\omega|_{\wedge^d(V_\omega)} \neq 0.$$

Remark 6.0.1. (a) When $\mathfrak{g}$ is not of type D_ℓ (with ℓ even), given $\omega \in P_d$, there exists a unique irreducible submodule V of dimension d in $\mathfrak{g}$. This can be seen by Proposition 4.2.1 and by consulting a table of exponents of $\mathfrak{g}$. Thus, V_ω is uniquely determined.

If $\mathfrak{g}$ is of type D_ℓ (with ℓ even), unless $d = 2\ell - 1$, given $\omega \in P_d$, there is a unique irreducible submodule V of dimension d in $\mathfrak{g}$. Thus, again V_ω is uniquely determined (for $d \neq 2\ell - 1$).

(b) A different choice of principal TDS results in the irreducible submodules being equal to $\mathrm{Ad}g \cdot V$, for some $g \in G$, and some irreducible submodule V for the original principal TDS (cf. Proposition 4.2.1). But, since we are only considering forms $\omega \in \wedge^d(\mathfrak{g}^*)^{\mathfrak{g}}$ (which are by definition $\mathrm{Ad}\, G$-invariant), $\omega|_{\wedge^d(\mathrm{Ad}g \cdot V)} \neq 0$ if and only if $\omega|_{\wedge^d(V)} \neq 0$.

Now, we come to the main results of part two of this dissertation (cf. [4]).

Theorem 6.0.1. *If Hitchin's conjecture is valid for any simply-laced simple Lie algebra $\mathfrak{g}$, then it is valid for any simple Lie algebra.*

More precisely, if Hitchin's conjecture is valid for $\mathfrak{g}$ of type $(A_{2\ell-1}; A_{2\ell}; D_4; E_6)$, then it is valid for $\mathfrak{g}$ of type $(C_\ell; B_\ell; G_2; F_4)$ respectively.

Proof: Let $\mathfrak{k}$ be a non simply-laced simple Lie algebra. Then, there exists a simply-laced simple Lie algebra $\mathfrak{g}$ together with a diagram automorphism σ (i.e., an automorphism σ of $\mathfrak{g}$ induced from a diagram automorphism of its Dynkin diagram) such that $\mathfrak{k}$ is the σ-fixed point $\mathfrak{g}^\sigma$ of $\mathfrak{g}$. Moreover, given $\mathfrak{k}$, we can choose $\mathfrak{g}$ to be of type given in the statement of the theorem (cf. §4.1). In particular, we never need to take $\mathfrak{g}$ of type D_ℓ except D_4.

Choose a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{b}$ such that they both are stable under σ. let $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset \mathfrak{t}^*$ be the set of simple roots of $\mathfrak{g}$, where ℓ is the rank of $\mathfrak{g}$. Since σ keeps $\mathfrak{b}$ and $\mathfrak{t}$ stable, σ permutes the simple roots. Let $\{\widetilde{\beta}_1, \dots, \widetilde{\beta}_{\ell_\mathfrak{k}}\}$ be a set of simple roots taken exactly one simple root from each orbit of σ in Δ. Then, the fixed subalgebra $\mathfrak{b}_\mathfrak{k} := \mathfrak{b}^\sigma$ is a Borel subalgebra of $\mathfrak{k}$, $\mathfrak{t}_\mathfrak{k} := \mathfrak{t}^\sigma$ is a Cartan subalgebra of $\mathfrak{k}$ and $\{\beta_1, \dots, \beta_{\ell_\mathfrak{k}}\}$ is the set of simple roots of $\mathfrak{k}$, where $\beta_i := \widetilde{\beta}_i|_{\mathfrak{t}_\mathfrak{k}}$ (cf. [28]). In particular, $\ell_\mathfrak{k}$ is the rank of $\mathfrak{k}$.

For any $1 \leq n \leq \ell_\mathfrak{k}$, choose a nonzero element $x_n \in \mathfrak{g}_{\widetilde{\beta}_n}$, where $\mathfrak{g}_{\widetilde{\beta}_n}$ is the root space of $\mathfrak{g}$ corresponding to the root $\widetilde{\beta}_n$. Define

$$y_n = \sum_{i=1}^{\operatorname{ord}(\sigma)} \sigma^i(x_n),$$

where $\operatorname{ord}(\sigma)$ is the order of σ (which is 2 except when $\mathfrak{g}$ is of type D_4 and $\mathfrak{k}$ is of type G_2, in which case it is 3). If $\widetilde{\beta}_n$ is fixed by σ, then σ acts trivially on $\mathfrak{g}_{\widetilde{\beta}_n}$ (cf. [28]), hence y_n is never zero. Of course, $y_n \in \mathfrak{k}$ and, in fact, $y_n \in \mathfrak{k}_{\beta_n}$. Define the element $y \in \mathfrak{k}$ by

$$y = \sum_{n=1}^{\ell_\mathfrak{k}} y_n.$$

By [16], Theorem 5.3, y is a principal nilpotent element of $\mathfrak{k}$ and hence there exists a principal TDS in $\mathfrak{k}$:

$$\varphi : \mathfrak{sl}_2 \to \mathfrak{k} \quad \text{such that} \quad \varphi(X) = y.$$

Moreover, since

$$y = \sum_{n=1}^{\ell_\mathfrak{k}} \sum_{i=1}^{\operatorname{ord}(\sigma)} \sigma^i(x_n),$$

again using [16], Theorem 5.3, we get that y is a principal nilpotent of $\mathfrak{g}$ as well. Hence, φ is a principal TDS of $\mathfrak{g}$ also. Decompose $\mathfrak{g}$ under the adjoint action of $\mathfrak{sl}_2$ via φ:

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_{\ell_\mathfrak{k}} \oplus V_{\ell_\mathfrak{k}+1} \oplus \cdots \oplus V_\ell,$$

where $V_1 \oplus \cdots \oplus V_{\ell_{\mathfrak{k}}}$ is a decomposition of $\mathfrak{k}$.

Take a primitive element $\omega_d \in P_d(\mathfrak{k}) \subset \wedge^d(\mathfrak{k}^*)^{\mathfrak{k}}$, where $P_d(\mathfrak{k})$ is the space of primitive elements for $\mathfrak{k}$. By (subsequent) Theorem 6.0.2 , the canonical restriction map $\wedge^d(\mathfrak{g}^*) \to \wedge^d(\mathfrak{k}^*)$ induces a surjection

$$P_d(\mathfrak{g}) \to P_d(\mathfrak{k}), \quad \text{for any } d > 0.$$

Take a preimage $\widetilde{\omega}_d \in P_d(\mathfrak{g})$ of ω_d. By remarks 6.0.1, there exists a unique irreducible $\mathfrak{sl}_2$-submodule V_{ω_d} of $\mathfrak{k}$ of dimension d. Further, there exists a unique irreducible $\mathfrak{sl}_2$-submodule $V_{\widetilde{\omega}_d} \subset \mathfrak{g}$ of dimension d. (For any $\mathfrak{k}$ not of type G_2, the uniqueness of $V_{\widetilde{\omega}_d}$ follows since we have chosen $\mathfrak{g}$ not of type D_ℓ; for $\mathfrak{k}$ of type G_2, $P_d(\mathfrak{k})$ is nonzero if and only if $d = 3, 11$ (cf. equations (4.5)- (4.7). Again, for these values of d, $\dim P_d(D_4) = 1$.) Hence, $V_{\omega_d} = V_{\widetilde{\omega}_d}$. Assuming the validity of Hitchin's conjecture for $\mathfrak{g}$, we get that $\widetilde{\omega}_d|_{\wedge^d(V_{\widetilde{\omega}_d})} \neq 0$. Hence,

$$\omega_d|_{\wedge^d(V_{\omega_d})} = \widetilde{\omega}_d|_{\wedge^d(V_{\widetilde{\omega}_d})} \neq 0.$$

This proves the theorem. ■

We give the Lie algebra analogue of Theorem 5.0.2 as a corollary.

Corollary 6.0.1. *The canonical restriction map*

$$S(\mathfrak{g}^*)^{\mathfrak{g}} \to S(\mathfrak{k}^*)^{\mathfrak{k}}$$

is surjective.

Proof: By [30], section 6.4, for any connected semisimple algebraic group H over $\mathbb{C}$, the restriction map

$$r : \mathbb{C}[H//\mathrm{Ad}\ H] \cong \mathbb{C}[H]^H \to \mathbb{C}[T_H]^{W_H} \tag{6.1}$$

is an isomorphism of $\mathbb{C}$-algebras, where $T_H \subset H$ is a maximal torus and W_H is the Weyl group of H.

Similarly, by Chevalley's restriction Theorem, the restriction map

$$r_o : \mathbb{C}[\mathfrak{h}]^H \to \mathbb{C}[\mathfrak{t}_{\mathfrak{h}}]^{W_H} \tag{6.2}$$

is a graded algebra isomorphism, where $\mathfrak{h}$ (resp. $\mathfrak{t}_{\mathfrak{h}}$) is the Lie algebra of H (resp. T_H). Thus, to prove the corollary, it suffices to show that the canonical restriction map

$$\beta_o^* : \mathbb{C}[\mathfrak{t}]^W \to \mathbb{C}[\mathfrak{t}_{\mathfrak{k}}]^{W_K}$$

is surjective,where W (resp. W_K) is the Weyl group of G (resp. K). Since β_o^* is a graded algebra homomorphism induced from the $\mathbb{C}^*$-equivariant map $\beta_o : \mathfrak{t}_{\mathfrak{k}}/W_K \to \mathfrak{t}/W$ (where the $\mathbb{C}^*$-action is the standard homothety action), it suffices to show that the tangent map between the Zariski tangent spaces at 0:

$$(d\beta_o)_0 : T_0(\mathfrak{t}_{\mathfrak{k}}/W_K) \to T_0(\mathfrak{t}/W)$$

is injective. Also, to see that this injectivity is sufficient, note that $\mathfrak{t}/W$ and $\mathfrak{t}_{\mathfrak{k}}/W_K$ are affine spaces by another theorem of Chevalley (cf. [6]). Let T^{anal} denote the analytic tangent space. Then, the canonical map

$$T_x^{anal}(X) \to T_x(X)$$

is an isomorphism for any algebraic variety X and any point $x \in X$.

Consider the commutative diagram:

$$\begin{tikzcd}
\mathfrak{t}_{\mathfrak{k}}/W_K \arrow[r, "\beta_o"] \arrow[d, "\exp"] & \mathfrak{t}/W \arrow[d, "\exp"] \\
T_K/W_K \arrow[r, "\beta"'] & T/W,
\end{tikzcd}$$

where $T_K \subset K$ is the maximal torus with Lie algebra $\mathfrak{t}_{\mathfrak{k}}$ and $\beta : T_K/W_K \to T/W$ is the canonical map. Since T_K, T are tori, exp is a local isomorphism in the analytic category. In particular, there exist open subsets (in the analytic topology) $0 \in U_{\mathfrak{k}} \subset \mathfrak{t}_{\mathfrak{k}}/W_K, 0 \in U \subset \mathfrak{t}/W, 1 \in V_K \subset T_K/W_K$ and $1 \in V \subset T/W$ such that $\beta_o(U_K) \subset U$ and $\exp_{|U_{\mathfrak{k}}} : U_{\mathfrak{k}} \to V_K$ is an analytic isomorphism and so is $\exp_{|U} : U \to V$. Since, by Theorem 5.0.2 and the isomorphism (6.1), β is a closed embedding,

$$(d\beta)_1 : T_1^{anal}(T_K/W_K) \cong T_1(T_K/W_K) \to T_1^{anal}(T/W) \simeq T_1(T/W)$$

is injective and hence so is $T_0(\mathfrak{t}_{\mathfrak{k}}/W_K) \to T_0(\mathfrak{t}/W)$. This proves the corollary. ■

As a consequence of Corollary 6.0.1, we get the following.

Theorem 6.0.2. *With the notation and assumptions as in Theorem 5.0.2, the canonical restriction map $\gamma : H^*(G) \to H^*(K)$ is surjective. Moreover, this induces a surjective (graded) map*

$$\gamma_o : P(\mathfrak{g}) \to P(\mathfrak{k}),$$

where $P(\mathfrak{g}) \subset H^*(G)$ is the subspace of primitive elements.

Proof: From the definition of coproduct, it is easy to see that the following diagram is commutative:

$$\begin{tikzcd} H^*(G) \arrow[r, "\Delta_G"] \arrow[d, "\gamma"] & H^*(G) \otimes H^*(G) \arrow[d, "\gamma\otimes\gamma"'] \\ H^*(K) \arrow[r, "\Delta_K"'] & H^*(K) \otimes H^*(K). \end{tikzcd}$$

Thus, γ takes $P(\mathfrak{g})$ to $P(\mathfrak{k})$.

Let $\mathfrak{h}$ be a reductive Lie algebra. For any $v \in \mathfrak{h}$, define the derivation $i(v) : S(\mathfrak{h}^*) \to S(\mathfrak{h}^*)$ given by $i(v)(f) = f(v)$, for $f \in \mathfrak{h}^*$. Further, define an algebra homomorphism $\lambda : S(\mathfrak{h}^*) \to \wedge^{\text{even}}(\mathfrak{h}^*)$ by $\lambda(f) = df$, for $f \in \mathfrak{h}^* = S^1(\mathfrak{h}^*)$, where $d : \wedge^1(\mathfrak{h}^*) = \mathfrak{h}^* \to \wedge^2(\mathfrak{h}^*)$ is the standard differential in the Lie algebra cochain complex $\wedge^\bullet(\mathfrak{h}^*)$. Now, define the transgression map (cf. §4.4).

$$\tau = \tau_{\mathfrak{h}} : S^+(\mathfrak{h}^*)^{\mathfrak{h}} \to \wedge^+(\mathfrak{h}^*)^{\mathfrak{h}}, \;\; \tau(p) = \sum_j e_j^* \wedge \lambda(i(e_j)p),$$

for $p \in S^+(\mathfrak{h}^*)^{\mathfrak{h}}$, where $\{e_j\}$ is a basis of $\mathfrak{h}$ and $\{e_j^*\}$ is the dual basis of $\mathfrak{h}^*$.

By a result of Cartan (cf. [5], Théorème 2 and [21]), τ factors through

$$S^+(\mathfrak{h}^*)^{\mathfrak{h}}/(S^+(\mathfrak{h}^*)^{\mathfrak{h}}) \cdot (S^+(\mathfrak{h}^*)^{\mathfrak{h}})$$

to give an injective map

$$\bar{\tau} : S^+(\mathfrak{h}^*)^{\mathfrak{h}}/(S^+(\mathfrak{h}^*)^{\mathfrak{h}}) \cdot (S^+(\mathfrak{h}^*)^{\mathfrak{h}}) \to \wedge^+(\mathfrak{h}^*)^{\mathfrak{h}}$$

with image precisely equal to the space of primitive elements $P(\mathfrak{h})$. From the definition of τ, it is easy to see that the following diagram is commutative:

$$\begin{tikzcd} S^+(\mathfrak{g}^*)^{\mathfrak{g}} \arrow[r, "\tau_{\mathfrak{g}}"] \arrow[d] & \wedge^+(\mathfrak{g}^*)^{\mathfrak{g}} \arrow[d] \\ S^+(\mathfrak{k}^*)^{\mathfrak{k}} \arrow[r, "\tau_{\mathfrak{k}}"] & \wedge^+(\mathfrak{k}^*)^{\mathfrak{k}}, \end{tikzcd}$$

where the vertical maps are the canonical restriction maps. By using Corollary 6.0.1, this proves that $P(\mathfrak{g})$ surjects onto $P(\mathfrak{k})$. Since $P(\mathfrak{k})$ generates $\wedge^*(\mathfrak{k}^*)^{\mathfrak{k}} \cong H^*(K)$ as an algebra, we get that γ is surjective. This proves the theorem. ■

APPENDIX A: TOWARD SEMISTABILITY AND STABILITY

In this appendix, we provide some additional characterizations of when points are semistable or stable. Up to this point, none of these are particularly useful in explicitly determining the semistable locus, but our hope is that useful criterion can be developed from these results in the future. In section one, we take a direct approach. In section two, we restrict to $G = SL(n)$ and identify $(G/B)^3$ with a subvariety of products of grassmanians to obtain an alternative characterization of semistability and stability.

A.1 Direct Approach

Lemma A.1.1. *$(B, g_2B, g_3B) \in (G/B)^3$ is semistable with respect to $\mathcal{L}(\lambda) \boxtimes \mathcal{L}(\mu) \boxtimes \mathcal{L}(\nu)$ if and only if $v^+_{N\lambda} \otimes g_2 v^+_{N\mu} \otimes g_3 v^+_{N\nu}$ has a non-zero G-invariant component.*

Proof: Assume (B, g_2B, g_3B) is semistable. By definition (B, g_2B, g_3B) is semistable if and only if there is

$$\sigma \in H^0((G/B)^3, \mathcal{L}(N\lambda) \boxtimes \mathcal{L}(N\mu) \boxtimes \mathcal{L}(N\nu))^G$$

$$\cong \left[H^0(G/B, \mathcal{L}(N\lambda)) \otimes H^0(G/B, \mathcal{L}(N\mu)) \otimes H^0(G/B, \mathcal{L}(N\nu))\right]^G$$

such that $\sigma(B, g_2B, g_3B) \neq 0$ for some $N > 0$. The Borel-Weil isomorphism is G-equivariant, so

$$\left[V(N\lambda)^* \otimes V(N\mu)^* \otimes V(N\nu)^*\right]^G \cong \left[H^0(G/B, \mathcal{L}(N\lambda)) \otimes H^0(G/B, \mathcal{L}(N\mu)) \otimes H^0(G/B, \mathcal{L}(N\nu))\right]^G$$

under the map with diagonal action on all tensor products.

If $v^+_{N\lambda} \otimes g_2 v^+_{N\mu} \otimes g_3 v^+_{N\nu}$ has a nonzero G-invariant component, then there must be some G-invariant element in

$$\sum \alpha_i \otimes \beta_i \otimes \gamma_i \in \left[V(N\lambda)^* \otimes V(N\mu)^* \otimes V(N\nu)^*\right]^G.$$

The previous statement follows since if α is not a G-invariant dual element, then for some g, $g\alpha$ lives in some weight space $\lambda \neq 1$. Choose some $t \in T$ such that $\lambda(t) \neq 1$ and consider

$$\alpha(v) = \alpha(g^{-1}t^{-1}v) = \lambda(t)(g\alpha)(v) = \lambda(t)\alpha(g^{-1}v) = \lambda(t)\alpha(v).$$

This shows that every non-G-invariant dual element vanishes on G-invariants. On the other hand, there must be some dual element that is non-vanishing on every G-invariant.

Define regular functions $\widetilde{\alpha_i}, \widetilde{\beta_i}, \widetilde{\gamma_i} : G \to \mathbb{C}$ by $\widetilde{\alpha_i}(g) = \alpha_i(gv^+_{N\lambda})$, and similarly for others. Observe that $\widetilde{\alpha_i}(gb) = \alpha_i(gbv^+_{N\lambda}) = e^{N\lambda}(b)\alpha_i(gv^+_{N\lambda}) = e^{N\lambda}(b)\widetilde{\alpha_i}(g)$. So, these functions define elements of $H^0(G/B, \mathcal{L}(N\lambda))$ by $[g, \widetilde{\alpha_i}(g)]$. We define $\sigma \in H^0(G/B, \mathcal{L}(N\lambda)) \otimes H^0(G/B, \mathcal{L}(N\mu)) \otimes H^0(G/B, \mathcal{L}(N\nu))$ by

$$\sigma(h_1B, h_2, B, h_3) = \sum [h_1, \widetilde{\alpha_i}(h_1)] \otimes [h_2, \widetilde{\beta_i}(h_2)] \otimes [h_3, \widetilde{\beta_i}(h_3)].$$

Then, σ is the image of $\sum \alpha_i \otimes \beta_i \otimes \gamma_i$ under the G-equivariant Borel-Weil isomorphism. Hence, σ must also be G-equivariant.

To show that $\sigma(B, g_1B, g_2B) \neq 0$ it suffices to show that σ defines a non-vanishing section σ^* of the pull-back bundle $\pi^*(\mathcal{L}) = G^3 \times C$ over G^3 via the natural quotient. For $(e, g_1, g_2) \in G^3$ lying over (B, g_1B, g_2B)

$$\sigma^*(1, g_1, g_2) = \sum \widetilde{\alpha_i}(1) \otimes \widetilde{\beta_i}(g_2) \otimes \widetilde{\beta_i}(g_3)$$

$$= \sum \alpha_i \otimes \beta_i \otimes \beta_i(v^+_{N\lambda} \otimes g_1 v^+_{N\mu} \otimes g_2 v^+_{N\nu}) \neq 0$$

Conversely, given a G-invariant section σ nonvanishing at (B, g_1B, g_2B), there is some $\sum_i(\alpha_i \otimes \beta_i \otimes \gamma_i) \in \left[V(N\lambda)^* \otimes V(N\mu)^* \otimes V(N\nu)^*\right]^G$ corresponding to σ under the Borel-Weil isomorphism. Hence

$$0 \neq \sum \alpha_i \otimes \beta_i \otimes \beta_i(v^+_{N\lambda} \otimes g_1 v^+_{N\mu} \otimes g_2 v^+_{N\nu})$$

Clearly, $v^+_{N\lambda} \otimes g_1 v^+_{N\mu} \otimes g_2 v^+_{N\nu}$ must have a nonzero G-invariant component.

■

A.2 Embedding in Grassmanians for $G = SL(n)$

We now restrict to the case when $G = SL(n)$. Let $E_i = \mathbb{C}\langle e_1, \ldots, e_i\rangle$ be the standard i-dimensional subspace of $\mathbb{C}^n$, where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{C}^n$. G acts naturally on E_i, consider the following G-equivariant embedding. $G/B \hookrightarrow \prod_{k=1}^{n-1} \mathrm{Gr}_k(\mathbb{C}^n)$, given by

$$gB \mapsto (gE_1, gE_2, \ldots, gE_{n-1}).$$

Injectivity is apparent since for any g_1, g_2 with the same image, $g_1^{-1}g_2$ will fix the flag $E_1 \subset E_2 \subset \cdots \subset E_{n-1}$, and hence will live in B.

This induces the G-equivariant embedding $j : (G/B)^3 \hookrightarrow \prod_{i=1}^{3} \prod_{k_i=1}^{n-1} \mathrm{Gr}_{k_i}(\mathbb{C}^n)$. Now, for any k we have the Plücker embedding $\mathrm{Gr}_k(\mathbb{C}^n) \hookrightarrow \mathbb{P}(\wedge^k \mathbb{C}^n)$ given by

$$\mathrm{span}(v_1, \ldots, v_k) \mapsto [v_1 \wedge \cdots \wedge v_k].$$

For some $a \in \mathbb{Z}_{>0}$ we also have the Veronese embedding $\mathbb{P}(\wedge^k \mathbb{C}^n) \hookrightarrow \mathbb{P}(\mathrm{Sym}^a(\wedge^k \mathbb{C}^n))$ given by

$$[v_1 \wedge \cdots \wedge v_k] \mapsto [(v_1 \wedge \cdots \wedge v_k)^a],$$

where the latter denotes the a-th symmetric power. Given any $3(n-1)$-tuple of $a_i^{k_i} \in \mathbb{Z}_{>0}$, by composing products of these two embeddings we get the following embedding

$$(G/B)^3 \hookrightarrow \prod_{i=1}^{3} \prod_{k_i=1}^{n-1} \mathbb{P}(\mathrm{Sym}^{a_i^{k_i}}(\wedge^{k_i} \mathbb{C}^n)),$$

given by in the (i, k_i)-th image coordinate

$$(g_1B, g_2B, g_3B) \mapsto [(g_ie_1 \wedge g_ie_2 \wedge \cdots \wedge g_ie_{k_i})^{a_i^{k_i}}].$$

If we follow this with a Segre embedding, we get

$$\phi : (G/B)^3 \hookrightarrow \mathbb{P}\big[\otimes_{i=1}^{3} \otimes_{k_i=1}^{n-1} \mathrm{Sym}^{a_i^{k_i}}(\wedge^{k_i} \mathbb{C}^n)\big],$$

given by

$$(g_1B, g_2B, g_3B) \mapsto$$

$$[(g_1e_1)^{a_1^1} \otimes \cdots \otimes (g_ie_1 \wedge g_ie_2 \wedge \cdots \wedge g_ie_{k_i})^{a_i^{k_i}} \otimes \cdots \otimes (g_3e_1 \wedge g_3e_2 \wedge \cdots \wedge g_3e_{n-1})^{a_3^{n-1}}].$$

Now, on the other hand, given three regular, dominant weights, we have the standard embedding

$$\psi : (G/B)^3 \hookrightarrow \mathbb{P}\big[V(\chi_1) \otimes V(\chi_2) \otimes V(\chi_3)\big]$$

given by $(g_1B, g_2B, g_3B) \mapsto [g_1v^+_{\chi_1} \otimes g_2v^+_{\chi_2} \otimes g_3v^+_{\chi_3}]$.

Now, if $\chi_1 = \sum_{k_1=1}^{n-1} a_1^{k_1}\varpi_{k_1}$, $\chi_2 = \sum_{k_2=1}^{n-1} a_2^{k_2}\varpi_{k_2}$, and $\chi_3 = \sum_{k_3=1}^{n-1} a_3^{k_3}\varpi_{k_3}$, then

$$V(\chi_1) \subset \otimes_{k_1=1}^{n-1}\mathrm{Sym}^{a_1k_1}(\wedge^{k_1}\mathbb{C}^n)$$

with highest weight vector

$$v^+_{\chi_1} = e_1^{a_1^1} \otimes (e_1 \wedge e_2)^{a_1^2} \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{n-1})^{a_1^{n-1}}$$

and similarly for $V(\chi_2)$ and $V(\chi_3)$. Thus, as G-modules we have

$$V(\chi_1) \otimes V(\chi_2) \otimes V(\chi_3) \subset \otimes_{i=1}^{3} \otimes_{k_i=1}^{n-1} \mathrm{Sym}^{a_i^{k_i}}(\wedge^{k_i}\mathbb{C}^n),$$

which gives the G-equivariant embedding

$$i : \mathbb{P}\big[V(\chi_1) \otimes V(\chi_2) \otimes V(\chi_3)\big] \subset \mathbb{P}\big[\otimes_{i=1}^{3} \otimes_{k_i=1}^{n-1}\mathrm{Sym}^{a_i^{k_i}}(\wedge^{k_i}\mathbb{C}^n)\big].$$

Then, the following proposition is immediate.

Proposition A.2.1. *With the notation as above, $i \circ \psi = \phi$.*

Let $V = \otimes_{i=1}^{3} \otimes_{k_i=1}^{n-1} \mathrm{Sym}^{a_i^{k_i}}(\wedge^{k_i}\mathbb{C}^n)$ and $W = V(\chi_1) \otimes V(\chi_2) \otimes V(\chi_3)$.

Lemma A.2.1. $\phi^*\mathcal{O}_{\mathbb{P}(V)}(1) = \mathcal{L}(\chi_1, \chi_2, \chi_3)$.

Proof:

First, Since, the tautological bundle on $\mathbb{P}(V)$ pulls back to the tautological bundle on $\mathbb{P}(W)$, we see that $i^*\mathcal{O}_{\mathbb{P}(V)}(1) = \mathcal{O}_{\mathbb{P}(W)}(1)$. Hence, it suffices to observe that $\psi^*(\mathcal{O}_{\mathbb{P}(W)}(1)) = \mathcal{L}(\chi_1, \chi_2, \chi_3)$ and this is well known. ■

The following proposition is a generalization of [23], §4.4, and [32].

Proposition A.2.2. *Consider the same notation as above. Then,*

$$(L_1^1, \ldots, L_1^{n-1}, L_2^1, \ldots, L_2^{n-1}, L_3^1, \ldots, L_3^{n-1}) \in \prod_{i=1}^{3}\prod_{k_i=1}^{n-1} Gr_{k_i}(\mathbb{C}^n)$$

is stable, or semistable, with respect to the embedding determined by χ_1, χ_2, χ_3 *as above if and only if*

$$\begin{aligned} &\textstyle\sum_{j=1}^{n-1} a_1^j\big(\dim(F)j - n\dim(L_1^j \cap F)\big) + a_2^j\big(\dim(F)j - n\dim(L_2^j \cap F)\big) \\ &+a_3^j\big(\dim(F)j - n\dim(L_3^j \cap F)\big) > 0, \end{aligned} \tag{A.1}$$

for every non-zero, proper subspace F *of* $\mathbb{C}^n$*, with* ≥ 0 *for the statement on semistability.*

Proof:

We will proceed by proving this formula for the embedding of the product of Grassmanians. In particular, we consider $\mathrm{Gr}_k(\mathbb{C}^n) \hookrightarrow \mathbb{P}(\mathrm{Sym}^a(\wedge^k\mathbb{C}^n))$. On basis vectors, this embedding looks like $\mathrm{span}(e_{i_1}, \ldots, e_{i_k}) \mapsto [(e_{i_1} \wedge \cdots \wedge e_{i_k})^a] = [p^a_{i_1,\ldots,i_k}]$, where $p_{i_1,\ldots,i_k} = e_{i_1} \wedge \cdots \wedge e_{i_k}$. Let $\mathcal{O}_k(1)$ on $\mathrm{Gr}_k(\mathbb{C}^n)$ be the pull-back of $\mathcal{O}_{\mathbb{P}(V)}(1)$ via this embedding. Then, $\mathcal{O}_k(1)$ is a G-linearized line bundle over $\mathrm{Gr}_k(\mathbb{C}^n)$.

Now, consider $\sigma(t) = \mathrm{diag}(t^{r_1}, \ldots, t^{r_n})$ and observe the diagonalized action of $\sigma(t)$ on the natural basis for $\mathrm{Sym}^a(\wedge^k(\mathbb{C}^n))$

$$\sigma(t) \cdot (p_{i_1^1,\ldots,i_k^1} p_{i_1^2,\ldots,i_k^2} \cdots p_{i_1^a,\ldots,i_k^a}) = t^{\sum_{j=1}^a (r_{i_1^j} + \cdots + r_{i_k^j})} (p_{i_1^1,\ldots,i_k^1} p_{i_1^2,\ldots,i_k^2} \cdots p_{i_1^a,\ldots,i_k^a}),$$

where we take $i_1^j < \cdots < i_k^j$.

For any $L \in \mathrm{Gr}_k(\mathbb{C}^n)$, take $p_{i_1^1,\ldots,i_k^1}(L)$ to be the coefficient of $p_{i_1^1,\ldots,i_k^1}$ for the image of L under the Plücker embedding. Then using Proposition 2.3 of [23],

$$\mu^{\mathcal{O}_k(1)}(L, \sigma) = \max\{-\sum_{j=1}^{a} (r_{i_1^j} + \cdots r_{i_k^j}) \mid p_{i_1^1,\ldots,i_k^1}(L) p_{i_1^2,\ldots,i_k^2}(L) \cdots p_{i_1^a,\ldots,i_k^a}(L) \neq 0\}. \tag{A.2}$$

Let $L_{(n+1)} = \mathbb{C}^n$, $L_{(n)} = \{x_n = 0\}$, $\ldots$, $L_{(i,i+1,\ldots,n)} = \{x_i = x_{i+1} = \cdots = x_n = 0\}$, $\ldots$, $L_{(1,\ldots,n)} = \{0\}$. For each $L \in \mathrm{Gr}_k(\mathbb{C}^n)$, there exist unique integers $\alpha_1 < \cdots < \alpha_k$ such that

$$\dim(L \cap L_{(\alpha_i+1,\ldots,n)}) = i, \text{ and } \dim(L \cap L_{(\alpha_i,\alpha_i+1,\ldots,n)}) = i - 1.$$

The definition of $\{\alpha_1, \ldots, \alpha_k\}$ requires that for each $i = 1, \ldots, k$, there is a point $q_i \in L$ such that

$$q_i \in L_{(\alpha_i+1,\ldots,n)} \setminus L_{(\alpha_i,\ldots,n)}.$$

Since there are k of these, we see that

$$L = \operatorname{span}(q_1, \ldots, q_n).$$

If we assume that $\sigma(t)$ is such that $r_1 > \cdots > r_n$, then $\sigma(t) \cdot \mathbb{C}q_i = \mathbb{C}e_{\alpha_i} + t^\beta \mathbb{C}(\sum_{j<\alpha_i} b_j e_j)$, where $\beta > 0$ and b_j could be zero. Thus

$$\lim_{t\to 0} \sigma(t) \cdot L = \mathbb{C}e_{\alpha_1} + \mathbb{C}e_{\alpha_2} + \cdots + \mathbb{C}e_{\alpha_k} =: L_0.$$

Then, $\mu^{\mathcal{O}_k(1)}(L, \sigma) = \mu^{\mathcal{O}_k(1)}(L_0, \sigma)$, so it suffices to compute the latter. But $p_{i_1,\ldots,i_k}(L_0) \neq 0$ if and only if $i_j = \alpha_j$ for all j. Thus, by equation (A.2)

$$\begin{aligned} \mu^{\mathcal{O}_k(1)}(L,\sigma) &= -a \textstyle\sum_{i=1}^k r_{\alpha_i} \\ &= -a \textstyle\sum_{i=1}^n \big[\dim(L \cap L_{(i+1,\ldots,n)}) - \dim(L \cap L_{(i,i+1,\ldots,n)})\big] r_i \\ &= -a\big[k r_n + \textstyle\sum_{i=2}^n \dim(L \cap L_{(i,i+1,\ldots,n)})(r_{i-1} - r_i)\big] \\ &= -akr_n + a \textstyle\sum_{i=2}^n \dim(L \cap L_{(i,i+1,\ldots,n)})(r_i - r_{i-1}). \end{aligned} \tag{A.3}$$

By, the same justification used in §4.4 of [23], we conclude that this formula must also hold for $\sigma(t)$ of the form $r_1 \geq r_2 \geq \cdots \geq r_n$.

Then, take a point $\vec{L} := (L_1^1, \ldots, L_1^{n-1}, L_2^1, \ldots, L_2^{n-1}, L_3^1, \ldots, L_3^{n-1}) \in \prod_{i=1}^3 \prod_{k_i=1}^{n-1} \mathrm{Gr}_{k_i}(\mathbb{C}^n)$ and χ_1, χ_2, χ_3 as above with associated embedding ϕ (taken on the product of Grassmannians), and let $\underline{\mathcal{O}}(1)$ be the pullback of $\mathcal{O}_{\mathbb{P}(V)}(1)$ under the embedding

$$\prod_{k_i=1}^{n-1} \mathrm{Gr}_{k_i}(\mathbb{C}^n) \hookrightarrow \mathbb{P}(V)$$

determined by χ_1, χ_2, χ_3. Then by equation $(A.3)$ and Proposition 1.2.1 (iii), we have

$$\mu^{\underline{\mathcal{O}}(1)}(\vec{L}, \sigma) = \sum_{s=1}^{3} \sum_{j=1}^{n-1} \Big(- a_s^j j r_n + a_s^j \sum_{i=2}^{n} \dim(L_s^j \cap L_{(i,i+1,\ldots,n)})(r_i - r_{i-1}) \Big) \tag{A.4}$$

Now, the subset of $\mathbb{Z}^n$ with $r_1 \geq \cdots \geq r_n$ and $\sum_{i=1}^n r_i = 0$ is $\mathbb{Q}_{\geq 0}$-spanned by the *extremal* σ_p, $p = 1, \ldots, n-1$, of the form

$$(n-p) = r_1 = \cdots = r_p \geq r_{p+1} = \cdots = r_n = -p.$$

Since the equation $(A.4)$ is a linear function on the r_i's, it follows that $\mu^{\underline{\mathcal{O}}(1)}(\vec{L}, \sigma) > 0$ or ≥ 0 for any sigma of the form $r_1 \geq \cdots \geq r_n$ if and only if the corresponding inequality holds for the all extremal σ_p. So, we calculate equation $(A.4)$ for the extremal σ_p.

$$\mu^{\underline{\mathcal{O}}(1)}(\vec{L}, \sigma_p) = \sum_{s=1}^{3}\sum_{j=1}^{n-1} a_s^j\big(pj - n\dim(L_s^j \cap L_{(p+1,\ldots,n)})\big) \tag{A.5}$$

On the other hand, every $\delta \in \mathrm{OPS}(G)$ is conjugate to $\sigma \in \mathrm{OPS}(T)$ of the form $r_1 \geq \cdots \geq r_n$ and $\mu^{\underline{\mathcal{O}}(1)}(\vec{L}, g\sigma g^{-1}) = \mu^{\underline{\mathcal{O}}(1)}(g^{-1}\cdot\vec{L}, \sigma)$. Moreover, $\dim(g^{-1}L_s^j \cap L_{(i,i+1,\ldots,n)}) = \dim(L_s^j \cap gL_{(i,i+1,\ldots,n)})$. Thus, $\mu^{\underline{\mathcal{O}}(1)}(\vec{L}, \delta) > 0$, or ≥ 0, for all $\delta \in \mathrm{OPS}(G)$ if and only if

$$\sum_{s=1}^{3}\sum_{j=1}^{n-1} a_s^j\big(pj - n\dim(L_s^j \cap gL_{(p+1,\ldots,n)})\big) > 0,$$

or ≥ 0, for every $p = 1, \ldots, n-1$ and every $g \in G$. Noting that any proper, non-zero, subspace of dimension p is obtainable as a G translate of $L_{(p+1,\ldots,n)}$, we have that $\vec{L}$ is stable or semistable if and only if

$$\sum_{s=1}^{3}\sum_{j=1}^{n-1} a_s^j\big(\dim(F)j - n\dim(L_s^j \cap F)\big) > 0,$$

or ≥ 0, for every proper, non-zero, subspace $F \subset \mathbb{C}^n$. ■

Now, if $j : (G/B)^3 \hookrightarrow \prod_{i=1}^3 \prod_{k_i=1}^{n-1} \mathrm{Gr}_{k_i}(\mathbb{C}^n)$ is as described above, by lemma $(A.2.1)$ and recalling the definition of $\underline{\mathcal{O}}(1)$ given in the proof of the previous proposition, $j^*\underline{\mathcal{O}}(1) = \phi^*\mathcal{O}_{\mathbb{P}(V)}(1) = \mathcal{L}(\chi_1, \chi_2, \chi_3)$. Thus, for $x \in (G/B)^3$,

$$\mu^{\mathcal{L}}(x, \delta) = \mu^{\phi^*\mathcal{O}_{\mathbb{P}(V)}(1)}(x, \delta) = \mu^{j^*\underline{\mathcal{O}}(1)}(x, \delta) = \mu^{\underline{\mathcal{O}}(1)}(j(x), \delta).$$

In particular, we have the following corollary.

Corollary A.2.1. *$(B, g_1B, g_2B) \in ((G/B)^3)^s$, or $((G/B)^3)^{ss}$, if and only if*

$$\begin{aligned}&\textstyle\sum_{j=1}^{n-1} a_1^j\big(\dim(F)j - n\dim(E_j \cap F)\big) + a_2^j\big(\dim(F)j - n\dim(g_1E_j \cap F)\big)\\ &+a_3^j\big(\dim(F)j - n\dim(g_2E_j \cap F)\big) > 0,\end{aligned} \tag{A.6}$$

with ≥ 0 for $((G/B)^3)^{ss}$, for every proper, non-zero, subspace $F \subset \mathbb{C}^n$.

If we let

$$\begin{aligned} x_j &= \dim(F)j - n\dim(E_j \cap F)\\ y_j &= \dim(F)j - n\dim(g_1E_j \cap F)\\ z_j &= \dim(F)j - n\dim(g_2E_j \cap F),\end{aligned}$$

then the expression $(A.6)$ is simply

$$\sum_{j=1}^{n-1} a_1^j x_j + a_2^j y_j + a_3^j z_j.$$

In particular, since the set of possible x_j, y_j, z_j is a finite list of integer values, this criterion for stabiliy/semistability says that $((G/B)^3)^{ss} \neq ((G/B)^3)^s$ implies (χ_1, χ_2, χ_3) lies in a finite union of hyperplanes in $((X(T))^{++})^3$.

Example A.2.1. Let's see what this condition is when $G = SL(2)$. Then, $\dim(F) = 1$ and $x_1, y_1, z_1 = 1, -1$, with the former exactly when $g_iE_1 = F$, $i = 0, 1, 2$ (take $g_0 = e$). If there are any semistable points, then these will certainly occur off the diagonal where $g_iE_1 \neq g_jE_1$ for $i \neq j$, so semistable points exist if and only if

$$a_1 + a_2 - a_3 \geq 0$$

$$a_1 - a_2 + a_3 \geq 0$$

$$-a_1 + a_2 + a_3 \geq 0.$$

WLOG we can assume that $a_1 \geq a_2 \geq a_3$, in which case the first two inequalities are irrelevant. Thus,

$\mathcal{L}(a_1, a_2, a_3)$ has a semistable point if and only if

$$a_1 \leq a_2 + a_3.$$

Now, because the saturation property holds for $SL(n)$ (cf. [15]), $\mathcal{L}(a_1, a_2, a_3)$ has a semistable point if and only if $[V(2a_1) \otimes V(2a_2) \otimes V(2a_3)]^G \neq 0$. Thus, $V(2a_1)$ is a component in $V(2a_2) \otimes V(2a_3)$ if and only if the inequality above holds. But, the above inequality is equivalent to

$$2a_2 - 2a_3 \leq 2a_1 \leq 2a_2 + 2a_3.$$

This is the Clebsch-Gordon condition.

Now, note that $g_i E_1 = g_j E_1$ if and only if $g_i B = g_j B$. We know that the diagonal (B, B, B) is never semistable. So we cannot have $x_1 = y_1 = z_1 = -1$ corresponding to a semistable point. Also, if we have any two of x_1, y_1, z_1 taking -1 corresponding to a semistable point, then when the corresponding expressions are set to zero, we get the same equations as above. Thus, the three inequalities given above is a full condition for stability or semistability. In particular, $((G/B)^3)^{ss} \neq (G/B)^3)^s$ if and only if one of these attains zero.

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