STOCHASTIC MODELS FOR RESOURCE ALLOCATION, SERIES PATIENTS SCHEDULING, AND INVESTMENT DECISIONS

Siyun Yu

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Approved by:

Vidyadhar G. Kulkarni

Nilay Argon

Vinayak Deshpande

Nur Sunar

Serhan Ziya

ABSTRACT

SIYUN YU: STOCHASTIC MODELS FOR RESOURCE ALLOCATION,
SERIES PATIENTS SCHEDULING, AND INVESTMENT DECISIONS
(Under the direction of Vidyadhar G. Kulkarni.)

We develop stochastic models to devise optimal or near-optimal policies in three different areas: resource allocation in virtual compute labs (VCL), appointment scheduling in healthcare facilities with series patients, and capacity management for competitive investment.

A VCL consists of a large number of computers (servers), users arrive and are given access to severs with user-specified applications loaded onto them. The main challenge is to decide how many servers to keep "on", how many of them to preload with specific applications (so users needing these applications get immediate access), and how many to be left flexible so that they can be loaded with any application on demand, thus providing delayed access. We propose dynamic policies that minimize costs subject to service performance constraints and validate them using simulations with real data from the VCL at NC State.

In the second application, we focus on healthcare facilities such as physical therapy (PT) clinics, where patients are scheduled for a series of appointments. We use Markov Decision Processes to develop the optimal policies that minimize staffing, overtime, overbooking and delay costs, and develop heuristic secluding policies using the policy improvement algorithm. We use the data from a local PT center to test the effectiveness of our proposed policies and compare their performance with other benchmark policies.

In the third application, we study a strategic capacity investment problem in a duopoly model with an unknown market size. A leader chooses its capacity to enter a new market. In a continuous-time Bayesian setting, a competitive follower dynamically learns about the favorableness of the new market by observing the performance of the leader, and chooses its capacity and timing of investment. We show that an increase in the probability of a favorable market can strictly decrease the leaders expected discounted profit due to non-trivial interplay between leaders investment capacity and timing of the dynamically-learning follower.

ACKNOWLEDGEMENTS

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My gratitude extends to Dr. Deshpande and Dr. Sunar of UNC Kenan-Flagler Business School, and Dr. Shen of HKU, for their enthusiasm, advice and support in the three projects. Without their expertise, I would not have been able to continue and complete the three projects presented in this dissertation.

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CHAPTER 1: Introduction

Stochastic models are extensively used in devising resource allocation policies in many fields, such as call centers, healthcare systems, cloud computing, production systems, etc. Resource allocation plays a critical role in balancing the demand and supply, with the goal of optimizing the economical or social benefits.

Discrete and continuous stochastic models provide appropriate tools to quantify the performance of policies in these applications, including service quality, management profit and cost, etc. Queueing models are useful in service systems where queues arise from the mismatch of supply and demand. They can be used to obtain implementable policies to manage these systems. Continuous models such as Brownian models are useful in describing the uncertain behaviour of system evolution in response to control policies, and can be used in devising optimal control policies to manage such systems.

In this dissertation we use both discrete and continuous stochastic models to allocate resources in three different applications. In the rest of this chapter, we introduce the background and motivation for each one of them.

1.1 Application 1. Server Configuration in Cloud Computing.

A cloud computing system provides computing service (such as sharing data and processing resources) through internet to clients. It allows companies to decrease the IT related infrastructure and management costs, at the same time get access to storage or a flexible collection of software applications through a simple web-based interface. In such systems, the service providers are computer servers, while the customers are the clients who generate requests of accessing the computing resources. The clients may belong to different classes and their arrival rates may vary with time. The configuration of the computer capacity allocated to the client for a given type of computing service can be controlled to guarantee the service quality.

In Chapter 2, we study a special case of cloud computing system: the campus-wide Virtual

Computing Lab (VCL). Such a system provides users remote access to their desired set of software applications. There are usually hundreds or even thousands of software applications that the users may choose from. Some examples of application are "Matlab on Windows 7", "Maple on Windows XP", and "Arena and CPLEX OPL" etc. Currently UNC-Chapel Hill and NC State both host hundreds of computer servers for students, faculty, and researchers. Each user is granted full control of the assigned server. If two users request the same software application, they will need two different servers loaded with that application.

The servers in the VCL may be on or off. The on servers may be preloaded with specific applications, or left flexible. When a user requests a server with a specific application, and one such server preloaded with that application is available, the user request can be satisfied immediately. If such a server is not available, we can load the application on a flexible server and make it available to this user. However, this loading operation creates a delay and degrades service somewhat. Turning an off server on usually takes too long to do in response to a user request, and hence if all servers are busy, the user leaves without service, which is the worst outcome for the user. The number of on servers can be changed during the day in an exogenous fashion in response to anticipated demand.

The main issue for the VCL is to decide on how many servers should be kept on (the rest are left off to save energy). Further, we need to decide how many of the on servers should be preloaded with which applications, and how many servers should be left flexible. The service quality can be measured in terms of the fraction of the users who get immediate access, the fraction that gets delayed access, and the fraction that gets rejected.

We use machine learning algorithms to predict the demand rates, queueing models to do the resource allocation, and simulation to evaluate the performance of several policies using real data collected from the NC state VCL over three years. We conclude that under the proposed policy, the required service quality can be satisfied while more than half of the servers can be turned off thus significantly reducing the operating costs.

1.2 Application 2. Appointment Systems for Series Patients

In a health care unit, the service providers are nurses, doctors, etc, while the customers are patients. The nurses and doctors can be specialized to serve multiple classes of patients. The number of nurses and doctors on duty can be controlled to satisfy the patients demands so as to provide a certain quality assurance. The patients can belong to multiple classes, and their arrival rates may vary with time; they may need multiple visits to the clinic, (as in a physical therapy (PT) setting); they may have preferences about appointment schedules, and may display no-show and cancellation behaviors.

In Chapter 3 and 4, we focus on special clinics such as physical therapy, where patients are scheduled for repeated visits at the time of admission. Such patients are called series patients. Patients in need of physical therapy are referred by the physician to the physical therapy clinic. A clinic administrator schedules a first appointment for an initial examination. Based on the diagnosis of the patient, a plan of care is determined by the physical therapist. Such a plan includes the frequency of the visits, total number of visits, and duration of each visit. Based on the plan of care, an administrator in the clinic generates an appointment schedule. The frequency, duration and length of the appointment time for the follow up visits vary greatly depending upon the patient's diagnosis and/or specific needs as well as the referring physician's orders.

In Chapter 3, we consider a model where the plan of care for a patient is set up at beginning, so once the patient is evaluated, the number of visits is known. Under this assumption, the randomness comes from the initial evaluation, after which the series of appointments are scheduled without uncertainty. In Chapter 4, we study an alternate model where the randomness occurs after each visit of a patient. Specifically, whether a patient needs the next visit is evaluated after every appointment. These two models capture different features of the series patients, depending on whether their treatment can be predetermined or needs to be modified based on their recovery status.

The major challenge in such health units is to appropriately and consistently match supply and demand of physical therapy services. Each new patient creates a string of follow up visits, which creates additional demands on the capacity of the clinic.We develop a Markov

Decision Process model that can be used to compute an optimal appointment schedule for each patient. Unfortunately, the large state space makes this method impractical computationally. However, this model helps us devise a simple heuristic policy, called the index policy, that can be obtained by using policy improvement algorithm. We use simulation to show that the index policy provides better performance than other benchmark policies used in practice.

1.3 Application 3. Investment Capacity and Timing Management

When companies plan to introduce a new product or a new technology to the market, the actual demand is uncertain at the very beginning. A common scenario is that, after one firm (called the leader) initiates an investment, the competitors (called the followers) start observing the performance of this leader firm and make their decisions accordingly. In particular, there are two most critical decisions for the follower firms to make, the first is the investment capacity, that is how much to invest so as to better suit the unknown market demand, the second is the investment timing, that is when to enter the market so as to make profit with enough knowledge about the market.

We are particularly interested in competitive relations between the leader and follower firms. Once the follower enters the market, it takes part of the market share from the leader, and after that the overflow from either of the firms cannot be fulfilled by the other party, even if the other party has extra supply. This is an appropriate model for products or technologies that have a high customer loyalty so that once the customers subscribe to one of the brands, the chance for them to switch brands is very small. One example is the competitive relation between Apple and Amazon. Both firms recently invested in the development of smart speakers, a smart device controlled by human voice so that the users can play music, control echo-home systems, call taxis, etc. Since the two companies use different operating systems, all the built-in applications and other supportive devices (cell phone, smart TVs) are not compatible with each other. Therefore it is costly for the customers to switch the brands.

In Chapter 5, we build models for the duopoly game of investment with capacity and timing decision options. We assume that the investment is costly, and the firms' earning processes are based on their initial investment capacity. The uncertain demand of the new market could be one of the binary status: high and low. A leader enters the market at the beginning of the horizon, and a follower starts its observation of the leader's earning process and decides on when and how much to invest. The leader, knowing that the follower will adopt the optimal investment strategy based on the leader's action, optimizes its strategy about how much to invest at the beginning.

The following chapters present the details of each of the above three problems. The review of the relevant literature is included in each chapter separately. We create stochastic models of the physical problems and discuss the collection and analysis of data that is needed to seed the models. We use tools from Markov decision processes, index policies, queueing theory, stochastic control, Bayesian learning to formulate the models and analyze them. We also use statistical tools from machine learning to analyze the data and estimate the parameters of the models. We then describe our current results and our plan for the future work.

CHAPTER 2: Statistical Forecasting and Queueing Models for Virtual Computing Labs

2.1 Introduction

The Virtual Computing Lab (VCL) is a cloud computing service that provides users remote access to their desired set of software applications. There are usually hundreds or even thousands of software applications that the users may choose from. Some examples of applications are "Matlab on Windows 7", "Maple on Windows XP", "Matlab and MS Excel on Windows 7", "Arena and CPLEX OPL", etc. The VCL was first developed at North Carolina State University (NC State) and is now an open-source project at the Apache Software Foundation - *http://vcl.apache.org*. An increasing number of institutions are hosting VCL servers, for example, UNC-Chapel Hill and NC State currently have hundreds of such servers for students, faculty, and researchers. From the perspective of modeling, we do not distinguish between the terms - "virtual computer" and "virtual machine", instead we use the generic term "server". Each user is granted full control of the assigned server. If two users request the same software application, they will need two different servers loaded with that application.

A server in the VCL may be preloaded with a specific application, or left flexible. A user gets immediate access to a server preloaded with the desired application if one is available. Otherwise, the user has to wait for several minutes until a flexible server is loaded with the desired application. In this chapter we are concerned with the issue of deciding how many servers should be preloaded with which applications, and how many servers should be left flexible. We call the preloaded and flexible servers the *on* servers, and the rest of the servers the *off* servers that are turned off for cost saving (both energy and management). The service quality is measured by the fraction of the users who get immediate access, and the fraction that get delayed access, while the system cost is measured by the number of *on* servers.

Although our research is motivated by the VCL application, the methodology developed here is of general applicability in any service center with sufficiently flexible servers. It is also applicable to *software as a service* (SaaS) in cloud computing. In such large-scale computing environments, the energy consumption is an important issue, both economically and environmentally. Hence, it is important and relevant to dynamically allocate "the right number of on servers with the right capabilities".

In many VCLs such as those at UNC-Chapel Hill and NC State, all servers are always on and there are no flexible servers. For example, the VCL at NC State has about 800 servers and 1600 applications; the current policy ranks the applications by the frequency with which an application is requested, and each of the 400 most popular applications are preloaded on two servers (each preloaded server has exactly one application). In general, assume the VCL has a total of M servers and is capable of handling N types of application. A user who desires type n ($1 \leq n \leq N$) application is called a type n user. A type n user arriving at the system receives instant service if a server preloaded with application n is available. Otherwise, the user is delayed and needs to wait until the system manager (an automated software) chooses a k-preloaded server ($k \neq n$), removes application k and loads application n. If no server is available then this user is blocked (rejected). After a user finishes the session, the system manager wipes the server clean, and then reloads it with the same or another application.

A server allocation plan (SAP) decides how many servers should be pre-loaded with which applications, how many servers should be kept flexible, and how many should be turned off. An SAP is called dynamic if these numbers change with time; otherwise it is called static. It makes sense to consider dynamic SAPs to accommodate time-varying demand rates. Such time-varying demand is an inherent feature to the VCL system. This is caused by the seasonal demand induced by the semesters, the days of week, and the time of the day. In this chapter, we propose a dynamic SAP with the objective of minimizing the system cost while achieving the targeted service quality.

We shall begin with a static SAP assuming constant arrival rates. We keep d_n servers preloaded with application n (i.e. the dedicated type n server pool, $1 \leq n \leq N$), f servers flexible (i.e. the flexible pool), and $M - f - \sum_{n=1}^{N} d_n$ servers off. Figure 2.1 illustrates the server network diagram. When a type n arrival occurs (with rate λ_n), this user is given a server from the dedicated pool for application n, if one is available. Otherwise a server from the flexible pool is loaded with application n and assigned to this user. In the latter case the user needs to

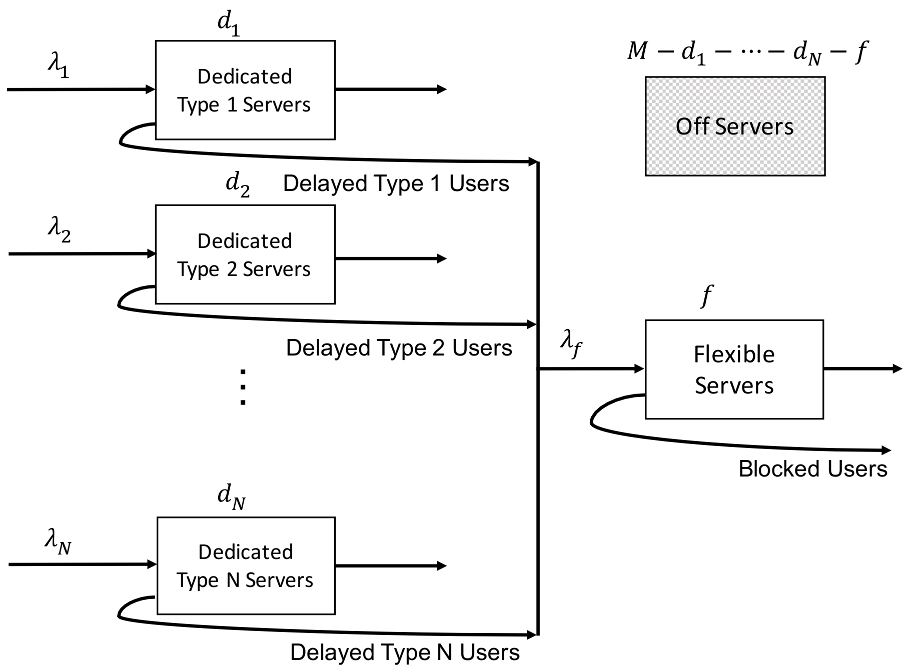


Figure 2.1: VCL Servers Network Diagram

wait a few extra minutes for the loading operation. If no flexible servers are available, the user leaves the system without service, even if there are idle servers in other dedicated pools. When a type n user finishes service from the dedicated pool, the released server is wiped clean and is reloaded with the same application to keep d_n a constant. Similarly, when a user finishes service from the flexible pool, the released server is left flexible to keep f a constant.

We then design a dynamic SAP based on the above static SAP. This is accomplished by dividing the whole time horizon into small periods, and implementing the static SAP over each period. This is called the stationary independent period by period (SIPP) approach. Under dynamic SAP, the parameters d_n $(1 \leq n \leq N)$ and f vary from period to period, but remain constant within each period. Thus the number of off servers will vary from period to period. We use three queueing models to quantify the probabilities that an incoming user is delayed or blocked. The three models are of increasing complexity and account for different statistical aspects of the arrival processes. We also propose a modified SIPP (MSIPP) approach which uses a better estimate of the arrival rates. In Section 2.7 we recommend the most beneficial combination of a queueing model and statistical forecasting technique that results in a desirable dynamic SAP.

Our proposed policy uses a dynamic SAP (see Section 2.7 for details) under which at most 10% users are delayed and at most 0.1% of the service requests are blocked. The policy only uses a maximum of 300 on servers at any time, resulting in substantial energy savings. In contrast, Lee (2013) shows that, under the current policy followed by the VCL at NC State, over 11% of the users are delayed and no users are blocked but all the available servers are always kept on. This makes our proposed policy extremely attractive.

It is clear that we need the forecasts of the arrival rates λ_n $(1 \leq n \leq N)$ for each period in order to execute our SAP. We explore two statistical methods to forecast the future arrival rates: the moving average (MA) method and the singular value decomposition (SVD) method. These arrival rate forecasts are then used as inputs to the queueing models to estimate the blocking probabilities. For the input parameters to the queueing models, we use either the mean arrival rate forecast or the upper bound of the 95% prediction interval of the mean arrival rate.

The remainder of the chapter is as follows. We provide a literature review on the related topics in Section 2.2. Section 2.3 introduces the structure of the data, and highlights the challenges posed by time-varying demands. We formulate our static SAP model in Section 2.4.1 and construct three static queueing models in Section 2.4.2. Algorithms are developed there to quantify the constraints in the allocation model. Section 2.4.3 explains our procedure of creating the dynamic SAP based on the static SAP introduced in Section 2.4.2. Section 2.5 introduces two ways of forecasting future arrival rates, and Section 2.6 describes how we conduct discrete event simulations using real data and presents the results. Section 2.7 makes recommendations about the most implementable policy as the combination of the best forecasting method, the estimation approach, and the queueing model. Finally, Section 2.8 summarizes the chapter and discusses how we can extend the current work to applications beyond VCL.

2.2 Literature Review

One can think of the VCL as a server farm. There is a large literature on the topic of resource allocation in server farms. Gandhi et al. [2010, 2009, 2011] defined four states of the servers: off, setup, idle, and on (busy). Comparing with our system, their idle servers are similar to our dedicated idle servers, where the user receives immediate service; their servers in setup state (switching from off to on) are similar to our flexible idle servers, where the user has to wait for some extra setup time. Two performance measurements are usually considered in a server farm setting: waiting time and power consumption. In their work, Gandhi et al. used queueing model and simulations to derive these metrics. One of their conclusions is that keeping the servers idle is superior for reducing waiting time, and turning the servers off is superior for reducing power. Adan et al. [2013] used a constant setup cost instead of a setup time to discourage switching between off and on, which simplified the state space and resulted in a switching-curve structure of the optimal policy.

Our model differs from the above server farm models in two important aspects. First, in the papers mentioned above, the users are homogeneous and the systems are stationary, while in our system, there are multiple types of users with time-varying arrival rates and service times. Second, the server farm literature deals with waiting times and power consumption, while our model focuses more on the fraction of delayed or blocking users. The use of dedicated servers is essential to reduce the fraction of delayed servers.

In this chapter we address three main features of the VCL systems: (1) time-varying demands, (2) multi-type demand structure, and (3) availability of data to forecast future demand. We shall review the relevant literature below in each of these areas.

The phenomenon of time-dependent arrival is commonly seen in many service systems, and it is critical to staff them at appropriate levels to cope with this variation. There is a large literature dedicated to this problem. For an in-depth review on determining the staffing levels in the presence of time-varying demand, see Green et al. [2007], Whitt [2007], Liu and Whitt [2011, 2012]; Liu et al. [2014].

One approach to modeling the time-varying demand is to use stationary models in a non-stationary manner. It is achieved by dividing the working period (workday or workweek) into shifts, hours, quarter-hours, etc, and then applying a series of stationary queueing models over each planning period. This method is called the stationary independent period by period (SIPP) approach in Green et al. [2001]. However its performance highly depends on the system parameters such as the arrival rate, the mean service time and the service quality, see Thompson [1993] and Puhalskii et al. [2010]. As a counter example in Green et al. [2001], when the Markovian model with sinusoidal arrival rates is considered in the simulation, the SIPP approach underestimates the staffing levels. Thompson [1993] and Green et al. [2001, 2007] have addressed this issue and discussed several solutions such as a lagged SIPP, which essentially shifts the arrival rate curve to the right by a fixed amount.

In our work, we apply approaches similar to SIPP to deal with the time-varying demand issue. Furthermore, we propose stationary dependent period by period (SDPP) approach that takes into account the customers that remain in the system from the previous period. The SDPP approach outperforms the regular SDPP approach in the service quality while using fewer servers.

The second feature is the existence of the multiple types of users. This heterogeneity in the sources of demands creates the critical issue of whether to use dedicated (specialized) or flexible resources. When the servers have sufficiently overlapping capabilities and work as a single super-server, the best possible performance can be achieved by the complete resource pooling strategy. See, for example, Harrison [1988, 2000]. Between the extremes of full-flexibility and full-specialization, different limited-flexibility structures can be constructed. Jordan and Graves [1995] are the first to show that well-designed limited flexibility can be as good as full flexibility. These principles are further justified by Akcsin and Karaesmen [2007], Iravani et al. [2007] and Bassamboo et al. [2008]. They also propose methods to evaluate different flexibility structures. Our model considers the combination of dedicated server pools and a flexible server pool, in order to provide immediate service as much as possible while guaranteeing an overall service quality.

Finally we address the statistical features of the VCL system. As pointed out by Chen and Henderson [2001], where the staffing problem is studied under a Police Communication Center setting, designing a staffing level policy needs an accurate forecast of arrival rates. A handful of efficient forecasting approaches have been developed for call centers. For a comprehensive review, see Aksin et al. [2007] and Ibrahim et al. [2016]. Typically, arrival data in call centers are aggregated within each short time periods, such as 15-minute or 30-minute intervals, and the target of forecasting is implemented over such periods, see Jongbloed and Koole [2001]. This is consistent with our SIPP and SDPP modeling approaches. Similar techniques are used in Weinberg et al. [2007], where a multiplicative effects model is constructed to forecast Poisson arrival rates over intervals of 15, 30, or 60 minutes length, with a one-day lead time. More recently, Shen and Huang [2008] propose a statistical model for forecasting call volumes within short time periods of a given day and also provide approaches to account for intraday forecast updating. Their singular value decomposition (SVD) based method outperforms existing forecasting methods. Our work adopts their SVD forecasting model to the VCL setting. Numerical experiments show that it leads to better service quality over the standard moving average (MA) method.

2.3 Data

The VCL data set from NC State University contains information about all user requests from August 1, 2008 to July 31, 2011. In total there are 595,000 service requests for 1,643 different applications. For the ease of presentation, we sort the applications by their frequency of use in descending order. The usages vary considerably for different applications, where the top two applications account for 18.88%, the top ten applications account for 42.63%, and the top 400 applications account for 97.30% of the total requests. On the other hand, each of the bottom eight hundred applications is used no more than ten times over the three-year period that we consider. The detailed information is presented in Table 2.1. The VCL has around 700 to 900 servers. (The information about the exact number of servers is not given in our data set. The real time information about the number of on/off servers is given on the VCL website.) We present below some details of the arrival and service time data.

Cumulative Relative Frequency (%)	
Top 1	9.57%
Top 2	18.88%
Top 10	42.63%
Top 50	71.15%
Top 100	82.27%
Top 200	91.66%
Top 400	97.30%
Top 800	99.38%
Top 1643	100.00%

Table 2.1: Cumulative Relative Frequency of Arrivals

2.3.1 Arrivals

Figure 2.2 (a) plots the aggregated hourly arrivals from August 1, 2008 to July 31, 2011. To provide a better idea of arrival pattern in finer time scales, we use Panels (b) and (c) to show the average hourly arrivals in each hour of the week (starting with Sunday midnight) and each hour of the day (starting with midnight) respectively. We also present the same set of graphs for individual applications as comparison. For illustration, Figure 2.3 is shown here for Application 1, while Figures A.1 and A.2 (see Appendix A.1.1) are for Applications 10 and 100.

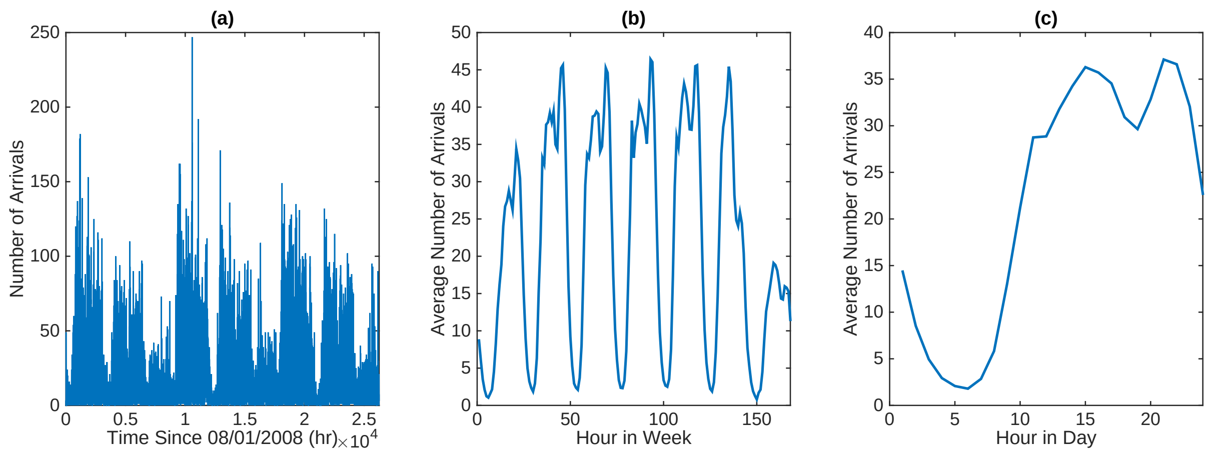


Figure 2.2: Aggregated Arrivals

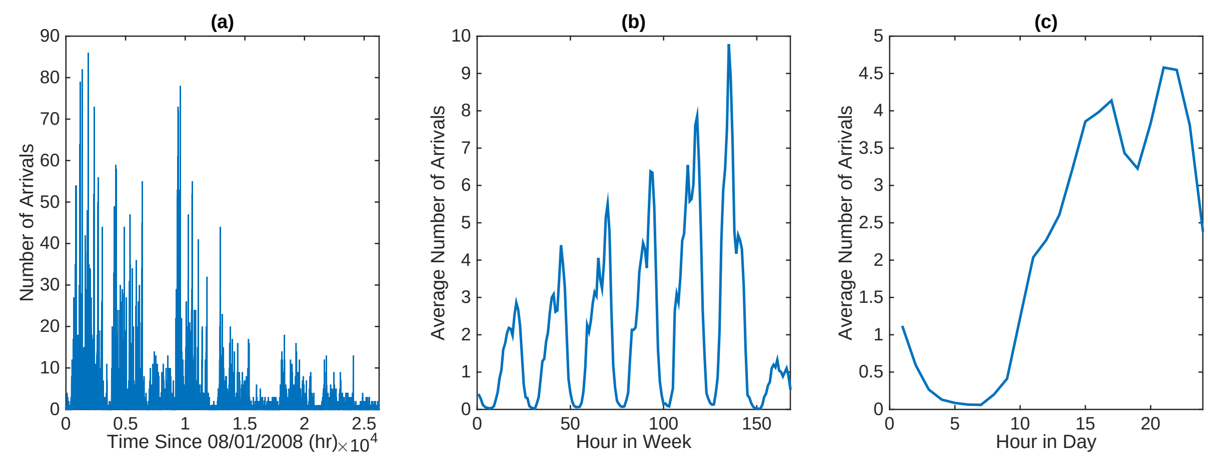


Figure 2.3: Arrivals of Application 1

We observe that arrivals show a predictable, repeating pattern. The semesterly, weekly, and daily cycles are quite clear on both aggregated and individual application levels. In Figure 2.3(b) we see that for application 1 the arrival volumes are low on Saturdays and high on Fridays. During the day, the first peak occurs around 3PM, followed by a second peak around 10PM. This is true in the aggregated case (Figure 2.3(c)) and for most of the applications. We observe that not all the applications were available in the VCL on the initial date of 08/01/2008. For example, application 10 was not available until 08/18/2010 (Figure A.1 in the appendix).

Using the aggregated arrival data we also plot the mean and standard deviation against the time of day, grouped by the day of week, as shown in Figure 2.4. We observe the heteroscedasticity phenomenon - both the mean and the standard deviation depends on time. Besides, the magnitude of the standard deviation is almost at the same level of the mean; hence the variance exceeds the mean, which implies that the observations are over-dispersed with respect to a Poisson distribution.

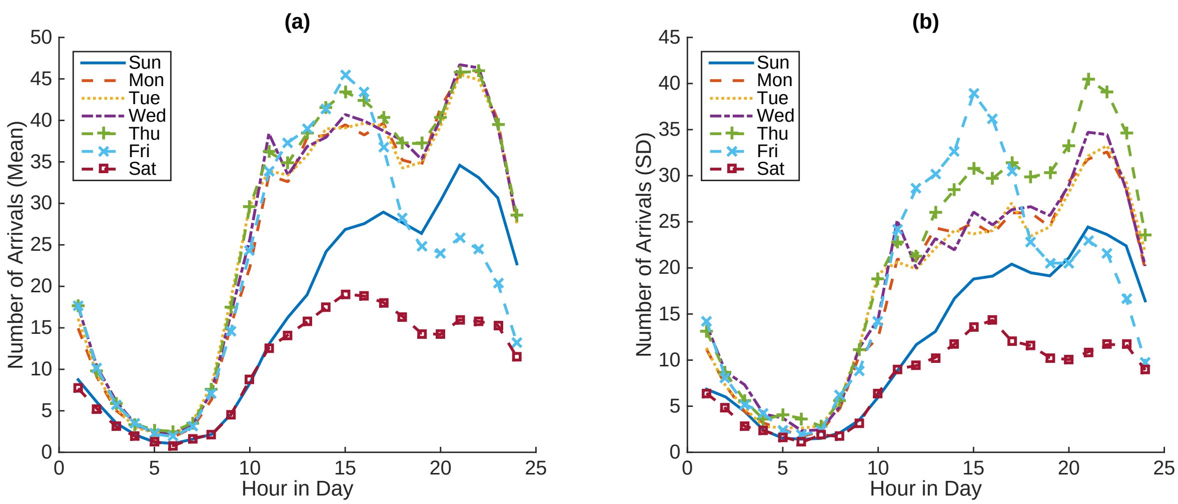


Figure 2.4: Mean and Standard Deviation of Aggregated Arrivals by Day of Week

2.3.2 Service Times

Running the analysis on the service time data, we find that about 3% of the service times (i.e. 1327 arrivals) are less than 1 minute. The very short service times are questionable in the setting of software online service. One possible explanation is the accidental shut down of the system which would force the users to log off. Besides, there is a four-hour check initiated by the server, and if the user is not active, he/she is automatically logged off. If the user is active he can request extensions in two hour increments.

In Figure 2.5 we plot the empirical cumulative distribution function (CDF) of the service times for applications 1, 10, 100, along with the corresponding CDF of the exponential distribution with the same mean. The plots suggest that an exponential assumption on the service time distribution is reasonable. (Though we do not need this exponential assumption in most of our queueing models.)

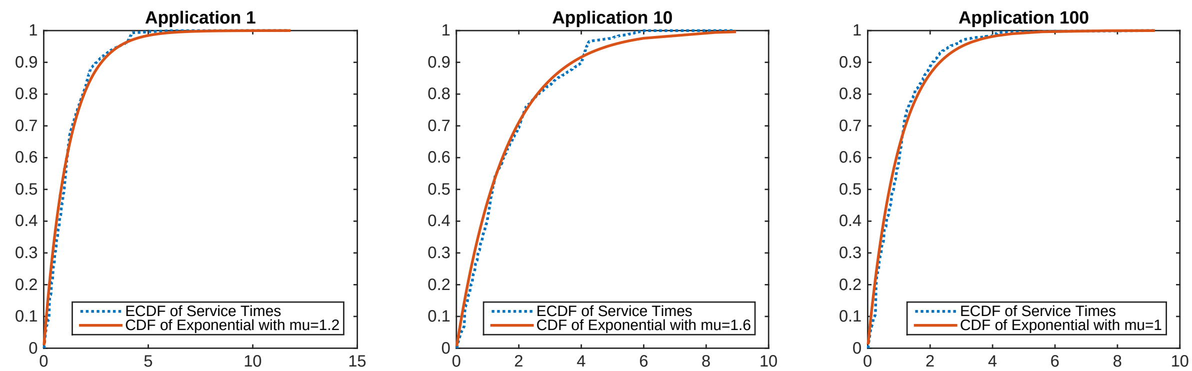


Figure 2.5: Empirical CDF of Service Times and Exponential CDF

2.4 Queueing Models and Staffing Policies

As described in Section 2.1, we consider a VCL with M servers, N applications, and N dedicated server pools (of sizes $d_1, d_2, \cdots, d_N$), one flexible server pool of size f, and the rest being off servers (Figure 2.1). In this section we develop server allocation plans (SAP) for the VCL. In Section 2.4.1, we first assume that the arrival rates for the applications are constant, and present the formulation of a static SAP. Then in Section 2.4.2, we consider three queueing models for the VCL system and derive the corresponding static SAP. In Section 2.4.3, we extend the static SAP to dynamic SAP using stationary independent period by period (SIPP) and a modification version that allows dependence between periods of (SDPP).

2.4.1 Problem Formulation - Static SAP

Under the static SAP, we are interested in policies which assign d_n dedicated servers preloaded with application n $(1 \leq n \leq N)$, allow f servers to be left flexible, and turn off the remaining $M - f - \sum_{n=1}^{N} d_n$ servers. The trade-off here is between minimizing energy consumption cost, which is directly reflected by the number of on servers, and guaranteeing certain service quality (to be quantified later).

Note that when a user gets a dedicated server, the wiping and reloading operations take place at the end of the service (after the user leaves the system); if a user gets a flexible server, the loading operation takes place before the service starts and the wiping operation takes place after the service finishes. Thus in both cases the service times are augmented by the wiping and loading operations.

Let the delay probability α be the fraction of the users who do not receive immediate service upon arrival, and the blocking probability β be the fraction of the users who leave the system without service. Our aim is to identify the smallest $d_1, \cdots, d_N$ and f that can satisfy the following service level constraints:

$$\alpha < \alpha^*, \tag{2.1}$$

$$\text{and } \beta < \beta^*, \tag{2.2}$$

where α^* and β^* are the design parameters. For example, we consider $\alpha^* = 0.05$ and $\beta^* = 0.001$. Below we describe three queueing models to quantify the above constraints.

2.4.2 Queueing Models for Static SAP

In this section, we consider three queueing models of increasing complexity to help size the dedicated pools and the flexible pool: $M/G/c/c$ that assumes Poisson arrivals, $M^X/M/\infty$ that uses compound Poisson arrivals and accounts for overdispersion, and finally $M(t)/G/\infty$ that uses non-homogeneous Poisson arrivals and captures the continuously time-varying arrival rate feature. We compare their performances using simulation in Section 2.6.

2.4.2.1 $M/G/c/c$ Queue.

In this model, we assume that the type n requests form a Poisson process (PP) in the dedicated server pool with rate λ_n. The service time for each specific application is generally distributed with mean s_n. If a type n preloaded dedicated server is available when a type n user arrives, its service starts immediately and lasts for an amount of time with mean s_n plus the wiping and loading time with mean u_n. If such a preloaded server is not available, then the user is delayed (or lost). Thus we model the dedicated and flexible pools as $M/G/c/c$ queues. The details are given below.

- *Dedicated Pool.*

Let d_n be the number of dedicated servers assigned to application n. Thus we think of the number of type n users in the dedicated pool as an $M/G/d_n/d_n$ queue. The offered load of type n over this interval is then given by

$$a_n = \lambda_n(s_n + u_n).$$

We assume that the system is in steady state so that the probability of type n users not receiving immediate service is equal to the probability of type n users being blocked from the dedicated pool for application n, which is given by the well known Erlang-B formula (see, for example, Cooper (1972)):

$$P_d(d_n, a_n) = \frac{(a_n)^{d_n}}{d_n!} \Bigg/ \sum_{k=0}^{d_n} \frac{(a_n)^k}{k!}. \tag{2.3}$$

We call $P_d(d_n, a_n)$ the *probability of delay.* Now, the probability that an arrival is of type n is $\lambda_n / \sum_{k=1}^{N} \lambda_k$. Hence the overall probability of delay is given by

$$\frac{\sum_{n=1}^{N} \lambda_n P_d(d_n, a_n)}{\sum_{n=1}^{N} \lambda_n}, \tag{2.4}$$

which provides an approximation of α. Hence we formulate the following optimization problem for the dedicated pools:

Problem P1

$$\begin{aligned} \text{minimize} \quad & \sum_{n=1}^{N} d_n, \\ \text{subject to} \quad & \frac{\sum_{n=1}^{N} \lambda_n P_d(d_n, a_n)}{\sum_{n=1}^{N} \lambda_n} < \alpha^*. \end{aligned}$$

Note that the Erlang-B formula in Eq.2.3 is a convex function in d_n (see Messerli (1972)).

In addition, it can also be expressed recursively as follows (Cooper (1972)):

$$P_d(0, a_n) = 1,$$
$$P_d(k, a_n) = \frac{a_n P_d(k-1, a_n)}{a_n P_d(k-1, a_n) + k}, \quad \forall k = 1, \cdots, d_n.$$

These two properties yield the following greedy algorithm to solve P1.

Algorithm 1: Greedy Algorithm to Solve P1

for $n = 1 : N$ **do**

$$d_n \leftarrow 0;$$
$$P_d(d_n, a_n) \leftarrow 1;$$
$$\delta(n) \leftarrow \lambda_n[1 - P_d(1, a_n)];$$

end for loop

$$\alpha \leftarrow \frac{\sum_{n=1}^{N} \lambda_n P_d(d_n, a_n)}{\sum_{k=1}^{N} \lambda_k};$$

while $\alpha \geq \alpha^*$ **do**

$$n^* \leftarrow \arg\max_n \delta(n);$$
$$d_{n^*} \leftarrow d_{n^*} + 1;$$
$$\delta(n^*) \leftarrow \lambda_{n^*}[P_d(d_{n^*}, a_n) - P_d(d_{n^*} + 1, a_{n^*})]$$
$$\alpha \leftarrow \alpha - \delta(n^*) / \sum_{n=1}^{N} \lambda_n;$$

end while loop

This yields the optimum $(d_1, \cdots, d_N)$ that satisfies Eq 2.1.

- *Flexible Pool.*

We see that the arrival process to the flexible pool is a superposition of the overflows from all the dedicated pools. The overflow process from an $M/M/c/c$ queue can be accurately approximated by an interrupted Poisson process (Kuczura (1973)). However, each dedicated pool we consider in Section 2.4.2.1 is an $M/G/c/c$ queue. In addition, the system we consider has hundreds of dedicated pools. The approximation of the aggregate process of hundreds of interrupted PP would be intractable. Hence, we choose to simply approximate the aggregate arrival process to the flexible pool by a PP. Let λ_n^f be the rate of overflow from nth dedicated server pool, which can be computed as

$$\lambda_n^f = \lambda_n P_d(d_n, a_n).$$

Hence the arrival rate of the aggregate process to the flexible pool is given by

$$\lambda_f = \sum_{n=1}^{N} \lambda_n^f.$$

The mean service time in the flexible pool is a weighted sum of the mean service times of each type n plus the mean loading time:

$$s_f = \frac{\sum_{n=1}^{N} \lambda_n^f (s_n + u_n)}{\sum_{n=1}^{N} \lambda_n^f}. \tag{2.5}$$

Then the offered load to the flexible pool is given by

$$a_f = \lambda_f s_f = \sum_{n=1}^{N} \lambda_n^f (s_n + u_n).$$

Again, assuming the system is in steady state, the probability of a user being blocked from the flexible pool is given by the Erlang-B formula:

$$P_b(f, a_f) = \frac{(a_f)^f}{f!} \bigg/ \sum_{k=0}^{f} \frac{(a_f)^k}{k!}.$$

Note that the probability that a user leaves the system without service is equal to the probability that a user is blocked from the flexible pool. We call it *blocking probability.* Hence, to satisfy Eq.2.2, we solve the following optimization problem:

Problem P2

$$\begin{aligned} &\text{minimize} \quad f, \\ &\text{subject to} \quad P_b(f, a_f) = \frac{(a_f)^f}{f!} \Bigg/ \sum_{k=0}^{f} \frac{(a_f)^k}{k!} < \beta^*. \end{aligned} \tag{2.6}$$

Determining the minimum f that satisfies Eq.2.6 is straightforward since $P_b(m, a)$ is a decreasing function in m.

2.4.2.2 $M^X/M/\infty$ Queue.

Recall the overdispersion issue with the arrival data discussed in Section 2.3. To address this issue, we use Compound Poisson Process (CPP) to model the arrival process for both the dedicated server pool and the flexible server pool. We construct the CPP arrival process $\{Y(t), t \geq 0\}$ as follows:

$$Y(t) = \sum_{k=1}^{Z(t)} D_k$$

where $\{D_k : k \geq 1\}$ are positive independent and identically distributed (i.i.d.) Geometric random variables with parameter p, and $\{Z(t)\}$ is a PP with rate θ, which is also independent of $\{D_k\}$. Then we have

$$E(Y(t)) = \theta t E(D) = \theta t/p, \tag{2.7}$$

and

$$\text{Var}(Y(t)) = \theta t E\left(D^2\right) = \theta t(2-p)/p^2. \tag{2.8}$$

Thus the variance of the process is $(2-p)/p$ times the expectation. Since $(2-p)/p$ is always greater than one for $0 < p < 1$, this is consistent with the overdispersion seen in the real data.

Now we assume that the service times are i.i.d. exponential with mean s and use $M^X/M/\infty$ queue to model the number of users in a dedicated pool. This model has been analyzed in the Chapter 6 of Chaudhry and Templeton (1983), where it has been shown that in steady state, the number of users in this queue follows a negative binomial distribution $NB(r, 1-p)$, where

$r = \theta s/(1-p)$ (which may not be an integer). To be precise, if $X \sim NB(r, 1-p)$ we have

$$P(X = k) = \binom{k+r-1}{k} p^r (1-p)^k, \quad k \geq 0.$$

We see that instead of a Poisson distribution of the number of users in the $M/M/\infty$ queue, our $M^X/M/\infty$ model produces a negative binomial distribution for the number of users. In fact, in the case of modest overdispersion, the negative binomial distribution is a standard model and a good alternative to the Poisson distribution, as introduced in studies like Cameron and Trivedi (2013).

- *Dedicated Pool.*

Assume the dedicated pool for application n forms an $M^X/M/\infty$ queue with parameters θ_n and p_n. Let $r_n = \theta_n s_n/(1-p_n)$. Inspired by the Erlang-B loss formula, which is in fact a truncated Poisson distribution, we propose to use the truncated negative binomial distribution to approximate the probability of delay for type n users:

$$Q_d(d_n, r_n) = \binom{d_n + r_n - 1}{d_n}(1-p_n)^{d_n} \bigg/ \sum_{k=0}^{d_n} \binom{k + r_n - 1}{k}(1-p_n)^k, \tag{2.9}$$

where the numerator is the steady state probability that there are exactly d_n users and the denominator is the probability that there are at most d_n users in the $M^X/M/\infty$ queue.

The overall probability that a typical arrival not receiving immediate service can be obtained by replacing $P_d(d_n, a_n)$ in Eq.2.4 with $Q_d(d_n, r_n)$, which leads to an optimization problem similar to *P1*. To use the same algorithm for solving this truncated negative binomial version of *P1*, we need the convexity of the function in Eq.2.9 in d_n. Our numerical studies show that this convexity holds, although we have not been able to prove this analytically.

- *Flexible Pool.*

The overdispersion issue in the flexible pool is even more critical, because the arrival process is the aggregate overflow process from the dedicated pools. We therefore assume that the arrival process to the flexible pool is a CPP with parameters θ_f and p_f. They can be approximated as follows:

$$\theta_f = \sum_{n=1}^{N} \theta_n Q_n(d_n, r_n), \tag{2.10}$$

$$\frac{1}{p_f} = \frac{1}{\theta_f} \sum_{n=1}^{N} \frac{\theta_n}{p_n} Q_n(d_n, r_n). \tag{2.11}$$

Also we have $r_f = \theta_f s_f/(1-p_f)$, where s_f can be obtained by Eq. 2.5. Then the blocking probability is given by

$$Q_b(f, r_f) = \binom{f + r_f - 1}{f}(1-p_f)^f \Bigg/ \sum_{k=0}^{f} \binom{k + r_f - 1}{k}(1-p_f)^k.$$

It is straightforward to determine the minimum number of flexible servers to satisfy the criterion 2.6 because we can show that the truncated negative binomial version of $Q_b(m, a)$ is a decreasing function in m (see Appendix A.1.2).

2.4.2.3 $M(t)/G/\infty$ Queue.

In Section 2.3 we see that the arrival rates are time dependent, therefore it would be appropriate to model the arrival process with a non-homogeneous Poisson process (NHPP), thus yielding an $M(t)/G/c/c$ queue. However, we have analytical solution to this system only when $c = \infty$. Hence we use the $M(t)/G/\infty$ model, and then truncate the steady state distribution at c to get an approximation for the steady state distribution of the $M(t)/G/c/c$ model. With the assumption that the arrival rate function over the horizon has a finite upper bound, we show that there exists an upper bound of the mean number of users over the infinite horizon, and we choose this bound as the parameter to quantify α, the probability of delay in the dedicated pool. However, this model is not feasible in the flexible pool, since it induces the time dependence of service times.

- *Dedicated Pool.*

Let $\lambda_n(t)$ be the time-varying arrival rate of application n defined on $[0, \infty)$, with an upper bound of Λ. Assume the service time for type n users follows a CDF $G_n(\cdot)$ that is time independent and has mean s_n. This CDF can be approximated by the empirical CDF using the corresponding service times data, as shown in Section 2.3. The main $M(t)/G/\infty$ result, due to Palm (1988) and Khinchin et al. (2013), is that the number of type n users at time t has a Poisson distribution with mean given by

$$m_n(t) = \int_0^t (1 - G_n(u))\, \lambda_n(t-u) du. \tag{2.12}$$

Note that if the actual service time has an upper bound (in our VCL case the service times do not go beyond 12 hours), the above mean is in fact an integral over a finite interval, say from $t-12$ to t. Also with the assumption of the finite upper bound on $\lambda_n(t)$, we see that

$$m_n(t) \le \int_0^\infty (1 - G_n(u)) \cdot \Lambda du = s_n \cdot \Lambda, \quad \text{for } \forall t \in [0, \infty).$$

Therefore, there exists an upper bound on $m_n(t)$, defined by

$$m_n = \max_{t \in [0,\infty)} m_n(t).$$

The probability of delay for type n users at time t can be approximated by the truncated Poisson distribution, which is again given by the Erlang-B formula:

$$P_d(d_n, m_n) = \frac{(m_n)^{d_n}}{d_n!} \Bigg/ \sum_{k=0}^{d_n} \frac{(m_n)^k}{k!}. \tag{2.13}$$

Note that using m_n in the above equation ensures that the probability of delay is the maximum.

The sizing problem for the dedicated pool is exactly the same as the one discussed in Section 2.4.2.1, only with m_n instead of the offered load a_n. We do not repeat the details here.

- *Flexible Pool.*

In the flexible pool, since the composition of the user types varies with time, the service time distribution is also time-dependent. Hence we use the $M/G/c/c$ approach introduced in Section 2.4.2.1 to estimate the blocking probability and solve the sizing problem.

2.4.3 Dynamic SAP

As observed in Section 2.3, the user arrival rates are highly time dependent. Hence we introduce a dynamic version of SAP, which adjusts the sizes of the server pools periodically. The main idea is to divide the entire time horizon into small planning periods. To estimate the arrival rates, we first apply the traditional stationary independent period-by-period (SIPP) methodology by assuming that the smaller planning periods are independent, and the system is in steady state over each period (see Section 2.4.3.1). Due to the relatively longer service times (compared to the lengths of the periods) at VCL, these assumptions can be violated easily. Hence we modify this approach to account for the dependence between the consecutive periods (see Section 2.4.3.2).

We introduce the following notation for the description of the dynamic SAP. Let $\{\tau_i, i \geq 1\}$ be a given increasing sequence starting with $\tau_1 = 0$. For example we use $\tau_i = i - 1$ ($i \geq 1$, in hours). We call the interval $[\tau_i, \tau_{i+1})$ the ith period. Let $d_{n,i}$, $1 \leq n \leq N$ be the number of servers in the dedicated pool for application n over the time interval $[\tau_i, \tau_{i+1})$, and f_i be the corresponding number of servers in the flexible pool. Therefore our dynamic SAP is defined by $\{(f_i, d_{1,i}, \cdots, d_{N,i}), i \geq 1\}$.

For the $M/G/c/c$ queue of Section 2.4.2.1, we replace the notations such as λ_n, s_n, a_n with the notations $\lambda_{n,i}$, $s_{n,i}$, $a_{n,i}$, which indicate the arrival rate, service time, and offered load respectively, of application n over the ith period. Then the sizes of the dedicated server pools $(d_{1,i}, \cdots, d_{N,i})$ are determined by applying *Algorithm 1* over the ith period. Similarly, for the flexible pool, we use notation $\lambda^f_{n,i}$, $\lambda_{f,i}$, $s_{f,i}$ and $a_{f,i}$ instead of λ^f_n, λ_f, s_f and a_f for the ith period, and we are able to determine the size f_i period by period.

For the $M^X/M/\infty$ queue of Section 2.4.2.2, we use the parameters $\theta_{n,i}$, $p_{n,i}$ and $r_{n,i}$ separately over the ith period. Again we apply *Algorithm 1* over each period to determine the sizes of the dedicated pools by simply replacing all $P_d(d_{n,i}, a_{n,i})$ by $Q_d(d_{n,i}, r_{n,i})$. For the flexible pool, we replace notations θ_f, p_f and r_f with the periodic version of $\theta_{f,i}$, $p_{f,i}$ and $r_{f,i}$, and the size f_i can be obtained period by period.

For the $M(t)/G/\infty$ queue of Section 2.4.2.3, the upper bound of m_n is computed by maximizing over the ith period. That is

$$m_{n,i} = \max_{t \in [\tau_i, \tau_{i+1})} m_n(t),$$

and we replace the notation m_n with $m_{n,i}$ over the ith period. The period-by-period sizing of the dedicated pools follows naturally using Eq. 2.13. We size the flexible pool using the same procedure of the $M/G/c/c$ queue.

2.4.3.1 Traditional SIPP.

The traditional SIPP approach approximates the arrival rate $\lambda_{n,i}$ of application n over the ith period by

$$\hat{\lambda}_{n,i} = \frac{1}{\tau_{i+1} - \tau_i} \int_{\tau_i}^{\tau_{i+1}} \lambda_n(t) dt.$$

Since we assume that $\lambda_n(t)$ is a constant equal to $\lambda_{n,i}$ over $[\tau_i, \tau_{i+1})$, we have

$$\hat{\lambda}_{n,i} = \lambda_{n,i}. \tag{2.14}$$

One drawback of the SIPP method is that it requires the independence between adjacent periods. Thus the efficacy of this method is highly dependent on the system parameters such as the magnitude of the arrival and service rates (Green et al. (1991)), which can result in mis-sizing the service systems. We find a similar issue in our numerical experiments. Thus we introduce the following modification to SIPP (denoted by SDPP).

2.4.3.2 Modified SIPP (SDPP).

As discussed in Section 2.3, typical service times range from 30 minutes to four hours. Hence many of the arrivals in one interval would continue to be in the system over the next interval due to relatively long service times compared to the length of the interval. This contradicts the SIPP assumption. Hence, we propose the Stationary Dependent Period by Period (SDPP) approach that accounts for the dependence between the consecutive intervals (see similar treatment in Massey and Whitt (1994)).

Let $B_n(t)$ be the number of type n users who are using servers from their dedicated pool at time t, and $F_n(t)$ be the number of type n users who are using servers from the flexible pool at time t. Also $B_n(t) + F_n(t)$ is the number of type n users in the system at time t. Let S_n be the representative service time of a type n user. If a user arrives at the system during $[\tau_i - S_n, \tau_{i+1})$, the sojourn time in the system will overlap $[\tau_i, \tau_{i+1})$. Hence we adjust the arrival rate of type n over the ith period to

$$\begin{aligned}
\lambda_{mod(n,i)} &= \int_{\tau_i - S_n}^{\tau_{i+1}} \lambda_n(t)dt \Bigg/ (\tau_{i+1} - \tau_i + S_n) \\
&= \left[\int_{\tau_i - S_n}^{\tau_i} \lambda_n(t)dt + \int_{\tau_i}^{\tau_{i+1}} \lambda_n(t)dt\right] \Bigg/ (\tau_{i+1} - \tau_i + S_n) \\
&= \left[\int_{\tau_i - S_n}^{\tau_i} \lambda_n(t)dt + (\tau_{i+1} - \tau_i)\lambda_{n,i}\right] \Bigg/ (\tau_{i+1} - \tau_i + S_n)
\end{aligned}$$

Since we do not assume any specific distribution of service times, even the expectation of the integral in the above equation

$$\int_{\tau_i - S_n}^{\tau_i} \lambda_n(t)dt \tag{2.15}$$

is intractable in general. However, we can interpret Eq.2.15 as the total number of arrivals over $[\tau_i - S_n, \tau_i)$, and since S_n is a representative service time, all these arrivals are expected to be in the system at time τ_i. Hence we approximate the integral by $B_n(\tau_i) + F_n(\tau_i)$, the actual number of type n users in the system at time τ_i. More specifically we approximate $\lambda_{mod(n,i)}$ by

$$\hat{\lambda}_{mod(n,i)} = [\lambda_{n,i} + B_n(\tau_i) + F_n(\tau_i)] \Bigg/ (\tau_{i+1} - \tau_i + s_n), \tag{2.16}$$

where s_n is again the mean service time of application n.

2.5 Forecasting Future Arrival Demand

To implement the dynamic SAP in practice, we need to forecast future arrival demand, denoted as $\lambda_{n,i}$ in the above sections. Here we introduce two forecasting methods that will be compared in our numerical study section. The baseline method is the moving average (MA) method, and the more sophisticated one is the singular value decomposition (SVD) method introduced by Shen and Huang (2008).

We introduce the common notation used in this section. Let T be the number of days. Each day is divided into P periods. For example, we use $P = 24$ hourly periods in our numerical experiments. Let $x_{t,i}$ be the number of arrivals during the i-th period of day t, $t = 1, \cdots, T$, $i = 1, \cdots, P$. The vector $x_t = [x_{t1}, \cdots, x_{tP}]$ records the hourly arrival volumes on day t. Given the historical data $x_1, \cdots, x_T$, the following sections discuss the two methods to forecast the next-day demands x_{T+1}.

2.5.1 Moving Average Forecasting

Let w be the rolling horizon, which is the number of historical days used for forecasting. Specifically, considering the daily patterns in our data, we forecast the arrival demand over each ith period of day $T+1$ as the average of the same time periods of the previous w days. That is

$$\hat{x}^{(1)}_{T+1,i} = \frac{1}{w} \sum_{t=T-w+1}^{T} x_{t,i}, \quad i = 1, \cdots, P, \quad T > w.$$

To accommodate the variation in the demand and achieve a higher service quality, we also propose an inflated estimate using the sample standard deviation:

$$\hat{x}^{(2)}_{T+1,i} = \hat{x}^{(1)}_{T+1,i} + 1.96 \sqrt{\frac{1}{w-1} \sum_{t=T-w+1}^{T} (x_{t,i} - \hat{x}^{(1)}_{T+1,i})^2}.$$

The above estimate will be an approximation of the upper bound of the 95% prediction interval of the mean arrival rate forecast.

One important issue in the MA method is to decide the size of the rolling horizon w. The larger the window, the less influence the short term daily fluctuation will have, and more clearly we can see the long term effects. However, a larger w would make the MA method less sensitive to the non-stationary phenomenon. Our numerical studies use $w = 30$ days.

2.5.2 SVD Forecasting

Here we apply the inter-day forecasting method introduced in Shen and Huang (2008). To be consistent with the MA method, we also use w historical days. Let $X = [x^\top_{T-w+1}, \cdots, x^\top_T]^\top$ be the historical data matrix used in the forecasting of arrivals on day $T+1$ (here the sign $^\top$

stands for transpose). The singular value decomposition (SVD) of the X can be expressed as $X = USV^\top$, where U records the row information (the daily (inter-day) pattern of the original X), and V records the column information (the time of day (intra-day) pattern of the arrivals). We write the three decomposition matrices in the form of column vectors and diagonal elements: $U = (u_1, \cdots, u_P)$, $V = (v_1, \cdots, v_P)$, $S = diag(s_1, \cdots, s_P)$, where $s_1 \geq s_2 \geq \cdots \geq s_P$. Then it follows

$$x_t = (s_1 u_{t,1}) v_1^\top + \cdots + (s_P u_{t,P}) v_P^\top.$$

To summarize the inter-day features while reducing the dimension of data profiles, we extract the first K singular vectors from V, which is similar to the techniques used in the principle component analysis. By setting $s_k u_{t,k} = \beta_{t,k}$, we get the following approximation

$$\begin{aligned} x_t &\cong (s_1 u_{t,1}) v_1^\top + \cdots + (s_K u_{t,K}) v_K^\top \\ &= \beta_{t,1} v_1^\top + \cdots + \beta_{t,K} v_K^\top. \end{aligned}$$

The last step is to forecast arrivals on day $T + 1$. Since there are obvious day of week patterns shown in the data, we include $z_i = 1, \cdots, 7$ as a categorical covariate controlling for the day of week effects in the AR(1) time series model to obtain $\beta_{T+1,k}$, $1 \leq k \leq K$:

$$\beta_{T+1,k} = a_k(z_{T+1}) + b_k \beta_{T,k} + \epsilon_{T+1,k}. \tag{2.17}$$

Next, the arrival vector on day $T + 1$ is modeled as

$$x_{T+1} = \beta_{T+1,1} v_1^\top + \cdots + \beta_{T+1,K} v_K^\top + \epsilon_{T+1}, \tag{2.18}$$

whose mean can be estimated by

$$\hat{x}_{T+1}^{(3)} = \beta_{T+1,1} v_1^\top + \cdots + \beta_{T+1,K} v_K^\top.$$

We use the first semester arrival rates of application 1, 3 and 4 as test data to obtain the scree plots (Shen and Huang, 2008) in Figure 2.6, and find that $K = 2$ or 3 gives good forecasting results.

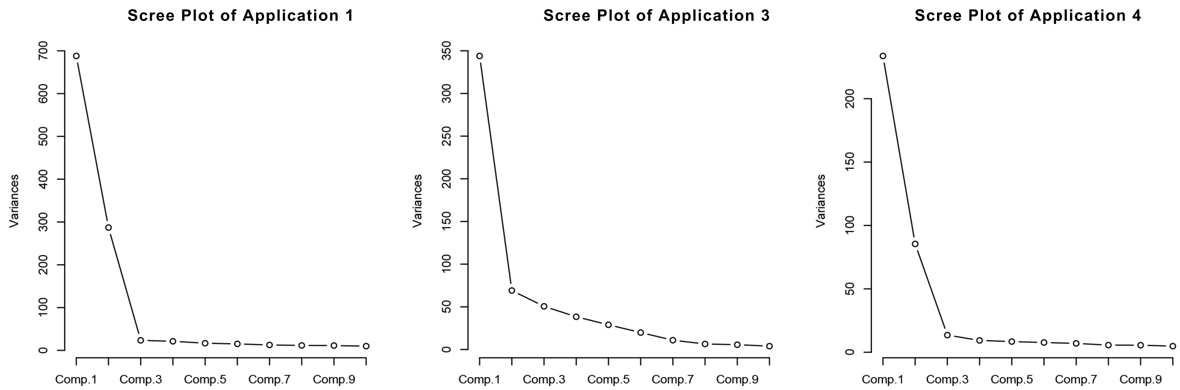


Figure 2.6: Scree Plots

Because the SVD forecasting approach assumes no probabilistic models, Shen and Huang (2008) propose a nonparametric bootstrap method to derive the prediction interval of the forecast. Using our data, we apply the same method by first bootstraping the errors from time series model (2.17), and obtain the series $\{\hat{\beta}^b_{T+1,1}, \cdots, \hat{\beta}^b_{T+1,K}\}$, $1 \leq b \leq B$, where B is the number of bootstrap samples. Then by bootstrapping the errors in the forecasting model (2.18), we obtain B forecasts of x_{T+1}. For an upper bound of the 95% prediction interval, we use the end point of the 97.5% empirical percentiles of $\{x^1_{T+1}, \cdots, x^B_{T+1}\}$, and denote that estimate by $\hat{x}^{(4)}_{T+1}$.

Note that the SVD method involves regression on the decomposed data matrix, which is not efficient when the historical arrival volumes are close to zero. This happens to be the case for applications ranked 100 and lower in terms of total requests, whose hourly arrival rates are usually less than one. We therefore pool the lower ranked applications into few groups, and forecast the aggregated arrival rates within each group. We then split the forecasted group total arrival rates using the Bernoulli splitting rule, and obtain the hourly arrival rates for each individual application.

In the rest of the chapter, we use $\lambda^{(j)}_{n,i}$ to denote the forecasted arrival rate of application n over the ith period using the jth forecasting method illustrated above ($j = 1, 2, 3, 4$ as in the notations $\hat{x}^{(j)}_{T+1}$).

2.6 Numerical Experiment

In this section, we report on the numerical experiments conducted to implement and compare the performance of the dynamic SAP using the three queueing models. The first queueing model ($M/G/c/c$) assumes that the arrival process is Poisson over each period. We use statistical tests from Brown et al. [2005] to validate this assumption, and show that overall there is no evidence to reject the null hypothesis that the arrival process is NHPP with piecewise constant (PC) arrival rates. See Appendix A.1.3 for details. This PC arrival rate function can also be used in the dynamic SAP calculation with the $M(t)/G/\infty$ model.

To implement the dynamic SAP, we forecast arrival rates $\lambda_{n,i}$ using the methods introduced in Section 2.5. We assume that the mean service times $s_{n,i}$ are time-independent and simply take the sample mean s_n for application n as the estimate of $s_{n,i}$ for all the periods. The approaches we use to estimate other parameters such as $\theta_{n,i}$, $p_{n,i}$ of the $M^X/M/\infty$ queue are introduced later.

We simulate the system with the trace data from the VCL to compare the efficacy of all the methods introduced earlier. Each simulation setup can be described by four attributes described below:

Attribute 1. Forecasting Method: MA vs. SVD;

Attribute 2. Forecasting Level: Mean vs. upper bound of 95% prediction interval;

Attribute 3. Estimation Method: SIPP vs. SDPP;

Attribute 4. Queueing Model: $M/G/c/c$ queue vs. $M^X/M/\infty$ queue vs. $M(t)/G/\infty$ queue.

Thus there are a total of $2 \times 2 \times 2 \times 3 = 24$ simulation setups.

As a general parameter setting, we assume the targeted global probability of delay α^* is 5%, and the targeted global blocking probability β^* is 0.1%. There are 400 dedicated pools and one flexible pool. To forecast the arrival rates, we choose the length of rolling horizon $w = 30$ days, and assume that there are $P = 24$ hourly periods in a day. Applying methods described in Section 2.5 for each application, we obtain four different sets of forecasts $\lambda_{n,i}^{(1)}$ and $\lambda_{n,i}^{(2)}$ using the MA method, $\lambda_{n,i}^{(3)}$ and $\lambda_{n,i}^{(4)}$ using the SVD approach ($\lambda_{n,i}^{(1)}$ and $\lambda_{n,i}^{(3)}$ are at the mean forecasting level, and $\lambda_{n,i}^{(2)}$ and $\lambda_{n,i}^{(4)}$ are at the upper bound of 95% prediction interval level). Besides, we estimate the mean service time s_n by the sample mean of all the recorded service times of application n, and we set loading time $u_n = 5$ mins for all the applications.

To determine the number of servers assigned over the ith interval, we let $\tau_i = i - 1$ ($i \geq 1$, in hours), and record the number of type-n users using the dedicated servers ($B_n(i-1)$) and the flexible servers ($F_n(i-1)$) at the end of the earlier period. Then the SIPP method and the SDPP method yield the following estimates:

$$\hat{\lambda}_{n,i} = \lambda_{n,i}^{(j)},$$

$$\hat{\lambda}_{mod(n,i)} = [\lambda_{n,i}^{(j)} + B_n(i-1) + F_n(i-1)]/(1 + s_n).$$

2.6.1 $M/G/c/c$ Queue.

In this case, all of the estimated arrival rates are used directly as the input to the greedy algorithm of Section 2.4.2.1. We record the numbers of users blocked from all of the dedicated pools (overflows) and use them to forecast the arrivals to the flexible pool. The same procedures are used as those applied to the dedicated pools. The simulation results are presented in Figure 2.7.

The three plots on the left show the results using the mean forecasting level, while that right ones use the upper bound of 95% prediction interval. Panels (a) and (b) show the average probability of delay over the 24 hours in a day. We see that using the SVD forecasting method produces lower probabilities, and SDPP outperforms SIPP. In panel (a) we find all of the results are above the target level of $\alpha^* = 5\%$, which means our policies underestimate the number of servers needed. With the upper bound of 95% prediction interval, in panel (b), the two curves using the MA forecasting methods again fail to meet the target level. The other two curves using the SVD forecasting method are very close to, and hit our target 5% during the off-peak hours (3am-8am), but during the peak hours, the probability of delay goes up to 9%.

In Figure 2.7, Panels (c) and (d) plot the average blocking probability over the 24 hours. At the mean forecasting level, we see that policies using SDPP have the best performance, though none of them hit the target $\beta^* = 0.1\%$. In Panel (d), all of the blocking probabilities are much lower, even though only the policy using MA combined with SDPP method frequently hits the target.

We also plot the average number of on servers (dedicated + flexible) over the 24 hours in Figure 2.7(e) and (f). Policies using the SVD forecasting method usually allocate more servers during the off-peak hour, which explains the lower probabilities of delay in (a) and (b). Besides, SDPP methods require less on servers than SIPP, while achieving the similar performance as shown in panel (b). Hence, for this queueing model, considering Attributes 1-3, we recommend the upper bound of 95% prediction interval as forecasting level, SDPP method, combined with SVD for dedicated pools and MA for flexible servers.

2.6.2 $M^X/M/\infty$ Queue.

Let $\sigma_{n,i}^2$ be the variance of the number of arrivals. From Eq. 2.7 and 2.8 we see that

$$p_{n,i} = 2/(\frac{\sigma_{n,i}^2}{\lambda_{n,i}} + 1), \tag{2.19}$$

$$\theta_{n,i} = \lambda_{n,i} p_{n,i},$$

which suggests,

$$r_{n,i} = \lambda_{n,i} s_n p_{n,i}/(1 - p_{n,i}).$$

Note that $Q_d(d_{n,i}, r_{n,i})$ is a function of $r_{n,i}$ and $p_{n,i}$, and hence can be viewed as a function of $\lambda_{n,i}$ and $p_{n,i}$. Again we estimate $\lambda_{n,i}$ using one of the combinations of our forecasting and estimation methods, while we estimate $p_{n,i}$ separately by computing the sample mean and variance of the corresponding hourly arrivals of the preceding 30 days, before plugging it into Eq. 2.19.

Furthermore, for a small fraction of the cases where $p_{n,i} > 0.99$, we use the Erlang-B loss formula to approximate the blocking probability instead of Eq. 2.9 to avoid computational instability. This makes sense since the Poisson distribution is the limiting case of the negative binomial distribution $NB(r, 1-p)$ when $p \to 1$ (or equivalently, as $r \to \infty$).

We collect the historical hourly overflow to the flexible pool and estimate $\lambda_{f,i}$ and $p_{f,i}$ in the same way as for the dedicated pools. These parameters can also be computed using Eq. 2.10 and 2.11.

We do not include here the complete simulation results to compare Attributes 1-3 since they are quite similar to those shown in Figure 2.7. (See Appendix A.1.4 for the corresponding figure.) Again, we find that the best policy uses 95% forecasting level, SDPP, combined with SVD forecasting for the dedicated pools and MA forecasting for the flexible pool. We will present the differences among the various queueing models (Attribute 4) later in Section 2.7, by using the same Attributes 1-3 for all queueing models.

2.6.3 $M(t)/G/\infty$ Queue.

In this case, we first obtain the arrival function $\lambda_{n,i}(t)$ by formulating a piecewise linear function using the hourly estimates $\hat{\lambda}_{n,i}$ (or $\hat{\lambda}_{mod(n,i)}$). Then we approximate the CDF of the service time of each application by its own empirical CDF. Finally we compute the integral in Eq. 2.12 numerically, and obtain $m_n(t)$, the mean number of busy servers of application n at time t. We use the same techniques for the flexible pool as in the case of $M/G/c/c$ queue. We arrive at the same conclusion in comparing Attributes 1-3. See Section 2.7 for the comparison among the three queueing models.

2.7 Recommendations

Combining the results from the 24 simulation setups, we make the following recommendations about the server allocation plan for the VCL:

(1) The SVD forecasting method results in a higher service quality than the MA for the dedicated pools. This can be explained by the time series model used in the SVD method, which include the day of week effects and the correlation within days. However for the more complicated arrival process in the flexible pool, the MA method simplifies the forecasting procedure and achieves better service quality.

(2) We notice the problem of the underestimation of the number of server allocated when we use the average forecasts. We believe the potential sources of this are the non-stationarity and over-dispersion in the arrival data. We therefore use a conservative forecasting level at the upper bound of the 95% prediction interval to allow for the potential variations. The simulated fraction of the users not receiving immediate service (probability of delay) at this level is closer to the target level. The sizing of the flexible pool at this level successfully achieves the target

service level (probability of blocking).

(3) The SDPP approach is highly recommended over the standard SIPP: our results show that SDPP always provides lower probability of delay while using significantly fewer servers (see Panel (f) in Figure 2.7). This is expected since SDPP incorporates the most recent data from the system to update the decision for the next period. Fortunately, SDPP is only slightly more computationally intensive than SIPP.

(4) To compare the three queueing models, we plot their performance in Figure 2.8 using the above recommended approaches: the upper bound of the 95% prediction interval, SDPP, combined with SVD for the dedicated pools and MA for the flexible pool. The figure shows that for the dedicated pools (Panel (a)), the performances of the three queueing models are quite similar. Since the $M/G/c/c$ queue is the most computationally efficient model, we recommend to use this model to quantify the probabilities of delay in the dedicated pools. For the flexible pool (Panel (b)), we recommend the $M^X/M/\infty$ queueing model, which is a good approximation to the $M^X/M/c/c$ queue. Since the flexible pool is essentially accommodating the aggregate overflows from all of the dedicated pools, the over-dispersion issue is more obvious, which is taken care of by the CPP of the $M^X/M/\infty$ queue model. The observations from the simulation results indeed show lower blocking probabilities than those obtained using the other queueing models.

2.8 Summary and Extensions

Motivated by the server allocation problem in the VCL, we model the system using two types of server pools – the dedicated pools where pre-determined types of applications are preloaded on certain dedicated servers, and the flexible pool where different applications are loaded on demand. We formulate an optimization problem to minimize the number of on servers, subject to pre-specified service level constraints. The service level includes the probabilities of a user being delayed or blocked from the system. We construct three queueing models to quantify the service quality constraints, and develop algorithms to identify the corresponding static SAP which is further extended to a dynamic SAP. We extend the traditional SIPP approach and propose a modified versoin SDPP to handle the time-varying demand.

We then consider two methods – MA and SVD – to forecast future arrival rates given historical data, at two forecasting levels: the mean level and the upper bound of the 95% prediction interval level. We evaluate the performance of the dynamic SAP by conducting discrete event simulation experiments. We run statistical tests to justify the assumption of the piecewise constant arrival rate function over the planning horizon.

Overall, our recommended dynamic SAP keeps no more than 300 servers on during the peak hours and about 150 servers on during the off-peak hours, which is a significant saving over the 700-900 servers currently being kept on by the VCL using its policy. Furthermore, under our policy, at least 90% of the users receive immediate service from the dedicated server pools, and 99.9% of them are guaranteed a service from the flexible pool with no more than 5 extra minutes of waiting.

For future work, we are interested in modeling the VCL system using Markovian decision processes, where decisions on the allocation of the servers and the admission of the users can be chosen to optimize a long run cost. In that case, we can allow switching between different types of dedicated servers, or between dedicated servers and flexible servers; we can even allow rejecting a user even when servers are available. Costs of switching servers and rejecting different types of users can also be incorporated.

Figure 2.7: Simulation Results of $M/G/c/c$ Queue

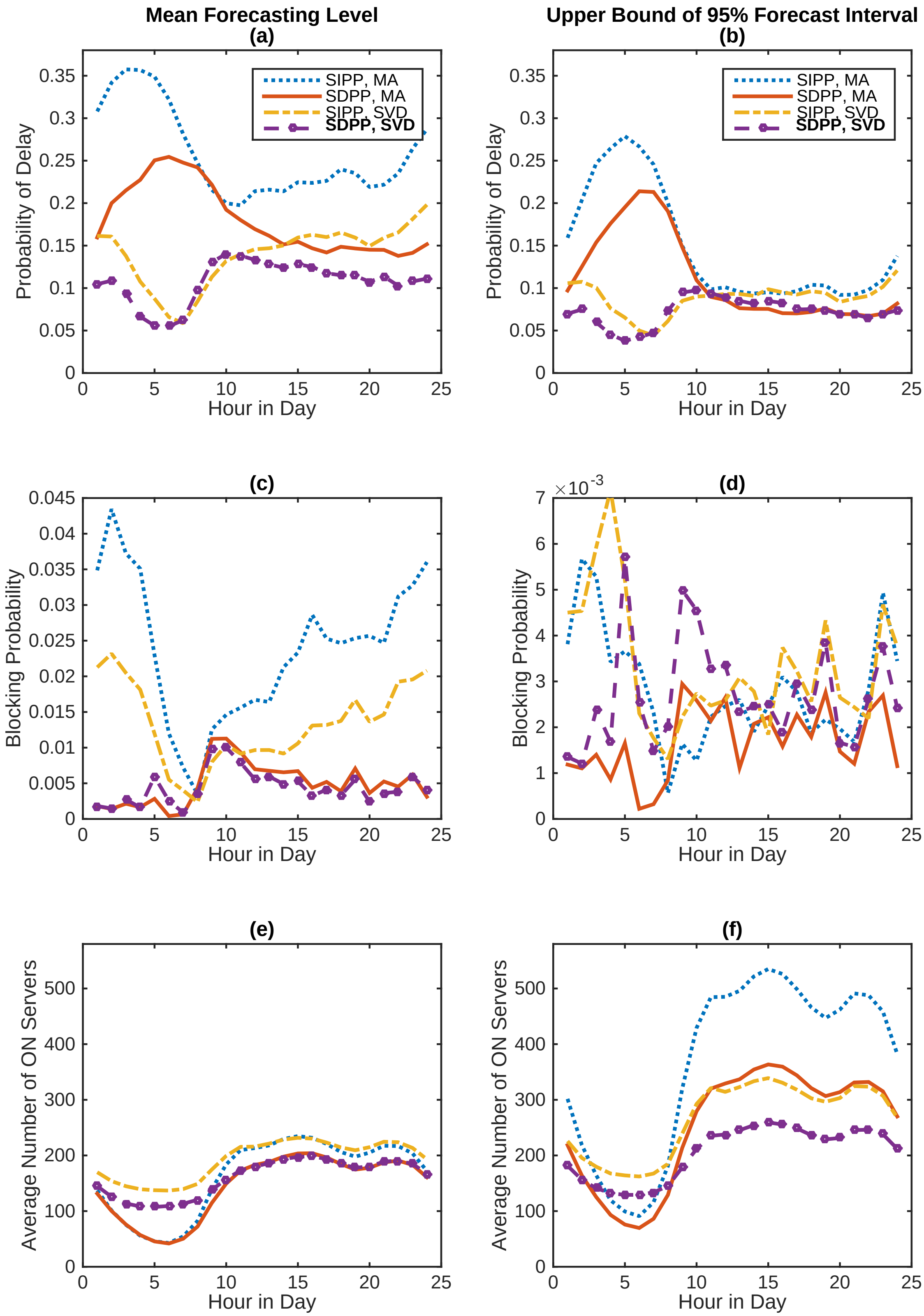
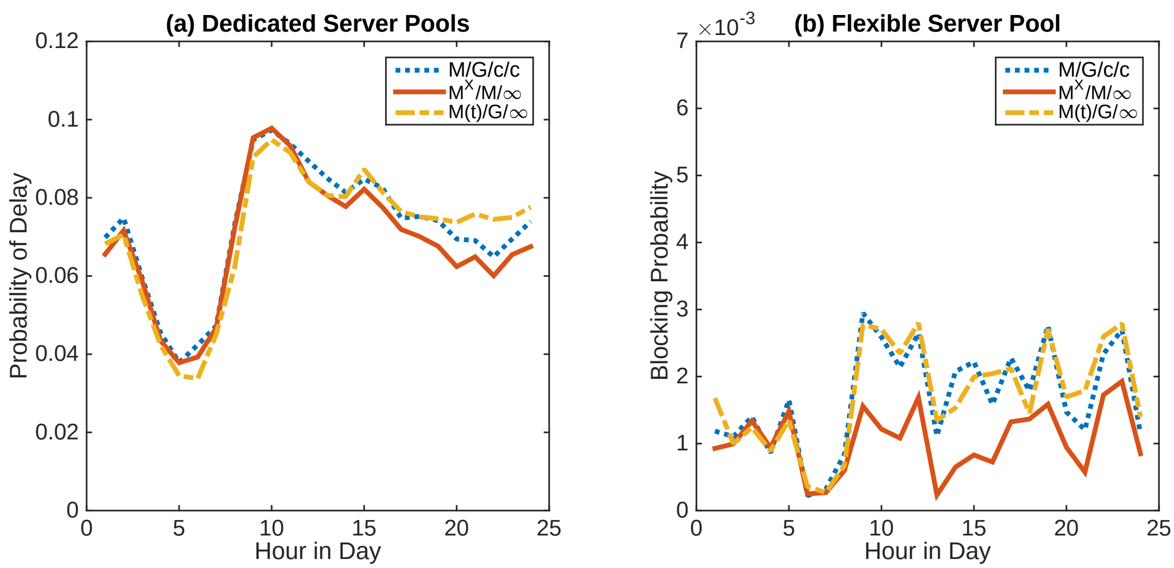


Figure 2.8: Comparison of Three Queueing Models



CHAPTER 3: Staffing and Scheduling for Health Care Facilities with Series Patients: Model I

3.1 Introduction

The United States spends more per capita on healthcare than many other developed countries, and also has one of the highest growth rates on health care spending. The health care sector recently accounted for 17.3% of the GDP. The Institute of Medicine, in a recent report, estimated $750 billion in unnecessary health spending in 2009 alone, of which $130 billion has been attributed to inefficient delivery of care. There is a large opportunity for health care organizations to improve efficiency, quality, and timeliness of delivery of health care.

A healthcare organization's appointment scheduling system can affect both timeliness of access to health services, as well as the efficiency of the healthcare operation. Timely access to a healthcare service is important from a clinical perspective, as well as from a patient satisfaction perspective. Scheduling systems also have the goal of matching demand with capacity in order to utilize resources efficiently. An appointment schedule also regulates the demand, i.e., patients visits, so as to minimize cost of operation.

While there has been significant research in the area of patient scheduling (see Smith-Daniels et al. (1988), Cayirli and Veral (2003), and Gupta and Denton (2008) for excellent surveys), most of prior literature has focused on scheduling patients with single appointments. Repeat visits by patients are usually treated in such models as distinct and independent visits. Our research focuses on determining capacity and scheduling "series" patients. Series patients are patients who need to be scheduled for multiple visits. In our research, we focus on specialty care clinics with such series patients. The issue of series patients arises in several specialty health services such as physical therapy, radiotherapy/chemotherapy for cancer, kidney dialysis, diabetes treatment, orthodontic treatment (braces), etc.

Our research is motivated by our interaction with a large health network in Indiana. In many specialty care clinics such as physical therapy, patients at the time of admission are scheduled for repeated visits. At the beginning of the day, a batch of patient referrals is delivered from hospitals to a specialty care clinic. A clinic administrator contacts each patient and tries to schedule a first appointment for an initial examination. The health network with whom we interacted with requires that an initial evaluation examination should be scheduled within 48 hours from referral. In the case that there is appointment slot available for a new patient in this time window, the patient's first examination is scheduled. Once he or she is evaluated, the plan of care for the patient is specified by a physician. The plan of care determines days between subsequent visits, total number of visits, and duration of each visit. Based on the plan of care, an administrator in the clinic generates an appointment schedule. The frequency, duration and length of the appointment time for the follow up visits vary greatly depending upon the patient's diagnosis and/or specific needs as well as the physician order. Follow up visits are to be scheduled in a timely manner as well (2-3 days after the evaluation).

The health network we interacted with is currently unable to appropriately and consistently match supply and demand of physical therapy services. Each new patient creates a string of follow up visits. The follow up visits absorb a lot of the capacity on the schedule, making it difficult to get new patients in within the 24-48 hour required time frame at this health network. Thus, they are not able to consistently get new patients scheduled within 24-48 hours of referral. Many times, they are not able to consistently offer timely follow up appointments to their patients.

The primary goal of a our work is to devise a model for matching supply and demand through scheduling, as well as determining capacity, for a health service area that has series patients. Our model determines appropriate capacity to meet the demand of new patients (in the required time frame) as well as recurring visits (in the required time frame). We incorporate both revenue considerations as well as regular and overtime staffing costs in our model.

The key contributions of this work are as follows: This is one of the first work that provides an analytical characterization of a health care facility with *series* patients to the best of our knowledge. This chapter focuses on determining the staffing and scheduling for healthcare facilities with series patients who need to be scheduled for multiple visits. We use a newsvendor type model for the optimal staffing problem, and show that the optimal number of service slots scheduled each day satisfies a simple equation. The series patients' scheduling problem is formulated as an MDP model. A key analytical contribution is to prove it to be uni-chain. This yields the existence of an optimal policy that maximizes the long run average profit, and also an algorithm to compute this optimal policy. Due to the large state space, instead of solving for the optimal policy, we propose several implementable policies. To test the effectiveness of our proposed policies in a real-world setting, we use data from a local PT center to compare the performance of the proposed policies. We find that the Index policy provides the best improvement in the average profit and requires minimal effort in implementing, and hence, is recommended for practice.

One critical assumption we make in this chapter is that we know the exact number of visits a patient will need before we do the scheduling. We call this Model I. In the next chapter we shall study Model II, where we do not know the number of visits before scheduling. Model I is appropriate when the patients arrive with a recommendation that they be seen for a fixed number of visits (which may vary from patient to patient). This is commonly the case when the patients are seeking the treatment for work related injury and the number of visits is proscribed by labor regulations. It is also common in many types of chemotherapy where the patient is scheduled for a fixed number of repeat visits, say three or six. Model II is appropriate when the number of visits of a patient depends upon the progress the patients shows as a result of a treatment, and hence the number is a random variable whose distribution is known, but the realized value is not known.

The remainder of this chapter is organized as follows. Section 3.2 reviews the relevant literature. In Section 3.3, we introduce the structure of the model with series patients, and describe the problems involved with staffing and scheduling policy. In Section 3.4, the staffing model is formulated as a newsvendor type problem, and in Section 3.5 the scheduling model is formulated as an MDP. We then propose several implementable policies in Section 3.6, where some properties of the MDP models are proved and the procedures of the MDP based policies are given in full detail. Using the data from local physical therapy (PT) center, we conduct numerical experiments and compare the performance of different policies via simulation in Section 3.7. A conclusion and future extensions of our work are provided in Section 3.8. All proofs are available in the Appendices.

3.2 Literature Review

The operations research literature on healthcare management on facility staffing and appointment scheduling is extensive. Cayirli and Veral [2003] provides a comprehensive survey of research on appointment scheduling in outpatient services. In the review paper by Gupta and Denton (2008), the most common types of health care delivery systems are described. They also discuss the appointment scheduling issues that can arise due to the impacts of arrival and service time variability, patient and provider preferences, available information technology, and so on. For the review of the related field of network revenue management, see Talluri and Van Ryzin [2006] and Chiang et al. [2006].

Typically the problems of health care system staffing and scheduling have been studied separately. From the staffing perspective, we are particularly interested in using newsvendor type models considering the uncertain demands and limited capacity. In a very complete work on nurse staffing problem by Kao and Queyranne [1985], a single period, time-varying demand model is solved as a newsvendor type problem. To solve the problem of determining nurse staffing levels in a hospital environment, Green et al. [1991] combined the empirical investigation of the factors affecting nurse absenteeism rates with an analytical treatment of nurse staffing decisions using a variant of the newsvendor model. Davis et al. [2014] applied asymmetric cost functions representing overstaffing and understaffing nursing costs. In our staffing model, we use a variation of newsvendor model for daily profit, considering an upper bound on overtime slots. When it comes to the sizing of inpatient care units, most of the above literature assumes that a patient requires a single medical or nursing unit throughout the entire stay. We make the same assumption that all the series patients require a single service slot per visit. See section 3.4. Health care facility staffing and scheduling problems under patients no-shows or cancellations are studied in Luo et al. [2012]; Feldman et al. [2014]; Qu et al. [2007]; Green and Savin [2008], etc. We do not consider patient no-shows/cancellations in our model

From the scheduling perspective, the problem is generally more complicated, since we need to think both from a patient perspective (demand) with concerns of quality and time of service, and from a hospital perspective (supply) with concerns of revenue and cost. Luo et al. [2015] modeled the appointment system by two tandem queues, and derived performance measures such as service utilization and customer long-run average waiting times in both queues. Kolesar [1970] investigated Markov chains and linear programming to determine number of admissions to schedule to maximize average occupancy or minimize overflows. One popular topic discussed in scheduling problems is patient classification. Different studies used different methods of classification for scheduling purposes such as new/return, variability of service times, and type of treatment procedure. These factors are studied in Klassen and Rohleder [1996]; Rohleder and Klassen [2000], and Bosch and Dietz [2000]. In our scheduling model, the patients will be classified by the number of total appointments needed, and the algorithm of scheduling suggests that it is better to always make the scheduling decision for the patients with the least appointments requests first.

Moreover, there is also a significant body of literature integrating staffing and scheduling problems together. For example, Abernathy et al. [1973] developed an integrated model for staffing and scheduling nursing units using stochastic programming with chance constraints for demand uncertainty and service level objectives. Using a linear programming model, Ittig [1978] demonstrated how to decide the number of physicians of each type, patient group-physician assignments, and population acceptance rate to maximize services provided subject to operating budget and physician supply. Hancock et al. [1978] studied the interrelationships between admissions scheduling, hospital occupancy, and facility size with a simulation model that assumed Poisson emergency arrival rates. Rath et al. [2015] analyze the joint staffing and scheduling decision for surgery operating rooms. In this chapter, we first propose the staffing policy and scheduling policy separately, then we integrate the two parts in the numerical experiments.

3.3 The Model

In this section we describe the model of a health care clinic with series patients. New patients call ahead to make a series of appointments. Let B_n be the number of new arrivals (calls) on day n, and let V_i be the number of repeat visits needed by the ith patient. We shall assume that $\{B_n, n \geq 0\}$ are independent and identically distributed (i.i.d.) random variables with common probability mass function (pmf)

$$\phi_j = P(B_n = j), \quad j \geq 0, \tag{3.1}$$

and $\{V_i, i \geq 1\}$ is a sequence of i.i.d. random variables with common pmf

$$p_k = P(V_i = k), \quad k \geq 1. \tag{3.2}$$

and common complementary cdf (cumulative distribution function)

$$f_k = P(V_i \geq k) = \sum_{j=k}^{\infty} p_j, \quad k \geq 1.$$

Furthermore, $\{V_i, i \geq 1\}$ are independent of $\{B_n, n \geq 0\}$.

Let $B_{n,k}$ be the number of new arrivals on day n who require exactly k repeat visits. Then, given $B_n = b$, we see that $[B_{n,1}, B_{n,2}, B_{n,3} \cdots]$ is a multinomial random variable with parameters b and $p = [p_1, p_2, p_3, \cdots]$, that is,

$$P\left([B_{n,1}, B_{n,2}, B_{n,3} \cdots] = [b_1, b_2, b_3, \cdots] \mid B_n = b\right) \; = \; b! \prod_{k=1}^{\infty} \frac{p_k^{b_k}}{b_k!}, \tag{3.3}$$

if $\sum b_k = b$, and zero otherwise. We shall find it convenient to introduce the following notation to describe the arrival stream. Let

$$A_{n,k} = \sum_{i=k}^{\infty} B_{n,i}, \quad k \geq 1$$

be number of new arrivals on day n who need *at least* k repeat visits ($k = 1, 2, \ldots; n = 1, 2, \cdots$). Let

$$A_n = [A_{n,1}, A_{n,2}, A_{n,3}, \cdots].$$

Thus $\{A_n, n \geq 0\}$ is a sequence of i.i.d. random vectors, taking values in $\mathcal{A}$ given by

$$\mathcal{A} = \{a = [a_1, a_2, a_3, \cdots] : a_k \geq a_{k+1}, \; a_k \in \mathbb{Z}^+ \text{ for } k \geq 1\}, \tag{3.4}$$

where $\mathbb{Z}^+$ as the set of non-negative integers. From the definition of $A_{n,k}$, it is obvious that $A_{n,1} = B_n$ is the total number of arrivals on day n. Furthermore, from Equation 3.3, we get

$$A_{n,k} - A_{n,k+1} \sim \mathcal{B}(B_n, p_k),$$

and

$$A_{n,k} \sim \mathcal{B}(B_n, f_k),$$

where $\mathcal{B}(m,p)$ denotes the binomial distribution with parameters m and p.

When a patient arrives (calls) on day n, he/she has to be given the first appointment on day $m \in \{n+1, n+2, \cdots, n+T\}$, where T is a given positive integer, and if he/she needs k repeat visits, they are scheduled on days $m, m+T, m+2T, \cdots, m+(k-1)T$. Thus there are no same day appointments and no walk-ins. Also, once the day of the first appointment for a patient is decided, it automatically fixes the entire appointment schedule for that patient for all her repeat visits. The rule that dictates how the first appointment is scheduled is called the scheduling policy.

Let D_n be the total number of patients that are scheduled to be seen on day n. We assume that each of these patients requires exactly one one-hour appointment (called the slot). Thus D_n is the demand for the number of slots on day n, which depends on the scheduling policy. The detail of scheduling model will be given in Section 3.5.

Next we describe the staffing policy. We shall assume that the clinic has a fixed number of permanent employees that together can offer q slots on each day, and this cannot be changed dynamically in response to the daily demand. The clinic does have the flexibility of asking the employees to put in overtime to handle the random variations in demand for the slots. However, industry regulations specify an upper bound on overtime. For example, one cannot schedule more that 15% overtime for a physical therapist. We introduce a parameter $\gamma \geq 1$ with the following interpretation: if the clinic has q slots from its regular employees, it can get a total of γq (regular + overtime) slots from them. Thus for a physical therapy clinic we have $\gamma = 1.15$. The staffing policy decides the q. We refer to Section 3.4 for the detail our staffing model.

Clearly, both the optimal staffing and scheduling policies depend on the costs and revenues involved. Hence we now describe the cost structure. Let c be the cost of offering one regular slot. Note that we always incur a fixed cost cq per day, regardless of how many regular slots are actually used to serve the patients. Let p be the revenue from one appointment. Every patient served in the overtime slot costs $c' > c$. If the total demand for the day exceeds the total slots γq, the excess demand (called the overflow demand) produces no net revenue (that is, the cost of serving one patient from the overflow demand is p, the same as the revenue). We also assume that there are delay costs incurred as follows: if a new customer arriving on day n is given her first appointment on day $n + t$ $(1 \le t \le T)$ it costs $c_d(t)$. We assume that

$$0 = c_d(1) \le c_d(2) \le \cdots \le c_d(T).$$

This delay cost function can be used to represent many realistic scenarios, including the healthcare facility that motivated our problem context. Let $p(q, \pi)$ be the long run net profit per day (assuming it exists) if staffing level is q and scheduling policy is π. The goal is to find the optimal (staffing level, scheduling policy) pair (q^*, π^*) (assuming it exists) such that

$$p(q^*, \pi^*) \ge p(q, \pi)$$

for all pairs (q, π). This is an intractable problem to solve in one go. Hence in Section 3.4 we assume that π is fixed and solve for the optimal staffing level $q^* = q^*(\pi)$ such that

$$p(q^*, \pi) \ge p(q, \pi)$$

for all staffing levels q. Then, in Section 3.5, we assume that q is fixed and solve for the optimal scheduling policy $\pi^* = \pi^*(q)$ such that

$$p(q, \pi^*) \ge p(q, \pi)$$

for all scheduling policies π. The problem of finding the optimal staffing and scheduling simultaneously is discussed in Section 3.7.

3.4 The Optimal Staffing Policy

In this section we describe a newsvendor type formulation to determine the optimal staffing policy q, assuming that a given scheduling policy π is followed. Let D_n be the demand on day n generated by this policy. In Section 3.5 we shall describe in detail how D_n depends on the scheduling policy π. Let F be the steady state distribution of D_n as $n \to \infty$. Clearly F will depend on the scheduling policy. Let D be a generic random variable with cdf F. We shall choose q to maximize the expected daily profit in steady state.

Note that we always incur a fixed cost cq, regardless of how many regular slots are actually used. When the $q < D \leq \gamma q$, we satisfy the $D - q$ demands by using the overtime slots. If $D > \gamma q$, we do not get any revenue from that overflow demand. Let $\rho(D, q)$ be the net daily profit if the staffing level is q, and (random) demand is D. Based on the above description, we see that, $\rho(D, q)$ is given by

$$\rho(D, q) = p \min(D, \gamma q) - cq - c' \max(\min(D, \gamma q) - q, 0)). \tag{3.5}$$

In Figure 3.1, we see that the marginal profit of serving one slot is p when $D < q$. Then the marginal profit decreases by c' when $q \leq D < \gamma q$, since we are using overtime slots and this will cost c'. After the demand exceeds the total capacity $D > \gamma q$, we see that there is no marginal gains by admitting more patients. This case can be thought as those patients are referred to other clinic due the limited capacity.

Let the expected daily profit be denoted by:

$$G(q) = E(\rho(D, q)) = E\left(p \min(D, \gamma q) - cq - c' \max(\min(D, \gamma q) - q, 0)\right).$$

Note that

$$p(q, \pi) = G(q) - \text{expected delay cost under } \pi.$$

Since the delay cost does not depend on q, finding the q that maximizes $p(q, \pi)$ is the same as finding the q that maximizes $G(q)$. We proceed to that next.

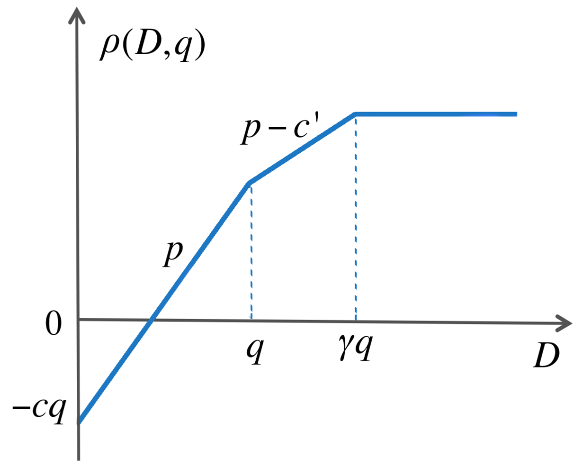


Figure 3.1: Daily Net Profit as Function of D

Using the cdf F of D, $G(q)$ can be expressed explicitly as follows:

$$G(q) = p\int_0^{\gamma q}(1 - F(x))\,dx - cq - c'\int_q^{\gamma q}(1 - F(x))\,dx. \tag{3.6}$$

The optimal staffing level q^* is the value of q that maximizes $G(q)$. Although D and q are integer valued, it is easier to treat them as continuous quantities to obtain the results about q^* as in the theorem below (see proof in Appendix A.2.1).

Theorem 3.1. Assume the pdf of D exists and $p \geq c' \geq c$, then profit function $G(\cdot)$ is concave. Then $G(\cdot)$ is maximized at q^*, where q^* be the unique solution to

$$\frac{(p-c')\gamma}{(p-c')\gamma + c'}F(\gamma q) + \frac{c'}{(p-c')\gamma + c'}F(q) = \zeta, \tag{3.7}$$

where

$$\zeta = \frac{(p-c')\gamma + c' - c}{(p-c')\gamma + c'}. \tag{3.8}$$

Furthermore, q^* satisfies

$$\frac{F^{-1}(\zeta)}{\gamma} \leq q^* \leq F^{-1}(\zeta). \tag{3.9}$$

The hard part in finding q^* is the computation of the steady state demand distribution F. In Section 3.6 we shall study a class of policies for which this is relatively easy to do.

Remark 3.1. Since q is an integer, we can rewrite Equation 3.9 as

$$\lceil \frac{F^{-1}(\zeta) - \gamma + 1}{\gamma} \rceil \leq q^* \leq \lfloor F^{-1}(\zeta) \rfloor.$$

Remark 3.2. When $\gamma = 1$ (that is, no over time allowed), the result in Theorem 3.1 reduces to the standard newsvendor formula.

Remark 3.3. Let the expectation of demand be μ. We can show that if $\zeta < F(\mu)$ then $q^* < \mu$, that is, the optimal number of slots assigned each day is less then the expected daily demand. On the other hand, if we have $\zeta > F(\gamma\mu)$, then $q^* > \mu$, which means we need more slots then the expected demand.

3.5 Optimal Scheduling Policies

In this section we assume that the staffing level q is fixed. Our aim is to find a policy that maximizes the long run expected net profit per day. We do this by formulating the problem as a discrete time Markov decision process (MDP).

The state of the appointment schedule at the beginning of day $n \geq 0$ is described by a matrix $X_n = [X_n(k,t)]$, where $X_n(k,t)$ is the number of patients who have appointments (not necessarily their first) on day $n+t$, and have *at least* k visits remaining. X_n takes values in the set $\mathcal{X}$:

$$\mathcal{X} = \left\{ x = [x(k,t)] : x(k,t) \geq x(k+1,t),\ x(k,t) \in \mathbb{Z}^+ \text{ for } k \geq 1;\ t = 1, \ldots, T \right\}. \tag{3.10}$$

Let A_n be the arrival vector during day n. Note that in practice, the scheduling decisions are made for each new patient at the time of her arrival, depending on the state of the appointment schedule at that time. This, unfortunately, leads to more complicated models. Hence we simplify the situation by assuming that the entire A_n vector is observable at the beginning of day n (every new patient calls early in the morning) and we decide their schedule based on X_n and A_n. This creates a tractable model and it leads to policies that can be implemented on a patient by patient basis. Thus the scheduling problem reduces to deciding how many of the $A_{n,k}$ new patients should be given their first appointments on day $n+t$, for $1 \leq t \leq T$.

Accordingly, we define $Y_n(k,t)$ as the number of patients who arrived on day n, asked for *at least* k repeat visits, and are given their first appointment on day $n+t$. Let $Y_n = [Y_n(k,t)]$ be the matrix representing the scheduling action taken on day n. For the given realization $a \in \mathcal{A}$, let

$$\Delta(a) = \{y \in \mathcal{X} :\ ye = a\},$$

where e is a column vector of all ones. Clearly, given $A_n = a$, we have $Y_n \in \Delta(a)$.

The state of the system at the beginning of day n is the combination of scheduling state and the arrival vector, namely, (X_n, A_n). The decision on day n is $Y_n \in \Delta(a)$ given that $A_n = a$. Next we describe the transition of state under cyclic scheduling.

For $z \in \mathcal{X}$, define the matrix $T_z = [T_z(k,t)]$ as follows:

$$T_z(k,t) = \begin{cases} z(k,t+1) & \text{if } t = 1,\dots,T-1,\ k \geq 1; \\ z(k+1,1) & \text{if } t = T, k \geq 1. \end{cases}$$

We see that, given $X_n = x$ and $Y_n = y$, the state of the scheduling on day $n+1$ is given by

$$X_{n+1} = T_{x+y}.$$

To compute the immediate reward, we first need to know the demand for day n, which is the number of appointments scheduled on day $n+1$, and is given by

$$D_n = X_{n-1}(1,1) + Y_{n-1}(1,1), \quad n \geq 1. \tag{3.11}$$

Note that demand on day 0 is undefined, and is not given by the initial state X_0. The immediate reward for day n, given the state $X_n = x$, and the decision $Y_n = y$, is given by:

$$r(x,y) = \rho(x(1,1) + y(1,1), q) - \sum_{t=1}^{T} c_d(t) y(1,t), \tag{3.12}$$

where the first term gives revenue minus the staffing cost using the function ρ from Equation 3.5, and the second term gives the delay cost incurred from assigning $y(1,t)$ number of new arrivals their first appointment t days later. Note that we suppress the dependence of $r(x,y)$ on q to simplify the notation.

Since $\{A_n, n \geq 0\}$ are iid vectors, we see from the above description that $\{((X_n, A_n), Y_n), n \geq 0\}$ is a discrete time MDP. The objective is to maximize the long-run net profit per day. Let π be a stationary policy, and the long-run average profit under π is given by (assuming the limit exists)

$$p^\pi(x,a) = \lim_{n\to\infty} \frac{1}{n} E^\pi \left[\sum_{j=1}^{n} r(X_j, Y_j) | X_1 = x, A_1 = a \right].$$

Let

$$p^*(x,a) = \sup_\pi p^\pi(x,a), \quad \forall (x,a) \in \mathcal{X} \times \mathcal{A},$$

where the supremum is taken among all stationary policies. We call π^* the optimal policy if it achieves the above supremum:

$$p^{\pi^*}(x,a) = p^*(x,a), \quad \forall (x,a) \in \mathcal{X} \times \mathcal{A}.$$

Next, we restate the known results from Tijms (2004) as to when such an optimal policy exists and how to obtain it. It is known that an average profit optimal policy exists if there is a constant g and a function $v : \mathcal{X} \times \mathcal{A} \to (-\infty, \infty)$ satisfying the following equation, called the Bellman equation:

$$v(x,a) + g = \max_{y \in \Delta(a)} \{r(x,y) + E(v(T_{x+y}, A))\}, \quad (x,a) \in \mathcal{X} \times \mathcal{A}. \tag{3.13}$$

If there exists a solution to the above equation, the $p^*(x,a) = g$ for all states (x,a). Let

$$S(x,a) = \operatorname*{argmax}_{y \in \Delta(a)} \{r(x,y) + E(v(T_{x+y}, A))\}, \quad (x,a) \in \mathcal{X} \times \mathcal{A}. \tag{3.14}$$

Then the policy that chooses any decision from $S(x,a)$ in state (x,a) is optimal. We next state an important property of our MDP (see proof in Appendix A.2.2).

Theorem 3.2. *Suppose*

$$\phi_0 = P(B_n = 0) > 0.$$

Then the MDP $\{((X_n, A_n), Y_n), n \geq 0\}$ *is uni-chain and aperiodic.*

It is known that if an MDP is unichain, there exists a solution to the Bellman equation, and hence the optimal average profit policy exists. It can be numerically computed as follows:

Let $v^0(x,a) = 0$ for all $(x,a) \in \mathcal{X} \times \mathcal{A}$, and

$$v^n(x,a) = \max_{y \in \Delta(a)} \left\{ r(x,y) + E(v^{n-1}(T_{x+y}, A)) \right\}, \quad (x,a) \in \mathcal{X} \times \mathcal{A}, \;\; n \geq 1.$$

Here the expectation on the right hand side is taken with respect to A which is a generic random vector representing the new arrivals on a day. Next define

$$m_n = \min\{|v^n(x,a) - v^{n-1}(x,a)| : (x,a) \in \mathcal{X} \times \mathcal{A}\},$$

$$M_n = \max\{|v^n(x,a) - v^{n-1}(x,a)| : (x,a) \in \mathcal{X} \times \mathcal{A}\}.$$

Then it can be shown that

$$v^n(x,a) - ng \to v(x,a), \quad (x,a) \in \mathcal{X} \times \mathcal{A} \tag{3.15}$$

and

$$m_n \leq g \leq M_n.$$

These results yield a simple algorithm than can be used to numerically compute the optimal average profit g with a margin of error bounded by an $\epsilon > 0$, and the optimal policy π^*. It terminates in a finite number of steps if the state space is finite.

3.5.1 Structural Properties of the Optimal Policy

In this subsection we study the structural properties of the optimal policy. We ignore the integrality constraints inherent in the above formulation to simplify our results. So we abuse the notation in this subsection and use $\mathcal{X}$ and $\mathcal{A}$ as given by the Equation 3.4 and Equation 3.10 without the integrality constraints.

Recall that the immediate reward in Equation 3.12 can be written as

$$r(x,y) = R(x+y) + f(y)$$

where $R(x+y) = \rho(x(1,1)+y(1,1),q)$ is concave in $x+y$, and $f(y) = -\sum_{t=1}^{T} c_d(t)y(1,t)$ is linear in y.

We can further simplify the notation by letting $g^n(x+y) = E(v^{n-1}(T_{x+y}, A))$. Then in step 2 of the algorithm, we are in fact solving the following optimization problem $P(n)$:

$$P(n): \quad v^n(x,a) = \max_y \quad \{g^n(x+y) + R(x+y) + f(y)\}$$

$$\text{s.t.} \quad y \in \mathcal{X}, \tag{3.16}$$

$$ye = a \tag{3.17}$$

Note that the constraint 3.17 is linear in y, so the feasible set given by 3.16 and 3.17 is convex, by which we can establish the concavity of v^n for any $n \geq 0$ below (see proof in Appendix A.2.3):

Theorem 3.3. For any $n \geq 0$ and a given arrival vector $a \in \mathcal{A}$, the value function $v^n(x,a)$ is concave in $x \in \mathcal{X}$.

It follows from Equation 3.15 that $v(x,a)$ is concave in x. We have stated earlier that a stationary Markovian policy π^* that chooses an action $y(x,a) \in S(x,a)$ in state (x,a) is an optimal policy. The next theorem gives the structural properties of π^* that follow from the concavity of v. In particular it gives a result about how an optimal solution in state (x,a) changes if the state (x,a) is perturbed to $(x+h,a)$ (see prove in Appendix A.2.4).

Theorem 3.4. Let $x_1 \in \mathcal{X}$, $a \in \mathcal{A}$ and $y_1 \in S(x_1,a)$. Let h satisfy:

(a) $x_2 = x_1 + h \in \mathcal{X}$, and

(b) $y_1 \pm h \in \mathcal{X}$, and

(c) $he = 0$.

Then there exists an optimal decision $y_2 \in S(x_2,a)$ satisfying

$$y_2 = y_1 - h. \tag{3.18}$$

Note that we consider the decision variable in the equivalent optimization problem to be $x+y$ as a whole, and show that under certain conditions, the change in x will not affect the optimal $x+y$. For example, consider h with $h(1,t) = +1,\ h(1,s) = -1$ for some $s, t \in \{1, \cdots, T\}$ (and all the other components of h are zero). Then x_2 is obtained by shifting one of the patients on the schedule x_1 from day s to day t. If we also have $y_1(1,t) > 0,\ y_1(1,s) > 0$ then Theorem 3.4 implies that $y_2(1,t) = y_1(1,t) - 1,\ y_2(1,s) = y_1(1,s) + 1$, that is the optimal policy will assign one less patient on day t but one more on day s to keep $x_1 + y_1 = x_2 + y_2$.

Remark 3.4. In the Theorem 3.4, we assume the change only happens in x, while the arrival vector a does not change. The next theorem gives a result about how the optimal solution changes if the state changes from (x, a) to $(x + h, b)$.

Theorem 3.5. Let $x_1 \in \mathcal{X}$, $a \in \mathcal{A}$ and $y_1 \in S(x_1, a)$. Let h satisfy:

(a) $x_2 = x_1 + h \in \mathcal{X}$, and

(b) $y_1 \pm h \in \mathcal{X}$, and

(c') $b = a - he \in \mathcal{A}$.

Then there exists an optimal decision $y_2 \in S(x_2, b)$ satisfying

$$y_2 = y_1 - h.$$

As an example, denote $a = [a_1, a_2 \cdots]$, and $b = [b_1, b_2 \cdots]$. Let $b_1 = a_1 - 1$ while $h(1,t) = +1$ for some t (and all the other components of h are zero), that is we have one less of arrival but one more appointment scheduled on day t. Again if $y_1(1,t) > 0$, then by the above conclusion, the optimal decision with state x_2 will simply be achieved by assigning one less patient on day t, that is $y_2(1,t) = y_1(1,t) - 1$ and $x_1 + y_1 = x_2 + y_2$ still holds.

It is clear that it is not feasible to derive the ϵ-optimal policy π for any realistically sized problem, since the size of $\mathcal{X} \times \mathcal{A}$ is infinite. Even if we make it finite by assuming no patient requires more than K number of repeat visits, and we never keep more than M patients on the schedule, the size is still too huge to compute the policy numerically, for reasonable values of T, K, N and M. For example, even if each patient visits the facility no more than twice and if the schedule repeats every five days, it would still require a solution to a 10 dimensional problem. Hence, in the following section, we study several heuristic policies that are easy to implement in practice. This section derived structural results which gave insights on the properties of the optimal policy. Many of the implementable heuristic policies studied in the next section are also shown to have the same structural properties.

3.6 Implementable Policies

In this section, we describe several implementable policies including Next Day policy (as a special case of Randomized policy), Shortest Queue policy, Max-Marginal Profit policy, Min-cost Flow policy, and Index policy. Except Index policy, the other four policies share the same structural property of the optimal policy, and use increasingly more information regarding the system state and the future demand in decision making. Both Min-cost Flow policy and Index policy are based on our MDP formulation, and Index policy leads to a good approximation to Min-cost Flow policy while requiring less computation efforts.

3.6.1 Randomized policy (RP)

We start with a baseline policy that schedules the first appointment of every new patient arriving on day n to day $n+t$ $(1 \leq t \leq T)$ with probability of β_t, where $\sum_{t=1}^{T} \beta_t = 1$. This is done independently of the arrivals and the current state of the appointment schedule. Under this policy, given an arrival vector A_n, we have

$$Y_n(k,t) \sim \mathcal{B}(A_{n,k}, \beta_t), \quad \text{for } \; t = 1, \cdots, T; \; k \geq 1,$$

as the allocation decision made for day $n \geq 0$. Since $A_{n,k} \sim \mathcal{B}(B_n, f_k)$, it can be shown that

$$Y_n(k,t) \sim \mathcal{B}(B_n, f_k \beta_t), \quad \text{for } \; t = 1, \cdots, T; \; k \geq 1. \tag{3.19}$$

We begin by evaluating the distribution of the demand D_n under RP (see Equation 3.11). The main result is given in the following theorem. First we introduce some relevant notation. For any $n \geq 1$, there is a unique $t = t_n \in \{1, 2, \cdots, T\}$ and $k = k_n \geq 1$ such that

$$n = (k-1)T + t.$$

They are given by

$$k_n = \lceil n/T \rceil, \quad t_n = n - (k_n - 1)T.$$

We also find the following set notation useful:

$$S_n = \{(k,t) : k \geq 1, 1 \leq t \leq T, (k-1)T + t \leq n\}.$$

Theorem 3.6. Suppose $X_0 = x$ and let D_n be the demand on day n under the RP with parameters $[\beta_1, \beta_2, \cdots, \beta_T]$. Let

$$\Phi(z) = E(z^{B_n})$$

be the generating function of B_n. Then

$$D_n = x(k_n, t_n) + Z_n, \quad n \geq 1, \tag{3.20}$$

where the random variable Z_n has the generating function given below:

$$E(z^{Z_n}) = \prod_{(k,t) \in S_n} \Phi(z f_k \beta_t - f_k \beta_t + 1), \tag{3.21}$$

Corollary 3.1. Suppose $\{B_n, n \geq 0\}$ are i.i.d. random variables with mean λ. Also assume $\{V_i, i \geq 1\}$ are i.i.d. random variables with mean m. If the number of patients on schedule on day 0 is finite, we have

$$\lim_{n \to \infty} E(D_n | X_0 = x) = \lim_{n \to \infty} E(Z_n) = \lambda m.$$

This is to be expected: the expected number of new arrivals in a day is λ, and each of them requires m repeat visits on average. Hence the expected total number of new slots requested by all the new arrivals in a day is λm. It holds under all the stationary policies that the expected demand for appointment slots per day in steady state must be λm.

Corollary 3.2. Suppose $\{B_n, n \geq 0\}$ are i.i.d. Poisson random variables with parameter λ, denoted as $\mathcal{P}(\lambda)$, and the $\{V_i, i \geq 1\}$ are i.i.d. random variables with mean m. Let

$$\lambda_n = \lambda \sum_{(k,t) \in S_n} f_k \beta_t. \tag{3.22}$$

Then

$$Z_n \sim \mathcal{P}(\lambda_n).$$

Furthermore, if there are a finite number of patients on schedule on day 0, Z_n, and D_n converge to a $\mathcal{P}(\lambda m)$ random variable as $n \to \infty$.

As a special case, we now introduce the next day (ND) policy under which all of the new patients arriving on day n are given their first appointments on day $n+1$. It is obtained from RP by setting $\beta_1 = 1$. Under ND policy, we have

$$Y_n(k,t) = \begin{cases} A_{n,k}, & \text{if } \ t = 1; \\ 0, & \text{if } \ 2 \le t \le T. \end{cases}$$

From Theorem 3.6 we see that

$$E(z^{Z_n}) = \prod_{k=1}^{k_n} \Phi(z f_k - f_k + 1).$$

In particular, if the daily arrivals are i.i.d. $\mathcal{P}(\lambda)$, we can use Corollary 3.2 to see that

$$D_n = x(k_n, t_n) + \mathcal{P}(\lambda \sum_{k=1}^{k_n} f_k), \tag{3.23}$$

and, in the limit when $n \to \infty$, D_n converges to a $\mathcal{P}(\lambda m)$ random variable.

Since $c_d(1) = 0$, the long run expected delay cost per day for ND policy is zero. Thus, among all the state-independent RP's, the ND policy minimizes the delay cost.

3.6.2 Shortest Queue (SQ) policy

Note that the decisions under the RP's are independent of the schedule state. In this section we consider the shortest queue (SQ) policy that uses the information of the current scheduling state to a limited extent. Suppose the state of the schedule is x when a new patient arrives. Then $x(1,t)$ represents the total number of patients already scheduled on t days into future,

$1 \leq t \leq T$. The SQ policy assigns the arrival to a slot $m(x)$ days into the future, where

$$m(x) \in \text{argmin}\{x(1,t) : 1 \leq t \leq T\}.$$

(If the argmin set on the right hand side of above equation has more than one element, $m(x)$ can be chosen to be any one of them.) Hence, SQ policy chooses a day with the smallest $x(1,t)$ among $1 \leq t \leq T$. Since the SQ policy is state dependent, the distribution of the daily demand in steady state is very hard to compute even under further assumptions on the arrival stream such as i.i.d. Poisson or Geometric arrivals. However, it can be easily implemented in practice as well as in simulation.

3.6.3 Max-Marginal Profit (MP) policy

One clear drawback of the SQ policy is that it ignores the revenue or cost from serving patients, as well as the delay costs. In this section we develop a policy that addresses this drawback. Suppose the current day is n and state of the schedule is x. Assume a new patient arrives and we need to assign this patient to day $n+t$ $(1 \leq t \leq T)$, and no more new patients arrive. This changes the demand on day $n+t$ from $x(1,t)$ to $x(1,t)+1$, and incurs a delay cost of $c_d(t)$. To begin with we ignore the increase in the demand in later cycles due to this assignment.

The marginal increase in net profit over the first cycle from this assignment is thus

$$\psi(x,t) = \rho(x(1,t)+1, q) - \rho(x(1,t), q) - c_d(t).$$

Using the definition of ρ in Equation 3.5, we can show that

$$\psi(x,t) = p\delta(x(1,t) < \gamma q) - c'\delta(q \leq x(1,t) < \gamma q) - c_d(t), \tag{3.24}$$

where $\delta(A) = 1$ if the condition in A is satisfied and zero otherwise. This policy assigns the new arrival to day $n + m(x)$, where

$$m(x) \in \text{argmax}\{\psi(x,t) : 1 \leq t \leq T\}.$$

That is, the policy selects the t that maximizes the function $\psi(x,t)$ for the given state x of the schedule and schedules the first appointment on day $n+t$. We call this policy the Max-marginal Profit (MP) policy. We would expect a better performance of this policy than ND and SQ policies, since it considers the current schedule status as well as the revenue and cost.

As in the SQ policy, the demand distribution under the MP policy is intractable, and needs to be estimated by simulation.

In the above policy, we do not use the information about the type of the new arrival. If we know the arriving patient is of type k, we can further modify the MP policy as follows. We continue to assume that there are no more arrivals after this patient. Thus when we assign the incoming patient to day $n+t$, it increases the demand on day $n+t+(j-1)T$ $(1 \leq j \leq k)$ from $x(j,t)$ to $x(j,t)+1$. Hence the marginal increase in net profit over the next k cycles from this assignment is given by

$$\begin{aligned}\psi_k(x,t) =& \sum_{j=1}^{k}(\rho(x(j,t)+1,q) - \rho(x(j,t),q)) - c_d(t) \\ =& p\sum_{j=1}^{k}\delta(x(j,t) < \gamma q) - c'\sum_{j=1}^{k}\delta(q \leq x(j,t) < \gamma q) - c_d(t). \qquad (3.25)\end{aligned}$$

As before we choose to assign the new arrival to day $n+m_k(x)$, where

$$m_k(x) \in \text{argmax}\{\psi_k(x,t) : 1 \leq t \leq T\}.$$

We call this policy the modified Max-marginal Profit (MMP) policy. It is unclear if MMP is better than MP. Indeed, we show in Section 3.7 that we do not benefit more from MMP than MP policy.

3.6.4 Policy Improvement Heuristics: Index policy

The MP and MMP policies both account for the current state of the schedule as well as the cost structure. However, they ignores the future arrivals and the impact of the current decision on future admissions. Now we develop policies that address this concern. This leads us to consider the procedure of policy improvement.

We use f_y to denote the policy that takes action y at the beginning of day 0 but from day 1 on uses the ND policy. Let f' be the policy if we choose y optimally. Its expected total profit with n periods to go is given by

$$v^n_{f'}(x,a) = \max_{y\in\Delta(a)} v^n_{f_y}(x,a) = \max_{y\in\Delta(a)} \left\{ r(x,y) + E(v^{n-1}_{ND}(T_{x+y},A)) \right\} \tag{3.26}$$

where the last expectation is taken over all the random arrivals A.

Using Equation 3.20, and utilizing the fact that the ND policy has zero delay cost, we see that

$$\begin{aligned} E(v^{n-1}_{ND}(x,A)) &= E_{ND}\left(\sum_{j=1}^{n-1} \rho(D_j,q) \mid X_0 = x, A_0 = A \right) \\ &= \sum_{j=1}^{n-1} E\left(\rho(x(k_j,t_j) + Z_j, q) \mid X_0 = x, A_0 = A\right) \\ &= \sum_{j=1}^{n-1} g_{k_j,t_j}(x(k_j,t_j)), \end{aligned} \tag{3.27}$$

where

$$g_{k,t}(z) = E(\rho(z + Z_{(k-1)T+t})), \tag{3.28}$$

We state the important property of the g functions in the next theorem.

Proposition 3.1. Assume $p \geq c'$. Then, for $k \geq 1$ and $t = 1, \cdots, T$, $g_{k,t}(z)$ of Equation 3.28 is a concave function in z.

From Equation 3.27, maximizing the right hand side of 3.26 is equivalent to maximizing:

$$\begin{aligned} v^n_{f_y}(x,a) &= \rho(x(1,1) + y(1,1), q) - \sum_{t=1}^{T} c_d(t) y(1,t) + \sum_{j=1}^{n-1} g_{k_j,t_j}(T_{x+y}(k_j,t_j)) \\ &= \sum_{j=1}^{n} \tilde{g}_{k_j,t_j}\left(x(k_j,t_j), y(k_j,t_j)\right) \end{aligned}$$

where

$$\tilde{g}_{k,t}(w,z) = \begin{cases} \rho(w+z,q) & \text{if } k=1,\ t=1; \\ g_{k-1,T}(w+z) & \text{if } k \geq 2,\ t=1; \\ g_{k,t-1}(w+z) - c_d(t)z\delta(k=1) & \text{if } k \geq 1,\ 2 \leq t \leq T. \end{cases}$$

Therefore, given the initial state of $X_0 = x$ and $A_1 = a$, the optimization problem discussed above can be formulated explicitly as follows:

$$\begin{aligned} \max_{y} \quad & \sum_{j=1}^{n} \tilde{g}_{k_j,t_j}\left(x(k_j,t_j), y(k_j,t_j)\right) \\ \text{s.t.} \quad & y \in \Delta(a) \end{aligned} \tag{3.29}$$

It's obvious that all the $\tilde{g}_{k,t}(w,z)$ are concave functions in z from Proposition 3.1.

It is possible to solve the above optimization problem by converting it to a min cost flow problem on directed networks with general costs (positive as well as negative). We show that formulation in the next subsection. Here we describe a greedy procedure as a suboptimal but simple implementable policy, which (as we see in Section 3.7) works very well in practice.

We need the following notation:

$$\tilde{y}(k,t) = y(k,t) - y(k+1,t), \quad k \geq 1, t = 1, \cdots, T$$

$$\tilde{a}_k = a_k - a_{k+1}, \quad k \geq 1.$$

Note that we can construct y from $\tilde{y}$ and a from $\tilde{a}$. The greedy procedure starts with $\tilde{y}$ and the corresponding y being the zero matrices, and in a_1 steps builds a matrix $y \in \Delta(a)$. In each step one of the elements of $\tilde{y}$ increases by one. The procedure is "greedy" in that at each step it chooses to increment the element that is feasible and yields the maximum change in the the objective function. The procedure is described in detail below. First we define the index function:

$$I_k(x,t) = \sum_{j=1}^{k} \tilde{g}_{j,t}(x(j,t),1) - \tilde{g}_{j,t}(x(j,t),0),$$

which gives the change of the objective function in Equation 3.29 if $\tilde{y}(k,t)$ is incremented by

1. In general, we have the explicit expression for the index function:

$$I_k(x,t) = \sum_{j=1}^{k} (p-c')F_{j,t}(\gamma q - x(j,t) - 1) + c'F_{j,t}(q - x(j,t) - 1) - c_d(t), \tag{3.30}$$

where $F_{j,t}(\cdot)$ is the cdf of the random variable $\tilde{Z}_{j,t}$ given by:

$$\tilde{Z}_{j,t} = \begin{cases} 0 & \text{if } j = 1,\ t = 1; \\ Z_{(j-2)T+t}, & \text{if } j \geq 2,\ t = 1; \\ Z_{(j-1)T+t-1}, & \text{if } j \geq 1,\ t \geq 2. \end{cases}$$

Note that the generating function of variable Z_n is given in Equation 3.21.

This index function is especially easy to compute when the new arrivals are i.i.d. Poisson. By Equation 3.23, we can show that $\tilde{Z}_{k,t}$ are Poisson random variables with rate of $\lambda_{k,t}$ given by:

$$\lambda_{k,t} = \begin{cases} 0 & \text{if } k = 1,\ t = 1; \\ \lambda \sum_{j=1}^{k-1} f_j, & \text{if } k \geq 2,\ t = 1; \\ \lambda \sum_{j=1}^{k} f_j, & \text{if } k \geq 1,\ t \geq 2; \end{cases}$$

Using the index function we describe the following greedy procedure:

Greedy Procedure:

- Given x, a, set $K = \max\{k : a_k > 0\}$;
- Initialize $y,\ \tilde{y}$ to be 0;
- For $k = 1$ to K do:
  - While $\sum_{t=1}^{T} \tilde{y}(k,t) < \tilde{a}_k$ do:
    - Determine the index t^* as follows:

$$t^* \in \operatorname{argmax}\{I_k(x,t) : 1 \leq t \leq T\}. \tag{3.31}$$

    - Set $\tilde{y}(k,t^*) = \tilde{y}(k,t^*) + 1$. Update y accordingly, and set $x = x + y$.

- End while loop;

- End for loop.

Thus, when the above algorithm terminates, the final y is in $\Delta(a)$.

Remark 3.5. The above algorithm indeed produces the optimal solution if $x(k,t) = 0$ for all $k \geq 2$, and all $1 \leq t \leq T$. See Ibaraki and Katoh (1988), and Bretthauer and Shetty (1995). However, in general the greedy algorithm produces a suboptimal solution.

We can use the above algorithm to construct a scheduling policy that operates as follows: when an arrival of type k occurs, we first observe x, the current state of the schedule. Then we compute $I_k(x,t)$ for all $1 \leq t \leq T$ and then find the t^* as in Equation 3.31. Then we schedule the first appointment of this new arrival t^* days later. This is called the Index policy. It is clear that this policy uses the information about the schedule, the current arrival, the cost structure, and accounts for the future arrivals, and the influence of the current decision on future costs. Hence we expect this policy to do better than those considered so far in the earlier part of this section.

Proposition 3.2. The index in Equation 3.30 has the following properties:
p1: If $k_1 \leq k_2$, then $I_{k_1}(x,t) \leq I_{k_2}(x,t)$, for $\forall x,\ t$;
p2: If $t_1 \leq t_2$ and for $\forall k$ we have $x(k,t_1) = x(k,t_2)$, then $I_k(x,t_1) \geq I_k(x,t_2)$, for $\forall k$;
p3: If $x_1(k,t) \leq x_2(k,t)$ for $\forall k,\ t$, then $I_k(x_1,t) \geq I_k(x_2,t)$;

The interpretation for **p1** is straightforward: given the same current scheduling state, scheduling one more patient of type k_2 to t days later is more profitable than that of a type k_1 to the same day, since $k_1 < k_2$ and type k_2 patient needs more appointments. Property **p2** suggests that, if we have the same number of patients already scheduled on some day $n + t_1$ and day $n + t_2$, it would be more profitable to assign the incoming patient to the earlier day, considering the delay costs in non-increasing in t. From **p3** we see that the marginal profit of admitting one more patient is greater if we have a lighter schedule.

3.6.5 Policy Improvement Heuristics: Min-Cost Flow (MF) policy

Equation 3.29 can be solved to optimality by formulating it as a min-cost network flow problem with $KT + K + 1$ nodes and $2KT$ arcs. The schedule x provides the constraints, the arrival a provides the flow, and $\tilde{g}_{k,t}$ provides the costs. This can be solved by the successive shortest path algorithm of Bazaara and Jarvis (1977) in $O(a_1 K^3 T^2)$ steps. The details of the network and the algorithm are given in the Appendix A.2.11.

We can show that when there is only one patient (of any type) arrives, the successive shortest path procedure reduces to the greedy procedure. In this trivial case, MF policy makes the same decision as the Index policy.

Moreover, MF policy assumes that the entire arrival vector for the current day is known before first appointments are assigned. This is typically not practical in real applications, where the appointment decision has to be made once a new patient arrives (calls). We study it nevertheless, because the theory of MDP states that it is provably better than the ND policy in maximizing the expected profit per day. We cannot make any such claim for the Index policy. One would naturally expect this policy to outperform the Index policy, but at the cost of higher computational load.

3.6.6 Structural Properties of the Implementable Policies

Now we study the structural properties of the implementable policies introduced above. It is interesting to note that, except for the MF policy, all the other policies can be thought of as index policies with their own index functions. For example, the ND policy uses the index function

$$I_k^{ND}(x,t) = I^{ND}(t) = -c(t), \quad 1 \le t \le T,$$

which does not depend on k or x. The SQ policy uses the index function

$$I_k^{SQ}(x,t) = -x(1,t), \quad 1 \le t \le T,$$

which does not depend on k, and uses only the first row of x. The MP policy uses the index function

$$I_k^{MP}(x,t) = \psi(x,t), \quad 1 \le t \le T,$$

where $\psi(x,t)$ is defined in Equation 3.24. Note that this function is independent of k and uses only the first row of x, and also incorporates the revenue and cost. The MMP policy uses the index function

$$I_k^{MMP}(x,t) = \psi_k(x,t), \quad 1 \le t \le T,$$

where $\psi_k(x,t)$ is defined in Equation 3.25. Note that this function depends on k and uses the first k rows of x, and also involves the revenue and cost. Each of these policies f chooses an

$$m_k(x) \in \text{argmax}\{I_k^f(x,t) : 1 \le t \le T\},$$

and gives the first appointment to the type k arrival $m_k(x)$ days into the future. We reserve the term "Index policy" to mean the policy that uses the index defined in Equation 3.30. Note that the policies ND, SQ, MP, and MMP use increasingly more information about the state (x,a) in making decisions. Hence we expect that they will perform increasingly closer to the optimal policy. We shall numerically evaluate this hypothesis in the next section.

In the remainder of this section we shall investigate if these policies have structural properties similar to that of the optimal policy as studied in in Section 3.5.1.

For the policies such as SQ, MP and Index policies, note that each decision is made right after each new arrival, hence any optimal y derived accordingly has *at most* one positive component in each row. However, the conditions (a) and (b) in the Theorem 3.4 implying that as long as $h \neq 0$, at least two components in the same row of the initial optimal solution y have to be positive.

In order to explore the structure for implementable policies, we need to force the arrival vector a to be with more than one new arrival ($a_1 > 1$). Accordingly, we fix a randomly arriving order of the a_1 patients, and assume that we update the scheduling state for each arrival, without changing the earlier assignments. Let $y_f(x,a)$ be the final allocation obtained by adding up all the a_1 successive single-arrival decisions made by the policy f. Finally, we compare the decision $y_f(x_1, a)$, $y_f(x_2, a)$ based on the same arriving order. The main result is given in the theorem below.

Theorem 3.7. The conclusions in Theorem 3.4 and 3.5 hold under SQ, MP, MMP and MF policies.

It is possible to construct a numerical counter-example to show that the Index policy does not have this structural property. However the Index policy requires less computation efforts, and we will show numerically that it is a good approximation of MF policy.

In the next section we study a real-life small-scale physical therapy clinic and compare the performance of the polices of this section numerically.

3.7 Numerical Examples with Real Data

To evaluate and compare the performances under different implementable policies, we run simulations with the data from a local PT center. We collect the exact appointment dates of 395 patients who started their service between July 1st 2013 and June 30th 2014 at this center.

We first estimate $\phi_j (j = 0, 1, 2, \cdots)$, the pmf of the number of daily new arrivals (see Equation 3.1). In Figure 3.2, we find that there are at most six new patients admitted on any given day in this data set, while admitting one or two new patients per day are the most common cases. We ignore the days on which the clinic was closed while computing the number of days with no new arrivals. The data show that the daily average number of new patients to the PT center is about 1.2. We also find that the empirical distribution is very close a Poisson distribution with rate 1.2, so we assume that the number of new arrivals are Poisson random variables. We do not test the independence, but simply assume it. We use this assumption in the simulation.

Next we estimate $p_k, (k = 1, 2, 3, \cdots)$, the pmf of the number of visits by a patient (see Equation 3.2). The data shows that only 9 out of 395 patients required more than 20 appointments. We ignore them so that the maximum number of visits for any patient is $K = 20$. In Figure 3.3, we find that more than 70% of patients need more than one appointments, and of these, most need 2 to 4 visits in total, while only a few patients need as many as 15 to 20 visits. We use these frequencies of patients who need k visits as the estimate of $p_k, (k = 1, 2, 3, \cdots)$.

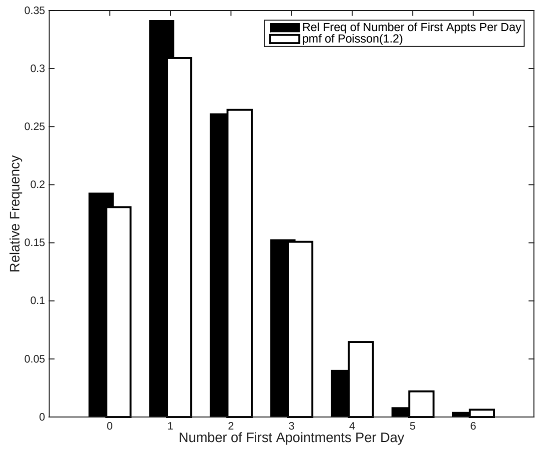


Figure 3.2: Distribution of Number of Daily (First) Appointments

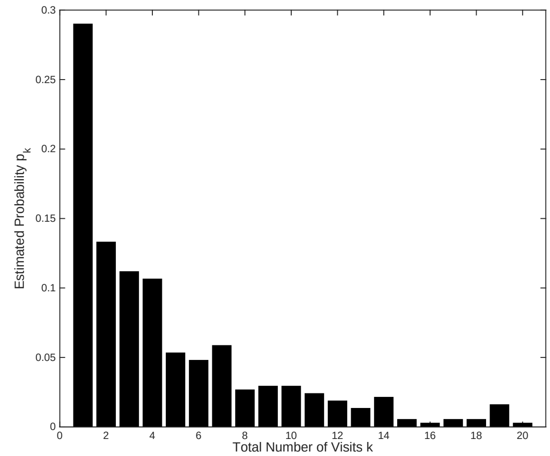


Figure 3.3: Distribution of Total Number of Appointments Per Patient

We do not fit any parametric distribution to this pmf, since it is not needed.

In all our simulations, we use a weekly cycle model with $T = 5$, and assume 15% overtime slots are allowed (that is, $\gamma = 1.15$). We also consider a linear delay penalty in the number of days delayed, namely $c_d(t) = 25(t-1)$, for $t = 1, \cdots, 5$. Specifically, we simulate two systems: the actual system, with $\lambda = 1.2$ and a larger (fictitious) system with with $\lambda = 6$. In both systems we assume that the pmf of the number of visits is as found in data. Note that this leads to the expected number total visits per day in the two systems to be $\mu = 5.6043$ for $\lambda = 1.2$ and $\mu = 28.0213$ for $\lambda = 6$.

We consider two choices for the reward/cost parameters:

Case A: $p = 100$, $c = 20$, and $c' = 30$, since we usually pay 50% higher than the regular salary for overtime work. This yields $\zeta = 0.8190$ (see Equation 3.8.) We numerically solve Equation 3.7 to obtain $q^* = 6$ when $\lambda = 1.2$ and $q^* = 29$ when $\lambda = 6$. Thus we have $q^* > \mu$, which is consistent with Remark 3.

Case B: $p = 100$, $c = 70$, and $c' = 80$. This yields $\zeta = 0.320$ which yields $q^* = 3$ when $\lambda = 1.2$ and $q^* = 24$ when $\lambda = 6$. Thus we have $q^* < \mu$, which is again consistent with Remark 3.

For each of these four cases, we simulate six implementable policies: Min-cost Flow policy (MF), Index policy (IP), Max-Marginal Profit policy (MP), Modified Max-Marginal Profit policy (MMP), Shortest Queue policy (SQ) and the Next Day policy (ND). Note that the staffing level q is optimal for ND, but need not be so for the other policies.

We run 100 replications of the system, each replication is for 350 working days (including 100 days of warmup period.) We compute the average daily profit under all the policies, using the same sample path of the arrival stream for all the policies in each replication. All the results described above are given in Tables 3.1 and 3.2 in cases A and B. In each table, the 'Mean' column records the average daily profit under the six policies. The 'Dif Mean' gives the difference of the average daily profit under the MF policy and each other policy. The 'LB' and 'UB' columns give the lower and upper bounds on the 95% confidence interval of this difference.

Set	$\lambda = 1.2, \quad q = 6$				$\lambda = 6, \quad q = 29$			
Policy	Mean	Dif Mean	LB	UB	Mean	Dif Mean	LB	UB
MF	414.92	0	0	0	2155.82	0	0	0
IP	414.11	0.81	0.44	1.18	2155.79	0.03	-0.33	0.38
MP	414.71	0.22	-0.33	0.76	2137.68	18.19	16.24	20.14
MMP	413.94	0.99	0.40	1.57	2137.35	18.47	16.45	20.49
SQ	391.42	12.50	21.82	25.19	2126.89	28.93	26.19	31.67
ND	364.24	50.68	49.74	51.63	1681.66	474.16	471.81	476.51

Table 3.1: Daily Average Profits and Differences from MF Policy in Case A

Set	$\lambda = 1.2, \quad q = 3$				$\lambda = 6, \quad q = 24$			
Policy	Mean	Dif Mean	LB	UB	Mean	Dif Mean	LB	UB
MF	100.81	0	0	0	740.14	0	0	0
IP	100.77	0.04	-0.15	0.22	740.73	-0.59	-1.034	-0.14
MP	97.54	3.27	2.79	3.75	604.57	135.58	130.66	140.50
MMP	82.37	18.45	18.03	18.84	587.40	152.74	149.10	156.38
SQ	95.62	5.19	4.35	6.03	711.07	29.07	25.90	32.25
ND	40.50	60.31	58.95	61.67	273.76	466.38	462.62	470.14

Table 3.2: Daily Average Profits and Differences from MF Policy in Case B

We conclude from the above tables that the IP and MF policy perform almost the same, and both are better than SQ policy and ND policy. In the small system (left half of the tables), the MP policy is almost as good as the IP and MF policies, since the 95% confidence interval of the mean difference is close to or includes 0. In the large system (right half of tables), the MP policy does not perform as well as IP and MF policies, the difference is not that clearcut in Case A (about 0.8%), but is significant in Case B (about 22.3%). Note that the MMP is always no better than MP, one explanation for this situation is that, even though under MMP, we know more information about the new patient type k than that under MP, we have to make the assumption that there won't be any new arrivals for the following k weeks, which is more unrealistic than the assumption that there won't be new arrivals for the next week under MP.

The superiority of the IP and MF policies is more pronounced in Case B as compared to Case A. That is, when the staffing level is less than the mean demand, the IP and MF policies are more superior to other policies. With all these considerations, we recommend the use of the IP in general. MF policy is also as good, but it is not implementable in practice, since it requires the knowledge the entire day's demand to be known before any appointment scheduling is done, and also, it involves a lot more computation in the form of min-cost flow problem.

We can also use simulation to find the optimal staffing level for each policy as follows. We choose a scheduling policy, say the Index policy, run simulations with different the staffing levels q, and find the one that maximizes the average daily profit. In Figure 3.4 and 3.5, we show the simulated results of applying this procedure in the larger system ($\lambda = 6$). In case A, we vary q in the range $\{24, 25, \cdots, 34\}$ and find that the simulated estimate of the average daily profit is maximized at $q = 28$, which is one less than the $q^* = 29$ derived as optimal for the ND policy. Similarly in case B, we vary q in the range $\{20, 21, \cdots, 29\}$ and find average daily profit is maximized at $q = 25$, which is one more than the $q^* = 24$ derived as optimal for the ND policy.

So we conclude that, in Case A ($q^* > \mu$), the optimal staffing level q selected by simulation is smaller than, but close to the q^* obtained by the staffing model in Section 3.4 using the demand distribution from the ND policy. The reduction in profit from using this approximate staffing level is rather small (less than 1%). On the contrary, in Case B ($q^* < \mu$) , the simulated optimal q is greater than, but still close to the estimated q^* by staffing model under ND policy. Similar phenomenon is observed under all the other policies. In brief, the simulation results show that the actual optimal staffing levels are closer to the expected demand μ, compared with the estimated q^*.

A key contribution of this chapter is that we model "series" patients. An alternate solution to this problem is to ignore the "series" nature of patient appointments and to treat each appointment as an independent arrival. We ran numerical experiments to demonstrate the

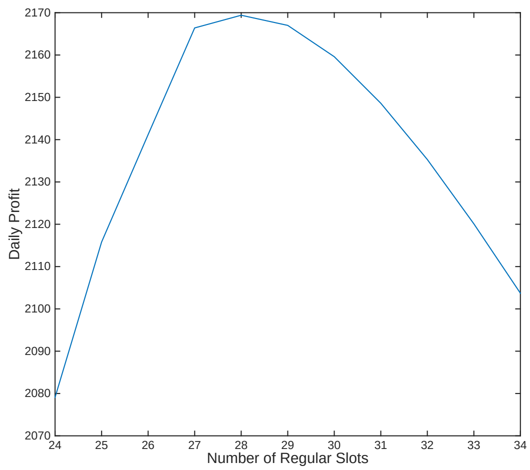


Figure 3.4: Average Profits under Index Policy as a Function of q in Case A with $\lambda = 6$.

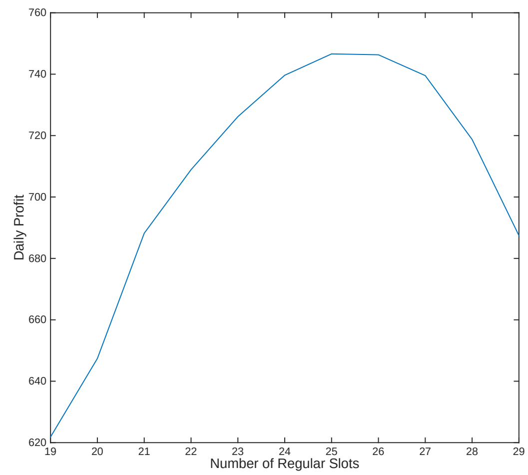


Figure 3.5: Average Profits under Index Policy as a Function of q in Case B with $\lambda = 6$

benefit of taking into account the "series" nature of patient appointments. To do so, we ran simulations with identical settings, but using the Index policy assuming that the daily demand comes from patients with single appointment visits ($K = 1$ instead of $K = 20$ under the current setting) and inflating the arrival rate correspondingly to keep the total appointments requested identical to the system with series appointments. The results show that in both Case A and B, treating the series appointments as independent arrivals lowers the daily profit by 2% - 5% compared to the index policy that explicitly considers the series nature of appointments. The optimal staffing levels q are also slightly different by ignoring the series nature of appointments, in Case A the optimal q decreases by 1, while in Case B the optimal q increases by 1. Thus, capturing the "series" nature of patient appointments is important from a practical perspective.

3.8 Conclusions and Extensions

This work is motivated by the phenomenon that in many specialty care clinics patients need a series of appointments to fully complete their treatments. We studied the staffing and scheduling problems for these series patients. We used a variation of newsvendor model for the optimal staffing problem. We showed that the optimal number of slots assigned each day satisfies a simple equation and can be obtained numerically. With different parameter sets, this optimal staffing level can be either greater or smaller than the average daily demand.

We further assumed that each patient needs a random number of total visits with a constant inter-visit time. We modeled the scheduling system by an MDP, with the state being a combination of the current schedule and the arrival vector. Since computing the optimal policy is intractable due to the size of state space, we proposed the following implementable policies: Next Day policy (as a special case of Randomized policy), Shortest Queue policy, Max-Marginal Profit policy, Min-cost Flow policy, and Index policy. Except Index policy, the other four policies share the same structural property of the optimal policy, and use increasingly more information regarding the system state and future demand in decision making. However the Index policy leads to a good approximation to the Min-cost Flow policy, can be used on a patient by patient basis, and it is computationally efficient.

Using the data collected from a local PT center, our simulation study showed that the Index policy performs very well compared with the other policies under different parameter settings. Hence, we recommend the Index policy for general use, since it generates the most average profit while requiring less computational efforts to implement.

There are a number of ways to extend our results and our model. First, in this chapter we assumed that all the patients visit the clinic exactly once per T days, and request the same service session length of one slot per visit. However, in practice the inter-visit time and the length of service session can vary from patient to patient (as we have observed in our PT center data), and can even be random. Thus it would be interesting to develop and analyze the models that account for these aspects. In next chapter, we modify the scheduling model by assume random number of visit and the patient will be assigned next visit after each visit. We obtain the similar Index policy that achieve significant improvement from Next Day policy.

Second, our current scheduling decision is made under the assumption that we accept all the newly admitted patients, assuming there will always be "external" physical therapists available when the demand exceeds the maximum"internal" supply of regular plus overtime slots. One possibility is to allow rejection of new patients at each epoch. If we accept a patient, we need to decide the optimal appointment schedule for her. Finally, we could include patients' cancellations and rescheduling in the future work, because we do observe these phenomena in the data.

CHAPTER 4: Staffing and Scheduling for Health Care Facilities with Series Patients: Model II

4.1 Introduction

In this chapter, we modify both the staffing and scheduling models based on those constructed in Chapter 3. We first introduce the overflow penalty, which is the cost of admitting more patients than the total supply available (regular and overtime slot). For the scheduling model, instead of assigning the fixed number of visits at the beginning of each patient's initial visit, now we assume the number of visits are random. Based on the evaluation of the patient after each visit, the physical therapist decides if the patient needs the next appointment, or discharge from the system. Due to this modification, we need a new scheduling state to describe the system, and a different formulation of Index policies and other heuristic policies. We do not repeat the literature review section since this chapter shares the same background of Chapter 3. In the rest of this chapter, we modify the revenue and staffing cost function in Section 4.2, and formulate the new scheduling model in Section 4.3. The heuristic policies based on the MDP models are introduced and analyzed in Section 4.4, and the numerical experiments are conducted in Section 4.5. The summary and conclusion are given in Section 4.6.

4.2 The Model II

Most of the model assumptions and notation in Model II are the same as in Model I. Here we only mention the differences. We use A_n to denote the number of new patients who arrive (or call) on day n to make appointments. In model I we used A_n as a vector, but here it is a scalar. This change of notation is made so that the state of the system will still be denoted by X_n, A_n, as explained below. We shall assume that $\{A_n, n \geq 0\}$ are i.i.d. random variables with common probability mass function (pmf)

$$\phi_j = P(A_n = j), \quad j \geq 0. \tag{4.1}$$

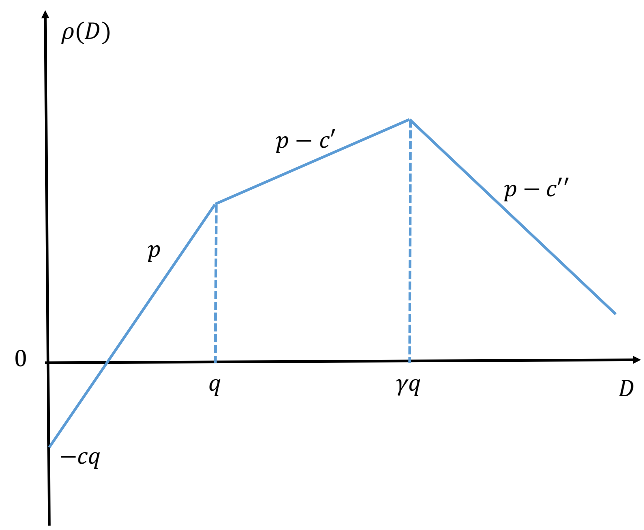


Figure 4.1: Daily Net Profit as Function of D

When a patient arrives (calls) on day n, he/she has to be given the first appointment on day $m \in \{n+1, n+2, \cdots, n+T\}$, where T is a given positive integer, without knowing the number of visits the patient will need. The rule that dictates how the first appointment is scheduled is called the scheduling policy.

In this chapter we are interested in finding a scheduling policy that maximizes the long run average net profit. The net profit can be thought of as the revenue from the patients minus the staffing cost, overtime cost, overbooking cost and the delay cost. The first three costs are the same as in Model I. We describe the last one below.

The revenue p, and the costs c and c' are the same as in model I. Here we introduce an additional cost called the overbooking cost: If the total number of of slots demanded for the day exceeds the total regular plus overtime slots γq, each excess slot needed beyond γq is called an overbooking, and has to be routed to another clinic or handled in some other way. This costs $c'' > c'$. We call c'' the overbooking cost.

With this modification, the net profit $\rho(D, q)$ of Equation 4.2, with generic demand D and fixed staffing level q, becomes

$$\rho(D, q) = pD - cq - c' \max(D - q, 0) - (c'' - c') \max(D - \gamma q, 0). \qquad (4.2)$$

We plot this function in Figure 4.1, with D being a general demand.

4.3 Optimal Scheduling Policies

In this section we shall formulate the scheduling problem as a discrete time Markov decision process (MDP) that can be used to find a policy that maximizes the long run expected net profit per day.

We first describe the state space of the MDP. Recall that each patient needs one slot every T days. Let K be the maximum number of revisits a patient can have. The state of the appointment schedule at the beginning of day $n \geq 0$ and before the arrivals of new patients is described by a matrix $X_n = [X_n(k,t)]$, where $X_n(k,t)$ $(k = 1, \cdots, K,\ t = 1, \cdots, T)$ is the number of patients who have their kth appointment on day $n+t$. Let $\mathbb{N} = \{0, 1, 2, \cdots\}$, and

$$\mathcal{X} = \{x = [x(k,t)] : x(k,t) \in \mathbb{N} \text{ for } k = 1, \ldots, K;\ t = 1, \ldots, T\},$$

Thus $X_n \in \mathcal{X}$ for all $n \geq 0$. (Note that this definition differs from that in Model I.)

We assume that all new patient calls occur at the beginning of the day. Thus A_n, the number of new calls on day n, is known at the beginning of day n. This simplification creates a tractable model and leads to policies that can also be implemented on a one-by-one arrival basis. Thus the state of the system on day n is given by (X_n, A_n), and the state space is $\mathcal{X} \times \mathbb{N}$.

Next we describe the action space. The scheduling problem is to decide how many of the A_n new patients should be given their first appointments on day $n+t$, for $1 \leq t \leq T$. It is convenient to introduce the follwing notation. Let $Y_n(1,t)$ be the number of *new* patients who are given their first appointments on day $n+t$, and $Y_n(k,t) = 0$ for $2 \leq k \leq K$. Hence $Y_n = [Y_n(k,t)]$ is a matrix representing the scheduling action taken on day n. For a given realization of $A_n = a \in \mathbb{N}$, we have $Y_n \in \Delta(a)$, where

$$\Delta(a) = \left\{ y \in \mathcal{X} : \sum_{t=1}^{T} y(1,t) = a, \quad y(k,t) = 0 \ \ for\ \ k = 2, \ldots, K, \ \ t = 1, \ldots, T \right\}.$$

Next we describe the transition probabilities of the MDP. For this we need a probabilistic model of the number of visits needed by a patient. We assume that the number of visits needed by the patients are i.i.d. random variables taking values in $\mathbb{N}$. Let V be a generic random variable representing the number of visits needed by a patient. Let

$$f_k = P(V \geq k+1), \quad 0 \leq k \leq K-1. \tag{4.3}$$

Let $\alpha_k = f_k/f_{k-1}, \quad 1 \leq k \leq K-1$. Thus, if on day n, z number of patients had their k-th visit, the number of those who will return on day $n+T$ for their $(k+1)$st visit is a $\mathrm{Bin}(z, \alpha_k)$ random variable.

Suppose $(X_n, A_n) = (x, a)$, and the action taken on day n is $Y_n = y \in \Delta(a)$. At the end of day n, the scheduling state is updated to $x+y$, and at the beginning of day $n+1$ the new scheduling state is $X_{n+1} = T_{x+y}$, where the (k,t)th entry of the matrix T_{x+y} is given by:

$$T_{x+y}(k,t) = \begin{cases} x(k,t+1) + y(k,t+1), & \text{if } t = 1, \dots, T-1, \ k = 1, \dots, K; \\ 0, & \text{if } t = T, \ k = 1; \\ \mathrm{Bin}(x(k-1,1) + y(k-1,1), \alpha_k), & \text{if } t = T, \ k = 2, \dots, K. \end{cases}$$

Note that $T_{x+y}(1,T) = 0$ because X_{n+1} is the state before the new arrivals come in and all the existing patients have completed the first visits on day n.

Finally, we compute the expected one-step reward. We shall use the notation $X(\cdot, t)$ to mean the sum of the tth column of the matrix X, that is

$$X(\cdot, t) = \sum_{k=1}^{K} X(k,t).$$

We first look at demand on day n, which is given by

$$D_n = Y_{n-1}(1,1) + X_{n-1}(\cdot, 1), \quad n \geq 1. \tag{4.4}$$

The immediate reward for day n is defined to be the expected net profit on day $n+1$, that is, $r(X_n, Y_n)$ is the difference between $\rho(D_{n+1})$ and delay cost. This shifting of the rewards does not cause any problem since we are interested in long run average net profit per day. Thus, the one-step reward, given the state $X_n = x$ and the decision $Y_n = y$, is given by:

$$r(x, y) = \rho\left(y(1, 1) + x(\cdot, 1)\right) - \sum_{t=1}^{T} c_d(t) y(1, t). \tag{4.5}$$

Since $\{A_n, n \geq 0\}$ are i.i.d., and the number of visits by the patients are i.i.d., we see from the above description that $\{((X_n, A_n), Y_n), n \geq 0\}$ is a discrete time MDP. The objective is to maximize the long-run average profit. As in Chapter 3 we get the following

Theorem 4.1. *Suppose $\phi_0 = P(A_n = 0) > 0$ and the number of visits is at most K for each patient, then the MDP $\{((X_n, A_n), Y_n), n \geq 0\}$ is uni-chain and aperiodic.*

Proof of Theorem 4.1. Let π be any stationary policy and $(x, a) \in \mathcal{X} \times \mathbb{N}$ be any initial state. Note that the number of patients in the system can be modeled as a discrete time infinite server queue, hence the system is recurrent. This implies that state $(0, 0)$ is recurrent. Now we show that with a positive probability, the system can reach state $(0, 0)$ from any state (x, a) in a finite number of steps. With probability ϕ_0^{KT} there are no arrivals over the next KT days, and since $\alpha_K = 0$ all of the existing patients leave the system by KT days. Besides, under any policy we have $\phi_0^{KT} > 0$. This completes the proof of uni-chain. Furthermore the probability of transitioning from state $(X_n, A_n) = (0, 0)$ to $(X_{n+1}, A_{n+1}) = (0, 0)$ in one step is the probability that $A_{n+1} = 0$, which is again $\phi_0 > 0$. This proves aperiodicity. $\square$

It is known that if an MDP is unichain, there exists a solution to the Bellman equation, and hence the optimal average reward policy exists. It can be numerically computed by classic algorithms such as value iteration and policy iteration.

However neither of these algorithms are feasible to derive the optimal or ϵ-optimal policy for any realistically sized problem in our case. With reasonable values of T and K, the size of the space state is too huge to compute the policy numerically. For example, even if each patient visits the facility no more than twice and if the schedule repeats every five days, it would still require a solution to a 10 dimensional problem. Hence, in Section 4.4, we study several heuristic policies that are easy to implement in practice. Specifically, we focus on a policy-improvement heuristic, which utilizes the policy improvement algorithm and improves on a tractable policy.

4.3.1 Special Case: Geometric Number of Visits

Now we consider a special case by assuming that the number of visits needed by the patients are iid Geometric random variables with parameter $\bar{\alpha}$. That is

$$P(V = k) = \bar{\alpha}^k (1 - \bar{\alpha}), \quad k = 1, 2, \cdots .$$

Here we let $K = \infty$. Note that this implies that if there are z patients visiting today, $\text{Bin}(z, \bar{\alpha})$ of them will return after T days, and the rest are discharged. This simplifies the model a lot since we don't need to record the number of completed visits, instead all we need to track is the total number of patients on the schedule for each day $t \in \{(1, \cdots, T\}$. We abuse the notation and let $X_n(t)$ be the number of patients who are scheduled for a visit on day $n+t$, $(1 \leq t \leq T)$, and let $X_n = [X_n(1), X_n(2), \cdots, X_n(T)]$. We have

$$X_n \in \mathcal{X} = \{x = [x(t)] : x(t) \in \mathbb{N} \quad \text{for } t = 1, \cdots, T\},$$

The decision vector is now given by $Y_n = [Y_n(1), Y_n(2), \cdots, Y_n(T)]$, where $Y_n(t)$ is the number of new arrivals on day n who are given their first appointment on day $n + t$. The transition vector $T_{x+y} \in \mathcal{X}$ is given by

$$T_{x+y}(t) = \begin{cases} x(t+1) + y(t+1), & t = 1, \ldots, T-1, \\ \text{Bin}(x(1) + y(1), \bar{\alpha}), & t = T. \end{cases}$$

Similarly, Eq. 4.4 and 4.5 reduce to

$$D_n = X_{n-1}(1) + Y_{n-1}(1),$$

$$r(X_{n-1}, Y_{n-1}) = \rho \left(X_{n-1}(1) + Y_{n-1}(1) \right) - \sum_{t=1}^{T} c_d(t) Y_{n-1}(t).$$

Unfortunately, even with this simplification, solving numerically for the optimal policy is still not feasible. But in the heuristic policies, it significantly reduces the computation costs without degrading the performance measurement as the full model with different α_k. The details are given in Section 4.5.

4.4 Implementable Policies

In this section, we describe several implementable policies including the Next Day Policy (NDP), the Shortest Queue Policy (SQP), and the Index Policy (IP), with or without geometric assumption for the number of visits. The first two policies are straightforward and easily implementable, require less information of the current state and the revenue structure. The state-independent policies like NDP are attractive because it is easier to compute the demand distribution in steady state under them, and hence one can easily find an optimal staffing level q. IP utilizes the MDP model, performs one-step improvement on the NDP, and achieves higher profits at a modest increase in the computational effort, especially with geometric assumption.

4.4.1 Next Day Policy (NDP)

The Next Day Policy (NDP) simply assigns all of the new arrivals to the next day, therefore it does not need any information about the current state of the system. It minimizes the delay cost among all the policies. However, it ignores the overtime and overbooking costs.

First, we introduce some relevant notation. For any $1 \leq n \leq KT$, there is a unique $t = t_n \in \{1, 2, \cdots, T\}$ and $k = k_n \in \{0, 2, \cdots, K-1\}$ such that

$$n = kT + t.$$

They are given by

$$k_n = \lfloor n/T \rfloor, \quad t_n = n - k_n T.$$

Now we evaluate the distribution of demand D_n under such policy.

Lemma 4.1. *Suppose $X_0 = x$ and the daily arrivals are iid Poisson with rate λ. Let D_n be the demand on day n under the NDP. Also assume that the number of visits by the patients* *are i.i.d. random variables with complementary cdf given by Eq. 4.3. Then*

$$D_n = \begin{cases} Poisson\left(\lambda \sum_{i=0}^{k_n} f_i\right) + \sum_{i=1}^{K-k_n} Bin\left(x(i,t), \quad f_{k_n+i-1}/f_{i-1}\right), & 0 \leq k_n < K; \\ Poisson\left(\lambda \sum_{i=0}^{K-1} f_i\right), & k_n \geq K. \end{cases} \tag{4.6}$$

It is clear from the above that for $n > KT$, the distribution of D_n is Poisson with mean $\lambda E(V)$, independent of X_0. This is to be expected: the expected number of new arrivals in a day is λ, and each of them requires EV repeat visits on average. Hence the expected total number of new slots requested by all the new arrivals in a day is $\lambda E(V)$. In fact, these observations are true for any state-independent scheduling policy.

Let D be a Poisson random variable with mean $\lambda E(V)$. Then, using Eq. 4.2, it is easy to see that the long run average profit under NDP is given by

$$G(q) = E(\rho(D)) = pE(D) - cq - c'E(\max(D-q,0) - (c''-c')E(\max(D-\gamma q,0)). \tag{4.7}$$

This is a variation of a newsvendor type objective function. If D is a continuous random variable, and we treat q as a real-valued variable, it is easy to show that this is a convex function of q. However, when q and D are integer valued, it need not be convex. However, it is straight forward to find the q that maximizes it. Thus for the NDP (or for any state-independent policy) it is easy to find the optimal static staffing level.

In the special case, when V follows a Geometric distribution, we have $\alpha_k = \bar{\alpha}$ for all $k \geq 1$, hence the probability parameter f_{k_n+i-1}/f_{i-1} reduces to $\bar{\alpha}^{k_n}$ for every $i \geq 1$. Therefore D_n reduces to the sum of one Poisson and one Binomial random variable:

$$D_n = \text{Poisson}\left(\lambda \frac{1-\bar{\alpha}^{k_n+1}}{1-\bar{\alpha}}\right) + \text{Bin}\left(x(t), \quad \bar{\alpha}^{k_n}\right),$$

In this case D_n converges to D as $n \to \infty$. The long run average expected profit, as a function of q, is still given by Eq. 4.7.

4.4.2 Shortest Queue Policy (SQP)

The shortest queue policy (SQP) that uses the information of the current scheduling state to a limited extent. Suppose the state of the schedule is x when a new patient arrives. Then $x(\cdot, t)$ represents the total number of patients already scheduled on t days into future, $1 \leq t \leq T$. The SQP assigns the arrival to a slot $t_{sq}(x)$ days into the future, where

$$t_{sq}(x) \in \text{argmin}_t \left\{ x(\cdot, t) : 1 \leq t \leq T \right\},$$

and then updates the matrix x. If the argmin set on the right hand side of above equation has more than one element, $t_{sq}(x)$ can be chosen to be any one of them. Thus, SQP chooses a day with the smallest currently scheduled appointments among the next T days. This policy attempts to minimize the overtime and overbooking costs by evening out the load among the next T days, but it ignores the delay cost. Since the scheduling decisions under SQP are state dependent, the distribution of the daily demand in steady state (and hence the long run average profit) depend on q, and is very hard to compute even under further assumptions such as i.i.d. Poisson arrivals or Geometric number of visits. Finding the optimal q is complicated for the same reason. However, SQP can be easily implemented in practice as well as in simulation.

4.4.3 Index Policy (IP)

The aforementioned SQP and NDP are straightforward and easily implementable, but they ignore the revenue structure, future arrivals and the impact of the current decision on future costs and revenues. To develop policies that address this concern, we consider the procedure of policy improvement based on the tractable value function under NDP.

We first need the bias function v^{NDP} and long run average revenue g^{NDP} for NDP. We have already computed the long run average revenue in Eq. 4.7, namely

$$g^{NDP} = G(q). \tag{4.8}$$

The next lemma gives the expression for the bias function.

Lemma 4.2.

$$v^{NDP}(x,a) = \sum_{n=1}^{KT} E(\rho(D_n)|X_0 = x, A_0 = a) - g^{NDP}KT, \tag{4.9}$$

where D_n is as given in Eq. 4.6.

Proof: Let

$$\phi^m(x,a) = \sum_{n=1}^{m} E(\rho(D_n)|X_0 = x, A_0 = a),$$

be the total expected profit over the first m days under the NDP policy. Let D be a Poisson random variable with mean $\lambda E(V)$. Since $D_n \sim D$ for $n > KT$, we have, for $m \geq KT$

$$\begin{aligned}
\phi^m(x,a) &= \sum_{n=1}^{KT} E(\rho(D_n)|X_0 = x, A_0 = a) + (m - KT)E(\rho(D)), \\
&= \sum_{n=1}^{KT} E(\rho(D_n)|X_0 = x, A_0 = a) + (m - KT)G(q), \quad \text{(from Eq. 4.7)} \\
&= \sum_{n=1}^{KT} E(\rho(D_n)|X_0 = x, A_0 = a) - KTg^{NDP} + mg^{NDP}, \quad \text{(from Eq. 4.8).}
\end{aligned}$$

From Tijms [2003] we know that

$$\phi^m(x,a) = v^{NDP}(x,a) + mg^{NDP} + o(m),$$

where $o(m)$ is a function that goes to zero as $m \to \infty$. This implies the Lemma. □

The policy improvement step involves computing the following

$$v(x,a) + g = \max_{y \in \Delta(a)} \left\{ r(x,y) + h(x,y) - KTg^{NDP} \right\}, \tag{4.10}$$

where

$$h(x,y) = E\left(\sum_{n=1}^{KT} \rho(D_n)|X_0 = T_{x+y}, A_0 = A \right).$$

Here the expectation is taken over all the random arrivals A and the random transitions in T_{x+y}.

Next, we show how to solve for the optimal decision that maximizes Eq. 4.10 by greedy algorithm. Note that can write the right hand side above as a sum of separable functions of

$y(1,t)$,

$$r(x,y)+h(x,y)=\sum_{t=1}^{T} g_t(y(1,t),x).$$

We need the following notation to show g_t explicitly. First define the following random variables:

$$P_k \sim \text{Poisson}(\lambda \sum_{i=0}^{k} f_i) \tag{4.11}$$

$$B_k^1(x) \sim \sum_{i=1}^{K-k-1} \text{Bin}(x(i,1), f_{k+i}/f_{i-1}), \tag{4.12}$$

$$B_k^t(x) \sim \sum_{i=1}^{K-k} \text{Bin}(x(i,t), f_{k+i-1}/f_{i-1}), \quad 2 \le t \le T, \tag{4.13}$$

$$R_k^1(w) \sim \text{Bin}(w, f_{k+1}/f_0), \tag{4.14}$$

$$R_k^t(w) \sim \text{Bin}(w, f_k/f_0), \quad 2 \le t \le T. \tag{4.15}$$

Here $w \in \mathbb{N}$ is the value that decision variable $y(1,t)$ can take. For $t=1$ we have

$$g_1(w,x)=\sum_{k=0}^{K-2} E[\rho(P_k+B_k^1(x)+R_k^1(w))]+\rho\left(w+x(\cdot,1)\right)-c_d(1)w, \tag{4.16}$$

and, for $2 \le t \le T$,

$$g_t(w,x)=\sum_{k=0}^{K-1} E[\rho(P_k+B_k^t(x)+R_k^t(w))]-c_d(t)w. \tag{4.17}$$

Note that the difference in $g_t(w,x)$ for $t=1$ and $t \ge 2$ is a result of the random initial state T_{x+y}.

We first prove an important property of the g_t functions in the lemma below.

Lemma 4.3. *For a fixed $x \in \mathcal{X}, g_t(w,x)$ is a concave function of w on $\mathbb{N}$, that is,*

$$g_t(w+1,x)-g_t(w,x) \ge g_t(w+2,x)-g_t(w+1.x). \tag{4.18}$$

Proof: Using Eq. 4.2, we have

$$E(\rho(D)) = pE(D) - cq - c' \sum_{n=q+1}^{\infty} P(D \geq n) - (c'' - c') \sum_{n=\lfloor \gamma q \rfloor + 1}^{\infty} P(D \geq n).$$

Hence

$$E(\rho(D+1) - \rho(D)) = p - c'P(D \geq q) - (c'' - c')P(D \geq \lfloor \gamma q \rfloor).$$

Now, consider random variable R_w that is a sum of a general random variable R_0 and an independent Binomial random variable $\mathrm{Bin}(w, \alpha)$ for a fixed α. Then we can construct a common probability space on which

$$R_{w+1} = R_w + Z$$

where $Z \sim \mathrm{Bin}(1, \alpha)$. Thus, under this coupling,

$$P(R_{w+1} = R_w + 1) = \alpha, \;\; P(R_{w+1} = R_w) = 1 - \alpha.$$

Hence

$$\begin{aligned} E(\rho(R_{w+1}) - \rho(R_w)) =& \alpha E(\rho(R_w + 1) - \rho(R_w)) \\ =& \alpha p - c'P(R_w \geq q) - (c'' - c')P(R_w \geq \lfloor \gamma q \rfloor). \end{aligned} \tag{4.19}$$

This is a decreasing function of w, since R_w is stochastically increasing in w. Thus, $E(\rho(R_w))$ is concave in w. Now it follows from Equations 4.16 and 4.17 that $g_t(w, x)$ consists of sums of terms of the form $E(\rho(R_w))$ for appropriately chosen R_w's. Hence $g_t(w, x)$ is concave in w. $\square$

As a consequence of the above lemma, we see that the optimization problem on the right hand side of Eq. 4.10 reduces to the maximization of a sum of separable concave functions as follows: Given $x \in \mathcal{X}$ and $a \in \mathbb{N}$,

$$\max \sum_{t=1}^{t} g_t(y(1,t), x)$$

subject to

$$\sum_{t=1}^{T} y(1,t) = a.$$

It is well known that this problem can be solved by a greedy algorithm in a steps. We need some notation to describe the algorithm. Define

$$I_t(w,x) = g_t(w+1,x) - g_t(w,x), \quad 1 \le t \le T. \tag{4.20}$$

Given the number of new arrivals $A = a$, the greedy procedure starts with $y = 0$ and in a steps builds the first row of the matrix $y \in \Delta(a)$. In each step one of the elements in the first row of y increases by one. The procedure is "greedy" in that at each step it chooses to increment the element that is feasible and yields the maximum change in the objective function. The procedure is described below.

Greedy Procedure:

- Given x, a, initialize $y = 0$;
- While $\sum_{t=1}^{T} y(1,t) < a$ do:
  - Determine the index t^* as follows:

$$t^* \in \operatorname{argmax} \{I_t(y(1,t),x) : 1 \le t \le T\}. \tag{4.21}$$

  - Set $y(1,t^*) = y(1,t^*) + 1$.
- End while loop;

One can use the above algorithm to construct a scheduling policy that operates on an arrival by arrival basis as follows: when an arrival occurs, we first observe x, the current state of the schedule. Then we compute $I_t(x) = I_t(0,x)$ for all $1 \le t \le T$ and then find the $t = t^*$ for which the index is the maximum. Then we schedule the first appointment of this new arrival t^* days later and update the x matrix by setting $x(1,t^*) = x(1,t^*) + 1$. Note that we do not need to know the number of new arrivals at the beginning of the day to implement this algorithm and we do not keep track of y.

The function

$$I_t(x) = I_t(0, x)$$

is called the index of day t in state x. The policy simply assigns an incoming arrival to the day with the highest index. Hence we call this policy an index policy (IP). To complete the description of IP, we we need the expressions for $I_t(x)$. Using Eq. 4.19, we get:

$$\begin{aligned} I_1(x) = \sum_{k=0}^{K-2} f_{k+1}/f_0 \Big[&(p - c'') + (c'' - c')P(P_k + B_k^1(x) \leq \lfloor \gamma q \rfloor - 1) \\ &+ c' P(P_k + B_k^1(x) \leq q - 1) \Big] \\ &+ (p - c'') + (c'' - c')\mathbb{1}_{\{x(\cdot,1) \leq \lfloor \gamma q \rfloor - 1\}} + c' \mathbb{1}_{\{x(\cdot,1) \leq q-1\}} - c_d(1), \end{aligned}$$

and for $t \geq 2$,

$$\begin{aligned} I_t(x) = \sum_{k=0}^{K-1} f_k/f_0 \Big[&(p - c'') + (c'' - c')P(P_k + B_k^t(x) \leq \lfloor \gamma q \rfloor - 1) \\ &+ c' P(P_k + B_k^t(x) \leq q - 1) \Big] - c_d(t). \end{aligned}$$

4.4.4 Index Policy with Geometric Assumption (GIP)

In this subsection we consider the case of Geometric($\bar{\alpha}$) number of visits, as describe in Section 4.3.1, and show the simplified expressions for the index. Note that the state is now described by a vector X. For a given vector $X = x$, the equations 4.11 - 4.13 simplify to

$$P_k \sim \text{Poisson}(\lambda \frac{1 - \bar{\alpha}^{k+1}}{1 - \bar{\alpha}}) \tag{4.22}$$

$$B_k^1(x) \sim \text{Bin}(x(1), \bar{\alpha}^{k+1}), \tag{4.23}$$

$$B_k^t(x) \sim \text{Bin}(x(t), \bar{\alpha}^k), \tag{4.24}$$

Using these random variables, the index function calculations now simplify to

$$
\begin{aligned}
I_1(x) = &\sum_{k=0}^{\infty} \bar{\alpha}^{k+1} \Big[(p - c'') + (c'' - c')P(P_k + B_k^1(x) \le \lfloor \gamma q \rfloor - 1) \\
&+ c' P(P_k + B_k^1(x) \le q - 1)\Big] \\
&+ (p - c'') + (c'' - c')\mathbb{1}_{\{x(\cdot,1) \le \lfloor \gamma q \rfloor - 1\}} + c' \mathbb{1}_{\{x(\cdot,1) \le q-1\}} - c_d(1),
\end{aligned}
$$

and for $t \ge 2$,

$$
\begin{aligned}
I_t(x) = &\sum_{k=0}^{\infty} \bar{\alpha}^{k} \Big[(p - c'') + (c'' - c')P(P_k + B_k^t(x) \le \lfloor \gamma q \rfloor - 1) \\
&+ c' P(P_k + B_k^t(x) \le q - 1)\Big] - c_d(t).
\end{aligned}
$$

We call the policy using the above index function the Geometric Index Policy (GIP).

Note that the summation of the index function here has infinite terms. Numerically, one can truncate the summation at a k such that $\bar{\alpha}^k$ is small enough.

It is clear that both IP and GIP use the information of current scheduling state, arrival volume, and the profit structure. Also they account for the future arrivals, as well as the influence of current decision on future profits. Hence we expect IP and GIP to do better than SQP and NDP. (IP is guaranteed to do better than NDP.) We can also use GIP as an approximation to IP, by approximating a non-geametric distribution by a geometric distribution with the same mean. We expect GIP to perform similarly to IP with less computational efforts. In the next Section 4.5, we show the performances under the four heuristic policies using discrete event simulation.

4.5 Numerical Examples with Real Data

In this section use the data and parameter estimation from Chapetr 3 and compare of the performance of policies, evaluate of the benefits of accounting for the series nature of the patients, and find optimal staffing levels.

4.5.1 Comparison of Policies

In this section we compare the long run average profits per day earned by the four policies: NDP, SQP, IP and GIP. We can compute this exactly for the NDP. For the other policies we estimate this by using simulation. In all our simulations, we use a weekly cycle model with $T = 5$, and assume 15% overtime slots are allowed (that is, $\gamma = 1.15$). We also consider a linear delay penalty in the number of days delayed, namely $c_d(t) = 15(t-1)$, for $t = 1, \cdots, 5$. Specifically, we simulate two systems: the actual system, with $\lambda = 1.2$ and a larger (fictitious) system with with $\lambda = 6$. In both systems we assume that the pmf of the number of visits is as found in data. When simulating the GIP, we approximate this empirical distribution by a Geometric distribution with parameter $\bar{\alpha} = .82364$, so as to keep the means same as the empirical mean. Note that this leads to the expected total number of visits per day in the two systems to be $\lambda EV = 5.6043$ for $\lambda = 1.2$ and $\lambda EV = 28.0213$ for $\lambda = 6$.

We consider two choices for the reward/cost parameters:

Case A: $p = 100$, $c = 20$, $c' = 30$, and $c'' = 200$, since the clinics usually pay 50% higher than the regular salary for overtime work. We assume $c'' = 200$ to account for the loss of revenue, plus the cost of rerouting the overbooked customers to an alternate physical therapy unit. We numerically compute the optimal staffing level q under NDP as the initial guess: $q = 8$ when $\lambda = 1.2$ and $q = 32$ when $\lambda = 6$. Thus we have $q > \lambda EV$ for both cases.

Case B: $p = 100$, $c = 70$, $c' = 80$, and $c'' = 200$. Again, we obtain numerically the optimal staffing level under NDP as the initial guess: $q = 7$ when $\lambda = 1.2$ and $q = 28$ when $\lambda = 6$. Compared with Case A, the costs of providing service are higher, which leads to the smaller optimal value of q under NDP. Now we have $q < \lambda EV$ for the large system.

Note that the staffing level q used in the simulations is optimal for the NDP, but need not be so for the other policies.

We use batch means method to build the 95% confidence intervals for the average profits under different policies. For each policy, we run the simulation for 5000 days, and the first 1000 days are taken as a warm-up period based on the Welch's method. (Heidelberger and Welch (1983)). The batch size is selected to be 100 days. The results are given in Tables 4.1 and 4.2 for cases A and B respectively. In each table, the parameters λ, q and g^{NDP} are given in the top row. the 'Mean' column records the average daily profit under the three policies. The 'Diff' gives the average increase in profit of each policy over that of NDP. The 'LB' and 'UB' columns give the lower and upper bounds on the 95% confidence interval of this difference.

Setting	$\lambda = 1.2$, $q = 8$, g^{NDP}=375.280				$\lambda = 6$, $q = 32$, g^{NDP}=2109.400			
Policy	Mean	Diff	LB	UB	Mean	Diff	LB	UB
IP	398.881	23.601	2.352	44.851	2170.411	61.010	15.173	106.847
GIP	399.733	24.453	3.134	45.771	2171.963	62.561	16.488	108.635
SQP	365.473	-9.807	-30.370	10.756	1881.858	-227.544	-273.647	-181.440

Table 4.1: Average Profits and Improvement from NDP in Case A

Setting	$\lambda = 1.2$, $q = 7$, g^{NDP}=9.498				$\lambda = 6$, $q = 28$, g^{NDP}=585.907			
Policy	Mean	Diff	LB	UB	Mean	Diff	LB	UB
IP	56.288	46.792	29.196	64.388	709.763	123.855	104.301	143.410
GIP	56.536	47.041	29.578	64.503	710.195	124.288	104.703	143.873
SQP	29.268	19.772	2.049	37.495	450.320	-135.587	-156.093	-115.082

Table 4.2: Average Profits and Improvement from NDP in Case B

We conclude from the tables that both IP and GIP perform significantly better than SQP and NDP, since the confidence intervals do not contains 0. The GIP performance is almost the same as that of IP, and hence GIP is preferable considering its reduced computation cost.

In Case A, we observe that in the small system (left half of the tables), the average profits under GIP and IP are close, both achieve an improvement of about 6.4% over NDP. In the large system (right half of tables), the improvements are about 2.9% over NDP. The superiority of the GIP and IP policies is more pronounced in Case B as compared to Case A. The improvement is about 494.7% in the small system, and 21.2% in the large system. That is, when the staffing level is low due to the higher service cost, the GIP and IP are more superior to other policies, especially in small systems.

The SQP does not perform well compared to IP and GIP, and in most cases does even worse than NDP. This is due to the fact that SQP ignores the profit structure, especially the delay cost.

With all these considerations, we recommend the GIP policy in general. IP policy also performs well but it involves a lot more computation in the form of convolution of multiple distributions.

4.5.2 Optimal Staffing

In this subsection we use simulation to find the optimal static staffing level q, assuming we follow GIP for scheduling. We first compute the optimal q for NDP (see Section 4.4.1) and then search a small neighborhood of this q and plot the long run average profits of GIP as a function of q and pick the q that maximizes the profits.

In Figures 4.2 $\sim$ 4.5, we show the simulated results of applying this procedure in the two systems for both Case A and Case B. The graphs show the average profits with varying q and the vertical error bars show the 95% confidence interval. In Case A, the simulated estimate of the average profits under GIP are maximized at $q = 7$ and $q = 30$, for the small and the large system respectively, which are less that than the optimal $q = 8$ and $q = 32$ computed under NDP. Similarly for the small system in Case B we get an optimal q of 6, while that for NDP is $q = 8$. For the large system in Case B, the profits under GIP are maximized at $q = 28$ which is the same as that under NDP. Note that this is the only case out of the four with $q < \lambda EV$. So when the optimal q under NDP is higher than the actually expected demand, we expect a smaller q to maximize the profit under GIP. In fact the profits using the optimal q under GIP are very close to those using the optimal q found under NDP. Thus, the simulation results show that the initial guess of optimal staffing levels which achieves optimality under NDP are good approximations to the real optimal staffing level.

Finally, recall in Section 4.5.1 we compare the performance of GIP to that of NDP using q values that are optimal for NDP, and we find that GIP offers a significant improvement over NDP. This improvement can only become better if we had carried out the comparison using the staffing level that is optimal for GIP. This makes our case for GIP even stronger.

4.5.3 Value of Modeling Series Patients

A key contribution of this chapter is that we explicitly model "series" nature of patients. An alternate solution to this problem is to ignore the "series" nature of the patient appointments and treat each appointment as an independent new arrival. We run numerical experiments to evaluate the benefit of taking into account the "series" nature of patient appointments. To do so, we apply GIP assuming that the daily demand comes from patients with single

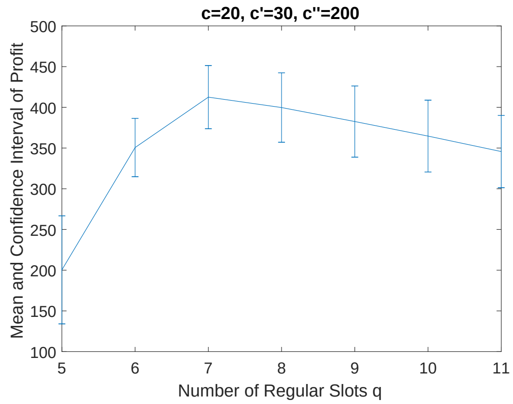


Figure 4.2: Case A, $\lambda = 1.2$

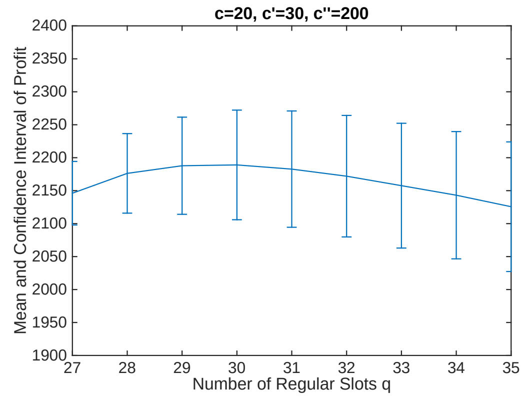


Figure 4.3: Case A, $\lambda = 6$

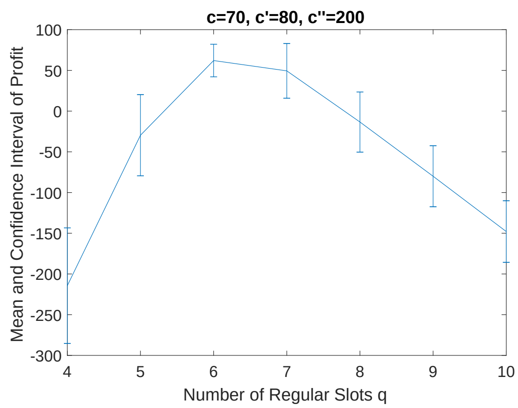


Figure 4.4: Case B, $\lambda = 1.2$

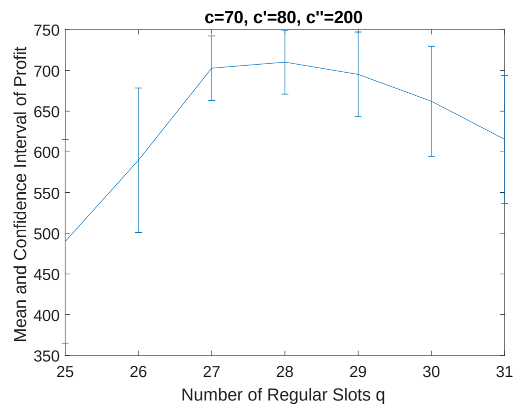


Figure 4.5: Case B, $\lambda = 6$

appointment visits (that is, Geometric with $\bar{\alpha} = 0$ instead of Geometric with $\bar{\alpha} = .8236$) and inflating the arrival rate correspondingly to keep the total expected demand for appointments per day identical to the system with series appointments. In Table 4.3, we compare the profits under GIP with series appointment assumption ($\bar{\alpha} = .8236$, fourth column) and under GIP without such assumption but inflating λ by multiplying EV ($\bar{\alpha} = 0$, fifth column). Simulations are performed under 4 parameter settings as the previous subsection. We find that treating the series appointments as independent arrivals reduces the average daily profit by at least 2.7% in Case A and 25.8% in Case B, compared to the GIP that explicitly considers the series nature of appointments (Reduction% column). Note that the choice of q in these experiments is a result of the optimization discussed in the previous subsection.

Thus, capturing the "series" nature of patient appointments leads to a significant improvement in the performance of the system, and hence is important from a practical perspective.

Table 4.3: Value of Modeling Series Patients

Case	λ	q	GIP ($\bar{\alpha} = .823$)	GIP ($\bar{\alpha} = 0$)	Reduction%
A	1.2	7	412.51	401.40	2.7%
A	6	30	2189.05	2079.89	5.0%
B	1.2	6	50.63	-21.28	142.0%
B	6	28	710.20	527.12	25.8%

4.6 Conclusions and Extensions

This work is motivated by the phenomenon that patients need a series of appointments in many specialty care clinics to fully complete their treatment.

We initially assumed a fixed staffing level, and that each patient needs a random number of total visits with a constant inter-visit time. We modeled the scheduling system as an MDP, with the state being a combination of the current schedule and the number of new arrivals. Since computing the optimal policy is intractable due to the size of state space, we proposed a heuristic policy called the Index Policy. Setting up the problem as an MDP helps us use the policy iteration algorithm to design the Index policy. As a result, finding an index in our model is considerably easier since we can use a greedy algorithm to solve the optimization problem exactly.

We compared the performance of the Index policy to two other commonly used policies: the Next Day policy and the Shortest Queue policy. Using the data collected from a local PT center, we used a simulation study to show that the Index policy performs very well compared to commonly used policies under different parameter settings. Furthermore, the Index policy (and its variation called the Geometric Index Policy) can be used for scheduling appointments on arrival of each patient, i.e., it does not require the knowledge about the total number of patients arriving on a day while scheduling a patient. Hence, we recommend the Geometric Index policy for use in practice, due to its superior performance and ease of implementation. We established that explicitly modeling the series nature of patient appointments leads to more beneficial policies than approaches that treat each arrival as an independent new arrival. We also showed that the optimal static staffing level for the GIP can be well approximated by the optimal static optimal staffing level for the NDP, which can be analytically computed.

We can study the same extensions to Model II as we mentioned in Model I in Chapter 3.

CHAPTER 5: Optimal policies for Investment Capacity and Timing

5.1 Introduction

When investors or companies plan to introduce a new product or a new technology to the market, the actual demand is uncertain at the very beginning. After a leader firm initiates an investment, the competitors start observing the performance of this leader firm and making their decisions accordingly. In particular, there are two most critical decisions for the follower firms to make, the first is the investment timing, that is when to enter the market so as to make profit with enough knowledge about the market; and the second is the investment capacity, that is how much to invest so as to better suit the unknown market demand.

With a competitive relationship between the leader and the follower firms, the timing and the capacity decisions interplay with each other and affects the decisions for both firms. For example, the follower firm may benefit from a late investment time if it can gather more valuable information from the leader and hence make a wiser decision on how much to invest. But entering the market earlier can also be beneficial for the follower since waiting does not provide profits that can otherwise earned from investment. On the other hand, the leader can invest a smaller amount at the beginning so as to share less information for the follower, and therefore delay or mislead the follower's investment decisions. Of course the leader firm may also earn more profits if the belief in the market is very positive and the leader can do nothing to delay the follower's entrance. The existing literature mostly focuses on one of the two decision making process, investment timing and capacity, separately. To the best of our knowledge, this is one of the first papers studying the interplay of optimal investment timing and capacity.

In this chapter, we build models for a duopoly game of investment with timing and capacity decision options. We assume that the investment is costly, and the firms' earning processes are based on their initial investment capacity, that is the maximum earning rates are no more than the capacities invested at the beginning. The uncertain demand of the new market could be one of the binary status: high and low. A leader enters the market at the beginning of the horizon, and a follower starts its observation of the leader's earning process and decides on when and how much to invest. The leader, known that the follower shall adopt the optimal investment strategy based on the leader's action, shall also optimize its strategy about how much to invest.

The two firms are competitive, such that once the follower enters the market, it shall take part of the market share from the leader, and since then the overflow from either of the firms cannot be fulfilled by the other party, even the other party has extra supply. Therefore we consider products or technologies that have a high customer loyalty so that once the customers subscribe to one of the brands, the chance for them to switch brand is very small. For example, Amazon is one of the first firms to develop and lunch the smart speaker "Amazon Echo", which is a smart device that users can command with their voice, interact and control other devices around the home (Forbes 2017). Most customers of this smart speaker are Amazon Prime members, which is another paid service offered by Amazon. To play music with the speaker, they need Amazon Music accounts. Almost all of the functions are also compatible certain devices, such that once a customer buys an Amazon Echo and builds a echo-home system, it will be costly to switch to another brand.

Another competitor, Apple Inc. plans to enter the market in December 2017. After purchasing their smart speaker, it is obvious that the customers will purchase or already have purchased relevant products from Apple to support the smart speaker, such as Apple TV, iPhone, iWatch, etc. (New York Times 2017). Since the two companies use different operation systems (Apple uses IOS, and Amazon uses Fire), plus the $200 value of the smarter speaker only, converting from one device to another will cost a lot for the customer.

Another example that our model can be applied to, is the market of single-serve coffee machine. Once the customer purchases a machine from the company Keurig, they start to constantly purchase the k-cup coffee, and therefore there is little chance for them to switch brand since the k-cup coffee is only fitted into a Keurig machine. Similarly, our model can help investors of sports clubs to make their investment decisions. For examples, the two football teams Manchester City and Manchester United share one sports market: the football fans in Manchester, UK. Back in 1894, Manchester City football club was founded and acted as a leader in the sports market. By observing the financing situation of Manchester City, the investor of the Manchester United could learn and optimize the capacity and timing of investment as a follower. In this example, once a customer decides on supporting one of the team, the team loyalty would prevent the switching between the two teams, which is similar to a brand loyalty in the smart speaker market.

Motivated by these real-life examples, we model the firms' earning process in a continuous time setting with Brownian model, and update the belief in the uncertain market with Bayesian learning method. This chapter aims to answer the following questions: (1) When shall the follower enter the market, and how much shall it invest; (2) Assume the leader firm enters the market at the beginning of horizon, how much shall it invest to optimize its profit while delaying the entrance of the follower as much as possible; (3) How does the initial belief in the uncertain demand affect the follower and leader's decisions on both investment timing and capacity. We shall give the main contribution of this chapter in the next subsection, which includes the answers to the above questions.

5.1.1 Summary of Main Results

We analyze the structural properties of the optimal policies for both follower and leader firms. In Section 5.3.1 we develop the follower's policy in equilibrium assuming the investment capacity of the leader has been given. Proposition 5.2 describes a two-threshold policy for the follower, which achieves the optimality under given condition. The policy works as follows: when the initial belief for the uncertain market is no less than an upper threshold, the follower shall invest immediately with a large capacity; when the initial belief is more than a lower threshold, the follower shall also invest immediately but with a smaller capacity. If otherwise the initial belief is between the two thresholds, it is optimal for the follower to wait and observe the performance of the leader.

In Section 5.3.2, we first analyze the leader's policy in equilibrium under two limits: if the noise level is 0 or infinity. Proposition 5.4 shows that the leader shall follow a one threshold policy under the two limits. By formulating the value function of the leader given the aforementioned two thresholds of the follower, we identify the sufficient conditions under which it is optimal for the leader to invest with a capacity such that the follower shall always wait and observe as long as the initial belief is within its lower and upper thresholds.

One of the most important findings of this chapter is in Proposition 5.6, where we show the sufficient conditions under which the equilibrium value function of the leader in fact decreases even with the increasing initial belief of the market. We find such sufficient conditions imply an unfavorable investment environment for the leader, such as a smaller investment cost of the follower, and a smaller fraction of market after the follower's entrance. Under those conditions, the leader has to make very costly changes to delay follower's entrance. Numerical examples are given after the proposition to illustrate such special phenomenon with explanations regarding the impact of follower's investment timing on the leader's value function.

In Section 5.5 we propose a simpler model by assuming a lower bound on the investment capacity for both leader and follower. Under this assumption, the follower's policy in equilibrium reduces to a one-threshold policy, therefore we are able to derive the explicit formula of this threshold. Again, we obtain the sufficient conditions under which the leader's value function can be decreasing in the initial belief. This is given in Proposition 5.9. Another interesting finding of this chapter is presented in Proposition 5.11 and Corollary 5.1. There we found that the follower's threshold is in fact increasing in the initial belief under certain conditions. Basically, by observing higher initial belief, the follower tends to be more ambitious and sets a higher investment threshold.

5.1.2 Literature Review

This chapter is closely related to two main streams of literature: learning process with stopping and capacity management. Optimal stopping problem is an extension of the classic sequential testing problem. With the objective of minimizing the cost of choosing the incorrect hypothesis, the probability of correct hypothesis is updated over continuous observation. The decision maker can choose to reject the null hypothesis, accept the null hypothesis, or continue with the test for another observation. The pioneer work of this subject has been done by Wald [1945], where the problem is introduced under a discrete time setting. In the paper by Jensen [1982], the observation updating is achieved by Bayesian decision models when the profitability is uncertain during the learning process.

There are a few papers about learning process with stopping times that consider a single decision maker. For example, Kwon and Lippman [2011] use Brownian model with Bayesian updating, study how a single firm expand or exist a market after learning a pilot project. The learning process is about the unknown market and the firm can only gain information from the proceeds from the pilot project. Harrison and Sunar [2015] consider a learning process where the information observed and the cost are both dependent on the discrete learning modes (similar to the term capacity in this chapter). There is a single firm considered in their model, whose complete strategy consists of the learning mode plus a stopping time, when the firm decides either to invest or abandon the project. They study both discounted and undiscounted problems, and derive the effect of problem parameters on optimal critical beliefs. In the paper by Décamps et al. [2005], a single firm is involved in the learning process and he needs to decide on when to stop learning and invest. The investment cost is fixed and the learning process is independent of how much to invest.

Unlike their work, we have two firms in our model competing with each other. This differs from their models and results in two closely correlated policies for each firm. In particular, since we consider a duopoly problem, the critical beliefs of the follower are dependent of the leader's investment capacity and hence dependent of the initial belief. But in their papers, the critical beliefs are independent of initial belief.

When multiple decision makers are planning to invest in the same project with uncertain profitability, they can learn from the competitors who enter the market earlier. Reinganum [1989] and Fudenberg and Tirole [1986] are two of the earlier work that apply tools from economics and game theory in optimal timing problems. One example of the most recent work in this topic is Kwon et al. [2015]. They examine a duopoly Brownian investment model, where the second firm can learn about the profitability of the investment by observing the performance of the first firm. Here both firms optimize each investment timing. They study both negative and positive externalities between the two firms. Our work is further differentiated from this literature because we design a learning process that is dependent on the capacity the firms invest. Below we review literature in this area.

The second stream that is relevant to our work is capacity investment and management. Van Mieghem and Dada [1999] study a model with a downstream firm (manufacturer) and an upstream firm (subcontractor), both of them have the option to decide on the two-stage adjustment to the investment capacities, production quantities and prices. They emphasize the impact of price over the sensitivities of capacity and production on uncertainty. Qi et al. [2017] consider a model with one supplier and two competitive firms to invest the new product from the supplier. There are two stage of the activity: firms first invest in the supplier, then the capacity is realized, and firms place orders and serve the market. They focus on the equilibrium in terms of the number of investing firms and capacity levels. In another example also by Qi et al. [2015], they study two scenarios for a single firm's optimal strategy to adjust its capacity: to adjust once or multiple times, depending on the adjustment cost and managerial hurdles. Two decisions can be made for the firm: when and how much to invest. They consider multiple unknown demands over each period without using Brownian process. In Wang [2012], two market entry strategies are examined and their influences on the optimal capacity levels are evaluated. But no competition or learning is considered in their model. Overall, the main distinction of our work from the above papers is that we incorporate timing decisions with continuous learning process in our capacity management.

The following capacity management related papers also consider the timing factors. Boyacı and Özer [2010] consider a manufacturer who can postpone the capacity investment to acquire advance sales information through advance selling. Hence the manufacturer decides on when and how much to invest, as well as the sale price during the normal sales season (after advance selling). They impose a cost of delaying investment by assuming a nondecreasing investment cost in time. Unlike our model, they use discrete time stochastic models, and the learning process is accomplished by updating the cumulative demands during the advance selling period by period. Swinney et al. [2011] analyze a competitive investment problems with a leader firm who invests in capacity early when the market is uncertain, and a follower who invest pater when the uncertainty has been resolved. The two firms include an established firm and a start-up firm. Therefore their goal is to maximize the probability of survival, while our work focuses on maximizing the total profits.

Other relevant fields include war of attrition. For example, Thijssen et al. [2006] focus on a preemption attrition equilibrium with Bayesian learning about the profitability of the project. Our model is different from their work since they assume that the true market information is available right after the leaders investment. In the paper by Décamps and Mariotti [2004], they also consider a war of attrition game. One difference of their model from ours is that, besides the shared public information between the leader the follower's firm, there are private information about the investment cost within each firm.

In the review paper by Van Mieghem [2003], many topics related to the strategic capacity management are included, for example, determining the capacity sizes, types, and timing of capacity adjustments under uncertainty, and partially irreversible capacity investment under uncertainty. Particularly, our work is most relevant to two sections of this review paper: Section 4.3, where he introduces game-theoretic capacity investments by multiple agents but with stationary constant investment; and Section 5, where the literature with stopping times is included. Our model is differentiated from these two streams of literature because we incorporate capacity management and optimal investment timing strategy by looking at the interactivity of duopoly firms.

5.2 The Model

There are two competing firms, each deciding a one-time capacity investment to enter a new market with unknown earning potential. Let $i = 1$ be the index for the first-mover firm, i.e., the *leader*, and $i = 2$ be the index for the *follower*. The leader chooses its capacity $K_1 \geq 0$ and invests at $t = 0$, while the follower decides both its investment timing $\tau \geq 0$ and capacity $K_2 \geq 0$. The investment is costly: Firm i incurs a lump sum cost of $c_i K_i$ at the time of its investment.

We take a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which independent standard Brownian motions $B_1 \doteq \{B_{1,t}, t \geq 0\}$ and $B_2 \doteq \{B_{2,t}, t \geq 0\}$, and a binary random variable μ independent of these Brownian motions are defined.

The leader and the follower sell substitutable products. After the follower's investment, firm i's market size is $\mu_i > 0$, $i = 1, 2$; before the follower's investment, the leader's market size is equal to the total market size $\mu \doteq \mu_1 + \mu_2$ because the leader is the only firm in the market for $t \in [0, \tau]$.

The leader's cumulative earning process is represented by $X_1 \doteq \{X_{1,t}, t \geq 0\}$ such that

$$dX_{1,t} = \min(K_1, \mu)dt + \sigma dB_{1,t}, \quad t \leq \tau, \tag{5.1}$$

and

$$dX_{1,t} = \min(K_1, \mu_1)dt + \sigma dB_{1,t}, \quad t > \tau, \tag{5.2}$$

where σ is a positive constant. The follower's cumulative earning process is $X_2 \doteq \{X_{2,t}, t \geq \tau\}$ where

$$dX_{2,t} = \min(K_2, \mu_2)dt + \sigma dB_{2,t}, \quad t > \tau. \tag{5.3}$$

Note that firm i's expected earning rate is driven by the minimum of its market size and capacity, which is the selling quantity of firm i at any given time instant.

The caveat is that the favorableness of the market is unknown to both firms; the market can be either favorable or unfavorable for firms. If the market is favorable, $\mu_1 = \mu_{1,H}$ and $\mu_2 = \mu_{2,H}$; if it is unfavorable, $\mu_1 = \mu_{1,L}$ and $\mu_2 = \mu_{2,L}$ where $\mu_{i,j}$ is a positive constant and $\mu_{i,H} > \mu_{i,L}$ for $i = 1, 2$ and $j = H, L$. This implies that the total market size μ could be either $\mu_H \doteq \mu_{1,H} + \mu_{2,H}$ or $\mu_L \doteq \mu_{1,L} + \mu_{2,L}$. Firms initially believe that the market is favorable, i.e., $\mu = \mu_H$, with probability π_0. This probability will be referred to as the *initial belief*. The follower dynamically updates its initial belief based on the leader's cumulative earning process X_1. Let $\mathcal{F}_t^{X_1} \subseteq \mathcal{F}$ be the smallest sub-σ-algebra that satisfies the standard properties such as completeness and right-continuity, and $\{X_{1,s}, 0 \leq s \leq t\}$ are measurable with respect to $\mathcal{F}_t^{X_1}$. Then, $\mathcal{F}^{X_1} \doteq \{\mathcal{F}_t^{X_1}, t \geq 0\}$ will be called the *natural filtration* of the X_1 process. In light of this, we define the follower's posterior belief process $\{\pi_t, t \geq 0\}$ as

$$\pi_t \doteq \mathbb{P}(\mu = \mu_H | \mathcal{F}_t^{X_1}), \ t \geq 0.$$

By Theorem 9.1 of Lipster and Shiryayev [1977], the follower's posterior belief process evolves as follows.

Lemma 5.1. *For any given capacity $K_1 \geq 0$ of the leader, the posterior belief process $\pi = \{\pi_t, t \geq 0\}$ satisfies the stochastic differential equation*

$$d\pi_t = \sqrt{q(K_1)}\pi_t(1 - \pi_t)dW_t, \quad t \geq 0,$$

where

$$q(K_1) \doteq \frac{1}{\sigma^2}\left[\min(K_1,\mu_H) - \min(K_1,\mu_L)\right]^2 \tag{5.4}$$

and $W = \{W_t, t \geq 0\}$ *is a standard Brownian motion with respect to the filtration* $\left\{\mathcal{F}_t^{X_1}, t \geq 0\right\}$. *In particular,*

$$W_t = \frac{1}{\sigma}X_{1,t} - \frac{1}{\sigma}\int_0^t \left[\pi_s \min(K_1,\mu_H) + (1-\pi_s)\min(K_1,\mu_L)\right] ds.$$

The term $q(K_1)$ in (5.4) can be interpreted as the *signal quality* or *information quality* of the follower's observation process X_1. By (5.1) and (5.2), the leader's capacity K_1 affects the evolution of the process X_1 which, in turn, impacts the evolution of the follower's posterior belief process π through the signal quality. Hereafter, we will use y_0 and y as generic notations for initial and posterior beliefs, respectively. The posterior belief process is defined for the entire time horizon $[0,\infty)$. This is just for mathematical convenience; our analysis will show that eventually, only the posterior belief process in $[0,\tau]$ will be relevant.

We now formulate the leader's and the follower's expected discounted profits for any discount rate $\lambda > 0$. A stopping time τ is called *admissible* for the follower if it is adopted to the natural filtration $\mathcal{F}^{X_1}$. Thus, the follower's feasible strategy is the combination of an admissible stopping time τ at which the follower stops learning about the favorableness of the market and invests, and the capacity $K_2 \geq 0$.

Based on this, for any given $K_1 \geq 0$, the follower chooses a feasible strategy (τ, K_2) to maximize its expected discounted profit:

$$\max_{(\tau,K_2)} \mathbb{E}\left[\int_\tau^\infty e^{-\lambda t} dX_{2,t} - e^{-\lambda\tau} c_2 K_2 \mid \pi_\tau = y\right]. \tag{5.5}$$

The follower's expected discounted profit under a feasible strategy (τ, K_2) is called its *value function* under that strategy. Considering the follower's best response for any $K_1 \geq 0$, the leader chooses $K_1 \geq 0$ to maximize its expected discounted profit with the initial belief y_0:

$$\max_{K_1} \mathbb{E}\left[\int_0^\infty e^{-\lambda t} dX_{1,t} - c_1 K_1 \middle| \pi_0 = y_0\right]. \tag{5.6}$$

Observe from (5.1) and (5.2) that the evolution of X_1 depends on τ, which is a decision variable to the follower. We will also establish later that τ depends on the leader's capacity choice.

DEFINITION 1. The strategy profile $(\widetilde{K}_1, \widetilde{K}_2, \widetilde{\tau})$ is an equilibrium if and only if

$$\widetilde{K}_1 = \arg\max_{K_1 \geq 0} \mathbb{E}\left[\int_0^\infty e^{-\lambda t} dX_{1,t}^{(K_1,\widetilde{\tau})} - c_1 K_1 \middle| \pi_0 = y_0\right] \tag{5.7}$$

$$\text{subject to} \quad (\widetilde{\tau}, \widetilde{K}_2) = \arg\max_{(\tau,K_2)} \mathbb{E}\left[\int_\tau^\infty e^{-\lambda t} dX_{2,t} - e^{-\lambda\tau} c_2 K_2 \mid \pi_\tau^{\widetilde{K}_1} = y\right], \tag{5.8}$$

where $\pi^{\widetilde{K}_1}$ stands for the belief process when the leader's capacity investment is $\widetilde{K}_1$, and $X_1^{(K_1,\widetilde{\tau})}$ represents the leader's cumulative earning process with the capacity K_1 and the follower's investment timing $\widetilde{\tau}$.

5.3 Analysis

First, we assume that $c_2\lambda < 1$ otherwise the follower shall never invest. To characterize equilibrium strategies of firms, we shall solve backwards. Section 5.3.1 identifies the follower's optimal policy for a given K_1. Based on this analysis, Section 5.3.2 characterizes equilibrium strategies of both firms. We begin with the analysis of the follower's problem.

5.3.1 Follower's Problem

For any given $K_1 \geq 0$, define the follower's optimal value function as

$$V_2(y; K_1) \doteq \max_{(\tau,K_2)} \mathbb{E}\left[\int_\tau^\infty e^{-\lambda t} dX_{2,t} - e^{-\lambda\tau} c_2 K_2 \mid \pi_\tau = y\right]. \tag{5.9}$$

If $V_2(\cdot; K_1)$ is C^1 and piecewise C^2, $V_2(\cdot; K_1)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$0 = \max\Big\{ \max_{K_2 \geq 0} \{g(y, K_2)\} - V_2(y; K_1),\ \frac{1}{2} q(K_1) y^2 (1-y)^2 V_2''(y; K_1) - \lambda V_2(y; K_1)\Big\}, \tag{5.10}$$

where $g(y, K_2)$ is the follower's total expected discounted profit at τ with capacity K_2 and belief $\pi_\tau = y$, and equal to

$$\mathbb{E}\left[\int_0^\infty e^{-\lambda t} dX_{2,t} - c_2 K_2 \mid \pi_\tau = y\right] = \left[\frac{1}{\lambda}[y \min(K_2, \mu_{2,H}) + (1-y)\min(K_2, \mu_{2,L})] - c_2 K_2\right]. \tag{5.11}$$

Hereafter, we will use the following notation for brevity:

$$g(y) \doteq \max_{K_2 \geq 0} \{g(y, K_2)\}.$$

The heuristic derivation of (5.10) can be found in Appendix A.3.1. We will not directly use the HJB equation (5.10) in our analysis. Rather, we will use (5.10) to come up with a guess for the follower's optimal policy. The structural properties of the HJB equation suggest that a candidate for the follower's optimal policy is a *two-threshold* policy.

For a given K_1, a two-threshold investment policy is summarized by an upper belief threshold $h(K_1)$, a lower belief threshold $\ell(K_1)$, and the follower's capacity investment. Under such a policy, the investment time is given by

$$\tau \doteq \inf\{t \geq 0 : \pi_t \in [0, \ell(K_1)] \cup [h(K_1), 1]\},$$

and the follower's investment capacity is

$$K_2 = \begin{cases} \mu_{2,L}, & \pi_\tau \in [0, \ell(K_1)], \\ \mu_{2,H}, & \pi_\tau \in [h(K_1), 1]. \end{cases}$$

Note that both lower and upper belief thresholds depend on K_1. Below, we will show the existence of a two-threshold policy whose value function satisfies certain properties.

Proposition 5.1. *If $K_1 > \mu_L$, there exists a two-threshold investment policy $(\ell^*(K_1), h^*(K_1))$ under which the follower's value function $v(\cdot; K_1)$ is C^1, piecewise C^2, and satisfies the following* *conditions:*

$$g(y) \leq v(y; K_1), \quad y \in (0,1), \tag{5.12}$$

$$\frac{1}{2} q(K_1) y^2 (1-y)^2 v''(y; K_1) \leq \lambda v(y; K_1), \quad y \in (0,1) \setminus \{\ell^*(K_1), h^*(K_1)\}. \tag{5.13}$$

Proposition 5.2 proves the optimality of the policy analyzed in Proposition 5.1 by showing that its value function is an upper bound on the value function of any other feasible policy.

Proposition 5.2. (***a***) *For any given $0 \leq K_1 \leq \mu_L$, it is optimal to invest at $\tau = 0$ with the following capacity:*

$$K_2^* = \begin{cases} \mu_{2,L}, & y_0 \in [0, \ \ c_2\lambda], \\ \mu_{2,H}, & y_0 \in (c_2\lambda, \ 1]. \end{cases}$$

(***b***) *For any given $K_1 > \mu_L$, the two-threshold policy analyzed in Proposition 5.1 is an optimal policy for the follower. Specifically, for the follower, it is optimal to wait whenever $\pi_t \in (\ell^*(K_1), h^*(K_1))$; otherwise, it is optimal to invest the following capacity*

$$K_2^* = \begin{cases} \mu_{2,L}, & \pi_\tau \in [0, \ \ \ell^*(K_1)], \\ \mu_{2,H}, & \pi_\tau \in [h^*(K_1), \ 1]. \end{cases}$$

Proposition 5.2 shows that the structure of the follower's optimal policy depends on K_1. For K_1 in part (**a**), the signal quality $q(K_1) = 0$, meaning that the follower does not obtain any information by waiting and observing the leader's cumulative earning process. As a result, it is optimal for the follower to invest immediately to avoid any loss due to discounting. Part (**b**) shows that if the leader invests in a capacity that provides information for the follower, it is optimal for follower to use a two-threshold policy; thus, it is optimal for the follower to wait as long as the belief is moderate.

In both parts (a) and (b), the follower optimally invests a large (respectively, small) capacity if its belief is sufficiently large (respectively, small). This is because the follower invests more if it believes that the market size is larger (i.e., $\mu_2 = \mu_{2,H}$).

As verified in Proposition A.1, $\ell^*(K_1)$ decreases with the signal quality $q(K_1)$ and $h^*(K_1)$ increases with $q(K_1)$. (A similar insight is identified by Harrison and Sunar [2015] in a different setting that focuses on a single firm.) These findings demonstrate that, as the signal quality increases, the belief region $(\ell^*(K_1), h^*(K_1))$ becomes wider and the follower is more likely to follow a two-threshold policy.

The following proposition analyzes the convergence rates of $\ell^*(K_1)$ and $h^*(K_1)$, which will be used later in obtaining the structures of the leader's policy.

Proposition 5.3. *The belief thresholds $\ell^*(K_1)$ and $h^*(K_1)$ of the two-threshold policy identified in Proposition 5.2-(b) satisfy the following properties:*

(***a***) *As $\sigma \to \infty$,*

$$c_2\lambda - \ell^*(K_1) = o(\sigma^{-1}) \quad \textit{and} \quad h^*(K_1) - c_2\lambda = o(\sigma^{-1}).$$

(***b***) *As $\sigma \to 0$,*

$$\ell^*(K_1) = O(\sigma^2) \quad \textit{and} \quad 1 - h^*(K_1) = O(\sigma^2).$$

5.3.2 Leader's Problem and Equilibrium Strategies

Using the follower's optimal policy identified in Proposition 5.2 for any given $K_1 \geq 0$, in this section, we will identify firms' equilibrium strategies. To do so, first, we shall provide more details about the leader's objective function. Recall from (5.6) that the leader's equilibrium profit is given by

$$V_1(y_0) \doteq \max_{K_1 \geq 0} H(K_1, y_0) \doteq \max_{K_1 \geq 0} \mathbb{E}\left[\int_0^\infty e^{-\lambda t} dX_{1,t} - c_1 K_1 \middle| \pi_0 = y_0\right].$$

Proposition 5.2 proves that under the follower's optimal policy, the pair (K_1, y_0) determines whether $\tau = 0$ or $\tau > 0$, which, in turn, impacts the form of $H(K_1, y_0)$ through the leader's expected earning rate. We will analyze $H(K_1, y_0)$ under two different cases: $\tau = 0$ and $\tau > 0$. We begin our analysis with the case $\tau = 0$. By Proposition 5.2, $\tau = 0$ when (a) $(K_1, y_0) \in [0, \mu_L] \times [0, 1]$ because the signal quality is 0 in that region, and (b) $(K_1, y_0) \in (\mu_L, \infty) \times [0, \ell^*(K_1)) \cup (h^*(K_1), 1]$ by the definition of two-threshold policy. Then, for $(K_1, y_0) \in \{[0, \mu_L] \times [0,1]\} \cup \{(\mu_L, \infty) \times [0, \ell^*(K_1)) \cup (h^*(K_1), 1]\}$, we have

$$\begin{aligned} H(K_1, y_0) &= H_0(K_1, y_0) \\ &\doteq \mathbb{E}\left[\int_0^\infty e^{-\lambda t} \min\{K_1, \mu_1\} dt + \int_0^\infty e^{-\lambda t} \sigma dB_{1,t} - c_1 K_1 \mid \pi_0 = y_0\right] \\ &= \mathbb{E}\left[\frac{\min\{K_1, \mu_1\}}{\lambda} - c_1 K_1 \mid \pi_0 = y_0\right] \\ &= y_0 \frac{\min\{K_1, \mu_{1,H}\}}{\lambda} + (1 - y_0) \frac{\min\{K_1, \mu_{1,L}\}}{\lambda} - c_1 K_1. \end{aligned} \tag{5.14}$$

On the other hand, if $(K_1, y_0) \in (\mu_L, \infty) \times (\ell^*(K_1), h^*(K_1))$, the follower will wait for a positive time until its posterior belief hits either $\ell^*(K_1)$ or $h^*(K_1)$, which implies $\tau > 0$. Thus this parameter region leads to a different structure of $H(K_1, y_0)$, which we denote as $H_1(K_1, y_0)$. Specifically, if $(K_1, y_0) \in (\mu_L, \infty) \times (\ell^*(K_1), h^*(K_1))$,

$$\begin{aligned} &H(K_1, y_0) \\ &= H_1(K_1, y_0) \\ &\doteq \mathbb{E}\left[\int_0^\tau e^{-\lambda t} \min\{K_1, \mu\} dt + \int_\tau^\infty e^{-\lambda t} \min\{K_1, \mu_1\} dt + \int_0^\infty e^{-\lambda t} \sigma dB_{1,t} - c_1 K_1 \middle| \pi_0 = y_0\right] \end{aligned} \tag{5.15}$$

$$\begin{aligned} &= -\mathbb{E}\left[\frac{\min\{K_1, \mu\} - \min\{K_1, \mu_1\}}{\lambda} e^{-\lambda \tau} \middle| \pi_0 = y_0\right] \\ &\quad + \frac{1}{\lambda}\left(y_0 \min\{K_1, \mu_H\} + (1 - y_0) \min\{K_1, \mu_L\}\right) - c_1 K_1. \end{aligned} \tag{5.16}$$

Here,

$$\begin{aligned} &\mathbb{E}\left[\frac{\min\{K_1, \mu\} - \min\{K_1, \mu_1\}}{\lambda} e^{-\lambda \tau} \middle| \pi_0 = y_0\right] \\ &= \frac{\xi_1(\ell^*(K_1)) - \xi_2(\ell^*(K_1))}{\lambda} \mathbb{E}\left[e^{-\lambda \tau} \mathbb{1}_{\{\pi_\tau = \ell^*(K_1)\}} \mid \pi_0 = y_0\right] \\ &+ \frac{\xi_1(h^*(K_1)) - \xi_2(h^*(K_1))}{\lambda} \mathbb{E}\left[e^{-\lambda \tau} \mathbb{1}_{\{\pi_\tau = h^*(K_1)\}} \mid \pi_0 = y_0\right], \end{aligned} \tag{5.17}$$

where

$$\xi_1(y) \doteq y \min\{K_1, \mu_H\} + (1-y)\min\{K_1, \mu_L\}, \tag{5.18}$$

$$\xi_2(y) \doteq y \min\{K_1, \mu_{1,H}\} + (1-y)\min\{K_1, \mu_{1,L}\}. \tag{5.19}$$

Based on (5.17), let

$$\phi_{\ell^*}(y; K_1) \doteq \mathbb{E}\left[e^{-\lambda\tau}\mathbb{1}_{\{\pi_\tau = \ell^*(K_1)\}} \mid \pi_0 = y\right]. \tag{5.20}$$

Then, by Itô's lemma, $\phi_{\ell^*}(y; K_1)$ satisfies the following ODE

$$\lambda\phi_{\ell^*}(y; K_1) = \frac{1}{2}q(K_1)y^2(1-y)^2\phi''_{\ell^*}(y; K_1), \tag{5.21}$$

subject to these boundary conditions:

$$\phi_{\ell^*}(\ell^*(K_1); K_1) = 1 \quad \text{and} \quad \phi_{\ell^*}(h^*(K_1); K_1) = 0. \tag{5.22}$$

The derivation of (5.21) can be found in Appendix A.3.6, and the boundary conditions follow from the fact that the follower stops learning and invests whenever its belief is $\ell^*(K_1)$ or $h^*(K_1)$. Solving for $\phi_{\ell^*}(y; K_1)$ using (5.21) and (5.22), we have

$$\phi_{\ell^*}(y; K_1) = a(\ell^*(K_1), h^*(K_1))y^{\frac{1+\eta}{2}}(1-y)^{\frac{1-\eta}{2}} + b(\ell^*(K_1), h^*(K_1))y^{\frac{1-\eta}{2}}(1-y)^{\frac{1+\eta}{2}} \tag{5.23}$$

for $y \in [\ell^*(K_1), h^*(K_1)]$ where

$$\eta \doteq \sqrt{1 + 8\lambda/q(K_1)} \tag{5.24}$$

$$a(x, y) = \left(x^{\frac{1+\eta}{2}}(1-x)^{\frac{1-\eta}{2}} - x^{\frac{1-\eta}{2}}(1-x)^{\frac{1+\eta}{2}}\left(\frac{y}{1-y}\right)^{\eta}\right)^{-1}, \tag{5.25}$$

$$b(x, y) = \left(x^{\frac{1-\eta}{2}}(1-x)^{\frac{1+\eta}{2}} - x^{\frac{1+\eta}{2}}(1-x)^{\frac{1-\eta}{2}}\left(\frac{1-y}{y}\right)^{\eta}\right)^{-1}. \tag{5.26}$$

Similarly, let

$$\phi_{h^*}(y;K_1) \doteq \mathbb{E}\left[e^{-\lambda\tau}\mathbb{1}_{\{\pi_\tau=h^*\}} \mid \pi_0 = y\right]. \tag{5.27}$$

Then, $\phi_{h^*}(y;K_1)$ satisfies the ODE in (5.21) and the following boundary conditions:

$$\phi_{h^*}(\ell^*(K_1);K_1) = 0 \quad \text{and} \quad \phi_{h^*}(h^*(K_1);K_1) = 1.$$

Based on these, for $y \in [\ell^*(K_1), h^*(K_1)]$, we have

$$\phi_{h^*}(y;K_1) = a(h^*(K_1),\ell^*(K_1))y^{\frac{1+\eta}{2}}(1-y)^{\frac{1-\eta}{2}} + b(h^*(K_1),\ell^*(K_1))y^{\frac{1-\eta}{2}}(1-y)^{\frac{1+\eta}{2}}, \tag{5.28}$$

where functions $a(h^*(K_1),\ell^*(K_1))$ and $b(h^*(K_1),\ell^*(K_1))$ are given in (5.25) and (5.26). Combining (5.16) and (5.17), $H(K_1,y_0)$ is equal to $H_1(K_1,y_0)$ for $(K_1,y_0) \in (\mu_L,\infty)\times(\ell^*(K_1),h^*(K_1))$, where

$$\begin{aligned} &H_1(K_1,y_0)\\ &= -\frac{\xi_1(\ell^*(K_1)) - \xi_2(\ell^*(K_1))}{\lambda}\phi_{\ell^*}(y_0;K_1) - \frac{\xi_1(h^*(K_1)) - \xi_2(h^*(K_1))}{\lambda}\phi_{h^*}(y_0;K_1)\\ &\quad + \frac{\xi_1(y_0)}{\lambda} - c_1K_1. \end{aligned} \tag{5.29}$$

Note from (5.18), (5.19),(5.23) and (5.28) that the functions $\xi_1(\cdot)$, $\xi_2(\cdot)$ ϕ_{ℓ^*} and ϕ_{h^*} all depend on K_1 through $\min\{K_1,\mu_L\}$, $\min\{K_1,\mu_H\}$, $\min\{K_1,\mu_{1,L}\}$ and $\min\{K_1,\mu_{1,H}\}$, as well as through the dependence of ℓ^* and h^* on K_1.

Remark 5.1. Observe from (5.14) and (5.29) that for $K_1 > \mu_H$, ξ_1, ξ_2, ℓ^*, h^*, ϕ_{ℓ^*}, ϕ_{h^*} do not change with respect to K_1, and $H(K_1,y_0)$ is affected by K_1 only via the last term $-c_1K_1$. This implies that the leader shall never invest more than μ_H in order to maximize its discounted profit.

Therefore the leader's equilibrium profit is the maximum of the optimal values of H_0 and H_1:

$$V_1(y_0) = \max\left\{\max_{K_1} H_0(K_1,y_0), \max_{K_1} H_1(K_1,y_0)\right\}.$$

If $V_1(y_0) = \max_{K_1} H_0(K_1,y_0)$ it means that it is optimal for the leader to select an investment capacity K_1 that leads to the follower's investment at $\tau = 0$. If $V_1(y_0) = \max_{K_1} H_1(K_1, y_0)$, it means that the leader should select an investment capacity K_1 that induces the follower to adopt a two-threshold policy.

Figure1 5.1 shows the feasible regions for H_0 and H_1 separated by $\ell^*(K_1)$ and $h^*(K_1)$.

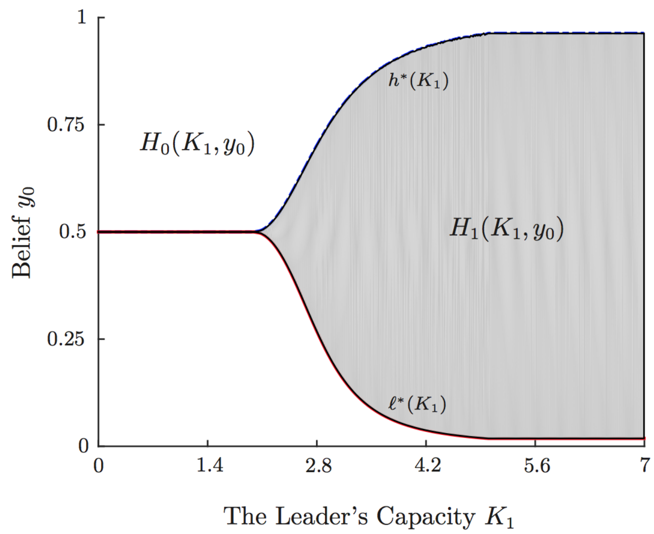


Figure 5.1: Feasible Regions for H_0 and H_1

Parameters used in this figure: $\lambda = 0.1$, $\sigma = 1$, $c_1 = c_2 = 5$, $\mu_L = 2$, $\mu_H = 5$, $\mu_{1,L} = 1$, $\mu_{1,H} = 2.5$, $\mu_{2,L} = 1$, $\mu_{2,H} = 2.5$.

5.3.2.1 Equilibrium

We begin our analysis of the equilibrium investment strategies under the following two limits of the volatility parameter σ: $\sigma \to \infty$ and $\sigma \to 0$.

Proposition 5.4. *Suppose that $\sigma \to 0$ or $\sigma \to \infty$. Then, in equilibrium, the leader's capacity investment is*

$$\widetilde{K}_1 = \begin{cases} \mu_{1,L}, & y_0 \leq c_1\lambda, \\ \mu_{1,H}, & y_0 > c_1\lambda, \end{cases}$$

and the follower enters the market at $\widetilde{\tau} = 0$ and invests

$$\widetilde{K}_2 = \begin{cases} \mu_{2,L}, & y_0 \leq c_2\lambda, \\ \mu_{2,H}, & y_0 > c_2\lambda. \end{cases}$$

Thus, as $\sigma \to 0$ or $\sigma \to \infty$, the follower always invests immediately, and both firms follow a one-threshold policy: firm i invests a larger capacity $\mu_{i,H}$ if the initial belief y_0 is high, or invests a smaller capacity $\mu_{i,L}$ if the initial belief is low.

Next, we show that, with a moderate σ, there exists a belief region such that $\tau > 0$ with probability 1 in equilibrium, and the optimal capacity for the leader can take values other than $\mu_{1,L}$ and $\mu_{1,H}$.

Proposition 5.5. *Consider $\sigma \in (0, \infty)$ and suppose that μ_L is not extremely large (i.e., $\mu_L < \mu_{1,H}$). Then, the leader's equilibrium value function is*

$$V_1(y_0) = \begin{cases} \max_{K_1 \geq 0} \ H_0(K_1, y_0) & \text{if} \quad y_0 \leq \underline{\underline{\alpha}}, \\ \max_{K_1 \in \mathcal{C}} H_1(K_1, y_0) & \text{if} \quad y_0 \in (\underline{\alpha}, \overline{\alpha}), \\ \max_{K_1 \geq 0} \ H_0(K_1, y_0) & \text{if} \quad y_0 > \overline{\overline{\alpha}}, \end{cases} \tag{5.30}$$

and the leader's equilibrium capacity investment is

$$\widetilde{K}_1 = \begin{cases} \mu_{1,L} & \text{if} \quad y_0 \leq \min\{\underline{\underline{\alpha}}, c_1\lambda\}, \\ \arg\max_{K_1 \in \mathcal{C}} \ H_1(K_1, y_0) & \text{if} \quad y_0 \in (\underline{\alpha}, \overline{\alpha}), \\ \mu_{1,H} & \text{if} \quad y_0 > \max\{\overline{\overline{\alpha}}, c_1\lambda\}, \end{cases} \tag{5.31}$$

for some belief thresholds $\underline{\alpha}$, $\overline{\alpha}$, $\overline{\overline{\alpha}}$ and $\underline{\underline{\alpha}}$ such that $\underline{\underline{\alpha}} \leq \underline{\alpha}$ and $\overline{\overline{\alpha}} \geq \overline{\alpha}$.

Remark 5.2. If the investment cost per unit capacity is the same for both firms, i.e., $c_1 = c_2$, then $\max\{\overline{\overline{\alpha}}, c_1\lambda\} = \overline{\overline{\alpha}}$ and $\min\{\underline{\underline{\alpha}}, c_1\lambda\} = \underline{\underline{\alpha}}$ in (5.31).

Remark 5.3. From Proposition 5.5 we see that there exists a belief region such that $\tau > 0$ w.p.1, therefore $\widetilde{K}_1(y_0) \in (\mu_L, \mu_H]$ over this region. We also have $\widetilde{K}_1(y_0) = \mu_{1,L}$ for small y_0 close to 0, which indicates that there is jump from $\mu_{1,L}$ to some value above μ_L ($\mu_L > \mu_{1,L}$) as y_0 increasing from 0.

One interesting finding of this chapter is that, under certain conditions, the optimal profit of the leader as function of initial belief y_0 is not necessarily non-decreasing. We demonstrate in the next proposition such detailed conditions, which are unfavorable to the leader and thus leads to the costly behavior of the leader to delay the follower's entrance.

Proposition 5.6. *Suppose that μ_L is not extremely large (i.e., $\mu_L < \mu_{1,H}$). Then, with a moderately large value of σ, there exists an initial belief region such that the leader's equilibrium value function $V_1(y_0)$ decreases in y_0, if $c_2\lambda < 1/2$ and the following condition is satisfied:*

$$\frac{\mu_H - \mu_L}{\mu_{2,L}} < \frac{1 - 2c_2\lambda}{2c_2\lambda}.$$

Note that the conditions require small values of c_2, $\mu_{1,L}$, and large value of $\mu_{2,L}$, both indicating a setting that is unfavorable to the leader. In Figure 5.2, the left panel shows the leader's equilibrium value function against y_0, and we select three initial beliefs labeled in red stars as examples within the decreasing region. In the right panel, the 3 curves show the LST of τ defined as $\mathbb{E}_{K_1}(e^{-\lambda\tau}|\pi_0 = y_0)$ corresponding to 3 fixed y_0 labeled in the left panel. The red stars in the right panel also display the equilibrium capacity under the corresponding initial beliefs.

From the right panel, we find that as y_0 increases, the leader has to increase the capacity so as to delay the follower's entrance to the market. However, as the curve becoming less sensitive to the capacity, the leader has to take costly actions and therefore the profits earned is decreasing even with an increasing belief in market.

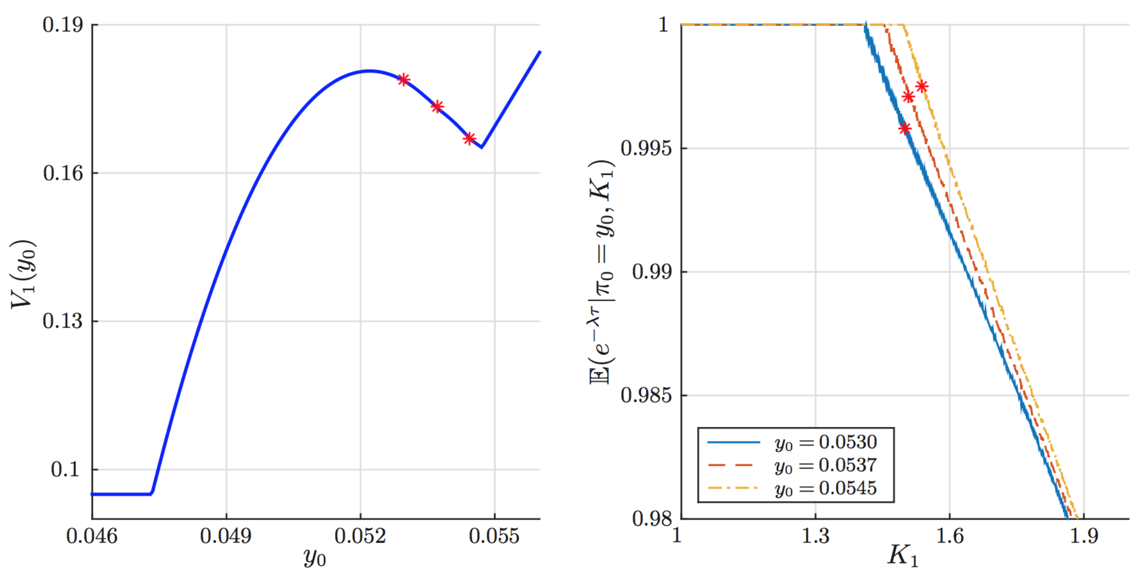


Figure 5.2: V_1 as Functions of y_0, and LST of τ as Function of K_1
Parameters used in this figure: $\lambda = 0.1, \quad c_1 = c_2 = 0.5, \quad \sigma = 1.5, \quad \mu_L = 1, \quad \mu_H = 9, \quad \mu_{1,L} = 0.01, \quad \mu_{1,H} = 1.5.$

5.4 Numerical Experiments

In this section, we focus on several numerical examples to show the structure of the leader's equilibrium capacity as function of y_0, also the leader and follower's value function in equilibrium, both of which could decrease when y_0 increases.

Here we fix the following parameters $\sigma = 1, \lambda = 0.1, c_1 = c_2 = 5, \mu_L = 1, \mu_{1,H} = 10, \mu_{1,L} = 0.5$, and display 3 figures of $\widetilde{K}_1(y_0)$ with 3 different structures by varying $\mu_{1,H}$ (note that $\mu_{2,H} = \mu_H - \mu_{1,H}$ is also changing).

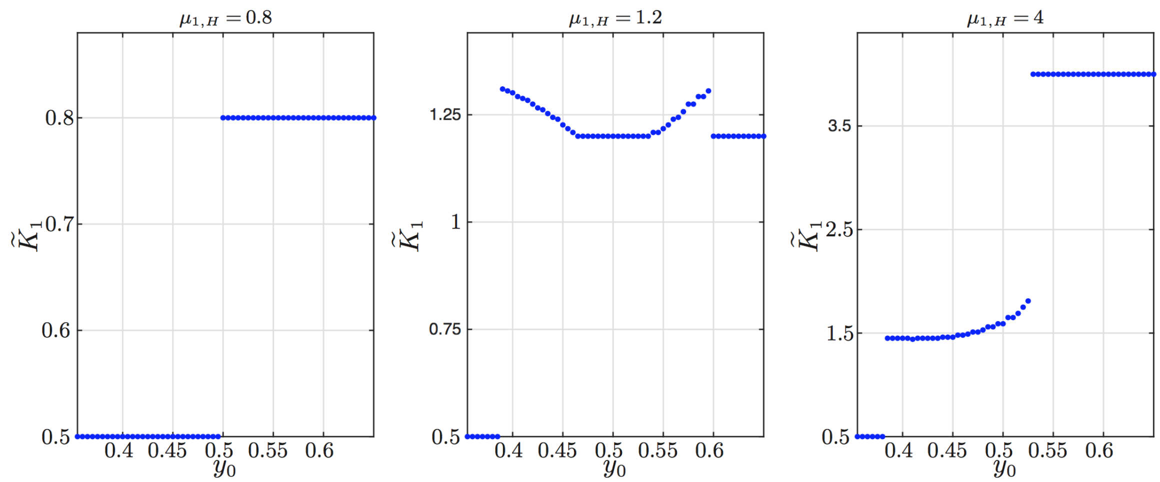


Figure 5.3: Leader's equilibrium capacity $\widetilde{K}_1$ as functions of y_0 by different values of $\mu_{1,H}$
Parameters used in this figure: $\lambda = 0.1,\ \ c_1 = c_2 = 5,\ \ \sigma = 1,\ \ \mu_L = 1,\ \ \mu_H = 10, \mu_{1,L} = 0.5$.

In the left panel, $\mu_{1,H} = 0.8$ and the leader is following a Bang-Bang type of policy: invest at a lower level ($\mu_{1,L}$) when the initial belief is low, and a higher level ($\mu_{1,H}$) when the initial belief is high. The type of structure happens when $\mu_{1,H}$ is very small. Under this condition, the region $(\underline{\alpha}, \overline{\alpha})$ shrinks to an empty set, hence $\max_{K_1 ge 0 H_0(K_1,y_0)}$, hence the leader's optimal policy reduces to a one-threshold policy.

In the middle panel we have $\mu_{1,H} = 1.2$. When y_0 is very small or very large, the leader followers the policy with the similar Bang-Bang structure shown in the left panel. When y_0 is in a moderate region, the leader's optimal investment capacity decreases first before it hits $\mu_{1,H}$ and stay $\mu_{1,H}$ on a small region around $y_0 = c_2\lambda$, and then increases again. This policy structure happens more often for $mu_{1,H}$ close to μ_L. Recall in the proof of Proposition 5.5, we show $H(K_1, y_0)$ as function of K_1 by different values of y_0 in Figure A.10. Note that $H(K_1, y_0)$

contains two pieces: piecewise linear part for $K_1 \leq K_0(y_0)$, and concave part for $K_1 > K_0(y_0)$. As $mu_{1,H}$ getting closer to μ_L, the maximum point at $\mu_{1,H}$ of the piecewise part is also closer to the peak point of the concave part under moderate value of y_0, therefore it is easier to observe the optimality achieved above $\mu_{1,H}$. Besides, we can show that $K_0(y_0)$ is decreasing for moderately small $y_0 < c_2\lambda$ and increasing for moderately large $y_0 > c_2\lambda$, this leads to the first decreasing then increasing pattern of the equilibrium capacity.

In the right panel we have a relatively larger $\mu_{1,H} = 4$, the equilibrium capacity shows an increasing pattern on the region with moderate y_0. We first observe that $H(K_1, y_0)$ drops rapidly after $\mu_{1,H}$, hence the maximum can only be achieved at of before $\mu_{1,H}$. Also as we increase $\mu_{1,H}$ the concavity of $H(K_1, y_0)$ is reduced for a fixed y_0 comparing with earlier smaller values of $\mu_{1,H}$. Thus the maximum values of $H(K_1, y_0)$ is more relying on y_0, and the equilibrium capacity can never be decreasing in y_0.

5.5 Minimum Capacity Investment Requirement

In this section, unlike in Section 5.2, there is a minimum capacity investment requirement $\underline{\mu}$ for both firms to enter the market. Thus, unlike Section 5.3, the analysis of this model requires considering following capacity constraints: $K_1 \geq \underline{\mu}$ and $K_2 \geq \underline{\mu}$. To study a meaningful setting, this section considers $\underline{\mu}$ that is not extremely small or extremely large, i.e.,

$$\max\left\{\mu_L, \frac{\mu_{2,L}}{c_2\lambda}, \mu_{2,H}\right\} < \underline{\mu} < \min\left\{\mu_{1,H}, \frac{\mu_{2,H}}{c_2\lambda}\right\}. \tag{5.32}$$

The lower bound on $\underline{\mu}$ ensures that it is profitable for the follower to eventually enter the market, and the follower can learn about the favorableness of the market by observing the leader's earning process. The upper bound on $\underline{\mu}$ guarantees that the leader's investment strategy is a non-trivial one. Similar to Section 5.3.1, $V_2(\cdot; K_1)$ represents the follower's optimal value function for a given $K_1 \geq \underline{\mu}$. If $V_2(\cdot; K_1)$ is $\mathcal{C}^1$ and piecewise $\mathcal{C}^2$, $V_2(\cdot; K_1)$ satisfies the following HJB:

$$0 = \left\{\max_{K_2 \geq \underline{\mu}} g_2(y, K_2) - V_2(y; K_1), \frac{1}{2}q(K_1)y^2(1-y)^2 V_2''(y; K_1) - \lambda V_2(y; K_1)\right\} \tag{5.33}$$

Proposition 5.7. *Given any fixed capacity of the leader* $K_1 \geq \underline{\mu}$*, it is optimal for the follower to invest at*

$$\tau = \inf\{t \geq 0 : \quad \pi_t \geq \theta^*(K_1)\},$$

where $\theta^*(K_1)$ *is a belief threshold and equal to*

$$\theta^*(K_1) = \left(1 + \frac{\sqrt{1 + \frac{8\lambda\sigma^2}{(\min(K_1,\mu_H)-\min(K_1,\mu_L))^2}} - 1}{\sqrt{1 + \frac{8\lambda\sigma^2}{(\min(K_1,\mu_H)-\min(K_1,\mu_L))^2}} + 1} \frac{\mu_{2,H} - c_2\lambda\underline{\mu}}{c_2\lambda\underline{\mu} - \mu_{2,L}}\right)^{-1}. \tag{5.34}$$

Thus, it is optimal for the follower to invest immediately whenever the belief $\pi_t \geq \theta^*(K_1)$*, and wait and learn from the leader's cumulative earning process whenever* $\pi_t < \theta^*(K_1)$*.*

Remark 5.4. The belief threshold $\theta^*(K_1)$ is increasing in K_1. We will later prove in this section that the leader's equilibrium capacity $\widetilde{K}_1$ is dependent on the initial belief y_0, which implies that in equilibrium, belief threshold $\theta^*(\widetilde{K}_1)$ is a *function* that changes with the initial belief y_0. The optimal policy structure we identify has been identified in the literature; in the literature (see, for instance, Harrison and Sunar [2015]), in different contexts, the belief threshold for investment is shown to be a scalar that does not change with respect to the initial belief y_0.

Proposition 5.7 establishes the follower's optimal policy for a given $K_1 \geq \underline{\mu}$. Given this, we can now study the equilibrium investment strategies of firms.

Recall from (5.16) that the leader's value function for a given pair of (K_1, y_0) is

$$\begin{aligned} H(K_1, y_0) = &- \mathbb{E}\left[\frac{\min\{K_1, \mu\} - \min\{K_1, \mu_1\}}{\lambda} e^{-\lambda\tau} \middle| \pi_0 = y_0\right] \\ &+ \frac{1}{\lambda}\left(y_0 \min\{K_1, \mu_H\} + (1 - y_0)\min\{K_1, \mu_L\}\right) - c_1 K_1. \end{aligned} \tag{5.35}$$

As explained in Section 5.3.1, for a given pair (K_1, y_0), the leader's value function can take two different forms depending on whether $\mathbb{E}[\tau] > 0$ or $\mathbb{E}[\tau] = 0$ with that pair. If (K_1, y_0) is such that $\mathbb{E}[\tau] = 0$, then $H(K_1, y_0) = H_0(K_1, y_0)$ where $H_0(K_1, y_0)$ is as in (5.14). Note that from Proposition 5.7, a fixed K_1 implies a belief threshold $\theta^*(K_1)$ such that $\tau = 0$ with probability

1 if $y_0 \geq \theta^*(K_1)$. That is, $H(K_1, y_0) = H_0(K_1, y_0)$ for $(K_1, y_0) \in [\underline{\mu}, \infty) \times (\theta^*(K_1), 1]$.

On the other hand, if $(K_1, y_0) \in [\underline{\mu}, \infty) \times (0, \theta^*(K_1)]$, which implies $\tau > 0$ with probability 1, the leader's value function $H(K_1, y_0)$ takes another form. Specifically, by (5.32), the first term of $H(K_1, y_0)$ in (5.35) is given by

$$\mathbb{E}\left[\frac{\min\{K_1, \mu\} - \min\{K_1, \mu_1\}}{\lambda} e^{-\lambda\tau} \middle| \pi_0 = y_0\right] = \frac{1}{\lambda}\left(\xi_1(\theta^*(K_1)) - \xi_2(\theta^*(K_1))\right) \mathbb{E}\left[e^{-\lambda\tau} \middle| \pi_0 = y_0\right], \tag{5.36}$$

where $\xi_1(\cdot)$ and $\xi_2(\cdot)$ are as in (5.18) and (5.19). Note that in (5.36),

$$\mathbb{E}\left[e^{-\lambda\tau} \middle| \pi_0 = y_0\right] = \frac{\phi(y_0)}{\phi(\theta^*(K_1))}, \tag{5.37}$$

where

$$\phi(z) = z^{\frac{1+\eta}{2}}(1-z)^{\frac{1-\eta}{2}}. \tag{5.38}$$

Thus, for any given $(K_1, y_0) \in [\underline{\mu}, \infty) \times (0, \theta^*(K_1)]$, the leader's value function $H(K_1, y_0)$ is equal to

$$H_1(K_1, y_0) \tag{5.39}$$

$$= -\left[\xi_1(\theta^*(K_1)) - \xi_2(\theta^*(K_1))\right] \frac{\phi(y_0)}{\lambda\phi(\theta^*(K_1))} + \frac{1}{\lambda}\Bigg(y_0 \min\{K_1, \mu_H\}$$

$$+ (1-y_0)\min\{K_1, \mu_L\}\Bigg) - c_1 K_1. \tag{5.40}$$

Below, we will identify the leader's *equilibrium* value function, and show that in equilibrium, there exists a region in which $V_1(y_0) = \max_{K_1 \in \mathcal{C}} H_1(K_1, y_0)$.

Proposition 5.8. *The leader's equilibrium value function is*

$$V_1(y_0) = \begin{cases} \max_{K_1 \in \mathcal{C}} \ H_1(K_1, y_0) & \text{if} \quad y_0 \in [0, \theta^*(\underline{\mu})] \cup (\underline{y}, \theta^*(\mu_H)), \\ \max_{K_1 \geq 0} \ H_0(K_1, y_0) & \text{if} \quad y_0 > \theta^*(\mu_H), \end{cases} \tag{5.41}$$

where

$$\mathcal{C} \doteq \begin{cases} \{K_1 : K_1 \in \big((\theta^*)^{-1}(y_0), \mu_H\big)\} & \textit{if} \quad y_0 \geq \theta^*(\underline{\mu}), \\ (\underline{\mu}, \mu_H) & \textit{if} \quad y_0 \leq \theta^*(\underline{\mu}). \end{cases}$$

Remark 5.5. If c_1 is not too large, i.e.,

$$c_1 < \frac{1}{\lambda}\left(1 + \frac{\sqrt{1+\frac{8\lambda\sigma^2}{(\underline{\mu}-\mu_L)^2}} - 1}{\sqrt{1+\frac{8\lambda\sigma^2}{(\underline{\mu}-\mu_L)^2}} + 1} \frac{\mu_{2,H} - c_2\lambda\underline{\mu}}{c_2\lambda\underline{\mu} - \mu_{2,L}}\right)^{-1},$$

$\underline{y} < \theta^*(\underline{\mu})$ and thus the initial belief interval $[0, \theta^*(\underline{\mu})] \cup (\underline{y}, \theta^*(\mu_H))$ in (5.41) reduces to $[0, \theta^*(\mu_H))$.

The following proposition identifies the conditions under which the leader's equilibrium value function decreases with the initial belief.

Proposition 5.9. *There exists a belief region such that the leader's equilibrium value function $V_1(y_0)$ decreases in the initial belief y_0 if*

$$\sqrt{1 + \frac{8\lambda\sigma^2}{(\underline{\mu} - \mu_L)^2}}\left(\frac{U}{V} + 1\right) - \frac{U}{V} > \frac{\mu_H - \mu_L}{\mu_{2,L}},$$

where

$$U \doteq \mu_{2,H} - c_2\lambda\underline{\mu} \quad \textit{and} \quad V \doteq c_2\lambda\underline{\mu} - \mu_{2,L}. \tag{5.42}$$

Note that by fixing the other parameter, the condition above can be satisfied more easily with larger value of $\mu_{2,L}$ and $\mu_{2,H}$, or smaller values of c_2. All of these indicate that the market is not in favor of the leader. In Figure 5.2, the left panel shows the leader's equilibrium value function against y_0, and we select 4 initial beliefs labeled in red stars as examples within the decreasing region. In the right panel, the 4 curves show the LST of τ defined as $\mathbb{E}_{K_1}(e^{-\lambda\tau}|\pi_0 = y_0)$ corresponding to 4 fixed y_0 labeled in the left panel. The red stars in the right panel also display the equilibrium capacity under the corresponding initial beliefs.

From the right panel, we find that as y_0 increases, the leader has to increase the capacity so as to delay the follower's entrance to the market. However, as the curve becoming less sensitive to the capacity, the leader has to take costly actions and therefore the profits earned is decreasing even with an increasing belief in market.

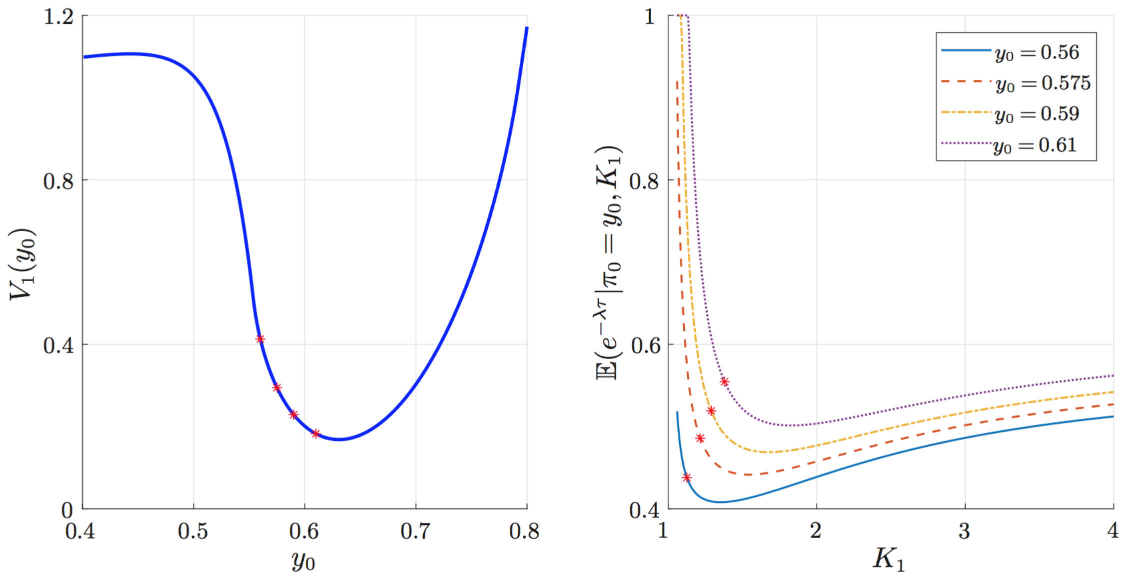


Figure 5.4: V_1 as Functions of y_0, and LST of τ as Function of K_1
Parameters used in this figure: $\lambda = 0.16$, $c_1 = c_2 = 5$, $\mu_L = 1$, $\mu_H = 5, \mu_{1,L} = 0.4$, $\mu_{1,H} = 3.95$, $\underline{\mu} = 1.06$.

Proposition 5.10 characterizes the leader's equilibrium capacity for a given initial belief α_0.

Proposition 5.10. *If the initial belief y_0 is relatively high, then the leader's equilibrium capacity is either $\mu_{1,H}$ or $\underline{\mu}$. That is,*

$$\widetilde{K}_1(y_0) = \begin{cases} \mu_{1,H} & \textit{if} \quad y_0 \geq \max\left\{\left(1 + \frac{U}{V}\frac{\sqrt{1+\frac{8\lambda\sigma^2}{(\mu_H-\mu_L)^2}}-1}{\sqrt{1+\frac{8\lambda\sigma^2}{(\mu_H-\mu_L)^2}}+1}\right)^{-1}, c_1\lambda\right\} \\ \underline{\mu} & \textit{if} \quad c_1\lambda \geq y_0 \geq \left(1 + \frac{U}{V}\frac{\sqrt{1+\frac{8\lambda\sigma^2}{(\mu_H-\mu_L)^2}}-1}{\sqrt{1+\frac{8\lambda\sigma^2}{(\mu_H-\mu_L)^2}}+1}\right)^{-1}, \end{cases}$$

where U and V are as defined in (5.42)*. On the other hand, if the initial belief y_0 is relatively low, that is,*

$$y_0 \leq \left(1 + \frac{U}{V}\frac{\sqrt{1+\frac{8\lambda\sigma^2}{(\underline{\mu}-\mu_L)^2}}-1}{\sqrt{1+\frac{8\lambda\sigma^2}{(\underline{\mu}-\mu_L)^2}}+1}\right)^{-1},$$

the leader's equilibrium capacity is

$$\widetilde{K}_1(y_0) = \max\{K_1^*, \underline{\mu}\}, \tag{5.43}$$

where K_1^* *is the solution of* $\partial H_1(K_1^*, y_0)/\partial K_1 = 0$.

Note that the optimal K_1 falls in $[\underline{\mu}, \mu_H]$, hence the threshold θ falls in$[\theta(\eta_{max}), \theta(\eta_{min})]$. For any $y_0 \geq \alpha_H$, we know the initial belief is always higher than the threshold, that is $V_1(y_0) = \max_{K_1} H_0(K_1, y_0)$ and it is optimal for the leader to invest $\mu_{1,H}$. On the other hand, if $y_0 \leq \alpha_L$, we know the initial belief is always below the threshold, hence $V_1(y_0) = \max_{K_1} H_1(K_1, y_0)$.

Next proposition describes the monotonicity of the optimal policy $\widetilde{K}_1(y_0)$ as function of y_0.

Proposition 5.11. *The leader's equilibrium capacity* $\widetilde{K}_1(y_0)$ *is non-decreasing in* y_0 *if conditions in one of the two cases are satisfied:*
Case 1: *if* $\widetilde{K}_1 \leq \mu_{1,H}$, *and*

$$1 - \frac{\mu_{2,L}}{\underline{\mu} - \mu_L} \frac{\eta_{max}^2 - 1}{2\eta_{max}} \left(\frac{U}{V} + \frac{\eta_{max} - 1}{\eta_{max} + 1} \right) > 0, \tag{5.44}$$

Or,
Case 2: *if* $\widetilde{K}_1 > \mu_{1,H}$, $\mu_{1,H} \leq \mu_L + 3 \cdot \mu_{2,L} \cdot \frac{\eta_0 - 1}{\eta_0 + 1} \cdot \frac{U}{V}$, *and*

$$1 - \frac{\eta_0^2 - 1}{2\eta_0} \cdot \frac{1}{\mu_{1,H} - \mu_L} \cdot \left[\mu_{2,H} \left(\frac{\eta_0 + 1}{\eta_0 - 1} + \frac{V}{U} \right) + \mu_{2,L} \left(\frac{\eta_0 - 1}{\eta_0 + 1} + \frac{U}{V} \right) \right] > 0, \tag{5.45}$$

where η_{max} *and* η_0 *are given in* (A.107) *and* (A.109).

In Case 1, we find that the LHS of the inequality (5.44) is decreasing in the ratio U/V. Hence the condition can be easily satisfied with a larger c_2 and smaller $\mu_{2,H}$. In Case 2, the LHS of the inequality (5.45) is decreasing in η_0, hence the condition can be easily satisfied with a larger $\mu_{1,H}$. All of these imply a positive investment environment for the leader, therefore the optimal value function of the leader shall increase as long as the initial belief is increasing. We can again use Fig 5.4 especially panel (d) to explain such phenomenon. Here we plot the LST of τ by varying y_0 and K_1. The solid dots on each curve are the optimal $\widetilde{K}_1$ with the corresponding y_0. As we increase y_0 in the region $(0.63, 0.80)$, where $\widetilde{K}_1(y_0)$ is increasing as shown in panel (a), we observe that the leader can delay the follower's entrance by increasing K_1. Besides, the cost of expanding capacity can be easily offset by the higher earning rate and initial belief.

Followed by this result, we show in the following corollary how the threshold θ changes with respect to y_0 under the optimal policy.

Corollary 5.1. *Under the optimal policy and the conditions in Proposition 5.11, the follower's investment threshold $\theta^*(\widetilde{K}_1(y_0))$ is non-decreasing in the initial belief y_0.*

It is common in the literature that a investment threshold is stationary and not affected by the market information. Here due to our special setting of variable capacity, the initial belief impacts the follower's optimal policy by first affecting the leader's investment capacity. Clearly, as the market information providing more positive prediction of the new market, the follower tends to be more ambitious and sets a higher threshold in the belief that it can be reached in a short time horizon.

5.6 Summary

We analyze optimal investment timing and capacity for both the leader and follower firms in a duopoly game with uncertain market. The follower firm update its belief with a Bayesian approach based on the performance of the leader firm, and decide on the optimal time to invest and how much to invest. The leader can impact the follower's decision through the "signal quality" by twisting its investment capacity. We study the problem backwards, and first establish a two-threshold policy for the follower and show its existence and optimality. Such policy can be characterized by a lower bound ℓ^* and a upper bound h^* of the posterior belief: when the belief is lower than ℓ^* or higher than h^*, it is optimal to invest immediately without further learning; otherwise it is valuable to continue the learning process and update the belief until it hit either of the bounds.

Next we focus on the leader's problem and first derive the policy in equilibrium under the two limits of the noise level σ. With a positive finite σ we establish the sufficient conditions for the leader to invest certain amount so that the follower adopts the non-trivial two-threshold policy and starts the learning process at the beginning. Furthermore we find the sufficient conditions under which the leader's equilibrium value function in decreasing in the initial belief y_0. These sufficient conditions imply unfavorable investment opportunities for the leader, hence result in costly decisions for the leader to delay the follower.

Lastly we add a lower bound constraint on the investment capacity for both leader and follower. This simplifies the model since the follower's policy in equilibrium reduces to a one-threshold policy, and such threshold is established explicitly as function of the leader's investment capacity. We find the similar results as in the original model, and show the sufficient conditions under which the leader's value function is decreasing in the initial belief. More important, we characterize the leader's policy in equilibrium explicitly, and show that under certain conditions the equilibrium capacity is increasing in the initial belief. Another contribution of this chapter is to show how the threshold of the follower with respect to y_0 under the leader's policy in equilibrium, while most of the literature introduces the thresholds that cannot be affected by the market status.

APPENDIX A: APPENDICES

A.1 Appendices for Chapter 2

A.1.1 Additional Plots for Sample Applications

Arrival patterns for applications 10 and 100, in comparison to Figures 2.2 and 2.3 in the main paper.

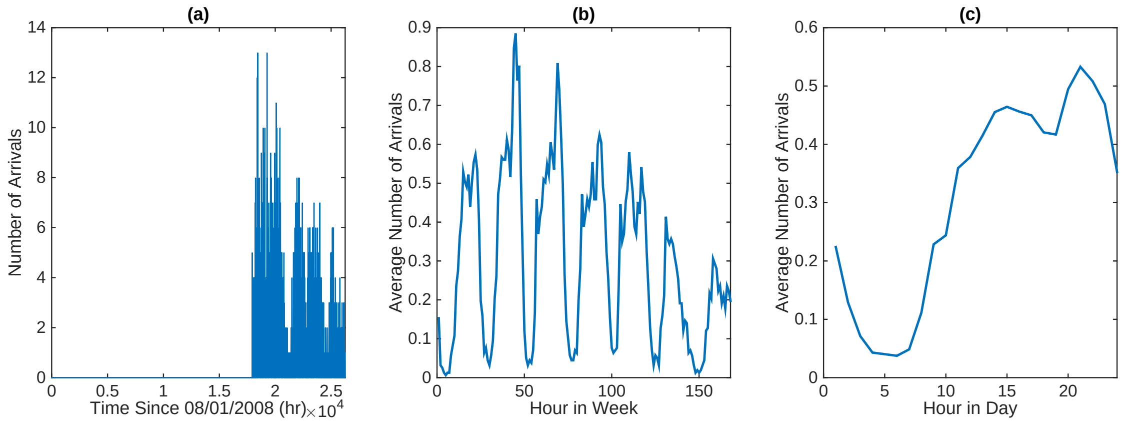


Figure A.1: Arrivals of Application 10

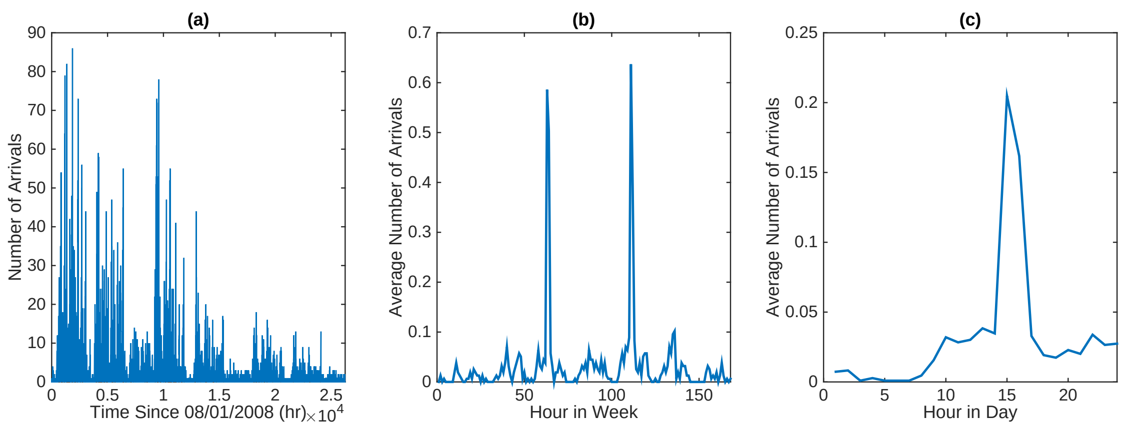


Figure A.2: Arrivals of Application 100

A.1.2 Proof of monotonicity of truncated negative binomial distribution

Let $X_r \sim NB(r, 1-p)$ with probability mass function (pmf)

$$P_{k,r} = P(X_r = k) = \binom{k+r-1}{k} p^r (1-p)^k, \quad k = 0, \quad 1, \cdots,$$

and let

$$f(k,r) = P_{k,r} / \sum_{i=0}^{k} P_{i,r}.$$

Theorem A.1. *For a fixed $r > 0$, $f(k,r)$ is a decreasing function of k.*

Proof: Let $\delta_k = f(k,r) - f(k+1,r)$. We show that $\delta_k \geq 0$ for $\forall k = 0, \quad 1, \cdots$. We see that

$$\begin{aligned}
\delta_k &= \frac{\binom{k+r-1}{k} p^r (1-p)^k}{\sum_{i=0}^{k} \binom{i+r-1}{i} p^r (1-p)^i} - \frac{\binom{k+r}{k+1} p^r (1-p)^{k+1}}{\sum_{i=0}^{k+1} \binom{i+r-1}{i} p^r (1-p)^i} \\
&= \binom{k+r-1}{k} (1-p)^k \left(\frac{1}{\sum_{i=0}^{k} \binom{i+r-1}{i} (1-p)^i} - \frac{\frac{k+r}{k+1}(1-p)}{\sum_{i=0}^{k+1} \binom{i+r-1}{i} (1-p)^i} \right) \\
&= C_k \left(\sum_{i=0}^{k+1} \binom{i+r-1}{i} (1-p)^i - \frac{k+r}{k+1} \sum_{i=0}^{k} \binom{i+r-1}{i} (1-p)^{i+1} \right),
\end{aligned}$$

where

$$C_k = \frac{\binom{k+r-1}{k} (1-p)^k}{\sum_{i=0}^{k} \binom{i+r-1}{i} (1-p)^i \cdot \sum_{i=0}^{k+1} \binom{i+r-1}{i} (1-p)^i}.$$

Now $C_k \geq 0$ for $\forall k \geq 0$. Hence to show $\delta_k \geq 0$, it is suffice to show that:

$$\Delta_k = \sum_{i=0}^{k+1} \binom{i+r-1}{i} (1-p)^i - \frac{k+r}{k+1} \sum_{i=0}^{k} \binom{i+r-1}{i} (1-p)^{i+1} \geq 0.$$

We first look at the case where $0 < r \leq 1$, and thus $(k+r)/(k+1) \leq 1$. Then we have

$$\begin{aligned}
\Delta_k &= \binom{k+r}{k+1} (1-p)^{k+1} + \sum_{i=0}^{k} \binom{i+r-1}{i} (1-p)^i \left(1 - \frac{k+r}{k+1}(1-p)\right) \\
&\geq \binom{k+r}{k+1} (1-p)^{k+1} + \sum_{i=0}^{k} \binom{i+r-1}{i} (1-p)^i \left(1 - (1-p)\right) \geq 0.
\end{aligned}$$

Now consider the case $r \geq 1$. We have

$$\begin{aligned}
\Delta_k &= \sum_{i=0}^{k} \binom{i+r-1}{i}(1-p)^i + \binom{k+r}{k+1}(1-p)^{k+1} \\
&\quad - \frac{k+r}{k+1}\sum_{i=0}^{k-1}\binom{i+r-1}{i}(1-p)^{i+1} - \frac{k+r}{k+1}\binom{k+r-1}{k}(1-p)^{k+1} \\
&= \sum_{i=0}^{k}\binom{i+r-1}{i}(1-p)^i - \frac{k+r}{k+1}\sum_{i=0}^{k-1}\binom{i+r-1}{i}(1-p)^{i+1} \\
&= 1 + \sum_{i=1}^{k}\binom{i+r-1}{i}(1-p)^i - \frac{k+r}{k+1}\sum_{i=1}^{k}\binom{i+r-2}{i-1}(1-p)^i \\
&= 1 + \sum_{i=1}^{k}\binom{i+r-2}{i-1}(1-p)^i\left(\frac{i+r-1}{i} - \frac{k+r}{k+1}\right) \\
&= 1 + \sum_{i=1}^{k}\frac{k-i+1}{k+1}\cdot\frac{r-1}{i}\binom{i+r-2}{i-1}(1-p)^i \\
&= \frac{1}{k+1}\sum_{i=0}^{k}(k-i+1)\binom{i+r-2}{i}(1-p)^i \\
&\geq \frac{1}{k+1}\sum_{i=0}^{k}\binom{i+r-2}{i}(1-p)^i \geq 0.
\end{aligned}$$

This completes the proof. $\square$

A.1.3 A3. NHPP Tests on the Arrival Data

Using the results of Brown et al. (2005) we construct the Kolmogorov-Smirnov test for the null hypothesis that arrivals of given types of application form a NHPP. Their approach addresses the time-varying arrival rate by converting the problem into a standard statistical test to determine whether inter-arrival times data can be regarded as a sample from a sequence of independent and identically distributed (i.i.d.) random variables with a specified distribution (see Kim and Ward (2014)). The first step is to approximate the NHPP by a piecewise-constant (PC) NHPP, by assuming that the arrival rates are constant on each interval. The second step is to apply the conditional-uniform (CU) transformation to transform the PC NHPP into a sequence of i.i.d. random variables uniformly distributed on $[0, 1]$. Because of the assumption of PC, the NHPP now can be regarded as a homogeneous Poisson process (PP) over each interval. For a PP on $[0, T]$, conditioned on the total number of arrivals in that interval, the arrival times divided by T are distributed as the order statistics of i.i.d. random variables uniformly distributed on $[0, 1]$. Finally, following Brown et al. (2005) we use a scaled logarithmic transformation of the data, which under the Poisson null hypothesis produces a sequence of i.i.d. mean-one exponential random variables. Then we apply the KS test with $F(x) = 1 - e^{-x}$.

Using our data, the first test example includes all the arrivals for application 1 arriving on every Monday from 14:00 to 14:59, August 1, 2008 to July 31, 2011. In total we have 150 such Monday one-hour intervals, and 470 arrivals. The respective Kolmogorov-Smirnov statistic has a value of $K = 0.0374$ (p-value = 0.3811). The second example includes 278 arrivals requesting for type 2 application on Wednesday Oct 14th, 2009. The interval length is half hour. For this case we have Kolmogorov-Smirnov statistic $K = 0.0648$ (p-value = 0.18). These results are typical of those we have obtained from various selections of intervals of the various types of requests. Thus, overall there is no evidence in this data set to reject the null hypothesis that the arrival process of application requests is NHPP with PC arrival rates. We also apply the root-unroot method from Brown et al. (2005) to stabilize the variance.

A.1.4 Additional Simulation Results

Figures A.3 and A.4 show the simulation results obtained under the $M^X/M/\infty$ and the $M(t)/G/\infty$ queue. These figures can be compared with Figure 2.7 in Section 3.7 of the main paper.

A.2 Appendices for Chapter 3

A.2.1 Proof of Theorem 3.1

Taking the first and second derivatives of Equation 3.6 with regard to q, we get:

$$\begin{aligned} G'(q) &= (p - c')\gamma(1 - F(\gamma q)) - c'F(q) + c' - c, \\ G''(q) &= -(p - c')\gamma^2 f(\gamma q) - c' f(q). \end{aligned} \tag{A.1}$$

When $p > c'$, it is obvious that $G''(q) \leq 0$, hence $G(\cdot)$ is concave, which implies that there is a unique point q^* where $G(\cdot)$ is maximized. Setting the first derivative in Equation A.1 to be 0, we get Equation 3.7.

Since $F(q^*) \leq F(\gamma q^*)$ for $\gamma \geq 1$, by substituting q^* by γq^* and conversely in Equation A.1 we have:

$$(p - c')\gamma F(\gamma q^*)) + c'F(\gamma q^*) \geq c' - c + (p - c')\gamma.$$

$$(p - c')\gamma F(q^*)) + c'F(q^*) \leq c' - c + (p - c')\gamma.$$

Solving the above inequalities, get Equation 3.9. □

A.2.2 Proof of Theorem 3.2

Let π be any stationary policy and $(x, a) \in \mathcal{X} \times \mathcal{A}$ be any initial state. Since the capacity is infinite, and a patient who needs k appointments stays in the system for at most kT days, we can bound the number of patients in the system by a recurrent infinite server queue. This implies that state $(0, 0)$ is recurrent. Then it is straightforward to prove that it is possible to reach state $(0, 0)$ from any state (x, a) in a finite number of steps: suppose $K = \max\{k : x(k, t) > 0,\ t = 1, \ldots, T\}$. Then $K < \infty$ with probability 1, and if there are no arrivals in the next KT days, all the existing appointments will be no longer on the schedule and the system state would be $(0, 0)$ after KT days. The probability of this is $\phi_0^{KT} > 0$, under any policy π. Therefore, we have proved the MDP is indeed uni-chain.

For the similar reason, it is possible to go from state $(X_n, A_n) = (0, 0)$ to $(X_{n+1}, A_{n+1}) = (0, 0)$ in one step, as long as $B_n = 0$ (with probability $\phi_0 > 0$). This proves aperiodicity. □

A.2.3 Proof of Theorem 3.3

First, we state a useful known result below (see Section 3.2.5 in Boyd and Vandenberghe (2004)).

Concavity Theorem

Let $g(x, y)$ be concave in (x, y), and C be a convex nonempty set. Then the function

$$v(x) = \max_{y \in C} g(x, y)$$

is concave in x, provided $v(x) > -\infty$ for all x.

Next we prove Theorem 3.3 by induction on n. For $n = 0$, we have $v^0(x, a) = 0$, which is trivially concave in x.

Now we assume that $v^{n-1}(x, a)$ is concave in x for some $n \geq 1$. Hence $v^{n-1}(T_{x+y}, a)$ is concave in $x+y$ for any given $A = a$, since the transformation T simply rearranges components of $x + y$. Hence

$$g^n(x + y) = E(v^{n-1}(T_{x+y}, A)) = \sum_{a \in \mathcal{A}} \left(v^{n-1}(T_{x+y}, a)\right) P(A = a)$$

is concave, since it is a convex combination of concave functions. Using the above Concavity Theorem for optimization problem $P(n)$, we see that $v^n(x, a)$ is concave in x, and hence by induction, it is concave for all $n \geq 0$. $\square$

A.2.4 Proof of Theorem 3.4

Let $g(x+y) = \lim_{n \to \infty} g^n(x+y) = E\left(v(T_{x+y}, A)\right)$, then $P(n)$ can be written as the following optimization problem P:

$$v(x, a) = \max\{g(x + y) + R(x + y) + f(y)\}. \tag{P}$$

$$\text{s.t.} \quad y \in \mathcal{X}, \tag{3.16}$$

$$ye = a. \tag{3.17}$$

Note that the above P with decision variable y is equivalent to the following optimization problem P* with decision variable $z = x + y$

$$
\begin{aligned}
v(x, a) + f(x) = \max \quad & \{g(z) + R(z) + f(z)\} && \text{P*} \\
\text{s.t.} \quad & z - x \in \mathcal{X}, && \text{(A.5)} \\
& ze = a + xe. && \text{(A.6)}
\end{aligned}
$$

Let the above objective function be $\tilde{g}(z) = g(z)+R(z)+f(z)$. It follows from Theorem 3.3 that $\tilde{g}(z)$ is concave, and the constraints A.5 and A.6 are equivalent to 3.16 and 3.17 respectively. Let $\mathcal{A}_i$ be the feasible set given by A.5 and A.6 given $x = x_i$, $i = 1,\ 2$. We see that $\mathcal{A}_i$ is a convex set for $i = 1,\ 2$.

For $i = 1, 2$, suppose the optimal solutions to P* with $x = x_i$ is $z_i = x_i + y_i$. We prove the theorem by showing $z_1 \in \mathcal{A}_2$, and $\tilde{g}(z_2) = \tilde{g}(z_1)$. Therefore we can simply take $z_2 = z_1$.

First, by condition (c), we see that

$$x_2 e = (x_1 + h)e = x_1 e,$$

hence the right hand side of constraint A.6 remains the same for $x = x_1$ and $x = x_2$. Also condition (b) implies that $z_1 - x_2 = z_1 - x_1 - h = y_1 - h \in \mathcal{X}$, so we have $z_1 \in \mathcal{A}_2$. We must have $\tilde{g}(z_2) \geq \tilde{g}(z_1)$ since the objective function is maximized at z_2 in $\mathcal{A}_2$. The proof will be completed if we show that $\tilde{g}(z_2) \leq \tilde{g}(z_1)$. We do this by contradiction.

So suppose $\tilde{g}(z_2) > \tilde{g}(z_1)$, and let $w = \frac{1}{2}z_2 + \frac{1}{2}z_1$. Since $\tilde{g}$ is concave, we have

$$\tilde{g}(w) \geq \frac{1}{2}\tilde{g}(z_1) + \frac{1}{2}\tilde{g}(z_2) > \tilde{g}(z_1). \tag{A.2}$$

Note that the linear constraint A.6 satisfied by z_1 and z_2 will also be satisfied by w.

We also have

$$
\begin{aligned}
w &= \frac{1}{2}(x_1 + h + y_2) + \frac{1}{2}(x_1 + y_1) \\
&= x_1 + \frac{1}{2}y_2 + \frac{1}{2}(y_1 + h).
\end{aligned}
$$

From (b), we see that $y_1 + h \in \mathcal{X}$. Since $y_2 \in \mathcal{X}$, we have $w - x_1 \in \mathcal{X}$, that is w satisfies constraint A.5 given $x = x_1$. Hence $w \in \mathcal{A}_1$. However, $z_1 \in \mathcal{A}_1$ is the optimal solution for $x = x_1$, hence $\tilde{g}(w) \le \tilde{g}(z_1)$. This contradicts A.2, and thus completes the proof. □

A.2.5 Proof of Theorem 3.5

This follows because in the proof of Theorem 3.4, condition (c) is needed such that constraint A.6 remains to be unchanged, and (c') is a general version of (c).

A.2.6 Proof of Theorem 3.6

We know that the demand on day $n \ge 1$ in Equation 3.11 is accumulated from the scheduling decisions over times $\{0, 1, \cdots, n-1\}$.

Note that a new arrival on day j who is given the first appointment on day $j + t$ creates a demand for one slot on day n if $n - j - t = (k-1)T$ for some $k \ge 1$, and he/she needs at least k repeat visits. Recall that $Y_j(k, t)$ is precisely the number of such patients for $0 < j < n$, and they contribute to the demand on day n if $(k, t) \in S_n$. Besides, $X_0(k_n, t_n) = x(k_n, t_n)$ of the patients on the initial schedule also have an appointment on day n. Hence

$$D_n = x(k_n, t_n) + \sum_{(k,t) \in S_n} Y_{n-(k-1)T-t}(k, t). \tag{A.3}$$

We see that the only element in the initial state $X_0 = x$ involved in the demand on day n is $x(k_n, t_n)$.

From Equation 3.19, we see that the random variables Y's in Equation A.3 are independent, since $\{B_j, 0 \le j < n\}$ are i.i.d.. Therefore the generating function for Z_n has the product form. Further, we can compute

$$\begin{aligned} E(z^{Y_n(k,t)}) &= E(z^{\mathcal{B}(B_n, f_k\beta_t)}) = E((zf_k\beta_t - f_k\beta_t + 1)^{B_n}) \\ &= \Phi(zf_k\beta_t - f_k\beta_t + 1), \end{aligned} \tag{A.4}$$

Finally, Equation 3.21 is obtained by plugging Equation A.4 into Equation A.3. □

A.2.7 Proof of Corollary 3.1

We are given that

$$\lambda = E(B_n),$$

and

$$E(V_i) = \sum_{k=1}^{\infty} kp_k = \sum_{k=1}^{\infty} f_k.$$

In the steady state, combining Equation 3.19 and Equation A.3, taking expectations and letting $n \to \infty$, we get:

$$\begin{aligned}\lim_{n\to\infty} E(Z_n) &= \lim_{n\to\infty} E\left(\sum_{(k,t)\in S_n} \mathcal{B}\left(B_{(n-(k-1)T-t),k}, f_k\beta_t\right)\right)\\ &= \lim_{n\to\infty} \sum_{(k,t)\in S_n} \lambda f_k\beta_t = \sum_{k=1}^{\infty}\sum_{t=1}^{T} \lambda f_k \beta_t = \lambda \sum_{k=1}^{\infty} f_k = \lambda m.\end{aligned}$$

Next, suppose the total number of patients on the schedule on day 0 is finite, that is,

$$\sum_{t=1}^{T} x(1,t) < \infty.$$

Let K be the maximum number of repeat visits needed by any of these initial patients. Then $x(x,t) = 0$ for $k > K$ for all $1 \le t \le T$. Hence

$$\lim_{n\to\infty} x(k_n, t_n) = 0.$$

Hence we get

$$\lim_{n\to\infty} E(D_n | X_0 = x) = \lim_{n\to\infty} x(k_n, t_n) + E(Z_n) = \lambda m.$$

□

A.2.8 Proof of Corollary 3.2

Since $\{B_n, n \ge 0\}$ are i.i.d. $\mathcal{P}(\lambda)$, we have

$$\Phi(z) = \exp(-\lambda(1-z)).$$

Substituting in Equation 3.21, we have

$$E(z^{Z_n}) = \prod_{(k,t)\in S_n} \exp(-\lambda(1 - zf_k\beta_t + f_k\beta_t - 1) = \exp(-(\sum_{(k,t)\in S_n} f_k\beta_t)(1-z)).$$

Hence $Z_n \sim \mathcal{P}(\lambda_n)$, where λ_n is as given in Equation 3.22.

We have $\lim_{n\to\infty} \lambda_n = \lambda m$. Hence Z_n converges to a $\mathcal{P}(\lambda m)$ random variable. Finally, following the proof of Corollary 3.1, we see that finiteness of the number of patients on schedule on day 0 implies that D_n and Z_n have the same limit in distribution. $\square$

A.2.9 Proof of Proposition 3.1

First note that the function $\rho(z,q)$ is a concave function in z, which can be seen directly from the definition and Figure 3.1, as long as $p \geq c'$. Since $E\left(\rho(z+Z,q)\right)$ is a convex combination of $\rho(z+w,q)$, $w \geq 0$, it follows that $g_{k,t}(z)$ is concave in z. $\square$

A.2.10 Proof of Proposition 3.2

Recall Equation 3.30:

$$I_k(x,t) = \sum_{j=1}^{k} (p-c')F_{j,t}(\gamma q - x(j,t) - 1) + c'F_{j,t}(q - x(j,t) - 1) - c_d(t),$$

p1 and **p2** can be derived easily. To prove **p3**, note that $F_{k,t}(\cdot)$ is a cdf and therefore non-decreasing, thus for $x_1(k,t) \leq x_2(k,t)$ we have

$$F_{k,t}(\gamma q - x_1(k,t) - 1) \geq F_{k,t}(\gamma q - x_2(k,t) - 1),$$

and similarly

$$F_{k,t}(q - x_1(k,t) - 1) \geq F_{k,t}(q - x_2(k,t) - 1).$$

Hence $I_k(x_1,t) \geq I_k(x_2,t)$ follows directly. $\square$

A.2.11 Solution to Min-Cost Flow Problem

Now we show how the problem of Equation 3.29 can be solved to optimality by formulating it as a min-cost flow problem on a network with positive and negative costs. As in the first step of *greedy procedure* in Section 3.6, we assume that the current state (x, a) is given and let $K = \max\{k \geq 1 : a_k > 0\}$.

The network model for this problem is provided in Figure A.5. This network has K source nodes, labelled $s_1, \ s_2, \ \cdots, \ s_K$, with node s_k having supply $\tilde{a}_k$. There are K nodes for each day t labeled $(1,t), \cdots, (K,t)$ and a single sink node t_0 with demand $a_1 = \sum_{k=1}^{K} \tilde{a}_k$. The network contains arcs $(s_k, (k,t))$ with flow $\tilde{y}(k,t)$ and zero cost for $k = 1, \cdots, K$ and $t = 1, \cdots, T$; arcs $((k,t),(k-1,t))$ with flow $x(k,t) + y(k,t)$ and cost $-\tilde{g}_{kt}(x(k,t), y(k,t))$ for $i = 2, \cdots, K$ and $t = 1, \cdots, T$; and arcs $((1,t), t_0)$ with flow $x(1,t) + y(1,t)$ and cost $-\tilde{g}_{kt}(x(1,t), y(1,t))$ for $t = 1, \cdots, T$.

With this formulation, we see that solving the optimization problem is same as finding the min-cost flow on this network that satisfies all the supply and demand constraints. We can apply the successive shortest path algorithm for the minimum convex-cost network flow problem to this network with $KT + K + 1$ nodes and $2KT$ arcs to solve this problem. (See Bazaraa et al. [2011].) At each step we construct a residual network $G(x+y, \tilde{y})$ in which the cost of an arc with flow $x(k,t) + y(k,t)$ is

$$\partial(-\tilde{g})(x(k,t), y(k,t)+1) = -\tilde{g}(x(k,t), y(k,t)+1) + \tilde{g}(x(k,t), y(k,t)),$$

and the cost of the reverse arc with flow $x(k,t) + y(k,t) > x(k,t)$ is

$$-\partial(-\tilde{g})(x(k,t), y(k,t)) = \tilde{g}(x(k,t), y(k,t)) - \tilde{g}(x(k,t), y(k,t)-1).$$

If $x(k,t)+y(k,t) = x(k,t)$, there is no reverse arc, since the existing scheduling matrix $x(k,t)$ cannot be rearranged. The approach to solve the optimization problem of Equation 3.29 is to push flow from s_k to t in the reduced network G_k, which omits the source nodes s_K down to s_{k+1} and their incident arcs. This procedure begins by pushing flow from s_1 to t and continues until we push flow s_K to t over the full network. We outline the algorithm below:

Successive Shortest Path Procedure:

- Initialize $y,\ \tilde{y} = 0$;
- For $k = 1$ to K fo
  - While $\sum_{t=1}^{T} \tilde{y}(k,t) < \tilde{a}_k$ do
    - Determine the shortest path P from node s_k to node t in the residual network $G_k(x+y, \tilde{y})$ constructed from the reduced network G_k;
    - Augment 1 unit of flow along the path P;
    - Update y, $\tilde{y}$ and $G_k(x+y, \tilde{y})$;
  - End while loop;
- End for loop.

The convexity of $-\tilde{g}_{k,t}(w, \cdot)$ ensures that there is no negative cycle in the graph. Considering the cost of arcs can be negative, we use FIFO modified label correcting algorithm (see Cherkassky et al. [1996]) to find the shortest path for each repetition in the while loop, which has a complexity of $O((KT)^2)$. Because of the nice structure of each residual graph, the implementation of the successive shortest paths algorithm has a polynomial complexity of $O(a_1 K^3 T^2)$.

A.2.12 Proof of Theorem 3.7

We first study SQ policy. It is obvious that the order, and hence the type of arrivals does not affect the final decision since each decision is dependent only on the first row of current state, that is the total number of existing appointments on each of the T days. Therefore, we can formulate the SQ policy as the following optimization problem with decision variable $z = x + y$:

$$\begin{aligned} \min \quad & \max\{z(1,1), \cdots, z(1,T)\} \\ \text{s.t.} \quad & z - x \in \mathcal{X}, && \text{(A.5)} \\ & ze = a + xe. && \text{(A.6)} \end{aligned}$$

It can be shown straightforwardly that the function $\max\{z(1,1), \cdots, z(1,T)\}$ is convex in z, therefore the same structural property in Equation 3.18 holds for SQ policy under the same the conditions (a) - (c) (or (c')).

Similarly, the decision under MP policy does not depend on the type of arriving patient either. The equivalent optimization problem under MP policy is:

$$\begin{aligned} \max \quad & \sum_{t=1}^{T} \rho(z(1,t), q) - c_d(t) z(1,t) \\ \text{s.t.} \quad & z - x \in \mathcal{X}, && \text{(A.5)} \\ & ze = a + xe. && \text{(A.6)} \end{aligned}$$

where the objective function is also concave in $z = x + y$. And for MMP policy, the equivalent problem with concave objective function is:

$$\begin{aligned} \max \quad & \sum_{j=1}^{k} \sum_{t=1}^{T} \rho(z(j,t), q) - c_d(t) z(j,t) \\ \text{s.t.} \quad & z - x \in \mathcal{X}, && \text{(A.5)} \\ & ze = a + xe. && \text{(A.6)} \end{aligned}$$

We conclude that, under conditions (a) - (c) (or (c')), structural property in Equation 3.18 holds for MP and MMP policies.

For the MF policy, recall that it is the optimal solution to the maximization problem in Equation 3.26, and if we consider the corresponding long run expected net profit v_{MF}, the problem can be formulated as

$$v_{MF}(x,a) = \max_{y \in \Delta(a)} v_{f_y}(x,a) = \max_{y \in \Delta(a)} \left\{ r(x,y) + E(v_f(T_{x+y}, A)) \right\},$$

where v_{ND} is the long run expected net profit under ND policy. From equation 3.27, we know that $E(v_{ND}^{n-1}(T_{x+y}, A))$, and hence the limiting $E(v_{ND}(T_{x+y}, A))$ is also a concave function in $x+y$. Therefore the same proof shows that if the change in the initial scheduling state satisfies

(a) - (c) (or (c')), the optimal decision made under MF policy also satisfies Equation 3.18. $\square$

A.3 Appendices for Chapter 5

A.3.1 The Heuristic Derivation of the HJB Equation (5.10)

Fix any $K_1 \geq 0$ for the leader's capacity. Take a small $h > 0$, and consider the time interval $[0, h]$. The follower can take one of the two possible actions during that time interval: The follower either waits and learns from the leader's cumulative earnings about the favorableness of the market without making an investment, or stops learning and invests at $t = 0$.

First, suppose that during $[0, h]$, the follower decides to wait and learn from the leader's cumulative earnings about the favorableness of the market without making an investment. Then, by Itô's lemma, we have

$$\begin{aligned}
\mathbb{E}\left[V_2(\pi_h; K_1) \mid \pi_0 = y\right] &= \mathbb{E}\left[V_2(y; K_1) + \int_0^h V_2'(\pi_s; K_1) d\pi_s + \frac{1}{2}\int_0^h V_2''(\pi_s; K_1)(d\pi_s)^2 \mid \pi_0 = y\right] \\
&= V_2(y; K_1) + \frac{1}{2} q(K_1) y^2 (1-y)^2 V_2''(y; K_1) h + o(h).
\end{aligned}$$

Based on this, if the follower decides not to enter the market during $[0, h]$, its expected discounted profit is

$$\begin{aligned}
V_2(y; K_1) &= \mathbb{E}\left[e^{-\lambda h} V_2(\pi_h; K_1) \mid \pi_0 = y\right] \\
&= (1 - \lambda h)\mathbb{E}\left[V_2(\pi_h; K_1) \mid \pi_0 = y\right] + o(h) \\
&= V_2(y; K_1) + \left[\frac{1}{2} q(K_1) y^2 (1-y)^2 V_2''(y; K_1) - \lambda V_2(y; K_1)\right] h + o(h).
\end{aligned}$$

Subtracting $V_2(y; K_1)$ from both sides, dividing by h, and letting $h \to 0$ in the above equation, we obtain

$$0 = \frac{1}{2} q(K_1) y^2 (1-y)^2 V_2''(y; K_1) - \lambda V_2(y; K_1). \tag{A.7}$$

On the other hand, if the follower decides to invest $t = 0$, it receives

$$
\begin{aligned}
&\max_{K_2 \geq 0} \left\{ \mathbb{E}\left[\int_0^\infty e^{-\lambda t} dX_{2,t} - c_2 K_2 \mid \pi_0 = y_0 \right] \right\} \\
&= \max_{K_2 \geq 0} \left\{ \frac{1}{\lambda} [y \min(K_2, \mu_{2,H}) + (1-y)\min(K_2, \mu_{2,L})] - c_2 K_2 \right\}.
\end{aligned} \tag{A.8}
$$

Note that the maximizer K_2^* in (A.8) depends on y, c_2, and λ:

$$
K_2^* = \begin{cases} \mu_{2,L}, & 0 \leq y \leq c_2 \lambda, \\ \mu_{2,H}, & c_2 \lambda < y \leq 1. \end{cases} \tag{A.9}
$$

Plugging the above expression for K_2^* in (A.8), we obtain the follower's optimal discounted profit at the stopping:

$$
g(y) = \begin{cases} (\frac{1}{\lambda} - c_2)\mu_{2,L}, & 0 \leq y \leq c_2 \lambda, \\ \frac{1}{\lambda}(\mu_{2,H} - \mu_{2,L})y + \frac{1}{\lambda}\mu_{2,L} - c_2 \mu_{2,H}, & c_2 \lambda < y \leq 1. \end{cases} \tag{A.10}
$$

The follower chooses the action (among stopping and waiting) that results in larger payoff. Thus, by (A.7) and (A.10), we obtain

$$
V_2(y; K_1) = \max\Big\{ g(y), \quad V_2(y; K_1) + \frac{1}{2} q(K_1) y^2 (1-y)^2 V_2''(y; K_1) - \lambda V_2(y; K_1) \Big\}.
$$

Subtracting $V_2(y; K_1)$ from both sides in the above equation, we obtain the HJB equation for a given $K_1 \geq 0$:

$$
0 = \max\Big\{ g(y) - V_2(y; K_1), \quad \frac{1}{2} q(K_1) y^2 (1-y)^2 V_2''(y; K_1) - \lambda V_2(y; K_1) \Big\}.
$$

□

A.3.2 Proof of Proposition 5.1

Fix capacity $K_1 > \mu_L$ for the leader. The outline of our proof is as follows. First, for any candidate lower critical belief $\ell \in (0, 1)$, we consider a function $f_\ell(\cdot)$, and show the structural properties of this function in Lemmas X and Y. And then we will show that there exists a particular $\ell^*(K_1) \in (0,1)$ such that $f_{\ell^*}(\cdot)$ is $\mathcal{C}^2$ on (ℓ^*, ∞) and $f_{\ell^*}(\cdot)$ is tangent to $g(\cdot)$ at a unique point $h^*(K_1)$. Using $f_{\ell^*}(\cdot)$, we will construct a function $v(\cdot; K_1)$, which we will show to be the value function of the two-threshold policy $(\ell^*(K_1), h^*(K_1))$. Finally, using the structural properties of $v(\cdot; K_1)$, we will show that $v(\cdot; K_1)$ satisfies said properties in Proposition 5.1.

Fix any $\ell \in (0,1)$. Define the function $f_\ell(\cdot) : [\ell, 1] \to \mathbb{R}$ as the solution of the following ODE and boundary conditions:

$$\lambda f_\ell(y) = \frac{1}{2} q(K_1) y^2 (1-y)^2 f_\ell''(y), \tag{A.11}$$

$$f_\ell(\ell) = g(0) > 0 \quad \text{and} \quad f_\ell'(\ell) = 0. \tag{A.12}$$

Existence of a solution to (A.11) and (A.12) follows from elementary ODE theory. Below, we will show certain structural properties of $f_\ell(\cdot)$ in Lemmas A.1 and A.2.

Lemma A.1. *For any $0 < \ell < 1$, $f_\ell'(y) > 0$ and $f_\ell''(y) > 0$ for $y \in (\ell, 1]$.*

Proof of Lemma A.1: We first prove the strict convexity of $f_\ell(\cdot)$. Suppose for a contradiction that there exists at least one belief value at which $f_\ell''(\cdot) \le 0$. Let $y_a \doteq \min\{y \in [\ell, 1] : f_\ell''(y) \le 0\}$. (The minimum is achieved as $f_\ell''(\cdot)$ is continuous.) Because $f_\ell''(\ell+) > 0$ by (A.11), $y_a > \ell$. From (A.11), $f_\ell''(y_a) \le 0$ implies that

$$f_\ell(y_a) \le 0. \tag{A.13}$$

Then, because $f_\ell(\ell) > 0$ and $f_\ell'(\ell) = 0$, there exists a $y_b \in (\ell, y_a)$ such that $f_\ell'(y_b) < 0$. Furthermore, since $f_\ell'(\ell) = 0$ and $f_\ell'(y_b) < 0$, there exists a $y_c \in (\ell, y_b)$ such that $f_\ell''(y_c) < 0$. As a result, $y_c < y_b < y_a$ contradicts the definition of y_a. Thus, $f_\ell(\cdot)$ is strictly convex on $(\ell, 1]$. Because $f_\ell'(\ell) = 0$ and $f_\ell''(y) > 0$ on $(\ell, 1]$, it immediately follows that $f_\ell'(y) > f_\ell'(\ell) = 0$ on $(\ell, 1]$. □

Lemma A.2. *Consider any given $\ell \in (0,1)$. Then,*

(a) $\lim_{y \to 1} f_\ell(y) = \infty$,

(b) for $\epsilon > 0$ sufficiently close to zero, $f_\epsilon(y) > g(y)$ for $y > \epsilon$, and

(c) $f_{c_2\lambda}(y)$ crosses $g(y)$ at some point $\tilde{y} > c_2\lambda$.

Proof of Lemma A.2: Part (a): Recall (A.11) and (A.12). By these equations, we have

$$f''_\ell(y) = \lambda f_\ell(y)/\left(q(K_1)y^2(1-y)^2\right), \quad y \geq \ell. \tag{A.14}$$

Take any $\bar{y} \in (0,1)$. Then, $f''_\ell(y) > \lambda f_\ell(\bar{y})/\left(q(K_1)y^2\right) > 0$ for $y \in [\bar{y}, 1)$. This implies that $f_\ell(y) > f_\ell(\bar{y}) + \lambda f_\ell(\bar{y})\left[log(\bar{y}) - \log(y)\right]/q(K_1)$ for $y > \bar{y}$. Then, because $\lim_{y\to 1} - \log(y) = \infty$, it follows that $\lim_{y\to 1} f_\ell(y) = \infty$.

Part (b): If $\ell > 0$ is sufficiently close to zero and $y > \ell$ is sufficiently close to ℓ, $f''_\ell(y) > c/y^2$ for some positive constant $c > 0$. This implies that for such y and ℓ values, $f'_\ell(y) > c(y-\ell)/\ell^2 > k$ where k is the slope of the upward portion of $g(\cdot)$. Then, because $f_\ell(\cdot)$ is strictly convex and increasing, and by (A.12), $f_\ell(y)$ lies above $g(y)$ for $y > \epsilon$ if $\ell > 0$ is sufficiently close to zero. An example of this can be seen Figure A.6.

Part (c): We already know by (A.12) that $f_{c_2\lambda}(y)$ lies below the upward portion of $g(\cdot)$ for $y > c_2\lambda$ sufficiently close to $c_2\lambda$. This, the fact that $f_{c_2\lambda}(y)$ strictly increases for $y > c_2\lambda$ by Lemma A.1 and $g(\cdot)$ is bounded by a finite constant,and part (a) imply part (c). Figure A.7 demonstrates an example of this. $\square$

For each for each $\ell \in [0, c_2\lambda]$, define y_ℓ such that $f'_\ell(y_\ell) = k$. (The existence of y_ℓ follows from similar arguments in the proof of Lemma A.2-(a).) An example of y_ℓ is illustrated for $\ell = 0.3$ in Figure A.8. By definition, f_ℓ has the same derivative as the upward portion of $g(\cdot)$ at $y = y_\ell$. We now show that there exists a $\ell^* < c_2\lambda$ such that $f_{\ell^*}(\cdot)$ is tangent to $g(\cdot)$ at a unique point $h^* \doteq y_{\ell^*} > \ell^*$ and $h^* > c_2\lambda$. Specifically, we will show that there exists a pair (ℓ^*, h^*) and a function $f_{\ell^*}(\cdot)$ such that

$$\lambda f_{\ell^*}(y) = \frac{1}{2}q(K_1)y^2(1-y)^2 f''_{\ell^*}(y), \tag{A.15}$$

$$f_{\ell^*}(\ell^*) = g(\ell^*), \quad f'_{\ell^*}(\ell^*) = g'(\ell^*) = 0, \tag{A.16}$$

$$f_{\ell^*}(h^*) = g(h^*), \quad f'_{\ell^*}(h^*) = g'(h^*) = k > 0. \tag{A.17}$$

This claim immediately follows from Lemma A.4 below. We now state and prove another lemma which we will use in the proof of Lemma A.4.

Lemma A.3. *Consider $K_1 > \mu_L$ that implies $q(K_1) > 0$. The general solution to the ODE*

$$\frac{1}{2}q(K_1)y^2(1-y)^2 f''(y) = \lambda f(y) \tag{A.18}$$

is given by

$$f(y) = \kappa_1 y^{\frac{1+\eta}{2}}(1-y)^{\frac{1-\eta}{2}} + \kappa_2 y^{\frac{1-\eta}{2}}(1-y)^{\frac{1+\eta}{2}}, \tag{A.19}$$

where κ_1 and κ_2 are two constants, and η is as in (5.24).

Proof of Lemma A.3: In the context of XYZ, the form of the general solution of a similar ODE is stated in Alvares (2003) without a proof. Here, we will verify that (A.19) is indeed the general solution of (A.18). Suppose the solution to the ODE (A.18) is in the form of

$$f(y) = y^a(1-y)^b$$

Therefore, $f'(y)$ and $f''(y)$ are given by

$$f'(y) = ay^{a-1}(1-y)^b - by^a(1-y)^{b-1},$$

$$f''(y) = a(a-1)y^{a-2}(1-y)^b - 2aby^{a-1}(1-y)^{b-1} + b(b-1)y^a(1-y)^{b-2}.$$

By replacing $f(y)$ and $f''(y)$ with the above expressions in (A.18), and dividing both sides of that modified ODE by $y^a(1-y)^b$, we obtain

$$\frac{1}{2}q(K_1)\left[a(a-1)(1-y)^2 - 2aby(1-y) + b(b-1)y^2\right] = \lambda.$$

Rearranging the above equation, we obtain

$$[a(a-1) + 2ab + b(b-1)]y^2 - [2a(a-1) + 2ab]y + [a(a-1) - \frac{2\lambda}{q(K_1)}] = 0.$$

For (A.18) to be a solution, the above equation must hold, meaning that all three equations below must be satisfied:

$$\begin{cases} a(a-1)+2ab+b(b-1)=0 \\ 2a(a-1)+2ab=0 \\ a(a-1)-\frac{2\lambda}{q(K_1)}=0. \end{cases}$$

The solutions of these equations are given by

$$(a,b)\in\left\{\left(\frac{1+\eta}{2},\frac{1-\eta}{2}\right),\left(\frac{1-\eta}{2},\frac{1+\eta}{2}\right)\right\},$$

which complete the proof. □

To state the following lemma, define functions

$$\bar{G}(\ell)\doteq f_\ell(y_\ell) \quad \text{and} \quad \underline{G}(\ell)\doteq g(y_\ell), \quad \text{for} \quad \ell\in[0,c_2\lambda).$$

Lemma A.4. *(a) y_ℓ is continuous with respect to ℓ, hence $\Delta(\ell)\doteq\bar{G}(\ell)-\underline{G}(\ell)$ is a continuous function with respect to ℓ.*

(b) There exists an $\epsilon\in(0,c_2\lambda)$ such that $\bar{G}(\epsilon)-\underline{G}(\epsilon)>0$.

(c) $\bar{G}(c_2\lambda)-\underline{G}(c_2\lambda)<0$.

Proof of Lemma A.4: Part (a): The general solution to the ODE (A.11) is given by

$$f_\ell(y)=\kappa_1 y^{\frac{1+\eta}{2}}(1-y)^{\frac{1-\eta}{2}}+\kappa_2 y^{\frac{1-\eta}{2}}(1-y)^{\frac{1+\eta}{2}}. \tag{A.20}$$

Then, based on (A.15) through (A.17), we have four unknowns l^*, h^*, κ_1, κ_2 that should satisfy the following four equations:

$$
\begin{aligned}
&\kappa_1 \ell^{\frac{1+\eta}{2}}(1-\ell)^{\frac{1-\eta}{2}} + \kappa_2 \ell^{\frac{1-\eta}{2}}(1-\ell)^{\frac{1+\eta}{2}} = \left(\frac{1}{\lambda} - c_2\right)\mu_{2,L} \\
&\kappa_1 h^{\frac{1+\eta}{2}}(1-h)^{\frac{1-\eta}{2}} + \kappa_2 h^{\frac{1-\eta}{2}}(1-h)^{\frac{1+\eta}{2}} = \frac{\mu_{2,H}-\mu_{2,L}}{\lambda}h + \frac{1}{\lambda}\mu_{2,L} - c_2\mu_{2,H} \\
&\kappa_1 \left(\frac{1+\eta}{2}\ell^{\frac{-1+\eta}{2}}(1-\ell)^{\frac{1-\eta}{2}} - \frac{1-\eta}{2}\ell^{\frac{1+\eta}{2}}(1-\ell)^{\frac{-1-\eta}{2}}\right) \\
&+\kappa_2 \left(\frac{1-\eta}{2}\ell^{\frac{1+\eta}{2}}(1-\ell)^{-\frac{1-\eta}{2}} - \frac{1+\eta}{2}\ell^{\frac{1-\eta}{2}}(1-\ell)^{\frac{-1+\eta}{2}}\right) = 0 \\
&\kappa_1 \left(\frac{1+\eta}{2}h^{\frac{-1+\eta}{2}}(1-h)^{\frac{1-\eta}{2}} - \frac{1-\eta}{2}h^{\frac{1+\eta}{2}}(1-h)^{\frac{-1-\eta}{2}}\right) \\
&+\kappa_2 \left(\frac{1-\eta}{2}h^{\frac{1+\eta}{2}}(1-h)^{-\frac{1-\eta}{2}} - \frac{1+\eta}{2}h^{\frac{1-\eta}{2}}(1-h)^{\frac{-1+\eta}{2}}\right) = \frac{\mu_{2,H}-\mu_{2,L}}{\lambda}.
\end{aligned}
$$

Solving for κ_1 and κ_2, we obtain

$$
\kappa_1 = -\frac{1}{\eta}\left\{(\frac{1}{\lambda} - c_2)\mu_{2,L}\left[\frac{1-\eta}{2}\ell^{\frac{-1-\eta}{2}}(1-\ell)^{\frac{1+\eta}{2}} - \frac{1+\eta}{2}\ell^{\frac{1-\eta}{2}}(1-\ell)^{\frac{-1+\eta}{2}}\right]\right\},
$$

and

$$
\kappa_2 = \frac{1}{\eta}\left\{(\frac{1}{\lambda} - c_2)\mu_{2,L}\left[\frac{1+\eta}{2}\ell^{\frac{-1+\eta}{2}}(1-\ell)^{\frac{1-\eta}{2}} - \frac{1-\eta}{2}\ell^{\frac{1+\eta}{2}}(1-\ell)^{\frac{-1-\eta}{2}}\right]\right\},
$$

Note that both κ_1 and κ_2 are continuous with respect to ℓ. Then, it is trivial to show that $f'_\ell(y)$ is thus continuous with respect to ℓ, and $y_\ell = f'^{-1}_\ell(k)$ is also continuous with respect to ℓ. Besides, both $f_\ell(y)$ and $g(y)$ are continuous with respect to y, thus the continuity of $\underline{G}(\cdot)$ and $\bar{G}(\cdot)$ follows.

Part (b): This part immediately follows from Lemma A.2-(b). **Part (c):** This part immediately follows from Lemma A.2-(c). □

Finally, with the existence of solutions (ℓ^*, h^*) and f_{ℓ^*}, we define function

$$
v(y; K_1) = \begin{cases} f_{\ell^*}(y), & y \in (\ell^*, h^*), \\ g(y), & y \in [0, \ell^*] \cup [h^*, 1]. \end{cases} \tag{A.21}
$$

By definition, v clearly satisfies (5.12), (5.13) and other mentioned properties. Lemma A.5 shows that $v(\cdot, K_1)$ is the value function under the two-threshold policy (ℓ^*, h^*). Note that these lower and upper critical thresholds are both functions of K_1; for brevity, we do not include K_1 as an argument of K_1. This completes the proof of Proposition 5.1.

Lemma A.5. *$v(\cdot; K_1)$ is the follower's value function (i.e., expected discounted profit) under the two-threshold policy (ℓ^*, h^*).*

Proof of Lemma A.5: By definition $A.21$, $v(\cdot, K_1)$ is $\mathcal{C}^1$, piecewise $\mathcal{C}^2$ and satisfies the following relation:

$$\frac{1}{2}q(K_1)v''(y; K_1)y^2(1-y)^2 - \lambda v(y; K_1) = 0. \tag{A.22}$$

It follows from standard probability arguments that the follower's expected time to invest is strictly positive under the two -threshold policy (ℓ^*, h^*). Then, by Itô's lemma, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\tau d\left(e^{-\lambda s}v(\pi_s, K_1)\right)\right] \\
&= \mathbb{E}\left[-\int_0^\tau \lambda e^{-\lambda s}v(\pi_s, K_1)ds + \int_0^\tau e^{-\lambda s}\left(v'(\pi_s, K_1)d\pi_s + \frac{1}{2}q(K_1)v''(\pi_s, K_1)(d\pi_s)^2\right)\right] \\
&= \mathbb{E}\left[\int_0^\tau e^{-\lambda s}\left(-\lambda v(\pi_s, K_1) + \frac{1}{2}q(K_1)v''(\pi_s, K_1)\right)ds\right] = 0.
\end{aligned} \tag{A.23}$$

The last equation above follows from (A.22). In addition,

$$\mathbb{E}\left[\int_0^\tau d\left(e^{-\lambda s}v(\pi_s; K_1)\right)\right] = \mathbb{E}\left[e^{-\lambda\tau}v(\pi_\tau; K_1)\right] - v(\pi_0; K_1).$$

Combining this with (A.23), we have

$$v(\pi_0; K_1) = \mathbb{E}\left[e^{-\lambda\tau}v(\pi_\tau; K_1)\right]. \tag{A.24}$$

We already know that $v(\ell^*; K_1) = g(\ell^*)$ and $v(h^*; K_1) = g(h^*)$. From Section 3.1, we also know that

$$g(y) = \mathbb{E}\left[\int_0^\infty e^{-\lambda t}dX_{2,t} - c_2K_2^* \mid \pi_\tau = y\right], \quad y \in \{\ell^*, h^*\}.$$

This and (A.24) immediately imply that $v(\cdot; K_1)$ is the expected discounted profit under the two-threshold policy (ℓ^*, h^*). □

Lemma A.6. *For any given $0 < \ell_1 < \ell_2 < 1$, $f_{\ell_1}(y) > f_{\ell_2}(y)$ for $y \geq \ell_2$.*

Proof of Lemma A.6: Suppose for a contradiction that there exists at least one belief at which $f_{\ell_1}(\cdot) \leq f_{\ell_2}(\cdot)$. Let $y_1 \doteq \min\{y : f_{\ell_1}(y) \leq f_{\ell_2}(y)\}$. The aforementioned minimum is achieved by continuity of f_{ℓ_1} and f_{ℓ_2}. By Lemma A.1, we know that $f_{\ell_1}(y)$ is strictly increasing in y, hence $f_{\ell_1}(\ell_2) > f_{\ell_1}(\ell_1) = f_{\ell_2}(\ell_2)$. Thus, $y_1 > \ell_2$. By definition of y_1 and the fact that $f_{\ell_1}(\ell_2) > f_{\ell_2}(\ell_2)$, there exists a $y_2 \in (\ell_2, y_1)$ such that $f'_{\ell_1}(y_2) < f'_{\ell_2}(y_2)$. By strict convexity of f_{ℓ_1}, we have $f'_{\ell_1}(\ell_2) > f'_{\ell_1}(\ell_1) = f'_{\ell_2}(\ell_2)$. This, $f'_{\ell_1}(y_2) < f'_{\ell_2}(y_2)$, and the continuity of f'_{ℓ_1} and f'_{ℓ_2} imply that there exists a $y_3 \in (\ell_2, y_2)$ such that $f''_{\ell_1}(y_3) < f''_{\ell_2}(y_3)$. Then, by ((A.11)), we have $f_{\ell_1}(y_3) < f_{\ell_2}(y_3)$. This contradicts the definition of y_1 as $y_3 < y_1$. $\square$

A.3.3 Proof of Proposition 5.2

We begin with the proof of part (a). Suppose that $K_1 \leq \mu_L$. Then, by (5.4), $q(K_1) = 0$, meaning that the leader's cumulative earning process is uninformative to the follower, and hence $\pi_t = \pi_0$ for $t \geq 0$. Then, because of discounting, it is optimal for the follower to invest immediately, i.e., $\tau = 0$. Thus, $V_2(y, K_1) = g(y)$ where $g(y) \doteq \max_{K_2 \geq 0} g(y, K_2)$ and $g(y, K_2)$ is as in (5.11). As a result, by (A.9), $K_2^* = \operatorname{argmax}_{K_2 \geq 0} g(y, K_2)$ is as in part (a).

We now prove part (b). Suppose that $K_1 > \mu_L$. Under this condition, we will show the optimality of the two-threshold policy constructed in the proof of Proposition 5.1. Recall that the value function of the policy considered in Proposition 5.1 is denoted by $v(y; K_1)$. Then, for any stopping time T such that $\mathbb{E}(T) < \infty$, we have the following by Ito's lemma:

$$\mathbb{E}[e^{-\lambda T} v(\pi_T; K_1)] = v(\pi_0; K_1) + \mathbb{E}\left[\int_0^T e^{-\lambda T}\left(\frac{1}{2} q(K_1)\pi_t^2(1-\pi_t)^2 v''(\pi_t; K_1) - \lambda v(\pi_t, K_1)\right)\right], \tag{A.25}$$

where $\{\pi_t, t \geq 0\}$ is the belief process under any given capacity $K_1 > \mu_L$. From (5.13), the value function $v(y; K_1)$ satisfies

$$\frac{1}{2} q(K_1) y^2 (1-y)^2 v''(y; K_1) - \lambda v(y; K_1) \leq 0, \quad y \in (0,1) \setminus \{\ell^*(K_1), h^*(K_1)\}. \tag{A.26}$$

Note that because $|\{\ell^*(K_1), h^*(K_1)\}| < \infty$, we can assign a value to $v''(\ell^*(K_1); K_1)$ and $v''(h^*(K_1); K_1)$ such that (A.26) holds for any $y \in (0, 1)$. Then, (A.25) and (A.26) lead to

$$\mathbb{E}\left[e^{-\lambda T}v(\pi_T;K_1)\right] \leq v(\pi_0;K_1). \tag{A.27}$$

Take $T = \tau$ under the two-threshold policy $(\ell^*(K_1), h^*(K_1))$, and let $(\hat{\tau}, K_2)$ be an arbitrary admissible policy for the follower. Then, from (A.27),(5.13), and the definition of $g(\cdot)$ in Section 3.1, we have

$$\mathbb{E}[e^{-\lambda\hat{\tau}}g(\pi_{\hat{\tau}}, K_2)] \leq \mathbb{E}[e^{-\lambda\hat{\tau}}g(\pi_{\hat{\tau}})] \leq v(\pi_0;K_1). \tag{A.28}$$

Noting that $\mathbb{E}[e^{-\lambda\hat{\tau}}g(\pi_{\hat{\tau}}, K_2)]$ is the follower's expected net profit under the policy $(\hat{\tau}, K_2)$ completes our proof. □

A.3.4 Proof of Proposition 5.3

We first prove the monotonicity of ℓ^* and h^* as function of q, which guarantees the existence of the limits. Then we study the asymptotic solutions under the two limits in σ.

Proposition A.1. *ℓ^* is strictly decreasing in q, and h^* is strictly increasing in q.*

Proof: Let $q_2 > q_1$ be two values that $q(k_1)$ can take. For $i \in \{1, 2\}$, let l_i and h_i be the two thresholds under q_i, and $f_\ell^{(i)}$ be the function defined in (A.11) when $q(K_1) = q_i$. We complete the proof in two steps:

1) We show $\ell_2 < \ell_1$ by comparing two functions $f_{\ell_1}^{(1)}$ and $f_{\ell_1}^{(2)}$.

2) We show $h_2 > h_1$ by comparing two functions $f_{\ell_1}^{(1)}$ and $f_{\ell_2}^{(2)}$.

proof of step 1): We first show by contradiction that

$$f_{\ell_1}^{(1)\prime}(y) > f_{\ell_1}^{(2)\prime}(y), \quad y \in (\ell_1, 1), \tag{A.29}$$

(see details below). From this it is obvious that $f_{\ell_1}^{(1)}(y) > f_{\ell_1}^{(2)}(y)$ holds on $y \in (\ell_1, 1)$, and since $f_{\ell_1}^{(1)}(y)$ is tangent to $g(y)$ at h_1, $f_{\ell_1}^{(2)}(y)$ has to intersect with $g(y)$ on (ℓ_1, h_1). From Lemma A.6, for any $\ell > \ell_1$, $f_\ell^{(2)}(y)$ lies below $f_{\ell_1}^{(2)}(y)$ and thus will also intersect with $g(y)$. Hence it has to be $\ell_2 < \ell_1$ for $f_{\ell_2}^{(2)}(y)$ to be tangent to $g(y)$.

Now we prove (A.29). We see from the ODE that

$$f_{\ell_1}^{(1)\prime\prime}(\ell_1) > f_{\ell_1}^{(2)\prime\prime}(\ell_1)$$

since $f_{\ell_1}^{(1)}(\ell_1) = f_{\ell_1}^{(2)}(\ell_1)$ and $q_1 < q_2$. Thus $\exists \eta > 0$ such that

$$f_{\ell_1}^{(1)\prime\prime}(y) > f_{\ell_1}^{(2)\prime\prime}(y), \quad y \in (\ell_1, \ell_1 + \eta),$$

and therefore

$$f_{\ell_1}^{(1)\prime}(y) > f_{\ell_1}^{(2)\prime}(y), \quad y \in (\ell_1, \ell_1 + \eta).$$

To prove by contradiction, we assume $\exists y_n \in (\ell_1, 1)$ such that $y_n = \sup\{y : \ f_{\ell_1}^{(1)\prime}(y) < f_{\ell_1}^{(2)\prime}(y)\}$. Otherwise (A.29) holds naturally. Therefore we have

$$f_{\ell_1}^{(1)\prime}(y_n) = \int_{\ell_1}^{y_n} f_{\ell_1}^{(1)\prime\prime}(y)dy + f_{\ell_1}^{(1)\prime\prime}(\ell_1) < \int_{\ell_1}^{y_n} f_{\ell_1}^{(2)\prime\prime}(y)dy + f_{\ell_1}^{(2)\prime\prime}(\ell_1) = f_{\ell_1}^{(2)\prime}(y_n).$$

Since we have $f_{\ell_1}^{(1)\prime\prime}(\ell_1) > f_{\ell_1}^{(2)\prime\prime}(\ell_1)$, combing with the above inequality we arrive at

$$\int_{\ell_1}^{y_n} f_{\ell_1}^{(1)\prime\prime}(y)dy < \int_{\ell_1}^{y_n} f_{\ell_1}^{(2)\prime\prime}(y)dy.$$

Therefore $\exists y_p \in (\ell_1, y_n)$ such that $f_{\ell_1}^{(1)\prime\prime}(y_p) < f_{\ell_1}^{(2)\prime\prime}(y_p)$. We use the ODE one more time and consider the fact that $q_1 < q_2$, we have

$$f_{\ell_1}^{(1)}(y_p) < f_{\ell_1}^{(2)}(y_p).$$

Again we look at the integrals

$$f_{\ell_1}^{(1)}(y_p) = \int_{\ell_1}^{y_p} f_{\ell_1}^{(1)\prime}(y)dy < \int_{\ell_1}^{y_p} f_{\ell_1}^{(2)\prime}(y)dy = f_{\ell_1}^{(2)}(y_p),$$

and find that $\exists y_q \in (\ell_1, y_p)$ such that $f_{\ell_1}^{(1)\prime}(y_q) < f_{\ell_1}^{(2)\prime}(y_q)$, which contradicts the definition of y_n in the beginning.

proof of step 2): We prove by contradiction. Assume $h_2 < h_1$, since $f_{\ell_1}^{(1)}(y) > g(y)$ for $y < h_1$ we have

$$f_{\ell_2}^{(2)}(h_2) = g(h_2) < f_{\ell_1}^{(1)}(h_2).$$

Also we have

$$f_{\ell_2}^{(2)}(\ell_1) > f_{\ell_2}^{(2)}(\ell_2) = f_{\ell_1}^{(1)}(\ell_1).$$

Because both functions are strictly increasing, $\exists y_r \in (\ell_1, h_2)$ uniquely where two functions intersect. Precisely we have

$$\begin{cases} f_{\ell_2}^{(2)}(y) > f_{\ell_2}^{(2)}(y), & y \in [\ell_2, y_r) \\ f_{\ell_2}^{(2)}(y) = f_{\ell_2}^{(2)}(y), & y = y_r \\ f_{\ell_2}^{(2)}(y) < f_{\ell_2}^{(2)}(y), & y \in (y_r, h_2) \end{cases} \tag{A.30}$$

Furthermore we have

$$f_{\ell_2}^{(2)\prime}(h_2) = g'(h_2) = g'(h_1) = f_{\ell_1}^{(1)\prime}(h_1) > f_{\ell_1}^{(1)\prime}(h_2),$$

from which we have

$$\int_{y_1}^{h_2} f_{\ell_2}^{(2)\prime\prime}(y)dy + f_{\ell_2}^{(2)\prime\prime}(y_r) > \int_{y_1}^{h_2} f_{\ell_1}^{(1)\prime\prime}(y)dy + f_{\ell_1}^{(1)\prime\prime}(y_r).$$

Note that with $f_{\ell_1}^{(1)}(y_r) = f_{\ell_2}^{(2)}(y_r)$, $q_1 < q_2$ and the ODE each function satisfies, we have

$$f_{\ell_2}^{(2)\prime\prime}(y_r) < f_{\ell_1}^{(1)\prime\prime}(y_r),$$

we arrive at the inequality

$$\int_{y_1}^{h_2} f_{\ell_2}^{(2)\prime\prime}(y)dy > \int_{y_1}^{h_2} f_{\ell_1}^{(1)\prime\prime}(y)dy.$$

Therefore $\exists y_s \in (y_r, h_2)$ such that

$$f_{\ell_2}^{(2)\prime\prime}(y_s) > f_{\ell_1}^{(1)\prime\prime}(y_s),$$

and with $q_2 > q_1$ and the ODEs we have

$$f_{\ell_2}^{(2)}(y_s) > f_{\ell_1}^{(1)}(y_s),$$

which contradicts (A.30).

$\square$

Figure A.9 shows how ℓ^* and h^* change as function of K_1, where $c_2\lambda = 0.5$ and $\mu_L = 1$, $\mu_H = 5$, $\sigma = 1$.

A.3.5 Proof of the Main Result

Proof of Part (a): From Lemma A.1 and the monotone convergence theorem, $\lim_{\sigma\to\infty}\ \ell^*(K_1)$ and $\lim_{\sigma\to\infty}\ h^*(K_1)$.

We first show that $\lim_{\sigma\to\infty}\ \ell^*(K_1) = c_2\lambda$ by contradiction. Suppose for a contradiction that $\lim_{\sigma\to\infty} \ell^*(K_1) = l_0 < c_2\lambda$. Select a large enough σ such that the corresponding signal quality $q(K_1) = q_1$ satisfies

$$q_1 < \frac{4\lambda^2 g(0)(c_2\lambda - l_0)}{\mu_{2,H} - \mu_{2,L}},$$

and let l_1 be the optimal lower belief threshold $\ell^*(K_1)$ and h_1 be the optimal higher threshold $h^*(K_1)$ when $q(K_1) = q_1$. Then, by Lemma A.1, we have $l_1 \le l_0$, and by the proof of Proposition 5.1, the follower's value function under the optimal policy satisfy (A.15) through (A.17). Then, by (A.15), we have the following for $y > \ell_1$

$$\begin{aligned} f_{l_1}''(y) &= \frac{\lambda f_{l_1}(y)}{q_1 y^2(1-y)^2} \\ &\ge \frac{\lambda g(0)}{q_1 y^2(1-y)^2} \qquad\qquad (A.31) \\ &> \frac{\mu_{2,H} - \mu_{2,L}}{4\lambda^2 g(0)(c_2\lambda - l_0)} \frac{\lambda g(0)}{y^2(1-y)^2} \\ &\ge \frac{\mu_{2,H} - \mu_{2,L}}{(c_2\lambda - l_0)\lambda}. \qquad\qquad (A.32) \end{aligned}$$

Note that (A.31) holds because $f_{\ell_1}(\cdot)$ is strictly increasing by Lemma A.1 and $f_{l_1}''(l_1) = g(0)$

by (A.16); (A.32) holds because $y(1-y) \leq 0.5$. Therefore, for any $y \geq c_2\lambda$ we arrive at

$$f'_{\ell_1}(y) = \int_{\ell_1}^{y} f''_{\ell_1}(u)du + f'_{\ell_1}(\ell_1) \geq \int_{\ell_1}^{c_2\lambda} f''_{\ell_1}(u)du > \frac{\mu_{2,H} - \mu_{2,L}}{\lambda},$$

which is equivalent to $g'(y)$ for $y > c_2\lambda$ by (A.10). However, this means that $f_{\ell_1}(\cdot)$ can never be tangent to $g(y)$ at any $y > c_2\lambda$. This contradicts (A.17). Hence the limit of $\ell^*(K_1)$ as $\sigma \to \infty$ can only be $c_2\lambda$.

We now prove that $\lim_{\sigma\to\infty} h^*(K_1) = c_2\lambda$ by contradiction. Suppose for a contradiction that $\lim_{\sigma\to\infty} h^*(K_1) = h_0 > c_2\lambda$ and choose a large enough σ such that the corresponding signal quality, which is denoted by q_2, satisfies

$$q_2 < \frac{4\lambda^2 g(0)(h_0 - c_2\lambda)}{\mu_{2,H} - \mu_{2,L}}.$$

Let ℓ_2 and h_2 be the optimal lower and upper belief thresholds when $q(K_1) = q_2$. Then, by Lemma A.1, $h_2 \geq h_0$. From (A.15), for $y > \ell_2$, we have

$$\begin{aligned}
f''_{\ell_2}(y) &= \frac{\lambda f_{\ell_2}(y)}{q_2 y^2(1-y)^2} \\
&> \frac{\lambda g(0)}{q_2 y^2(1-y)^2} \\
&> \frac{\mu_{2,H} - \mu_{2,L}}{4\lambda^2 g(0)(h_0 - c_2\lambda)} \frac{\lambda g(0)}{y^2(1-y)^2} \\
&\geq \frac{\mu_{2,H} - \mu_{2,L}}{(h_0 - c_2\lambda)\lambda}.
\end{aligned}$$

Therefore, we arrive at

$$\begin{aligned}
f'_{\ell_2}(h_2) &= \int_{\ell_2}^{h_2} f''_{\ell_2}(u)du + f'_{\ell_2}(\ell_2) \\
&\geq \int_{c_2\lambda}^{h_2} f''_{\ell_2}(u)du \\
&> \frac{\mu_{2,H} - \mu_{2,L}}{\lambda}
\end{aligned}$$

which means f_{ℓ_2} is not tangent to $g(y)$ at any $h_2 > c_2\lambda$. Hence, the limit of $h^*(K_1)$ can only be $c_2\lambda$ as $\sigma \to \infty$.

Now we show the speeds at which $\ell^*(K_1)$ and $h^*(K_1)$ converge to $c_2\lambda$. Recall from (A.16) and (A.17) that the value function $f_{\ell^*(K_1)}(\cdot)$ and the optimal belief thresholds $\ell^*(K_1)$ and $h^*(K_1)$ satisfy the following equations:

$$f_{\ell^*(K_1)}(\ell^*(K_1)) = g(\ell^*(K_1)) = \left(\frac{1}{\lambda} - c_2\right)\mu_{2,L}, \tag{A.33}$$

$$f_{\ell^*(K_1)}(h^*(K_1)) = g(h^*(K_1)) = \frac{\mu_{2,H} - \mu_{2,L}}{\lambda}h^*(K_1) + \frac{1}{\lambda}\mu_{2,L} - c_2\mu_{2,H}, \tag{A.34}$$

$$f'_{\ell^*(K_1)}(\ell^*(K_1)) = g'(\ell^*(K_1)) = 0, \tag{A.35}$$

$$f'_{\ell^*(K_1)}(h^*(K_1)) = g'(h^*(K_1)) = \frac{\mu_{2,H} - \mu_{2,L}}{\lambda}. \tag{A.36}$$

where

$$f_{\ell^*(K_1)}(y) = \kappa_1 y^{\frac{1+\eta}{2}}(1-y)^{\frac{1-\eta}{2}} + \kappa_2 y^{\frac{1-\eta}{2}}(1-y)^{\frac{1+\eta}{2}}. \tag{A.37}$$

Define

$$u \doteq \frac{1+\eta}{2}, \quad v \doteq \frac{1-\eta}{2} = 1-u, \quad \text{and} \quad w(y) \doteq \frac{y}{1-y}. \tag{A.38}$$

Note that we have the following as $\sigma \to \infty$

$$\frac{u}{\sigma} = O(1).$$

Because $u/\sigma = O(1)$ and $\ell^*(K_1)$ and $h^*(K_1)$, in short, ℓ^* and h^*, depend on σ only through u by (A.33) through (A.38), analyzing the rate at which ℓ^* and h^* converge to $c_2\lambda$ with respect to u will give us the rate at which ℓ^* and h^* converge to $c_2\lambda$ with respect to σ.

Take a large σ such that ℓ^* and h^* are sufficiently close to $c_2\lambda$, and suppose that these threshold beliefs satisfy the following relations for some finite constants γ_ℓ and γ_h:

$$h^* = c_2\lambda + \gamma_h/u + o(u^{-1}), \quad \ell^* = c_2\lambda - \gamma_\ell/u + o(u^{-1}).$$

Then, by (A.38),

$$w(h^*) = \frac{c_2\lambda + \gamma_h/u + o(u^{-1})}{(1-c_2\lambda)\left(1 - \frac{\gamma_h/u}{1-c_2\lambda} + o(u^{-1})\right)} = \frac{c_2\lambda + \gamma_h/u + o(u^{-1})}{1-c_2\lambda}\left(1 + \frac{\gamma_h/u}{1-c_2\lambda} + o(u^{-1})\right)$$
$$= w_0\left(1 + \frac{w_h}{u}\right) + o(u^{-1}), \tag{A.39}$$

where

$$w_0 \doteq \frac{c_2\lambda}{1-c_2\lambda} \quad \text{and} \quad w_h \doteq \frac{\gamma_h}{c_2\lambda(1-c_2\lambda)}. \tag{A.40}$$

Similarly,

$$w(\ell^*) = \frac{c_2\lambda - \gamma_\ell/u + o(u^{-1})}{(1-c_2\lambda)\left(1 + \frac{\gamma_\ell/u}{1-c_2\lambda} + o(u^{-1})\right)} = \frac{c_2\lambda - \gamma_\ell/u + o(u^{-1})}{1-c_2\lambda}\left(1 - \frac{\gamma_\ell/u}{1-c_2\lambda} + o(u^{-1})\right)$$
$$= w_0\left(1 - \frac{w_\ell}{u}\right) + o(u^{-1}). \tag{A.41}$$

where

$$w_\ell \doteq \frac{\gamma_\ell}{c_2\lambda(1-c_2\lambda)}. \tag{A.42}$$

Assume also that there exist finite coefficients $k_{10} > 0$ and $k_{20} > 0$ such that

$$\kappa_1 = k_{10}w_0^{-u} \quad \text{and} \quad \kappa_2 = k_{20}w_0^{u}. \tag{A.43}$$

Below we will show that there exists γ_ℓ, γ_h, k_{10} and k_{20} that satisfy (A.33) through (A.36), which proves that (A.41), (A.39) and (A.43) provide the perturbation solution of (A.33) through (A.36). Note that, by (A.37) and (A.38), we have

$$f_{\ell^*}(y) = \kappa_1(1-y)w(y)^u + \kappa_2 y w(y)^{-u},$$

and

$$f'_{\ell^*}(y) = \kappa_1 w(y)^u\left(\frac{u}{w(y)} - (1-u)\right) - \kappa_2 w(y)^{-u}\left(uw(y) - (1-u)\right).$$

Using these relations and the expressions in (A.41), (A.39) and (A.43), (A.35) is equivalent to

$$\begin{aligned}
f'_{\ell^*}(\ell^*) &= k_{10}w_0^{-u}\left(w_0(1-w_\ell/u)+o(u^{-1})\right)^u\left(\frac{u}{w_0(1-w_\ell/u)+o(u^{-1})}-1+u\right)\\
&\quad - k_{20}w_0^u\left(w_0(1-w_\ell/u)+o(u^{-1})\right)^{-u}\left([w_0(1-w_\ell/u)+o(u^{-1})]u-1+u\right)\\
&= 0.
\end{aligned}$$

Diving both sides of the above equation by u and letting $u \to \infty$, we get

$$k_{10}\frac{e^{-w_\ell}}{c_2\lambda} - k_{20}\frac{e^{w_\ell}}{1-c_2\lambda} = 0. \tag{A.44}$$

Similarly, we rewrite (A.36) as the following:

$$\begin{aligned}
f'_{\ell^*}(h^*) &= k_{10}w_0^{-u}\left(w_0(1+w_h/u)+o(u^{-1})\right)^u\left(\frac{u}{w_0(1+w_h/u)+o(u^{-1})}-1+u\right)\\
&\quad - k_{20}w_0^u\left(w_0(1+w_h/u)+o(u^{-1})\right)^{-u}\left([w_0(1+w_h/u)+o(u^{-1})]u-1+u\right)\\
&= \frac{\mu_{2,H}-\mu_{2,L}}{\lambda}.
\end{aligned}$$

In the above equation, dividing both sides by u and letting $u \to \infty$, we get

$$k_{10}\frac{e^{w_h}}{c_2\lambda} - k_{20}\frac{e^{-w_h}}{1-c_2\lambda} = 0. \tag{A.45}$$

These lead to

$$e^{2w_\ell} = e^{-2w_h} = \frac{k_{10}(1-c_2\lambda)}{k_{20}c_2\lambda},$$

hence $w_\ell = -w_h$. This implies by (A.40) and (A.42) that

$$\frac{\gamma_\ell}{c_2\lambda(1-c_2\lambda)} = -\frac{\gamma_h}{c_2\lambda(1-c_2\lambda)}.$$

Then, because $\gamma_\ell \geq 0$, $\gamma_h \geq 0$, it follows that

$$\gamma_\ell = 0 \quad \text{and} \quad \gamma_h = 0. \tag{A.46}$$

Furthermore, we rewrite (A.33) and (A.34) as follows.

$$\begin{aligned} f_{\ell^*}(\ell^*) &= k_{10}w_0^{-u}\left(w_0(1-w_\ell/u)+o(u^{-1})\right)^u\left(1-c_2\lambda+o(u^{-1})\right) \\ &\quad + k_{20}w_0^u\left(w_0(1-w_\ell/u)+o(u^{-1})\right)^{-u}\left(c_2\lambda+o(u^{-1})\right) \\ &= \left(\frac{1}{\lambda}-c_2\right)\mu_{2,L} \end{aligned}$$

$$\begin{aligned} f_{\ell^*}(h^*) &= k_{10}w_0^{-u}\left(w_0(1+w_h/u)+o(u^{-1})\right)^u\left(1-c_2\lambda+o(u^{-1})\right) \\ &\quad + k_{20}w_0^u\left(w_0(1+w_h/u)+o(u^{-1})\right)^{-u}\left(c_2\lambda+o(u^{-1})\right) \\ &= \left(\frac{1}{\lambda}-c_2\right)\mu_{2,L}+o(u^{-1}) \end{aligned}$$

By letting $u \to \infty$ and using the fact that $w_\ell = w_h = 0$ by (A.46), both equations above reduce to

$$k_{10}(1-c_2\lambda)+k_{20}c_2\lambda = \left(\frac{1}{\lambda}-c_2\right)\mu_{2,L}. \tag{A.47}$$

Note from (A.44), (A.45) and (A.46) that we also have

$$\frac{k_{10}}{c_2\lambda}-\frac{k_{20}}{1-c_2\lambda}=0.$$

Using this and (A.47), we obtain the following solution for k_{10} and k_{20}:

$$k_{10}=\frac{\mu_{2,L}}{2\lambda} \quad \text{and} \quad k_{20}=\frac{1-c_2\lambda}{\lambda}\frac{\mu_{2,L}}{2c_2\lambda}.$$

This completely characterizes the asymptotic solution as $u \to \infty$. This and the fact that $u/\sigma = O(1)$ complete the proof of part (a).

Proof of Part (b): We will use (A.33) through (A.37) to prove the claim. Recall from the definition of η in (5.24) that

$$\eta = \sqrt{1+\frac{8\lambda\sigma^2}{(\min(K_1,\mu_H)-\min(K_1,\mu_L))^2}},$$

which is equivalent to the following by Taylor series expansion around 1 when $\sigma > 0$ is sufficiently small:

$$1+\frac{4\lambda\sigma^2}{\left(\min(K_1,\mu_H)-\min(K_1,\mu_L)\right)^2}+O(\sigma^4), \tag{A.48}$$

Define

$$\epsilon \doteq \frac{2\lambda\sigma^2}{\left(\min(K_1,\mu_H)-\min(K_1,\mu_L)\right)^2}, \tag{A.49}$$

which is $O(\sigma^2)$ for $\sigma > 0$ sufficiently small. Based on this and (A.48), we can replace η with $1+2\epsilon+O(\epsilon^2)$ in $f_{\ell^*}(y)$ and $f'_{\ell^*}(y)$:

$$\begin{aligned}
f_{\ell^*}(y) &= \kappa_1 y^{1+\epsilon+O(\epsilon^2)}(1-y)^{1-\epsilon-O(\epsilon^2)} + \kappa_2 y^{-\epsilon-O(\epsilon^2)}(1-y)^{1+\epsilon+O(\epsilon^2)} \\
&= \kappa_1 y\left(\frac{y}{1-y}\right)^{\epsilon+O(\epsilon^2)} + \kappa_2(1-y)\left(\frac{1-y}{y}\right)^{\epsilon+O(\epsilon^2)} \qquad \text{(A.50)}\\
&= \kappa_1 y\left(1+\log\left(\frac{y}{1-y}\right)\epsilon+O(\epsilon^2)\right) + \kappa_2(1-y)\left(1-\log\left(\frac{y}{1-y}\right)\epsilon+O(\epsilon^2)\right) \\
&= \kappa_1 y+\kappa_2(1-y)+\left[\kappa_1 y-\kappa_2(1-y)\right]\log\left(\frac{y}{1-y}\right)\epsilon+O(\epsilon^2), \quad \text{and} \qquad \text{(A.51)}\\
f'_{\ell^*}(y) &= \kappa_1-\kappa_2+\left[\frac{\kappa_1}{1-y}-\frac{\kappa_2}{y}+(\kappa_1+\kappa_2)\log\left(\frac{y}{1-y}\right)\right]\epsilon+O(\epsilon^2). \qquad \text{(A.52)}
\end{aligned}$$

We now characterize a solution to (A.33) through (A.37) when $\epsilon > 0$ is sufficiently small. To do so, we first consider a candidate solution that has a particular relationship with ϵ, and then we will show that there exists a unique solution in the class of candidate solutions. This and the fact that there exists a unique solution to (A.33) through (A.37) prove that the candidate solution is indeed the solution of(A.33) through (A.37) when $\epsilon > 0$ is sufficiently small.

Consider the following candidate solution to (A.33) through (A.37) for some finite constants α_{01}, α_{02}, α_{11}, α_{12}, H, L, l_0 and h_0 when $\epsilon > 0$ is sufficiently small:

$$\kappa_1 = \alpha_{01}+\alpha_{11}\epsilon+O(\epsilon^2) \tag{A.53}$$

$$\kappa_2 = \alpha_{02}+\alpha_{12}\epsilon+O(\epsilon^2) \tag{A.54}$$

$$\ell^* = L+l_0\epsilon+O(\epsilon^2) \tag{A.55}$$

$$h^* = H-h_0\epsilon+O(\epsilon^2). \tag{A.56}$$

Because $\lim_{\epsilon\to 0}\ell^* \in [0, c_2\lambda]$ and $\lim_{\epsilon\to 0} h^* \in [0, c_2\lambda]$ and $\lim_{\epsilon\to 0} h^* \in [c_2\lambda, 1]$, there can be four main cases related to L and H. Later, we will analyze the candidate solution in said four cases. Before the analysis of the cases, we first introduce some preliminary analysis that will be repeatedly used in the remainder of the proof.

Using (A.53) and (A.54) in (A.51) and (A.52), $f_{\ell^*}(\cdot)$ and $f'_{\ell^*}(\cdot)$ reduce to

$$\begin{aligned}
f_{\ell^*}(y) &= (\alpha_{01} + \alpha_{11}\epsilon)\, y + (\alpha_{02} + \alpha_{12}\epsilon)\,(1-y) \\
&\quad + \left[(\alpha_{01} + \alpha_{11}\epsilon)\, y - (\alpha_{01} + \alpha_{11}\epsilon)\,(1-y)\right] \log\left(\frac{y}{1-y}\right)\epsilon + O(\epsilon^2) \\
&= \alpha_{01}y + \alpha_{02}(1-y) + \left[\alpha_{11}y + \alpha_{12}(1-y) + (\alpha_{01}y - \alpha_{02}(1-y))\log\left(\frac{y}{1-y}\right)\right]\epsilon + O(\epsilon^2),
\end{aligned} \tag{A.57}$$

$$\begin{aligned}
f'_{\ell^*}(y) &= (\alpha_{01} + \alpha_{11}\epsilon) - (\alpha_{02} + \alpha_{12}\epsilon) \\
&\quad + \left[\frac{(\alpha_{01} + \alpha_{11}\epsilon)}{1-y} - \frac{(\alpha_{02} + \alpha_{12}\epsilon)}{y} + (\alpha_{01} + \alpha_{02} + (\alpha_{11} + \alpha_{12})\epsilon)\log\left(\frac{y}{1-y}\right)\right]\epsilon + O(\epsilon^2) \\
&= \alpha_{01} - \alpha_{02} + \left[\alpha_{11} - \alpha_{12} + \frac{\alpha_{01}}{1-y} - \frac{\alpha_{02}}{y} + (\alpha_{01} + \alpha_{02})\log\left(\frac{y}{1-y}\right)\right]\epsilon + O(\epsilon^2).
\end{aligned}$$

Moreover, the limit of $f'_{\ell^*}(x)$ as $\epsilon \to 0$ at ℓ^* and h^* satisfy the following relations:

$$\begin{aligned}
\lim_{\epsilon\to 0} f'_{\ell^*}(\ell^*) &= \lim_{\epsilon\to 0}\left\{\alpha_{01} - \alpha_{02} + \left[\alpha_{11} - \alpha_{12} + \frac{\alpha_{01}}{1-\ell^*} - \frac{\alpha_{02}}{\ell^*} + (\alpha_{01} + \alpha_{02})\log\left(\frac{\ell^*}{1-\ell^*}\right)\right]\epsilon\right\} \\
&= \lim_{\epsilon\to 0}\left\{\alpha_{01} - \alpha_{02} + \left[-\frac{\alpha_{02}}{\ell^*} + (\alpha_{01} + \alpha_{02})\log\left(\frac{\ell^*}{1-\ell^*}\right)\right]\epsilon\right\} \\
&= \lim_{\epsilon\to 0}\left\{\alpha_{01} - \alpha_{02} + \left[-\alpha_{02} + (\alpha_{01} + \alpha_{02})\ell^*\log\left(\frac{\ell^*}{1-\ell^*}\right)\right]\frac{\epsilon}{\ell^*}\right\}
\end{aligned} \tag{A.58}$$

The second equation above holds because $\alpha_{11} - \alpha_{12} + \frac{\alpha_{01}}{1-\ell^*}$ is bounded on $[\alpha_{11} - \alpha_{12} + \alpha_{01}, \alpha_{11} - \alpha_{12} + \frac{\alpha_{01}}{1-c_2\lambda}]$. Similarly, since $\alpha_{11} - \alpha_{12} - \frac{\alpha_{02}}{h^*}$ is bounded on $\left[\alpha_{11} - \alpha_{12} - \alpha_{02},\ \ \alpha_{11} - \alpha_{12} - \frac{\alpha_{02}}{c_2\lambda}\right]$, we also have

$$\lim_{\epsilon\to 0} f'_{\ell^*}(h^*) = \lim_{\epsilon\to 0}\left\{\alpha_{01} - \alpha_{02} + \left[\alpha_{11} - \alpha_{12} + \frac{\alpha_{01}}{1-h^*} - \frac{\alpha_{02}}{h^*} + (\alpha_{01}+\alpha_{02})\log\left(\frac{h^*}{1-h^*}\right)\right]\epsilon\right\} \tag{A.59}$$

$$= \lim_{\epsilon\to 0}\left\{\alpha_{01} - \alpha_{02} + \left[\frac{\alpha_{01}}{1-h^*} + (\alpha_{01}+\alpha_{02})\log\left(\frac{h^*}{1-h^*}\right)\right]\epsilon\right\} \tag{A.60}$$

$$= \lim_{\epsilon\to 0}\left\{\alpha_{01} - \alpha_{02} + \left[\alpha_{01} + (\alpha_{01}+\alpha_{02})(1-h^*)\log\left(\frac{h^*}{1-h^*}\right)\right]\frac{\epsilon}{1-h^*}\right\} \tag{A.61}$$

We will also use the following limits in the analysis of cases.

$$\lim_{x\to 0} x\log\left(\frac{x}{1-x}\right) = \lim_{x\to 0}\frac{\log\left(\frac{x}{1-x}\right)}{1/x} = \lim_{x\to 0}\frac{\frac{1-x}{x}\frac{1}{(1-x)^2}}{-1/x^2} = \lim_{x\to 0}\frac{-x}{1-x} = 0, \tag{A.62}$$

and

$$\lim_{x\to 1}(1-x)\log\left(\frac{x}{1-x}\right) = \lim_{x\to 1}\frac{\log\left(\frac{x}{1-x}\right)}{1/(1-x)} = \lim_{x\to 1}\frac{\frac{1-x}{x}\frac{1}{(1-x)^2}}{1/(1-x)^2} = \lim_{x\to 1}\frac{1-x}{x} = 0, \tag{A.63}$$

We now study the four different cases for L and H below to conclude that the only possibility is that $L = 0$ and $H = 1$.

Case I: Suppose that $L > 0$ and $H < 1$. Then, (A.35), (A.36) and the limits of $f'_{\ell^*}(\cdot)$ in (A.58) and (A.61) produce:

$$\lim_{\epsilon\to 0} f'_{\ell^*}(\ell^*) = \alpha_{01} - \alpha_{02} = 0, \text{ and}$$

$$\lim_{\epsilon\to 0} f'_{\ell^*}(h^*) = \alpha_{01} - \alpha_{02} = \frac{\mu_{2,H} - \mu_{2,L}}{\lambda} > 0,$$

which is a contradiction.

Case II: Suppose that $L > 0$ and $H = 1$. Then, by (A.58),(A.61), (A.35), (A.36) and (A.63), we have

$$\lim_{\epsilon\to 0} f'_{\ell^*}(\ell^*) = \alpha_{01} - \alpha_{02} = 0,$$

$$\lim_{\epsilon\to 0} f'_{\ell^*}(h^*) = \alpha_{01} - \alpha_{02} + \alpha_{01}\lim_{\epsilon\to 0}\frac{\epsilon}{1-h^*} = \alpha_{01} - \alpha_{02} + \alpha_{01}\frac{1}{h_0} = \frac{\mu_{2,H} - \mu_{2,L}}{\lambda}.$$

Also from (A.57), (A.33) and (A.34), we have

$$\lim_{\epsilon\to 0} f_{\ell^*}(\ell^*) = \alpha_{01}L + \alpha_{02}(1-L) = \alpha_{01} = \left(\frac{1}{\lambda} - c_2\right)\mu_{2,L}$$

$$\begin{aligned}
\lim_{\epsilon\to 0} f_{\ell^*}(h^*) &= \lim_{\epsilon\to 0}\Bigg\{\alpha_{01}h^* + \alpha_{02}(1-h^*) \\
&\qquad + \left[\alpha_{11}h^* + \alpha_{12}(1-h^*) + (\alpha_{01}h^* - \alpha_{02}(1-h^*))\log\left(\frac{h^*}{1-h^*}\right)\right]\epsilon\Bigg\} \\
&= \lim_{\epsilon\to 0}\left\{\alpha_{01} + \alpha_{01}(1-h_0\epsilon + O(\epsilon^2))\log\left(\frac{1-h_0\epsilon+O(\epsilon^2)}{h_0\epsilon+O(\epsilon^2)}\right)\epsilon\right\} \\
&= \alpha_{01} \\
&= \left(\frac{1}{\lambda} - c_2\right)\mu_{2,H}.
\end{aligned}$$

This leads to the contradiction since $\mu_{2,H} > \mu_{2,L}$.

Case III: Suppose that $L = 0$ and $H < 1$. Then, by (A.61), (A.35), (A.36) and (A.63), we have:

$$\lim_{\epsilon\to 0} f'_{\ell^*}(\ell^*) = \alpha_{01} - \alpha_{02} - \alpha_{02}\lim_{\epsilon\to 0}\frac{\epsilon}{\ell^*} = \alpha_{01} - \alpha_{02} - \alpha_{02}\frac{1}{l_0} = 0,$$

$$\lim_{\epsilon\to 0} f'_{\ell^*}(h^*) = \alpha_{01} - \alpha_{02} = \frac{\mu_{2,H} - \mu_{2,L}}{\lambda}. \tag{A.64}$$

Also, from (A.57), (A.33) and (A.34), it follows that

$$\begin{aligned}
\lim_{\epsilon\to 0} f_{\ell^*}(\ell^*) &= \lim_{\epsilon\to 0}\Bigg\{\alpha_{01}\ell^* + \alpha_{02}(1-\ell^*) \\
&\qquad + \left[\alpha_{11}\ell^* + \alpha_{12}(1-\ell^*) + (\alpha_{01}\ell^* - \alpha_{02}(1-\ell^*))\log\left(\frac{\ell^*}{1-\ell^*}\right)\right]\epsilon\Bigg\} \qquad \text{(A.65)} \\
&= \lim_{\epsilon\to 0}\left\{\alpha_{02} - \alpha_{02}(1-l_0\epsilon + O(\epsilon^2))\log\left(\frac{l_0\epsilon+O(\epsilon^2)}{1-l_0\epsilon+O(\epsilon^2)}\right)\epsilon\right\} \\
&= \alpha_{02} \\
&= \left(\frac{1}{\lambda} - c_2\right)\mu_{2,L}, \qquad \text{(A.66)}
\end{aligned}$$

and

$$\lim_{\epsilon\to 0} f_{\ell^*}(h^*) = \alpha_{01}H + \alpha_{02}(1-H) = \frac{\mu_{2,H}-\mu_{2,L}}{\lambda}H + \frac{1}{\lambda}\mu_{2,L} - c_2\mu_{2,H}. \tag{A.67}$$

Note from (A.64) and (A.66) that

$$(\alpha_{01}-\alpha_{02})H + \alpha_{02} = \frac{\mu_{2,H}-\mu_{2,L}}{\lambda}H + \left(\frac{1}{\lambda} - c_2\right)\mu_{2,L} > \frac{\mu_{2,H}-\mu_{2,L}}{\lambda}H + \frac{1}{\lambda}\mu_{2,L} - c_2\mu_{2,H},$$

which contradicts (A.67).

Case IV: Suppose that $L = 0$ and $H = 1$. Then, by (A.35), (A.36), (A.62) and (A.63), we have:

$$\lim_{\epsilon\to 0} f'_{\ell^*}(\ell^*) = \alpha_{01} - \alpha_{02} - \alpha_{02}\lim_{\epsilon\to 0}\frac{\epsilon}{\ell^*} = \alpha_{01} - \alpha_{02} - \alpha_{02}\frac{1}{l_0} = 0,$$

$$\lim_{\epsilon\to 0} f'_{\ell^*}(h^*) = \alpha_{01} - \alpha_{02} + \alpha_{01}\lim_{\epsilon\to 0}\frac{\epsilon}{1-h^*} = \alpha_{01} - \alpha_{02} + \alpha_{01}\frac{1}{h_0} = \frac{\mu_{2,H}-\mu_{2,L}}{\lambda}.$$

Also, by (A.57), (A.33) and (A.34), we have

$$\begin{aligned}
\lim_{\epsilon\to 0} f_{\ell^*}(\ell^*) &= \alpha_{02} = \left(\frac{1}{\lambda} - c_2\right)\mu_{2,L},\\
\lim_{\epsilon\to 0} f_{\ell^*}(h^*) &= \alpha_{01}\\
&= \lim_{\epsilon\to 0}\frac{\mu_{2,H}-\mu_{2,L}}{\lambda}(1-h_0\epsilon) + \frac{\mu_{2,L}}{\lambda} - c_2\mu_{2,H} + O(\epsilon^2)\\
&= \left(\frac{1}{\lambda} - c_2\right)\mu_{2,H}
\end{aligned}$$

The above equations lead to the following set of solutions for the coefficients α_{01}, α_{02}, ℓ_0 and h_0:

$$\begin{aligned}
\alpha_{01} &= \left(\frac{1}{\lambda} - c_2\right)\mu_{2,H},\\
\alpha_{02} &= \left(\frac{1}{\lambda} - c_2\right)\mu_{2,L},\\
\ell_0 &= \frac{\mu_{2,L}}{\mu_{2,H}-\mu_{2,L}},\\
h_0 &= \left(\frac{1}{c_2\lambda} - 1\right)\frac{\mu_{2,H}}{\mu_{2,H}-\mu_{2,L}}.
\end{aligned}$$

Next we derive α_{11} and α_{12} as follows. Note that from (A.50) we have

$$f_{\ell^*}(y) = \kappa_1 y \left(\frac{y}{1-y}\right)^{\epsilon+O(\epsilon^2)} + \kappa_2(1-y)\left(\frac{1-y}{y}\right)^{\epsilon+O(\epsilon^2)},$$

and with $\ell^* = \ell_0\epsilon + O(\epsilon^2)$ and $h^* = 1 - h_0\epsilon + O(\epsilon^2)$, we have the following two limits:

$$\begin{aligned}\left(\frac{\ell_0\epsilon}{1-\ell_0\epsilon}\right)^{\epsilon} &= {\ell_0}^{\epsilon}\epsilon^{\epsilon}(1-\ell_0\epsilon)^{-\epsilon} \\ &= 1 + \epsilon\log\ell_0 + o(\epsilon);\end{aligned}$$

Similarly

$$\begin{aligned}\left(\frac{h_0\epsilon}{1-h_0\epsilon}\right)^{\epsilon} &= {h_0}^{\epsilon}\epsilon^{\epsilon}(1-h_0\epsilon)^{-\epsilon} \\ &= 1 + \epsilon\log h_0 + o(\epsilon).\end{aligned}$$

Hence, we have

$$\begin{aligned}f_{\ell^*}(\ell^*) &= (\alpha_{01}+\alpha_{11}\epsilon)\ell_0\epsilon(1+\epsilon\log\ell_0) + (\alpha_{02}+\alpha_{12}\epsilon)(1-\ell_0\epsilon)(1-\epsilon\log\ell_0) + o(\epsilon) \\ &= \alpha_{02} + (\alpha_{01}\ell_0 - \alpha_{02}\ell_0 - \alpha_{02}\log\ell_0 + \alpha_{12})\epsilon + o(\epsilon) \\ &= \left(\frac{1}{\lambda} - c_2\right)\mu_{2,L},\end{aligned}$$

which gives us

$$\begin{aligned}\alpha_{12} &= -\alpha_{01}\ell_0 + \alpha_{02}\ell_0 + \alpha_{02}\log\ell_0 \\ &= -\left(\frac{1}{\lambda} - c_2\right)\left(1 - \log\frac{\mu_{2,L}}{\mu_{2,H}-\mu_{1,L}}\right)\mu_{2,L}.\end{aligned}$$

Similarly,

$$\begin{aligned}f_{\ell^*}(h^*) &= (\alpha_{01}+\alpha_{11}\epsilon)(1-h_0\epsilon)(1-\log h_0\epsilon) + (\alpha_{02}+\alpha_{12}\epsilon)(h_0\epsilon)(1+\log h_0\epsilon) + o(\epsilon) \\ &= \alpha_{01} + (-\alpha_{01}h_0 + \alpha_{02}h_0 - \alpha_{01}\log h_0 + \alpha_{11})\epsilon + o(\epsilon) \\ &= \left(\frac{1}{\lambda} - c_2\right)\mu_{2,H} - \frac{\mu_{2,H}-\mu_{2,L}}{\lambda}h_0\epsilon + o(\epsilon),\end{aligned}$$

which leads to

$$\alpha_{11} = -\frac{\mu_{2,H} - \mu_{2,L}}{\lambda} h_0 + \alpha_{01} h_0 - \alpha_{02} h_0 + \alpha_{01} \log h_0$$
$$= \left(\frac{1}{\lambda} - c_2 \right) \left(-1 + \log \frac{\mu_{2,H}(1 - c_2\lambda)}{(\mu_{2,H} - \mu_{1,L}) c_2 \lambda} \right) \mu_{2,H}.$$

Then, based on the analysis in Cases I through IV, when $\epsilon > 0$ is sufficiently small, the solution of (A.33) through (A.36) is in the form (A.53) through (A.56). This and the fact that ϵ is $O(\sigma^2)$ complete the proof of part (b). □

A.3.6 The Heuristic Derivation and the Solution of (5.21)

For notational brevity, we use the following notation by suppressing the dependence of the function ϕ_{ℓ^*} on K_1:

$$\phi_{\ell^*}(y) = \phi_{\ell^*}(y; K_1).$$

Take an interval $[0, h]$ for a small $h > 0$. By an application of the Ito's lemma, we have

$$\mathbb{E}\left[\phi_{\ell^*}(\pi_h) \mid \pi_0 = y\right]$$
$$= \mathbb{E}\left[\phi_{\ell^*}(y) + (d\pi_h)\phi'_{\ell^*}(y) + \frac{1}{2}(d\pi_h)^2 \phi''_{\ell^*}(y)\right] + o(h)$$
$$= \phi_{\ell^*}(y) + \frac{1}{2} q(K_1) y^2 (1-y)^2 \phi''_{\ell^*}(y) h + o(h).$$

Note that the expected discounted earning is

$$\phi_{\ell^*}(y) = \mathbb{E}\left[e^{-\lambda h}\phi_{\ell^*}(\pi_h) \mid \pi_0 = y\right]$$
$$= (1 - \lambda h)\mathbb{E}\left[\phi_{\ell^*}(\pi_h) \mid \pi_0 = y\right] + o(h)$$
$$= \phi_{\ell^*}(y) + \left[\frac{1}{2} q(K_1) y^2 (1-y)^2 \phi''_{\ell^*}(y) - \lambda \phi_{\ell^*}(y)\right] h + o(h).$$

Subtracting $\phi_{\ell^*}(y)$ from both sides , dividing by h, and letting $h \to 0$, we obtain the ODE in (5.21).

From Lemma A.3, the general solution to the ODE (5.21) is of the following form:

$$\phi_{\ell^*}(y) = Ay^{\frac{1+\eta}{2}}(1-y)^{\frac{1-\eta}{2}} + By^{\frac{1-\eta}{2}}(1-y)^{\frac{1+\eta}{2}},$$

with A and B are coefficients to be determined from the boundary conditions. From the boundary conditions in (5.22), we have

$$\begin{cases} \phi_{\ell^*}(\ell^*) = A\ell^{*\frac{1+\eta}{2}}(1-\ell^*)^{\frac{1-\eta}{2}} + B\ell^{*\frac{1-\eta}{2}}(1-\ell^*)^{\frac{1+\eta}{2}} = 1, \\ \phi_{\ell^*}(h^*) = Ah^{*\frac{1+\eta}{2}}(1-h^*)^{\frac{1-\eta}{2}} + Bh^{*\frac{1-\eta}{2}}(1-h^*)^{\frac{1+\eta}{2}} = 0. \end{cases}$$

Solving for A and B using the above equations, we obtain $A = a(\ell^*, h^*)$ and $B = b(\ell^*, h^*)$ where $a(\ell^*, h^*)$ and $b(\ell^*, h^*)$ are as in (5.23). $\square$

A.3.7 Proof of Proposition 5.4

To prove the claim, we will first state and prove Lemma A.7 below. As $\sigma \to \infty$ or $\sigma \to 0$, the follower invests at $t = 0$ in equilibrium by Lemma A.7. Thus, $\widetilde{K}_2$ is as in Proposition 5.2-(a) and the leader's equilibrium value function with capacity K_1 is (5.14). Then, from standard optimization arguments, $\widetilde{K}_1$ is as in Proposition 5.4.

Lemma A.7. *In equilibrium, as $\sigma \to \infty$ or $\sigma \to 0$, the follower's investment timing $\tilde{\tau} \to 0$ with probability 1.*

Proof of Lemma A.7: We will first prove the lemma when $\sigma \to \infty$. Consider any finite $\sigma > 0$. With such $\sigma > 0$, there can be two possibilities related to the leader's capacity investment in equilibrium: $0 \leq \widetilde{K}_1 \leq \mu_L$ or $\mu_L < \widetilde{K}_1 \leq \mu_H$. (We have already explained right after Proposition 5.2 that it is never optimal for the leader to invest in a capacity more than μ_H.)

Because the leader do not invest more than μ_H in equilibrium, the leader's equilibrium capacity investment with any σ is bounded and takes a value in $[0, \mu_H]$. This and Proposition 5.3-(a) that proves $\lim_{\sigma\to\infty} \ell^*(K_1) = \lim_{\sigma\to\infty} h^*(K_1) = c_2\lambda$ for any K_1 immediately imply that $\lim_{\sigma\to\infty} \tilde{\tau} = 0$ for any given initial belief $\pi_0 \in [0, 1]$.

We now analyze the case when $\sigma \to 0$. Recall from Lemma 5.1 that for any given K_1, the posterior belief process π_t satisfies the following differential equation:

$$d\pi_t = \sqrt{q(K_1)}\pi_t(1-\pi_t)dW_t, \quad t \geq 0.$$

Then, by Itô's formula, we have

$$d\left(\log\left(\frac{\pi_t}{1-\pi_t}\right)\right) = -\frac{1}{2}q(K_1)(1-2\pi_t)dt + \sqrt{q(K_1)}dW_t.$$

Taking the integral on both sides above, we obtain

$$\log\left(\frac{\pi_t}{1-\pi_t}\right) - \log\left(\frac{\pi_0}{1-\pi_0}\right) = -\frac{1}{2}q(K_1)\int_0^t (1-2\pi_u)du + \sqrt{q(K_1)}W_t, \tag{A.68}$$

where π_0 is the initial belief.

Take a small $t = s > 0$. Then, by (A.68),

$$\log\left(\frac{\pi_s}{1-\pi_s}\right) = -\frac{1}{2}q(1-2\pi_0)s + I(s, q(K_1)) + \sqrt{q(K_1)}Z_s + \log\frac{\pi_0}{1-\pi_0},$$

where $I(s, q(K_1))$ is the integral approximation error due to approximating $\frac{1}{2}q(K_1)\int_0^t (2\pi_u - 1)du$. Therefore,

$$\pi_s = 1 - \left[1 + \frac{\pi_0}{1-\pi_0}\exp\left(-\frac{1}{2}q(K_1)(1-2\pi_0)s + I(s, q(K_1)) + \sqrt{q(K_1)}W_s\right)\right]^{-1}.$$

Note that $W_s \sim N(0, s)$, and hence

$$\pi_s = 1 - \left[1 + \frac{\pi_0}{1-\pi_0}\exp(Z)\right]^{-1},$$

where $Z \sim N\left(-\frac{1}{2}q(K_1)(1-2\pi_0)s + I(s, q(K_1)), \sqrt{q(K_1)s}\right)$. In equilibrium, if $\lim_{\sigma\to 0}\widetilde{K}_1 \leq \mu_L$, then it immediately follows from Proposition 5.2-(a) that $\lim_{\sigma\to 0}\widetilde{\tau} = 0$ with probability 1. We now analyze the alternative case in which $\lim_{\sigma\to 0}\widetilde{K}_1 \in (\mu_L, \mu_H]$. In this case, according to Proposition 5.2-(b), the follower uses a two-threshold policy (ℓ^*, h^*). We state and prove below that $\lim_{\sigma\to 0}\ \mathbb{P}(\pi_s \in (\ell^*(K_1), h^*(K_1))|\pi_0 \in (\ell^*(K_1), h^*(K_1))) = 0$ for $K_1 \in (\mu_L, \mu_H]$. This and the fact that $\widetilde{K}_1$ is bounded immediately imply that $\lim_{\sigma\to 0}\widetilde{\tau} = 0$ with probability 1.

We now show our claim that $\lim_{\sigma\to 0}\ \mathbb{P}(\pi_s \in (\ell^*(K_1), h^*(K_1))|\pi_0 \in (\ell^*(K_1), h^*(K_1))) = 0$ for $K_1 \in (\mu_L, \mu_H]$. Take any $K_1 \in (\mu_L, \mu_H]$ and a sufficiently small $\sigma > 0$. Suppose also that the initial belief $\pi_0 \in (\ell^*(K_1), h^*(K_1))$. Then, we have

$$\begin{aligned}
\mathbb{P}(\pi_s \in (\ell^*(K_1), h^*(K_1))) &= \mathbb{P}\left(\ell^*(K_1) < 1 - \left[1 + \frac{\pi_0}{1-\pi_0}\exp(Z)\right]^{-1} < h^*(K_1)\right) \\
&= \mathbb{P}\left(\frac{1}{1-\ell^*(K_1)} < 1 + \frac{\pi_0}{1-\pi_0}\exp(Z) < \frac{1}{1-h^*(K_1)}\right) \\
&= \mathbb{P}\left(\log\left(\frac{\ell^*(K_1)(1-\pi_0)}{(1-\ell^*(K_1))\pi_0}\right) < Z < \log\left(\frac{h^*(K_1)(1-\pi_0)}{(1-h^*(K_1))\pi_0}\right)\right) \\
&= \int_{\log\left(\frac{\ell^*(K_1)(1-\pi_0)}{(1-\ell^*(K_1))\pi_0}\right)}^{\log\left(\frac{h^*(K_1)(1-\pi_0)}{(1-h^*(K_1))\pi_0}\right)} \frac{1}{\sqrt{2\pi q(K_1)s}} e^{-\frac{\left(z+\frac{1}{2}q(K_1)(1-2\pi_0)s + I(s,q(K_1))\right)^2}{2q(K_1)s^2}} dz
\end{aligned}$$

Note that

$$\begin{aligned}
\mathbb{P}(\pi_s \in (\ell^*(K_1), h^*(K_1))) &\le \int_{\log\left(\frac{\ell^*(K_1)(1-\pi_0)}{(1-\ell^*(K_1))\pi_0}\right)}^{\log\left(\frac{h^*(K_1)(1-\pi_0)}{(1-h^*(K_1))\pi_0}\right)} \frac{1}{s\sqrt{2\pi q(K_1)}} dz \\
&= \frac{1}{s\sqrt{2\pi q(K_1)}}\left[\log\left(\frac{h^*(K_1)(1-\pi_0)}{(1-h^*(K_1))\pi_0}\right) - \log\left(\frac{\ell^*(K_1)(1-\pi_0)}{(1-\ell^*(K_1))\pi_0}\right)\right] \\
&= \frac{1}{s\sqrt{2\pi q(K_1)}}\left[\log\left(\frac{h^*(K_1)}{1-h^*(K_1)}\right) - \log\left(\frac{\ell^*(K_1)}{1-\ell^*(K_1)}\right)\right] \\
&\le \frac{1}{s\sqrt{2\pi q(K_1)}}\left[\log\left(\frac{1}{1-h^*(K_1)}\right) - \log \ell^*(K_1)\right].
\end{aligned}$$

Using the same ϵ defined in (A.49) and the solution of $\ell^*(K_1)$ and $h^*(K_1)$ in (A.55) and (A.56) with $L = 0$ and $H = 1$ (from Case IV of the proof of Proposition 5.3-(b)), we have

$$\begin{aligned}
\lim_{\sigma\to 0}\mathbb{P}(\pi_s \in (\ell^*(K_1), h^*(K_1))) &\le \lim_{\epsilon\to 0}\sqrt{\frac{\epsilon}{4\lambda}}\frac{1}{s}\left[\log\left(\frac{1}{1-h^*(K_1)}\right) - \log \ell^*(K_1)\right] \\
&= \lim_{\epsilon\to 0}\sqrt{\frac{\epsilon}{4\lambda}}\frac{1}{s}\left[-\log(\,h_0\epsilon + O(\epsilon^2)\,) - \log\left(l_0\epsilon + O(\epsilon^2)\right)\right] \\
&= 0.
\end{aligned}$$

The last equation holds because

$$\lim_{x\to 0}\sqrt{x}\log x = \lim_{x\to 0}\frac{\log x}{x^{-1/2}} = \lim_{x\to 0}\frac{x^{-1}}{-0.5x^{-3/2}} = \lim_{x\to 0} -2x^{1/2} = 0, \tag{A.69}$$

Therefore, for any $K_1 \in (\mu_L, \mu_H]$ and $s > 0$,

$$\lim_{\sigma \to 0} \mathbb{P}(\pi_s \in (\ell^*(K_1), h^*(K_1))) = 0.$$

$\square$

This completes the proof of the proposition. $\square$

A.3.8 Proof of Proposition 5.5

We first state and prove Lemma A.8. That result will be used in the remainder of the proof.

Lemma A.8. ***(a)*** *$H_1(K_1, y_0) > H_0(K_1, y_0)$ for $(K_1, y_0) \in (\mu_L, \infty) \times (\ell^*(K_1), h^*(K_1))$.*

(b) *For a given y_0, the unconstrained optimizer of $H_0(K_1, y_0)$ is*

$$\arg\max_{K_1 \geq 0} \ H_0(K_1, y_0) = \begin{cases} \mu_{1,H} & \text{if} \quad y_0 > c_1 \lambda, \\ \mu_{1,L} & \text{if} \quad y_0 \leq c_1 \lambda. \end{cases}$$

Proof of Lemma A.8: We begin with the proof of part (a). By proposition 5.2, the investment timing of the follower, that is, τ, is strictly positive with probability 1. Then, because $e^{-\lambda t} \min\{K_1, \mu\} > e^{-\lambda t} \min\{K_1, \mu_1\}$ with probability 1 for any $t > 0$, it follows from (5.15) and (5.14) that $H_1(K_1, y_0) > H_0(K_1, y_0)$ for $y_0 \in (\ell^*(K_1), h^*(K_1))$. We now show part (b). For any fixed y_0, $H_0(K_1, y_0)$ as a function of K_1 is as follows by (5.14).

$$H_0(K_1, y_0) = \begin{cases} \left(\frac{1}{\lambda} - c_1\right) K_1 & \text{if} \quad 0 < K_1 \leq \mu_{1,L} \\ \left(\frac{1}{\lambda} y_0 - c_1\right) K_1 + (1 - y_0) \frac{\mu_{1,L}}{\lambda} & \text{if} \quad \mu_{1,L} < K_1 \leq \mu_{1,H} \\ -c_1 K_1 + y_0 \frac{\mu_{1,H}}{\lambda} + (1 - y_0) \frac{\mu_{1,L}}{\lambda} & \text{if} \quad K_1 > \mu_{1,H}. \end{cases}$$

Because $\partial H_0(K_1, y_0)/\partial K_1 > 0$ for $0 < K_1 \leq \mu_{1,L}$ and $\partial H_0(K_1, y_0)/\partial K_1 > 0$ for $\mu_{1,L} < K_1 \leq \mu_{1,H}$ if and only if $y_0 > c_1 \lambda$, part (b) follows. $\square$

Lemma A.9. *There exist initial beliefs $\underline{\underline{\alpha}}$ and $\overline{\overline{\alpha}}$ such that the leader's equilibrium value function is*

$$V_1(y_0) = \max_{K_1 \geq 0} \ H_0(K_1, y_0), \quad \textit{for} \quad y_0 \in [0, \underline{\underline{\alpha}}) \cup (\overline{\overline{\alpha}}, 1], \tag{A.70}$$

and the leader's equilibrium capacity is

$$\widetilde{K}_1 = \begin{cases} \mu_{1,L} & \textit{if} \quad y_0 \le \min\{c_1\lambda, \underline{\underline{\alpha}}\}, \\ \mu_{1,H} & \textit{if} \quad y_0 \ge \max\{c_1\lambda, \overline{\overline{\alpha}}\}. \end{cases} \tag{A.71}$$

Proof of Lemma A.9: We claim and show below that $\ell^*(K_1) > 0$ and $h^*(K_1) < 1$ for any given K_1. This and the fact that $\widetilde{K}_1 \in [\mu_L, \mu_H]$ imply that $\ell^*(\widetilde{K}_1)$ is bounded below and $h^*(\widetilde{K}_1)$ is bounded above in equilibrium. We already know that $\ell^*(K_1)$ and $h^*(K_1)$ depend on K_1 only through the signal quality $q(K_1)$, which is increasing in K_1. In addition, recall from the proof of Proposition 5.2 that ℓ^* decreases in the signal quality while h^* increases in that. Thus, $\ell^*(\widetilde{K}_1) \ge \underline{\underline{\alpha}} \doteq \ell^*(\mu_H) > 0$ and $h^*(\widetilde{K}_1) \le \overline{\overline{\alpha}} \doteq h^*(\mu_H) > 0$. Then, by Proposition 5.2, in equilibrium, the follower invests at $\tau = 0$ for $y_0 \in [0, \underline{\underline{\alpha}}) \cup (\overline{\overline{\alpha}}, 1]$ regardless of the leader's capacity K_1. Thus, for $y_0 \in [0, \underline{\underline{\alpha}}) \cup (\overline{\overline{\alpha}}, 1]$, the leader's value function with capacity K_1 is $H_0(K_1, y_0)$, and hence the leader's equilibrium value function is as in (A.70). This and Lemma A.8-(b) immediately imply (A.71).

It only remains to prove our above claim that $\ell^*(K_1) > 0$ and $h^*(K_1) < 1$ for any given $K_1 > \mu_L$. Recall the function $f_\ell(\cdot)$ we used to construct the optimal value function in the proof of Proposition 5.2. We already know from Lemma A.2-(b) that $\ell^*(K_1)$ cannot be 0 or close to 0 because for ℓ sufficiently close to zero $f_\ell(\cdot)$ cannot touch $g(\cdot)$ at a point in the increasing portion of $g(\cdot)$. Similarly, by Lemma A.2-(c), $h^*(K_1)$ cannot be 1 or very close to 1 because it $h^*(K_1)$ were 1, neither the value-matching condition or the smooth-pasting condition would have been met at $y = h^*(K_1)$. This completes the proof of our claim. □

Given Lemma A.9, it only remains to show statements related to $y_0 \in (\underline{\alpha}, \overline{\alpha})$ in Proposition 5.5. To characterize the leader's equilibrium value function, we will evaluate $H_0(K_1, y_0)$ and $H_1(K_1, y_0)$ by fixing an initial belief y_0. As explained in the proof of Lemma A.9, $\ell^*(K_1)$ decreases continuously and monotonically from $c_2\lambda$ to $\ell^*(\mu_H)$, and $h^*(K_1)$ increases continuously and monotonically from $c_2\lambda$ to $h^*(\mu_H)$ as K_1 increases from μ_L to μ_H. Also, by the proof of Proposition 5.2-(b), we have $0 < \ell^*(\mu_H) < c_2\lambda$ and $c_2\lambda < h^*(\mu_H) < 1$. Therefore, if $y_0 \in (\ell^*(\mu_H), c_2\lambda]$, there exists a $K_0(y_0)$ such that $\ell^*(K_0(y_0)) = y_0$, and hence $\tau = 0$ with probability 1 for $K_1 \le K_0(y_0)$ and $\tau > 0$ with probability 1 for $K_1 > K_0(y_0)$. Similarly, when

$y_0 \in [c_2\lambda, h^*(\mu_H))$, there exists a $K_0(y_0)$ such that $h^*(K_0(y_0)) = y_0$, and thus $\tau = 0$ with probability 1 for $K_1 \le K_0(y_0)$ and $\tau > 0$ with probability 1 for $K_1 > K_0(y_0)$. That is

$$H(K_1, y_0) = \begin{cases} H_0(K_1, y_0) & 0 < K_1 \le K_0(y_0), \\ H_1(K_1, y_0) & K_1 > K_0(y_0), \end{cases}$$

Therefore, the leader's equilibrium capacity $\widetilde{K}_1(y_0)$ is determined by comparing the maximum of $H_0(K_1, y_0)$ over $K_1 \le K_0(y_0)$ and the maximum of $H_1(K_1, y_0)$ over $K_1 > K_0(y_0)$.

We show in Figure A.10 as an example that how the value function $H(K_1, y_0)$ varies by K_1 for different fixed y_0.

Recall the unconstrained optimizer of $H_0(K_1, y_0)$ from Lemma A.8-(b). Suppose that $K_0(y_0) < \mu_{1,H}$. We claim and show below that this condition is equivalent to $y_0 \in (y_a, y_b)$ for some y_a and y_b.

The condition $K_0(y_0) < \mu_{1,H}$ implies the leader's value function $H(K_1, y_0) = H_1(K_1, y_0)$ in a region including $\mu_{1,H}$. Hence, by Lemmas A.8-(a) and A.8-(b), we have the following for $y_0 > c_1\lambda$:

$$\max_{K_1 > K_0(y_0)} H_1(K_1, y_0) \ge H_1(\mu_{1,H}, y_0) > H_0(\mu_{1,H}, y_0) > \max_{K_1 \le K_0(y_0)} H_0(K_1, y_0). \tag{A.72}$$

Note that H_1 is continuous in y_0, $H_0(\mu_{1,L}, y_0)$ does not change with y_0, and $H_0(K_1, y_0)$ increases in y_0 for any given $K_1 \in (\mu_{1,L}, \mu_{1,H}]$. Then, from (A.72), there exists $\underline{y} < c_1\lambda$ such that

$$\max_{K_1 > K_0(\underline{y})} H_1(K_1, \underline{y}) = H_0(\mu_{1,H}, c_1\lambda) = \left(\frac{1}{\lambda} - c_1\right) \mu_{1,L} = \max_{K_1 \le K_0(\underline{y})} H_0(K_1, \underline{y}),$$

and

$$\max_{K_1 > K_0(y_0)} H_1(K_1, y_0) > \max_{K_1 \le K_0(y_0)} H_0(K_1, y_0)$$

for $y_0 > \underline{y}$. Combining this and the fact that $K_0(y_0) < \mu_{1,H}$ is equivalent to $y_0 \in (y_a, y_b)$ for some y_a and y_b, we conclude that if $y_0 \in (\underline{\alpha}, \overline{\alpha})$ where $\underline{\alpha} \doteq \max\{y_a, \underline{y}\}$ and $\overline{\alpha} \doteq y_b$, then the leader's equilibrium capacity $\widetilde{K}_1(y_0) > K_0(y_0)$ and hence $\tau > 0$ with probability 1.

It only remains to prove our above claim that $K_0(y_0) < \mu_{1,H}$ can be written as $y_0 \in (\underline{\alpha}, \overline{\alpha})$

for some $\underline{\alpha}$ and $\overline{\alpha}$. For $y_0 \in (c_2\lambda, h^*(\mu_H))$, $K_0(y_0) < \mu_{1,H}$ is equivalent to $y_0 = h^*(K_0(y_0)) < h^*(\mu_{1,H})$. On the other hand, for $y_0 \in (\ell^*(\mu_H), c_2\lambda)$, $K_0(y_0) < \mu_{1,H}$ is equivalent to $y_0 = \ell^*(K_0(y_0)) > \ell^*(\mu_{1,H})$. Letting $y_a \doteq \ell^*(\mu_{1,H})$ and $y_b \doteq h^*(\mu_{1,H})$ completes the proof of the claim. $\square$

A.3.9 Proof of Proposition 5.6

We will prove this proposition in three steps. In Step A, we will derive an expression for $\frac{\partial H_1(K_1,y_0)}{\partial y_0}$, that will be used in the remainder of the proof. In Step B, using Proposition 5.3, we get alternative expressions for various terms of $\frac{\partial H_1(K_1,y_0)}{\partial y_0}$ for large σ. Using the expressions derived in Steps A and B, in Step C, we will drive sufficient conditions stated in the proposition.

Step A: Recall from (5.15) the definition of $H_1(K_1, y_0)$. Then,

$$\begin{aligned}&\frac{\partial H_1(K_1,y_0)}{\partial y_0}\\ &= -\frac{\xi_1(\ell^*(K_1)) - \xi_2(\ell^*(K_1))}{\lambda}\frac{\partial \phi_{\ell^*}(y_0;K_1)}{\partial y_0} - \frac{\xi_1(h^*(K_1)) - \xi_2(h^*(K_1))}{\lambda}\frac{\partial \phi_{h^*}(y_0;K_1)}{\partial y_0} + \frac{K_1-\mu_L}{\lambda},\end{aligned} \tag{A.73}$$

where, by (5.18) and (5.19),

$$\begin{aligned}&\xi_1(\ell^*(K_1)) - \xi_2(\ell^*(K_1))\\ &= \begin{cases}\mu_{2,L}(1-\ell^*(K_1)) & \text{if } \mu_{1,H} > \mu_L \text{ and } \mu_L < K_1 \le \mu_{1,H}\\ \mu_{2,L}(1-\ell^*(K_1)) + (K_1-\mu_{1,H})\ell^*(K_1) & \text{if } K_1 > \mu_{1,H} > \mu_L \text{ or } \ \mu_{1,H} \le \mu_L,\end{cases}\end{aligned}$$

and

$$\begin{aligned}&\xi_1(h^*(K_1)) - \xi_2(h^*(K_1))\\ &= \begin{cases}\mu_{2,L}(1-h^*(K_1)) & \text{if } \mu_{1,H} > \mu_L \text{ and } \mu_L < K_1 \le \mu_{1,H}\\ \mu_{2,L}(1-h^*(K_1)) + (K_1-\mu_{1,H})h^*(K_1), & \text{if } K_1 > \mu_{1,H} > \mu_L \text{ or } \mu_{1,H} \le \mu_L.\end{cases}\end{aligned}$$

Let

$$f_1(y) \doteq y^{\frac{1+\eta}{2}}(1-y)^{\frac{1-\eta}{2}} \quad \text{and} \quad f_2(y) \doteq y^{\frac{1-\eta}{2}}(1-y)^{\frac{1+\eta}{2}}.$$

Then, we have

$$f_1'(y) = \frac{\eta+1}{2}\left(\frac{y}{1-y}\right)^{\frac{\eta-1}{2}} + \frac{\eta-1}{2}\left(\frac{y}{1-y}\right)^{\frac{\eta+1}{2}} > 0,$$
$$f_2'(y) = -\frac{\eta-1}{2}\left(\frac{1-y}{y}\right)^{\frac{\eta+1}{2}} - \frac{\eta+1}{2}\left(\frac{1-y}{y}\right)^{\frac{\eta-1}{2}} < 0.$$

Recall (5.23) through (5.27). Then,

$$\frac{\partial \phi_{\ell^*}(y_0; K_1)}{\partial y_0} = a(\ell^*(K_1), h^*(K_1))f_1'(y_0) + b(\ell^*(K_1), h^*(K_1))f_2'(y_0),$$
$$\frac{\partial \phi_{h^*}(y_0; K_1)}{\partial y_0} = a(h^*(K_1), \ell^*(K_1))f_1'(y_0) + b(h^*(K_1), \ell^*(K_1))f_2'(y_0).$$

Recall also (5.24) and define

$$\beta \doteq \frac{\ell^*(K_1)(1-h^*(K_1))}{h^*(K_1)(1-\ell^*(K_1))} < 1, \tag{A.74}$$

$$\begin{aligned} A_1 &\doteq \ell^*(K_1)a(\ell^*(K_1), h^*(K_1)) + h^*(K_1)a(h^*(K_1), \ell^*(K_1)) \\ &= \left(\beta^{\frac{-\eta}{2}} - \beta^{\frac{\eta}{2}}\right)^{-1}\left(\frac{\ell^*(K_1)}{1-\ell^*(K_1)}\right)^{\frac{1}{2}}\left(\frac{1-h^*(K_1)}{h^*(K_1)}\right)^{\frac{\eta}{2}}\left[\beta^{\frac{-1-\eta}{2}} - 1\right] > 0, \end{aligned}$$

$$\begin{aligned} B_1 &\doteq \ell^*(K_1)b(\ell^*(K_1), h^*(K_1)) + h^*(K_1)b(h^*(K_1), \ell^*(K_1)) \\ &= \left(\beta^{\frac{-\eta}{2}} - \beta^{\frac{\eta}{2}}\right)^{-1}\left(\frac{\ell^*(K_1)}{1-\ell^*(K_1)}\right)^{\frac{1}{2}}\left(\frac{1-h^*(K_1)}{h^*(K_1)}\right)^{-\frac{\eta}{2}}\left[1 - \beta^{\frac{-1+\eta}{2}}\right] > 0, \end{aligned}$$

$$\begin{aligned} A_2 &\doteq (1-\ell^*(K_1))a(\ell^*(K_1), h^*(K_1)) + (1-h^*(K_1))a(h^*(K_1), \ell^*(K_1)) \\ &= \left(\beta^{\frac{-\eta}{2}} - \beta^{\frac{\eta}{2}}\right)^{-1}\left(\frac{1-\ell^*(K_1)}{\ell^*(K_1)}\right)^{\frac{1}{2}}\left(\frac{1-h^*(K_1)}{h^*(K_1)}\right)^{\frac{\eta}{2}}\left[\beta^{\frac{1-\eta}{2}} - 1\right] > 0, \end{aligned}$$

$$
\begin{aligned}
B_2 &\doteq (1-\ell^*(K_1))b(\ell^*(K_1),h^*(K_1)) + (1-h^*(K_1))b(h^*(K_1),\ell^*(K_1)) \\
&= \left(\beta^{\frac{-\eta}{2}} - \beta^{\frac{\eta}{2}}\right)^{-1} \left(\frac{1-\ell^*(K_1)}{\ell^*(K_1)}\right)^{\frac{1}{2}} \left(\frac{1-h^*(K_1)}{h^*(K_1)}\right)^{-\frac{\eta}{2}} \left[1-\beta^{\frac{1+\eta}{2}}\right] > 0.
\end{aligned}
$$

Using these and (A.73), we get

$$
\frac{\partial H_1(K_1,y_0)}{\partial y_0} = \begin{cases} \frac{-\mu_{2,L}}{\lambda}\left(A_2 f_1'(y_0) + B_2 f_2'(y_0)\right) + \frac{1}{\lambda}(K_1-\mu_L), \\ \qquad \text{if } \mu_{1,H} > \mu_L \text{ and } \mu_L < K_1 \le \mu_{1,H} \\ \frac{-\mu_{2,L}}{\lambda}\left(A_2 f_1'(y_0) + B_2 f_2'(y_0)\right) - \frac{(K_1-\mu_{1,H})\left(A_1 f_1'(y_0)+B_1 f_2'(y_0)\right)}{\lambda} + \frac{(K_1-\mu_L)}{\lambda}, \\ \qquad \text{if } K_1 > \mu_{1,H} > \mu_L \text{ or } \mu_{1,H} \le \mu_L. \end{cases}
$$

For notational brevity, hereafter, we will drop the argument K_1 from ℓ^* and h^*.

Step B: Recall Proposition 5.3 and the notation β from (A.74). Let $\ell^* = c_2\lambda - \delta_\ell$ and $h^* = c_2\lambda + \delta_h$ where $\delta_\ell > 0$ and $\delta_h > 0$. Then,

$$
\begin{aligned}
\beta &= \frac{\ell^*(1-h^*)}{h^*(1-\ell^*)} = \frac{(c_2\lambda-\delta_\ell)(1-c_2\lambda-\delta_h)}{(c_2\lambda+\delta_h)(1-c_2\lambda+\delta_\ell)} \\
&= \frac{c_2\lambda-\delta_\ell}{(1-c_2\lambda)(1+\frac{\delta_\ell}{1-c_2\lambda})} \cdot \frac{1-c_2\lambda-\delta_h}{c_2\lambda(1+\frac{\delta_h}{c_2\lambda})} \\
&= \frac{1}{c_2\lambda(1-c_2\lambda)}(c_2\lambda-\delta_\ell)(1-c_2\lambda-\delta_h)\left(1-\frac{\delta_\ell}{1-c_2\lambda}+O(\delta_\ell^2)\right)\left(1-\frac{\delta_h}{c_2\lambda}+O(\delta_h^2)\right) \\
&= 1-\frac{\delta_\ell+\delta_h}{c_2\lambda(1-c_2\lambda)} + O(\delta_\ell^2) + O(\delta_h^2) + O(\delta_\ell\delta_h).
\end{aligned}
$$

This, the generalized binomial theorem and the fact that $\eta = O(\sigma)$ imply that as $\sigma \to \infty$, we have

$$\beta^{-\eta/2} = 1 + \frac{\delta_\ell + \delta_h}{2c_2\lambda(1-c_2\lambda)}\eta + O(\sigma\delta_\ell^2) + O(\sigma\delta_h^2) + O(\sigma\delta_\ell\delta_h) \tag{A.75}$$

$$\beta^{\eta/2} = 1 - \frac{\delta_\ell + \delta_h}{2c_2\lambda(1-c_2\lambda)}\eta + O(\sigma\delta_\ell^2) + O(\sigma\delta_h^2) + O(\sigma\delta_\ell\delta_h) \tag{A.76}$$

$$\beta^{-(\eta+1)/2} = 1 + \frac{\delta_\ell + \delta_h}{2c_2\lambda(1-c_2\lambda)}(\eta+1) + O(\sigma\delta_\ell^2) + O(\sigma\delta_h^2) + O(\sigma\delta_\ell\delta_h) \tag{A.77}$$

$$\beta^{(-1+\eta)/2} = 1 - \frac{\delta_\ell + \delta_h}{2c_2\lambda(1-c_2\lambda)}(\eta-1) + O(\sigma\delta_\ell^2) + O(\sigma\delta_h^2) + O(\sigma\delta_\ell\delta_h) \tag{A.78}$$

$$\beta^{(1-\eta)/2} = 1 + \frac{\delta_\ell + \delta_h}{2c_2\lambda(1-c_2\lambda)}(\eta-1) + O(\sigma\delta_\ell^2) + O(\sigma\delta_h^2) + O(\sigma\delta_\ell\delta_h) \tag{A.79}$$

$$\beta^{(\eta+1)/2} = 1 - \frac{\delta_\ell + \delta_h}{2c_2\lambda(1-c_2\lambda)}(\eta+1) + O(\sigma\delta_\ell^2) + O(\sigma\delta_h^2) + O(\sigma\delta_\ell\delta_h). \tag{A.80}$$

Step C: Using these expressions, we now prove the main result via Envelope Theorem. Recall that the function $H_1(\cdot,\cdot)$ is only defined for the initial belief in the continuation region. We also know from the proof of Proposition 5.2 that $c_2\lambda$ always lies in the continuation region. We already know from the proof of Proposition 5.5 that $\widetilde{K}_1(c_2\lambda) > \mu_L$ when $\mu_L < \mu_{1,H}$. Thus, under that condition, $H(\widetilde{K}_1(c_2\lambda), c_2\lambda) = H_1(\widetilde{K}_1(c_2\lambda), c_2\lambda)$ and $\widetilde{K}_1(c_2\lambda)$ is an optimizer of $H_1(K_1, c_2\lambda)$. Then, if $H_1(K_1; c_2\lambda)$ is differentiable at $\widetilde{K}_1(c_2\lambda)$, then by Envelope Theorem,

$$V_1'(c_2\lambda) = \left.\frac{\partial H_1(K_1, c_2\lambda)}{\partial y_0}\right|_{K_1=\widetilde{K}_1(c_2\lambda)}.$$

We claim and show at the end of the proof that $H_1(K_1; c_2\lambda)$ is differentiable for any $K_1 \in (\mu_L, \mu_H)$ except at most one point, that is, $\mu_{1,H}$. Because the parameter set in which $\widetilde{K}_1(c_2\lambda) = \mu_{1,H}$ is Lebesgue measure zero, in our analysis we focus on the case where $\widetilde{K}_1(c_2\lambda) \in (\mu_L, \mu_H) \setminus \{\mu_{1,H}\}$. Below, we will analyze $\left.\frac{\partial H_1(K_1,c_2\lambda)}{\partial y_0}\right|_{K_1=\widetilde{K}_1(c_2\lambda)}$ under two cases and show that in each case, $\left.\frac{\partial H_1(K_1,c_2\lambda)}{\partial y_0}\right|_{K_1=\widetilde{K}_1(c_2\lambda)} < 0$ under the sufficient conditions stated in Proposition 5.6:

Case I: Suppose that $\mu_L < \widetilde{K_1}(c_2\lambda) < \mu_{1,H}$. Under this case, we have

$$\begin{aligned}
&\partial H_1(K_1, c_2\lambda)/\partial y_0\Big|_{K_1=\widetilde{K}_1(c_2\lambda)} \\
&= -\frac{\mu_{2,L}}{\lambda}\left(A_2 f_1'(c_2\lambda) + B_2 f_2'(c_2\lambda)\right) + \frac{1}{\lambda}(K_1 - \mu_L) \\
&= \frac{(\widetilde{K}_1(c_2\lambda) - \mu_L)}{\lambda} - \frac{\mu_{2,L}}{2\lambda}\left(\frac{1-\ell^*}{\ell^*}\right)^{1/2} \\
&\Bigg[\left(\beta^{-\eta/2} - \beta^{\eta/2}\right)^{-1}\left(\beta^{\frac{1-\eta}{2}} - 1\right)\left(\frac{1-h^*}{h^*}\right)^{\eta/2}\left(\frac{c_2\lambda}{1-c_2\lambda}\right)^{\eta/2} \\
&\quad \times \left((\eta+1)\left(\frac{c_2\lambda}{1-c_2\lambda}\right)^{-1/2} + (\eta-1)\left(\frac{c_2\lambda}{1-c_2\lambda}\right)^{1/2}\right) &\text{(A.81)}\\
&\quad - \left(\beta^{-\eta/2} - \beta^{\eta/2}\right)^{-1}\left(1 - \beta^{\frac{1+\eta}{2}}\right)\left(\frac{h^*}{1-h^*}\right)^{\eta/2}\left(\frac{1-c_2\lambda}{c_2\lambda}\right)^{\eta/2} &\text{(A.82)}\\
&\quad \times \left((\eta-1)\left(\frac{1-c_2\lambda}{c_2\lambda}\right)^{1/2} + (\eta+1)\left(\frac{1-c_2\lambda}{c_2\lambda}\right)^{-1/2}\right)\Bigg]. &\text{(A.83)}
\end{aligned}$$

Note from (A.75), (A.76) and (A.79), we have

$$\begin{aligned}
&\lim_{\sigma\to\infty}\left(\beta^{-\eta/2} - \beta^{\eta/2}\right)^{-1}\left(\beta^{\frac{1-\eta}{2}} - 1\right) \\
&= \lim_{\sigma\to\infty}\left(\frac{(\delta_\ell + \delta_h)\eta}{c_2\lambda(1-c_2\lambda)} + O(\sigma\delta_\ell^2) + O(\sigma\delta_h^2) + O(\sigma\delta_\ell\delta_h)\right)^{-1} \\
&\qquad \left(\frac{(\delta_\ell + \delta_h)(\eta-1)}{2c_2\lambda(1-c_2\lambda)} + O(\sigma\delta_\ell^2) + O(\sigma\delta_h^2) + O(\sigma\delta_\ell\delta_h)\right) &\text{(A.84)}\\
&= \frac{1}{2}. &\text{(A.85)}
\end{aligned}$$

Similarly, it follows from (A.75), (A.76) and (A.80) that

$$\begin{aligned}
&\lim_{\sigma\to\infty} -\left(\beta^{-\eta/2} - \beta^{\eta/2}\right)^{-1}\left(1 - \beta^{\frac{1+\eta}{2}}\right) \\
&= \lim_{\sigma\to\infty} -\left(\frac{(\delta_\ell + \delta_h)\eta}{c_2\lambda(1-c_2\lambda)} + O(\sigma\delta_\ell^2) + O(\sigma\delta_h^2) + O(\sigma\delta_\ell\delta_h)\right)^{-1} \\
&\qquad \times \left(\frac{(\delta_\ell + \delta_h)(\eta+1)}{2c_2\lambda(1-c_2\lambda)} + O(\sigma\delta_\ell^2) + O(\sigma\delta_h^2) + O(\sigma\delta_\ell\delta_h)\right) &\text{(A.86)}\\
&= -\frac{1}{2}. &\text{(A.87)}
\end{aligned}$$

In addition,

$$\lim_{\sigma\to\infty}\left(\frac{1-h^*}{h^*}\frac{c_2\lambda}{(1-c_2\lambda)}\right)^{\eta/2}=\lim_{\sigma\to\infty}\left(1+\frac{c_2\lambda-h^*}{(1-c_2\lambda)h^*}\right)^{\frac{\eta}{2}}=\lim_{\sigma\to\infty}\left(1+\frac{\eta}{2}\frac{c_2\lambda-h^*}{(1-c_2\lambda)h^*}\right)=1. \tag{A.88}$$

The last equation above is because $\eta=O(\sigma)$ and, by Proposition 5.4, $h^*=o(\sigma^{-1})$. Combining (A.82) through (A.88), we have

$$\begin{aligned}
&\lim_{\sigma\to\infty}\ \partial H_1(K_1,c_2\lambda)/\partial y_0\Big|_{K_1=\widetilde{K}_1(c_2\lambda)}\\
&=\lim_{\sigma\to\infty}\ \frac{(\widetilde{K}_1(c_2\lambda)-\mu_L)}{\lambda}-\frac{\mu_{2,L}}{4\lambda}\left[(\eta+1)\left(\frac{1-2c_2\lambda}{c_2\lambda}\right)+(\eta-1)\left(\frac{2c_2\lambda-1}{c_2\lambda}\right)\right]\\
&=\lim_{\sigma\to\infty}\ \frac{(\widetilde{K}_1(c_2\lambda)-\mu_L)}{\lambda}-\frac{(1-2c_2\lambda)\mu_{2,L}}{2c_2\lambda^2}.
\end{aligned}$$

Note that the above limit is strictly negative if

$$\frac{\mu_H-\mu_L}{\mu_{2,L}}<\frac{1-2c_2\lambda}{2c_2\lambda}. \tag{A.89}$$

Case II: Suppose that $\widetilde{K_1}(c_2\lambda)\in(\mu_{1,H},\mu_H)$. Under this case, we have

$$\begin{aligned}
&\partial H_1(K_1,c_2\lambda)/\partial y_0\Big|_{K_1=\widetilde{K}_1(c_2\lambda)}\\
&=-\frac{1}{\lambda}\left[\mu_{2,L}\left(A_2f_1'(c_2\lambda)+B_2f_2'(c_2\lambda)\right)+(\widetilde{K}_1(c_2\lambda)-\mu_{1,H})\left(A_1f_1'(c_2\lambda)+B_1f_2'(c_2\lambda)\right)\right]\\
&\quad+\frac{1}{\lambda}(\widetilde{K}_1(c_2\lambda)-\mu_L).
\end{aligned}$$

Note that compared to Case I, under this case, the only term that differs in $\partial H_1(\cdot,\cdot)/\partial y_0|_{K_1=\widetilde{K}_1(c_2\lambda)}$

is $-(\widetilde{K}_1(c_2\lambda) - \mu_{1,H})\left(A_1 f_1'(c_2\lambda) + B_1 f_2'(c_2\lambda)\right)/\lambda$. Thus, below we will analyze that term:

$$
\begin{aligned}
& -\left(\widetilde{K}_1(c_2\lambda) - \mu_{1,H}\right)\left(A_1 f_1'(c_2\lambda) + B_1 f_2'(c_2\lambda)\right)/\lambda \\
& = -\frac{\widetilde{K}_1(c_2\lambda) - \mu_{1,H}}{2\lambda}\left(\frac{\ell^*}{1-\ell^*}\right)^{1/2} \\
& \Bigg[\left(\beta^{-\eta/2} - \beta^{\eta/2}\right)^{-1}\left(\beta^{\frac{-1-\eta}{2}} - 1\right)\left(\frac{1-h^*}{h^*}\right)^{\eta/2}\left(\frac{c_2\lambda}{1-c_2\lambda}\right)^{\eta/2} \\
& \qquad \times\left((\eta+1)\left(\frac{c_2\lambda}{1-c_2\lambda}\right)^{-1/2} + (\eta-1)\left(\frac{c_2\lambda}{1-c_2\lambda}\right)^{1/2}\right) && \text{(A.90)} \\
& \qquad -\left(\beta^{-\eta/2} - \beta^{\eta/2}\right)^{-1}\left(1 - \beta^{\frac{-1+\eta}{2}}\right)\left(\frac{h^*}{1-h^*}\right)^{\eta/2}\left(\frac{1-c_2\lambda}{c_2\lambda}\right)^{\eta/2} && \text{(A.91)} \\
& \qquad \times\left((\eta-1)\left(\frac{1-c_2\lambda}{c_2\lambda}\right)^{1/2} + (\eta+1)\left(\frac{1-c_2\lambda}{c_2\lambda}\right)^{-1/2}\right)\Bigg]. && \text{(A.92)}
\end{aligned}
$$

Then, it follows from (A.75), (A.76), (A.77) and (A.78) that

$$
\lim_{\sigma\to\infty}\left(\beta^{-\eta/2} - \beta^{\eta/2}\right)^{-1}\left(\beta^{\frac{-1-\eta}{2}} - 1\right) = \frac{1}{2}, \quad \text{and} \quad \lim_{\sigma\to\infty} -\left(\beta^{-\eta/2} - \beta^{\eta/2}\right)^{-1}\left(1 - \beta^{\frac{-1+\eta}{2}}\right) = -\frac{1}{2}.
$$

As a result,

$$
\begin{aligned}
& \lim_{\sigma\to\infty} -(\widetilde{K}_1(c_2\lambda) - \mu_{1,H})\left(A_1 f_1'(c_2\lambda) + B_1 f_2'(c_2\lambda)\right)/\lambda \\
& = \lim_{\sigma\to\infty} -\frac{\widetilde{K}_1(c_2\lambda) - \mu_{1,H}}{4\lambda}\left(\frac{c_2\lambda}{1-c_2\lambda}\right)^{1/2} \\
& \quad \Bigg[\left((\eta+1)\left(\frac{c_2\lambda}{1-c_2\lambda}\right)^{-1/2} + (\eta-1)\left(\frac{c_2\lambda}{1-c_2\lambda}\right)^{1/2}\right) \\
& \quad -\left((\eta-1)\left(\frac{1-c_2\lambda}{c_2\lambda}\right)^{1/2} + (\eta+1)\left(\frac{1-c_2\lambda}{c_2\lambda}\right)^{-1/2}\right)\Bigg] && \text{(A.93)} \\
& = -\frac{\widetilde{K}_1(c_2\lambda) - \mu_{1,H}}{2\lambda}\left(1 - \frac{c_2\lambda}{1-c_2\lambda}\right). && \text{(A.94)}
\end{aligned}
$$

Note that the expression on the last line is negative if

$$
c_2\lambda < 1/2. \tag{A.95}
$$

Then, the analysis in Cases I and II shows that if $\mu_{1,H} > \mu_L$, and (A.89) and (A.95) hold, except on a set of measure zero, $V_1'(c_2\lambda) < 0$ for large values of σ. This and the continuity of $H_1(\cdot,\cdot)$ in y_0 immediately imply the existence of a initial belief region that contains $c_2\lambda$ such that $V_1'(y_0) < 0$.

It only remains to show our earlier claim that $H_1(K_1, y_0)$ in (5.29) is differentiable at all $K_1 \in (\mu_L, \mu_H)$ except at $\mu_{1,H}$. Observe from (5.29) that both ϕ_{ℓ^*} and ϕ_{h^*} are differentiable in K_1 for $K_1 \in (\mu_L, \mu_H]$ based on the fact that $\ell^*(K_1)$ and $h^*(K_1)$ are differentiable in K_1. Then, by (5.18), it follows that $\xi_1(y)$ is also differentiable on $K_1 \in (\mu_L, \mu_H)$. Finally, from (5.19), we know that $\xi_2(y)$ is differentiable on $K_1 \in (\mu_L, \mu_{1,H}) \cup (\mu_{1,H}, \mu_H]$, therefore $H_1(K_1, y_0)$ is differentiable in K_1 on $(\mu_L, \mu_{1,H}) \cup (\mu_{1,H}, \mu_H]$, that is $\mu_H 1, H$ is the only non-differentiable point for any y_0. $\square$

A.3.10 Proof of Proposition 5.7

Let $f(\cdot) : [0,1] \to \mathbb{R}$ and $\theta^* \in (0,1)$ be the solutions to the following ODE

$$0 = \frac{1}{2} q(K_1) y^2 (1-y)^2 f''(y) - \lambda f(y) \tag{A.96}$$

subject to these boundary conditions:

$$f(\theta^*) = g(\theta^*) \quad \text{and} \quad f'(\theta^*) = g'(\theta^*), \tag{A.97}$$

where

$$g(y) \doteq \max_{K_2 \geq \underline{\mu}} g_2(y, K_2) = \frac{\mu_{2,H} - \mu_{2,L}}{\lambda} y + \frac{\mu_{2,L}}{\lambda} - c_2 \underline{\mu}.$$

Note that the following function satisfies (A.96):

$$f(y) = \kappa y^{\frac{1+\eta}{2}} (1-y)^{\frac{1-\eta}{2}}.$$

Using this general form and (A.97), we now solve for the two unknowns θ^* and $f(\cdot)$. It follows from (A.97) that

$$\kappa \theta^{*\frac{1+\eta}{2}} (1-\theta^*)^{\frac{1-\eta}{2}} = \frac{\mu_{2,H} - \mu_{2,L}}{\lambda} \theta^* + \frac{\mu_{2,L}}{\lambda} - c_2 \underline{\mu}, \tag{A.98}$$

$$\kappa \left(\frac{\eta+1}{2} \left(\frac{\theta^*}{1-\theta^*} \right)^{\frac{\eta-1}{2}} + \frac{\eta-1}{2} \left(\frac{\theta^*}{1-\theta^*} \right)^{\frac{\eta+1}{2}} \right) = \frac{\mu_{2,H} - \mu_{2,L}}{\lambda}. \tag{A.99}$$

The equation (A.98) is equivalent to

$$\kappa \left(\frac{\theta^*}{1-\theta^*} \right)^{\frac{\eta-1}{2}} = \frac{1}{\theta^*} \left(\frac{\mu_{2,H} - \mu_{2,L}}{\lambda} \theta^* + \frac{\mu_{2,L}}{\lambda} - c_2 \underline{\mu} \right)$$

$$\Leftrightarrow$$

$$\kappa \left(\frac{\theta^*}{1-\theta^*} \right)^{\frac{\eta+1}{2}} = \frac{1}{1-\theta^*} \left(\frac{\mu_{2,H} - \mu_{2,L}}{\lambda} \theta^* + \frac{\mu_{2,L}}{\lambda} - c_2 \underline{\mu} \right).$$

Combing these with (A.99), we obtain the solution θ^* as function of K_1:

$$\theta^*(K_1) = \left(1 + \frac{\eta-1}{\eta+1} \frac{\mu_{2,H} - c_2 \lambda \underline{\mu}}{c_2 \lambda \underline{\mu} - \mu_{2,L}} \right)^{-1},$$

which is equivalent to the following by (5.24):

$$\theta^*(K_1) = \left(1 + \left(\frac{\mu_{2,H} - c_2 \lambda \underline{\mu}}{c_2 \lambda \underline{\mu} - \mu_{2,L}} \right) \frac{\sqrt{1 + \frac{8\lambda\sigma^2}{\min(K_1,\mu_H) - \min(K_1,\mu_L)}} - 1}{\sqrt{1 + \frac{8\lambda\sigma^2}{\min(K_1,\mu_H) - \min(K_1,\mu_L)}} + 1} \right)^{-1}.$$

It is perhaps worth noting that $\theta^*(K_1) < 1$ because $\frac{\mu_{2,H} - c_2 \lambda \underline{\mu}}{c_2 \lambda \underline{\mu} - \mu_{2,L}} > 0$ by (5.32). Furthermore, by (A.99), we have

$$\kappa = \left(\frac{\eta+1}{2} \left(\frac{\theta^*(K_1)}{1-\theta^*(K_1)} \right)^{\frac{\eta-1}{2}} + \frac{\eta-1}{2} \left(\frac{\theta^*(K_1)}{1-\theta^*(K_1)} \right)^{\frac{\eta+1}{2}} \right)^{-1} \frac{\mu_{2,H} - \mu_{2,L}}{\lambda}.$$

Define

$$\widehat{f}(y) \doteq \begin{cases} f(y), & y < \theta^*(K_1), \\ g(y), & y \geq \theta^*(K_1). \end{cases} \tag{A.100}$$

Using similar arguments as in the proof of Proposition 5.1, it is straightforward to show that $\widehat{f}(\cdot)$ is the follower's value function under the single-threshold investment strategy, which consists of the follower's capacity investment $\underline{\mu}$ and investment timing

$$\tau = \inf\{t \geq 0 : \pi_t \geq \theta^*(K_1)\},$$

and $\widehat{f}(\cdot)$ satisfies the following conditions:

$$\begin{aligned}
&\widehat{f}(y) \geq g(y), \quad y \in [0,1] \\
&0 \leq \frac{1}{2}q(K_1)y^2(1-y)^2\widehat{f}''(y) - \lambda\widehat{f}(y), \quad y \in (0,1) \setminus \{\theta^*(K_1)\}.
\end{aligned}$$

Using these conditions, from similar arguments in the proof of Proposition 5.2, it follows that $\widehat{f}(y)$ is larger than the follower's value function at y under any other feasible investment policy. Hence, the aforementioned single-threshold policy is optimal for the follower. □

A.3.11 Proof of Proposition 5.8

This proof uses the similar arguments explained in the proof of Proposition 5.5. Note that the following variant of Lemma A.8 holds: $H_1(K_1, y_0) > H_0(K_1, y_0)$ for $(K_1, y_0) \in (\mu_L, \infty) \times (0, \theta^*(K_1))$.

Fix an initial belief $y_0 \in (0,1)$. We now evaluate $H_0(K_1, y_0)$ and $H_1(K_1, y_0)$. First note that $\theta^*(K_1)$ increases continuously and monotonically from $\theta^*(\underline{\mu})$ to $\theta^*(\mu_H)$ as K_1 increases.

If $y_0 < \theta^*(\underline{\mu})$, clearly, $\tau > 0$ with probability 1 whereas if $y_0 > \theta^*(\mu_H)$, the follower immediately invests, that is, $\tau = 0$ with probability 1.

We now focus on $y_0 \in (\theta^*(\underline{\mu}), \theta^*(\mu_H))$. From the structural properties of $\theta^*(\cdot)$, for any given such y_0, there exists a unique K_0 such that $\theta^*(K_0) = y_0$. Thus, if $K_1 \leq K_0(y_0)$, $\tau = 0$ with probability 1; otherwise, $\tau > 0$ with probability 1. This implies that

$$H(K_1, y_0) = \begin{cases} H_0(K_1, y_0) & \text{if} \quad \underline{\mu} < K_1 \leq K_0(y_0), \\ H_1(K_1, y_0) & \text{if} \quad K_1 > K_0(y_0), \end{cases}$$

Therefore, the equilibrium capacity $\widetilde{K}_1^*(y_0)$ is identified by comparing the maximum of $H_0(K_1, y_0)$ over $\underline{\mu} \leq K_1 \leq K_0(y_0)$ and the maximum of $H_1(K_1, y_0)$ over $K_1 > K_0(y_0)$. By

(5.14), for any fixed y_0,

$$H_0(K_1, y_0) = \begin{cases} \left(\frac{1}{\lambda}y_0 - c_1\right) K_1 + (1-y_0)\frac{\mu_{1,L}}{\lambda} & \text{if} \quad \underline{\mu} < K_1 \leq \mu_{1,H} \\ -c_1 K_1 + y_0 \frac{\mu_{1,H}}{\lambda} + (1-y_0)\frac{\mu_{1,L}}{\lambda} & \text{if} \quad K_1 > \mu_{1,H}. \end{cases}$$

Fix any initial belief $y_0 \geq c_1\lambda$. Then, by standard optimization arguments,

$$H_0(\mu_{1,H}, y_0) = \max_{K_1 \geq 0} H_0(K_1, y_0), \quad \text{for} \quad y_0 \in [c_1\lambda, 1].$$

If $\underline{\mu} < K_0(y_0) < \mu_{1,H}$ then $\mu_{1,H}$ lies in the region where $H_1(K_1, y_0) > H_0(K_1, y_0)$, and hence

$$\max_{K_1 > K_0(y_0)} H_1(K_1, y_0) \geq H_1(\mu_{1,H}, y_0) > H_0(\mu_{1,H}, y_0) \geq \max_{K_1 \leq K_0(y_0)} H_0(K_1, y_0).$$

We now consider the scenario where $y_0 < c_1\lambda$. There can be two cases related to that: Either there exists a least one initial belief $y_0 < c_1\lambda$ such that $\max_{K_1 > K_0(y_0)} H_1(K_1, y_0) < \max_{K_1 \leq K_0(y_0)} H_0(K_1, y_0)$ or $\max_{K_1 > K_0(y_0)} H_1(K_1, y_0) < \max_{K_1 \leq K_0(y_0)} H_0(K_1, y_0)$ for all $y_0 < c_1\lambda$. If it is the latter, then $\underline{y} \doteq \theta^*(\underline{\mu})$. If it is the former scenario, then by the continuity of $\max_{K_1 > K_0(y_0)} H_1(K_1, y_0)$, and from the fact that the capacity that maximizes $H_0(K_1, y_0)$ remains the same for $y_0 < c_1\lambda$, there exists an $\underline{y} < c_1\lambda$ such that

$$\max_{K_1 > K_0(y_0)} H_1(K_1, y_0) \geq \max_{K_1 \leq K_0(y_0)} H_0(K_1, y_0), \quad y_0 \geq \underline{y}.$$

Combining all, for $y_0 \in (0, \theta^*(\underline{\mu}) \cup [\underline{y}, \theta^*(\mu_H))$, we have $H(\widetilde{K}_1(y_0), y_0) = H_1(\widetilde{K}_1(y_0), y_0)$. $\square$

A.3.12 Proof of Proposition 5.9

We will use the Envelope Theorem to prove the result. Recall from (5.40) that

$$\begin{aligned} &H_1(K_1, y_0) \\ &= -\left[\xi_1(\theta^*(K_1)) - \xi_2(\theta^*(K_1))\right] \frac{\phi(y_0)}{\lambda\phi(\theta^*(K_1))} \\ &\quad + \frac{1}{\lambda}\left(y_0 \min\{K_1, \mu_H\} + (1-y_0)\min\{K_1, \mu_L\}\right) - c_1 K_1, \end{aligned}$$

where ξ_1 and ξ_2 are given in (5.18) and (5.19), and ϕ is as in (5.38). Hence, the partial derivative of H_1 with respect to y_0 is given by

$$\frac{\partial H_1(K_1, y_0)}{\partial y_0}$$
$$= -\frac{\xi_1(\theta^*(K_1)) - \xi_2(\theta^*(K_1))}{\lambda\phi(\theta^*(K_1))}\left(\frac{\eta+1}{2}\left(\frac{y_0}{1-y_0}\right)^{\frac{\eta-1}{2}} + \frac{\eta-1}{2}\left(\frac{y_0}{1-y_0}\right)^{\frac{\eta+1}{2}}\right) + \frac{K_1 - \mu_L}{\lambda}, \tag{A.101}$$

and H_1 is concave in y_0:

$$\frac{\partial^2 H_1(K_1, y_0)}{\partial y_0^2}$$
$$= -\frac{\xi_1(\theta^*(K_1)) - \xi_2(\theta^*(K_1))}{\lambda\phi(\theta^*(K_1))}\frac{1}{(1-y_0)^2}\left(\frac{\eta+1}{2}\frac{\eta-1}{2}\left(\frac{y_0}{1-y_0}\right)^{\frac{\eta-3}{2}} + \frac{\eta-1}{2}\frac{\eta+1}{2}\left(\frac{y_0}{1-y_0}\right)^{\frac{\eta-1}{2}}\right) < 0,$$

Recall that we would like to identify sufficient conditions under which $V_1'(y_0) < 0$, which is equivalent to the following by the Envelope Theorem if $\widetilde{K}_1(y_0)$ is an interior capacity in $(K_0(y_0), \mu_H)$:

$$V_1'(y_0) = \frac{\partial H_1(K_1, y_0)}{\partial y_0}\Big|_{K_1 = \widetilde{K}_1(y_0)}. \tag{A.102}$$

Because H_1 is concave in y_0, for any given K_1 and $y_0 \in (\theta^*(\underline{\mu}), \theta^*(\mu_H))$,

$$\frac{\partial H_1(K_1, y_0)}{\partial y_0} \in \left(\frac{\partial H_1(K_1, \theta^*(K_1))}{\partial y_0}, \frac{\partial H_1(K_1, \theta^*(\underline{\mu}))}{\partial y_0}\right).$$

This suggests that a sufficient condition for (A.102) is

$$\partial H_1(K_1, \theta^*(\underline{\mu}))/\partial y_0 < 0 \quad \text{for} \quad K_1 \in (\underline{\mu}, \mu_H),$$

which is equivalent to

$$\left[\frac{\eta+1}{2}\left(\frac{\theta^*(\underline{\mu})}{1-\theta^*(\underline{\mu})}\right)^{\frac{\eta-1}{2}}+\frac{\eta-1}{2}\left(\frac{\theta^*(\underline{\mu})}{1-\theta^*(\underline{\mu})}\right)^{\frac{\eta+1}{2}}\right]\left[\xi_1(\theta^*(\underline{\mu}))-\xi_2(\theta^*(\underline{\mu}))\right]$$
$$>(K_1-\mu_L)\,\phi(\theta^*(\underline{\mu})),$$

which, by expanding $\phi(\cdot)$, can be simplified to

$$\left[\frac{\eta+1}{2\theta^*(\underline{\mu})}+\frac{\eta-1}{2(1-\theta^*(\underline{\mu}))}\right]\left[\xi_1(\theta^*(\underline{\mu}))-\xi_2(\theta^*(\underline{\mu}))\right]>K_1-\mu_L. \tag{A.103}$$

Define $U\doteq\mu_{2,H}-c_2\lambda\underline{\mu}$ and $V\doteq c_2\lambda\underline{\mu}-\mu_{2,L}$. Then, by (5.34),

$$\frac{\eta+1}{2\theta^*(\underline{\mu})}+\frac{\eta-1}{2(1-\theta^*(\underline{\mu}))}=\eta+\frac{(\eta-1)}{2}\frac{U}{V}+\frac{(\eta+1)}{2}\frac{V}{U}, \tag{A.104}$$

and

$$\xi_1(\theta^*(\underline{\mu}))-\xi_2(\theta^*(\underline{\mu}))=\begin{cases}(1-\theta^*(\underline{\mu}))\mu_{2,L}, & \text{if} \quad K_1\le\mu_{1,H}\\ \theta(K_1-\mu_{1,H})+(1-\theta^*(\underline{\mu}))\mu_{2,L}, & \text{if} \quad K_1>\mu_{1,H}.\end{cases}$$
$$\ge(1-\theta^*(\underline{\mu}))\mu_{2,L}. \tag{A.105}$$

In addition, because $K_1<\mu_H$,

$$K_1-\mu_L<\mu_H-\mu_L. \tag{A.106}$$

Using (A.104) through (A.106), the following condition implies (A.103):

$$\frac{\eta+1}{2}\frac{(1-\theta^*(\underline{\mu}))}{\theta^*(\underline{\mu})}+\frac{\eta-1}{2}>\frac{\mu_H-\mu_L}{\mu_{2,L}},$$

which is guaranteed by the condition that

$$\sqrt{1+\frac{8\lambda\sigma^2}{(\underline{\mu}-\mu_L)^2}}\left(\frac{U}{V}+1\right)-\frac{U}{V}>\frac{\mu_H-\mu_L}{\mu_{2,L}}.$$

This completes our argument for this proof. $\square$

A.3.13 Proof of Proposition 5.11.

We first give the following definition for algebra convenience.

$$\eta_{max} := \sqrt{1 + \frac{8\lambda\sigma^2}{(\underline{\mu} - \mu_l)^2}}, \tag{A.107}$$

$$\eta_{min} := \sqrt{1 + \frac{8\lambda\sigma^2}{(\mu_h - \mu_l)^2}}. \tag{A.108}$$

$$\eta_{min} := \sqrt{1 + \frac{8\lambda\sigma^2}{(\mu_{1,h} - \mu_l)^2}}. \tag{A.109}$$

Assume the optimal $\widetilde{K}_1(y_0)$ takes value in an interior region of the initial belief, let

$$F(K_1, y_0) = \frac{\partial H_1(K_1, y_0)}{\partial K_1},$$

the first order condition gives us $F(\widetilde{K}_1(y_0), y_0) = 0$. From the implicit function theorem, we have

$$\frac{d\widetilde{K}_1}{dy_0} = -\frac{\frac{\partial F(K_1,y_0)}{\partial y_0}}{\frac{\partial F(K_1,y_0)}{\partial K_1}}\ \Big|_{\widetilde{K}_1(y_0)} = -\frac{\frac{\partial^2 H(K_1,y_0)}{\partial y_0 \partial K_1}}{\frac{\partial^2 H(K_1,y_0)}{\partial K_1^2}}\ \Big|_{\widetilde{K}_1(y_0)}$$

Note that since $\widetilde{K}_1(y_0)$ is the maximum point, we have

$$\frac{\partial^2 H(\widetilde{K}_1(y_0), y_0)}{\partial {K_1}^2} < 0,$$

therefore the sign of $\frac{d\widetilde{K}_1(y)}{dy_0}$ has is same as that of the numerator $\frac{\partial^2 H(K_1,y)}{\partial y \partial K_1}$. We derive the explicit formulation of the cross derivative $\frac{\partial^2 H_1}{\partial y \partial K_1}$ and prove that it is positive.

Recall (A.101), the partial derivative of H_1 with respect to y_0 is given by

$$\frac{\partial H_1(K_1, y_0)}{\partial y_0} = -\frac{\xi_1(\theta) - \xi_2(\theta)}{\lambda\phi(\theta)}\phi'(y_0) + \frac{K_1 - \mu_L}{\lambda},$$

where θ is function of K_1 via the link η, and

$$\phi(y) = y^{\frac{1+\eta}{2}}(1-y)^{\frac{1-\eta}{2}}$$

and hence

$$\phi'(y) = \left(\frac{\eta+1}{2}\left(\frac{y}{1-y}\right)^{\frac{\eta-1}{2}} + \frac{\eta-1}{2}\left(\frac{y}{1-y}\right)^{\frac{\eta+1}{2}}\right).$$

Besides, in (5.18) and (5.19), we have

$$\xi_1(\theta) - \xi_2(\theta) := \delta_m(K_1,\theta) = \begin{cases} (1-\theta)\mu_{2,L}, \quad K_1 \le \mu_{1,H}; \\ \theta(K_1 - \mu_{1,H}) + (1-\theta)\mu_{2,L}, \quad K_1 > \mu_{1,H}. \end{cases}$$

Hence δ_m is a function of θ and K_1. Besides θ can be written as function of η, hence we have

$$\frac{\partial}{\partial\eta}\left[\frac{\delta_m(K_1,\theta)}{\phi(\theta)}\phi'(y_0)\right] = \frac{\partial}{\partial\eta}\left[\frac{\delta_m(K_1,\theta)}{\phi(\theta)}\right]\phi'(y_0) + \left[\frac{\delta_m(K_1,\theta)}{\phi(\theta)}\right]\frac{\partial\phi'(y_0)}{\partial\eta}. \tag{A.110}$$

To obtain (A.110), we need the following partial derivatives:

- (1)
$$\frac{\partial}{\partial\eta}\left[\frac{1}{\phi(\theta)}\right]$$

- (2)
$$\frac{\partial\delta_m(K_1,\theta)}{\partial\eta}$$

- (3)
$$\frac{\partial}{\partial\eta}\left[\frac{\delta_m(K_1,\theta)}{\phi(\theta)}\right]$$

- (4)
$$\frac{\partial\phi'(y_0)}{\partial\eta}$$

Derivation of (1)

$$
\begin{aligned}
\frac{\partial \phi(\theta)^{-1}}{\partial \eta} &= \frac{\partial}{\partial \eta}\left(\theta(1-\theta)\right)^{-1/2}\left(\frac{1-\theta}{\theta}\right)^{\eta/2} \\
&= \left(\frac{1-\theta}{\theta}\right)^{\eta/2}\left[-\frac{1}{2}(1-2\theta)(\theta(1-\theta))^{-3/2}\theta'(\eta)\right] \\
&+ \left(\frac{1-\theta}{\theta}\right)^{\eta/2}(\theta(1-\theta))^{-1/2}\left[-\frac{1}{\theta^2}\frac{\theta}{1-\theta}\frac{\eta}{2}\theta'(\eta)+\frac{1}{2}\ln\left(\frac{1-\theta}{\theta}\right)\right] \\
&= -\frac{1}{2}\phi(\theta)^{-1}\left[\frac{1-2\theta+\eta}{\theta(1-\theta)}\theta'(\eta)-\ln\left(\frac{1-\theta}{\theta}\right)\right].
\end{aligned}
$$

Here we need the generalized power rule for the second equation.

Derivation of (2)

$$
\begin{aligned}
\frac{\partial \delta_m(K_1,\theta)}{\partial \eta} &= \begin{cases} -\mu_{2,L}\theta'(\eta), & K_1 \le \mu_{1,H}; \\ (K_1-\mu_{1,H}-\mu_{2,L})\theta'(\eta), & K_1 > \mu_{1,H}. \end{cases} \\
&:= \delta(K_1)\theta'(\eta),
\end{aligned}
$$

where

$$
\delta(K_1) = \begin{cases} -\mu_{2,L}, & K_1 \le \mu_{1,H}; \\ (K_1-\mu_{1,H}-\mu_{2,L}), & K_1 > \mu_{1,H}. \end{cases}
$$

Derivation of (3)

$$
\begin{aligned}
\frac{\partial}{\partial \eta}\left[\frac{\delta_m(K_1,\theta)}{\phi(\theta)}\right] &= -\frac{\delta_m(K_1,\theta)}{2\phi(\theta)}\left[\frac{1-2\theta+\eta}{\theta(1-\theta)}\theta'(\eta)-\ln\left(\frac{1-\theta}{\theta}\right)\right]+\frac{\delta(K_1)\theta'(\eta)}{\phi(\theta)} \\
&= \frac{\delta_m(K_1,\theta)}{2\phi(\theta)}\ln\left(\frac{1-\theta}{\theta}\right)+\frac{\theta'(\eta)}{\phi(\theta)}\left[-\frac{\delta_m(K_1,\theta)}{2}\frac{1-2\theta+\eta}{\theta(1-\theta)}+\delta(K_1)\right],
\end{aligned}
$$

Derivation of (4)

$$
\begin{aligned}
&\frac{\partial \phi'(y_0)}{\partial \eta} \\
=&\frac{1}{2}\Bigg[\left(\frac{y_0}{1-y_0}\right)^{\frac{\eta-1}{2}} \\
&\quad + \frac{\eta+1}{2}\left(\frac{y_0}{1-y_0}\right)^{\frac{\eta-1}{2}}\ln\left(\frac{y_0}{1-y_0}\right) + \left(\frac{y_0}{1-y_0}\right)^{\frac{\eta+1}{2}} + \frac{\eta-1}{2}\left(\frac{y_0}{1-y_0}\right)^{\frac{\eta+1}{2}}\ln\left(\frac{y_0}{1-y_0}\right)\Bigg] \\
=&\frac{1}{2}\left[\left(\frac{y_0}{1-y_0}\right)^{\frac{\eta+1}{2}} + \left(\frac{y_0}{1-y_0}\right)^{\frac{\eta-1}{2}}\right] + \frac{1}{2}\ln\left(\frac{y_0}{1-y_0}\right)\phi'(y_0).
\end{aligned}
$$

Combing (1) - (4), we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial \eta}\left[\frac{\delta_m(K_1,\theta)}{\phi(\theta)}\phi'(y_0)\right] \\
=&\frac{\partial}{\partial \eta}\left[\frac{\delta_m(K_1,\theta)}{\phi(\theta)}\right]\phi'(y_0) + \left[\frac{\delta_m(K_1,\theta)}{\phi(\theta)}\right]\frac{\partial \phi'(y_0)}{\partial \eta} \\
=&\frac{\delta_m(K_1,\theta)}{2\phi(\theta)}\ln\left(\frac{1-\theta}{\theta}\right)\phi'(y_0) + \frac{\theta'(\eta)}{\phi(\theta)}\left[-\frac{\delta_m(K_1,\theta)}{2}\frac{1-2\theta+\eta}{\theta(1-\theta)} + \delta(K_1)\right]\phi'(y_0) \\
&+ \frac{\delta_m(K_1,\theta)}{2\phi(\theta)}\left[\left(\frac{y_0}{1-y_0}\right)^{\frac{\eta+1}{2}} + \left(\frac{y_0}{1-y_0}\right)^{\frac{\eta-1}{2}}\right] + \frac{\delta_m(K_1,\theta)}{2\phi(\theta)}\ln\left(\frac{y_0}{1-y_0}\right)\phi'(y_0) \\
=&\frac{\phi'(y_0)\delta_m(K_1,\theta)}{2\phi(\theta)}\ln\left[\frac{(1-\theta)y_0}{\theta(1-y_0)}\right] + \frac{\delta_m(K_1,\theta)}{2\phi(\theta)}\left[\left(\frac{y_0}{1-y_0}\right)^{\frac{\eta+1}{2}} + \left(\frac{y_0}{1-y_0}\right)^{\frac{\eta-1}{2}}\right] \\
&+ \frac{\phi'(y_0)\theta'(\eta)}{\phi(\theta)}\left[-\frac{\delta_m(K_1,\theta)}{2}\frac{1-2\theta+\eta}{\theta(1-\theta)} + \delta(K_1)\right] \qquad \text{(A.111)}
\end{aligned}
$$

For the last term in the above equation, we use the explicit solution of θ and obtain

$$
-\frac{\xi_1(\theta)-\xi_2(\theta)}{2}\frac{1-2\theta+\eta}{\theta(1-\theta)} + \delta(K_1) = \begin{cases} -\frac{1}{2}(\eta+1)\frac{\mu_{2,L}}{\theta}, & K_1 \le \mu_{1,H}, \\ -\frac{1}{2}\left[(\eta-1)\frac{K_1-\mu_{1,H}}{1-\theta} + (\eta+1)\frac{\mu_{2,L}}{\theta}\right], & K_1 > \mu_{1,H}. \end{cases}
$$

Note that the above term, as well as $\delta_m(K_1,\theta)$ and $\delta(K_1)$ depends on the relation of K_1 and $\mu_{1,H}$ or $K_1 > \mu_{1,H}$.

Next we derive the following cross partial derivative.

$$
\frac{\partial^2 H_1(K_1,y_0)}{\partial y \partial K_1} = -\frac{\partial \eta}{\partial K_1} \times \frac{\partial}{\partial \eta}\left[\frac{\delta_m(K_1,\theta)}{\lambda\phi(\theta)}\phi'(y_0)\right] - \frac{\partial \delta_m(K_1,\theta)}{\partial K_1} \times \frac{\phi'(y_0)}{\lambda\phi(\theta)} + \frac{1}{\lambda}. \qquad \text{(A.112)}
$$

To complete, we need a few more partial derivatives:

- (5)
$$\frac{\partial \delta_m(K_1, \theta)}{\partial K_1}$$
- (6)
$$\theta'(\eta) = \frac{\partial}{\partial \eta}\left(1 + \frac{\eta - 1}{\eta + 1} \cdot \frac{U}{V}\right)^{-1}$$
- (7)
$$\frac{\partial \eta}{\partial K_1}$$

Derivation of (5)

$$\frac{\partial \delta_m(K_1, \theta)}{\partial K_1} = \begin{cases} 0, & K_1 \leq \mu_{1,H}, \\ \theta, & K_1 > \mu_{1,H}. \end{cases}$$

Derivation of (6)

$$\theta'(\eta) = -\left(1 + \frac{\eta - 1}{\eta + 1} \cdot \frac{U}{V}\right)^{-2} \cdot \frac{U}{V} \cdot \frac{2}{(\eta + 1)^2} = -\frac{2U\theta^2}{V(\eta + 1)^2} = -\frac{2\theta(1 - \theta)}{\eta^2 - 1}.$$

Derivation of (7)

$$\frac{\partial \eta}{\partial K_1} = \frac{\partial}{\partial K_1}\sqrt{1 + \frac{8\lambda\sigma^2}{(K_1 - \mu_L)^2}} = -\frac{\eta^2 - 1}{\eta(K_1 - \mu_L)}.$$

In the rest of the section, we study the cross derivative in (A.112) by looking at the two different cases: $K_1 \leq \mu_{1,H}$ and $K_1 > \mu_{1,H}$.

Case I. $\widetilde{K}_1(y_0) \leq \mu_{1,H}$

Under this assumption, (A.111) can be written as

$$
\begin{aligned}
&\frac{\partial}{\partial \eta}\left[\frac{\delta_m(K_1,\theta)}{\phi(\theta)}\phi'(y_0)\right]\\
=&\frac{\phi'(y_0)\delta_m(K_1,\theta)}{2\phi(\theta)}\ln\left[\frac{(1-\theta)y_0}{\theta(1-y_0)}\right]+\frac{\delta_m(K_1,\theta)}{2\phi(\theta)}\left[\left(\frac{y_0}{1-y_0}\right)^{\frac{\eta+1}{2}}+\left(\frac{y_0}{1-y_0}\right)^{\frac{\eta-1}{2}}\right]\\
&+\frac{\phi'(y_0)}{\phi(\theta)}\frac{-2\theta(1-\theta)}{\eta^2-1}\left(-\frac{\eta+1}{2}\cdot\frac{\delta_m(K_1,\theta)}{\theta(1-\theta)}\right)\\
=&\frac{\delta_m(K_1,\theta)}{2\phi(\theta)}\left[\left(\frac{\eta+1}{2}z^{\frac{\eta-1}{2}}+\frac{\eta-1}{2}z^{\frac{\eta+1}{2}}\right)\left(\ln\left[\frac{1-\theta}{\theta}z\right]+\frac{2}{\eta-1}\right)+z^{\frac{\eta+1}{2}}+z^{\frac{\eta-1}{2}}\right],
\end{aligned}
$$

here, for the algebra convenience, let

$$
z=\frac{y_0}{1-y_0}.
$$

Note that since $y_0 \in [0,\theta]$, we have $z \in [0, \frac{\theta}{1-\theta}]$.

Hence, the cross derivative in (A.112) (multiplied by a constant λ) can be expressed as

$$
\begin{aligned}
\lambda\frac{\partial^2 H(K_1,y)}{\partial y \partial K_1}=&-\frac{\partial \eta}{\partial K_1}\times\frac{\partial}{\partial \eta}\left[\frac{\delta_m(K_1,\theta)}{\phi(\theta)}\phi'(y_0)\right]-\frac{\partial \delta_m(K_1,\theta)}{\partial K_1}\times\frac{\phi'(y_0)}{\phi(\theta)}+1\\
=&1-\frac{\partial \eta}{\partial K_1}\frac{\delta_m(K_1,\theta)}{2\phi(\theta)}f_1(z),
\end{aligned}
$$

where

$$
f_1(z)=\left(\frac{\eta+1}{2}z^{\frac{\eta-1}{2}}+\frac{\eta-1}{2}z^{\frac{\eta+1}{2}}\right)\left(\ln\left[\frac{1-\theta}{\theta}z\right]+\frac{2}{\eta-1}\right)+z^{\frac{\eta+1}{2}}+z^{\frac{\eta-1}{2}},
$$

is the only part in the cross derivative that depends on z. Next we find the minimum of f_1 over z by looking at its first derivative. We obtain

$$
f_1'(z)=\left(z^{\frac{\eta-3}{2}}+z^{\frac{\eta-1}{2}}\right)\left[\frac{\eta^2-1}{4}\left(\ln\left[\frac{1-\theta}{\theta}z\right]+\frac{2}{\eta-1}\right)+\eta\right],
$$

and let z_1 satisfy

$$
\frac{\eta^2-1}{4}\left(\ln\left[\frac{1-\theta}{\theta}z_1\right]+\frac{2}{\eta-1}\right)+\eta=0,
$$

hence $f_1'(z)<0$ for $z<z_1$, and $f_1'(z)<0$ for $z>z_1$. The minimum of f_1 is thus achieved at

$z = z_1$. Note that z_1:

$$\ln\left[\frac{1-\theta}{\theta}z_1\right] = -\frac{2(3\eta+1)}{\eta^2-1} < 0. \tag{A.113}$$

Now let $z = z_1$, we have

$$f_{1min} = f_1(z_1) = -\left(\frac{\eta+1}{\eta-1}z_1^{\frac{\eta-1}{2}} + \frac{\eta-1}{\eta+1}z_1^{\frac{\eta+1}{2}}\right).$$

Hence, combing the result (7) we obtain

$$\begin{aligned}\left(\lambda\frac{\partial^2 H(K_1,y)}{\partial y\partial K_1}\right)_{min} =& 1 + \frac{\partial\eta}{\partial K_1}\frac{\delta_m(K_1,\theta)}{2\phi(\theta)}\left(\frac{\eta+1}{\eta-1}z_1^{\frac{\eta-1}{2}} + \frac{\eta-1}{\eta+1}z_1^{\frac{\eta+1}{2}}\right)\\ =& 1 - \frac{\eta^2-1}{2\eta}\frac{\mu_{2,L}}{K_1-\mu_L}\left(\frac{1-\theta}{\theta}z_1\right)^{\frac{\eta+1}{2}}\left(\frac{\eta+1}{\eta-1}z_1^{\frac{\eta-1}{2}} + \frac{\eta-1}{\eta+1}z_1^{\frac{\eta+1}{2}}\right)\\ =& 1 - \frac{\eta^2-1}{2\eta}\frac{\mu_{2,L}}{K_1-\mu_L}\left(\frac{\eta+1}{\eta-1}\cdot\frac{1-\theta}{\theta}\left(\frac{1-\theta}{\theta}z_1\right)^{\frac{\eta-1}{2}}\right.\\ &\left. + \frac{\eta-1}{\eta+1}\left(\frac{1-\theta}{\theta}z_1\right)^{\frac{\eta+1}{2}}\right).\end{aligned}$$

From (A.113) we see that $\frac{1-\theta}{\theta}z_1 < 1$, hence

$$\begin{aligned}\left(\lambda\frac{\partial^2 H(K_1,y)}{\partial y\partial K_1}\right)_{min} >& 1 - \frac{\eta^2-1}{2\eta}\frac{\mu_{2,L}}{K_1-\mu_L}\left(\frac{\eta+1}{\eta-1}\cdot\frac{1-\theta}{\theta} + \frac{\eta-1}{\eta+1}\right)\\ =& 1 - \frac{\mu_{2,L}}{K_1-\mu_L}\frac{\eta^2-1}{2\eta}\left(\frac{U}{V} + \frac{\eta-1}{\eta+1}\right).\end{aligned}$$

It is trivial to show that

$$\frac{\eta^2-1}{2\eta}\left(\frac{U}{V} + \frac{\eta-1}{\eta+1}\right) = \frac{\eta^2-1}{2\eta}\left(\frac{U}{V} + 1 - \frac{2}{\eta+1}\right)$$

is increasing in η, hence

$$\frac{\eta^2-1}{2\eta}\left(\frac{U}{V} + \frac{\eta-1}{\eta+1}\right) < \frac{\eta_{max}^2-1}{2\eta_{max}}\left(\frac{U}{V} + \frac{\eta_{max}-1}{\eta_{max}+1}\right),$$

where

$$\eta_{max} = \sqrt{1 + \frac{8\lambda\sigma^2}{(\underline{\mu}-\mu_L)^2}}.$$

Also we have

$$\frac{\mu_{2,L}}{K_1 - \mu_L} < \frac{\mu_{2,L}}{\underline{\mu} - \mu_L}.$$

Together we obtain a lower bound of the cross derivative:

$$\left(\lambda \frac{\partial^2 H(K_1, y)}{\partial y \partial K_1}\right)_{min} > 1 - \frac{\mu_{2,L}}{\underline{\mu} - \mu_L} \frac{\eta_{max}^2 - 1}{2\eta_{max}} \left(\frac{U}{V} + 1 - \frac{2}{\eta_{max} + 1}\right).$$

Therefore, under the condition that

$$1 - \frac{\mu_{2,L}}{\underline{\mu} - \mu_L} \frac{\eta_{max}^2 - 1}{2\eta_{max}} \left(\frac{U}{V} + 1 - \frac{2}{\eta_{max} + 1}\right) > 0,$$

we have the cross derivative being positive for any (K_1, y_0) including the equilibrium $(\widetilde{K}_1(y_0), y_0)$, hence the equilibrium capacity is increasing in y_0.

Case II. $\widetilde{K}_1(y_0) > \mu_{1,H}$

Under this assumption, (A.111) can be written as

$$\begin{aligned}
&\frac{\partial}{\partial \eta}\left[\frac{\delta_m(K_1,\theta)}{\lambda\phi(\theta)}\phi'(y_0)\right] \\
=&\frac{\phi'(y_0)\delta_m(K_1,\theta)}{2\phi(\theta)} \ln\left[\frac{(1-\theta)y_0}{\theta(1-y_0)}\right] + \frac{\delta_m(K_1,\theta)}{2\phi(\theta)}\left[\left(\frac{y_0}{1-y_0}\right)^{\frac{\eta+1}{2}} + \left(\frac{y_0}{1-y_0}\right)^{\frac{\eta-1}{2}}\right] \\
&+ \frac{\phi'(y_0)}{\phi(\theta)} \frac{-2\theta(1-\theta)}{\eta^2-1}\left(-\frac{\eta-1}{2}\cdot\frac{K_1-\mu_{1,H}}{1-\theta} - \frac{\eta+1}{2}\cdot\frac{\mu_{2,L}}{\theta}\right) \\
=&\frac{1}{2\phi(\theta)}\left(\theta(K_1-\mu_{1,H}) + (1-\theta)\mu_{2,L}\right)\left[\left(\frac{\eta+1}{2}z^{\frac{\eta-1}{2}} + \frac{\eta-1}{2}z^{\frac{\eta+1}{2}}\right)\ln\left(\frac{1-\theta}{\theta}z\right) + z^{\frac{\eta+1}{2}} + z^{\frac{\eta-1}{2}}\right] \\
&+ \frac{1}{2\phi(\theta)}\left(\frac{2}{\eta+1}\theta(K_1-\mu_{1,H}) + \frac{2}{\eta-1}(1-\theta)\mu_{2,L}\right)\left(\frac{\eta+1}{2}z^{\frac{\eta-1}{2}} + \frac{\eta-1}{2}z^{\frac{\eta+1}{2}}\right).
\end{aligned}$$

Then combining the above with results (5) and (7), we have

$$\begin{aligned}
\lambda\frac{\partial^2 H(K_1,y)}{\partial y \partial K_1} =& 1 - \frac{\partial \eta}{\partial K_1} \times \frac{\partial}{\partial \eta}\left[\frac{\delta_m(K_1,\theta)}{\phi(\theta)}\phi'(y_0)\right] - \frac{\partial \delta_m(K_1,\theta)}{\partial K_1} \times \frac{\phi'(y_0)}{\phi(\theta)} \\
=& 1 + \frac{1}{\phi(\theta)} f_2(z),
\end{aligned}$$

where

$$\begin{aligned}
f_2(z) =& \left\{ \frac{1}{2}\left(-\frac{\partial\eta}{\partial K_1}\right)\left[\left(\ln\left(\frac{1-\theta}{\theta}z\right)+\frac{2}{\eta+1}\right)\theta(K_1-\mu_{1,H})\right.\right.\\
&\left.\left. + \left(\ln\left(\frac{1-\theta}{\theta}z\right)+\frac{2}{\eta-1}\right)(1-\theta)\mu_{2,L}\right]-\theta\right\}\\
&\times\left(\frac{\eta+1}{2}z^{\frac{\eta-1}{2}}+\frac{\eta-1}{2}z^{\frac{\eta+1}{2}}\right)+\frac{1}{2}\left(-\frac{\partial\eta}{\partial K_1}\right)\\
&\times\left[\theta(K_1-\mu_{1,H})+(1-\theta)\mu_{2,L}\right]\times\left(z^{\frac{\eta-1}{2}}+z^{\frac{\eta+1}{2}}\right).
\end{aligned}$$

Furthermore we have

$$f_2'(z) = \left(z^{\frac{\eta-3}{2}}+z^{\frac{\eta-1}{2}}\right)f_3(z),$$

where

$$\begin{aligned}
f_3(z) =& \frac{\eta^2-1}{4}\left\{\frac{1}{2}\left(-\frac{\partial\eta}{\partial K_1}\right)\right.\\
&\left.\left[\left(\ln\left(\frac{1-\theta}{\theta}z\right)+\frac{2}{\eta+1}\right)\theta(K_1-\mu_{1,H})+\left(\ln\left(\frac{1-\theta}{\theta}z\right)+\frac{2}{\eta-1}\right)(1-\theta)\mu_{2,L}\right]-\theta\right\}\\
&+\eta\left[\frac{1}{2}\left(-\frac{\partial\eta}{\partial K_1}\right)\left[\theta(K_1-\mu_{1,H})+(1-\theta)\mu_{2,L}\right]\right].
\end{aligned}$$

Note that f_3 is increasing in z, hence if we assume at $z=z_2$, $f_3(z_2)=0$, then f_2 is decreasing for $z<z_2$ and increasing for $z>z_2$. Now we compare z_2 with $\frac{\theta}{1-\theta}$, which is the upper bound of z. We complete the comparison by evaluating $f_3\left(\frac{\theta}{1-\theta}\right)$. We show below that under the given sufficient conditions, this value is positive, hence $z_2<\frac{\theta}{1-\theta}$, thus the minimum of f_2 has definition and can be achieved at z_2.

$$\begin{aligned}
f_3\left(\frac{\theta}{1-\theta}\right) =& \frac{\eta^2-1}{4}\left\{\frac{1}{2}\left(-\frac{\partial\eta}{\partial K_1}\right)\left[\frac{2\theta(K_1-\mu_{1,H})}{\eta+1}+\frac{2(1-\theta)\mu_{2,L}}{\eta-1}\right]-\theta\right\}\\
&+\eta\left[\frac{1}{2}\left(-\frac{\partial\eta}{\partial K_1}\right)\left[\theta(K_1-\mu_{1,H})+(1-\theta)\mu_{2,L}\right]\right]\\
=&\frac{1}{2}\left(-\frac{\partial\eta}{\partial K_1}\right)\left[\frac{3\eta-1}{2}\theta(K_1-\mu_{1,H})+\frac{3\eta+1}{2}(1-\theta)\mu_{2,L}\right]-\frac{\eta^2-1}{4}\theta\\
=&\frac{\eta^2-1}{2\eta(K_1-\mu_L)}\left[\frac{3\eta-1}{2}\theta(K_1-\mu_{1,H})+\frac{3\eta+1}{2}(1-\theta)\mu_{2,L}-\frac{\eta}{2}\theta(K_1-\mu_L)\right].
\end{aligned}$$

For the last term being positive, it is equivalent to show that

$$K_1 > \frac{3\eta-1}{2\eta-1}\mu_{1,H} - \frac{3\eta+1}{2\eta-1}\frac{1-\theta}{\theta}\mu_{2,L} - \frac{\eta}{2\eta-1}\mu_L,$$

or it is sufficient to show the following:

$$\mu_{1,H} \geq \frac{3\eta-1}{2\eta-1}\mu_{1,H} - \frac{3\eta+1}{2\eta-1}\frac{1-\theta}{\theta}\mu_{2,L} - \frac{\eta}{2\eta-1}\mu_L.$$

Hence we obtain the first condition required for case II:

$$\mu_{1,H} \leq \mu_L + 3\cdot\mu_{2,L}\cdot\frac{\eta_0-1}{\eta_0+1}\cdot\frac{U}{V}.$$

Now we obtain the minimum of f_2 at z_2 as follows

$$f_{2min} = f_2(z_2) = \frac{\partial\eta}{\partial K_1}\cdot[\theta(K_1-\mu_{1,H})+(1-\theta)\mu_{2,L}]\cdot\left(\frac{\eta+1}{2(\eta-1)}z_2^{\frac{\eta-1}{2}} + \frac{\eta-1}{2(\eta+1)}z_2^{\frac{\eta+1}{2}}\right),$$

and

$$\begin{aligned}
\left(\lambda\frac{\partial^2 H(K_1,y)}{\partial y\partial K_1}\right)_{min} =&1+\frac{1}{\phi(\theta)}f_{2min}\\
=&1-\frac{\eta^2-1}{2\eta}\cdot\frac{1}{K_1-\mu_L}\cdot\frac{\theta(K_1-\mu_{1,H})+(1-\theta)\mu_{2,L}}{\theta^{\frac{1+\eta}{2}}(1-\theta)^{\frac{1-\eta}{2}}}\left(\frac{\eta+1}{\eta-1}z_2^{\frac{\eta-1}{2}}+\frac{\eta-1}{\eta+1}z_2^{\frac{\eta+1}{2}}\right)\\
=&1-\frac{\eta^2-1}{2\eta}\cdot\frac{1}{K_1-\mu_L}\cdot\left(\left(\frac{1-\theta}{\theta}\right)^{\frac{\eta+1}{2}}(K_1-\mu_{1,H})+\left(\frac{1-\theta}{\theta}\right)^{\frac{\eta-1}{2}}\mu_{2,L}\right)\\
&\times\left(\frac{\eta+1}{\eta-1}z_2^{\frac{\eta-1}{2}}+\frac{\eta-1}{\eta+1}z_2^{\frac{\eta+1}{2}}\right)\\
=&1-\frac{\eta^2-1}{2\eta}\cdot\frac{1}{K_1-\mu_L}\cdot\Bigg[(K_1-\mu_{1,H})\frac{\eta+1}{\eta-1}\left(\frac{1-\theta}{\theta}z_2\right)^{\frac{\eta-1}{2}}\\
&+\mu_{2,L}\frac{\eta-1}{\eta+1}\left(\frac{1-\theta}{\theta}z_2\right)^{\frac{\eta+1}{2}}+(K_1-\mu_{1,H})\frac{\eta-1}{\eta+1}\frac{\theta}{1-\theta}\left(\frac{1-\theta}{\theta}z_2\right)^{\frac{\eta+1}{2}}\\
&+\mu_{2,L}\frac{\eta+1}{\eta-1}\frac{1-\theta}{\theta}\left(\frac{1-\theta}{\theta}z_2\right)^{\frac{\eta-1}{2}}\Bigg]
\end{aligned}$$

Note that with $\frac{1-\theta}{\theta} z_2 < 1$, the above minimum value has the following lower bound:

$$\left(\lambda \frac{\partial^2 H(K_1, y)}{\partial y \partial K_1}\right)_{min} > 1 - \frac{\eta^2 - 1}{2\eta} \cdot \frac{1}{K_1 - \mu_L} \cdot \left[(K_1 - \mu_{1,H})\left(\frac{\eta+1}{\eta-1} + \frac{V}{U}\right) + \mu_{2,L}\left(\frac{\eta-1}{\eta+1} + \frac{U}{V}\right)\right]$$

Furthermore, note that $\frac{\eta^2-1}{2\eta}$ is increasing in η, and both

$$\frac{\eta^2 - 1}{2\eta} \cdot \frac{\eta+1}{\eta-1} = \frac{(\eta+1)^2}{2\eta} = \frac{1}{2}(\eta + 2 + \frac{1}{\eta})$$

and

$$\frac{\eta^2 - 1}{2\eta} \cdot \frac{\eta-1}{\eta+1} = \frac{(\eta-1)^2}{2\eta} = \frac{1}{2}(\eta - 2 + \frac{1}{\eta})$$

are increasing in η, with $\widetilde{K}_1 > \mu_{1,H}$ and $\eta < \eta_0$ we have

$$\begin{aligned}
&1 - \frac{\eta^2 - 1}{2\eta} \cdot \frac{1}{K_1 - \mu_L} \cdot \left[(K_1 - \mu_{1,H})\left(\frac{\eta+1}{\eta-1} + \frac{V}{U}\right) + \mu_{2,L}\left(\frac{\eta-1}{\eta+1} + \frac{U}{V}\right)\right] \\
>&1 - \frac{\eta_0^2 - 1}{2\eta_0} \cdot \frac{1}{K_1 - \mu_L} \cdot \left[(K_1 - \mu_{1,H})\left(\frac{\eta_0+1}{\eta_0-1} + \frac{V}{U}\right) + \mu_{2,L}\left(\frac{\eta_0-1}{\eta_0+1} + \frac{U}{V}\right)\right] \\
>&1 - \frac{\eta_0^2 - 1}{2\eta_0} \cdot \frac{1}{\mu_{1,H} - \mu_L} \cdot \left[\mu_{2,H}\left(\frac{\eta_0+1}{\eta_0-1} + \frac{V}{U}\right) + \mu_{2,L}\left(\frac{\eta_0-1}{\eta_0+1} + \frac{U}{V}\right)\right]
\end{aligned}$$

where η_0 is given in (A.109).

Therefore if we have

$$1 - \frac{\eta_0^2 - 1}{2\eta_0} \cdot \frac{1}{\mu_{1,H} - \mu_L} \cdot \left[\mu_{2,H}\left(\frac{\eta_0+1}{\eta_0-1} + \frac{V}{U}\right) + \mu_{2,L}\left(\frac{\eta_0-1}{\eta_0+1} + \frac{U}{V}\right)\right] > 0,$$

then the cross derivative is always positive. Now we have completed the proof. □

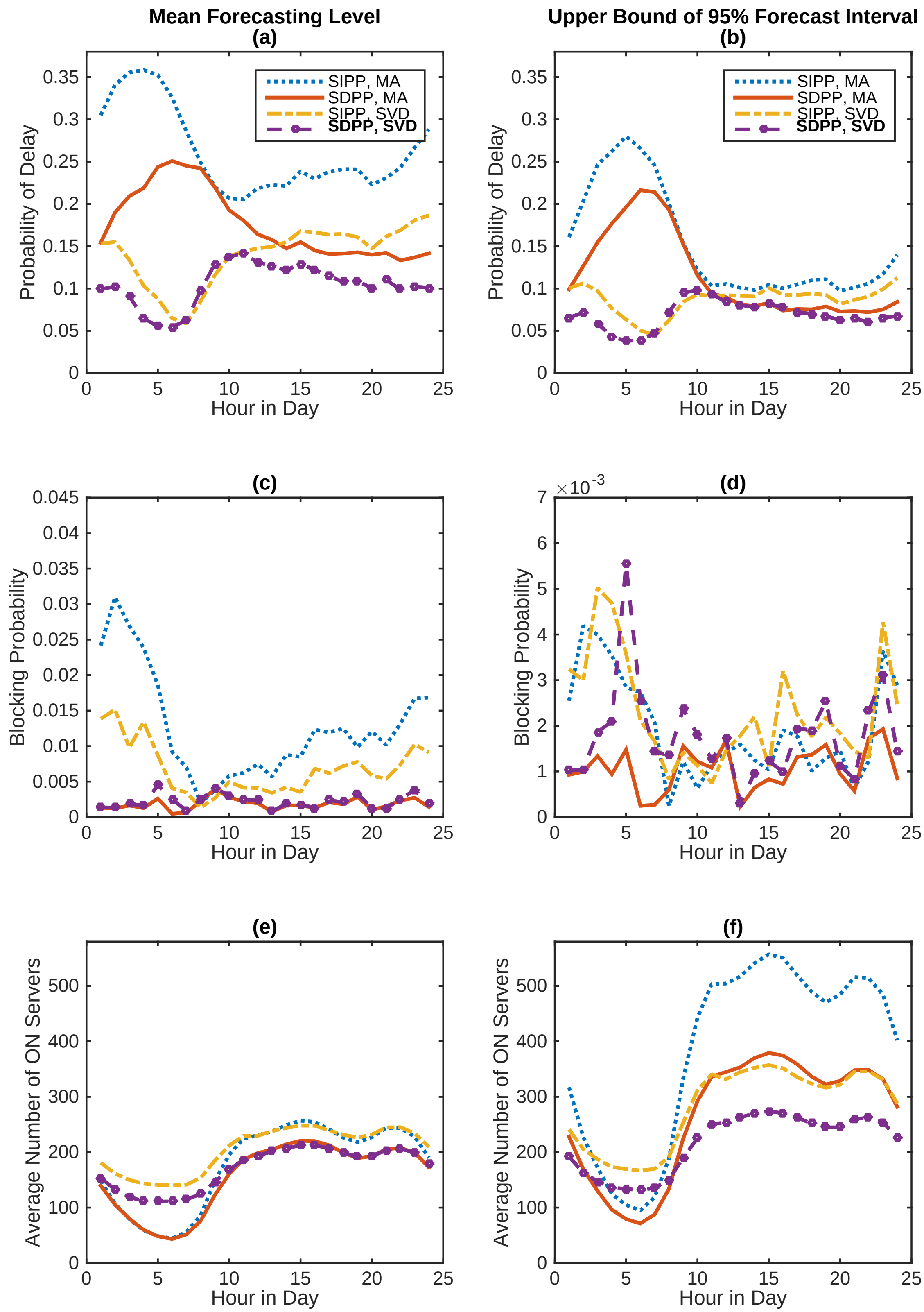


Figure A.3: Simulation Results of $M^X/M/\infty$ Queue

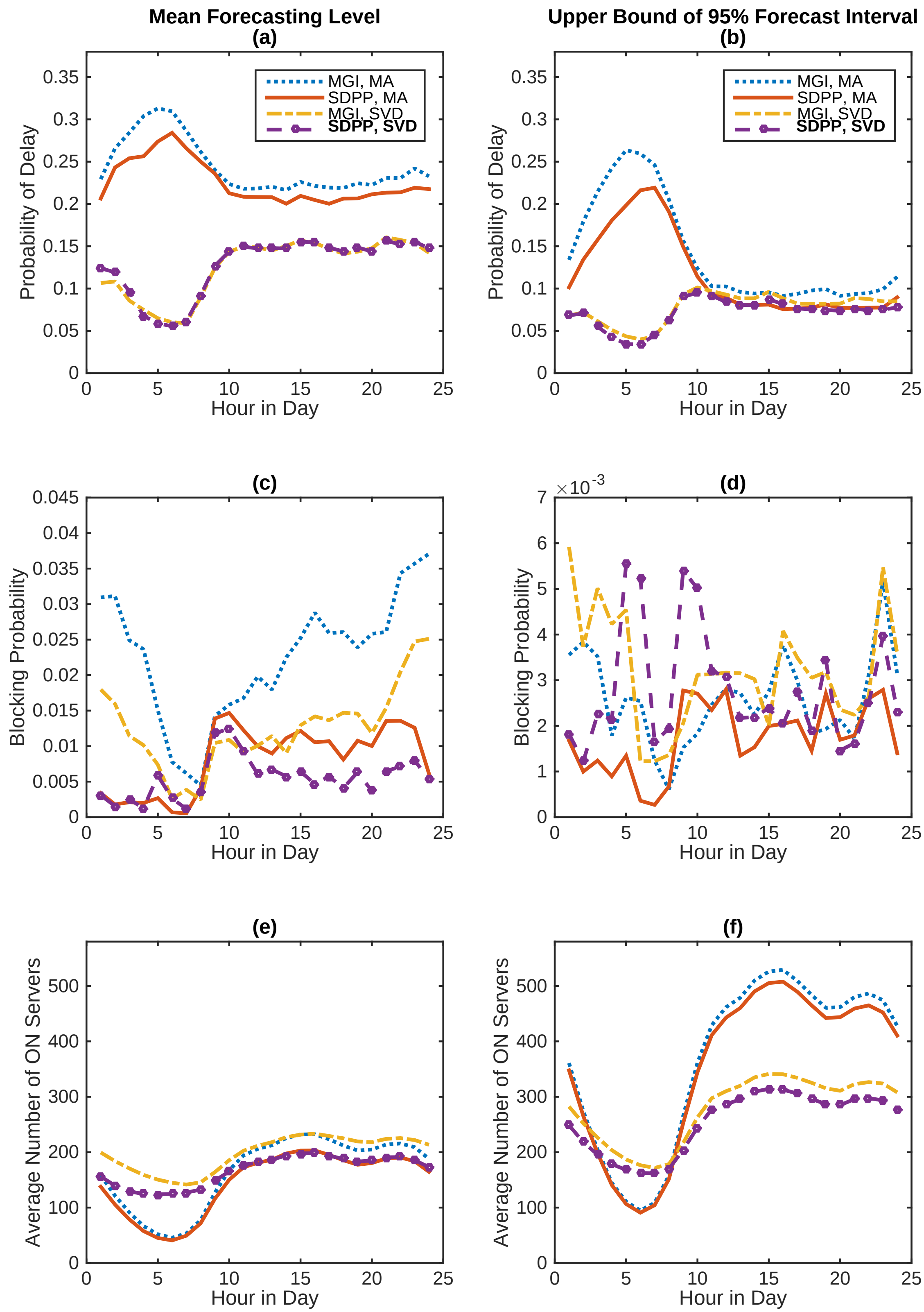


Figure A.4: Simulation Results of $M(t)/G/\infty$ Queue

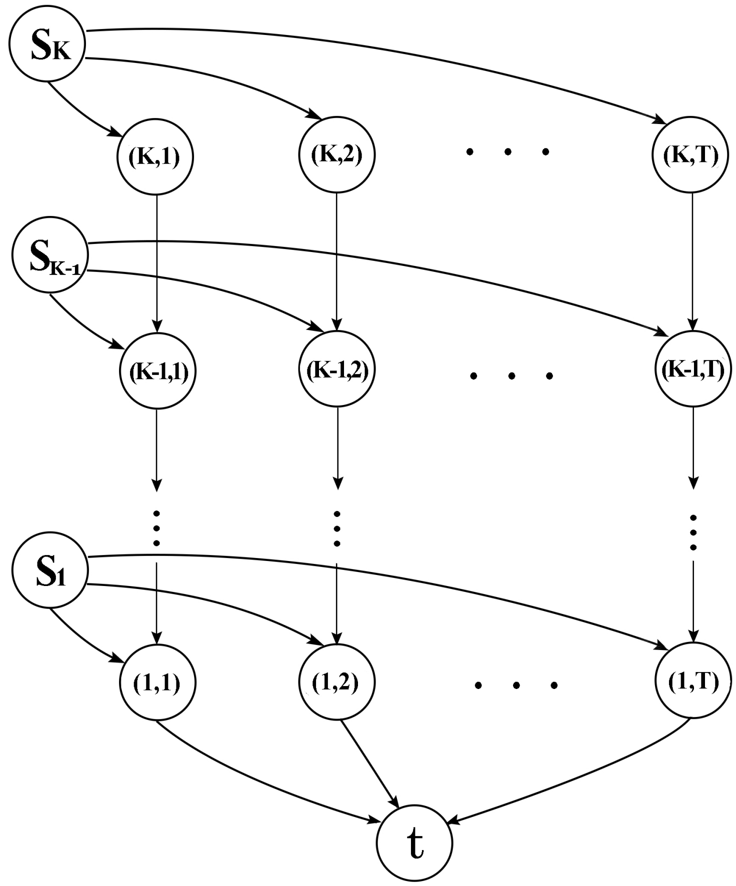


Figure A.5: Network Model of Optimization Problem

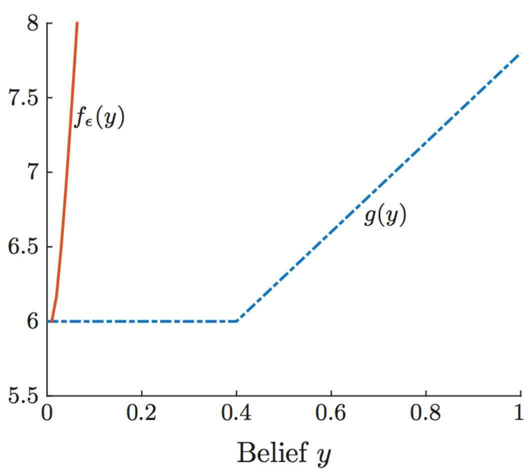


Figure A.6: $f_\ell(y)$ exploding with $\ell = 0.01$

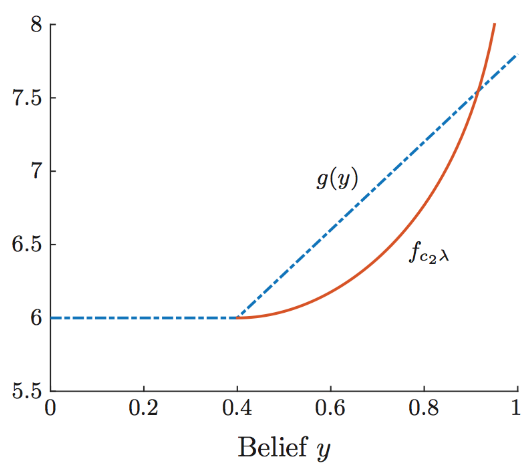


Figure A.7: $f_{c_2\lambda}(y)$ crossing $g(y)$.

Parameters used in Figure A.6 and A.7 $\lambda = 0.1$, $\sigma = 1$, $c_1 = 5$, $c_2 = 4$, $\mu_L = 2$, $\mu_H = 4.3$, $\mu_{1,L} = 1$, $\mu_{1,H} = 3$, $\mu_{2,L} = 1$, $\mu_{2,H} = 1.3$, $K_1 = 3.5$

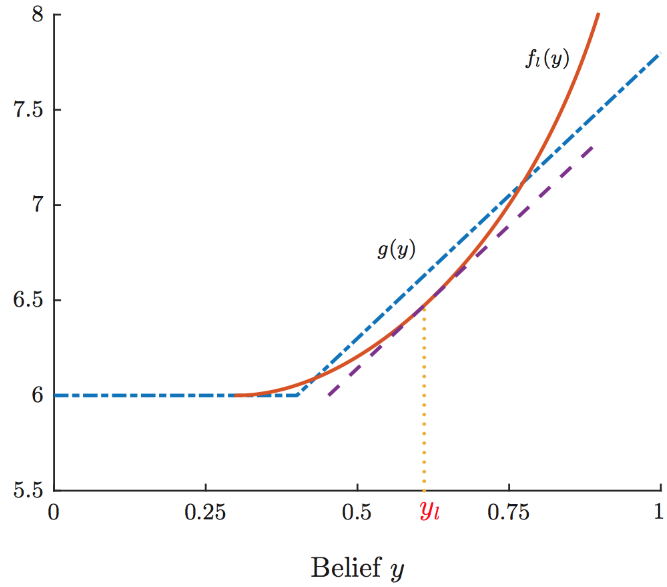


Figure A.8: Illustration of y_ℓ for $\ell = 0.3$ and $c_2\lambda = 0.4$. Parameters used in this figure: $\lambda = 0.1$, $\sigma = 1$, $c_1 = 5$, $c_2 = 4$, $\mu_L = 2$, $\mu_H = 4.3$, $\mu_{1,L} = 1$, $\mu_{1,H} = 3$, $\mu_{2,L} = 1$, $\mu_{2,H} = 1.3$, $K_1 = 3.5$

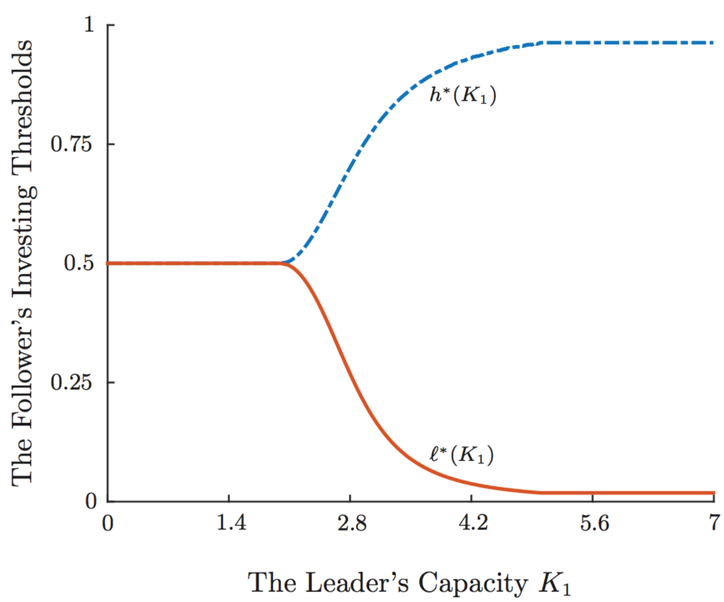


Figure A.9: Thresholds ℓ^* and h^* as function of K_1. Parameters used in this figure: $\lambda = 0.1$, $\sigma = 1$, $c_1 = c_2 = 5$, $\mu_L = 2$, $\mu_H = 5$, $\mu_{1,L} = 1$, $\mu_{1,H} = 2.5$, $\mu_{2,L} = 1$, $\mu_{2,H} = 2.5$.

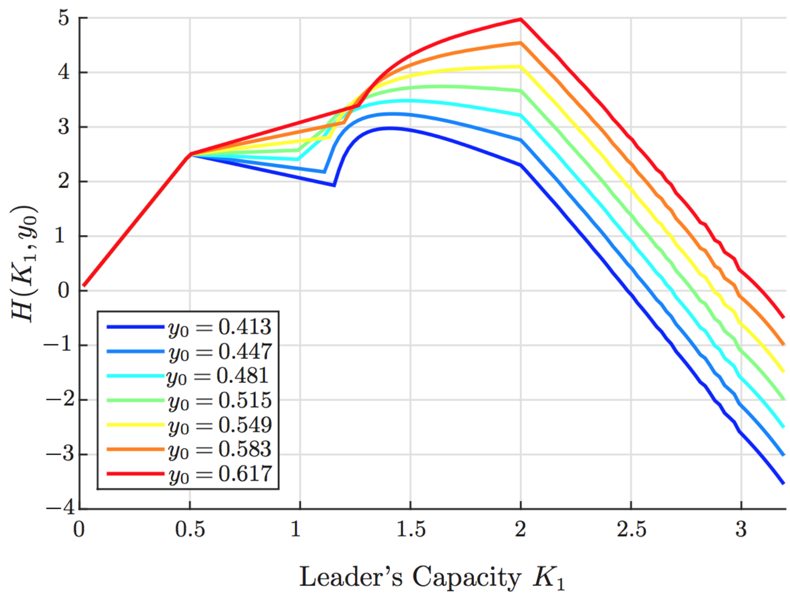


Figure A.10: The Leader's Value Function $H(K_1, y_0)$ as Function of K_1 for Different y_0. Parameters used in this figure: $\lambda = 0.1$, $\sigma = 1$, $c_1 = c_2 = 5$, $\mu_L = 1$, $\mu_H = 10$, $\mu_{1,L} = .5$, $\mu_{1,H} = 2$, $\mu_{2,L} = .5$, $\mu_{2,H} = 8$.

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