

**THE SPRINGER MORPHISM, POLYNOMIAL REPRESENTATION RINGS, AND THE  
COHOMOLOGY RING OF GRASSMANNIANS**

**Sean Rogers**

A dissertation submitted to the faculty of the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics.

Chapel Hill  
2018

Approved by:

Shrawan Kumar

Prakash Belkale

Jiuzu Hong

Robert Proctor

Richard Rimanyi

©2018  
Sean Rogers  
ALL RIGHTS RESERVED

## ABSTRACT

Sean Rogers: The Springer Morphism, Polynomial Representation Rings, and the Cohomology Ring of Grassmannians

(Under the direction of Shrawan Kumar)

To any almost faithful representation of a complex, connected, reductive algebraic group  $G$  of highest weight  $\lambda$  one can associate a dominant morphism from the group to its Lie algebra  $\mathfrak{g}$ . This map enjoys many nice properties. In particular, when restricted to a maximal torus it maps to the Cartan subalgebra. This map can be used to give a natural definition of polynomial representations for the classical groups of types B, C, and D. Given a parabolic subgroup  $P \subset G$ , Kumar showed there is a surjective algebra homomorphism from the polynomial representations of a Levi subgroup of  $P$  to the cohomology of  $G/P$  which extends a classical result relating the polynomial representations of  $GL(r)$  and the cohomology of the grassmannian of  $r$ -planes in  $n$ -space  $H^*(Gr(r, n))$ . In this work we give an explicit determination of the map  $\theta_\lambda$  for simple groups and consider Kumar's map for types B, C, and G.

## **ACKNOWLEDGEMENTS**

First I would like to thank my advisor Shrawan Kumar for his endless patience and encouragement. I would like to thank my parents for their support. I would like to thank Andrew, Sam, Colin, Ian, Dave, and Jacob for their camaraderie throughout graduate school. I would like to thank my parents for supporting me throughout my academic life. Finally, I would like to thank my significant other Emma. I would not have been able to finish this document without her support.

## TABLE OF CONTENTS

LIST OF TABLES .....	vii
1 Introduction .....	1
1.1 Historical Context .....	1
1.2 Concerning this work .....	4
2 Preliminaries .....	8
2.1 Reductive Groups, Root Data, and Representations .....	8
2.1.1 Weyl Groups .....	11
2.1.2 Dynkin Index .....	13
2.2 Bruhat Decomposition, Schubert varieties, and Parabolic Subgroups .....	14
2.3 Borel Characteristic map and BGG-operators .....	19
3 The Springer Morphism .....	25
3.1 Definition and Properties .....	26
3.2 An Explicit Determination of the Springer Morphism .....	30
3.2.1 Examples .....	39
4 Representation Ring of Levi Subgroups vs Cohomology Ring of Flag Varieties .....	42
4.1 Classical Result and Polynomials Invariants .....	42
4.1.1 Weyl Group Invariants .....	44

4.2	Main Result	45
4.3	Type A	48
4.3.1	Recovering the Classical Result	48
4.3.2	Cohomology of the Grassmannian	50
4.3.3	Result in the Inverse Limit	53
5	Types B,C, and G	58
5.1	Representation Ring of the Classical groups	58
5.2	Type C	62
5.2.1	Cohomology of $IG(n-k,2n)$	62
5.2.2	Theorem 4.2	67
5.2.3	Inverse Limit	75
5.3	Type B	77
5.3.1	Cohomology of $OG(n-k,2n+1)$	77
5.3.2	Theorem 4.2	79
5.3.3	Inverse Limit	83
5.4	$G_2$	84
5.4.1	Representation Ring of $G_2$	84
	BIBLIOGRAPHY	88

## LIST OF TABLES

4.1	Degrees of Basic Invariants .....	44
-----	-----------------------------------	----

## CHAPTER 1

### Introduction

#### 1.1 Historical Context

Schubert calculus as subject emerged from Schubert's work on the calculus of enumerative geometry [Sc1, Sc2]. A typical example of an enumerative problem is as follows (borrowed from [KL]). In three space, how many lines intersect four given lines? The solution for lines in general position turns out to be two. Schubert's approach to such problems was to work in the Grassmannian manifold. Let  $V = \mathbb{C}^n$ . Then the Grassmannian manifold  $Gr(m, V)$  as a set is the set of all  $m$ -dimensional subspaces of  $V$ . It can be given the structure of a complex manifold (or projective variety) of dimension  $n(m - m)$ . Let  $F_\bullet := \{0 = F_0 \subset F_1 \subset \dots \subset F_{n-1} \subset F_n = V\}$ , where  $\dim F_i = i$ , be a complete flag. The Grassmannian has a stratification of affine subsets given by geometric intersection conditions with respect to this flag. A partition  $\lambda = \{\lambda_1, \dots, \lambda_m\}$  is a sequence of weakly decreasing integers. Let  $|\lambda| = \sum_{i=1}^m \lambda_i$ . For a partition  $\lambda$  such that  $\lambda_1 \leq n - m$  we can define a subset of the Grassmannian,

$$\Omega_\lambda(F_\bullet) = \{X \in Gr(m, n) : \dim(X \cap V_{n-m+j-\lambda_j}) > j \forall i \leq j \leq m\}.$$

These subsets are isomorphic to  $\mathbb{C}^{m(n-m)-|\lambda|}$  and are called Schubert cells. They form a stratification the Grassmannian. Their closures are the so called Schubert varieties  $X_\lambda(F_\bullet)$ . Let  $[X_\lambda] \in H_{2(m(n-m)-|\lambda|)}(Gr(m, n))$  denote the fundamental class of  $X_\lambda$  in the singular homology of the Grassmannian. Then, take  $\sigma_\lambda \in H^{2|\lambda|}(Gr(m, n))$  be their Poincare dual classes. Then the Schubert basis theorem states that these classes, hereafter called Schubert classes,  $\sigma_\lambda$  form an integral basis of the singular cohomology ring of the Grassmannian  $H^*(Gr(m, n))$ . These classes are independent of the choice of flag  $F_\bullet$ . The fundamental insight of classical Schubert calculus is that problems in enumerative geometry and intersection theory can be solved by performing corresponding algebraic calculations in the ring  $H^*(Gr(m, n))$ . Let  $\sigma_i$  be the Schubert class corresponding to the partition  $\lambda = (i, 0, \dots, 0)$ . These are called the special Schubert classes. Pieri gave a



rule for expanding the cup product of a special Schubert class and a general Schubert class in the Schubert basis. Giambelli gave a formula for expressing any Schubert class  $\sigma_\lambda$  as a polynomial in the special Schubert classes  $\sigma_i$ . Solving the above problem in enumerative geometry amounts to computing  $\sigma_1^4 = 2\sigma_{(2,2)}$  in  $H^*(Gr(2, 4))$ .  $\sigma_{(2,2)}$  is the class of a point and we arrive at our answer of two. In general we have the structure constants of  $H^*(Gr(m, n))$  are

$$\sigma_\lambda \cdot \sigma_\mu = \sum c_{\lambda\mu}^\nu \sigma_\nu.$$

The constant  $c_{\lambda\mu}^\nu$  is known to be the number of points in the intersection of general translates of the Schubert varieties  $X_\lambda$ ,  $X_\mu$ , and  $X_{\check{\nu}}$ , where  $\check{\nu}$  is the dual partition  $\check{\nu}_i = m - n - \nu_{m+1-i}$ .

A combinatorial rule for computing the coefficients  $c_{\lambda\mu}^\nu$  in terms of the given partitions was given by Littlewood and Richardson. The context with which the coefficients arose however was not a priori related to intersections of Schubert cycles. Schur polynomials  $s_\lambda$  are a basis for the symmetric polynomials. Then one has  $s_\lambda s_\mu = \sum c_{\lambda\mu}^\nu s_\nu$ . It is well known that polynomial irreducible representations of  $GL(m)$  are indexed by partitions  $\lambda = (\lambda_1 \geq \dots \geq \lambda_m)$  where  $\lambda$  represents the highest weight of the representation. Denote this representation by  $V(\lambda)$ . The character of this representation is the Schur polynomial  $s_\lambda$ . As the character of a tensor product of representations is the product of the characters of the given representations, it holds that

$$V(\lambda) \otimes V(\mu) = \sum c_{\lambda\mu}^\nu V(\nu)$$

Note that  $c_{\lambda\mu}^\nu$  counts the dimension of the  $GL(m)$  invariant subspace of  $V(\lambda) \otimes V(\mu) \otimes V(\check{\nu})$ , where  $V(\check{\nu})$  is the dual representation of  $V(\nu)$ . Let  $Rep_{poly}(GL(m))$  be the polynomial representation ring. That the structure constants of  $H^*(Gr(m, n))$  and  $Rep_{poly}(GL(m))$  coincide as long as the partitions  $\lambda$ ,  $\mu$ ,  $\nu$  fit in a  $m \times m - n$  rectangle was proved by Lesieur [Les] in 1947. However, the proof is indirect. Essentially, they are both rings are governed by Schur functions. See [F6] for the history of this problem and other contexts in which the Littlewood-Richardson coefficients appear. Thus one wonders if there is a more natural explanation of this fact or a generalization to other classical groups ( $Sp(2m)$ ,  $SO(2m)$ ,  $SO(2m + 1)$ ). Possible explanations have been given by Tamvakis via the Chern-Weil theory of characteristic classes

[T1], by Belkale via tangent spaces to Schubert varieties [Be], and by Mukhin, Tarasov, and Varchenko via representations of the Bethe algebra [MTV]. One problem of generalization is how to adequately define polynomial representations for other classical groups, or more generally connected reductive groups. Kumar gives one attempt at generalization via the Springer morphism in [Ku2]. This work is the genesis for this thesis.

A modern formulation of Schubert calculus can be given in terms of generalized partial flag varieties. A generalized *flag variety* can be defined for an connected, complex, reductive algebraic group  $G$  as the projective variety  $G/B$ . Here  $B$  is a Borel subgroup, i.e. a maximal, connected, solvable Zariski closed subgroup. For  $G = GL(m)$ , the standard Borel subgroup is the set of upper triangular matrices and we have

$$G/B = \{F_\bullet : 0 = F_0 \subset F_1 \subset \cdots \subset F_{m-1} \subset F_m = \mathbb{C}^m\}$$

is the variety of complete flags  $F_\bullet$  with  $\dim(F_i) = i$ . Every reductive group has a maximal torus  $T \subset B \subset G$  and a Weyl group  $W = N(T)/T$ . Let  $\mathfrak{t}$ ,  $\mathfrak{b}$ ,  $\mathfrak{g}$  be the corresponding Lie algebras. The flag variety  $G/B$  has a cell decomposition called the Bruhat decomposition into affine open cells  $B_w$ , indexed by elements of the Weyl group  $w \in W$ . The closures of these cells  $X_w$  are also called Schubert varieties and a Schubert calculus can be defined on  $H^*(G/B)$ . Let  $\epsilon_w$  be the Kronecker dual to the fundamental homology class of  $X_w$  ( $\epsilon_w$  is then called a Schubert class). As before,  $H^*(G/B)$  has a basis of Schubert classes. Borel, [Bo], also gave a characterization of the cohomology of  $H^*(G/B)$  via the characteristic map

$$\beta : S(\mathfrak{t}^*) \rightarrow H^*(G/B).$$

Here  $S(\mathfrak{t}^*)$  is the symmetric algebra of the dual of the Cartan subalgebra  $\mathfrak{t}$ . The map is given by mapping characters to the first Chern class of the related line bundle. A Pieri-type formula is given by Chevalley, and a Giambelli type formula is given by the BGG Schubert polynomials defined by Bernstein, Gelfand, and Gelfand in their seminal work relating the Bruhat and Schubert presentations of the cohomology of  $G/B$  [BGG]. They identify a set of polynomials  $p_w$  in  $S(\mathfrak{t}^*)$  corresponding Schubert classes  $\epsilon_w$  using divided difference operators  $A_w : S(\mathfrak{t}^*) \rightarrow S(\mathfrak{t}^*)$  corresponding to elements of the Weyl group. A parabolic group  $P$  is any group containing a Borel subgroup  $B \subset P$ . Then  $G/P$  is a generalized partial flag variety and there is

an analagous picture of the Schubert calculus on  $G/P$ . For maximal parabolics  $P \subset GL(m)$ ,  $GL(m)/P$  is a Grassmannian.

Information about reductive groups and their Schubert calculus is discussed in more detail in Chapter 2 (Preliminaries).

## 1.2 Concerning this work

Here we describe the rest of the thesis. Let  $G$  be a connected reductive algebraic group over  $\mathbb{C}$  with Borel subgroup  $B$  and maximal torus  $T \subset B$  of rank  $n$  with character group  $X^*(T)$ . Let  $P$  be a standard parabolic subgroup with Levi subgroup  $L$  containing  $T$ . Let  $W$  (resp.  $W_L$ ) be the Weyl group of  $G$  (resp.  $L$ ). Let  $V_\lambda$  be an irreducible almost faithful representation of  $G$  with highest weight  $\lambda$ , i.e.  $\lambda$  is a dominant integral weight and the corresponding map  $\rho_\lambda : G \rightarrow GL(V_\lambda)$  has finite kernel. Then, Springer defined an adjoint-invariant regular map with Zariski dense image from the group to its Lie algebra,  $\theta_\lambda : G \rightarrow \mathfrak{g}$ , which depends on  $\lambda$  [BR]. Properties of this map are discussed in Chapter 3 Section 1. In particular, when restricted to the maximal torus we have  $\theta_{\lambda|_T} : T \rightarrow \mathfrak{t}$ . We note that this map can also be viewed as a generalization of the classical Cayley map. Furthermore, Kumar [Ku2] use this map to define the  $\lambda$ -polynomial representation ring of a group  $G$ ,  $\text{Rep}_\lambda^{\mathbb{C}}(G)$ . The ring  $\text{Rep}_\lambda^{\mathbb{C}}(G)$  is a subring of representation ring of  $G$  which is isomorphic to  $S(\mathfrak{t}^*)^W$ , the ring of Weyl group invariant polynomials. For any weights  $\lambda_1, \lambda_2$  the  $\lambda$ -polynomial representation ring are isomorphic but the isomorphism is different. For  $Sp(2n)$ ,  $SO(2n)$ , and  $SO(2n + 1)$  we define the polynomial representation ring to be  $\text{Rep}_{\omega_1}^{\mathbb{C}}(G)$ .

We can restate the classical result relating the polynomial representation ring of  $GL(r)$  to the singular cohomology ring of the Grassmannian  $H^*(Gr(r, n))$  as follows. There is an explicit surjective ring homomorphism

$$\xi : \text{Rep}_{\text{poly}}(GL(r)) \rightarrow H^*(Gr(r, n)).$$

In recent work by Kumar [Ku2], the Springer morphism is used in a crucial way to extend the above classical result relating the polynomial representation ring of the general linear group  $GL_r$  and the singular cohomology ring  $H^*(Gr(r, n))$  of the Grassmannian of  $r$ -dimensional complex linear subspaces of  $\mathbb{C}^n$  to the Levi subgroups of any reductive group  $G$  and the cohomology of the corresponding flag varieties  $G/P$ . Computing  $\theta_{\lambda|_T}$  is integral to this process. Importantly,  $\theta_\lambda$  takes the maximal torus  $T$  to its Lie algebra  $\mathfrak{t}$ ,

thus inducing an injective  $\mathbb{C}$ -algebra homomorphism  $(\theta_\lambda|_T)^* : \mathbb{C}[\mathfrak{t}] \rightarrow \mathbb{C}[T]$  between the corresponding affine coordinate rings. Let  $L$  be the Levi subgroup of the parabolic  $P$  which contains the torus  $T$ . The Springer morphism is equivariant under the adjoint action and thus  $(\theta_\lambda|_T)^*$  takes  $\mathbb{C}[\mathfrak{t}]^{W_L}$  to  $\mathbb{C}[T]^{W_L}$ . One can then define the  $\lambda$ -polynomial subring  $Rep_{\lambda-pol}^{\mathbb{C}}(L)$  to be the image of  $\mathbb{C}[\mathfrak{t}]^{W_L}$  under  $(\theta_\lambda|_T)^*$  (as  $Rep^{\mathbb{C}}(L) \simeq \mathbb{C}[T]^{W_L}$ ). Here  $Rep^{\mathbb{C}}(L)$  is the complex representation ring of  $L$ . This leads to a surjective  $\mathbb{C}$ -algebra homomorphism  $\xi_\lambda^P : Rep_{\lambda-pol}^{\mathbb{C}}(L) \rightarrow H^*(G/P, \mathbb{C})$ , as in [Ku2]. The map  $\theta_\lambda$  enjoys many nice properties (see [KM]). In this work we compute  $\theta_\lambda|_T$  in a uniform way for all simple algebraic groups  $G$  and any dominant integral weight  $\lambda$ .

As  $\theta_\lambda|_T$  maps  $T$  into  $\mathfrak{t}$ , we have that for a given simple group  $G$  and an irreducible representation  $V_\lambda$ , one may write

$$\theta_\lambda(t) = \sum_{i=1}^n c_i(\lambda) \check{\alpha}_i,$$

where we take the simple coroots  $\{\check{\alpha}_i\}$  as a basis for  $\mathfrak{t}$ . We give a complete determination for these coefficients  $c_i(t)$  for any simple, simply-connected algebraic group  $G$  as a sum over the weights of the torus action on  $V_\lambda$ .

For a given representation  $V_\lambda$ , let  $\Lambda_\lambda$  be the set of weights appearing in the weight space decomposition of  $V_\lambda = \bigoplus V_\lambda^\mu$ , listed with multiplicity. Let  $\omega_1, \dots, \omega_n$  be the fundamental weights in  $\mathfrak{t}^*$ , and consider the weights  $\mu \in \Lambda_\lambda$  written in the fundamental weight basis, i.e.  $\mu = (\mu_1, \dots, \mu_n) = \mu_1\omega_1 + \dots + \mu_n\omega_n$ . Let  $e^\mu(t) \in X^*(T)$  be the corresponding character of  $T$ . Then we find that,

**Theorem.** *The coefficients  $c_i(t)$  are determined by the following set of equations.*

$$\begin{pmatrix} \sum_{\mu \in \Lambda_\lambda} \mu_1 e^\mu(t) \\ \vdots \\ \sum_{\mu \in \Lambda_\lambda} \mu_n e^\mu(t) \end{pmatrix} = S(G, \lambda) \begin{pmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_n(t) \end{pmatrix},$$

where  $S(G, \lambda) = \{ \sum_{\mu \in \Lambda_\lambda} \mu_i \mu_j \}_{ij}$ .

The main result of [R] determines that

**Theorem.** *The above matrix*

$$S(G, \lambda) := \left\{ \sum_{\mu \in \Lambda_\lambda} \mu_i \mu_j \right\}_{ij} = \left( \frac{1}{2} \sum_{\mu \in \Lambda_\lambda} \mu_i^2 \right) S,$$

where  $S$  is a specific uniform symmetrization of the Cartan matrix  $A$  for  $G$ , and  $\mu_i$  is the coordinate of the fundamental weight corresponding to a long root (or any root in the simply-laced case).

In particular, for the simply-laced groups  $S(G, \lambda) = \left( \frac{1}{2} \sum_{\mu \in \Lambda_\lambda} \mu_1^2 \right) A$ . The determination of  $S(G, \lambda)$  relies on the fact that  $\Lambda_\lambda$  is invariant under the action of the Weyl group  $W$ , and moreover that if  $\sigma \in W$  then  $\dim(V_\mu) = \dim(V_{\sigma \cdot \mu})$ . The above results are discussed in Chapter 3.

In Chapters 4 and 5 we recall the work *Representation ring of Levi subgroups versus the cohomology ring of flag varieties* by Kumar [Ku2]. In particular, discuss the map

$$\xi_{\omega_1}^P(L) : \text{Rep}_{\lambda\text{-poly}}^{\mathbb{C}} \rightarrow H^*(G/P, \mathbb{C})$$

in the case of the classical complex algebraic groups  $GL(k)$ ,  $Sp(2k)$ , and  $SO(2k+1)$  and their maximal parabolics ( $V_{\omega_1}$  is the defining representation in each case). Note we do not analyze the type D case,  $SO(2n)$ . The analysis is more or less uniform. In types  $A$ ,  $B$ , and  $C$  quotienting by a maximal parabolic gives a Grassmannian. Consider the Dynkin diagram in classical type of rank  $n$  and take the maximal parabolic  $P^{n-k}$  corresponding to the  $(n-k)^{\text{th}}$  node of the Dynkin diagram. Then, the Grassmannians in question are:  
For type A,

$$Gr(n-k, n) = \{X \in \mathbb{C}^n : \dim(X) = n-k\},$$

For type C (let  $\vartheta$  be a skew-symmetric bilinear form on  $\mathbb{C}^{2n}$ ),

$$IG(n-k, 2n) = \{X \in \mathbb{C}^{2n} : \dim(X) = n-k, \vartheta(v, w) = 0 \forall v, w \in X\},$$

For type B (let  $\vartheta$  be a symmetric bilinear form on  $\mathbb{C}^{2n+1}$ ),

$$OG(n-k, 2n+1) = \{X \in \mathbb{C}^{2n+1} : \dim(X) = n-k, \vartheta(v, w) = 0 \forall v, w \in X\}.$$

Then for each of these spaces there is a short exact sequence of vector bundles

$$0 \rightarrow S \rightarrow V \rightarrow Q \rightarrow 0.$$

Where  $V$  is the trivial rank  $n$  (respectively  $2n, 2n+1$ ) bundle,  $S$  is the tautological subbundle (i.e. the fiber over  $X \in Gr(n-k, n)$  is  $X \subset \mathbb{C}^n$  for the type A case), and  $Q$  is the tautological quotient bundle. In the above cases the Chern classes  $c_i(Q)$  generate the cohomology ring  $H^*(G/P^{n-k})$  [BKT1]. Buch, Kresch, and Tamvakis gave Pieri and Giambelli formulas with the Chern classes  $c_i(Q)$  as special classes for both the classical and quantum cohomology of  $IG(n-k, 2n)$  and  $OG(n-k, 2n+1)$  in terms of  $k$ -strict partitions (see [BKT1, BKT2, BKT3, BKT4]). We rely heavily on their formalism and presentations of the cohomology rings.

For the parabolic group  $P^{n-k}$  above we have that the Levi subgroups of  $P^{n-k}$  for types A, B, and C are  $L_{n-k}^A = GL(n-k) \times GL(k)$ ,  $L_{n-k}^C = GL(n-k) \times Sp(2k)$ , and  $L_{n-k}^B = GL(n-k) \times SO(2k+1)$ . We then have maps  $\xi^{n-k}$ , from the the Levi subgroup  $L_{n-k}$  to the corresponding Grassmannian. Factoring through this map allows one to recover the classical map  $\xi$  [Ku2, Theorem 8]. We give explicit descriptions of these maps  $\xi^{n-k}$ , i.e. we describe the images of the generators of  $\text{Rep}_{poly}^C(L_{n-k})$  in terms of the Chern classes of  $S$  and  $Q$ . If we fix  $k$ , and allow  $n$  to go to infinity we get the stable cohomology rings. For example, the stable cohomology ring of type A,  $\mathbb{H}(Gr_k)$ , is the inverse limit in the category of graded rings of the system

$$\dots \leftarrow H^*(Gr(n-k, n), \mathbb{C}) \leftarrow H^*(Gr(n-k+1, n+1), \mathbb{C}) \leftarrow \dots$$

Then, factoring through  $\xi^{n-k}$  gives an isomorphism  $\text{Rep}_{\omega_1}^C(GL(k)) \simeq \mathbb{H}(Gr_k)$ . We prove that the analogous maps in types B and C are injective.

## CHAPTER 2

### Preliminaries

#### 2.1 Reductive Groups, Root Data, and Representations

Let  $G$  be a linear algebraic group over the complex numbers  $\mathbb{C}$ , i.e. a group which is also an algebraic variety such that the inverse and multiplication maps are morphisms. The radical of an algebraic group is the identity component of its maximal, closed, solvable subgroup and the subgroup of unipotent elements in this group is referred to as its unipotent radical. If the unipotent radical is trivial then  $G$  is called *reductive*. Further, if the radical of  $G$  is trivial then the group is called *semi-simple*. For the rest of this subsection we will assume  $G$  is reductive.

A subgroup that is isomorphic to  $(\mathbb{C}^*)^k$  for some  $k$  is called a torus. For a given maximal torus  $T$  we can define the Weyl group  $W = N(T)/T$ , where  $N(T)$  is the normalizer of  $T$  in  $G$ . Since all maximal tori are conjugate, different choices of  $T$  will lead to isomorphic Weyl groups. A Borel subgroup is a maximal, solvable, connected, Zariski closed subgroup. All Borel subgroups are conjugate and contain a maximal torus. For the rest of this section we will fix  $T \subset B \subset G$  and we fix  $\dim T = n$ , called the semisimple rank of  $G$  (or just rank). Let  $X(t)$  denote the character group of  $T$ , that is the set of morphisms  $T \rightarrow \mathbb{C}^*$

The tangent space at the identity of an algebraic group has the structure of a Lie algebra and is denoted using the lowercase gothic character  $\mathfrak{g}$ . Similarly, we let  $\mathfrak{t}$ ,  $\mathfrak{b}$  denote the Lie algebras of  $T$ , and  $B$ . There is a natural map  $\exp : \mathfrak{g} \rightarrow G$ . It follows that any representation of  $G$

$$\rho : G \rightarrow GL(V)$$

yields a Lie algebra representation by taking the differential

$$d\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$$

If we restrict the representation to  $\mathfrak{t}$  and note that all representations of  $\mathfrak{t}$  are completely reducible, we can right down a *weight* space decomposition for the representation  $V = \bigoplus V_\lambda$ . Here  $\lambda \in \mathfrak{t}^* = \text{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})$  and  $V_\lambda = \{v \in V \mid X \cdot v = \lambda(X)v \forall X \in \mathfrak{t}\}$

In particular,  $G$  naturally acts of  $\mathfrak{g}$  by the adjoint action  $Ad$ ,

$$Ad(g) \cdot X = \left. \frac{d}{dt} g \exp(tX) g^{-1} \right|_{t=0}$$

Differentiating this action give the adjoint representation of  $\mathfrak{g}$

$$ad : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$$

$$X \rightarrow [X, \cdot]$$

Then under the adjoint action of  $\mathfrak{t}$  we can decompose  $\mathfrak{g}$  as

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus \mathfrak{g}_\alpha$$

The nonzero weights  $\alpha \in \mathfrak{t}^*$  of the adjoint representation are called the roots of  $\mathfrak{g}$  (and  $G$  respectively). Denote the set of roots by  $R$ .  $R$  then forms a root system in  $\mathfrak{t}^*$ . Our choice of Borel  $B$  (hence  $\mathfrak{b}$ ) determines a base  $\Delta = \alpha_1, \dots, \alpha_n$  of simple roots. Every root in  $R$  can be written as a linear combination of simple roots with either all non-negative or all non-positive coefficients. Then let  $R^+$  be the set of positive roots and let  $R^- = -R^+$  be the set of negative roots. The action of  $W$  on  $T$  and the adjoint action of  $G$  on  $\mathfrak{g}$  induce an action of  $W$  on  $\mathfrak{t}$ . For any root  $\alpha$  and any  $\mu \in \mathfrak{t}^*$  we define the reflection in  $\mathfrak{t}^*$

$$s_\alpha(\mu) = \mu - \mu(\check{\alpha})\alpha = \mu - \frac{2\langle \alpha, \mu \rangle}{\langle \alpha, \alpha \rangle} \alpha$$

Then the simple reflections  $s_i$ , where  $s_i = s_{\alpha_i}$  for any simple root, generate  $W$  when  $W$  is identified with its action on  $\mathfrak{t}$

The Killing form on  $\mathfrak{g}$  is a symmetric, adjoint-invariant,  $W$ -invariant bilinear form given by  $\langle X, Y \rangle = \text{Tr}(ad(X)ad(Y))$ . This induces an identification of  $\mathfrak{t}$  and  $\mathfrak{t}^*$ . For any root  $\alpha \in R$ , let  $\check{\alpha} = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$  be the



corresponding coroot and let  $\check{R}$  denote the set of coroots. The simple coroots  $\check{\Delta}$  form a basis for  $\mathfrak{t}$ . Define the *fundamental weights*  $\omega_i$  by  $\langle \omega_i, \check{\alpha}_j \rangle = \omega_i(\check{\alpha}_j) = \delta_{ij}$ . Let  $\Lambda = \bigoplus_{i=1}^n \omega_i$  be the weight lattice in  $\mathfrak{t}^*$ , and let  $\Gamma = \bigoplus_{\alpha \in \Delta} \alpha$  be the root lattice. Identify  $X(T)$  with a lattice in  $\mathfrak{t}^*$  by differentiating. Then in general we have

$$\Gamma \subset X(T) \subset \Lambda$$

If  $G$  is simply connected we have  $X(T) = \Lambda$ .

A weight  $\lambda$  is called dominant if  $\langle \lambda, \alpha \rangle \geq 0 \forall \alpha \in \Delta$ . Any dominant weight can be written as a non-negative linear combination of fundamental weights. Denote the set of dominant weights by  $\Lambda^+$ . Given a representation  $V$  of  $G$ , a vector  $v \in V$  is called highest weight if (under the induced representation on  $\mathfrak{g}$ )  $v$  is an eigenvector of the action of  $\mathfrak{t}$  and is in the kernel of the action of  $\mathfrak{g}_\alpha$  for all roots  $\alpha$ . If the highest weight vector  $v$  is in the weight space  $V_\lambda$ , we say that  $\lambda$  is a highest weight for the representation  $V$ . It is highest in the sense that it will be the highest weight given by the following partial order on weights. We say that  $\lambda > \mu$  if  $\mu = \lambda - \sum_{\alpha \in \Delta} k_\alpha \alpha$  with  $k_\alpha \geq 0$  and integral. If  $V$  is irreducible then there is a unique highest weight. The following classification is a fundamental theorem in Lie theory,

**Theorem 2.1.** *For any dominant weight  $\lambda \in X(T) \cap \Lambda^+$  there is a unique irreducible finite-dimensional representation  $V_\lambda$  of  $G$  with highest weight  $\lambda$ .*

This will allow us to concretely describe the representation ring of a complex reductive group (See [FH, §23.2] and [BD, §II.7]). We form the representation ring  $Rep(G)$  by taking a free abelian group on the isomorphism classes  $[V]$  of finite dimensional representations  $V$ , modulo the relations  $[V] = [V'] + [V'']$  whenever  $V \simeq V' \oplus V''$ . Since  $G$  is reductive this is indeed a free abelian group on the classes of irreducible representations. The tensor product of representations turns this into a ring  $[V] \cdot [W] = [V \otimes W]$ . Elements such as  $[V] - [W]$  are called virtual representations, or virtual characters if we identify a representation with its character. We note that  $Rep(G \times H) \simeq Rep(G) \otimes Rep(H)$ . Recall that the  $k^{th}$  exterior power  $\bigwedge^k V$  of a representation has the following property

$$\bigwedge^k (V \oplus W) = \sum_{i+k=k} \bigwedge^i (V) \otimes \bigwedge^j (W)$$

Then the operators

$$\lambda^i : \text{Rep}(G) \rightarrow \text{Rep}(G)$$

$$[V] \rightarrow [\bigwedge^i V]$$

make  $\text{Rep}(G)$  into a special  $\lambda$ -ring. As we will see this structure will also hold for the subrings of polynomial representations for the classical groups.

Note that for a maximal torus  $T \subset G$ , the Weyl group  $W$  acts on  $T$  and thus on  $\text{Rep}(T)$ . The inclusion of the torus  $T \hookrightarrow G$  induces an isomorphism  $\text{Rep}(G) \simeq \text{Rep}(T)^W$ . If we consider the complexified representation ring  $\text{Rep}^{\mathbb{C}}(G)$  we have that

$$\text{Rep}^{\mathbb{C}}(G) \simeq \mathbb{C}[T]^W$$

. Given the representations  $V_{\omega_i}$  of highest weight  $\omega_i$  for the fundamental weights, we can also describe the representation ring and complex representation ring by  $\text{Rep}(G) = \bigoplus \mathbb{Z}V_{\omega_i}$  and  $\text{Rep}^{\mathbb{C}}(G) = \bigoplus \mathbb{C}V_{\omega_i}$  respectively.

### 2.1.1 Weyl Groups

Finally, we will collect some facts about the Weyl group which follow from the fact that it is also a Coxeter group (i.e.  $(W, S)$  is a Coxeter system). Again we note that  $W$  is generated by the simple reflections  $S = s_i$  associated to each simple root. These simple reflections obey several relations dependent on the root system. In particular they all obey

$$s_i^2 = 1$$

$$s_i s_j = s_j s_i \text{ if } |i - j| \geq 2$$

And for say  $G = SL(n, \mathbb{C})$ , where  $W = S_n$  the symmetric group we have the relation

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

. This allows us to define a length function  $l : W \rightarrow \mathbb{Z}_{\geq 0}$ , where  $l(w)$  is the smallest integer  $n$  such that  $w$  can be written as a product of  $n$  elements from  $S$ . Geometrically, the length of  $w$  is the cardinality of the set  $wR^+ \cap R^-$  (note the set of roots is invariant under the action of the Weyl group). A word, or decomposition,  $w = s_{i_1}s_{i_2}\dots s_{i_k}$  is said to be reduced if  $l(w) = k$ . A Weyl group has a unique longest element  $w_0$  where  $w_0R^+ = R^-$ . Then  $l(w_0w) = l(w_0) - l(w)$ . Note also that  $l(w^{-1}) = l(w)$ .

Any Coxeter system  $(W, S)$  admits a partial ordering  $\geq$  on  $W$  called the Bruhat order.

**Definition 2.1.** (Bruhat Order)

If  $w, w' \in W$  and there is a conjugate  $t$  of some  $s \in S$  such that  $w' = tw$  and  $l(w') = l(w) + 1$  then we say  $w'$  covers  $w$  (denoted  $w \rightarrow w'$ ). The Bruhat order is the transitive closure of  $\rightarrow$  (i.e.  $u < v$  with  $l(v) = l(u) + k$  then there is a sequence

$$u \rightarrow u_1 \rightarrow \dots \rightarrow u_{k-1} \rightarrow v$$

Note that if the Coxeter system  $(W, S)$  comes from a the Weyl group of some root system the the conjugate  $t$  of  $s$  corresponds to the reflection  $s_\beta$  for some positive root  $\beta \in R^+$ .

Consider a proper subset  $\theta \in S$ . The subgroup of  $W$  generated by  $\theta$  is called a parabolic subgroup, which we will denote  $W_\theta$ .  $(W_\theta, \theta)$  is itself a coxeter system. There is a natural, distinguished set of left coset representatives in  $W/W_\theta$  given by

$$W^\theta = \{w \in W : l(ws) = l(w) + 1, \text{ for all } s \in S\}$$

This gives the following decomposition [Hi, Chapter I, Section 5]

**Theorem 2.2.** *If  $w \in W, \theta \in S$ , then there is a unique expression  $w = w^\theta w_\theta$  with  $w^\theta \in W^\theta$  and  $w_\theta \in W_\theta$  with  $l(w) = l(w^\theta) + l(w_\theta)$ .*

As a corollary we see that the set  $W^\theta$  is precisely the set of minimal-length representatives in each coset  $wW_\theta$ .

### 2.1.2 Dynkin Index

We believe that for the  $\lambda$ -polynomials ring  $Rep_{\lambda}^{\mathbb{C}}(G)$  (to be defined in Chapter 4), the most appropriate weight to consider is the fundamental weight of minimum Dynkin index. For the classical groups of types  $A, B, C, D$  (that is  $GL_n, SL_n, SO_{2n+1}, Sp_{2n}, SO_{2n}$ ) this is just the defining representation. We define the Dynkin index and describe some of its properties here. This subsection follows [Ku3, §Appendix A]

**Definition 2.2.** Let  $f : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  be a Lie algebra homomorphism between finite-dimensional simple Lie algebras over  $\mathbb{C}$ . Then there exists a unique number  $d_f \in \mathbb{C}$  called the Dynkin index of  $f$ , satisfying

$$\langle f(x), f(y) \rangle = d_f \langle x, y \rangle, \text{ for all } x, y \in \mathfrak{g}_1$$

where  $\langle \cdot, \cdot \rangle$  is the nondegenerate, invariant, symmetric, bilinear form on  $\mathfrak{g}_i$  normalized so that  $\langle \theta_i, \theta_i \rangle = 2$  for the highest root  $\theta_i$  of  $\mathfrak{g}_i$ .

Note that if  $h : \mathfrak{g}_2 \rightarrow \mathfrak{g}_3$ , for  $\mathfrak{g}_3$  simple, then  $d_{f \circ h} = d_f \cdot d_h$ . Given a finite dimensional representation  $V$  of a simple Lie algebra  $\mathfrak{g}$ ,  $f_V : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ , we set

$$d_V = d_{f_V}$$

. Then for two representations  $V_1$  and  $V_2$ , taking their direct sum  $V_1 \oplus V_2$  we have

$$d_{V_1 \oplus V_2} = d_{V_1} + d_{V_2}$$

For their tensor product  $V_1 \otimes V_2$ , we have that

$$d_{V_1 \otimes V_2} = d_{V_1} \dim(V_2) + d_{V_2} \dim(V_1)$$

We have the following formula for the Dynkin index of an irreducible representation of highest weight  $\lambda$  of a simple Lie algebra [Ku3, Lemma A.2, A.3]

**Lemma 2.1.** *Let  $\mathfrak{g}$  be a simple Lie algebra and let  $V(\lambda)$  be an irreducible finite dimensional representation of  $\mathfrak{g}$  with highest weight  $\lambda$ . Then*

$$d_\lambda = d_{V(\lambda)} = \frac{\dim_{\mathbb{C}} V(\lambda)}{\dim_{\mathbb{C}} \mathfrak{g}} (\|\lambda + \rho\|^2 - \|\rho\|^2)$$

where  $\|\mu\|^2$  denotes  $(\mu, \mu)$  and  $2\rho$  represents the sum of all positive roots. Thus,  $d_\lambda$  is a strictly positive real number for any  $\lambda \neq 0$ . It is in fact true that  $d_\lambda$  is a positive integer.

**Lemma 2.2.** *Let  $\mathfrak{g}$  be a finite-dimensional simple Lie Algebra as before. Let  $V$  be a finite dimensional representation of  $\mathfrak{g}$  with its formal character given by*

$$ch V = \sum_{\lambda \in \mathfrak{t}^*} n_\lambda e^\lambda, \quad n_\lambda \in \mathbb{Z}$$

with  $\mathfrak{t} \subset \mathfrak{g}$  the Cartan subalgebra. Then,

$$d_V = \frac{1}{2} \sum_{\lambda} n_\lambda (\lambda(\check{\theta}^2))$$

where  $\check{\theta} \in \mathfrak{t}$  is the coroot associated to the highest root  $\theta$  of  $\mathfrak{g}$ .

Using Lemma 2.2 and the root data [?] one finds that the fundamental representations of minimal Dynkin index are the representations  $\omega_1$  of index 1 for the classical groups of types A,B,C, and D. For the exceptional groups we find that for  $G_2$  it is  $\omega_1$  (index 2), for  $F_4$  it is  $\omega_4$  (index 6), for  $E_6$  it is  $\omega_1$  or  $\omega_6$  (index 6), for  $E_7$  it is  $\omega_7$  (index 12), and for  $E_8$  it is  $\omega_8$  (index 60).

## 2.2 Bruhat Decomposition, Schubert varieties, and Parabolic Subgroups

Fix  $T \subset B \subset G$ . An arbitrary Borel group can be decomposed as  $B = T \rtimes U$ , where  $U$  is the set of unipotent elements contained in  $B$ . A choice of Borel  $B$  is equivalent to choice of positive roots  $R^+$ . The map  $exp$  restricted to  $\mathfrak{g}_\alpha$  is an isomorphism of varieties, and hence we write  $U_\alpha = exp(\mathfrak{g}_\alpha)$ . This is a

closed, one-dimensional subgroup of  $G$  isomorphic to  $\mathbb{C}$ . In particular,  $U \simeq \prod_{\alpha \in R^+} U_\alpha$ , with corresponding Lie algebra  $\mathfrak{u} = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$

For any Borel  $B$  there is always an opposite Borel  $B^-$  such that  $B \cap B^- = T$ . Then we can write  $B^- = T \rtimes U^-$ , where  $U^- \simeq \prod_{\alpha \in R^-} U_\alpha$ .  $U^-$  has Lie algebra  $\mathfrak{u}^- = \bigoplus_{\alpha \in R^-} \mathfrak{g}_\alpha$ . Define  $U_w^- = U \cap wU^-w^{-1}$  and its corresponding Lie algebra  $\mathfrak{u}_w^- = (Adw)\mathfrak{u}^- \cap \mathfrak{u} = \bigoplus_{\alpha \in wR^- \cap R^+} \mathfrak{g}_\alpha$ . Note that  $\mathfrak{u}_w^-$  is isomorphic as a variety to  $\mathbb{C}^{l(w)}$ . The following theorem [Borel, page 147] can be seen as a formal consequence of the fact that the data  $G, B, N(T), S$  form what is known as a BN-pair [Tits].

**Theorem 2.3.** (*Bruhat Decomposition*) *If  $G$  is a complex reductive group and  $T \subset B$ , then  $G$  is the disjoint union of double cosets  $BwB$ , i.e.*

$$G = \bigsqcup_{w \in W} BwB$$

. Further, there is an isomorphism of varieties  $U_w^- \times B \simeq BwB$ .

The homogenous space  $G/B$  is a projective variety. We have the following corollary of the Bruhat decomposition. The homogenous space  $G/B$  is a disjoint union of double cosets

$$G/B = \bigsqcup_{w \in W} BwB/B$$

Further, we have that  $BwB/B$  is a cell of complex dimension  $l(w)$  via the sequence of algebraic isomorphisms

$$\mathfrak{u}_w^- \xrightarrow{\exp} U_w^- \rightarrow U_w^- wB/B = UwB/B = BwB/B$$

We introduce the notation

$$C_w = BwB/B$$

This open affine variety, known as a Schubert Cell, is an orbit under the left translation action of  $B$  on  $G$ . We will denote the closure of  $C_w$  by  $X_w$ . This is known as a Schubert Variety. Note that  $X_w$  is also  $B$ -stable

and can thus be written as the disjoint union

$$X_w = \bigsqcup_{v \leq w} C_w$$

, where  $v, w \in W$  and  $v \leq w$  in the Bruhat order from the previous section. Use  $[X_w]$  to denote the the image of the fundamental class of  $X_w$  in the singular cohomology  $H_*(G/B)$ , where  $[X_w] \in H_{2l(w)}(G/B)$ . We will use  $PD(X_w)$  to denote the cohomology class of complementary dimension associated to  $X_w$  by Poincaré duality. The fact that  $G/B$  (also known as the flag variety) has a cellular decomposition with cells of only even real dimension has many consequences. In particular  $G/B$  is simply connected. Additionally, we have the following well know ([Reiner]):

**Theorem 2.4.** *The integral singular homology  $H_*(G/B)$  and cohomology  $H^*(G/B)$  are free  $\mathbb{Z}$  modules. They form dual lattices under the Kronecker pairing, having  $\mathbb{Z}$ -dual basis given by the cellular homology classes  $\{[X_w] : w \in W\}$  and their Kronecker dual cohomology classes denoted  $\{\epsilon_w : w \in W\}$ .*

Thus, for  $v, w \in W$  we have that

$$\langle \epsilon_w, X_v \rangle = \delta_{v,w}$$

where  $\langle \cdot, \cdot \rangle$  is the usual Kronecker pairing between homology and cohomology.

Note that  $\dim(G/B) = l(w_0)$ . Thus we have that  $\epsilon_w \in H^{2l(w)}(G/B)$  and  $PD(X_w) \in H^{2l(w_0)-2l(w)}(G/B)$ . The class  $PD(X_w)$  can be expressed in terms of the  $\{\epsilon_w : w \in W\}$  basis as follows [BGG]

**Theorem 2.5.** *For  $w \in W$  we have,*

$$PD(X_w) = \epsilon_{w_0 w}$$

In other words, for  $X_{w_0 w} = Bw_0 w B/B$  we have that  $PD(X_{w_0 w}) = \epsilon_w$

Now that we have a preferred basis for the cohomology ring  $H^*(G/B)$  we would like to describe the multiplication (i.e. cup product) with respect to this basis. That is, given  $v, w \in W$  we want a closed formula for the constants  $c_{vw}^u$  appearing in the decomposition of the product

$$\epsilon_v \cdot \epsilon_w = \sum_{u \in W} c_{vw}^u \epsilon_u$$

where  $\cdot$  is really the cup product. To this end we have the following Pieri-like formula due to Chevalley [BGG]

**Theorem 2.6.** (Chevalley Formula) For any  $w \in W$  and any simple root  $\alpha$ , with corresponding simple reflection  $s_\alpha$ , we have that

$$\epsilon_w \cdot \epsilon_{s_\alpha} = \sum (\omega_\alpha, \check{\beta}) \epsilon_{ws_\beta}$$

where the sum runs over the positive roots such that  $l(ws_\beta) = l(w) + 1$ .  $s_\beta$  is the reflection associated to a root given by  $s_\beta(\xi) = \xi - (\xi, \check{\beta})\beta$  for  $\xi \in \mathfrak{t}^*$ .

In the next section we will discuss a Giambelli-like formula due to [BGG]. Now we will discuss parabolic subgroups  $B \subset P$  and the generalized partial flag varieties  $G/P$ . A parabolic subgroup is any subgroup such that the quotient  $G/P$  can be realized as the orbit of the action of  $G$  on  $\mathbb{P}(V)$  for some representation  $V$  of  $P$ . In particular,  $G/P$  is a projective variety. Equivalently, parabolic subgroups are those subgroups that contain a conjugate of a Borel subgroup. So Borel subgroups are the minimal parabolic subgroups. Generally, we will fix a Borel and consider parabolics  $B \subset P$ . Parabolic subgroups are completely characterized by subsets of the simple roots  $\Delta$  up to conjugacy, and since the simple roots are in one-to-one correspondence to the vertices of the Dynkin diagram for  $G$  we have that parabolics are in one to one correspondence with subsets of vertices of the Dynkin diagram.

Consider a subset  $\theta \in S = \{\alpha_1, \dots, \alpha_n\}$ . Then we define the parabolic Lie algebra

$$\mathfrak{p}_\theta = \mathfrak{t} \oplus \bigoplus_{\beta \in T(\theta)} \mathfrak{g}_\beta$$

where  $T(\theta)$  is equal to the set of positive roots  $R^+$  and all roots generated by the negatives of  $\theta$ . Let  $P_\theta$  be the corresponding parabolic group. Note that  $P_\emptyset = B$  and  $P_S = G$ . Parabolic subgroups have a Levi decomposition  $P_\theta = L_\theta U_\theta$ , where  $U_\theta = \prod_{\alpha \in R^+ \setminus T(\theta)} U_\alpha$  is the unipotent radical of  $P_\theta$ , and  $L_\theta = \langle T, U_\alpha \mid \alpha \in R_\theta \rangle$  is a reductive group called the Levi subgroup and  $R_\theta$  is the root system generated by  $\theta$ . Let  $W_\theta$  be the Weyl group of  $L_\theta$ . If the parabolic  $P$  is understood this will sometimes be denoted  $W_P$ . Note that  $W_\emptyset$  is exactly the parabolic subgroup mentioned at the end of the previous section. We now have the following



Bruhat decompositions for  $G$  and  $G/P$

$$G = \bigsqcup_{w \in W^\theta} BwP_\theta$$

$$G/P = \bigsqcup_{w \in W^\theta} BwP_\theta/P_\theta$$

$W^\theta$  is the set of minimal length coset representatives in  $W/W_\theta$ . Geometrically,  $B$ -orbits in  $G/P$  are obtained by collapsing orbits in  $G/B$  if their  $w$ 's lie in the same  $W_\theta$  coset. Fix  $P_\theta = P$ . Again we have schubert cells  $C_w^P = BwP/P$  isomorphic to the affine space  $\mathbb{C}^{l(w)}$  for  $w \in W^P = W^\theta$ . Then the schubert variety  $X_w^P$ , fundamental class  $[X_w^P]$ , Poincare and Kronecker duals  $PD(X_w^P)$ ,  $\epsilon_w^P$  and all defined analogously as for  $G/B$ . These cohomology classes are related in the next theorem [Ku1, chapter 11].

**Theorem 2.7.** *Let  $\pi_P : G/B \rightarrow G/P$  be the natural projection. Then the induced map  $\pi_P^* : H^*(G/P) \rightarrow H^*(G/B)$  is injective with image equal to  $H^*(G/B)^{W_P}$ , the  $W_P$  invariants. In particular, for  $w \in W^P$ , we have*

$$\pi_P^*(\epsilon_w^P) = \epsilon_w$$

.

Let  $K \subset G$  be a maximal compact subgroups of  $G$ , with maximal compact torus  $T = K \cap H$ . Then there is a homoemorphism  $K/T \simeq G/B$ , and further we can identify  $W = N(H)/H = N(T)/T$ .  $W$  acts on  $K$  by conjugation and this action preserves  $T$  so there is an action of  $W$  on  $K/T$ . This induces an action of  $W$  on the homology and cohomology of  $K/T$  and hence on  $G/B$ .

So we can identify  $H^*(G/P)$  with a subring of  $H^*(G/B)$  and drop the  $P$  superscript from  $\epsilon_w^P$  when it is understood. Note that the action of  $W$  on  $H^*(G/B)$  is induced from the action of  $W$  on  $K/T = G/B$ . Recall that every element  $w \in W$  can be decomposed as  $w = w^\theta w_\theta$  where  $w_\theta \in W_\theta$ ,  $w^\theta \in W^\theta$ , and  $l(w) = l(w^\theta) + l(w_\theta)$ . Applying this to  $w_0$  we write  $w_0 = w^{0,P} w_{0,P}$  for  $P_\theta = P$ . Then  $w_{0,P}$  is the longest element of  $W_P$  and  $w^{0,P}$  is the longest minimal-length representative in  $W^P$ . Then Poincare duality between  $H_*(G/P)$  and  $H^*(G/P)$  is given by the following theorem (see [KLM, 2.1])

**Theorem 2.8.** Let  $w \in W^P$ . Define the involutive map  $\theta^P : W \rightarrow W$  by  $\theta^P(w) = w_0 w w_{0,P}$ . The  $\theta^P$  carries  $W^P$  into itself and we have

$$PD(X_w^P) = \epsilon_{w_0 w w_{0,P}}$$

The proof relies on the following key lemma. Let  $a \cdot b$  denote the intersection pairing on  $H_*(G/B)$  with  $a \in H_k(G/B)$  and  $b \in H_{l(w_0,P)-k}(G/P)$ . Then we have that

$$\langle PD(a), b \rangle = a \cdot v$$

**Lemma 2.3.** For  $v, w \in W^P$  with  $l(w) = l(v)$ ,

$$X_w^P \cdot X_{\theta^P(v)}^P = \delta_{v,w}$$

### 2.3 Borel Characteristic map and BGG-operators

The discussion above may be referred to as the Schubert picture of cohomology [Hi]. In this section we will discuss another point of view called the Borel picture of cohomology and we will discuss the results and formalism of Bernstein, Gelfand, Gelfand [BGG] and Demazure [D2] to connect the two pictures. Good resources for this material are the original papers [BGG], [D2], [Hi, chapter IV], [KLM, sections 2,3], [FP, appendix E], and [P4].

Let  $X$  be a variety that  $B$  acts freely on from the right such that the quotient  $X/B$  exists and the projection  $p : X \rightarrow X/B$  is a principal  $B$ -bundle. Let  $\rho : B \rightarrow GL(V)$  be a representation of  $B$ . Then consider the complex vector bundle  $\mathcal{L}_\rho = X \times^B V$  given by taking the quotient of  $X \times V$  by the relation

$$(x, v) \sim (xb, \rho(b)^{-1}v)$$

for  $x \in X$ ,  $b \in B$ ,  $v \in V$ . In particular let  $\lambda : B \rightarrow \mathbb{C}^*$  be a character of  $B$ , and let  $\mathcal{L}_\lambda$  be the complex line bundle described above. First define  $\beta : X(B) \rightarrow H^2(X/B)$  by taking a character of  $B$  to the first Chern class of the associated line bundle, i.e.  $\beta(\lambda) = c_1(\mathcal{L}_\lambda)$ . This can be extended symmetrically to a

homomorphism of graded rings (doubling degrees)

$$\beta : S(X(B)) \rightarrow H^*(X/B, \mathbb{Z})$$

where  $S^*(X(B))$  is the symmetric algebra of  $X(B)$ . This map is called the characteristic map of the fiber bundle  $p : X/B$ .

For our purposes, we let  $B$  act on  $G$  on the right and the fiber bundle under consideration is  $p : G \rightarrow G/B$ . Now, consider  $\mathfrak{t}^* = \text{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})$ . So  $\mathfrak{t}^*$  is just the characters of the Cartan subalgebra. Let us assume that  $G$  is simply connected. Then any  $\chi \in \mathfrak{t}^*$  lifts to a character  $\chi : T \rightarrow \mathbb{C}^*$  by  $(\exp(t)) = \exp(\chi(t))$  for  $t \in \mathfrak{t}$ . This character can be further extended to  $B = TU$  by setting  $\chi|_U = 1$  (and indeed the character group of  $B$  and  $T$  are euivalent under this identification). Then as above we can define a map  $\beta : \mathfrak{t}^* \rightarrow H^2(G/B)$  by lifting a character  $\chi$  to  $B$  and taking the first chern class of the associated complex line bundle  $\mathcal{L}_\chi$  over  $G/B$ . Again, we extend this symmetrically to obtain

$$\beta : S(\mathfrak{t}^*) \rightarrow H^*(G/B, \mathbb{Z})$$

This is known as the Borel characteristic map [Bo]. If we consider  $\beta \otimes \mathbb{C}$  then the map is surjective and has kernel  $J$  generated by the  $W$ -invariants with no constant term. So, letting  $S = S(\mathfrak{t}^*)$  and taking complex coefficients, we have an isomorphism  $\beta : S/J \rightarrow H^*(G/B, \mathbb{C})$ . Note that  $\beta$  commutes with the action of  $W$  on  $S$  and on  $H^*(G/B)$  [BGG, proposition 1.3(i)]. Also, since  $W$  acts as a finite complex reflection group on  $\mathfrak{t}^*$ , then by Chevalley's theorem the  $W$ -invariants  $S(\mathfrak{t}^*)^W$  are a polynomial subalgebra  $\mathbb{C}[f_1, \dots, f_n]$  where  $n$  is the rank of  $G$  [Hi, chapter II, section 3]. Thus, under Borel's presentation we see that  $H^*(G/B)$  is a complete intersection ring with  $n$  generators and as many relations.

Let  $P \subset G$  be a parabolic. Then we also have an isomorphism of graded rings as follows. Let  $S^{W_P}$  be the set of  $W_P$  invariants under the action of  $W_P$  on  $\mathfrak{t}^*$ . Then if we restrict we have

$$\beta : S^{W_P} \rightarrow H^*(G/P) \simeq H^*(G/B)^{W_P}$$

again with  $(S/J)^{W_P} \simeq H^*(G/P)$ .

In their seminal paper [BGG] Bernstein, Gelfand, Gelfand developed a connection between the Schubert and Borel pictures of the cohomology of  $H^*(G/B)$ . In particular they give polynomials  $p_w \in S^{l(w)}(\mathfrak{t}^*) \text{ mod } J$  such that

$$\beta(p_w) = \epsilon_w \in H^{2l(w)}(G/B)$$

The key algebraic operator used in this work is the following

**Definition 2.3.** For each root  $\alpha \in R$  define a divided difference operator  $A_\alpha : S^k(\mathfrak{t}^*) \rightarrow S^{k-1}(\mathfrak{t}^*)$  by

$$A_\alpha(f) = \frac{f - s_\alpha f}{\alpha_i}$$

These operators are also known as BGG or Demazure operators in the literature. We collect some properties of  $A_{s_i}$  in the following omnibus lemma [BGG, Lemma 3.3].

**Lemma 2.4.** Let  $\alpha \in S$  and  $w \in W$ . Let  $f, g \in S(\mathfrak{t}^*)$

- (i)  $A_\alpha^2 = 0$
- (ii)  $A_{-\alpha} = -A_\alpha$
- (iii)  $wA_\alpha w^{-1} = A_{w\alpha}$
- (iv)  $s_\alpha A_\alpha = A_\alpha$
- (v)  $s_\alpha = 1 - \alpha A_\alpha$
- (vi)  $A_\alpha(f) = 0 \leftrightarrow s_\alpha f = f$
- (vii)  $A_\alpha(fg) = A_\alpha(f)g + (s_\alpha f)A_\alpha(g)$
- (viii)  $A_\alpha J \subset J$

By (viii) above we see that  $A_\alpha$  induces an operator on  $S/J$ . For any  $w \in W$  we further define

$$A_w := A_{s_{\alpha_1}} \circ \cdots \circ A_{s_{\alpha_k}}$$

where  $w = s_{\alpha_1} \cdots s_{\alpha_k}$  is a reduced decomposition for  $w$ . Then we have that [Hi, Chapter IV, Proposition 1.7]

**Proposition 2.1.** *The operators  $A_w$  are well defined, i.e. they do not depend on the choice of reduced decomposition for  $w$ . Further, we have that  $A_w \circ A_v = A_{wv}$  if  $l(wv) = l(w) + l(v)$  and  $A_w \circ A_v = 0$  otherwise.*

The Borel characteristic map, the Schubert classes and the  $BGG$  operators are all related by the following equation [BGG, section 4].

**Proposition 2.2.** *Let  $\beta : S(\mathfrak{t}^*) \rightarrow H^*(G/B)$ . For  $f \in S^k$*

$$\beta(f) = \sum_{l(w)=k} A_w(f)\epsilon_w$$

*The above equation is valid for partial flag varieties as well if we restrict the summation to the set  $\{w \in W^P : l(w) = k\}$ .*

There is an analogue of the  $BGG$  operator  $D_{s_i}$  on  $H^*(G/B)$  which commutes with the Borel characteristic map, i.e. for  $f \in S$  we have that  $\beta(A_{s_i}f) = D_{s_i}\beta(f)$ . Hence we will just use  $A_w$  to refer to the  $BGG$  operator on both  $S$  and  $H^*(G/B)$ . For an explicit description of the geometric operator  $D_{s_i}$  see [KLM, section 3.3]. Then we have the following description of the action of  $A_w$  on the Schubert classes  $\epsilon_v$  [BGG, Theorem 3.14]

**Theorem 2.9.** *For  $v, w \in W$  such that  $l(vw^{-1}) = l(v) - l(w)$ , we have that*

$$A_w\epsilon_v = \epsilon_{vw^{-1}}$$

*and equals 0 otherwise.*

We also give the following formula for the Weyl group action on a Schubert class. For a simple root  $\alpha$  and  $w \in W$

$$s_\alpha\epsilon_w = \epsilon_w \quad \text{if } l(ws_\alpha) = l(w) + 1$$

$$s_\alpha\epsilon_w = -\epsilon_w - \sum (\alpha, \tilde{\gamma})\epsilon_{ws_\alpha s_\gamma} \quad \text{if } l(ws_\alpha) = l(w) - 1$$

where the sum is over all positive roots  $\gamma \neq \alpha$  such that  $l(ws_\alpha s_\gamma) = l(w)$ .

Now define elements  $p_w \in S$  as follows. Starting with the longest element  $w_0$  we let  $p_{w_0} = \frac{1}{|W|} \prod_{\gamma \in R^+} \gamma$ . Then for arbitrary  $w \in W$  recursively define  $p_w$  by  $p_w = A_{w^{-1}w_0} p_{w_0}$ . Then the main result of [BGG] is

**Theorem 2.10.**

$$\beta(p_w) = \epsilon_w$$

where really we are taking  $p_w \bmod J$  in  $S/J$ .

These are polynomial representatives in  $S$  of the Schubert classes. Lascoux and Schurzenburger [?] introduced another set of polynomial representatives in type  $A$ , called Schubert polynomials, which enjoy many nice combinatorial properties. These are obtained by applying divided difference operators to the monomial  $x_1^{n-1} x_2^{n-2} \dots x_{n-1}$  which represents the top class (here  $x_1, \dots, x_n$  are the coordinates of the standard representation of  $A_{n-1}$ ). There are natural analogues for the other classical types such as the Schubert polynomials of Billey and Haiman [BH], the theta and eta polynomials of Buch, Kresch, and Tamvakis [BKT1, BKT2].

We also note that the Chevalley formula (Theorem 2.6) is a partial solution to the general Littlewood-Richardson problem of describing the coefficients on the expansion

$$\epsilon_w \epsilon_v = \sum_{u \in W} c_{wv}^u \epsilon_w^u$$

These represent geometric intersections of the varieties  $X_{w_0u}$ ,  $X_{w_0v}$  and  $X_u$  and thus must be positive. We now give a brief description of these coefficients due to Pragacz [P3, P4]. Then by combining the above expansion with Theorem 2.9 we see that

$$c_{wv}^u = A_u(\epsilon_w \cdot \epsilon_v)$$

Now suppose that  $l(w) = k$  and that  $l(v) = l$ . Let  $u = s_{\alpha_1} \dots s_{\alpha_{k+l}}$  be a reduced decomposition. Then by iterating 2.2 (vii) we have

$$c_{wv}^u = A_{\alpha_1} \circ \dots \circ A_{\alpha_{k+l}}(\epsilon_w \cdot \epsilon_v) = \sum A_I(\epsilon_w) \cdot A_{\alpha}^I(\epsilon_v)$$

where the sum is over all subsequences  $I = (i_1, \dots, i_k) \subset [1, \dots, k+l]$  and  $A_I = A_{\alpha_{i_1}} \circ \dots \circ A_{\alpha_{i_k}}$  and  $A_\alpha^I$  is obtained by taking  $A_{\alpha_1} \circ \dots \circ A_{\alpha_{k+l}}$  and replacing each  $A_{\alpha_i}$  by  $s_{\alpha_i}$  for all  $i \in I$ . Then by Theorem 2.9 we can deduce that

$$c_{wv}^u = \sum A_\alpha^I(\epsilon_v)$$

where the sum is over all subsequences  $I$  such that  $s_{\alpha_{i_1}} \dots s_{\alpha_{i_k}}$  is a reduced decomposition for  $w$ . The Chevalley formula can then be derived from this rule.

## CHAPTER 3

### The Springer Morphism

We will first briefly set the notation for this chapter, primarily in §3.2

Let  $G$  be a simply-connected semi-simple algebraic group over  $\mathbb{C}$  (though the constructions of this §3.1 are valid in the more general case of a connected reductive complex group). Denote its Lie algebra  $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha} \mathfrak{g}_{\alpha}$  of rank  $n$ , and fixed base of simple roots  $\Delta = \{\alpha_j\}$ . Take the set of simple co-roots  $\check{\Delta} = \{\check{\alpha}_j\}$

as a basis for the Cartan subalgebra  $\mathfrak{t} \subset \mathfrak{g}$ . Then  $\mathfrak{t}_{\mathbb{Z}} = \bigoplus_{j=1}^n \mathbb{Z}\check{\alpha}_j$  is the co-root lattice. Further, the weight

lattice is  $\mathfrak{t}_{\mathbb{Z}}^* = \bigoplus_{i=1}^n \mathbb{Z}\omega_i$ , where  $\omega_i \in \mathfrak{t}^*$  is the  $i^{\text{th}}$  fundamental weight of  $\mathfrak{g}$  defined by  $\omega_i(\check{\alpha}_j) = \delta_{ij}$ . Then the

maximal torus  $T \subset G$  (with Lie algebra  $\mathfrak{t}$ ) can be identified with  $T = \text{Hom}_{\mathbb{Z}}(\mathfrak{t}_{\mathbb{Z}}^*, \mathbb{C}^*)$  as in [Sp]. Finally, let

$W$  be the Weyl group of  $G$ , generated by the simple reflections  $s_i$ . So for  $\mu \in \mathfrak{t}^*$ ,  $s_i(\mu) = \mu - \mu(\check{\alpha}_i)\alpha_i$ .

Let  $V_{\lambda}$  be the irreducible representation of  $G$  with highest weight  $\lambda$ . Then  $V_{\lambda}$  has weight space decomposition

$$V_{\lambda} = \bigoplus V_{\lambda}^{\mu}$$

where  $V_{\lambda}^{\mu} = \{v \in V_{\lambda} \mid t.v = ((\mu_1\omega_1 + \dots + \mu_n\omega_n)(t))v \ \forall v \in V_{\lambda}\}$  is the weight space with weight  $\mu = \mu_1\omega_1 + \dots + \mu_n\omega_n$ .

So for  $t \in T$  and  $v \in V_{\mu_1, \mu_2, \dots, \mu_n}$  we have that the action of  $t$  on  $v$  is given by

$$t.v = t(\mu_1, \dots, \mu_n)v = e^{\mu}(t)v$$

where  $(\mu_1, \dots, \mu_n) = \mu_1\omega_1 + \dots + \mu_n\omega_n$ . Additionally  $\check{\alpha}_j \in \mathfrak{t}$  acts on  $v$  by

$$\check{\alpha}_j.v = (\mu_1\omega_1 + \dots + \mu_n\omega_n)(\check{\alpha}_j)v = \mu_j v.$$



A representation  $\rho : G \rightarrow GL(V)$  is called almost faithful if it has finite kernel, i.e. the induced representation  $d\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  is injective.

### 3.1 Definition and Properties

In the literature this construction is also known as the Generalized Cayley Transform. Some references for this material are [BR, KM]. The following construction is in fact a special case of what [LPR1, §10] call a generalized Cayley map which is any dominant algebraic morphism  $G \rightarrow \mathfrak{g}$ .

Given a connected reductive group  $G$ , its Lie algebra  $\mathfrak{g}$ , and an almost faithful representation  $V_\lambda$ , the Springer morphism is a map

$$\theta_\lambda : G \rightarrow \mathfrak{g}$$

given by

$$\begin{array}{ccc} G & \longrightarrow & \text{Aut}(V(\lambda)) \subset \text{End}(V(\lambda)) = \mathfrak{g} \oplus \mathfrak{g}^\perp \\ & \searrow \theta_\lambda & \downarrow \pi_\lambda \\ & & \mathfrak{g} \end{array}$$

where  $\mathfrak{g}$  sits canonically inside  $\text{End}(V_\lambda)$  via the derivative  $d\rho_\lambda$ , the orthogonal complement  $\mathfrak{g}^\perp$  is taken via the adjoint invariant form  $\langle A, B \rangle = \text{tr}(AB)$  on  $\text{End}(V_\lambda)$ , and  $\pi$  is the projection onto the  $\mathfrak{g}$  component. So  $\theta_\lambda = \pi_\lambda \circ \rho_\lambda$ . By considering a compact form  $K \subset G$ , it is easy to see that the restriction of trace form to  $d\rho_\lambda(\mathfrak{g})$  is non degenerate and thus  $\mathfrak{g} \cap \mathfrak{g}^\perp = \{0\}$ . Note, that since  $\pi \circ d\rho_\lambda$  is the identity map,  $\theta_\lambda$  is a local diffeomorphism at 1, and hence has Zariski dense image. By construction,  $\theta_\lambda$  is an algebraic morphism.

Let  $d\theta_\lambda = \pi_\lambda \circ T\theta_\lambda : TG \rightarrow T\mathfrak{g} = \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  denote the differential of  $\theta_\lambda$ , so that  $d\theta_\lambda(g) = \pi_\lambda \circ T_g G \rightarrow \mathfrak{g}$ . We let  $X_1, \dots, X_n$  be a linear basis of  $\mathfrak{g}$  and let  $L_{X_1}, \dots, L_{X_n}$  denote the corresponding left-invariant vector fields on  $G$ . Let

$$\Psi_\lambda(g) = \det(d\theta_\lambda(g))$$

be the Springer determinant (or Cayley Determinant) for the representation  $V_\lambda$ . Note that  $\Psi_\lambda$  does not depend on choice of basis for  $\mathfrak{g}$ . We list the following basic properties of  $\theta_\lambda$  and  $\Psi_\lambda(g)$ .

**Theorem 3.1.** *Let  $G$  be a connected, reductive, complex algebraic group and let  $\theta_\lambda$  be the Springer morphism, where  $V_\lambda$  is an almost faithful representation. Let  $T \subset G$  be a maximal torus and let  $B_\lambda$  be the restriction of the inner product  $\langle A, B \rangle = \text{tr}(AB)$  on  $d\theta_\lambda(\mathfrak{g}) \subset \text{End}(V)$ . Then,*

1.  $\theta_\lambda \circ \text{Conj}_b = \text{Ad}_b \circ \theta_\lambda$
2.  $\theta_\lambda|_T : T \rightarrow \mathfrak{t}$
3.  $\Psi_\lambda$  is invariant under conjugation.
4.  $d\theta_\lambda(e) : \mathfrak{g} \rightarrow \mathfrak{g}$  is the identity mapping. So  $d\theta_\lambda(g)$  is invertible for  $g$  in the non-empty Zariski open dense subset  $\{h \in G : \theta_\lambda(h) \neq 0\}$  and is not invertible on the hypersurface  $\{h \in G : \theta_\lambda(h) = 0\}$
5. Let  $\chi_\lambda$  be the character of  $\rho_\lambda$ , i.e.  $\chi_\lambda(g) = \text{tr}(\rho_\lambda(g))$ . Then  $d\chi_\lambda(g)(T_e(\mu_g)X) = \text{tr}(d\rho(\theta_\lambda(g))d\rho_\lambda(X)) = B_\lambda(\theta_\lambda(g), X)$
6. The differential  $d\theta_\lambda(g).T_e(\mu_g).X \in \mathfrak{g}$  is given by the implicit equation  $\text{tr}(d\rho_\lambda(d\theta_\lambda(g)T_e(\mu_g)X)d\rho_\lambda(Y)) = \text{tr}(\theta_\lambda(g)d\rho_\lambda(X)d\rho_\lambda(Y))$  for  $Y \in \mathfrak{g}$
7. If  $\theta_\lambda(e) = 0$  and  $a \in G$  is such that  $\rho_\lambda(a) \in d\rho_\lambda(\mathfrak{g})$  then  $d\theta_\lambda(a^{-1})$  is not invertible.

*Proof.* We give a proof of (2) because of its importance to the rest of the paper. Let  $t \in T$ . We then write

$$\theta_\lambda(t) = h + \sum_{\alpha \in R} x_\alpha, \text{ for } h \in \mathfrak{t}, \text{ and } x_\alpha \in \mathfrak{g}_\alpha$$

Then by conjugation invariance (see (1) which follows from the invariance of trace) we have

$$\theta_\lambda(t) = \theta_\lambda(sts^{-1}) = h + \sum_{\alpha \in R} \text{Ad}_s(x_\alpha) \text{ for any } s \in T$$

Thus we see that  $x_\alpha = 0$ , i.e.,  $\theta_\lambda(t) \in \mathfrak{t}$  □

**Example 3.1.** The Springer morphism  $\theta_\lambda : G \rightarrow \mathfrak{g}$ , in general, indeed depends upon the choice of  $\lambda$ . For example, the Springer morphism  $\theta_{\omega_1} : Sl_2 \rightarrow \mathfrak{sl}_2$  restricted to the diagonal torus can easily be seen to be

$$\theta_{\omega_1} \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} \frac{z-z^{-1}}{2} & 0 \\ 0 & -\frac{z-z^{-1}}{2} \end{pmatrix}.$$

On the other hand, the Springer morphism  $\theta_{2\omega_1} : \mathbf{SL}_2 \rightarrow \mathfrak{sl}_2$  restricted to the diagonal torus is given by

$$\theta_{2\omega_1} \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} \frac{z^2-z^{-2}}{4} & 0 \\ 0 & -\frac{z^2-z^{-2}}{4} \end{pmatrix}.$$

◇

We also record the following theorems from Kostant and Michor [KM, 2.7,2.8]

**Theorem 3.2.** *Let  $G$  be semisimple and let  $\rho : G \rightarrow GL(V)$  be an almost faithful representation. Let  $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$  be the decomposition of  $\mathfrak{g}$  into simple ideals  $\mathfrak{g}_i$ . Let  $G_i, \dots, G_k$  be the corresponding connected subgroups of  $G$ . Then we have that*

$$\theta_\rho|_{G_i} = \theta_{\rho|_{G_i}}, \text{ for } i = 1, \dots, k$$

**Theorem 3.3.** *Let  $G$  be a simple algebraic group and let  $\rho_i : G \rightarrow GL(V_i)$  be non-trivial representations for  $i = 1, 2$ . The inner product  $B_{\rho_i}$  on  $\mathfrak{g}$  is a multiple of the Cartan Killing form  $B$  on  $\mathfrak{g}$ , so we write  $B_{\rho_i} = j_{\rho_i} B$ .*

*Then we have*

1. *For the direct sum representation  $\rho_1 \oplus \rho_2 : G \rightarrow GL(V_1 \oplus V_2)$  we have*

$$\theta_{\rho_1 \oplus \rho_2}(g) = \frac{j_{\rho_1}}{j_{\rho_1 \oplus \rho_2}} \theta_{\rho_1}(g) + \frac{j_{\rho_2}}{j_{\rho_1 \oplus \rho_2}} \theta_{\rho_2}(g) \in \mathfrak{g}$$

2. For the tensor representation  $\rho_1 \otimes \rho_2 : G \rightarrow GL(V_1 \otimes V_2)$  we have

$$\theta_{\rho_1 \otimes \rho_2}(g) = \frac{j_{\rho_1} \chi_{\rho_2}}{j_{\rho_1 \otimes \rho_2}} \theta_{\rho_1}(g) + \frac{j_{\rho_2} \chi_{\rho_1}}{j_{\rho_1 \otimes \rho_2}} \theta_{\rho_2}(g) \in \mathfrak{g}$$

3. For the  $n$ -fold tensor product representation  $\otimes^n \rho : G \rightarrow GL(\otimes^n V)$  we have

$$\theta_{\otimes^n \rho}(g) = \left( \frac{\chi_\rho(g)}{\dim(V)} \right)^{n-1} \theta_\rho(g)$$

4. For the contragredient representation  $\rho^T : G \rightarrow GL(V^*)$  given by  $\rho^T(g) = \rho(g^{-1})^T$  we have

$$\theta_{\rho^T}(g) = -\theta_\rho(g^{-1})$$

Where, for a complex simple algebraic group,  $j_{\rho_i}$  is a multiple of the Dynkin Index of the representation  $\rho_i$ . It is non-negative and satisfies

$$j_{\rho_1 \oplus \rho_2} = j_{\rho_1} + j_{\rho_2},$$

$$j_{\rho_1 \otimes \rho_2} = \dim(V_2)j_{\rho_1} + \dim(V_1)j_{\rho_2},$$

$$j_{\rho_\lambda} = \frac{\dim(V_\lambda)}{\dim(\mathfrak{g})} B(\lambda, \lambda + \rho)$$

where in the last line  $\rho$  is the half sum of all positive roots.

One motivation for studying such maps comes from a result of Springer which states that the Unipotent variety  $U \subset G$  of unipotent elements is isomorphic as an algebraic variety to the nilcone  $N \subset \mathfrak{g}$  of nilpotent elements in the lie algebra. Bardsley and Richardson [BR] used Springer morphisms, even in finite characteristic for good primes, to give examples of such isomorphisms. Kostant and Michor [KM, 4.5] then consider the complex case and generalize this to reductive algebraic groups to show

**Theorem 3.4.** *Let  $a \in G$  be regular and assume that  $d\theta_\lambda(s)$  is invertible. Then  $\theta_\lambda$  restricts to an isomorphism of affine varieties*

$$\theta_\lambda : \overline{\text{Conj}_G(a)} \rightarrow \overline{\text{Ad}_G(\theta_\lambda(a))}$$

Additionally, the Springer morphisms preserve the Jordan decomposition. Recall that any element  $a \in G$  has a multiplicative Jordan decomposition  $a = a_s a_u$ , where  $a_s$  and  $a_u$  are semisimple and unipotent elements. Similarly, for any  $X \in \mathfrak{g}$  we have that  $X = X_s + X_n$ , where  $X_s$  and  $X_n$  are semisimple and nilpotent elements respectively. Then we have [KM, 4.11] that  $\theta_\lambda(a_s) = \theta_\lambda(a)_s$  and  $\theta_\lambda(a_u) = \theta_\lambda(a)_u$ .

Finally we also want to consider the degree of the map  $\theta_\lambda$ . To that end we have the following theorems from [KM, 2.9,3.3] and [LPR2, Corrolary 2]

**Theorem 3.5.** *For the Springer morphism  $\theta_\lambda$  the induced mapping  $\theta_\lambda^* : \mathbb{C}[\mathfrak{g}] \rightarrow \mathbb{C}[G]$  between the algebra of regular functions is injective, equivariant, and maps the subalgebras of invariant regular functions to each other,  $\theta_\lambda^* : \mathbb{C}[\mathfrak{g}]^G \rightarrow \mathbb{C}[G]^G$ . Thus,  $\theta_\lambda : G \rightarrow \mathfrak{g}$  is a dominant algebraic morphism. By the algebraic Peter-Weyl theorem we have that  $\mathbb{C}[G] = \bigoplus_{\mu \in D} \mathbb{C}[G]_\mu$  where  $D$  is the set of dominant integral weights, and where*

$$\mathbb{C}[G]_\mu = \{f \in \mathbb{C}[G] : f(g) = \text{tr}(\rho_\mu(g)B) \text{ for some } B \in \text{End}(V_\mu)\}$$

For an irreducible representation  $\rho_\lambda$  we thus have  $\theta_\lambda^*(\mathfrak{g}^*) \in \mathbb{C}[G]_\lambda$ .

Finally, we have the following result about the degrees of Springer morphisms.

**Theorem 3.6.** *For a Springer morphism  $\theta_\lambda$  of a connected reductive group  $G$ . Then,*

$$\text{deg } \theta_\lambda = [Q(G) : Q(\mathfrak{g})] = [Q(G)^G : Q(\mathfrak{g})^G] = [Q(T)^W : Q(\mathfrak{t})^W]$$

We hope then that the results of the next section could help determine the degree of a Springer map for any semi-simple group.

### 3.2 An Explicit Determination of the Springer Morphism

Let  $V_\lambda$  be a  $d$  dimensional almost faithful irreducible representation of  $G$  of highest weight  $\lambda$ . Let  $\Lambda_\lambda = \{(\mu_1^i, \dots, \mu_n^i)\}_{i=1}^d$  be an enumeration of the set of weights considered with their multiplicity that appear in the weight space decomposition of  $V_\lambda$  (so  $\mu_j^i$  is the coordinate of the  $j^{\text{th}}$  fundamental weight for the  $i^{\text{th}}$  weight in the decomposition) Then we can take a basis of weight vectors  $\{v_{\mu_1^i, \dots, \mu_n^i}\}_{i=1}^d$  on which the torus  $T$  and hence each simple co-root acts diagonally. Thus,

$$\rho_\lambda(t) = \text{diag}\{e^{\mu^1}(t), \dots, e^{\mu^d}(t)\} \in \text{Aut}(V_\lambda)$$

and for a simple co-root  $\check{\alpha}_j$  we have that

$$d\rho_\lambda(\check{\alpha}_j) = \text{diag}\{\mu_j^1, \dots, \mu_j^d\} \in \text{End}(V_\lambda).$$

In order to compute the projection to  $\mathfrak{g} \in \text{End}(V_\lambda) \simeq \mathfrak{g} \oplus \mathfrak{g}^\perp$  we calculate  $d\rho_\lambda(\mathfrak{g})^\perp \in \text{End}(V_\lambda)$  with respect to the symmetric bilinear form  $\text{tr}(AB)$ . Recall that  $d\rho_\lambda$  is faithful so we identify  $\mathfrak{g}$  with its image under  $d\rho_\lambda$ . Let  $X = (x_{ij}) \in \text{End}(V_\lambda)$ . Then for  $X$  to be contained in  $d\rho_\lambda(\mathfrak{g})^\perp$  it follows that

$$\text{tr}(d\rho_\lambda(\check{\alpha}_j) \cdot X) = 0 \implies \sum_{i=1}^d \mu_j^i x_{ii} = 0$$

for all co-roots,  $\check{\alpha}_j \in \mathfrak{t}$ .

So  $\sum_{\mu \in \Lambda_\lambda} \mu_1^i x_{ii} = \sum_{\mu \in \Lambda_\lambda} \mu_2^i x_{ii} = \dots = \sum_{\mu \in \Lambda_\lambda} \mu_n^i x_{ii} = 0$ . Now to project  $\rho_\lambda(t)$  onto  $d\rho_\lambda(\mathfrak{t})$  we write  $\rho_\lambda$  as a sum

$$\rho_\lambda(t) = \sum_{j=1}^n c_j(t) d\rho_\lambda(\check{\alpha}_j) + X(t).$$

where  $c_j : T \mapsto \mathbb{C}$  is a function that depends on  $\lambda$ , and  $X(t) \in d\rho_\lambda(\mathfrak{g})^\perp$ . It follows then that

$$\theta_\lambda(t) = \sum c_j(t) \check{\alpha}_j$$

We aim to solve for the coefficients  $c_j(t)$ . Note that for the root space  $\mathfrak{g}_\alpha$ , we have that  $\mathfrak{g}_\alpha \cdot V_\mu \subset V_{\mu+\alpha}$ . Thus,  $d\rho_\lambda(e_\alpha)$  for  $e_\alpha \in \mathfrak{g}_\alpha$  will only have off diagonal entries, and as such the condition  $\text{tr}(d\rho_\lambda(e_\alpha) \cdot X) = 0$  will only add constraints to the off diagonal entries of  $X \in d\rho_\lambda(\mathfrak{g})^\perp$ . As the action of  $t$  and  $\check{\alpha}_j$  are both diagonal, by comparing coordinates we have the following set of  $d$  equations

$$e^{\mu^1}(t) = c_1(t) \mu_1^1 + \dots + c_n(t) \mu_n^1 + x_{11}(t)$$

$$e^{\mu^2}(t) = c_1(t)\mu_1^2 + \dots + c_n(t)\mu_n^2 + x_{22}(t)$$

⋮

$$e^{\mu^d}(t) = c_1(t)\mu_1^d + \dots + c_n(t)\mu_n^d + x_{dd}(t).$$

This can be reduced to  $n$  equations by utilizing the fact that  $\sum_{i=1}^d \mu_j^i x_{ii} = 0$ , as follows. Multiply each equation above by  $\mu_1^i$  and sum (then repeat with  $\mu_2^i, \dots, \mu_n^i$ )

$$\sum_{i=1}^d \mu_1^i e^{(\mu_1^i, \dots, \mu_n^i)}(t) = \sum_{i=1}^d (\mu_1^i)^2 c_1(t) + \sum_{i=1}^d \mu_1^i \mu_2^i c_2(t) + \dots + \sum_{i=1}^d \mu_1^i \mu_n^i c_n(t)$$

⋮

$$\sum_{i=1}^d \mu_n^i e^{(\mu_1^i, \dots, \mu_n^i)}(t) = \sum_{i=1}^d \mu_1^i \mu_n^i c_1(t) + \sum_{i=1}^d \mu_2^i \mu_n^i c_2(t) + \dots + \sum_{i=1}^d (\mu_n^i)^2 c_n(t)$$

More concisely this can be written as

$$\begin{pmatrix} \sum_{\mu \in \Lambda_\lambda} \mu_1 e^\mu(t) \\ \vdots \\ \sum_{\mu \in \Lambda_\lambda} \mu_n e^\mu(t) \end{pmatrix} = S(G, \lambda) \begin{pmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_n(t) \end{pmatrix}$$

where

$$S(G, \lambda) := \begin{pmatrix} \sum_{\mu \in \Lambda_\lambda} \mu_1^2 & \sum_{\mu \in \Lambda_\lambda} \mu_1 \mu_2 & \dots & \sum_{\mu \in \Lambda_\lambda} \mu_1 \mu_n \\ \sum_{\mu \in \Lambda_\lambda} \mu_1 \mu_2 & \sum_{\mu \in \Lambda_\lambda} \mu_2^2 & \dots & \sum_{\mu \in \Lambda_\lambda} \mu_2 \mu_n \\ \vdots & \ddots & & \vdots \\ \sum_{\mu \in \Lambda_\lambda} \mu_1 \mu_n & \dots & \sum_{\mu \in \Lambda_\lambda} \mu_{n-1} \mu_n & \sum_{\mu \in \Lambda_\lambda} \mu_n^2 \end{pmatrix}$$

In the next section we will show that  $S(G, \lambda)$  is a multiple of a symmetrization of the Cartan matrix for  $G$ , and is thus invertible. So, we have that

$$\begin{pmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_n(t) \end{pmatrix} = S^{-1}(G, \lambda) \begin{pmatrix} \sum_{\mu \in \Lambda_\lambda} \mu_1 e^\mu(t) \\ \vdots \\ \sum_{\mu \in \Lambda_\lambda} \mu_n e^\mu(t) \end{pmatrix}$$

We calculate the matrix  $S(G, \lambda)$  for the classical and exceptional simple algebraic groups. In the following sections, we continue the notation

$$\Lambda_\lambda = \{(\mu_1, \dots, \mu_n) \mid \mu_1 \omega_1 + \dots + \mu_n \omega_n \text{ is a weight of } V_\lambda\}$$

counted with multiplicity.

Our main result will be calculating the matrix  $S(G, \lambda)$  as defined in section 3, for the simple algebraic groups. We use the convention that the Cartan matrix associated to the root system of  $\mathfrak{g}$  is  $A = (A_{ij})$ , where  $A_{ij} = \alpha_i(\check{\alpha}_j)$ . Then  $A$  is a change-of-basis matrix for  $\mathfrak{t}^*$  between the fundamental weights and the simple roots. Furthermore,  $A$  satisfies the following properties

- For diagonal entries  $A_{ii} = 2$
- For non-diagonal entries  $A_{ij} \leq 0$
- $A_{ij} = 0$  iff  $A_{ji} = 0$
- $A$  can be written as  $DS$ , where  $D$  is a diagonal matrix, and  $S$  is a symmetric matrix.

Let  $D$  be the diagonal matrix defined by  $D_{ij} = \frac{\delta_{ij}}{2}(\alpha_i, \alpha_j)$ , where if we realize the root system  $R$  associated to  $\mathfrak{g}$  as a set of vectors in a Euclidean space  $E$ , then  $(\cdot, \cdot)$  is the standard inner product. In this framework we can write  $A_{ij} = \alpha_i(\check{\alpha}_j) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$ . Then, writing  $A = DS$ , we find that the matrix  $S$  has coordinate entries given by



$$S_{ij} = \frac{4(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)}$$

and is clearly symmetric.

$(\cdot, \cdot)$  is an invariant bilinear form on  $\mathfrak{t}^*$ , normalized so that so that  $(\alpha_i, \alpha_i) = 2$  where  $\alpha_i$  is the highest root. Note that under this formulation, if  $G$  is of simply-laced type then  $D$  is the identity matrix and  $S$  is the Cartan matrix. We find that in general for a given simple group  $G$  that  $S(G, \lambda)$  is a multiple of  $S$ . Before stating our result precisely we fix the following notation. If  $\alpha_j$  is any long simple root (for the simply laced case  $\alpha_j$  can be any simple root), consider the corresponding fundamental weight  $\omega_j$ . Let  $x_j(\lambda) := \sum_{\mu \in \Lambda_\lambda} \mu_j^2$ ,

where  $\mu_j$  is the  $j^{\text{th}}$  coordinate of the weight  $\mu \in \Lambda_\lambda$  in the fundamental weight basis.

**Proposition 3.1.** *Let  $G$  be a simple algebraic group. Let  $S(G, \lambda)$  be defined as in section 3. Set  $x_j(\lambda) := \sum_{\mu \in \Lambda_\lambda} \mu_j^2$  for a long root  $\alpha_j$ . Let  $S$  be a symmetrization of the Cartan matrix as above. Then  $S(G, \lambda)$  is a multiple of  $S$ . More precisely,*

$$S(G, \lambda) = \frac{1}{2} x_j(\lambda) \cdot S$$

and this is independent of the choice of long root  $\alpha_j$ .

*Proof.* The proof will rely on the fact that the set of weights  $\Lambda_\lambda$  of  $V_\lambda$  is invariant under the action of the Weyl group on  $\mathfrak{t}^*$ , i.e. for  $w \in W$ ,  $w \cdot \Lambda_\lambda = \Lambda_\lambda$ . The following Lemma is true for all simple groups. The following two lemmas are sufficient to prove the simply-laced case but also hold for the non-simply laced cases.

**Lemma 3.1.** *For a given simple group  $G$ , if the Cartan matrix entry  $A_{ij} = 0$ , i.e the nodes representing the simple roots  $\alpha_i$  and  $\alpha_j$  are not connected on the associated Dynkin diagram, then*

$$\sum_{\mu \in \Lambda_\lambda} \mu_i \mu_j = 0,$$

where  $\mu = (\mu_1, \dots, \mu_n)$ .

*Proof.* Consider the simple reflection  $s_i$  acting on a weight  $\mu = (\mu_1, \dots, \mu_n) \in \Lambda_\lambda$ . Then

$$s_i(\mu) = (\mu_1, \dots, \mu_n) - ((\mu_1, \dots, \mu_n)(\check{\alpha}_i))(\alpha_i)$$

where  $(\mu_1, \dots, \mu_n)(\check{\alpha}_i) = (\mu_1\omega_1 + \dots + \mu_n\omega_n)(\check{\alpha}_i) = \mu_i$ . Using the Cartan matrix to write the simple roots  $\alpha_i$  in the fundamental weight basis gives  $\alpha_i = (A_{i,1}, \dots, A_{i,n})$ . Then the above reflection yields

$$s_i(\mu) = (\mu_1, \dots, \mu_n) - \mu_i(A_{i,1}, \dots, A_{i,n}) = (\mu_1 - \mu_i A_{i,1}, \dots, \mu_n - \mu_i A_{i,n})$$

Now note that  $A_{ii} = 2$  and  $A_{ij} = 0$ . So the  $i^{\text{th}}$  coordinate of  $s_i(\mu)$  is  $[s_i(\mu)]_i = \mu_i - \mu_i A_{ii} = -\mu_i$  and the  $j^{\text{th}}$  coordinate of  $s_i(\mu)$  is  $[s_i(\mu)]_j = \mu_j - \mu_i A_{ij} = \mu_j$ . Thus we find that

$$\sum_{\mu \in \Lambda_\lambda} \mu_i \mu_j = \sum_{s_i(\mu) \in \Lambda_\lambda} \mu_i \mu_j = \sum_{\mu \in \Lambda_\lambda} [s_i(\mu)]_i \cdot [s_i(\mu)]_j = \sum_{\mu \in \Lambda_\lambda} -\mu_i \mu_j,$$

by invariance of  $\Lambda_\lambda$  under  $s_i$ . Thus, the result follows.  $\square$

**Lemma 3.2.** *If simple roots  $\alpha_i$  and  $\alpha_j$  of  $G$  are connected via the Dynkin diagram and have the same length then*

$$\sum_{\mu \in \Lambda_\lambda} \mu_i^2 = \sum_{\mu \in \Lambda_\lambda} \mu_j^2.$$

Furthermore,

$$\sum_{\mu \in \Lambda_\lambda} \mu_i \mu_j = -\frac{1}{2} \sum_{\mu \in \Lambda_\lambda} \mu_i^2$$

*Proof.* We have that  $A_{ij} = A_{ji} = -1$ . Then as above with  $\mu = (\mu_1, \dots, \mu_n) \in \Lambda_\lambda$ , we have that  $s_i(\mu) = (\mu_1 - \mu_i A_{i,1}, \dots, \mu_n - \mu_i A_{i,n})$ . Now consider

$$s_j s_i(\mu) = ((\mu_1 - \mu_i A_{i,1}) - (\mu_j - \mu_i A_{ij}) A_{j,1}, \dots, (\mu_n - \mu_i A_{i,n}) - (\mu_j - \mu_i A_{ij}) A_{j,n})$$

Thus,  $[s_j s_i(\mu)]_i = (\mu_i - \mu_i A_{ii}) - (\mu_j - \mu_i A_{ij}) A_{ji} = -\mu_i - (\mu_j + \mu_i)(-1) = \mu_j$ . Thus,

$$\sum_{\mu \in \Lambda_\lambda} \mu_i \mu_i = \sum_{\mu \in \Lambda_\lambda} [s_j s_i(\mu)]_i \cdot [s_j s_i(\mu)]_i = \sum_{\mu \in \Lambda_\lambda} \mu_j \mu_j$$

The second part of the lemma follows from the fact that  $[s_i(\mu)]_j = \mu_j - \mu_i A_{ij}$  with  $A_{ij} = -1$ . It follows that

$$\sum_{\mu \in \Lambda_\lambda} \mu_j^2 = \sum_{\mu \in \Lambda_\lambda} [s_i(\mu)]_j^2 = \sum_{\mu \in \Lambda_\lambda} (\mu_j + \mu_i)^2$$

Thus,  $\sum_{\mu \in \Lambda_\lambda} \mu_i \mu_i = -2 \sum_{\mu \in \Lambda_\lambda} \mu_i \mu_j$  □

Recall that the root systems of simple groups of type  $B_n, C_n, G_2, F_4$  contain long and short simple roots. Our convention will be the same as in [Bou]. That is, for  $B_n$  that  $\alpha_1, \dots, \alpha_{n-1}$  are the long roots and  $\alpha_n$  is short, for  $C_n$  that  $\alpha_1, \dots, \alpha_{n-1}$  are short and  $\alpha_n$  is long, for  $G_2$  that  $\alpha_1$  is short and  $\alpha_2$  is long, and for  $F_4$  that the first and second are long and that the third and fourth are short.

**Proposition 3.2.** *Let  $G$  be a rank  $n$  simple group of types  $B_n, C_n,$  or  $F_4$ . For any long root  $\alpha_j$ , set  $x_j(\lambda) = \sum_{\mu \in \Lambda_\lambda} \mu_j^2$ . If  $\alpha_i$  is a short root, then  $\sum_{\mu \in \Lambda_\lambda} \mu_i^2 = 2x_j(\lambda)$ . If either or both of  $\alpha_i$  and  $\alpha_j$  are short, then  $\sum_{\mu \in \Lambda_\lambda} \mu_i \mu_j = -x_j(\lambda)$*

*Proof.* Note that if  $\alpha_i$  and  $\alpha_j$  are both long roots, connected via the Dynkin diagram, then  $A_{ij} = A_{ji} = -1$

So Lemma 4.3 shows that

$$\sum_{\mu \in \Lambda_\lambda} \mu_i^2 = \sum_{\mu \in \Lambda_\lambda} \mu_j^2,$$

and that  $\sum_{\mu \in \Lambda_\lambda} \mu_i \mu_j = -\frac{1}{2} \sum_{\mu \in \Lambda_\lambda} \mu_i^2$ . The same is true for the short roots as  $A_{ij} = A_{ji} = -1$  for connected short roots. So we need to show that if  $\alpha_i$  and  $\alpha_j$  are short and long roots respectively and connected via the Dynkin diagram, then  $\sum_{\mu \in \Lambda_\lambda} \mu_i^2 = 2x_j(\lambda)$ , and that  $\sum_{\mu \in \Lambda_\lambda} \mu_i \mu_j = -x_j(\lambda)$ . To show this we first note that  $A_{ij} = -1$  and  $A_{ji} = -2$  and then compare  $[s_i(\mu)]_i, [s_j(\mu)]_j, [s_j(\mu)]_i$  and  $[s_i(\mu)]_j$ . Note that  $[s_i(\mu)]_i = -\mu_i$  and  $s_j(\mu_j) = -\mu_j$  as before. Also,  $[s_i(\mu)]_j = \mu_j - \mu_i A_{ij} = \mu_j + \mu_i$  and  $[s_j(\mu)]_i = \mu_i - \mu_j A_{ji} = \mu_i + 2\mu_j$ . Thus, we have that

$$\sum_{\mu \in \Lambda_\lambda} \mu_i \mu_j = \sum_{\mu \in \Lambda_\lambda} [s_j(\mu)]_i \cdot [s_j(\mu)]_j = \sum_{\mu \in \Lambda_\lambda} (\mu_i + 2\mu_j)(-\mu_j) = \sum_{\mu \in \Lambda_\lambda} -\mu_i \mu_j - 2\mu_j^2$$

Thus  $\sum_{\mu \in \Lambda_\lambda} \mu_i \mu_j = -\sum_{\mu \in \Lambda_\lambda} \mu_j^2 = -x_j(\lambda)$ . Applying,  $s_i$  to  $\mu$  gives

$$\sum_{\mu \in \Lambda_\lambda} \mu_i \mu_j = \sum_{\mu \in \Lambda_\lambda} [s_i(\mu)]_i \cdot [s_i(\mu)]_j = \sum_{\mu \in \Lambda_\lambda} -\mu_i \mu_j - \mu_i^2$$

Thus,  $\sum_{\mu \in \Lambda_\lambda} \mu_i^2 = 2x_j(\lambda)$  □

So it follows that with  $x_j(\lambda) = \sum_{\mu \in \Lambda_\lambda} \mu_j^2$ , where  $\alpha_j$  is a long root, then

$$S(B_n, \lambda) = \frac{x_j(\lambda)}{2} \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & \ddots & & & \\ & & & 2 & -1 & \\ & & & -1 & 2 & -2 \\ & & & & -2 & 4 \end{pmatrix}, S(C_n, \lambda) = \frac{x_j(\lambda)}{2} \begin{pmatrix} 4 & -2 & & & & \\ -2 & 4 & -2 & & & \\ & -2 & \ddots & & & \\ & & & 4 & -2 & \\ & & & -2 & 4 & -2 \\ & & & & -2 & 2 \end{pmatrix}$$

$$S(F_4, \lambda) = \frac{x_j(\lambda)}{2} \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 4 \end{pmatrix}$$

We give inverses of these matrices in the next section.

Let  $\alpha_1$  be the short root, and  $\alpha_2$  the long root of  $G_2$ .

**Proposition 3.3.**  $\sum_{\mu \in \Lambda_\lambda} \mu_1^2 = -2 \sum_{\mu \in \Lambda_\lambda} \mu_1 \mu_2 = 3 \sum_{\mu \in \Lambda_\lambda} \mu_2^2$

*Proof.* Let  $\mu = (\mu_1, \mu_2) \in \Lambda_\lambda$ . Then since  $A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ , we find that  $s_1(\mu) = (-\mu_1, \mu_1 + \mu_2)$  and

that  $s_2(\mu) = (\mu_1 + 3\mu_2, -\mu_2)$ . So,

$$\sum_{\mu \in \Lambda_\lambda} \mu_1^2 = \sum_{\mu \in \Lambda_\lambda} (\mu_1 + 3\mu_2)^2$$

from which it follows that  $\sum_{\mu \in \Lambda_\lambda} \mu_1 \mu_2 = -\frac{3}{2} \sum_{\mu \in \Lambda_\lambda} \mu_2^2$ . Additionally, we have that

$$\sum_{\mu \in \Lambda_\lambda} \mu_2^2 = \sum_{\mu \in \Lambda_\lambda} (\mu_1 + \mu_2)^2$$

from which we can see that  $\sum_{\mu \in \Lambda_\lambda} \mu_1^2 = -2 \sum_{\mu \in \Lambda_\lambda} \mu_1 \mu_2 = 3 \sum_{\mu \in \Lambda_\lambda} \mu_2^2$ . Thus,

$$S(G_2, \lambda) = \frac{1}{2} \sum_{\mu \in \Lambda_\lambda} \mu_2^2 \begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}$$

□

In particular, we can solve for  $c_1(t)$  and  $c_2(t)$  as

$$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = S(G_2, \lambda)^{-1} \begin{pmatrix} \sum_{\mu \in \Lambda_\lambda} \mu_1 e^{\mu(t)} \\ \sum_{\mu \in \Lambda_\lambda} \mu_2 e^{\mu(t)} \end{pmatrix}$$

then, letting  $x = \sum_{\mu \in \Lambda_\lambda} \mu_2^2$  we have that  $S^{-1}(G, \lambda) = \frac{2}{3x} \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$ . Thus,

$$c_1(t, \lambda) = \frac{2}{3x} \sum_{\mu \in \Lambda_\lambda} (2\mu_1 + 3\mu_2) e^\mu(t)$$

$$c_2(t, \lambda) = \frac{2}{3x} \sum_{\mu \in \Lambda_\lambda} (3\mu_1 + 6\mu_2) e^\mu(t)$$

.

### 3.2.1 Examples

Consider  $G = Sp(2n, \mathbb{C}) = \{A \in GL(2n) \mid M = A^t M A\}$  where  $M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$  where  $I_n$  is the

$n \times n$  identity matrix, and  $\mathfrak{sp}(2n, \mathbb{C}) = \{X \in \mathfrak{gl}(2n) \mid X^t M + M X = 0\}$ .

Let  $\lambda = \omega_1$ , the defining representation. Then we have that  $\Lambda_\lambda = \{\pm\omega_1 \text{ and } \pm(\omega_i - \omega_{i+1}) \text{ for } 1 \leq i \leq n-1\}$ . So,  $x = \sum_{\Lambda_\lambda} \mu_n^2 = 2$ . Let  $T = \text{diag}\{t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}\}$ . The simple roots are  $\alpha_i = \epsilon_i - \epsilon_{i+1}$

for  $1 \leq i \leq n-1$  and  $\alpha_n = 2\epsilon_n$ . The simple coroots in  $\mathfrak{t}$  are then  $\check{\alpha}_i = E_i - E_{i+1} - E_{n+i} + E_{n+i+1}$  for  $1 \leq i \leq n-1$  and  $\check{\alpha}_n = E_n - E_{2n}$  where  $E_i$  is the diagonal matrix with a 1 in the  $i^{\text{th}}$  slot and 0's elsewhere [FH]. In the orthogonal basis for  $\mathfrak{t}$ ,  $\omega_i = \epsilon_1 + \dots + \epsilon_i$ . Thus, the character  $e^\mu(t)$  is given by

$e^\mu(t) = t_1^{\mu_1 + \dots + \mu_n} \cdot t_2^{\mu_2 + \dots + \mu_n} \cdot \dots \cdot t_n^{\mu_n}$ . Then, we have that

$$\begin{pmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & \dots & 3 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 3 & \dots & n \end{pmatrix} \begin{pmatrix} t_1 - t_1^{-1} - t_2 + t_2^{-1} \\ t_2 - t_2^{-1} - t_3 + t_3^{-1} \\ \vdots \\ t_{n-1} - t_{n-1}^{-1} - t_n + t_n^{-1} \\ t_n - t_n^{-1} \end{pmatrix}$$

which gives

$$\begin{pmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} t_1 - t_1^{-1} \\ \vdots \\ t_{n-1} - t_{n-1}^{-1} \\ t_1 - t_1^{-1} + \dots + t_n - t_n^{-1} \end{pmatrix}$$

Thus,

$$\theta_\lambda(t) = c_1(t)\check{\alpha}_1 + \dots + c_n(t)\check{\alpha}_n = \text{diag}\left(\frac{t_1 - t_1^{-1}}{2}, \dots, \frac{t_n - t_n^{-1}}{2}, -\frac{t_1 - t_1^{-1}}{2}, \dots, -\frac{t_n - t_n^{-1}}{2}\right).$$

Note that this is equivalent to the Cayley transform as in §6 of [Ku2]. Similar results hold for  $\theta_{\omega_1}(t)$  for the standard maximal tori of  $SO(2n+1, \mathbb{C})$  and  $SO(2n, \mathbb{C})$ .  $\square$

The inverses of the Cartan matrices for  $A_n, D_n, E_6, E_7, E_8$  respectively have the form (as in [Ro])

$$\frac{1}{n+1} \begin{pmatrix} n & n-1 & n-2 & \dots & 3 & 2 & 1 \\ n-1 & 2(n-1) & 2(n-3) & \dots & 6 & 4 & 2 \\ n-2 & 2(n-2) & 3(n-2) & \dots & 9 & 6 & 3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 2 & 4 & 6 & \dots & (2n-2) & 2(n-1) & n-1 \\ 1 & 2 & 3 & \dots & n-2 & n-1 & n \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & 2 & \dots & 2 & 1 & 1 \\ 1 & 2 & 3 & \dots & 3 & \frac{3}{2} & \frac{3}{2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 3 & \dots & n-2 & \frac{n-2}{2} & \frac{n-2}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} & \dots & \frac{n-2}{2} & \frac{n}{4} & \frac{n-2}{4} \\ \frac{1}{2} & 1 & \frac{3}{2} & \dots & \frac{n-2}{2} & \frac{n-2}{4} & \frac{n}{4} \end{pmatrix}$$

$$\begin{pmatrix} \frac{4}{3} & 1 & \frac{5}{3} & 2 & \frac{4}{3} & \frac{2}{3} \\ 1 & 2 & 2 & 3 & 2 & 1 \\ \frac{5}{3} & 2 & \frac{10}{3} & 4 & \frac{8}{3} & \frac{4}{3} \\ 2 & 3 & 4 & 6 & 4 & 2 \\ \frac{4}{3} & 2 & \frac{8}{3} & 4 & \frac{10}{3} & \frac{5}{3} \\ \frac{2}{3} & 1 & \frac{4}{3} & 2 & \frac{5}{3} & \frac{4}{3} \end{pmatrix}, \begin{pmatrix} 2 & 2 & 3 & 4 & 3 & 2 & 1 \\ 2 & \frac{2}{2} & 4 & 6 & \frac{9}{2} & 3 & \frac{3}{2} \\ 3 & 4 & 6 & 8 & 6 & 4 & 2 \\ 4 & 6 & 8 & 12 & 9 & 6 & 3 \\ 3 & \frac{9}{2} & 6 & 9 & \frac{15}{2} & 5 & \frac{5}{2} \\ 2 & 3 & 4 & 6 & 5 & 4 & 2 \\ 1 & \frac{3}{2} & 2 & 3 & \frac{5}{2} & 2 & \frac{3}{2} \end{pmatrix}, \begin{pmatrix} 4 & 5 & 7 & 10 & 8 & 6 & 4 & 2 \\ 5 & 8 & 10 & 15 & 12 & 9 & 6 & 3 \\ 7 & 10 & 14 & 20 & 16 & 12 & 8 & 4 \\ 10 & 15 & 20 & 30 & 24 & 18 & 12 & 6 \\ 8 & 12 & 16 & 24 & 20 & 15 & 10 & 5 \\ 6 & 9 & 12 & 18 & 15 & 12 & 8 & 4 \\ 4 & 6 & 8 & 12 & 10 & 8 & 6 & 3 \\ 2 & 3 & 4 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}$$

The inverse of the matrix  $S$  for types  $C_n, B_n, G_2, F_4$  have the form

$$\frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & \dots & 3 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 3 & \dots & n \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 2 & 2 & 2 & \dots & 2 & 1 \\ 2 & 4 & 4 & \dots & 4 & 2 \\ 2 & 4 & 6 & \dots & 6 & 3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 2 & 4 & 6 & \dots & 2(n-1) & n-1 \\ 1 & 2 & 3 & \dots & n-1 & 2 \end{pmatrix}, \begin{pmatrix} \frac{2}{3} & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 3 & \frac{3}{2} \\ 1 & 2 & \frac{3}{2} & 1 \end{pmatrix}$$



## CHAPTER 4

### Representation Ring of Levi Subgroups vs Cohomology Ring of Flag Varieties

#### 4.1 Classical Result and Polynomials Invariants

Let  $Gr(r, n)$  be the Grassmanian of  $r$ -planes in  $\mathbb{C}^n$ . Then a classical result states that the tensor product of irreducible polynomial representations of the general linear group  $GL(r)$  over  $\mathbb{C}$  corresponds in a certain sense to the cup product in the cohomology of the flag manifold,  $H^*(Gr(r, n), \mathbb{Z})$ .

Note that the Lie group  $GL(r)$  is contained in its Lie algebra  $\mathfrak{gl}(r) = M_{r \times r}$ .

**Definition 4.1.** An irrep  $V(\lambda)$  of  $GL(r)$  is called a polynomial rep if its character lifts to a character on the Lie algebra  $\mathfrak{gl}(r)$

$$\begin{array}{ccc} GL(r) & \xrightarrow{\chi_\lambda} & \mathbb{C} \\ \downarrow i & \nearrow & \\ \mathfrak{gl}(r) & & \end{array}$$

Alternately, a finite dimensional representation  $\rho : GL(r) \rightarrow GL(V)$  is said to be polynomial if there exists a basis of  $V$  such that entries of  $\rho(g)$  are polynomials in the matrix entries of  $g$ . Every irreducible polynomial representation of  $GL(r)$  is indexed by a partition (its highest weight)

$$\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0\}$$

such that the action of the torus is given by

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_r \end{pmatrix} \rightarrow t_1^{\lambda_1} \dots t_r^{\lambda_r}$$

Note that  $Gr(r, n) = GL(r)/P_r$  where  $P_r$  is a maximal parabolic subgroups containing the the standard upper triangular Borel subgroup  $B \subset P_r \subset GL(r)$ .  $P_r$  is taken by deleting the  $r^{th}$  node of the Dynkin diagram for  $GL(r)$  (or in the language of Chapter 2,  $P_r = P_\theta$  with  $\theta = \Delta - \{\alpha_r\}$ ). Then we have the following Bruhat decomposition

$$\bigsqcup_{w \in W_G/W_{P_r}} BwP_r/P_r$$

where  $W_G = S_n$  and  $W_{P_r} = S_r \times S_{n-r-1}$  and  $W_G/W_{P_r} = W^\theta$  is the following set of length  $r$  subsequences of  $[n]$ ,  $S(r, n) = \{A : 1 \leq a_1 \leq a_2 < \dots < a_r \leq n\}$ . Any such tuple represents the permutation

$$\nu_A = (a_1, \dots, a_r, a_{r+1}, \dots, a_n), \quad i \mapsto a_i$$

Then we have that

$$H^*(Gr(r, n), \mathbb{Z}) = \bigoplus_{A \in S(r, n)} \mathbb{Z}_{\epsilon_{\nu(A)}^{P_r}}$$

where  $\epsilon_{\nu(A)}^{P_r} \in H^{2l(\nu(A))}(Gr(r, n))$ . This leads to the classical result that

**Theorem 4.1.** *The following map  $\xi$  is a surjective ring homomorphism*

$$\xi : Rep_{poly}(Gl(r)) \rightarrow H^*(Gr(r, n), \mathbb{Z})$$

where

$$[V(\lambda)] \rightarrow \epsilon_{\nu(A)}^{P_r} \text{ if } \lambda_1 \leq n - r$$

$$\rightarrow 0 \text{ otherwise}$$

and  $A(\lambda) = \{1 + \lambda_r < 2 + \lambda_{r-1} < \dots < r + \lambda_1\}$  is a surjective homomorphism.

The work [Ku2] (on which this thesis is largely based) aimed at generalizing the above classical result to the larger context of the Levi subgroups of reductive groups and the cohomology of the corresponding partial flag varieties. For another attempt at generalization, see [?]. We will use the Borel characteristic map of §2.3 and the Springer morphism of the previous chapter to do so in the next section. A historical difficulty for extending the above result to other classical types is that it is not clear how to define polynomial

representations for other groups. The polynomial ring of invariants  $S(\mathfrak{t}^*)^W$  for a Weyl group will serve as the model for the polynomial representations of a group with said Weyl group. We now give some basic facts about the ring  $S(\mathfrak{t}^*)^W$  and examples for the Weyl groups of simple groups.

### 4.1.1 Weyl Group Invariants

More generally, let  $G$  be a group acting linearly on a vector space  $V$ . If  $\mathbb{C}[V]$  is the space of polynomial functions on  $V$ , then there is an induced action of  $G$  on  $\mathbb{C}[V]$  given by  $(g \cdot f)(x) = f(g^{-1}(x))$ . Classical invariant theory was concerned itself with the structure of the space of invariant polynomials  $\mathbb{C}[V]^G = \{f \in \mathbb{C}[V] \mid g \cdot f = f \forall g \in G\}$ , particularly finiteness results [Hu]. For example, Hilbert and Noether showed that the ring of invariants is a finitely generated  $\mathbb{C}$ -algebra. A theorem of Chevalley-Shepard-Todd showed that the ring of invariants is a polynomial ring if and only if  $G$  is a complex reflection group. Furthermore the degrees of the generators are unique. As Weyl groups are complex reflection groups, their ring of invariants  $S(\mathfrak{t}^*)^W$  is a polynomial ring on  $\text{rank}(\mathfrak{t})$  generators. The degrees  $d_i$  of these generators are listed below.

Type	Degrees
$A_n$	2,3,...,n+1
$B_n$	2,4,6,...,2n
$C_n$	2,4,6,...,2n
$D_n$	2,4,6,...,2n-2,n
$G_2$	2,6
$F_4$	2,6,8,12
$E_6$	2,5,6,8,9,12
$E_7$	2,6,8,10,12,14,18
$E_8$	2,8,12,14,18,20,24,30

Table 4.1: Degrees of Basic Invariants

In particular, we also have that  $\prod_{i=1}^n d_i = |W|$  and  $\sum_{i=1}^n (d_i - 1)$  is the number of reflections. We can now describe the well-known polynomial invariants for the classical groups. For examples for the exceptional groups see [Lee, Me, Ts].

*Type  $A_n$*  : It is convenient to work in  $\mathbb{C}^{n+1}$  restricted to the hyperplane  $x_1 + \dots + x_{n+1} = 0$ . Then  $W_{A_n} = S_{n+1}$  acts on  $\mathbb{C}[x_1, \dots, x_{n+1}]$  by permuting the variables. Recall [Hu] that the simple roots are given by  $\Delta = \{e_i - e_{i+1} \mid i = 1, \dots, n\}$ . Then the simple reflections  $s_i$  act by permuting  $x_i$  with  $x_{i+1}$ . Then

we have the following set of basic invariants

$$f_i = e_i(x_1, \dots, x_{n+1})$$

for  $i = 2, 3, \dots, n + 1$ , where  $e_i$  is the  $i^{\text{th}}$  elementary symmetric polynomial (Note that  $e_1(x_1, \dots, x_{n+1}) = x_1 + \dots + x_{n+1} = 0$ ).

*Type  $B_n$  and  $C_n$*  : Note that  $C_n$  and  $B_n$  have the same Weyl Group. The simple roots of type  $B_n$  are  $\Delta = \{e_i - e_{i+1} \mid i = 1, \dots, n\} \cup \{e_n\}$ . So the simple reflections  $s_i$  act by permuting  $x_i$  and  $x_{i+1}$  and  $s_n$  acts by taking  $x_n$  to  $-x_n$ . In particular, the Weyl group  $W_{B_n} \simeq S_n \times \mathbb{Z}_2$  is the hyperoctahedral group. We have the following set of basic invariants

$$f_i = e_i(x_1^2, \dots, x_n^2)$$

for  $i = 1, \dots, n$ .

*Type  $D_n$*  The simple roots of type  $D_n$  are given by  $\Delta = \{e_i - e_{i+1} \mid i = 1, \dots, n - 1\} \cup \{e_{n-1} + e_n\}$ . The first  $n - 1$  simple reflections act as before and  $s_n$  acts by permuting  $x_{n-1}$  and  $x_n$  and changing their sign. The Weyl group  $W_{D_n}$  is the subgroup of  $W_{B_n}$  of elements with an even number of sign changes. We have the following set of basic invariants.

$$f_i = e_i(x_1^2, \dots, x_n^2)$$

for  $i = 1, \dots, n - 1$  and

$$f_n = e_n(x_1, \dots, x_n) = x_1 \dots x_n$$

## 4.2 Main Result

We are now ready to state the main result of [Ku2]. Let  $G$  be a connected reductive algebraic group over  $\mathbb{C}$  and  $P$  a standard parabolic subgroup with Levi subgroup  $L$  containing the chosen maximal torus  $T$ . Let  $W_L$  be the Weyl group of  $L$ .

Recall the surjective Borel morphism from §2.3,

$$S(\mathfrak{t}^*) \rightarrow H^*(G/B, \mathbb{C})$$

which takes a character  $\mu \in X(T)$  to the first Chern class of the line bundle  $\mathcal{L}(\mu)$ . We can realize  $X(T)$  as a lattice in  $\mathfrak{t}^*$  via taking derivative.  $W_L$  acts on both  $S(\mathfrak{t}^*)$  and  $H^*(G/B, \mathbb{C})$ , and restricting we get a surjective graded algebra homomorphism:

$$\beta^P : S(\mathfrak{t}^*)^{W_L} \rightarrow H^*(G/B, \mathbb{C})^{W_L} \simeq H^*(G/P, \mathbb{C}),$$

. where the last isomorphism is induced from the projection  $G/B \rightarrow G/P$ .

Take an almost faithful  $G$ -module  $V_\lambda$ . Let  $\theta_\lambda : G \rightarrow \mathfrak{g}$  be the associated Springer morphism from §3. Restricting  $\theta_{\lambda|T} : T \rightarrow \mathfrak{t}$  induces the corresponding  $W$ -equivariant injective algebra homomorphism on the affine coordinate rings:

$$\theta_{\lambda|T}^* : \mathbb{C}[\mathfrak{t}] = S(\mathfrak{t}^*) \rightarrow \mathbb{C}[T]$$

So, restricting to  $W_L$  invariants we get the following injective algebra homomorphism:

$$\theta_{\lambda|T}(P)^* : \mathbb{C}[\mathfrak{t}]^{W_L} = S(\mathfrak{t}^*)^{W_L} \rightarrow \mathbb{C}[T]^{W_L}$$

Now we let  $Rep(L)$  be the representation ring of  $L$  and let  $Rep^{\mathbb{C}}(L) = Rep(L) \otimes \mathbb{C}$  be its complexified representation ring. Then, recall from §2.1 that  $Rep^{\mathbb{C}}(L) \simeq \mathbb{C}[T]^{W_L}$  obtained by taking the character of an  $L$ -module restricted to  $T$ . Note again that a representation  $V$  of  $L$  is denoted by  $[V]$  as an element of  $Rep(L)$ .

Then we make the following definition inspired by the definition for a polynomial representation of  $GL(r)$  given earlier

**Definition 4.2.** A virtual character  $\chi \in Rep^{\mathbb{C}}(G)$  is called  $\lambda$ -poly if the following diagram commutes

$$\begin{array}{ccc} G & \xrightarrow{\chi} & \mathbb{C} \\ \downarrow \theta_\lambda & \nearrow & \\ \mathfrak{g} & & \end{array}$$

I.e.  $\chi \in Rep^{\mathbb{C}}(L)$  is  $\lambda$ -poly iff the corresponding function in  $\mathbb{C}[T]^{W_L}$  is in the image of  $\theta_{\lambda|T}(P)^*$ . The set  $Rep_{\lambda\text{-poly}}^{\mathbb{C}}(L)$  of all  $\lambda$ -polynomial characters is a subalgebra of  $Rep^{\mathbb{C}}(L)$  isomorphic to the algebra

$S(\mathfrak{t}^*)^{W_L}$  of Weyl polynomial invariants. Thus,  $\theta_{\lambda|T}(P)^*$  induces an isomorphism

$$\theta_{\lambda|T}(P)^* : S(\mathfrak{t}^*)^{W_L} \rightarrow \text{Rep}_{\lambda\text{-poly}}^{\mathbb{C}}(L)$$

Now, the main result is as follows by composing the above maps

**Theorem 4.2.** *Let  $V_\lambda$  be an almost faithful irreducible  $G$ -module and let  $P$  be any standard parabolic subgroup. Then, the above maps (specifically  $\beta^P \circ (\theta_{\lambda|T}(P)^*)^{-1}$ ) give rise to a surjective  $\mathbb{C}$ -algebra homomorphism*

$$\xi_\lambda^P : \text{Rep}_{\lambda\text{-poly}}^{\mathbb{C}}(L) \rightarrow H^*(G/P, \mathbb{C}).$$

Moreover, let  $Q$  be another standard parabolic subgroup with Levi subgroup  $R$  containing  $T$  such that  $P \subset Q$  (and hence  $L \subset R$ ). Then, we have the following commutative diagram:

$$\begin{array}{ccc} \text{Rep}_{\lambda\text{-poly}}^{\mathbb{C}}(R) & \xrightarrow{\xi_\lambda^Q} & H^*(G/Q, \mathbb{C}) \\ \downarrow \gamma & & \downarrow \pi^* \\ \text{Rep}_{\lambda\text{-poly}}^{\mathbb{C}}(L) & \xrightarrow{\xi_\lambda^P} & H^*(G/P, \mathbb{C}), \end{array}$$

where  $\pi^*$  is induced from the standard projection  $\pi : G/P \rightarrow G/Q$  and  $\gamma$  is induced from the restriction of representations.

**Example 4.1.** The subalgebra  $\text{Rep}_{\lambda\text{-poly}}^{\mathbb{C}}(G) \subset \text{Rep}^{\mathbb{C}}(G)$ , in general, indeed depends upon the choice of  $\lambda$ . For example, for  $G = \mathbf{SL}_2$ , following Example 3.1,

$$\text{Rep}_{\omega_1\text{-poly}}^{\mathbb{C}}(\mathbf{SL}_2) = \mathbb{C}[(z - z^{-1})^2],$$

whereas

$$\text{Rep}_{2\omega_1\text{-poly}}^{\mathbb{C}}(\mathbf{SL}_2) = \mathbb{C}[(z^2 - z^{-2})^2],$$

for the maximal torus in  $\mathbf{SL}_2$  given by

$$T = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbb{C}^* \right\}.$$

◇

### 4.3 Type A

In this section we will show how we can recover the classical result stated at the beginning of this section from [Theorem 4.2](#) and give more details on the Cohomology of the Grassmannian.

#### 4.3.1 Recovering the Classical Result

We adopt the same notation as from the beginning of the chapter and follow [[Ku2](#), §5]. The torus  $T \subset GL(n)$  is the set of diagonal matrices in  $GL(n)$  and the cartan subalgebra is then given by set of diagonal matrices  $\mathfrak{t} = \{diag(t_1, \dots, t_n) : t_i \in \mathbb{C}\}$ . Then the simple roots and coroots are given by

$$\alpha_i = t_i - t_{i+1} \text{ and } \check{\alpha}_i = diag(0, \dots, 0, 1, -1, 0, \dots, 0), \text{ for any } 1 \leq i \leq n - 1$$

where 1 is in the  $i^{th}$  place. And the fundamental weights are given by

$$\omega_i = t_1 + \dots + t_i$$

The Weyl group of type  $A_n$  is the symmetric group  $S_n$  generated by the reflections  $S_{\alpha_i}$  associated to the simple roots. Here,  $S_{\alpha_i} = s_i = (i, i + 1)$ . Now let  $P_r$  be the standard maximal parabolic associated to the subset  $\theta = \Delta - \{\alpha_r\}$  of simple roots. Then  $L_r$  is the unique Levi subgroup containing  $T$  such that its simple roots are  $\theta$ . Then as mentioned before  $GL(n)/P_r \simeq Gr(r, n)$ , the Grassmannian of  $n$ -planes in  $\mathbb{C}^n$ . Furthermore the set of minimal length cosets which index the Schubert classes in the cohomology of the grassmannian can be parametrized by the following set of strictly increasing sequences,

$$S(r, n) = \{I := 1 \leq a_1 < \dots < a_r \leq n\}.$$

These sequences represent the permutation  $\nu_A \in S_n$  given by

$$I = (a_1, \dots, a_r, a_{r+1}, \dots, a_n), i \mapsto a_i$$

where the above permutation is written in one-line notation and the  $\{a_{r+1}, \dots, a_n\} = [n] \setminus \{a_1, \dots, a_r\}$  are put in increasing order. Such permutations are said to have a descent at  $r$ , i.e.  $w(r+1) < w(r)$  or equivalently  $l(ws_r) < l(w)$ . There is also a parametrization by partitions  $\lambda = \{n - r \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_1 \geq 0\}$ . The partition  $\lambda_I$  corresponding to the sequence  $I$  is given by  $\lambda_i = I_{r+1-i} - (r + 1 - i)$ . So For example, if  $n=5$  and  $r=3$ , the sequence  $I = (1, 3, 5)$  corresponds to the one-line permutation  $w_I = (1, 3, 5, 2, 4)$  and to the partition  $\lambda_I = (2, 1, 0)$ . The corresponding Schubert variety will then be denoted by either  $X_I$ ,  $X_{w_I}$ , or  $X_{\lambda_I}$ .

Now as mentioned above, irreducible polynomial representations of  $GL(r)$  are also parametrized by permutations. The map  $\xi$  from [Theorem 4.1](#) can then be stated as mapping  $[V(\lambda)] \mapsto \epsilon_\lambda$ .

Let  $G = GL(n)$  and let  $\lambda = \omega_1$  so  $V_\lambda$  is the defining representation. Then we have that,

$$\theta_{\omega_1} : GL(n) \rightarrow \mathfrak{g}(n)$$

. Furthermore,  $Rep_{\omega_1-poly}(G)$  coincides with the usual notion of polynomial representation (where  $Rep_{\omega_1-poly}(G) = Rep_{\omega_1-poly}^{\mathbb{C}}(G) \cap Rep(G)$ ). For  $1 \leq r \leq n - 1$ , the Levi subgroup  $L_r$  is the subgroup

$$L_r \simeq GL(r) \times GL(n - r) \subset GL(n).$$

Then, from [Theorem 4.2](#) we have a  $\mathbb{C}$ -algebra homomorphism:

$$\xi_{\omega_1}^{P_r} : Rep_{\omega_1-poly}^{\mathbb{C}}(L_r) \rightarrow H^*(Gr(r, n), \mathbb{C})$$

where,

$$Rep_{\omega_1-poly}^{\mathbb{C}}(L_r) \simeq [Rep_{poly}(GL(r)) \otimes Rep_{poly}(GL(n - r))] \otimes_{\mathbb{Z}} \mathbb{C}$$



In order to get a map from  $\text{Rep}_{\omega_1}^{\mathbb{C}}(GL(r))$  we factor through the ring homomorphism

$$i : \text{Rep}_{\text{poly}}(GL(r)) \rightarrow \text{Rep}_{\text{poly}}(GL(r)) \otimes \text{Rep}_{\text{poly}}(GL(n-r))$$

where we tensor a  $GL(r)$  representation with the trivial one-dimensional  $GL(n-r)$  representation.

**Theorem 4.3.**  $\xi_{\omega_1}^{P_r} \circ i$  coincides with  $\xi$  from Theorem 4.1.

*Proof.* Note that since these are  $\mathbb{C}$  algebra homomorphisms, we need only check that they correspond on the fundamental representations  $[V_{\omega_1}]$  since they generate  $\text{Rep}_{\text{poly}}(GL_r)$ . Note that for  $\lambda = (1, \dots, 1, 0, \dots, 0)$  with  $i$  one's we have  $[V(\omega_i)] = [V(\lambda)]$ . Furthermore,  $w_\lambda = s_{r-i+1} \dots s_r$ . Thus by definition,

$$\xi([V(\omega_i)]) = \epsilon_{s_{r-i+1} \dots s_r}^{P_r}.$$

Moreover, the character of  $[V(\omega_i)]$  is the  $i^{\text{th}}$  elementary symmetric polynomial  $e_i(x_1, \dots, x_r)$  where  $x_i$  is the  $i^{\text{th}}$  coordinate map on  $\mathfrak{t}$ . Thus,

$$\xi_{\omega_1}^{P_r}([V(\omega_1)]) = \beta(e_i(x_1, \dots, x_r))$$

where  $\beta$  is the Borel characteristic map. Then, by [Hi, Chapter 4 Lemma 5.4] we have

$$\beta(e_i(x_1, \dots, x_r)) = \epsilon_{s_{r-i+1} \dots s_r}^{P_r}$$

completing the proof. □

### 4.3.2 Cohomology of the Grassmannian

As a model for what we will discuss in Chapter 5 for the other classical groups we will briefly discuss in more detail the structure of the cohomology of the Grassmannian. References for this material are [T2, T7]

From the previous section we know that  $X = Gr(r, n)$  can be realized as the homogenous space  $GL(n)/P_r$ . Then from chapter 2, sections 2 and 3, we saw that  $H^*(Gr(r, n), \mathbb{C})$  has an additive schubert basis indexed by the minimal length elements of  $W/W_{P_r}$ .

Fix the complete flag of vector subspaces  $(F_i = \langle e_1, \dots, e_i \rangle \subset \mathbb{C}^n)$

$$F_\bullet : 0 = F_0 \subset F_1 \subset \dots \subset F_n = \mathbb{C}^n$$

. Consider the set of index sets  $S(r, n) = \{I : 1 \leq i_1 < \dots < i_r \leq n\}$ . Now let  $\Omega \in X$ , then we can define a corresponding index set  $I(\Omega)$  by

$$I(\Omega) = \{(i_1, \dots, i_r) \mid \Omega \cap F_{i_j} \not\supseteq \Omega \cap F_{i_{j-1}}\}$$

Then we can define the following subvariety of  $X$  for a given index set  $I$ ,

$$X_I^\circ(F_\bullet) := \{\Omega \in X \mid I(\Omega) = I\}$$

$X_I^\circ(F_\bullet)$  is then isomorphic to an affine space of dimension  $\sum_{j=1}^r (i_j - j)$  and these give a familiar cell decomposition of the grassmannian

$$Gr(r, n) = \coprod_I X_I^\circ(F_\bullet)$$

These are exactly the open Bruhat cells (up to choice of full flag or Borel subgroup) and an index set corresponds to an element  $W/W_{P_r}$  as in the previous subsection. Let  $X_I(F_\bullet)$  be the closure. We can also parametrize subvarieties by partitions  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r)$  where we require  $n - r \geq \lambda_1$ . The subvarieties are

$$X_\lambda(F_\bullet) = \{\Omega \in X \mid \dim(\Omega \cap F_{n-r+j-\lambda_j}) \geq j, 1 \leq j \leq r\}$$

. We can associate a partition making the codimension of  $X_I$  apparent to an index set  $I$  by letting

$$\lambda_j = n - r + j - i_j.$$

Then  $X_\lambda(F_\bullet)$  is a subvariety of co-dimension  $|\lambda| = \sum_{j=1}^r \lambda_j$ . We can also associate the dual partition  $\tilde{\lambda}$  given by

$$\tilde{\lambda}_j = i_{r+1-j} - (r + 1 - j).$$

Then,  $X_{\tilde{\lambda}(F_{\bullet})}$  is a variety of dimension  $|\lambda|$ . Associate a permutation  $w_{\lambda}$  to a partition  $\lambda$  by

$$w_{\lambda}(i) = \lambda_{r+1-i} + i$$

. Now, let  $[X_{\lambda}]$  be the fundamental homology class of the subvariety  $X_{\lambda}$  (the class is independent of choice of flag, see [Br]). Associate  $\epsilon_{\lambda}$  and  $\epsilon_{\tilde{\lambda}}$  be cohomology classes associated to an index set  $I$ . Then  $\epsilon_{\tilde{\lambda}}$  and  $\epsilon_{\lambda}$  are related to each other as  $\epsilon_w$  and  $\epsilon_{w_0 w w_0^P}$  are from §2.2 ( $w_0 w_0^P$  is the longest element of  $W_P$ ).

For example, let  $w = (1, 2, 3, 6, 4, 6)$  be an element of  $S(4, 6)$ , i.e. the associated index set is  $I = (1, 2, 3, 6)$ . Then,  $\lambda = (2, 2, 2, 0)$  and  $\tilde{\lambda} = (2, 0, 0, 0)$  based of  $I$  as above. Then we have that

$$\epsilon_w = \epsilon_{\tilde{\lambda}},$$

and  $\epsilon_w = PD[X_{\lambda}]$  or equivalently that  $\epsilon_{w_0 w w_0^P} = \epsilon_{\lambda}$ .

The varieties indexed by partitions of a single part  $X_p := X_{(p, 0, \dots, 0)}$  for  $1 \leq p \leq n - r$  play a special role in determining the cohomology ring. They depend only on a single Schubert condition,

$$X_p(F_{\bullet}) = \{\Omega \in X \mid \Omega \cap F_{n+1-p} \neq 0\}.$$

Note,  $\epsilon_p \in H^p(X, \mathbb{C})$ . These are called the special Schubert classes.

We now want to give a presentation for the cohomology ring  $H^*(Gr(r, n), \mathbb{C})$ . The idea here is the same for all grassmannians of classical type [BKT1, §0]. Over the grassmannian there is a universal short exact sequence of vector bundles

$$0 \rightarrow S \rightarrow V \rightarrow Q \rightarrow 0.$$

$V$  is the trivial rank  $n$  bundle.  $S$  is the tautological rank  $r$  subbundle where the fiber over a point  $\Omega \in X$  is  $\Omega$ , and  $Q$  is the tautological rank  $d = n - r$  quotient bundle. Then we have that  $\epsilon_p = c_p(Q)$ , i.e.  $\epsilon_p$  is the  $p^{\text{th}}$  Chern class of the quotient bundle. Now, multiplication in the cohomology ring is determined by the classical Pieri rule, which is a type of Chevalley rule from §2.2. It states that the product of a Schubert class

$\epsilon_\lambda \in H^{|\lambda|}$  with a special Schubert class is given by,

$$\epsilon_\lambda \cdot \epsilon_p = \sum \epsilon_\mu,$$

where the sum is over all partitions  $\mu$  obtained from  $\lambda$  by adding  $p$  blocks to  $\lambda$  while adding no two in the same column. Additionally, any Schubert class  $\epsilon_\lambda$  can be expressed as a polynomial in the special Schubert classes. This is the classical Giambelli formula,

$$\epsilon_\lambda = \det(\epsilon_{\lambda_i+j-i})_{1 \leq i < j \leq n}$$

The Pieri formula implies that the special Schubert classes  $\epsilon_p$  generate the cohomology of the grassmannian. We can present the cohomology as a quotient of the polynomial ring  $\mathbb{C}[\epsilon_1, \dots, \epsilon_d]/I_{r,d}$  where  $I_{r,d}$  is generated by the determinantal relations

$$\det(\epsilon_{1+j-i})_{1, j \leq m} = 0, \quad r+1 \leq m \leq n$$

The Whitney sum formula applied to  $S$  and  $Q$ ,  $c(S)c(Q) = 1$  where  $c(Q)$  is the total Chern class of the bundle  $Q$ , can be used to show that these relations hold in  $H^*(X, \mathbb{C})$  and dimensional considerations show that they are sufficient. We will be able to give presentations for the cohomology of grassmannians of type  $B$  and  $C$  in terms of the Chern classes of a universal quotient bundle as well.

### 4.3.3 Result in the Inverse Limit

We will look at the situation from §4.3.1 again this time focusing on the second factor. Fix an integer  $d$ . We want to compare the ring  $\text{Rep}_{\text{poly}}(GL(d))$  to the cohomology ring  $H^*(Gr(n-d, n), \mathbb{C})$  of the grassmannian of codimension  $d$  subspaces of  $\mathbb{C}$ . Again we consider the maximal parabolic  $P_{n-d}$  associated to the subset of simple roots  $\Delta - \{\alpha_{n-d}\}$ . Then as before  $L_r = GL(n-d) \times GL(d)$  and theorem 4.2, and we have

$$\text{Rep}_{\omega_1}^{\mathbb{C}}(L_r) = \text{Rep}_{\omega_1}^{\mathbb{C}}(GL(n-d)) \otimes_{\mathbb{C}} \text{Rep}_{\omega_1}^{\mathbb{C}}(GL(d)) = \mathbb{C}_{\text{sym}}[t_1, \dots, t_{n-d}] \otimes \mathbb{C}_{\text{sym}}[t_{d+1}, \dots, t_n]$$

Here again  $\mathbb{C}_{sym}[x_1, \dots, x_k]$  is the subring of polynomials invariant under permutation of variable. The fundamental theorem of symmetric polynomials says that any symmetric polynomial can be written as a polynomial in the elementary symmetric polynomials  $e_i(x_1, \dots, x_k)$  for  $1 \leq i \leq k$ . The elementary symmetric polynomial is defined as

$$e_i(x_1, \dots, x_k) = \sum_{1 \leq j_1 < \dots < j_i \leq k} x_{j_1} \dots x_{j_i}$$

We saw in 4.3.1 that

$$\xi_{\omega_1}^{P_{n-d}}(e_i(t_1, \dots, t_{n-d})) = \epsilon_{s_{n-d-i+1} \dots s_{n-d}}$$

Consider  $e_1(t_{n-d+1}, \dots, t_n) = t_{n-d+1} + \dots + t_n$ . Let  $T = \{t = (t_1, \dots, t_n) | t_i \neq 0\}$  be the maximal torus in  $GL(n)$ . Then with  $\theta_{\omega_1}(t) = t \in \mathfrak{gl}_n$ . Let  $x_i : \mathfrak{t} \rightarrow \mathbb{C}$  be the linear map taking

$$\text{diag}(x_1, \dots, x_n) \in \mathfrak{t} \text{ to } x_i$$

Then since  $\theta_{\omega_1|_T}$  is just the inclusion  $T \subset \mathfrak{t}$ ,  $(\theta_{\omega_1|_T}^*)^{-1}(x_i) \mapsto t_i$ , so  $e_i(t_1, \dots, t_n) \mapsto e_i(x_1, \dots, x_n)$ . Then we just need to compute  $\beta(e_i(x_{n-d+1}, \dots, x_n))$ . Recall that  $\beta(\omega_i) = \epsilon_{s_i}$ . In the fundamental weight basis  $x_i = \omega_i - \omega_{i-1}$ , so

$$\beta(x_i) = \beta(\omega_i) - \beta(\omega_{i-1}) = \epsilon_{s_i} - \epsilon_{s_{i-1}},$$

except for  $\beta(x_n) = \beta(\omega_n - \omega_{n-1}) = -\epsilon_{s_{n-1}}$ . Then, finally

$$\beta(e_1(x_{n-d+1}, \dots, x_n)) = \sum_{i=1}^d \beta(\omega_{n-d+i} - \omega_{n-d+i-1}) = -\epsilon_{s_{n-d}}$$

In general, we claim

**Lemma 4.1.** *For any  $k = 1, \dots, d$ , the following equation holds in  $H^*(Gr(n-d, n), \mathbb{C})$*

$$\beta(e_k(x_{n-d+1}, \dots, x_n)) = (-1)^k (\epsilon_{s_{n-d+k-1} \dots s_{n-d}})$$

*Proof.* Recall that we can relate the Borel morphism to Schubert classes by

$$\beta(f) = \sum_{w \in W^P} A_w(f) \epsilon_w.$$

So we just need to show that  $A_w(e_k(x_{n-d+1}, \dots, x_n)) = (-1)^k$  if  $w = s_{n-d+k} \dots s_{n-d}$  and equals 0 otherwise.

This follows by inducting on the number of variables and noting the following properties of  $A_{s_i}$

$$A_{s_i}(e_k(x_j, \dots, x_n)) \neq 0 \text{ only if } i = j - 1$$

$$A_{s_j}(e_k(x_j, \dots, x_n)) = -e_{k-1}(x_{j+1}, \dots, x_n)$$

□

If we also consider the bijections between elements of  $W^{Pr}$  and  $S(n-d, d)$  we can associate  $\beta(e_k(x_{n-d+1}, \dots, x_n))$  to  $\epsilon_\lambda$  for some partition  $\lambda$ . The Weyl group element  $s_{n-d+k} s_{n-d+k-1} \dots s_{n-d}$  corresponds to the one-line permutation  $[1, 2, \dots, n-d-1, n-d+k, \dots]$  where  $w(n-d+1)$  through  $w(n)$  are taken from the remaining number and put in increasing order. This element then corresponds to the partition  $(k, 0, \dots, 0)$ . Thus,

$$\epsilon_{s_{n-d+k-1} \dots s_{n-d}} = \epsilon_k = c_k(Q).$$

We also note that  $\epsilon_{s_{n-d-k+1} \dots s_{n-d}} = \epsilon_{(1, \dots, 1, 0, \dots, 0)} = c_k(S)$ , where  $(1, \dots, 1, 0, \dots, 0)$  is the partition with  $k$  leading ones and  $c_k(S)$  is the  $k^{\text{th}}$  chern class of the tautological subbundle. We collect the above results and discussions into the following proposition

**Proposition 4.1.** *Let  $L_{n-d} = GL(n-d) \times GL(d) \subset GL(n) = G$  be the Levi subgroup of the maximal parabolic  $P_{n-d}$  associated to subset of simple roots  $\Delta - \{\alpha_{n-d}\}$ . Then the map*

$$\xi_{\omega_1}^{P_{n-d}} : \text{Rep}_{\omega_1}^{\mathbb{C}}(L_{n-d}) \rightarrow H^*(GL(n-d, n), \mathbb{C})$$

takes,

$$e_k(t_1, \dots, t_{n-d}) \mapsto \epsilon_{1^k} = c_k(S^*)$$

and

$$e_k(t_{n-d+1}, \dots, t_n) \mapsto \epsilon_k = (-1)^k c_k(Q)$$

Now, consider the map

$$l_2^n : \text{Rep}_{\text{poly}}(GL(d)) \rightarrow \text{Rep}_{\text{poly}}(GL(n-d)) \otimes \text{Rep}_{\text{poly}}(GL(d))$$

given by tensoring a  $GL(d)$  polynomials representation with the trivial  $GL(n-d)$  representation. Then this gives a map from  $\xi_{\omega_1}^{n,d} : \text{Rep}_{\text{poly}}(GL(d)) \mapsto H^*(Gr(n-d, n))$  by composing  $\xi_{\omega_1}^{P_{n-d}} \circ i_2$ . Now consider the following inclusion of varieties

$$\dots \rightarrow Gr(n-d, n) \rightarrow Gr(n+1-d, n+1) \rightarrow \dots$$

This yields a corresponding sequence

$$\dots \leftarrow H^*(Gr(n-d, n), \mathbb{C}) \leftarrow H^*(Gr(n-d+1, n+1), \mathbb{C}) \leftarrow \dots$$

Note the Chern classes of the universal quotient bundles are stable in this system, i.e. in the map  $H^*(Gr(n-d, n)\mathbb{C}) \leftarrow H^*(Gr(n-d+1, n+1), \mathbb{C})$  one has  $c_p(Q_n) \leftarrow c_p(Q_{n+1})$ . Then let

$$\mathbb{H}(Gr_d) = \varprojlim H^*(Gr(n-d, n), \mathbb{C}),$$

that is the inverse limit in the category of graded rings in the above system. Consider the diagram

$$\begin{array}{ccc}
 & & i_n^* \uparrow \\
 \text{Rep}_{\omega_1}^{\mathbb{C}}(GL(d)) & \xrightarrow{\xi_{\omega_1}^{n,d} \circ l_2^n} & H^*(Gr(n-d, n), \mathbb{C}) \\
 & \searrow \xi_{\omega_1}^{n+1,d} \circ l_2^{n+1} & \uparrow i_{n+1}^* \\
 & & H^*(Gr(n+1-d, n), \mathbb{C}) \\
 & & \uparrow i_{n+2}^*
 \end{array}$$

Then due to the stability of the chern classes, we have

$$i_{n+1}^* \circ \xi_{\omega_1}^{n+1,d}(e_k(x)) = i_{n+1}^*((-1)^k c_k(Q_{n+1})) = (-1)^k c_k(Q_n) = \xi_{\omega_1}^{n,d}(e_k(x))$$

. Thus, we have a map from

$$\text{Rep}_{\omega_1}^{\mathbb{C}}(GL(d)) \rightarrow \mathbb{H}(Gr_d)$$

. Looking at the presentation for the cohomology of  $H^*(Gr(n-d, n), \mathbb{C})$ , none of the relations hold in the inverse limit. This yields the following theorem

**Theorem 4.4.** *There is an graded algebra isomorphism between  $\text{Rep}_{poly}(GL(d))$  and  $\mathbb{H}(Gr_d) = \varprojlim H^*(Gr(n-d, n), \mathbb{C}) = \mathbb{C}[\epsilon_1, \dots, \epsilon_d]$ , given by mapping the  $k^{th}$  elementary symmetric polynomial to the class  $\epsilon_k = c_k(Q)$ .*

We will attempt to derive similar results fro other Lie types in the next chapter.



## CHAPTER 5

### Types B,C, and G

#### 5.1 Representation Ring of the Classical groups

In this section all results are due to and we follow closely [Ku2, §6]. In accordance with our expectation that the fundamental weight of minimal dynkin index is the most appropriate dominant weight for which to consider the  $\lambda$ -polynomial representation ring, we let  $\lambda = \omega_1$  (which is the defining representation for the classical groups  $Sp(2n, \mathbb{C})$ ,  $SO(2n+1, \mathbb{C})$ , and  $SO(2n, \mathbb{C})$ ).

Take symmetric forms on  $\mathbb{C}^{2n}$ ,  $\mathbb{C}^{2n+1}$  (resp. an alternating form on  $\mathbb{C}^{2n}$ ) so that  $SO(2n)$ ,  $SO(2n+1)$  (resp.  $Sp(2n)$ ) are given respectively by

$$SO(2n) = \{g \in SL_{2n} : (g^t)^{-1} = E_D g E_D^{-1}\}$$

$$SO(2n+1) = \{g \in SL_{2n+1} : (g^t)^{-1} = E_B g E_B^{-1}\}$$

$$Sp(2n) = \{g \in SL_{2n} : (g^t)^{-1} = E_C g E_C^{-1}\},$$

where  $E_D$  is the antidiagonal matrix with all its antidiagonal entries 1;  $E_B$  is the antidiagonal matrix with all its antidiagonal entries 1 except the  $(n+1, n+1)$ -th entry which is 2;  $E_C$  is the block matrix

$$E_C = \begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix},$$

where  $J_n$  is the antidiagonal  $n \times n$  matrix with all its antidiagonal entries 1. (The suffix  $D, B, C$  refers to the types of the corresponding groups.)

Depending upon the case, denote  $E_D$ ,  $E_B$  or  $E_C$  by the common symbol  $E$ . Consider the Springer morphism for these groups with  $\lambda = \omega_1$ , which is their defining representation. Then, Springer morphism in this case is just the Cayley transform [Ku2, Lemma 10].

**Lemma 5.1.** *The Springer morphism  $\theta : G \rightarrow \mathfrak{g}$  for  $G = So_{2n}$ ,  $So_{2n+1}$  or  $Sp_{2n}$  is given by*

$$g \mapsto \frac{g - E^{-1}g^tE}{2}, \text{ for } g \in G.$$

*Proof.* The lemma follows immediately since under the decomposition

$$\text{End}(V(\omega_1)) = \mathfrak{g} \oplus \mathfrak{g}^\perp,$$

any  $A \in \text{End}(V(\omega_1))$  decomposes as

$$A = \frac{(A - E^{-1}A^tE)}{2} + \frac{(A + E^{-1}A^tE)}{2}.$$

□

We choose the maximal tori in  $Sp(2n)$ ,  $SO(2n)$  and  $SO(2n + 1)$  respectively as follows:

$$T_C = T_D = \{ \mathfrak{t} = (t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1}) : t_i \in \mathbb{C}^* \} \quad (5.1)$$

$$T_B = \{ \mathfrak{t} = (t_1, \dots, t_n, 1, t_n^{-1}, \dots, t_1^{-1}) : t_i \in \mathbb{C}^* \}. \quad (5.2)$$

Their associated Cartan subalgebras are then given by

$$\mathfrak{t}_C = \mathfrak{t}_D = \{ \bar{\mathfrak{t}} = (x_1, \dots, x_n, -x_n, \dots, -x_1) : x_i \in \mathbb{C} \} \quad (5.3)$$

$$\mathfrak{t}_B = \{ \bar{\mathfrak{t}} = (x_1, \dots, x_n, 0, -x_n, \dots, -x_1) : x_i \in \mathbb{C} \}. \quad (5.4)$$

From the description of the Springer morphism given above, we immediately get the following:

**Lemma 5.2.** *Restricted to the maximal torus as above, we get the following description of the Springer morphism  $\theta_{\omega_1}$  (which can also easily be derived from Proposition 3.1 as the example in §3.2.1 was for type C) :*

$$(a) G = SO(2n) : \theta(\mathbf{t}) = \left( \frac{t_1 - t_1^{-1}}{2}, \dots, \frac{t_n - t_n^{-1}}{2}, -\left(\frac{t_n - t_n^{-1}}{2}\right), \dots, -\left(\frac{t_1 - t_1^{-1}}{2}\right) \right)$$

(b)  $G = Sp(2n)$  : Same as in the above case of  $G = So(2n)$ .

$$(c) G = So(2n) : \theta(\mathbf{t}) = \left( \frac{t_1 - t_1^{-1}}{2}, \dots, \frac{t_n - t_n^{-1}}{2}, 0, -\left(\frac{t_n - t_n^{-1}}{2}\right), \dots, -\left(\frac{t_1 - t_1^{-1}}{2}\right) \right).$$

Recall that  $\text{Rep}_{\omega_1}^{\mathbb{C}}(G)$  is isomorphic to the ring of Weyl group invariants  $S(\mathfrak{t}^*)^W$ . Then the above Lemma together with the description of  $S(\mathfrak{t}^*)^W$  given in §4.1 yields the following result

**Lemma 5.3.** • *Let  $G = SO(2n + 1)$  or  $Sp(2n)$ . Then the polynomial representation ring is given by*

$$\text{Rep}_{\omega_1}^{\mathbb{C}}(G) \simeq \mathbb{C}_{sym} \left[ \left( \frac{t_1 - t_1^{-1}}{2} \right)^2, \dots, \left( \frac{t_n - t_n^{-1}}{2} \right)^2 \right]$$

• *Let  $G = SO(2n)$ . Then the polynomial representation ring is given by*

$$\text{Rep}_{\omega_1}^{\mathbb{C}}(SO(n)) = \mathbb{C}_{sym} \left[ \left( \frac{t_1 - t_1^{-1}}{2} \right)^2, \dots, \left( \frac{t_{n-1} - t_{n-1}^{-1}}{2} \right)^2, \left( \frac{t_1 - t_1^{-1}}{2} \right) \dots \left( \frac{t_n - t_n^{-1}}{2} \right) \right]$$

Furthermore, like the standard representation rings  $\text{Rep}(G)$  the  $\omega_1$ -polynomial rings of type  $B$  and  $C$  also carry the structure of a special  $\lambda$ -ring as mentioned in §2.1. We give the following more complete definition, from [AT, §1].

**Definition 5.1.** *A special  $\lambda$ -ring is, by definition, a commutative ring  $R$  with identity with a map*

$$\lambda : R \rightarrow R[[q]], \quad x \mapsto \sum_{i \geq 0} \lambda^i(x) q^i,$$

which satisfies the following:

- (1)  $\lambda^0(x) = 1$
- (2)  $\lambda^1(x) = x$ , for all  $x \in R$
- (3)  $\lambda(x + y) = \lambda(x)\lambda(y)$ , for all  $x, y \in R$

(4)  $\lambda(1) = 1 + q$ , and

(5) There are universal (independent of  $R$ ) polynomials  $P_k$  and  $P_{k,l}$  over  $\mathbb{Z}$  such that

$$\lambda^k(xy) = P_k \left( \lambda^1(x), \dots, \lambda^k(x), \lambda^1(y), \dots, \lambda^k(y) \right)$$

$$\text{and } \lambda^k(\lambda^l x) = P_{k,l} \left( \lambda^1(x), \dots, \lambda^{kl}(x) \right), \text{ for all } k, l \geq 1.$$

Then as mentioned in Chapter 2, the operation

$$\lambda^i([V]) = [\wedge^i(V)]$$

turns  $\text{Rep}(G)$  into a special  $\lambda$ -ring and extend it to virtual representations by property (3). Then we have the following Theorem [Ku2, Lemma 15, Lemma 16, Lemma 18] for the classical groups.

**Theorem 5.1.** a) Let  $G = SO(2n + 1)$  or  $Sp(2n)$ . Then, the subring  $\text{Rep}(G) \subset \text{Rep}(G)$  of  $\omega_1$ -polynomial characters is a special  $\lambda$ -subring, where

$$\text{Rep}(G) := \text{Rep}^{\mathbb{C}}(G) \cap \text{Rep}(G).$$

b) Moreover, the character

$$\chi(\mathbf{t}) = \sum_{i=1}^n (t_i^2 + t_i^{-2}) \in (G), \text{ for } \mathbf{t} \in T_C \text{ given by (5.1) or } \mathbf{t} \in T_B \text{ given by (5.2)}$$

generates  $(G)$  as a  $\lambda$ -ring, i.e.,  $\chi(\mathbf{t}), \lambda^2(\chi(\mathbf{t})), \dots, \lambda^n(\chi(\mathbf{t}))$  generate the ring  $(G)$ .

In the case  $G = Sp(2n)$ ,  $\chi(\mathbf{t})$  is the character of the virtual representation  $[S^2V] - [\wedge V]$ , where  $[V]$  is the standard representation of  $Sp(2n)$ .

In the case  $G = SO(2n + 1)$ ,  $\chi(\mathbf{t})$  is the character of the virtual representation  $[S^2V] - [\wedge V] - [\epsilon]$ , where  $[V]$  is the standard representation of  $So(2n + 1)$  and  $[\epsilon]$  is the trivial one-dimensional representation.

- c) The ring  $\text{Rep}_{\text{poly}}(SO(2n))$  is not a  $\lambda$ -subring of  $\text{Rep}(G)$ . (Consider the function  $\prod_{i=1}^n (t_i - t_i^{-1}) \in \text{Rep}_{\text{poly}}(SO(2n))$ )
- d) For  $SO(2n)$  ( $n \geq 3$ ),  $\prod_{i=1}^n (t_i - t_i^{-1})$  is the character of the virtual representation  $[V(2\omega_n)] - [V(2\omega_{n-1})]$  where  $\omega_i$  - poly is the  $i^{\text{th}}$  fundamental representation of  $Spin(2n)$ .
- e) For  $G = Sp(2n)[n \geq 2]$ ,  $SO(2n+1)[n \geq 3]$ ,  $SO(2n)[n \geq 4]$ , no non-trivial irreducible representation  $[V(\lambda)]$  belong to  $\text{Rep}_{\omega_1}^{\mathbb{C}}(G)$ .

Note the contrast of Theorem 5.1(e) with the type  $A$  case in which  $V[\omega_i] \in \text{Rep}_{\omega_1}^{\mathbb{C}}(Gl(n))$ .

## 5.2 Type C

We try to generalize Theorem 4.4 to type C. We will describe the cohomology of isotropic grassmannians, how theorem 4.2 specializes to said case, and the extension to the inverse limit.

### 5.2.1 Cohomology of $IG(n-k, 2n)$

In this section we will describe the additive and multiplicative structure of the cohomology ring of isotropic grassmannians  $X = IG(n-k, 2n)$ . Again we fix an integer  $k$ , the reason for this will become apparent when we want to derive a partial analogue to [Theorem 4.4](#) in type  $C$ . As for type  $A$  we have parametrizations of Schubert varieties, classes, and Poincare dual classes via index sets, partitions, and minimal length coset representatives of  $W/W_{P_{n-k}}$ . There are special Schubert classes and Pieri and Giambelli formulas as well. The ring structure can also be described by Chern classes of certain tautological bundles.

References for the following parametrizations can be found in [[BK2](#), [BKT1](#), [LL](#), [PR1](#), [T2](#), [T7](#)].

Equip  $V = \mathbb{C}^{2n}$  with a non-degenerate skew-symmetric bilinear form  $\vartheta$ . Fix a complete *isotropic* flag  $F_{\bullet}$ ,

$$0 = F_0 \subset F_1 \subset \dots \subset F_{2n} = V$$

where  $F_i = F_{2n-i}^{\perp}$  with respect to  $\vartheta$ . Note,  $F_n$  is a maximal isotropic subspace. Then, we define the isotropic grassmannian  $IG(n-k, 2n)$  as,

$$IG(n-k, 2n) := \{\Omega \in Gr(n-k, 2n) : \vartheta(v, v') = 0, \forall v, v' \in \Omega\}$$

There exists numerous parametrizations of the Schubert varieties in the isotropic grassmannian. For one, they are parametrized by index sets  $\{I : 1 \leq p_{i_1} < \dots < p_{i_{n-k}} \leq 2n\}$  such that  $p_i + p_j \neq 2n + 1$ . The corresponding Schubert cell is given by

$$X_I^\circ = \{\Omega \in X \mid \Omega \cap F_p \not\supseteq \Omega \cap F_{p-1}\},$$

and the Schubert variety is given by

$$X_I(F_\bullet) = \{\Omega \in X \mid \dim(\Omega \cap F_{p_j}) \geq j, \forall 1 \leq j \leq n - k\}.$$

We also note that the dual index set is then given by  $\check{I}$  given by setting  $\check{p}_j = 2n + 1 - p_{n-k+1-j}$ . These index sets of course correspond to minimal length coset representatives of course.

Note that  $Sp(2n)$  can be realized as the fixed point subgroup  $G^\sigma$  of  $G = SL(2n)$  under the involution  $\sigma(A) = E(A^t)^{-1}E^{-1}$  where  $E = E_C$  as in §5.1. Here, we follow [BK2]. If  $T^A \subset B^A$  are the maximal torus and Borel subgroup of  $SL(2n)$ , then  $T^\sigma = T$ , and  $B^\sigma = B$  as in the previous section. Let  $\Delta_C = \{\beta_1, \dots, \beta_n\}$  be the simple roots of  $Sp(2n)$ . Then,  $\beta_i = \alpha_i | \mathfrak{t}$  where  $\{\alpha_1, \dots, \alpha_{2n-1}\}$  are the simple roots of  $SL(2n)$ . The corresponding simple coroots are given by

$$\check{\beta}_i = \check{\alpha}_i + \check{\alpha}_{2n-1}, \text{ for } 1 \leq i \leq n,$$

and

$$\check{\beta}_n = \check{\alpha}_n$$

Under the inclusion  $W_C \subset S_{2n}$  we have that the simple reflections of  $Sp(2n)$  are given by

$$s_i = r_i r_{2n-i} \text{ if } 1 \leq i \leq n - 1$$

$$= r_n \text{ if } i = n$$

where  $r_i$  is the  $i^{\text{th}}$  simple reflection for  $Sl(2n)$ . The Weyl group  $W_{C_n}$  can be identified with the subset of  $W_{A_{2n}}$  invariant under  $\sigma$ :

$$\{(a_1, \dots, a_{2n}) \in S_{2n} : a_{2n+1-i} = 2n + 1 - a_i \forall 1 \leq i \leq 2n\}$$

. Consider the parabolic weyl subgroup generated by  $\Delta_C - \{\beta_{n-k}\}$ . Then the minimal length coset representatives of  $W_C/W_{C, P_{n-k}}$  are can be identified with the set

$$I(n-k, 2n) = \{I := 1 \leq p_1 < \dots < p_n \leq 2n \text{ and } I \cap \bar{I} = \emptyset\},$$

where  $\bar{I} = \{2n + 1 - p_1, \dots, 2n + 1 - p_{n-k}\}$ . But this is just an index set. It represents the permutation in  $S_{2n}$  given by taking  $p_{n-k} + 1, \dots, p_n = [n] \setminus (I \sqcup \bar{I})$  and setting  $p_{2n+1-i} = 2n + 1 - p_i$

Finally we can also associate a  $k$ -strict permutation to an index set or Weyl group element. First, a partition  $\lambda$  is said to be  $k$ -strict if no part greater than  $k$  is repeated (i.e.  $\lambda_j > k \Rightarrow \lambda_{j+1} < \lambda_j$ ). This is the combinatorial object with which Buch, Kresch, and Tamvakis derive their Pieri and Giambelli rules in both the classical and quantum cohomology of the isotropic grassmannian. The bijection between index sets and  $k$ -strict partitions (contained in an  $(n-k) \times (n+k)$  rectangle) is defined as follows [BKT1, 4.1]. Let  $I = \{1 \leq i_1 < \dots < i_{n-k} \leq 2n\}$  be an index set. Then

$$\lambda_j(I) = n + k + 1 - I_j + \{i < j : I_i + I_j > 2n + 1\}.$$

In the reverse, given a  $k$ -strict partition  $\lambda = (\lambda_1, \dots, \lambda_{n-k})$  associates to an index set  $I(\lambda)$  by

$$I_j(\lambda) := n + k + 1 - \lambda_j + \{i < j : \lambda_i + \lambda_j \leq 2k + j - i\}.$$

The Schubert class of codimension  $|\lambda|$  is then simply  $X_\lambda = X_{I(\lambda)}$  where the Schubert variety with index set  $I$  is defined as above. The dual  $\check{\lambda}$  is the  $k$ -strict partition associated to the dual index set. The set of barred permutations in  $W/W_{P_{n-k}}$  can also be bijectively associated to  $k$ -strict partitions [T7, 4.2].

For an example, let  $n = 5, k = 2$ . Then  $s_2 s_3 \in W/W_{P_3}$  is associated to the index set  $(1, 3, 4)$ . The associated  $k$ -strict partition is then  $(7, 5, 4)$ . The dual index set is  $(7, 8, 10)$  and the dual partition

is  $(1, 1, 0)$ . For our purposes, we actually wish to associate to a Weyl group element  $w$  a partition  $\lambda_w$  such that  $l(w) = |\lambda|$ . So in the above case, we will write  $\epsilon_{s_3 s_2} = \epsilon_{(1,1,0)}$ . In general, when we write  $\epsilon_w = \epsilon_\lambda \in H^{l(w)=|\lambda|}(IG(n-k, n))$  the partition we refer to can be arrived at as follows. The index set is  $I_w = \{w(1), \dots, w(n-k)\}$ . Then take the dual index set  $\check{I}$  and take  $\lambda(\check{I})$ . We write  $\lambda(w)$  for this partition, and will interchange between writing  $\epsilon_w$  and  $\epsilon_\lambda$ . The classes  $\epsilon_i$  for  $1 \leq i \leq n+k$  are referred to as the special Schubert classes.

We will also need the Giambelli formula of [BKT2] and we follow closely some of the exposition given there. A fundamental insight of theirs is that classical Giambelli formulas can be restated in terms of Young's raising operators [?](see also [T3]). For any integer sequence  $(a_1, a_2, \dots)$  with finite support and  $i < j$ , they define  $R_{ij} = (\alpha_1, \dots, \alpha_i + 1, \dots, \alpha_j - 1, \dots)$ . Then a raising operator is any monomial in the  $R'_{ij}$ 's. Setting  $m_\alpha = \prod_i \epsilon_i$ , then  $Rm_\alpha = m_{R\alpha}$  for any raising operator. They show that the classical Giambelli formula for  $H^*(Gr(n-k, n))$  can be restated as

$$\epsilon_\lambda = \prod_{i < j} (1 - R_{ij}) m_\lambda.$$

For example, in  $H^*(Gr(3, 5))$  (with the convention  $\epsilon_0 = 1$  and  $\epsilon_i = 0$  for  $i < 0$  and  $i > 5$ ) one has

$$\begin{aligned} \epsilon_{(3,2,1)} &= (1 - R_{12})(1 - R_{13})(1 - R_{23})m_{321} \\ &= (1 - R_{12} - R_{23} - R_{13} + R_{12}R_{23} + R_{12}R_{13} + R_{13}R_{23} - R_{12}R_{13}R_{23})m_{321} \\ &= m_{321} - m_{411} - m_{4,2,0} + m_{4,2,0} + m_{5,1,-1} + m_{4,3,-1} - m_{5,2,-1} = \epsilon_3 \epsilon_2 \epsilon_1 - \epsilon_4 \epsilon_1^2 \end{aligned}$$

To any  $k$ -strict partition  $\lambda$  the associated raising operator is

$$R^\lambda = \prod_{i < j} (1 - R_{ij}) \prod_{\lambda_i + \lambda_k > 2k + j - i} \frac{1}{1 + R_{ij}}$$

. The Giambelli formula of Buch, Kresch, and Tamvakis can then be simply stated as



**Theorem 5.2.** [[BKT2](#), Theorem 1] For any  $k$ -strict partition  $\epsilon_\lambda$ , we have  $\epsilon_\lambda = R^\lambda m_\lambda$  in the cohomology ring of  $IG(n - k, 2n)$ .

Indeed, in the computations we need it for the partitions will only have 2 non-zero parts with  $\lambda_1 + \lambda_2 < 2k$  so the Giambelli formula reduces to  $\epsilon_{(\lambda_1, \lambda_2)} = \epsilon_{\lambda_1} \epsilon_{\lambda_2} - \epsilon_{\lambda_1+1} \epsilon_{\lambda_2-1}$

As in type A we have the following short exact sequence of vector bundles,

$$0 \rightarrow S \rightarrow V_C \rightarrow Q \rightarrow 0,$$

where  $V_C$  is the trivial bundle of rank  $2n$ ,  $S$  is the tautogological subbundle of rank  $n - k$ , and  $Q$  is the tautogological quotient bundle of rank  $n + k$ . Then the Schubert classes  $\epsilon_i$  equal to the  $i^{\text{th}}$  Chern class of the quotient bundle  $c_i(Q)$ , and these classes generate the cohomology ring. We give reduced decompositions so that  $\epsilon_i$  in the next section such that  $\epsilon_i = \epsilon_w \in H^*(IG(n - k, 2n), \mathbb{C})$ . Also, like the type A case there is a Pieri formula for the product of any Schubert class with that of a special Schubert class. Then we have the following presentation of the cohomology ring due to [[BKT2](#), Theorem 1.2]. By convention we set  $\epsilon_0 = 1$  and  $\epsilon_p = 0$  if  $p < 0$  or  $p > n + k$ .

**Theorem 5.3.** The cohomology ring  $H^*(IG(n - k, 2m), \mathbb{C})$  is presented as the quotient of the polynomial ring  $\mathbb{C}[\epsilon_1, \dots, \epsilon_{n+k}]$  by the relations

$$\det(\epsilon_{1+j-i})_{1 \leq i < j \leq r}, \quad n - k + 1 \leq r \leq n + k,$$

and

$$\epsilon_r^2 + \sum_{i=1}^{n+k-r} (-1)^i \epsilon_{r+i} \epsilon_{r-i} = 0, \quad k + 1 \leq r \leq n$$

As in type A, the determinantal relations come from the Whitney sum formula  $c(S)c(Q) = 1$ . The quadratic relations come from considering that the symplectic form gives a pairing  $SQ \rightarrow \mathcal{O}$  which yields an injection  $S \rightarrow Q^*$ . Chern classes  $c_j(Q^*/S)$  vanish for  $j > k$  and one can deduce that  $c(Q)c(Q^*)$  vanishes in degree  $> 2k$ .

### 5.2.2 Theorem 4.2

As for type  $A$ , we aim to explicitly determine the map in [Theorem 4.2](#) for  $G = Sp(2n)$  and any maximal parabolic  $P_{n-k}$ . Take  $V = \mathbb{C}^{2n}$  and  $IG(n-k, 2n)$  as in the previous section. Here we follow and expand on [\[Ku2, §7\]](#).

We take  $B_C := B \cap Sp_{2n}$  as the Borel subgroup of  $Sp_{2n}$ , where  $B$  is the standard Borel subgroup of  $SL_{2n}$  consisting of upper triangular matrices of determinant 1. Then,  $IG(n-k, 2n)$  is the quotient  $Sp_{2n}/P_{n-k}$  of  $Sp_{2n}$  by the standard maximal parabolic subgroup  $P_{n-k}$  with  $\Delta \setminus \{\alpha_r\{n-k\}\}$  as the set of simple roots of its Levi component  $L_{n-k}$ . (We take  $L_{n-k}$  to be the unique Levi subgroup of  $P_r$  containing  $T_C$ .)

$$L_{n-k} \simeq GL(n-k) \times Sp(2k).$$

From [Lemma 5.2](#), we have

$$\theta_{\omega_1}(\mathbf{t}_C) = \left( \frac{t_1 - t_1^{-1}}{2}, \dots, \frac{t_n - t_n^{-1}}{2}, -\left(\frac{t_n - t_n^{-1}}{2}\right), \dots, -\left(\frac{t_1 - t_1^{-1}}{2}\right) \right)$$

Also, recall that  $\text{Rep}_{\omega_1}^{\mathbb{C}}(L_{n-k}) \simeq S(\mathfrak{t}_C^*)^{W_P}$ . Using the fundamental invariants from §4.1.1 we find that the representation ring is given by,

$$\begin{aligned} \text{Rep}_{\omega_1}^{\mathbb{C}}(L_{n-k}) &\simeq \mathbb{C}_{\text{sym}} \left[ \left( \frac{t_1 - t_1^{-1}}{2} \right), \dots, \left( \frac{t_{n-k+1} - t_{n-k+1}^{-1}}{2} \right) \right] \\ &\otimes_{\mathbb{C}} \mathbb{C}_{\text{sym}} \left[ \left( \frac{t_{n-k+1} - t_{n-k+1}^{-1}}{2} \right)^2, \dots, \left( \frac{t_n - t_n^{-1}}{2} \right)^2 \right] \end{aligned}$$

where  $\mathbb{C}_{\text{sym}}$  denotes the subalgebra of the polynomial ring consisting of symmetric polynomials. Further, by [Theorem 5.1](#),

$$\mathbb{C}_{\text{sym}} \left[ \left( \frac{t_{n-k+1} - t_{n-k+1}^{-1}}{2} \right)^2, \dots, \left( \frac{t_n - t_n^{-1}}{2} \right)^2 \right]$$

is generated (as a  $\mathbb{C}$ -algebra) by the virtual representations:

$$\{\lambda^d ([S^2(V_{2(n-r)})] - [\Lambda^2(V_{2(n-r)})])\}_{1 \leq d \leq k},$$

where  $V_{2k} = \mathbb{C}^{2k}$  is the standard representation of  $Sp(2k)$  and  $\lambda$  is the  $\lambda$ -ring structure on  $Rep(G)$ .

The following theorem [Ku2, Proposition 19] partially determined the homomorphism of Theorem 4.2

**Theorem 5.4.** *The map  $\xi^{P_{n-k}} : Rep_{poly}^{\mathbb{C}}(L_{n-k}^{\mathbb{C}}) \rightarrow H^*(IG(n-k, 2n), \mathbb{C})$  of Theorem 4.2 takes*

$$\frac{1}{2}(t_1 - t_1^{-1} + \cdots + t_{n-k} + t_{n-k}^{-1}) \rightarrow \epsilon_{s_{n-k}},$$

and

$$\frac{1}{4}[(t_{n-k+1} - t_{n-k+1}^{-1})^2 + \cdots + (t_n + t_n^{-1})^2] \rightarrow \epsilon_{s_{n-k}}^2 + 2 \sum_{j=n-k+1}^{n-1} \epsilon_{s_j}^2 + \epsilon_{s_n}^2 - 2 \sum_{j=n-k}^{n-1} \epsilon_{s_j} \epsilon_{s_{j+1}}$$

*Proof.* For  $1 \leq n$ , let  $x_i : \mathfrak{t} \rightarrow \mathbb{C}$  be the linear map which takes

$$diag(x_1, \dots, x_n, -x_n, \dots, -x_1) \rightarrow x_i$$

. Then by Lemma 5.2, the homomorphism  $\theta_{\omega_1|T}^* : \mathbb{C}[\mathfrak{t}] \rightarrow \mathbb{C}[T]$  induced from the Springer morphism  $\theta_{\omega_1}$

takes

$$x_1 + \cdots + x_{n-k} \rightarrow \frac{1}{2}(t_1 - t_1^{-1} + \cdots + t_{n-k} + t_{n-k}^{-1}).$$

However, note that the weight  $x_1 + \dots + x_{n-k}$  is the first fundamental weight  $\omega_{n-k}$ . Thus,

$$\beta \circ (\theta_{\omega_1|T}^*)^{-1} \left( \frac{1}{2}(t_1 - t_1^{-1} + \cdots + t_{n-k} + t_{n-k}^{-1}) \right) = \beta(x_1 + \cdots + x_{n-k}) = \beta(\omega_{n-k}) = \epsilon_{s_{n-k}},$$

where the last inequality comes from the fact that by 2.2

$$\beta(\omega_{n-k}) = \sum_{\alpha \in \Delta} A_{\alpha}(\omega_{n-k}) \epsilon_{s_{\alpha}}$$

. Note that  $s_{n-k}(\omega_{n-k}) = \omega_{n-k} - \alpha_{n-k}$  and  $s_i(\omega_{n-k}) = 0$  otherwise. Then  $\beta(\omega_{n-k}) = A_{\alpha_{n-k}}(\omega_{n-k}) \epsilon_{s_{n-k}} = \epsilon_{s_{n-k}}$ . Indeed, in general for  $\beta : S(\mathfrak{t}^*) \rightarrow H^*(G/B)$ , one has that  $\beta(\omega_j) = \epsilon_{s_j}$

Similarly, under  $(\theta_{\omega_1|T}^*)^{-1}$ ,

$$\frac{1}{4} \left( (t_{n-k+1}^2 - t_{n-k+1})^{-1} + \cdots + (t_n + t_n^{-1})^2 \right) \rightarrow x_{n-k}^2 + \cdots + x_n^2$$

Since in type  $C$  we have  $\omega_i = \sum_{j=1}^i x_j$ , then we can right the coordinate functions in the fundamental weight basis  $x_i = \omega_i - \omega_{i-1}$ . So,  $x_{n-k+1}^2 + \cdots + x_n^2 = (\omega_{n-k+1} - \omega_{n-k})^2 + \cdots + (\omega_n - \omega_{n-1})^2$ . Then from the remark above it is clear that  $\xi^{P_{n-k}}$  takes

$$\frac{1}{4} \left( (t_{n-k+1}^2 - t_{n-k+1})^{-1} + \cdots + (t_n + t_n^{-1})^2 \right) \rightarrow (\epsilon_{s_{n-k+1}} - \epsilon_{s_{n-k}})^2 + \cdots + (\epsilon_{s_n} - \epsilon_{s_{n-1}})^2,$$

which expands to give the stated result. □

Note that the term

$$\epsilon_{s_{n-k}}^2 + 2 \sum_{j=n-k+1}^{n-1} \epsilon_{s_j}^2 + \epsilon_{s_n}^2 - 2 \sum_{j=n-k}^{n-1} \epsilon_{s_j} \epsilon_{s_{j+1}}$$

is not written in the basis  $\{\epsilon_w : w \in P^{n-k}\}$  of  $H^*(Sp(2n)P_{n-k}, \mathbb{C}) = H^*(IG(n-k, 2n), \mathbb{C})$ . Indeed,  $s_{n-k}$  is the only simple reflection in  $W_{n-k}^P$ . We can in theory use Chevalley's formula (§2.6 to expand the quadratic terms into the additive Schubert basis and all nonvanishing terms should be elements of  $W_{n-k}^P$ . Since the terms are all low degree this is feasible. We have the following lemma on products  $\epsilon_{s_i} \epsilon_{s_j}$  in  $H^*(Sp(2n), \mathbb{C})$  which is just a corollary to Chevalley's theorem.

**Lemma 5.4.** 1. If  $|i - j| \geq 2$ , then

$$\epsilon_i \epsilon_j = \epsilon_{s_i s_j}$$

2. If  $i, i + 1 \neq n$ ,

$$\epsilon_{s_i} \epsilon_{s_{i+1}} = \epsilon_{s_i s_{i+1}} + \epsilon_{s_{i+1} s_i}$$

3. If  $i \neq 1, n$ ,

$$\epsilon_i^2 = \epsilon_{s_{i-1} s_i} + \epsilon_{s_i s_{i+1}}$$

4. If  $i = 1$ ,

$$\epsilon_{s_1}^2 = \epsilon_{s_2 s_1}$$

5. If  $i = n$ ,

$$\epsilon_{s_n}^2 = 2\epsilon_{s_{n-1} s_n}$$

Using the Lemma 5.4, the quadratic term simplifies to

$$\epsilon_{s_{n-k}}^2 + 2 \sum_{j=n-k+1}^{n-1} \epsilon_{s_j}^2 + \epsilon_{s_n}^2 - 2 \sum_{j=n-k}^{n-1} \epsilon_{s_j} \epsilon_{s_{j+1}} = \epsilon_{s_{n-k-1} s_{n-k}} - \epsilon_{s_{n-k+1} s_{n-k}},$$

with  $\{s_{n-k+1} s_{n-k}, s_{n-k-1} s_{n-k}\} \in W_{n-k}^P$ . In order to fully realize the map  $\xi_{n-k\omega_1}^P$ , the images of the elementary symmetric polynomials  $e_k$  in the above torus variables must be determined. The above strategy using the Chevalley formula would be difficult as the degrees get large. Rather, as in type  $A$  we want to determine the image of  $\xi_{\omega_1}^{P_{n-k}}$  in terms of the Chern classes  $c_k(Q)$  of the tautological quotient bundle. Indeed, we have the following expansion of the previous theorem.

**Theorem 5.5.** The map  $\xi^{P_{n-k}} : \text{Rep}_{\text{poly}}^{\mathbb{C}}(L_{n-k}^{\mathbb{C}}) \rightarrow H^*(IG(n-k, 2n), \mathbb{C})$  of Theorem 4.2 takes

$$e_i \left[ \left( \frac{t_1 - t_1^{-1}}{2} \right), \dots, \left( \frac{t_{n-k} - t_{n-k}^{-1}}{2} \right) \right] \rightarrow c_i(S) = \epsilon_{(1)^i}(S) = \epsilon_{s_{n-k+i-1} \dots s_{n-k}}$$

For  $1 \leq i \leq n - k$ . Now, let  $\epsilon_i = c_i(Q)$  and let  $\epsilon_0 = 1$  and  $\epsilon_p = 0$  for  $p < 0$  or  $p > n + k$ . Then

$$e_i \left[ \left( \frac{t_{n-k+1} - t_{n-k+1}^{-1}}{2} \right)^2, \dots, \left( \frac{t_n - t_n^{-1}}{2} \right)^2 \right] \rightarrow \epsilon_i^2 + 2 \sum_{j=1}^{n+k-r} (-1)^i \epsilon_{i+j} \epsilon_{i-j}$$

for  $1 \leq i \leq k$

*Proof.* The 'Type A' part follows exactly as before. We see that

$$(\theta_{\omega_1|T}^*)^{-1} \left( e_i \left[ \left( \frac{t_1 - t_1^{-1}}{2} \right), \dots, \left( \frac{t_{n-k} - t_{n-k}}{2} \right) \right] \right) = e_i(x_1, \dots, x_{n-k})$$

Assume  $k > 1$ , then

$$\beta(e_i(x_1, \dots, x_{n-k})) = \sum_{l(w)=i} A_w(e_i(x_1, \dots, x_{n-k})) \epsilon_w.$$

Note that we are taking  $\Delta = \{e_1^* - e_2^*, \dots, e_{n-2}^* - e_{n-1}^*, 2e_n^*\}$  to be the simple roots.

Then, just as in §4.3.1, we have  $A_w(e_i(x_1, \dots, x_{n-k})) = \delta_{w, w_1^i}$ , where  $w_1^i = s_{n-k-1+i} \dots s_{n-k-1} s_{n-k}$ .

Furthermore,  $\epsilon_{w_1^i}$  is the  $i^{\text{th}}$  Chern class of the tautological subbundle [LL, 4.1].

Now let  $k = 0$ . So  $IG(n, 2n)$  is the variety of maximal isotropic planes in  $\mathbb{C}^{2n}$ . Then its clear  $A_{s_m}(e_1(x_1, \dots, x_n)) = 0$  if  $s_m \neq s_n$ . Otherwise,

$$A_{s_n}(e_1(x_1, \dots, x_n)) = \frac{e_1(x_1, \dots, x_n) - e_1(x_1, \dots, -x_n)}{2x_n}$$

Recall the following useful identity for elementary symmetric polynomials,

$$e_i(x_1, \dots, x_m) = e_i(x_1, \dots, x_{n-1}) + x_n(e_{i-1}(x_1, \dots, x_{n-1}))$$

. Then,

$$A_{s_n}(e_i(x_1, \dots, x_n)) = \frac{e_i(x_1, \dots, x_{n-1}) + x_n e_{i-1}(x_1, \dots, x_{n-1}) - e_i(x_1, \dots, x_n) + x_n e_{i-1}(x_1, \dots, x_{n-1})}{2x_n}$$

$$= \frac{2x_n e_{i-1}(x_1, \dots, x_n)}{2x_n} = e_{i-1}(x_1, \dots, x_{n-1})$$

From here the result follows as before.

The 'type C' part of the above theorem is trickier. First, we examine the polynomials associated to the Chern classes of the tautological quotient bundle. We note [LL, Remark 4.3] that for  $1 \leq i \leq k$ ,  $c_i(S) = \epsilon_{s_{n-k+i-1} \dots s_{n-k}}$  as before. For  $i \geq k+1$ , then  $c_i(S) = \epsilon_{s_{n-i+k+1} \dots s_{n-1} s_n s_{n-1} \dots s_{n-k}}$ . Let  $1 \leq i \leq k$ , then  $\beta(e_i(x_{n-k+1}, \dots, x_n)) = \epsilon_{s_{n-k+1-1} \dots s_{n-k}} = c_i(Q)$ . This is a type A element and so the proof is the same as for the quotient bundle over the grassmannian. The polynomials mapping to elements  $c_i(Q)$  for  $i \geq k$  are certain interpolations of Schur-Q and elementary symmetric polynomials called theta functions. These were developed in [BKT2, §5] (see also [W, TW]). Nevertheless, we observe how to compute  $\beta(e_i(x_{n-k+1}^2, \dots, x_n^2))$ . Under the association  $\epsilon_w = \epsilon_\lambda$  described in §5.2.1. The idea is to show that  $A_w(e_i(x_{n-k+1}^2, \dots, x_n^2)) = \pm 1$  if  $\lambda(w) = (\lambda(w)_1, \lambda(w)_2) = (2i-j+1, j-1)$  for  $1 \leq j \leq i+1$ , and  $A_w(e_i(x_{n-k}^2, \dots, x_n^2)) = 0$  otherwise. For example, if  $i = 3$  then the partitions which show up are  $(6, 0)$ ,  $(5, 1)$ ,  $(4, 2)$ ,  $(3, 3)$ . Then in general we have

$$\beta(e_i(x_{n-k+1}^2, \dots, x_n^2)) = \sum_{j=1}^{i+1} (-1)^{i+j-1} \epsilon_{(2i-j+1, j-1)}$$

Furthermore, applying to the Giambelli formula,

$$\epsilon_{(2i+1-j, j-1)} = \epsilon_{2i+1-j} \epsilon_{j-1} - \epsilon_{2i+2-j} \epsilon_{j-2}$$

and simplifying then yields

$$(e_i(x_{n-k}^2, \dots, x_n^2)) = \epsilon_i^2 + 2 \sum_{j=1}^{j=i} \epsilon_{i-j} \epsilon_{i+j}.$$

To simplify notation, we let  $l = n - k$ . Then we want to evaluate  $A_w(e_i(x_{l+1}^2, \dots, x_n^2))$  for words  $w \in W^P$  with  $l(w) = i^2$ . We have

$$A_{s_l} e_i(x_{l+1}^2, \dots, x_n^2) = (-x_l - x_{l+1}) e_{i-1}(x_{l+2}^2, \dots, x_n^2)$$

and  $A_{s_j}e_i(x_{l+1}^2, \dots, x_n^2) = 0$  if  $j \neq l$ . From here the options are  $A_{s_{l+1}}$  or  $A_{s_{l-1}}$ . If we apply  $A_{s_{l-1}}$ , using the Leibniz formula we get.

$$A_{s_{l-1}s_l}(e_i(X^2)) = A_{s_{l-1}}(-x_l - x_{l+1})e_{i-1}(x_{l+2}^2, \dots, x_n^2) + s_{l-1}(-x_{l-1} - x_l)A_{s_{l-1}}e_{i-1}(x_{l+2}^2, \dots, x_n^2)$$

But  $A_{l-1}e_{i-1}(x_{l+2}^2, \dots, x_n^2) = 0$  and we have

$$A_{s_{l-1}s_l}e_i(x_{l+1}^2, \dots, x_n^2) = e_{i-1}(x_{l+2}^2, \dots, x_n^2).$$

From here one must apply  $A_{s_{l+1}}$  and we are essentially back where we started. We remind the reader that if a word  $\tilde{w}$  is not reduced, then  $A_{\tilde{w}}=0$ . We also adopt a preferred reduced decomposition in which 'lower' reflections are moved to the right if possible. I.e., if  $i < j$  and  $s_i$  and  $s_j$  commute we will move  $s_i$  to the right of  $s_j$  if possible via the commutation or braid relations.

Now, given the above, we will prove the theorem for  $k = 2$  and then proceed by induction. So, we want to consider find  $w \in W$  such that  $A_w e_2(x_{n-1}^2, x_n^2) \neq 0$ . From above we have

$$\beta(e_1(x_{n-1}, x_n)) = \epsilon_{s_{n-3}s_{n-2}} - \epsilon_{s_{n-1}s_{n-2}}$$

The first operator applied to  $e_2(x_{n-1}^2, x_n^2)$  must be  $A_{s_{n-2}}$  as above which gives

$$A_{s_{n-2}}e_2(x_{n-1}^2, x_n^2) = (-x_{n-2} - x_{n-1})(x_n^2).$$

Choosing the lowest reduced decomposition, apply  $A_{s_{n-3}}$ ,

$$A_{s_{n-3}s_{n-2}}e_2(x_{n-1}^2, x_n^2) = x_n^2.$$

From here only  $A_{s_{n-1}}$  can be applied to give

$$A_{s_{n-1}s_{n-3}s_{n-2}}e_2(x_{n-1}^2, x_n^2) = -x_{n-1} - x_n.$$



From here apply  $A_{s_{n-2}}$  to get

$$A_{s_{n-2}s_{n-1}s_{n-3}s_{n-2}}e_2(x_{n-1}^2, x_n^2) = 1,$$

or apply  $A_{s_n}$  to get

$$A_{s_n s_{n-1} s_{n-3} s_{n-2}} e_2(x_{n-1}^2, x_n^2) = \frac{-x_{n-1} - x_n + x_{n-1} - x_n}{2x_n} = -1.$$

If instead after  $A_{s_{n-2}}$  we were to apply  $A_{s_{n-1}}$  we would get

$$A_{s_{n-1}s_{n-2}}e_2(x_{n-1}^2, x_n^2) = -x_n^2 + (x_{n-2} + x_n)(x_{n-1} + x_n).$$

Then the only choice (which we have not seen before under preferred reduced decomposition) is  $A_n$ ,

$$A_{s_n s_{n-1} s_{n-2}} e_2(x_{n-1}^2, x_n^2) = x_{n-1} + x_{n-2}.$$

Finally again our only choice to produce a new word is  $A_{s_{n-1}}$ ,

$$A_{s_{n-1}s_n s_{n-1} s_{n-2}} e_2(x_{n-1}^2, x_n^2) = 1$$

. Thus, collecting the above gives

$$\beta(e_2(x_{n-1}^2, x_n^2)) = \epsilon_{s_{n-2}s_{n-1}s_{n-3}s_{n-2}} - \epsilon_{s_n s_{n-1} s_{n-3} s_{n-2}} + \epsilon_{s_{n-1} s_n s_{n-1} s_{n-2}}.$$

In terms of  $k$ -strict partitions this give

$$\beta(e_2(x_{n-1}^2, x_n^2)) = \epsilon_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} - \epsilon_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array}} + \epsilon_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}}$$

Then, under the Giambelli formula of [BKT2] one has in this case  $\epsilon_\lambda = (1 - R_{12})m_\lambda$ , or

$$\beta(e_2(x_{n-1}^2, x_n^2)) = (\epsilon_2^2 - \epsilon_3\epsilon_1) - (\epsilon_3\epsilon_1 - \epsilon_4) + \epsilon_4 = \epsilon_2^2 - 2\epsilon_3\epsilon_1 + 2\epsilon_4$$

□

Now we are ready to state our (weaker) analog to [Theorem 4.4](#).

### 5.2.3 Inverse Limit

As in type A, define the stable cohomology ring [\[BKT3, §1.3\]](#) as

$$\mathbb{H}(IG_k) = \varprojlim H^*(IG(n - k, 2n), \mathbb{C})$$

as the inverse limit in the category of graded rings of the inverse system

$$\dots \leftarrow H^*(IG(n - k, 2n), \mathbb{C}) \leftarrow H^*(IG(n - k + 1, 2n + 2), \mathbb{C}) \leftarrow \dots$$

This ring has an additive basis of Schubert classes  $\epsilon_\lambda$  for each  $k$ -strict partition  $\lambda$ . There is a natural surjective ring homomorphism  $\mathbb{H}(IG_k) \rightarrow H^*(IG(n - k, 2n), \mathbb{C})$  given by mapping  $\epsilon_\lambda$  to  $\epsilon_\lambda$  whenever  $\lambda$  fits in a  $(n - k) \times (n + k)$  rectangle and to zero otherwise. Furthermore, from the presentation of the ring  $H^*(IG(n - k, n), \mathbb{C})$  ([Theorem 5.3](#)), none of the determinantal relations hold in the inverse limit. So,  $\mathbb{H}(IG_k)$  is isomorphic to the polynomial ring  $\mathbb{C}[\epsilon_1, \epsilon_2, \dots]$  modulo the relations

$$\epsilon_m^2 + 2 \sum_{i=1}^m (-1)^i \epsilon_{m+i} \epsilon_{m-i}$$

for  $m > k$ . To get a map from  $\text{Rep}_{\omega_1}^{\mathbb{C}}(SP(2k))$  to  $\mathbb{H}(IG_k)$  we map a polynomial  $f(h) \in \text{Rep}_{\omega_1}^{\mathbb{C}}(SP(2k)) = \mathbb{C}[(\frac{h_1 - h_1^{-1}}{2})^2, \dots, (\frac{h_k - h_k^{-1}}{2})^2]$  to  $1 \otimes f(t) \in \text{Rep}_{\omega_1}^{\mathbb{C}}(L_{n-k}^{\mathbb{C}})$ , where  $f(t)$  is the same polynomial written in the variables  $(\frac{t_i - t_i^{-1}}{2})^2$  for  $n - k + 1 < i < n$ . Then we have the map  $\xi_{n,k} := \xi^{P_{n-k}} \circ \iota_2 : \text{Rep}_{\omega_1}^{\mathbb{C}}(SP(2k)) \rightarrow$

$H^*(IG(n - k, 2n), \mathbb{C})$ . Consider the following diagram

$$\begin{array}{ccc}
 & & \uparrow \pi_n^* \\
 \text{Rep}_{\omega_1}^{\mathbb{C}}(Sp(2k)) & \xrightarrow{\xi_{n,k}} & H^*(IG(n - k + 1, 2n + 2), \mathbb{C}) \\
 & \searrow \xi_{n+1,k} & \uparrow \pi_{n+1}^* \\
 & & H^*(IG(n - k, 2n), \mathbb{C}) \\
 & & \uparrow \pi_{n+2}^*
 \end{array}$$

This map is compatible with the system since Chern classes are stable.

Then we have the following analog to 4.2,

**Theorem 5.6.** Define the map  $\xi_k : \text{Rep}_{\omega_1}^{\mathbb{C}}(Sp(2k)) \rightarrow \mathbb{H}(IG_k)$  by mapping generators

$$e_i(h_1^2, \dots, h_k^2) \rightarrow \epsilon_m^2 + 2 \sum_{i=1}^m (-1)^i \epsilon_{m+i} \epsilon_{m-i}.$$

Then this map is injective.

*Proof.* The above map is equivalent to the map  $\mathbb{C}[e_1, \dots, e_k] \rightarrow \mathbb{C}[\epsilon_1, \dots]/I$  where  $I$  is given by the relations

$$\epsilon_m^2 + 2 \sum_{i=1}^m (-1)^i \epsilon_{m+i} \epsilon_{m-i} \quad m > k.$$

Reduce coefficients to  $\mathbb{Z}$ . Note that  $\mathbb{H}(IG_k)$  over  $\mathbb{Z}$  is a free, torsion-free,  $\mathbb{Z}$  module. Reduce coefficients to  $\mathbb{Z}_2$ . Then the map becomes,

$$\mathbb{Z}_2[e_1, \dots, e_k] \rightarrow \mathbb{Z}_2[\epsilon_1, \dots, \epsilon_k] \otimes \mathbb{Z}_2[\epsilon_{k+1}, \dots] / \langle \epsilon_m^2 = 1 \rangle_{m > k}$$

, with  $e_i \rightarrow \epsilon_i$ . Then, clearly this map is injective. This suffices. □

### 5.3 Type B

The results here are nearly identical to those in type C.

#### 5.3.1 Cohomology of $OG(n-k, 2n+1)$

In this section we will describe the additive and multiplicative structure of the cohomology ring of orthogonal grassmannians  $X = OG(n - k, 2n)$ . Again we fix an integer  $k$ , the reason for this will become apparent when we want to derive a partial analogue to [Theorem 4.4](#) in type  $B$ . As for type  $A$  we have parametrizations of Schubert varieties, classes, and Poincare dual classes via index sets, partitions, and minimal length coset representatives of  $W/W_{P_{n-k}}$ . There are special Schubert classes and Pieri and Giambelli formulas as well. The ring structure can also be described by Chern classes of certain tautological bundles.

References for the following parametrizations can be found in [\[BK2, BKT1, LL, PR1, T2, T7\]](#).

Equip  $V = \mathbb{C}^{2n+1}$  with a non-degenerate symmetric bilinear form  $\vartheta$ . Fix a complete *orthogonal* flag  $F_\bullet$ ,

$$0 = F_0 \subset F_1 \subset \dots \subset F_{2n+1} = V$$

where  $F_i = F_{2n+1-i}^\perp$  with respect to  $\vartheta$ . Note,  $F_n$  is a maximal isotropic subspace. Then, we define the orthogonal grassmannian  $OG(n - k, 2n + 1)$  as,

$$OG(n - k, 2n + 1) := \{ \Omega \in Gr(n - k, 2n + 1) : \vartheta(v, v') = 0, \forall v, v' \in \Omega \}$$

Schubert varieties of  $OG(n - k, 2n + 1)$  are also parametrized by index sets  $\{I : 1 \leq p_{i_1} \leq \dots \leq p_{i_{n-k}} \leq 2n + 1\}$  such that  $p_i + p_j \neq 2n + 2$ . The open Schubert cell  $X_I^\circ$  and the closed Schubert variety  $X_I(F_\bullet)$  are defined in the same way as for type  $C$ .

Following [\[BK2\]](#), we can realize  $SO(2n + 1)$  as the fixed point subgroup  $G^\theta$  of  $G = SL(2n + 1)$  under the involution  $\sigma(A) = E^{-1}(A^t)^{-1}E$  where  $E = E_B$  as in [§5.1](#). If  $T^A \subset B^A$  are the maximal torus and Borel subgroup of  $SL(2n + 1)$ , then  $T^\sigma = T$ , and  $B^\sigma = B$  as in the [§5.1](#). Let  $\Delta_B = \{\beta_1, \dots, \beta_n\}$  be the simple roots of  $SO(2n + 1)$ . Then,  $\beta_i = \alpha_i|_{\mathfrak{t}}$  where  $\{\alpha_1, \dots, \alpha_{2n}\}$  are the simple roots of  $SL(2n + 1)$ . The

corresponding simple coroots are given by

$$\check{\beta}_i = \check{\alpha}_i + \check{\alpha}_{2n+1-i}, \text{ for } 1 \leq i \leq n,$$

and

$$\check{\beta}_n = 2\check{\alpha}_n + 2\check{\alpha}_{n+1}$$

Under the inclusion  $W_B \subset S_{2n}$  we have that the simple reflections of  $SO(2n)$  are given by

$$\begin{aligned} s_i &= r_i r_{2n+1-i} \text{ if } 1 \leq i \leq n-1 \\ &= r_n r_{n+1} r_n \text{ if } i = n \end{aligned}$$

where  $r_i$  is the  $i^{\text{th}}$  simple reflection for  $Sl(2n+1)$ . The Weyl group  $W_{B_n}$  can be identified with the subset of  $W_{A_{2n+1}}$  invariant under  $\sigma$ :

$$\{(a_1, \dots, a_{2n+1}) \in S_{2n+1} : a_{2n+2-i} = 2n+2 - a_i \forall 1 \leq i \leq 2n\}$$

. Consider the parabolic Weyl subgroup generated by  $\Delta_B - \{\beta_{n-k}\}$ . Then the minimal length coset representatives of  $W_B/W_{B, P_{n-k}}$  can be identified with the set

$$I(n-k, 2n+1) = \{I := 1 \leq p_1 < \dots < p_n \leq 2n+1 \text{ } p_j \neq n+1 \text{ for any } j \text{ and } I \cap \bar{I} = \emptyset\},$$

where  $\bar{I} = \{2n+2 - p_1, \dots, 2n+2 - p_{n-k}\}$ . But this is just an index set. It represents the permutation in  $S_{2n+1}$  given by taking  $p_{n-k} + 1, \dots, p_n = [n] \setminus (I \sqcup \bar{I})$  and setting  $p_{2n+1-i} = 2n+1 - p_i$ .

The Schubert varieties are also parametrized by the same set of  $k$ -strict partitions which fit in a  $(n-k) \times (n+k)$  rectangle. Following [BKT1, §4.2], to any  $k$ -strict partition, let  $I_j(\lambda) = n+k+1 - \lambda_j + \{i < j : \lambda_i + \lambda_j \leq 2k + j - i\}$ . Then the appropriate index set for the type  $B$  Schubert variety is given by  $\bar{I}$  where,

$$\bar{I}_j(\lambda) = \begin{cases} I_j(\lambda) + 1 & \text{if } \lambda_j \leq k, \\ I_j(\lambda) & \text{if } \lambda_j > k \end{cases}$$

Let  $\epsilon_\lambda$  be the Schubert class associated to a  $k$ -strict partition as before. Then, again,  $\epsilon_i$  are the special Schubert classes.

As in types  $A$  and  $C$ , there is a short exact sequence of vector bundles

$$0 \rightarrow S_B \rightarrow V_B \rightarrow Q_B \rightarrow 0$$

where  $V_B$  is the trivial bundle and  $S_B$  and  $Q_B$  are the tautological sub and quotient bundles. For a given  $k$ -strict partition  $\lambda$ , let  $l_k(\lambda)$  be the number of parts of  $\lambda$  which are strictly greater than  $k$ . Then a well known result of [BS], when translated into the language of  $k$ -strict partitions in [BKT4], says that the map taking  $c_p(Q_C)$  to  $c_p(Q_B)$  extends to a ring isomorphism  $\phi : H^*(IG(n-k, 2n), \mathbb{C}) \rightarrow H^*(OG(n-k, 2n+1), \mathbb{C})$  such that  $\phi(\epsilon_\lambda) = 2^{l_k(\lambda)} \epsilon_\lambda$ .

Buch, Kresch, and Tamkivakis have also shown that

$$c_i(Q_B) = \begin{cases} \epsilon_i & \text{if } i \leq k \\ 2\epsilon_i & \text{if } i > k \end{cases}$$

The Giambelli formula is then given by

$$\epsilon_\lambda = 2^{-l_k(\lambda)} R_\lambda m_\lambda$$

Note that  $m_\lambda = \prod_i c_{\lambda_i}$  is given in terms of the Chern classes (which unlike type  $C$  do not exactly match up the special Schubert classes  $\epsilon_i$ ). In terms of the variables  $c_i = c_i(Q_B)$  for  $1 \leq i \leq n-k$  (with  $c_0 = 1$  and  $c_i = 0$  if  $i < 0$  or  $n-k > 0$ ),  $H^*(OG(n-k, 2n+1), \mathbb{C})$  has the same presentation  $H^*(IG(n-k, 2n), \mathbb{C})$  from [Theorem 5.3](#).

### 5.3.2 Theorem 4.2

As for type  $A$ , we aim to explicitly determine the map in [Theorem 4.2](#) for  $G = So(2n+1)$  and any maximal parabolic  $P_{n-k}$ . Take  $V = \mathbb{C}^{2n+1}$  and  $OG(n-k, 2n+1)$  as in the previous section. Here we follow and expand on [Ku2, §8].

We take  $B_B := B \cap SO_{2n+1}$  as the Borel subgroup of  $SO_{2n+1}$ , where  $B$  is the standard Borel subgroup of  $SL_{2n+1}$  consisting of upper triangular matrices of determinant 1. Then,  $OG(n-k, 2n+1)$  is the quotient  $SO_{2n+1}/P_{n-k}$  of  $SO_{2n+1}$  by the standard maximal parabolic subgroup  $P_{n-k}$  with  $\Delta \setminus \{\alpha_{n-k}\}$  as the set of simple roots of its Levi component  $L_{n-k}$ . (We take  $L_{n-k}$  to be the unique Levi subgroup of  $P_r$  containing  $T_B$ ). Then,

$$L_{n-k} \simeq GL(n-k) \times SO(2k+1).$$

From [Lemma 5.2](#), we have

$$\theta_{\omega_1}(\mathbf{t}_C) = \left( \frac{t_1 - t_1^{-1}}{2}, \dots, \frac{t_n - t_n^{-1}}{2}, 0, -\left(\frac{t_n - t_n^{-1}}{2}\right), \dots, -\left(\frac{t_1 - t_1^{-1}}{2}\right) \right)$$

Also, recall that  $Rep_{\omega_1}^{\mathbb{C}}(L_{n-k}) \simeq S(\mathfrak{t}_B^*)^{W_{P_{n-k}}}$ . Using the fundamental invariants from §4.1.1 we find that the representation ring is given by,

$$\begin{aligned} \text{Rep}_{\omega_1}^{\mathbb{C}}(L_{n-k}) &\simeq \mathbb{C}_{sym} \left[ \left( \frac{t_1 - t_1^{-1}}{2} \right), \dots, \left( \frac{t_{n-k} - t_{n-k}^{-1}}{2} \right) \right] \\ &\otimes_{\mathbb{C}} \mathbb{C}_{sym} \left[ \left( \frac{t_{n-k+1} - t_{n-k+1}^{-1}}{2} \right)^2, \dots, \left( \frac{t_n - t_n^{-1}}{2} \right)^2 \right] \end{aligned}$$

where  $\mathbb{C}_{sym}$  denotes the subalgebra of the polynomial ring consisting of symmetric polynomials. Further, by [Theorem 5.1](#),

$$\mathbb{C}_{sym} \left[ \left( \frac{t_{r+1} - t_{r+1}^{-1}}{2} \right)^2, \dots, \left( \frac{t_n - t_n^{-1}}{2} \right)^2 \right]$$

is generated (as a  $\mathbb{C}$ -algebra) by the virtual representations:

$$\{\lambda^d ([S^2(V'_{2k+1})] - [\Lambda^2(V'_{2k+1})]) - [\epsilon]\}_{1 \leq d \leq k},$$

where  $V'_{2k+1} = \mathbb{C}^{2k}$  is the standard representation of  $SO(2k+1)$ ,  $[\epsilon]$  is the trivial one-dimensional representation, and  $\lambda$  is the  $\lambda$ -ring structure on  $Rep(G)$ .

The following theorem [Ku2, Proposition20] partially determined the homomorphism of Theorem 4.2

**Theorem 5.7.** *The map  $\xi^{P_{n-k}} : \text{Rep}_{\text{poly}}^{\mathbb{C}}(L_{n-k}^{\mathbb{C}}) \rightarrow H^*(OG(n-k, 2n+1), \mathbb{C})$  of Theorem 4.2 takes*

$$\frac{1}{2}(t_1 - t_1^{-1} + \cdots + t_{n-k} + t_{n-k}^{-1}) \rightarrow \epsilon_{s_{n-k}}, \text{ if } k > 0,$$

$$\frac{1}{2}(t_1 - t_1^{-1} + \cdots + t_{n-k} + t_{n-k}^{-1}) \rightarrow 2\epsilon_{s_{n-k}}, \text{ if } k = 0,$$

and

$$\frac{1}{4}[(t_{n-k+1} - t_{n-k+1}^{-1})^2 + \cdots + (t_n + t_n^{-1})^2] \rightarrow \epsilon_{s_{n-k}}^2 + 2 \sum_{j=n-k+1}^{n-1} \epsilon_{s_j}^2 + 4\epsilon_{s_n}^2 - 2 \sum_{j=n-k}^{n-1} \epsilon_{s_j} \epsilon_{s_{j+1}} - 4\epsilon_{s_{n-1}} \epsilon_{s_n}$$

*Proof.* For  $1 \leq n$ , let  $x_i : \mathfrak{t} \rightarrow \mathbb{C}$  be the linear map which takes

$$\text{diag}(x_1, \dots, x_n, 0, -x_n, \dots, -x_1) \rightarrow x_i$$

. Then by Lemma 5.2, the homomorphism  $\theta_{\omega_1|T}^* : \mathbb{C}[\mathfrak{t}] \rightarrow \mathbb{C}[T]$  induced from the Springer morphism  $\theta_{\omega_1}$  takes

$$x_1 + \cdots + x_{n-k} \rightarrow \frac{1}{2}(t_1 - t_1^{-1} + \cdots + t_{n-k} + t_{n-k}^{-1}).$$

However, note that the weight  $x_1 + \cdots + x_{n-k}$  is the first fundamental weight  $\omega_{n-k}$  if  $k > 0$ . Thus,

$$\beta \circ (\theta_{\omega_1|T}^*)^{-1} \left( \frac{1}{2}(t_1 - t_1^{-1} + \cdots + t_{n-k} + t_{n-k}^{-1}) \right) = \beta(x_1 + \cdots + x_{n-k}) = \beta(\omega_{n-k}) = \epsilon_{s_{n-k}}.$$

If  $k = 0$ , one has

$$x_1 + \cdots + x_n \rightarrow \frac{1}{2}(t_1 - t_1^{-1} + \cdots + t_n + t_n^{-1}).$$



Note that for type  $B$  that  $\omega_n = \frac{x_1 + \dots + x_n}{2}$ . Thus,

$$\beta \circ (\theta_{\omega_1|T}^*)^{-1} \left( \frac{1}{2} (t_1 - t_1^{-1} + \dots + t_n + t_n^{-1}) \right) = \beta(x_1 + \dots + x_n) = \beta(2\omega_n) = 2\epsilon_{s_n}.$$

Similiarly, under  $(\theta_{\omega_1|T}^*)^{-1}$ ,

$$\frac{1}{4} \left( (t_{n-k+1}^2 - t_{n-k+1})^{-1} + \dots + (t_n + t_n^{-1})^2 \right) \rightarrow x_{n-k}^2 + \dots + x_n^2$$

In type  $B$  we have  $\omega_i = \sum_{j=1}^i x_j$  for  $1 \leq i \leq n-1$ , and  $\omega_n = \frac{x_1 + \dots + x_n}{2}$ . Then we can right the coordinate functions in the fundamental weight basis  $x_i = \omega_i - \omega_{i-1}$  for  $1 \leq i \leq n-1$ , and  $x_n = 2\omega_n - \omega_{n-1}$ . So,  $x_{n-k+1}^2 + \dots + x_n^2 = (\omega_{n-k+1} - \omega_{n-k})^2 + \dots + (\omega_{n-1} - \omega_{n-2})^2 + (2\omega_n - \omega_{n-1})^2$ . Then from the remark above it is clear that  $\xi^{P_{n-k}}$  takes

$$\frac{1}{4} \left( (t_{n-k+1}^2 - t_{n-k+1})^{-1} + \dots + (t_n + t_n^{-1})^2 \right) \rightarrow (\epsilon_{s_{n-k+1}} - \epsilon_{s_{n-k}})^2 + \dots + (\epsilon_{s_{n-1}} - \epsilon_{s_{n-2}})^2 + (2\epsilon_{s_n} - \epsilon_{s_{n-1}})^2,$$

which expands to give the stated result. To write it in the  $W_B^{P_{n-k}}$  basis one can use the Chevalley formula or we note that

$$A_{s_j} (x_{n-k+1}^2 + \dots + x_n^2) = -x_{n-k} - x_{n+1} \text{ if } j = n-k \text{ and } 0 \text{ otherwise.}$$

Then we have  $A_{s_{n-k-1}}(-x_{n-k} - x_{n-k+1}) = 1$  and  $A_{s_{n-k+1}}(-x_{n-k} - x_{n-k+1}) = -1$ . Thus, we have exactly as in type  $C$ ,

$$\xi^{P_{n-k}} \left[ \frac{1}{4} \left[ (t_{n-k+1} - t_{n-k+1}^{-1})^2 + \dots + (t_n + t_n^{-1})^2 \right] \right] \rightarrow \epsilon_{s_{n-k-1}s_{n-k}} - \epsilon_{s_{n-k+1}s_{n-k}}$$

□

In terms of the Chern classes  $c_i$  we get essentially the same result as in Theorem 5.5

**Theorem 5.8.** The map  $\xi^{P_{n-k}} : \text{Rep}_{\text{poly}}^{\mathbb{C}}(L_{n-k}^B) \rightarrow H^*(OG(n-k, 2n+1), \mathbb{C})$  of Theorem 4.2 takes

$$e_i \left[ \left( \frac{t_1 - t_1^{-1}}{2} \right), \dots, \left( \frac{t_{n-k} - t_{n-k}}{2} \right) \right] \rightarrow c_i(S) = \epsilon_{(1)^i}(S) = \epsilon_{s_{n-k+i-1} \dots s_{n-k}} \text{ if } k > 0$$

$$e_i \left[ \left( \frac{t_1 - t_1^{-1}}{2} \right), \dots, \left( \frac{t_{n-k} - t_{n-k}}{2} \right) \right] \rightarrow c_i(S) = \epsilon_{(1)^i}(S) = 2\epsilon_{s_{n-k+i-1} \dots s_{n-k}} \text{ if } k=0$$

For  $1 \leq i \leq n-k$ , let  $c_i = c_i(Q_B)$ . Define  $c_0 = 1$  and  $c_p = 0$  for  $p < 0$  or  $p > n+k$ . Then,

$$e_i \left[ \left( \frac{t_{n-k+1} - t_{n-k+1}^{-1}}{2} \right)^2, \dots, \left( \frac{t_n - t_n}{2} \right)^2 \right] \rightarrow c_i^2 + 2 \sum_{j=1}^{n+k-i} (-1)^j c_{i+j} c_{i-j}$$

for  $1 \leq i \leq k$ .

*Proof.* For the type A part with  $k > 0$  the proof is the same as for Theorem 5.5. For  $k = 0$ , it is essentially the same except in this case

$$A_{s_n}(e_i(x_2, \dots, x_n)) = \frac{e_i(x_1, \dots, x_n) - e_i(x_1, \dots, -x_n)}{x_n},$$

and we see that

$$A_{s_n}(e_i(x_1, \dots, x_n)) = \frac{2x_n e_i - 1(x_1, \dots, x_{n-1})}{x_n} = 2e_{i-1}(x_1, \dots, x_{n-1}).$$

The result follows from here.

The type B part is analogous to the proof of Theorem 5.5 up to accounting for factors of two when applying  $A_{s_n}$  and using the type B Giambelli formula.  $\square$

### 5.3.3 Inverse Limit

Analogously, we define the stable cohomology ring [BKT3, §3.2] as

$$\mathbb{H}(OG_k) = \varprojlim H^*(OG(n-k, 2n+1), \mathbb{C})$$

as the inverse limit in the category of graded rings of the inverse system

$$\dots \leftarrow H^*(OG(n-k, 2n+1), \mathbb{C}) \leftarrow H^*(OG(n-k+1, 2n+3), \mathbb{C}) \leftarrow \dots$$

This ring has an additive basis of Schubert classes  $\epsilon_\lambda$  for each  $k$ -strict partition  $\lambda$ . There is a natural surjective ring homomorphism  $\mathbb{H}(OG_k) \rightarrow H^*(OG(n-k, 2n+1), \mathbb{C})$  given by mapping  $\epsilon_\lambda$  to  $\epsilon_\lambda$  whenever  $\lambda$  fits in a  $(n-k) \times (n+k)$  rectangle and to zero otherwise. In terms of the variables  $c_i$  we have that  $\mathbb{H}(OG_k)$  has the exact same presentation as  $\mathbb{H}(IG_k)$  from §5.2.3. Factoring through  $\xi^{P_{n-k}} : \text{Rep}_{\text{omega}_1}^{\mathbb{C}}(L_{n-k}) \rightarrow H^*(SO(n-k, 2n+1))$  as in §5.2.3, we get a map from  $\text{Rep}^{\mathbb{C}} SO(2k+1)$  to  $\mathbb{H}(OG_k)$ . Then we have the analogous theorem,

**Theorem 5.9.** *The map  $\xi_k : \text{Rep}_{\omega_1}^{\mathbb{C}}(SO(2k+1)) \rightarrow \mathbb{H}(OG_k)$  given by mapping generators*

$$e_i(h_1^2, \dots, h_k^2) \rightarrow c_i^2 + 2 \sum_{j=1}^i (-1)^i c_{i+j} c_{i-j}$$

*is injective.*

## 5.4 G2

### 5.4.1 Representation Ring of G2

Here we compute the  $\omega_1$ -polynomial representation ring of  $G_2$ . Note that  $\omega_1$  is the fundamental representation of minimal Dynkin index. We first need to compute the Springer morphism. We write  $\theta_{\omega_1}(t) = c_1(t)\check{\alpha}_1 + c_2(t)\check{\alpha}_2$ . From §3.2 we saw that

$$c_1(t, \lambda) = \frac{2}{3x} \sum_{\mu \in \Lambda_\lambda} (2\mu_1 + 3\mu_2) e^\mu(t)$$

$$c_2(t, \lambda) = \frac{2}{3x} \sum_{\mu \in \Lambda_\lambda} (3\mu_1 + 6\mu_2) e^\mu(t)$$

, for  $\theta_\lambda$ . In this case  $x = \sum_{\Lambda_\lambda} \mu_2^2$  where  $\mu_2$  is the coordinate of the second fundamental weight for a given weight  $\mu \in \Lambda_\lambda$ . For  $\lambda = \omega_1$  the weights are

$$\Lambda_{\omega_1} = \{(1, 0), (-1, 0), (1, -1), (-1, 1), (2, -1), (-2, 1)\}.$$

Let  $s$  be the simple reflection associated to the first simple root  $\alpha_1$  and  $t$  be the simple reflection for  $\alpha_2$ . To state the result more clearly we note the following from [A, Appendix A]. There is an embedding  $W_{G_2} \rightarrow W_{A_6}$  given by  $s \rightarrow r_{12}r_{35}r_{67}$  and  $t \rightarrow r_{23}r_{56}$  where  $r_{ij}$  is the transposition  $(i, j)$  in  $S_7$ . This inclusion corresponds to an inclusion  $G_2 \rightarrow GL(7)$ . The inclusion of tori is given by

$$(t_1, t_2) \rightarrow (t_1, t_2, t_1 t_2^{-1}, 1, t_1^{-1} t_2, t_2^{-1}, t_1^{-1}).$$

The inclusion on Cartan subalgebras is then given by

$$(h_1, h_2) \rightarrow (h_1, h_2, h_1 - h_2, 0, h_2 - h_1, -h_2, -h_1).$$

Let  $x_i$  be the  $i^{\text{th}}$  coordinate function on the Cartan subalgebra. We then have the following root data for  $G_2$ .

The simple roots are given by

$$\alpha_1 = x_1 - x_2, \quad \alpha_2 = -x_1 + 2x_2.$$

The simple coroots are given by

$$\check{\alpha}_1 = 3h_1 - 3h_2, \quad \check{\alpha}_2 = -h_1 + 2h_2.$$

The fundamental weights are given by

$$\omega_1 = x_1, \quad \omega_2 = x_1 + x_2.$$

We also note that  $\alpha_3 = \alpha_1 + \alpha_2 = x_2$  and  $\alpha_4 = 2\alpha_1 + \alpha_2 = x_1$ . We can also now write  $e^\mu(t) = e^{(\mu_1, \mu_2)}(t_1, t_2) = t_1^{\mu_1 + \mu_2} t_2^{\mu_2}$ . From the fundamental weights given above and the formulas from §3.2, we

have

$$c_1(t) = \frac{1}{6}(2e^{(1,0)}(t) + e^{(-1,1)}(t) + e^{(2,-1)}(t) - e^{(-2,1)}(t) - e^{(1,-1)}(t) - 2e^{(-1,0)}(t))$$

$$c_2(t) = \frac{1}{6}(3e^{(1,0)}(t) + 3e^{(-1,1)}(t) - 3e^{(1,-1)}(t) - 3e^{(-1,0)}(t)).$$

More explicitly using  $e^\mu(t) = t_1^{\mu_1} t_2^{\mu_2}$  we have,

$$c_1(t) = \frac{1}{6}(2t_1 + t_2 + t_1 t_2^{-1} - t_1^{-1} t_2 - t_2^{-1} - 2t_1^{-1})$$

$$c_2(t) = \frac{1}{2}(t_1 + t_2 - t_1^{-1} - t_2^{-1})$$

Then, we have  $\theta_{\omega_1}(t_1, t_2) = c_1(t)\check{\alpha}_1 + c_2(t)\check{\alpha}_2 = (3c_1(t) - c_2(t), -3c_1(t) + 2c_2(t))$  which gives

$$\theta_{\omega_1}(t_1, t_2) = \left( \frac{t_1 - t_1^{-1} + t_1 t_2^{-1} - t_1^{-1} t_2}{2}, \frac{t_2 - t_2^{-1} + t_1^{-1} t_2 - t_1 t_2^{-1}}{2} \right)$$

Now, recall that by definition  $\text{Rep}_{\omega_1}^{\mathbb{C}}(G) = S(\mathfrak{t}^*)^W$ . The Weyl group of  $G_2$  is the dihedral group  $D_6 = \{s, t | s^2 = t^2 = (st)^6 = 1\}$  and has fundamental invariants 2,6 (see [Table 4.1](#)). The following set of polynomial invariants is given by [\[Ts, 3.3\]](#)(other sets of invariants can be found in [\[Lee, Me\]](#)). in terms of the simple roots  $\alpha_1$  and  $\alpha_2$ :

$$f_{2k} = \alpha_1^{2k} + (\alpha_1 + \alpha_2)^{2k} + (2\alpha_1 + \alpha_2)^{2k}.$$

Then we have  $S(\mathfrak{t}^*) = \mathbb{C}[f_2, f_6]$ . Using the coordinates from above we can re-write these as

$$f_{2k} = (x_1 - x_2)^{2k} + x_1^{2k} + x_2^{2k}.$$

So,

$$f_2 = 2x_1^2 + 2x_2^2 - 2x_1 x_2$$

$$f_6 = 2x_1^6 - 6x_1^5 x_2 + 15x_1^4 x_2^2 - 20x_1^3 x_2^3 + 15x_1^2 x_2^4 - 6x_1 x_2^5 + 2x_2^6.$$

Then,

$$\begin{aligned} \text{Rep}_{\omega_1}^{\mathbb{C}}(G_2) &= \mathbb{C}[f_2\left(\frac{t_1 - t_1^{-1} + t_1 t_2^{-1} - t_1^{-1} t_2}{2}, \frac{t_2 - t_2^{-1} + t_1^{-1} t_2 - t_1 t_2^{-1}}{2}\right), \\ &f_6\left(\frac{t_1 - t_1^{-1} + t_1 t_2^{-1} - t_1^{-1} t_2}{2}, \frac{t_2 - t_2^{-1} + t_1^{-1} t_2 - t_1 t_2^{-1}}{2}\right)]. \end{aligned}$$

We have the following analogue of [Ku2, Proposition 24]. Under the coordinates of  $\theta_{\omega_1}(t_1, t_2)$  on the maximal torus  $T \subset G_2$ .

$$\text{Rep}_{\omega_1}^{\mathbb{C}}(T) = \mathbb{C}\left[\frac{t_1 - t_1^{-1} + t_1 t_2^{-1} - t_1^{-1} t_2}{2}, \frac{t_2 - t_2^{-1} + t_1^{-1} t_2 - t_1 t_2^{-1}}{2}\right]$$

where we think of  $T$  as the Levi subgroup of  $B$ .

**Theorem 5.10.** *Under the homomorphism  $\xi^B : \text{Rep}_{\omega_1}^{\mathbb{C}}(T) \rightarrow H^*(G_2/B, \mathbb{C})$ , we have*

$$\frac{t_1 - t_1^{-1} + t_1 t_2^{-1} - t_1^{-1} t_2}{2} \rightarrow \epsilon_{s_1}$$

and,

$$\frac{t_2 - t_2^{-1} + t_1^{-1} t_2 - t_1 t_2^{-1}}{2} \rightarrow \epsilon_{s_2} - \epsilon_{s_1}$$

*Proof.* Observe that  $x_1 = \omega_1$  and  $x_2 = \omega_2 - \omega_1$ , and  $\beta(\omega_i) = \epsilon_{s_i}$ . □

## BIBLIOGRAPHY

- [A] D. Anderson: *Chern class formulas for  $G_2$  schubert loci*, Transactions of the AMS. **363** no. 12 (2011), 6615–6646
- [AT] M.F. Atiyah and D.O. Tall, Group representations,  $\lambda$ -rings and the  $J$ -homomorphism, *Topology* **8** (1969), 253–297.
- [BR] P. Bardsley and R.W. Richardson, *Étale slices for algebraic transformation groups in characteristic  $p$* , Proc. London Math. Soc. **51** (1985), 295–317.
- [Be] P. Belkale : *Invariant theory of  $GL(n)$  and intersection theory of Grassmannians*, Int. Math. Res. Not. **2004**, no. 69, 3709–3721.
- [BK1] P. Belkale and S. Kumar : *Eigenvalue problem and a new product in cohomology of flag varieties*, Invent. Math. **166** (2006), 185–228.
- [BK2] P. Belkale and S. Kumar : *Eigencone, saturation, and Horn problems for symplectic and odd orthogonal groups*, J. Algebraic Geom. **19** (2010), no.2, 199–242.
- [BS] N. Bergeron and F. Sottile : *A Pieri-type formula for isotropic flag manifolds*, Trans. Amer. Math. Soc. **354** (2002), 4815–4829.
- [BGG] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand : *Schubert cells and cohomology of the spaces  $G/P$* , Russian Math. Surveys **28** (1973), 1–26.
- [Bi] S. Billey : *Kostant polynomials and the cohomology ring for  $G/B$* , Duke Math. J. **96** (1999), 205–224.
- [BH] S. Billey and M. Haiman : *Schubert polynomials for the classical groups*, J. Amer. Math. Soc. **8** (1995), 443–482.
- [BJS] S. Billey, W. Jockusch and R. P. Stanley : *Some combinatorial properties of Schubert polynomials*, J. Algebraic Combin. **2** (1993), 345–374.
- [Bo] A. Borel : *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. **57** (1953), 115–207.
- [Bou] N. Bourbaki, *Groupes et Algèbres de Lie*, Chap. 4–6, Masson, Paris, 1981.
- [Br] M. Brion : *Lectures on the geometry of flag varieties*, Topics in Cohomological Studies of Algebraic Varieties. Trends in Mathematics. Birkhäuser Basel (2005), 33–85.
- [BD] T. Brocker and T. tom Dieck, *Representations of Compact Lie Groups*, Springer-Verlag, 1985.

- [BKT1] A. S. Buch, A. Kresch and H. Tamvakis : *Quantum Pieri rules for isotropic Grassmannians*, Invent. Math. **178** (2009), 345–405.
- [BKT2] A. S. Buch, A. Kresch and H. Tamvakis : *A Giambelli formula for isotropic Grassmannians*, Selecta Math. (N.S.),
- [BKT3] A. S. Buch, A. Kresch and H. Tamvakis : *Quantum Giambelli formulas for isotropic Grassmannians*, Math. Annalen **354** (2012), 801–812.
- [BKT4] A. S. Buch, A. Kresch and H. Tamvakis : *A Giambelli formula for even orthogonal Grassmannians*, J. reine angew. Math. **708** (2015), 17–48.
- [C2] É. Cartan : *Sur les invariants intégraux de certains espaces homogènes clos et les propriétés topologiques de ces espaces*, Annales de la Société Polonaise de Mathématique **8** (1929), 181–225
- [D1] M. Demazure : *Invariants symétriques des groupes de Weyl et torsion*, Invent. Math. **21** (1973), 287–301.
- [D2] M. Demazure : *Désingularisation des variétés de Schubert généralisées*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 53–88.
- [FK2] S. Fomin and A. N. Kirillov : *The Yang-Baxter equation, symmetric functions, and Schubert polynomials*, Discrete Math. **153** (1996), 123–143.
- [FK3] S. Fomin and A. N. Kirillov : *Combinatorial  $B_n$ -analogs of Schubert polynomials*, Trans. Amer. Math. Soc. **348** (1996), 3591–3620.
- [FS] S. Fomin and R. P. Stanley : *Schubert polynomials and the nil-Coxeter algebra*, Adv. Math. **103** (1994), 196–207.
- [F3] W. Fulton : *Determinantal formulas for orthogonal and symplectic degeneracy loci*, J. Differential Geom. **43** (1996), 276–290.
- [F4] W. Fulton : *Young tableaux*, London Mathematical Society Student Texts **35**, Cambridge University Press, Cambridge, 1997.
- [F5] W. Fulton : *Intersection theory*, Second edition, Ergebnisse der Math. **2**, Springer-Verlag, Berlin, 1998.
- [F6] W. Fulton : *Eigenvalues, invariant factors, highest weights, and Schubert calculus*, Bulletin of the AMS. **37** (2000), 209–249.
- [FH] W. Fulton and J. Harris, *Representation Theory*, Graduate Texts in Mathematics, vol. **129**, Springer, 1991.
- [FP] W. Fulton and P. Pragacz : *Schubert varieties and degeneracy loci*, Lecture Notes in Math. **1689**, Springer-Verlag, Berlin, 1998.



- [G1] G. Z. Giambelli : *Risoluzione del problema degli spazi secanti*, Mem. R. Accad. Sci. Torino (2) **52** (1902), 171–211.
- [G2] G. Z. Giambelli : *Alcune proprietà delle funzioni simmetriche caratteristiche*, Atti Torino **38** (1903), 823–844.
- [Hi] H. Hiller : *Geometry of Coxeter groups*, Research Notes in Mathematics, **54**. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1982.
- [HB] H. Hiller and B. Boe : *Pieri formula for  $SO_{2n+1}/U_n$  and  $Sp_n/U_n$* , Adv. Math. **62** (1986), 49–67.
- [Hu] J.E. Humphreys : *Reflection and Coxeter Groups*, Cambridge studies in advanced mathematics, **29**. Cambridge University Press, 1990.
- [I] T. Ikeda : *Schubert classes in the equivariant cohomology of the Lagrangian Grassmannian*, Adv. Math. **215** (2007), 1–23.
- [IM] T. Ikeda and T. Matsumura : *Pfaffian sum formula for the symplectic Grassmannians*, Math. Z. **280** (2015), 269–306.
- [IMN] T. Ikeda, L. C. Mihalcea, and H. Naruse : *Double Schubert polynomials for the classical groups*, Adv. Math. **226** (2011), 840–886.
- [IN1] T. Ikeda and H. Naruse : *Excited Young diagrams and equivariant Schubert calculus*, Trans. Amer. Math. Soc. **361** (2009), 5193–5221.
- [Iv] V. N. Ivanov : *Interpolation analogues of Schur  $Q$ -functions*, Zap. Nauchn. J. Math. Sci. (N. Y.) **131** (2005), 5495–5507 .
- [K] Kajimoto, Hiroshi : *The Poincare duality and Gysin homomorphism for partial flag varieties*, Hiroshima Math J. **27** (1997), 189–207.
- [KL] S.L. Kleiman and D. Laksov : *Schubert calculus*, The American Mathematical Monthly. **79** (1972), 1061–1082.
- [KK] B. Kostant and S. Kumar :  *$T$ -equivariant  $K$ -theory of generalized flag varieties*, J. Differential Geom. **32** (1990), 549–603.
- [KM] B. Kostant and P. Michor, *The generalized Cayley map from an algebraic group to its Lie algebra*. Duval, Christian (ed.) et al., *The orbit method in geometry and physics. In honor of A. A. Kirillov. Papers from the international conference, Marseille, France, December 4–8, 2000*. Boston, MA: Birkhäuser, Prog. Math. 213, 259–296 (2003).
- [Kr] W. Kraśkiewicz : *Reduced decompositions in hyperoctahedral groups*, C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), 903–907.
- [KT1] A. Kresch and H. Tamvakis : *Double Schubert polynomials and degeneracy loci for the classical groups*, Ann. Inst. Fourier **52** (2002), 1681–1727.

- [KT3] A. Kresch and H. Tamvakis : *Quantum cohomology of orthogonal Grassmannians*, Compos. Math. **140** (2004), 482–500.
- [Ku1] S. Kumar, *Kac-Moody Groups, their Flag Varieties and Representation Theory*, Progress in Mathematics, vol. **204**, Birkhäuser, 2002.
- [Ku2] S. Kumar, *Representation ring of Levi subgroups versus cohomology ring of flag varieties*, S. Math. Ann.(2016) 366–395
- [Ku3] S. Kumar, *Conformal blocks, generalized theta functions and the Verlinde formula*, forthcoming book
- [KLM] Shrawan Kumar, Bernhard Leeb, and John Milsom : *The generalized triangle inequalities for rank 3 symmetric spaces of noncompact type*. Contemporary Mathematics. **332** (2003), 171–195
- [La1] T. K. Lam : *B and D analogues of stable Schubert polynomials and related insertion algorithms*, Ph.D. thesis, M.I.T., 1994; available at <http://hdl.handle.net/1721.1/36537>.
- [La2] T. K. Lam : *Bn Stanley symmetric functions*, Discrete Math. **157** (1996), 241–270.
- [LL] N.C. Leung and C.Li : *Quantum pieri rules for tautological subbundles*, Adv. Math. **248** (2013), 279–307.
- [LP1] A. Lascoux and P. Pragacz : *Operator calculus for  $\tilde{Q}$ -polynomials and Schubert polynomials*, Adv. Math. **140** (1998), 1–43.
- [LP2] A. Lascoux and P. Pragacz : *Orthogonal divided differences and Schubert polynomials,  $\tilde{P}$ -functions, and vertex operators*, Michigan Math. J. **48** (2000), 417–441.
- [Lee] C.Y. Lee : *Invariant polynomials of Weyl groups and applications to the centres of universal enveloping algebras*, Can. J. Math. **XXVI** (1974), 283–592.
- [LPR1] N. Lemire, V. Popov, Z. Reichstein: *Cayley groups*, J. Amer. Math. Soc. **19** (2006), 921–967
- [LPR2] N. Lemire, V. Popov, Z. Reichstein: *On the Cayley degree of an algebraic group*, Biblioteca de la Revista Matematica Iberoamericana, Actas del XVI Coloquio Latinoamericana de Algebra, Madrid (2007), 87–97.
- [Les] L. Lesieur : *Les probl‘emes d’intersection sur une vari‘et‘e de Grassmann*, C. R. Acad. Sci. **225** (1947), 916—917.
- [M] I. G. Macdonald : *Symmetric functions and Hall polynomials*, Second edition, The Clarendon Press, Oxford University Press, New York, 1995.
- [Me] M.L. Mehta : *Basic sets of invariants for finite reflection groups*, Communications in Algebra **16** (1988), 1083–1098.

- [MTV] E. Mukhin, V. Tarasov, and A. Varchenko : *Schubert and representations of the general linear group*, Journal of the AMS. **22** (2009), 909–940.
- [P1] P. Pragacz : *Enumerative geometry of degeneracy loci*, Ann. Sci. École Norm. Sup. (4) **21** (1988), 413–454.
- [P2] P. Pragacz : *Algebro-geometric applications of Schur  $S$ - and  $Q$ -polynomials*, Séminaire d'Algèbre Dubreil-Malliavin 1989-1990, Lecture Notes in Math. **1478** (1991), 130–191, Springer-Verlag, Berlin, 1991.
- [P3] P. Pragacz : *Symmetric polynomials and divided differences in formulas of intersection theory*, Parameter spaces (Warsaw, 1994), 125–177, Banach Center Publ. **36**, Polish Acad. Sci., Warsaw, 1996.
- [P4] P. Pragacz : *Multiplying Schubert Classes*, Topics in Cohomological Studies of Algebraic Varieties. Trends in Mathematics. Birkhäuser Basel, 163–174 2005
- [PR1] P. Pragacz and J. Ratajski : *A Pieri-type theorem for Lagrangian and odd orthogonal Grassmannians*, J. Reine Angew. Math. **476** (1996), 143–189.
- [PR2] P. Pragacz and J. Ratajski : *Formulas for Lagrangian and orthogonal degeneracy loci;  $\tilde{Q}$ -polynomial approach*, Compos. Math. **107** (1997), 11–87.
- [PR3] P. Pragacz and J. Ratajski : *A Pieri-type formula for even orthogonal Grassmannians*, Fund. Math. **178** (2003), 49–96.
- [R] S. Rogers : *An explicit determination of the springer morphism*, Comm. in Algebra (2018)
- [Ro] Boris Rosenfeld, *Geometry of Lie Groups*, Kluwer Academic Publishers, 1993.
- [Sc1] H. Schubert : *Kalkül der abzählenden Geometrie*, Teubner, Leipzig, 1879.
- [Sc2] H. Schubert : *Die  $n$ -dimensionalen Verallgemeinerungen der fundamentalen Anzahlen unseres Raums*, Math. Annalen **26** (1886), 26–51.
- [S1] I. Schur : *Über ein Klasse von Matrizen die sich einer gegebenen Matrix zuordnen lassen*, Gesammelte Abhandlungen **1**, 1-72, Springer-Verlag, Berlin-New York, 1973.
- [S2] I. Schur : *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, J. reine angew. Math. **139** (1911), 155–250.
- [SeC] Séminaire C. Chevalley; 2e année: 1958. *Anneaux de Chow et applications*, Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1958.
- [Sp] T.A. Springer, *Linear Algebraic Groups*, Modern Birkhäuser Classics, Birkhäuser, 1998.

- [Sta] R. P. Stanley : *On the number of reduced decompositions of elements of Coxeter groups*, European J. Combin. **5** (1984), 359–372.
- [St2] J. R. Stembridge : *Some combinatorial aspects of reduced words in finite Coxeter groups*, Trans. Amer. Math. Soc. **349** (1997), 1285–1332.
- [T1] H. Tamvakis : *The connection between representation theory and Schubert calculus*, Enseign. Math. **50** (2004), 267–286.
- [T2] H. Tamvakis : *Quantum cohomology of isotropic grassmannians*, Geometric Methods in Algebra and Number Theory, Progress in Math. **235** (2005), 311–338
- [T3] H. Tamvakis : *Giambelli, Pieri, and tableau formulas via raising operators*, J. reine angew. Math. **652** (2011), 207–244.
- [T4] H. Tamvakis : *The theory of Schur polynomials revisited*, Enseign. Math. **58** (2012), 147–163.
- [T5] H. Tamvakis : *A Giambelli formula for classical  $G/P$  spaces*, J. Algebraic Geom. **23** (2014), 245–278.
- [T6] H. Tamvakis : *A tableau formula for eta polynomials*, Math. Annalen **358** (2014), 1005–1029.
- [T7] H. Tamvakis : *Giambelli and degeneracy locus formulas for classical  $G/P$  spaces*, Moscow Mathematical Journal **16** (2016), 125–177
- [TW] H. Tamvakis and E. Wilson : *Double theta polynomials and equivariant Giambelli formulas*, Math. Proc. Cambridge Philos. Soc., to appear.
- [Ts] B. Tshishiku : *Pontyagin Classes of locally symmetric manifolds*, Algebr. Geom. Topol. **15** (2015), 2707–2754.
- [W] E. Wilson : *Equivariant Giambelli formulae for Grassmannians*, Ph.D. thesis, University of Maryland, 2010.