THE METHOD OF THE GEOMETRIC PHASE IN THE HOPF BUNDLE AS A REFORMULATION OF THE EVANS FUNCTION FOR REACTION DIFFUSION EQUATIONS

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ABSTRACT

Colin James Grudzien: The method of the geometric phase in the Hopf bundle as a reformulation of the Evans function for reaction diffusion equations
(Under the direction of Christopher Jones)

This thesis develops a stability index for the travelling waves of non-linear reaction diffusion equations using the geometric phase induced on the Hopf bundle, an odd dimensional sphere realized in an arbitrary complex vector space. This can be viewed as an alternative formulation of the winding number calculation of the Evans function, whose zeroes correspond to the eigenvalues of the linearization of reaction diffusion operators about a wave or, time invariant, coherent state. The stability of such a state can be determined by the existence of eigenvalues of positive real part for the linear operator associated to it. The method of geometric phase for locating and counting eigenvalues as demonstrated in this thesis is inspired by the numerical results in Way's Dynamics in the "Hopf bundle, the geometric phase and implications for dynamical systems," but it diverges on several important points. This thesis develops a detailed proof of the relationship between the phase and eigenvalues for dynamical systems defined in a simple case and sketches the proof of the generalized method of geometric phase for arbitrary systems on unbounded domains and its generalization to boundary-value problems. In addition it establishes novel links between the geometric phase generated in the Hopf bundle, and an equivalent phase generated by a path in the Stiefel bundle.

A demonstration of the numerical method is included for a simple bistable equation, and the Hocking-Stewartson Pulse of the Complex Ginzburg-Landau equation. These examples highlight the novel features of this formulation of the winding of the Evans function, namely the use of either the stable or unstable manifold, and the dependence on the wave parameter for the eigenvalue calculation. The continuous accumulation of the eigenvalue count is exhibited with a characteristic phase change, depending on the wave parameter. This thesis concludes with a discussion of open questions arising from the numerical implementation, regarding the phase transition, its link to the underlying wave structure and the possible formulation of the method of geometric phase with respect to a phase generated on the Stiefel bundle.

ACKNOWLEDGEMENTS

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CHAPTER 1

Introduction

When studying systems of non-linear reaction diffusion equations, coherent states such as travelling waves and solitons give important qualitative information about the system. While the system of differential equations may not be analytically solvable in general, the time invariant solutions can help one understand the long time behaviour of solutions of the full system. In particular, steady states that are stable will attract nearby solutions in the function space asymptotically in their evolution, and these solutions represent those which are most robust as model solutions in realistic, noisy conditions. Determining the stability of time invariant solutions to non-linear reaction diffusion equations has been long studied to simplify the analysis of complex systems and one of the major tools of analysis that has emerged is the Evans function.

The Evans function is a complex analytic function constructed from the linearization of a system of partial differential equations on one spatial variable, with zeros corresponding to the eigenvalues of the associated linear operator. With this correspondence of the zeros of the Evans function and the eigenvalues of the operator, one may determine the existence and location of eigenvalues via winding number arguments and root finding methods for the Evans function. The Evans function was first derived in a series of papers [2],[3],[4],[5] by Evans on nerve impulse equations, and was generalized by Alexander, Gardner & Jones [6] to general systems of reaction diffusion equations. The Evans function has been applied in many more situations and its development as well as the current state-of-the-art is well documented and explained in Kapitula & Promislow [7].

The major work of this thesis is to reformulate the winding number calculation with the Evans function into a new geometric setting, as was suggested by Way [1] in his PhD thesis. Way developed numerical results supporting the hypothesis that parallel translation in the Hopf bundle could be used to locate and measure the multiplicity of eigenvalues for linearizations of reaction-diffusion equations on the real line about travelling waves, but the central conjecture was left as an open question. This work builds on Way's by developing a precise methodology for this eigenvalue calculation for general systems and proving the connection between the geometric phase in the Hopf bundle and the Chern number calculation of Alexander, Gardner & Jones. The construction of the Evans function by Alexander, Gardner & Jones [6] utilizes the geometry of vector bundles, taking advantage of the unique classification of complex vector bundles over 2-spheres with their Chern number. By framing the discussion of the Evans function in the bundle setting developed by Alexander, Gardner & Jones, this thesis demonstrates the link between the Chern number, equal to the eigenvalue calculation, and the geometric phase in the Hopf bundle via a particular construction denoted the *relative phase* [8].

The total space of the Hopf bundle is an odd dimensional sphere, $S^{2n-1} \subset \mathbb{C}^n$—therefore, any non-zero vector in the space $\mathbb{C}^n$ can be mapped to the total space of the Hopf bundle, S^{2n-1}, simply via spherical projection. This realization of the S^{2n-1} as a subset of $\mathbb{C}^n$ allows one to consider an arbitrary complex dynamical system, such as that arising in the Evans function theory, and map non-zero solutions onto the Hopf bundle. Constructing the problem appropriately, one may develop a winding number through the displacement in the fibers of the Hopf bundle induced by the dynamics in the phase space. By defining the horizontal and vertical subspaces of the tangent space, any differentiable path in the sphere describes parallel transport. This choice of decomposition, called a connection, defines the movement of a path along the fiber—for the fiber S^1 which in turn describes a winding number.

One may consider, in particular, the eigenvalue problem for a reaction-diffusion operator, linearized about a steady state travelling wave. This operator will give rise to a dynamical system on $\mathbb{C}^n$, and for such linearizations, Way studied the winding in the fiber S^1 and its relationship to the eigenvalues of the operator. Projecting particular λ dependent solutions onto S^{2n-1}, the dynamics on $\mathbb{C}^n$ induce parallel translation in the Hopf bundle. As a property of linear systems, any non-zero solution will remain non-zero over finite integration scales and in this way the dynamics act naturally on the Hopf bundle. Loops of solutions in the phase space parametrized in the value λ will define parallel translation which, for closed contours in λ, generates a holonomy angle in S^1. The winding in the fiber is called the geometric phase, because of its relationship with Berry's phase in quantum mechanics (e.g. Berry [9], Way [1], Chruscinski & Jamiolkowski [10]). This thesis shows that particular choices of solutions pick up information from the dynamics on $\mathbb{C}^n$, and that the winding of these loops of particular solutions can be used to describe the spectrum of the linear operator.

This work is to be considered as an advancement of the Evans function, but it has intrinsic value in opening new modes of analysis. The general approach of calculating the dynamically accumulated winding in the Hopf bundle relative to some asymptotic value was denoted the **method of geometric phase** by Grudzien, Bridges & Jones in *Geometric Phase in the Hopf bundle and the stability of non-linear waves* [8]—this work considered the winding induced on a particular choice of solutions for a reaction diffusion equation, though the method of geometric phase does not seem limited to this setting. The method of geometric phase formalized by Grudzien, Bridges & Jones differs from Way's numerical method of geometric phase by realizing the necessity of computing the *relative phase* with respect to the asymptotic conditions for the dynamical system—the total accumulated phase of a loop of these particular solutions, relative to the asymptotic conditions, will yield the eigenvalue count. The work of Grudzien, Bridges & Jones formally proved that the asymptotic relative phase agrees with the Chern number calculation of Alexander, Gardner & Jones [6].

A sketch of the contents of this thesis is as follows: the basic framework for the thesis and essential background is included in Chapter 2; the full development of the method of geometric phase for scalar equations on unbounded domains is in Chapter 3, and a numerical example of the method of geometric phase in this setting is included in §3.5. For higher dimensional systems of equations, the method of geometric phase uses the exterior algebra and the determinant bundle construction, as in Alexander, Gardner & Jones [6]. Passing to the exterior algebra the proof of the relative phase calculation holds, and the method is explained and proven for **general systems** on unbounded domains in Chapter 4. In addition, Chapter 4 includes a demonstration of the general method on unbounded domains, as was performed in *The instability of the Hocking-Stewartson pulse and its geometric phase in the Hopf bundle* [11]. Grudzien, Bridges & Jones also formulated an adaptation of the method of geometric phase to calculate the winding of the Evans function for boundary value problems on finite domains, and this is treated in Chapter 5. The main results for demonstrating the method of geometric phase are stated in the Theorems 3.4.1, 4.1.9, 4.2.3 and 5.2.3. Finally Chapter 6 develops an original result concerning the analytic formulation of the connection of the Hopf bundle in the exterior algebra, and its relationship to canonical connection on the Stiefel bundle. This formulation of the geometric phase in the Stiefel bundle, described in Proposition 6.1.7, leaves open questions, both in its geometric implications and the numerical development of the method. These open questions will be discussed in the conclusion.

CHAPTER 2

The unstable bundle and the Hopf bundle

This chapter formally introduces the major objects of study for developing the method of geometric phase—in particular, the Evans function on unbounded domains, the unstable bundle construction for the Evans function and the generic Hopf bundle $S^{2n-1} \subset \mathbb{C}^n$. While this work does not treat the theory of connections or characteristic classes in a deep way, introducing basic terminology and properties of principal fiber bundles and vector bundles is necessary for the proof. After a brief treatment of connections of principal fiber bundles and vector bundles, and the Chern classes of vector bundles over spheres, this chapter concludes with a sketch of how these concepts will come together in the method of the geometric phase for a simple system.

2.1 Reaction diffusion equations on unbounded domains

Define a system of non-linear reaction diffusion equations,

$$\begin{aligned} U_t = U_{xx} + f(U)\,, \qquad U(x,0) = U_0(x) \in \mathbb{R}^m\,, \\ U : \mathbb{R}^2 \to \mathbb{R}^m \end{aligned} \tag{2.1.1}$$

where $f : \mathbb{R}^m \to \mathbb{R}^m$ is a smooth (at least C^2) non-linear mapping, and $x \in \mathbb{R}$. Assume that there exists a travelling wave solution, i.e., a solution of the single variable $\xi = x - ct$, so $U(\xi)$ satisfies:

$$-cU' = U'' + f(U) \qquad \left(' = \tfrac{d}{d\xi}\right) \cdot$$

The stability of travelling wave solutions for a system as above is determined by the existence of eigenvalues of positive real part for the linearized operator about the wave, as shown by Bates and Jones [12].

The system (2.1.1) is re-written in a moving frame as

$$U_t = U_{\xi\xi} + cU_\xi + f(U) \tag{2.1.2}$$

for which the travelling wave is a time independent solution. Linearizing equation (2.1.2) about the wave $U(\xi)$, one obtains the ξ dependent operator $\mathcal{L}$ such that:

$$\mathcal{L}(p) = p_{\xi\xi} + cp_\xi + F\Big(U(\xi)\Big)p \tag{2.1.3}$$

with $p \in \mathbb{B}(\mathbb{R}, \mathbb{R}^m)$, the bounded, uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}^m$, and F the Jacobian of f.

Let $\Omega \subset \mathbb{C}$ be an open, simply connected domain that contains only discrete spectrum of $\mathcal{L}$. For $\lambda \in \Omega$, consider the equation

$$(\mathcal{L} - \lambda I)(p) = 0$$

that has the equivalent formulation as the system

$$\begin{aligned} p' &= q \\ q' &= -cq + \Big(\lambda - F(U)\Big)p \end{aligned}$$

Let I be the $m \times m$ identity matrix—one can write the above as the linear system,

$$\begin{gathered} Y' = A(\lambda, \xi)Y \qquad\qquad Y = \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{C}^{2m} \\ A(\lambda, \xi) = \begin{pmatrix} 0 & I \\ \lambda - F(U) & -cI \end{pmatrix} \end{gathered} \tag{2.1.4}$$

where A is an $n \times n$ complex matrix with $n \equiv 2m$.

The matrix system (2.1.4) for the eigenvalue problem is non-autonomous with dependence on $U(\xi)$, but the travelling wave solution $U(\xi)$ must be bounded as $\xi \to \pm\infty$. Hence, consider systems such that the travelling wave (2.1.2) satisfies the following hypothesis.

Hypothesis 2.1.1. *Define the limits of the wave,* $\lim_{\xi\to\pm\infty} U(\xi) = U(\pm\infty)$. *Assume that there are positive* $a, C \in \mathbb{R}$ *for which*

$$\| U(\xi) - U(+\infty) \| \leq Ce^{-a\xi} \quad \textit{for} \quad \xi \geq 0 \tag{2.1.5}$$

$$\| U(\xi) - U(-\infty) \| \leq Ce^{a\xi} \quad \textit{for} \quad \xi \leq 0 \tag{2.1.6}$$

$$\| U'(\xi) \| \leq Ce^{-a|\xi|} \quad \textit{for all } \xi. \tag{2.1.7}$$

Under this hypothesis, one may define asymptotic, autonomous systems by the limiting values of the wave:

$$\begin{aligned} Y' &= A_{\pm\infty}(\lambda)Y \\ A_{\pm\infty}(\lambda) := \lim_{\xi\to\pm\infty} A(\lambda,\xi) &= \begin{pmatrix} 0 & I \\ \lambda - F\big(U(\pm\infty)\big) & -cI \end{pmatrix} \end{aligned} \tag{2.1.8}$$

Definition 2.1.2. *Let* $\mathcal{L}$ *be a linear operator derived as in equation (2.1.3) from a non-linear reaction diffusion equation. Suppose the equation* $(\mathcal{L} - \lambda)p = 0$ *defines a flow on* $\mathbb{C}^n$ *for* $\lambda \in \Omega \subset \mathbb{C}$*:*

$$\begin{aligned} Y' &= A(\lambda,\xi)Y \\ A_{\pm\infty}(\lambda) &:= \lim_{\xi\to\pm\infty} A(\lambda,\xi) \end{aligned} \tag{2.1.9}$$

System (2.1.9) is said to ***split in Ω*** *if* $A_{\pm\infty}$ *have no pure imaginary eigenvalues and each have exactly* k *eigenvalues of positive real part (**unstable eigenvalues**) and* $n-k$ *eigenvalues of negative real part (**stable eigenvalues**), including multiplicity, for every* $\lambda \in \Omega$.

Given a system as described above, the following hypotheses are sufficient to construct the Evans function on unbounded domains.

Hypothesis 2.1.3. *Assume Ω is open, simply connected and contains only discrete eigenvalues of $\mathcal{L}$. Note that under this hypothesis, equation (2.1.9) splits in the domain Ω.*

Hypothesis 2.1.4. *Let $K \subset \mathbb{C}$ be a contour in $\mathbb{C}$, describing a path for the spectral parameter λ. Assume that the contour K is a piecewise smooth, simple closed curve in $\Omega \subset \mathbb{C}$ such that there is no spectrum of $\mathcal{L}$ in K. Let K° be the region enclosed by K—assume K° is homeomorphic to the disk $D \subset \mathbb{R}^2$ and that K is parametrized by $\lambda(s) : [0,1] \hookrightarrow K$ with standard orientation.*

Recall that the eigenfunctions for $\mathcal{L}$, as in equation (2.1.3), are required to be bounded for all $\xi \in \mathbb{R}$. For the associated system of equations (2.1.8), the eigenvalues of $A_{\pm\infty}$ determine the asymptotic growth and decay rates of potential eigenfunctions. By a compactification of the ξ parameter one may define a dynamical system for $\left\{\xi \in [-\infty, +\infty]\right\}$ “capped” on the ends by these asymptotic, autonomous systems. The asymptotic systems have fixed points at 0, by linearity of the dynamics, and thus un/stable manifolds in the extended system. The un/stable eigenvectors of the system at $\pm\infty$ determine the asymptotic behavior of solutions that lie in the un/stable manifolds of the critical points of the asymptotic systems.

Definition 2.1.5. *Define the ξ dependent variable τ where*

$$\xi =: \frac{1}{2\kappa} \log\left(\frac{1+\tau}{1-\tau}\right)$$

for some $\kappa \in \mathbb{R}$. Appending τ yields the new, compacted system

$$\begin{aligned} Y' &= A(\lambda,\tau)Y \qquad A(\lambda,\tau) = \begin{cases} A(\lambda,\xi(\tau)) & \text{for } \tau \neq \pm 1 \\ A_{\pm\infty}(\lambda) & \text{for } \tau = \pm 1 \end{cases} \\ \tau' &= \kappa(1-\tau^2) \qquad\qquad\qquad\quad {}' = \tfrac{d}{d\xi} \end{aligned} \tag{2.1.10}$$

Lemma 2.1.6. *One may choose $\kappa > 0$ such that the flow defined by equation (2.1.10) is C^1 on the entire compact interval.*

Proof. On finite time scales the flow (2.1.10) is smooth by linearity, but Lemma 3.1 in Alexander, Gardner & Jones [6] shows that if $\kappa < \frac{a}{2}$, where a is defined in Hypothesis 2.1.1, then equation (2.1.10) C^1 on the entire compact interval. $\square$

Hypothesis 2.1.7. *Assume that for all systems under consideration,* $0 < \kappa < \frac{a}{2}$.

Within the invariant planes $\{\tau = \pm 1\}$ of system (2.1.10), the dynamics are governed by the linear, autonomous equations

$$\begin{aligned} Y' &= A_{\pm\infty} Y & \quad ' &= \tfrac{d}{d\xi} \\ \tau' &\equiv 0 & \quad (\tau &= \pm 1) \end{aligned}$$

so that solutions in these planes are determined entirely by the stable and unstable directions of the asymptotic systems. For $\left\{\tau \in (-1, +1)\right\}$, solutions are governed by the non-autonomous system and have limits in the invariant planes as $\xi \to \pm\infty$.

Consider the un/stable manifolds of the critical points

$$(0, \pm 1) \in \mathbb{C}^n \times \{\tau = \pm 1\}$$

The dynamics in the invariant planes are linear with k unstable directions and $n - k$ stable directions; with the appended τ equation, the system gains one real unstable/ stable direction at $\tau = \mp 1$ respectively. Standard invariant manifold theory dictates that there is a $2k + 1$ **(real)** dimensional local unstable manifold in some neighborhood of $(0, -1)$ that can be extended globally by taking its flow forward for all time. In the invariant plane $\tau = -1$, the unstable manifold is just the span of the unstable eigenvectors, but for $\tau > -1$, this becomes a τ dependent subspace of $\mathbb{C}^n$.

Lemma 2.1.8. *Under the above hypotheses 2.1.1, 2.1.3 and 2.1.7, a solution to the extended system is an eigenfunction for $\mathcal{L}$ corresponding to λ if and only if it is in the unstable manifold for $A_{-\infty}(\lambda)$ and the stable manifold of $A_{+\infty}(\lambda)$.*

Proof. This is proved by Alexander, Gardner & Jones [6], in Lemma 3.6. □

This geometric characterization of the eigenfunctions of the linear operator $\mathcal{L}$ allows a novel construction for the Evans function. Locating the eigenfunctions with the Evans function often relies on a matching of these unstable and stable manifolds and describing the Evans function through the Wronskian of the matched solutions. The Evans function was first derived in a series of papers [2],[3],[4],[5] by Evans on nerve impulse equations, and was generalized by Alexander, Gardner & Jones [6] to general systems of reaction diffusion equations—they characterized the eigenvalues for the operator $\mathcal{L}$ geometrically as the Chern number of a vector bundle, and this vector bundle formulation of the Evans function is the one considered in this work. The construction of this vector bundle, denoted the unstable bundle, will be described in the following section.

2.2 The Unstable bundle

The following definitions will introduce the terminology necessary to construct this trivial bundle and its sub-bundle, the unstable bundle. This exposition will follow from Morita Chapter 5 [13], and the reader is referred there for further discussion.

Definition 2.2.1. *Let M, E be a smooth manifold. An* ***n-dimensional complex vector bundle over*** $\mathbf{M}$*, (E, π, M), is defined*

$$\pi : E \to M \tag{2.2.1}$$

such that π is smooth and for each point $p \in M$,

- *$\pi^{-1}(p)$ is isomorphic to $\mathbb{C}^n$. The preimage $\pi^{-1}(p)$ is defined the* ***fiber*** *above p.*

- *there is a neighborhood $\mathcal{U}$ containing p, and a diffeomorphism $\phi_{\mathcal{U}}$, such that*

$$\phi_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \cong \mathcal{U} \times \mathbb{C}^n. \tag{2.2.2}$$

Moreover, this diffeomorphism restricted to any point $\pi^{-1}(q) \in \pi^{-1}(\mathcal{U})$ is a linear *isomorphism of*

$$\phi_{\mathcal{U}} : \pi^{-1}(q) \to \{q\} \times \mathbb{C}^n. \tag{2.2.3}$$

A smooth map $s : M \to E$ such that $\pi \circ s = Id$ is defined as a ***section*** *of the vector bundle.*

Definition 2.2.2. *For an n-dimensional vector bundle (E, π, M), let $\mathcal{U}, \mathcal{V} \subset M$ be neighborhoods such that*

$$\phi_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \cong \mathcal{U} \times \mathbb{C}^n \tag{2.2.4}$$

$$\phi_{\mathcal{V}} : \pi^{-1}(\mathcal{V}) \cong \mathcal{V} \times \mathbb{C}^n. \tag{2.2.5}$$

The diffeomorphisms $\phi_{\mathcal{U}}, \phi_{\mathcal{V}}$ are defined as ***local trivializations*** *of M. Over the intersection $\mathcal{U} \cap \mathcal{V}$ define the smooth map*

$$g_{\mathcal{U}\mathcal{V}} : \mathcal{U} \cap \mathcal{V} \to GL(\mathbb{C}, n) \tag{2.2.6}$$

pointwise via the mapping

$$\begin{aligned} \phi_{\mathcal{U}} \circ \phi_{\mathcal{V}}^{-1} \quad : (\mathcal{U} \cap \mathcal{V}) \times \mathbb{C}^n \quad &\cong \quad (\mathcal{U} \cap \mathcal{V}) \times \mathbb{C}^n \\ \big(p, V\big) \quad &\mapsto \quad \big(p, g_{\mathcal{U}\mathcal{V}}(p)(V)\big). \end{aligned} \tag{2.2.7}$$

The map $g_{\mathcal{U}\mathcal{V}}$ is defined as the ***transition map*** *of $\mathcal{U} \cap \mathcal{V}$.*

From the contour K and the τ variable, one may construct a “parameter sphere”. Above this parameter sphere, one can view solutions to the system in equation (2.1.10) as paths in an appended trivial $\mathbb{C}^n$ bundle representing the phase space, and the unstable bundle is constructed within the trivial bundle by the evolution of the unstable manifold. The trivializations and transition map of the unstable bundle will play an important role in the proof of the method of geometric phase, where the Chern number of the unstable bundle is related to the geometric phase of a particular choice of trivialization. The details of this construction are in Chapter 3.

Definition 2.2.3. *The set* $K \times \left\{\tau \in [-1,+1]\right\}$ *defines a topological cylinder as* K *is topologically equivalent to* S^1*. Gluing copies of the region enclosed by* K*,* K°*, to the cylinder one obtains a topological 2-sphere*

$$M \equiv K \times \left\{\tau \in [-1,+1]\right\} \cup K^\circ \times \{\tau = \pm 1\} \tag{2.2.8}$$

hereafter defined as the ***parameter sphere****. The trivial* $\mathbb{C}^n$ *bundle over the parameter sphere is defined as* $M \times \mathbb{C}^n$*.*

Solutions to the system in equation (2.1.10) can be tracked in the fibers of the trivial bundle, with their evolution defined by the flow and the parameter values in M. Alexander, Gardner & Jones [6] show that for fixed $\lambda \in K$, the unstable manifold of the critical point $(0,-1) \in \mathbb{C}^n \times [-1,1]$ converges to the unstable space of $A_{+\infty}(\lambda)$ for $\tau = 1$ in Grassmann norm. The unstable manifold is extended to the caps by foliating the unstable manifold over $\{\lambda \in K^\circ\} \times \{\tau = +1\}$ with fibers defined by the span of the unstable eigenvectors of $A_{+\infty}(\lambda)$.

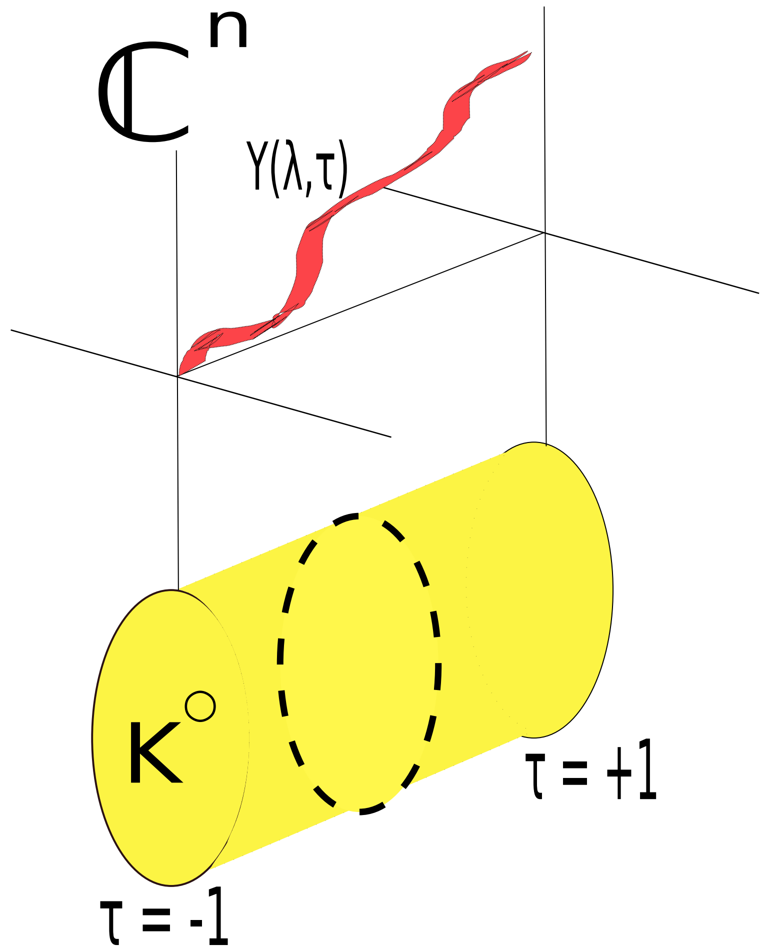

Figure 2.1: The trivial bundle over the parameter sphere.

Definition 2.2.4. *The unstable manifold of the critical point* $(0, -1) \in \mathbb{C}^n \times [-1, 1]$ *defines a subspace of* $\mathbb{C}^n$ *of trajectories that approach* $(0, -1)$*, exponentially decaying as* $\xi \to -\infty$ *for* $|\xi|$ *sufficiently large. For each fixed* (λ, τ) *let* $W^u(\lambda, \tau)$ *define the* ***unstable manifold*** *in* $\mathbb{C}^n$ *defined by the flow at* (λ, τ)*. The total space* E *defines a non-trivial bundle over* M *with projection* $\pi_E : E \to M$*,*

$$\begin{tikzcd} W^u \arrow[r] & E \arrow[d, "\pi_E"] \\ & M \end{tikzcd} \tag{2.2.9}$$

E is contained in the trivial bundle $M \times \mathbb{C}^n$, and is called the ***unstable bundle****.*

Lemma 2.2.5. *The unstable bundle is a k dimensional vector bundle over the sphere M.*

Proof. For a proof the reader is referred to the construction in §3 of Alexander, Gardner & Jones [6]. □

Chern numbers are topological invariants for a complex vector bundle, and there are several ways to treat their definition. This work provides only a cursory description of the Chern numbers, describing them in terms of connections on the vector bundle. Connections can be intuitively described as how the fibers above points in the base manifold M are glued together—more formally, this is constructed with the covariant derivative.

Definition 2.2.6. *Let $\mathcal{X}(M)$ be the space of vector fields on M. Then the **covariant derivative** is a mapping*

$$\begin{aligned} \nabla : \ \mathcal{X}(M) \times \mathcal{X}(M) &\rightarrow \mathcal{X}(M) \\ (X, Y) &\mapsto \nabla_X Y \end{aligned} \tag{2.2.10}$$

such that, for any $f \in C^\infty(M)$ and $X, Y, X_1, X_2, Y_1, Y_2 \in \mathcal{X}(M)$, the following relationships hold

$$\nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y \tag{2.2.11}$$

$$\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2 \tag{2.2.12}$$

$$\nabla_{fX} Y = f \nabla_X Y \tag{2.2.13}$$

$$\nabla_X (fY) = f \nabla_X Y + X(f) Y \tag{2.2.14}$$

The notion of a connection thus extends naturally from the covariant derivative, by applying the same framework to sections of the vector bundle.

Definition 2.2.7. *Let $\Gamma(E)$ define the space of sections for the complex vector bundle (E, π, M). A **connection** is a bilinear map*

$$\nabla : \mathcal{X}(M) \times \Gamma(M) \rightarrow \Gamma(M) \tag{2.2.15}$$

such that, for $f \in C^\infty(M)$, $X \in \mathcal{X}(M)$ and $s \in \Gamma(M)$, ∇ satisfies the following relations

$$\nabla_{fX} s = f \nabla_X s \tag{2.2.16}$$

$$\nabla_X(fs) = f\nabla_X s + X(f)s \tag{2.2.17}$$

Proposition 2.2.8. *Let ∇ be a connection defined for the n-dimensional vector bundle (E, π, M), then ∇ can be equivalently defined by a collection of n^2 1-forms, described collectively by an $n \times n$ matrix of 1-forms $\omega = (\omega^i_j)$. Given a connection ω, the curvature form for the bundle can be described by the relation*

$$d\omega = -\omega \wedge \omega + \Omega \tag{2.2.18}$$

*where Ω is the curvature form. Let $H^j(M, \mathbb{Z})$ be the j^{th} cohomology group of M with coefficients in $\mathbb{Z}$. The **Chern class of degree* j** *for the vector bundle is an element*

$$C_j(E) \in H^{2j}(M, \mathbb{Z}), \tag{2.2.19}$$

and is the j^{th} coefficient of the characteristic polynomial of the curvature form Ω, ie:

$$\det\left(I + \frac{t}{2\pi i}\Omega\right) = 1 + \sum_{j=1}^{n} t^j C_j(E) \tag{2.2.20}$$

Note, the Chern classes are independent of the choice of the connection ω.

Proof. The above proposition consists of classical results for deriving Chern classes—for a discussion of the results and a derivation of characteristic classes for general vector bundles consult Morita Chapter 5 [13]. □

Corollary 2.2.9. *For the k dimensional unstable bundle, $C_1(E)$ is the only non-trivial Chern class.*

Proof. Recall that the parameter sphere $M \cong S^2$ and the cohomology groups are given

$$H^j(S^2, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} \text{ if } j = 0, 2 \\ 0 \text{ otherwise} \end{cases} \tag{2.2.21}$$

□

Definition 2.2.10. *The* ***Chern number*** *of the unstable bundle is defined to be the integral of the Chern class over the sphere M.*

Lemma 2.2.11. *The Chern number of the unstable bundle equals the total multiplicity of the eigenvalues enclosed by the contour K.*

Proof. This is the result used by Alexander, Gardner & Jones [6] to construct the Evans function in their §6. □

The above lemma establishes the essential link in the existing Evans function constructions that will be utilized for validating the method of geometric phase. The next section will formally introduce principal fiber bundles and the Hopf bundle in particular.

2.3 The Hopf bundle

Following the exposition of Kobayashi & Nomizu [14], the following definition will allow the introduction of the Hopf bundle.

Definition 2.3.1. *Let M be a smooth manifold and G be a Lie group. A principal fiber bundle P over M with group G is described by the diagram*

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \Big\downarrow \pi \\ & & M \end{array} \tag{2.3.1}$$

such that the following hold

- *G acts freely on M on the right*

- *M is the quotient space of P with respect to the group action of G.*

- *Every point $p \in M$ has a neighborhood U and a diffeomorphism Ψ_U such that*

$$\begin{aligned} \Psi_U : \quad \pi_U^{-1}(U) \quad &\cong \quad U \times G \\ u \quad &\mapsto \quad \big(\pi(u), \phi_U(u)\big) \end{aligned} \tag{2.3.2}$$

  where $\phi_U : \pi^{-1}(U) \to G$ such that

$$\phi_U(ua) = \big(\phi_U(u)\big)a \tag{2.3.3}$$

  for any $u \in \pi^{-1}(U)$ and $a \in G$.

The Hopf bundle is a classical example of a principal fiber bundle, which has a total space with a realization in $\mathbb{C}^n$—this realization allows one to re-frame the winding of the unstable bundle in terms of the geometric phase induced in the fibers.

Definition 2.3.2. *The* ***Hopf bundle*** *is a principal fiber bundle with total space $P = S^{2n-1} \subset \mathbb{C}^n$, base space $M = \mathbb{C}P^{n-1}$, and fiber $G = S^1$. The fiber group S^1 acts naturally on S^{2n-1} by complex scalar multiplication; with respect to this action the quotient is $\mathbb{C}P^{n-1}$.*

Definition 2.3.3. *Let (P, G, M, π) define a principal fiber bundle and let $T_p(P)$ be the tangent space of P at p. The* ***vertical subspace*** *$V_p(P) \subset T_p(P)$ is canonically defined by the kernel of the derivative of the projection map*

$$D\pi : T_p(P) \to T_{\pi(p)}(M) \tag{2.3.4}$$

A connection on the principal fiber P is a choice of a ***horizontal subspace*** *$H_p(P) \subset T_p(P)$ for each $p \in P$ satisfying the following conditions*

- *$T_p(P) \cong V_p(P) \oplus H_p(P)$*

- *If R_g is the map defining the right action of $g \in G$, then*

$$(R_g)_* H_p(P) = H_{pg}(P) \tag{2.3.5}$$

- *$H_p(P)$ depends smoothly on $p \in P$*

The choice of a connection therefore defines a smooth decomposition of the tangent space for the principal fiber bundle. While the vertical subspace is canonically defined, the choice of the transverse horizontal subspace is not generally unique. A useful characterization of the horizontal subspace is through the use of 1-forms.

Definition 2.3.4. *Let (P, G, M, π) define a principal fiber bundle, and let $\mathcal{G}$ be the Lie algebra of G. Let A^* be the fundamental vector field of $A \in \mathcal{G}$ induced on P. A* ***connection 1-form*** *ω is defined*

$$\omega : T(P) \to \mathcal{G} \tag{2.3.6}$$

such that

- *$\omega(A^*) = A$ for all $A \in \mathcal{G}$*
- *$\omega\Big((R_a)_* X\Big) = ad(a^{-1}) \circ \omega(X)$ for every $a \in G$ and every vectorfield X on P, where ad denotes the adjoint representation of G in $\mathcal{G}$.*

Lemma 2.3.5. *The choice of a connection 1-form ω defines a connection of P, ie: $H(P)$ defined by the kernel of ω satisfies the conditions for a connection on a principal fiber bundle. Likewise, the choice of $H(P)$ defines a connection 1-form.*

Proof. This is a classical result and the reader is referred to Kobayashi & Nomizu Chapter 2 [14] for a full discussion of vertical and horizontal subspaces, and the theory of connections. □

For a generic Hopf bundle, of dimension $2n-1$, there exists an intuitive choice of connection between fibers. The realization of $S^{2n-1} \subset \mathbb{C}^n$ by spherical projection can be used to define the connection pointwise.

Definition 2.3.6. *For the Hopf bundle S^{2n-1}, viewed in coordinates for $\mathbb{C}^n$, define the connection 1-form ω pointwise for $p \in S^{2n-1}$ as a mapping of the tangent space of the Hopf bundle $T_p\left(S^{2n-1}\right) \subset T_p\left(\mathbb{C}^n\right)$*

$$\begin{array}{rccc} \omega_p: & T_p\left(S^{2n-1}\right) & \rightarrow & i\mathbb{R} \\ & V_p & \mapsto & \langle V_p, p\rangle_{\mathbb{C}^n} \end{array} \tag{2.3.7}$$

where $i\mathbb{R}$ is the Lie algebra of the fiber S^1 [1]. The connection defined by ω is defined to be the ***natural connection*** *on the Hopf bundle.*

Lemma 2.3.7. *The natural connection is a connection of the generic Hopf bundle S^{2n-1} and it is the unique connection for the S^3 Hopf bundle.*

Proof. This is proven by Way [1] in §3.5 and the reader is referred there for a full discussion. □

Given a differentiable path in the Hopf bundle, and a choice of connection, one may always choose a corresponding "horizontal lift", which will describe the displacement in the fiber. Here the horizontal lift is defined in a similar vein as Kobayashi & Nomizu page 64 [14].

Definition 2.3.8. *Let $v(s) : [0,1] \rightarrow S^{2n-1}$ be a differentiable path in the Hopf bundle. The* ***horizontal lift*** *of $v(s)$ is a path $w(s) : [0,1] \rightarrow S^{2n-1}$ for which*

$$\begin{aligned} w(0) &= v(0) \\ \pi\Big(w(s)\Big) &\equiv \pi\Big(v(s)\Big) \quad \forall s \\ \omega\left(\frac{d}{ds}(w(s)\right) &\equiv 0 \quad \forall s. \end{aligned}$$

ie: $\frac{d}{ds}w(s) \in H\left(S^{2n-1}\right)$ for all s.

Definition 2.3.9. *Let $v(s)$ be a differentiable path, $v : [0,1] \mapsto S^{2n-1}$, and let $w(s)$ be its horizontal lift. The* ***phase curve*** *$\theta(s)$ for $v(s)$ is defined by the equation*

$$v(s) = e^{i\theta(s)}w(s) \tag{2.3.8}$$

ie: the path in the fiber describing the displacement along $v(s)$ between $v(s)$ and its horizontal lift $w(s)$. The ***geometric phase*** *is the change in the phase curve, ie:*

$$GP\Big(v\big([0,1]\big)\Big) \equiv \frac{\theta(1)-\theta(0)}{2\pi} \tag{2.3.9}$$

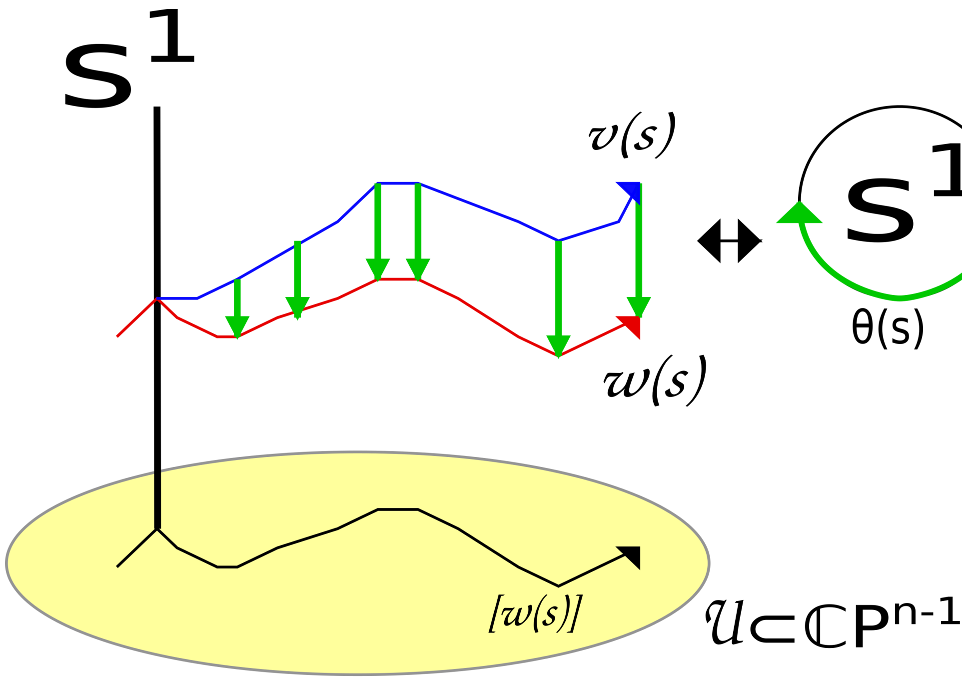

Figure 2.2: The phase curve defined by parallel translation.

Lemma 2.3.10. *Let $v(s) \subset S^{2n-1}$ parametrize a smooth path Γ in the Hopf bundle for $s \in [0,1]$, and let $\theta(s)$ be the phase curve with respect to the horizontal lift $w(s)$. Then the phase curve satisfies the differential equation*

$$\theta'(s) = -i\omega\Big(v'(s)\Big) \qquad\qquad \theta(0) = 0 \tag{2.3.10}$$

and the geometric phase can be computed as the pull back of the connection 1-form along Γ, *ie:*

$$\frac{\theta(1)}{2\pi} = \frac{1}{2\pi i}\int_{\Gamma}\omega \tag{2.3.11}$$

$$= \frac{1}{2\pi i}\int_0^1 \Big\langle v'(s), v(s)\Big\rangle ds \tag{2.3.12}$$

Proof. The general form of the differential equation describing the phase curve is derived by Kobayashi & Nomizu in Chapter 2 [14], and is formulated with respect to the natural connection on the Hopf bundle by Way in Chapter 3 [1]. □

Remark 2.3.11. *The geometric phase has important connections to the Berry phase in quantum mechanics, discussed by Way [1], and Chruscinski & Jamiolkowski [10].*

The method for computing eigenvalues with geometric phase utilizes general **non-zero**, differentiable paths in $\mathbb{C}^n$—a useful reformulation of the phase integral in equation (2.3.11) for non-zero paths is given in the following lemma.

Lemma 2.3.12. *Suppose for $s \in [0,1]$, $u(s)$ is a non-zero, differentiable path in $\mathbb{C}^n$. Then the connection of the tangent vector of its spherical projection, $\hat{u}(s) \in S^{2n-1}$, can be written*

$$\omega\left(\frac{d}{ds}\hat{u}(s)\right) = i\frac{Im\Big(\Big\langle u'(s), u(s)\Big\rangle\Big)}{\Big\langle u(s), u(s)\Big\rangle} \tag{2.3.13}$$

and the geometric phase along $\hat{u}(s)$ can be computed as

$$\frac{\theta(1)}{2\pi} = \frac{1}{2\pi}\int_0^1 \frac{Im\Big(\Big\langle u'(s), u(s)\Big\rangle\Big)}{\Big\langle u(s), u(s)\Big\rangle} ds \tag{2.3.14}$$

*If $u(s)$ is also a **closed curve**, then*

$$0 = \int_0^1 \frac{Re\Big(\Big\langle u'(s), u(s)\Big\rangle\Big)}{\Big\langle u(s), u(s)\Big\rangle} ds \tag{2.3.15}$$

and the geometric phase is equivalent to

$$\frac{\theta(1)}{2\pi} = \frac{1}{2\pi i}\int_0^1 \frac{\Big\langle u'(s), u(s)\Big\rangle}{\Big\langle u(s), u(s)\Big\rangle} ds \tag{2.3.16}$$

Proof. Consider the alternative form of the connection (2.3.13). If $\hat{u}(s)$ is the spherical projection of the path $u(s)$, then the natural connection is identically

$$\begin{aligned}
\omega\left(\frac{d}{ds}\hat{u}(s)\right) &= \left\langle \frac{d}{ds}\frac{u(s)}{\Big\langle u(s), u(s)\Big\rangle^{\frac{1}{2}}}, \frac{u(s)}{\Big\langle u(s), u(s)\Big\rangle^{\frac{1}{2}}} \right\rangle \\
&= \left\langle \frac{u'(s)\Big\langle u(s), u(s)\Big\rangle^{\frac{1}{2}}}{\Big\langle u(s), u(s)\Big\rangle} - \frac{u(s)Re\Big(\Big\langle u'(s), u(s)\Big\rangle\Big)}{\Big\langle u(s), u(s)\Big\rangle^{\frac{3}{2}}}, \frac{u(s)}{\Big\langle u(s), u(s)\Big\rangle^{\frac{1}{2}}} \right\rangle \\
&= \frac{\Big\langle u'(s), u(s)\Big\rangle}{\Big\langle u(s), u(s)\Big\rangle} - \frac{Re\Big(\Big\langle u'(s), u(s)\Big\rangle\Big)}{\Big\langle u(s), u(s)\Big\rangle} \\
&= i\frac{Im\Big(\Big\langle u'(s), u(s)\Big\rangle\Big)}{\Big\langle u(s), u(s)\Big\rangle}
\end{aligned}$$

which verifies the equations (2.3.13) and (2.3.14). Suppose that $u(s)$ is also a closed curve—then notice,

$$\begin{aligned}
0 &= \log\Big(\parallel u(s) \parallel^2\Big)\Big|_{s=0}^{s=1} \\
&= \int_0^1 \frac{\frac{d}{ds}\Big\langle u(s), u(s)\Big\rangle}{\Big\langle u(s), u(s)\Big\rangle} ds \\
&= 2\int_0^1 \frac{Re\Big(\Big\langle u'(s), u(s)\Big\rangle\Big)}{\Big\langle u(s), u(s)\Big\rangle} ds
\end{aligned}$$

which verifies equation (2.3.15)—combining this with equation (2.3.14) this verifies equation (2.3.16). □

Remark 2.3.13. *Given the formulation (2.3.14) of the geometric phase in terms of any non-zero path, one may unambiguously refer to the* ***geometric phase of a path*** $\mathbf{u}(\mathrm{s}) \in \mathbb{C}^{\mathbf{n}}$, *describing the geometric phase of its normalization.*

2.4 A sketch of the method of geometric phase on $\mathbb{C}^2$

The method of geometric phase may be easily understood in the case where the dynamical system is defined on $\mathbb{C}^2$, and the low dimension allows geometric intuition. This intuition is useful in proving the general technique, and much of the argument is identical for systems of larger dimension after introducing determinant bundle. The reader can consider the scalar bistable equation as a typical example of a PDE for which $\mathcal{L} - \lambda = 0$ defines a system on $\mathbb{C}^2$ satisfying the Hypotheses 2.1.1 and 2.1.3:

$$u_t = u_{xx} + f(u) \qquad\qquad f(u) = u(u+1)(u-1) \tag{2.4.1}$$

This PDE has steady localized solutions, and the spectral problem associated with the linearization about such a state can be formulated as in (2.1.4) with $Y \in \mathbb{C}^2$. This example is revisited in §3.5, with results demonstrating the numerical method. The method of geometric phase for such a PDE defining an ODE system on $\mathbb{C}^2$ is described as follows.

Table 2.1: The Method of Geometric Phase on $\mathbb{C}^2$

Step 1:	Choose a contour K in $\mathbb{C}$ that does not intersect the spectrum of the operator $\mathcal{L}$.
Step 2:	Varying $\lambda \in K$ define $X^+(\lambda)$ to be an analytic loop of eigenvectors for the $A_{+\infty}(\lambda)$ system in equation (2.1.9) where $X^+(\lambda)$ corresponds to the eigenvalue of positive real part.
Step 3:	Suppose $Z\Big(\lambda, \tau(\xi)\Big)$ is a solution to the system defined by equation (2.1.9), such that $\Big(Z\Big(\lambda, \tau(\xi)\Big), \tau\Big)$ is in the unstable manifold $(0, -1) \in \mathbb{C}^2 \times [-1, 1]$ for equation (2.1.10).
Step 4:	Calculate the **relative geometric phase** of $Z\Big(\lambda, \tau(\xi)\Big)$ with respect to $X^+(\lambda)$, ie: $GP\Big(Z\Big(K, \tau(\xi)\Big)\Big) - GP\Big(X^+\Big(K\Big)\Big)$, where $GP\Big(u\Big([0, 1]\Big)\Big)$ is the geometric phase of a non-zero, differentiable path in $\mathbb{C}^2$, defined in equation (2.3.14).

The main result. *The central theme of this work is demonstrating that, for an appropriate choice of $X^+(\lambda)$ and $Z(\lambda, \tau)$,* ***the asymptotic relative phase***

$$\lim_{\xi \to \infty} GP\Big(Z\Big(K, \tau(\xi)\Big)\Big) - GP\Big(X^+(K)\Big) \tag{2.4.2}$$

equals the total multiplicity of the eigenvalues enclosed by K.

Way's numerics supported the hypothesis that the geometric phase of $Z\Big(\lambda, \tau(\xi_1)\Big)$ should equal the total multiplicity of the eigenvalues for $\mathcal{L}$ in K° when ξ_1 is taken sufficiently large [1]. However, in this study the idea is reformulated with the **asymptotic relative phase** calculation, in equation (2.4.2), and the machinery of the determinant bundle. The dependence on the eigenvectors for $A_{+\infty}(\lambda)$ in the computation of the **relative phase** turns out to be an essential point in formulating the method, as is using the determinant bundle. The original numerical method studied the geometric phase of a single eigenvector corresponding to the strongest growing/decaying eigenvalue but in general the information of the full un/stable subspace is required. The method of geometric phase was proven in general by relating the Chern number and geometric phase of such a solution Z as above, treated as a trivialization of the determinant bundle [8]. The following chapter develops the proof of the method of geometric phase for systems defined on $\mathbb{C}^2$ and revisits the example (2.4.1) in §3.5 to demonstrate the technique.

CHAPTER 3

The method for scalar equations

The systems under consideration in this chapter will be restricted to the case of scalar equations, where $m = 1$ in equation (2.1.4), and to the case where the asymptotic system is symmetric,

$$\lim_{\xi \to \pm\infty} A(\lambda, \xi) \equiv A_{\pm\infty}(\lambda) \equiv A_{\infty}(\lambda)\,.$$

This restriction on the boundary conditions will give useful geometric intuition of the method, but the restriction is not necessary in general. The theory and proofs presented in Chapter 3 will be adapted to the general construction of the unstable bundle for n dimensions, k unstable directions, and non-symmetric asymptotic limits in Chapter 4.

3.1 Center-unstable manifold on $\mathbb{C}^2$

Let $\mathcal{L}$ be the linearization of a reaction diffusion equation about a steady state. From $(\mathcal{L} - \lambda)p = 0$, where $\lambda \in \Omega \subset \mathbb{C}$, one can derive the system on $\mathbb{C}^2$:

$$\begin{aligned} Y' &= A(\lambda, \tau)Y \qquad\qquad A_{\infty}(\lambda) := \lim\nolimits_{\xi \to \pm\infty} A(\lambda, \tau) \\ \tau' &= \kappa(1 - \tau^2) \\ A(\lambda, \tau) &= \begin{cases} A\Big(\lambda, \xi(\tau)\Big) & \text{for } \tau \neq \pm 1 \\ A_{\infty}(\lambda) & \text{for } \tau = \pm 1 \end{cases} \end{aligned} \tag{3.1.1}$$

where Ω is open and simply connected, and the system at infinity, $A_{\infty}(\lambda)$, has one stable and one unstable eigenvalue for every $\lambda \in \Omega$. Let K be a smooth, simple closed curve in $\Omega \subset \mathbb{C}$ that contains no spectrum of $\mathcal{L}$, let the enclosed region be denoted K° and let K be parametrized by $\lambda(s) : [0, 1] \hookrightarrow K$.

Denote the eigenvalues of $A_\infty(\lambda)$ by $\mu_1(\lambda)$,$\mu_2(\lambda)$ with

$$Re(\mu_1) < 0 < Re(\mu_2)$$

for each $\lambda \in \Omega$. The vector

$$X := e^{-\mu_2(\lambda)\xi} Y$$

is in $W^u(\lambda, \tau)$ provided $Y \in W^u(\lambda, \tau)$, because $W^u(\lambda, \tau)$ is a subspace. Let $\frac{d}{d\xi} ='$, then

$$\begin{aligned} X' =& -\mu_2(\lambda)e^{-\mu_2(\lambda)\xi}Y + e^{-\mu_2(\lambda)\xi}Y' \\ =&(A - \mu_2 I)X \end{aligned}$$

This motivates the following system on $\mathbb{C}^2$:

$$\begin{array}{rclc} X' & = & BX & B(\lambda,\tau) := \Big(A(\lambda,\tau) - \mu_2(\lambda)I\Big) \\ \tau' & = & \kappa(1-\tau^2) & B_\infty(\lambda) := \lim_{\xi\to\pm\infty} B(\lambda,\tau) \end{array} \tag{3.1.2}$$

The ξ dependent rescaling transforms the A system in equation (3.1.1) into the B system in equation (3.1.2) where it will be more convenient to work with the trajectories in the unstable manifold.

Definition 3.1.1. *Solutions to the A system (3.1.1) will be denoted with a $\sharp$. That is, if $Z^\sharp \in W^u(\lambda, \tau_0)$, then there is a ξ_0 for which $Z \equiv e^{-\mu_2(\lambda)(\xi-\xi_0)}Z^\sharp$ is the unique solution to the B system (3.1.2) that agrees with $Z^\sharp$ at (λ, τ_0). Similarly if Z is a solution to the B system, then $Z^\sharp \equiv e^{\mu_2(\lambda)(\xi-\xi_0)}Z$ is the unique solution to the A system that agrees with Z at ξ_0.*

Let $X^-(\lambda)$ be an unstable eigenvector for $A_{-\infty}(\lambda)$. Then in the B system (3.1.2), if $Z \in span_{\mathbb{C}}\Big\{X^-(\lambda)\Big\}$, then $(Z, \pm 1)$ is a fixed point. By the construction of B_∞, in $\tau = -1$, there is exactly one complex stable direction, one complex center direction corresponding to the line of fixed points, and the real unstable τ direction. One may thus construct the center-unstable manifold of a non-zero path of eigenvectors $X^-(\lambda)$ that correspond to the zero eigenvalue in the $B(\lambda)$ system (3.1.2).

3.2 The induced phase on the Hopf bundle

In order to measure the geometric phase of a solution which spans the unstable subspace $W^u(\lambda, \tau)$, the solution must project onto S^3. On finite timescales, ie: $\tau \in (-1, 1)$, this is not an issue. A non-zero solution to equation (3.1.1) may be viewed in hyper-spherical coordinates

$$\Big\{r \in (0, +\infty)\Big\} \times S^3 \times \Big\{\tau \in (-1, 1)\Big\}$$

because no solution reaches zero in finite time. However, to measure the phase over the entire bundle, one may appeal to solutions to the B system; Lemma 3.7 in Alexander, Gardner & Jones [6] demonstrates that a solution to equation (3.1.1) that is in W^u, is unbounded and converges to the unstable subspace of $(0, +1)$ in the Grassmann norm as $\xi \to +\infty$. Because the solutions of the system (3.1.1) in W^u approach 0 as $\xi \to -\infty$ and are unbounded as $\xi \to \infty$, the proof will appeal to solutions of the B system instead.

Lemma 3.2.1. *There exists a choice of unstable eigenvectors for $A_{\pm\infty}(\lambda)$, $X^{\pm}(\lambda)$, that are analytic in λ for $\lambda \in \Omega$.*

Proof. For a constructive algorithm for such bases the reader is referred to Humpherys, Sandstede & Zumbrun [15]. □

Note that under spherical projection, these bases may lose $\mathbb{C}$ differentiability, but will retain the differentiability in s, where $\lambda(s) : [0, 1] \hookrightarrow K$ and s is the path parameter.

Definition 3.2.2. *Let the contour $K \subset \mathbb{C}$ be given. A **reference path** for $\lambda \in K$, defined $X^{\pm}(\lambda)$ at $\tau = \pm 1$ respectively, is a loop of eigenvectors for $A_{\pm\infty}(\lambda)$ that corresponds to the eigenvalue of **largest, positive, real part** for $A_{\pm\infty}$.*

Definition 3.2.3. *Let $X^{\pm}(\lambda)$ be a reference path chosen analytically in λ over K that can be extended smoothly over K° without zeros. $X^{\pm}(\lambda)$ are defined **non-degenerate** as $X^{\pm}(\lambda)$ defines fibers compatible with the unstable bundle construction.*

Lemma 3.2.4. *Let $X^-(\lambda)$ be a non-degenerate reference path for $A_{-\infty}$. Let the center-unstable manifold of this line of critical points, in the B system (3.1.2), be parametrized by (λ, τ) as $Z(\lambda, \tau)$. Then $Z(\lambda, \tau)$ is non-singular and continuous in its limit $\xi \to +\infty$, and the span equals the unstable manifold $W^u(\lambda, \tau)$ for all $(\lambda, \tau) \in K \times [-1, 1]$.*

Proof. As in §4 of Alexander, Gardner & Jones [6], the center-unstable manifold of the path $X^-(\lambda)$ in the B system can parametrized by (λ, τ)

$$Z(\lambda, \tau) \qquad Z(\lambda, -1) \equiv X^-(\lambda)$$

such that it is $\mathbb{C}$ differentiable in λ for $\tau \in [-1, 1)$ fixed.

The ξ dependent scaling of Z

$$Z^\sharp(\lambda, \tau) = e^{\mu_2(\lambda)\xi} Z(\lambda, \tau)$$

yields a solution to the A system which is necessarily in W^u, by the exponential decay condition as $\xi \to -\infty$. Therefore $Z(\lambda, \tau)$ spans $W^u(\lambda, \tau)$ for each $\tau \in [-1, +1)$. Lemma 6.1 in [6] demonstrates that the limit of $Z(\lambda, \tau)$ as $\xi \to \infty$ is non-zero and continuous in λ. This means that $Z(\lambda, \tau)$ spans the unstable bundle for $\tau \in [-1, 1]$, and has a non-singular projection on to S^3 for all τ. □

Remark 3.2.5. *The above Lemma 3.2.4 holds for systems with non-symmetric asymptotic limits provided the appropriate scaling is used. The case of non-symmetric asymptotic limits will be treated in Chapter 4, in Proposition 4.2.2.*

Let Z and $X^\pm(\lambda)$ be defined as in Lemma 3.2.4, and $\hat{Z}$, $\hat{X}^\pm(\lambda)$ be their projections onto S^3, then $\hat{Z}$ defines a mapping to S^3 for which the following hold:

- $\hat{Z}(\lambda, \tau) \to \hat{X}^-(\lambda)$ as $\xi \to -\infty$
- $\hat{Z}(\lambda, \tau) \to \zeta(\lambda)\hat{X}^+(\lambda)$ as $\xi \to +\infty$ for some $\zeta(\lambda) \in \mathbb{C}$

- $span_{\mathbb{C}}\Big\{\hat{Z}(\lambda,\tau)\Big\} \equiv W^u(\lambda,\tau)$

Definition 3.2.6. *Let $X^{\pm}(\lambda)$ be reference paths for $A_{\pm\infty}(\lambda)$ respectively. The **induced phase**, with respect to $X^{\pm}(\lambda)$, is the complex scalar such that*

$$\zeta(\lambda)\hat{X}^{+}(\lambda) \equiv \hat{Z}(\lambda,+1).$$

Remark 3.2.7. *Note that by this construction, both $\hat{Z}$ and $\hat{X}^{+}$ are unit vectors, ie: $\zeta(\lambda) \in S^1$. In the simple case where $A_{-\infty}(\lambda) \equiv A_{+\infty}(\lambda)$ one may also take $X^{+}(\lambda) = X^{-}(\lambda)$ so that the induced phase is clearly a measure of the winding accumulated as the unstable manifold traverses M. For systems with non-symmetric asymptotic limits, the proof of the method must be adapted, but the intuition remains the same.*

Firstly the goal is to prove that, as a function of s, ζ is differentiable. Having this condition, the connection between $\zeta(s)$, the choice of reference paths, the total multiplicity of the eigenvalue in K° and the geometric phase will be established.

Proposition 3.2.8. *Let $X^{\pm}(\lambda)$ be non-degenerate reference paths for $A_{\pm\infty}(\lambda)$ respectively. For each $\lambda \in K$, define $\zeta(\lambda)$ such that $\hat{Z}(\lambda,+1) = \zeta(\lambda)\hat{X}^{+}(\lambda)$. If $\lambda(s)$ is a smooth parametrization of K, then*

$$\zeta\Big(\lambda(s)\Big) : [0,1] \to S^1$$

is a differentiable function.

Proof. As in Lemma 3.2.4 the limit $Z(\lambda,\tau) \to Z(\lambda,+1)$ is non-zero for each λ and $Z(\lambda,+1)$ is continuous. Moreover, Lemma 3.7 in Alexander, Gardner & Jones [6] demonstrates that the convergence of the manifold $Z(\lambda,\tau) \to Z(\lambda,+1)$ is locally uniform outside of the spectrum of $\mathcal{L}$ and thus uniform on K. Lemma 4.1 of [6] demonstrates that the solutions $Z(\lambda,\tau)$ are analytic in λ for $\tau \in [-1,1)$. But $\lim_{\tau\to 1} Z(\lambda,\tau)$ converges uniformly for $\lambda \in K$, so the limiting function of λ, $Z(\lambda,+1)$, is also analytic in λ. The spherical projection $\hat{Z}(\lambda,\tau)$ is not $\mathbb{C}$ analytic, but it will be real differentiable as a map from $\mathbb{R}^4 \to S^3$. This means the composition function $\hat{Z}\Big(\lambda(s), +1\Big)$ is differentiable with respect to the real parameter $s \in [0,1]$. The quantity $\zeta(\lambda)$ is given as the ratio of components of $\hat{Z}(\lambda, 1)$ and $\hat{X}^{+}(\lambda)$ and is therefore differentiable in s. $\square$

Definition 3.2.9. *Let Z and $X^{\pm}(\lambda)$ be defined as in Lemma 3.2.4 and fix some $\tau_0 \in [-1, 1]$. The **relative phase** of $Z(\lambda, \tau_0)$ is defined*

$$GP\Big(Z(K, \tau_0)\Big) - GP\Big(X^{+}(K)\Big) \tag{3.2.1}$$

Lemma 3.2.10. *For non-degenerate reference paths $X^{\pm}(\lambda)$ for $A_{\pm\infty}(\lambda)$ and $Z, \hat{Z}$ as defined in Lemma 3.2.4 above, the **relative phase** of $\hat{Z}(\lambda, +1)$ equals the winding of the induced phase.*

Proof. The natural connection on the Hopf bundle, S^3, is given by the 1-form

$$\omega(V_p) \equiv \langle V_p, p \rangle_{\mathbb{C}^2}, \qquad V_p \in T_p\left(S^3\right) \subset T_p\left(\mathbb{C}^2\right)$$

so that to calculate the geometric phase of $\hat{Z}\Big(\lambda(s), +1\Big)$, consider

$$\begin{aligned}
& \hat{Z}\Big(\lambda(s), +1\Big) &=& \quad \zeta\Big(\lambda(s)\Big)\hat{X}^{+}\Big(\lambda(s)\Big) \\
\Rightarrow \quad & \tfrac{d}{ds}\hat{Z}\Big(\lambda(s), +1\Big) &=& \quad \zeta'\Big(\lambda(s)\Big)\lambda'(s)\hat{X}^{+}\Big(\lambda(s)\Big) + \zeta\Big(\lambda(s)\Big)\tfrac{d}{ds}\hat{X}^{+}\Big(\lambda(s)\Big) \\
\Rightarrow \quad & \omega\left(\tfrac{d}{ds}\hat{Z}\Big(\lambda(s), +1\Big)\right) &=& \quad \overline{\zeta(s)}\zeta'\Big(\lambda(s)\Big)\lambda'(s) + \omega\left(\tfrac{d}{ds}\hat{X}^{+}\Big(\lambda(s)\Big)\right)
\end{aligned}$$

because $\hat{X}^{+}\Big(\lambda(s)\Big)$ is a unit vector and $\zeta(s) \in S^1$. But the geometric phase of $\hat{Z}(\lambda, +1)$ is given by

$$
\begin{aligned}
GP\Big(Z(K,+1)\Big) &= \frac{1}{2\pi i}\int_0^1 \omega\left(\frac{d}{ds}\hat{Z}\Big(\lambda(s),+1\Big)\right) ds \\
&= \frac{1}{2\pi i}\int_0^1 \left[\overline{\zeta(s)}\zeta'\Big(\lambda(s)\Big)\lambda'(s) + \omega\left(\frac{d}{ds}\hat{X}^+\Big(\lambda(s)\Big)\right)\right] ds \\
&= \frac{1}{2\pi i}\int_0^1 \frac{\zeta'\Big(\lambda(s)\Big)}{\zeta\Big(\lambda(s)\Big)}\lambda'(s)ds + \frac{1}{2\pi i}\int_0^1 \omega\left(\frac{d}{ds}\hat{X}^+\Big(\lambda(s)\Big)\right) ds \quad (3.2.2) \\
&= \frac{1}{2\pi i}\int_0^1 \frac{\zeta'\Big(\lambda(s)\Big)}{\zeta\Big(\lambda(s)\Big)}\lambda'(s)ds + GP\Big(X^+(K)\Big), \quad (3.2.3)
\end{aligned}
$$

so that the relative phase of $Z(\lambda(s),+1)$ equals the winding of the induced phase. □

The following two lemmas elaborate the dependence of the relative phase upon the reference paths.

Lemma 3.2.11. *Given the contour K, let $V_1(\lambda)$ be a non-degenerate reference path and $V_2(\lambda)$ be a meromorphic reference path for $A_{+\infty}(\lambda)$. Then*

$$GP\Big(V_1(K)\Big) = GP\Big(V_2(K)\Big) + \mathrm{Ind}(V_2) \tag{3.2.4}$$

where $\mathrm{Ind}(V_2)$ *is plus or minus multiplicity of any zero or pole for V_2 in K°.*

Proof. Suppose V_2 has no essential singularity in K°. This is a generic choice as V_2 is an eigenvector of $A_{+\infty}(\lambda)$; λ appears linearly in $\mathcal{L} - \lambda$ so that the only generic degeneracy of V_2 in K° is a pole or a zero. As eigenvectors, there must be some smooth scaling $\sigma : K \to \mathbb{C}^*$ such that $V_1(\lambda) \equiv \sigma(\lambda)V_2(\lambda)$. Moreover, $\sigma(\lambda)$ can be extended over K° up to any zeros or poles enclosed by K. Consider the connection of $V_1\Big(\lambda(s)\Big)$, for some parametrization $\lambda(s)$,

$$\omega\left(\frac{d}{ds}\hat{V}_1\Big(\lambda(s)\Big)\right) = \frac{d}{ds}\hat{\sigma}\Big(\lambda(s)\Big)\overline{\hat{\sigma}}\Big(\lambda(s)\Big) + \omega\left(\frac{d}{ds}\hat{V}_2\Big(\lambda(s)\Big)\right)$$

where $\hat{\sigma}\Big(\lambda(s)\Big) \equiv \frac{\sigma\big(\lambda(s)\big)}{\left|\sigma\big(\lambda(s)\big)\right|}$. Therefore the geometric phase of V_1 equals that of V_2 plus the winding of $\hat{\sigma}\Big(\lambda(s)\Big)$; this agrees with $Ind(V_2)$ by the argument principle. $\square$

Lemma 3.2.12. *Let $V(\lambda)$ be a reference path for $A_{-\infty}(\lambda)$, with corresponding solution $V(\lambda,\tau)$, such that $V(\lambda)$ has a pole or zero in K°. Then the geometric phase of $V(\lambda,+1)$ equals the geometric phase of a solution evolved from a non-degenerate reference path plus the index of its degeneracy.*

Proof. By definition $V(\lambda)$ is an eigenvector and therefore there must be some smooth scaling $\alpha : K \to \mathbb{C}^*$ and non-degenerate reference path $X^-(\lambda)$ such that

$$V^-(\lambda) \equiv \alpha(\lambda)X^-(\lambda) \tag{3.2.5}$$

Let V and Z denote solutions in the center unstable manifolds for these reference paths respectively, then by linearity of the flow the connection of the solution corresponding to $V(\lambda)$ is given

$$\begin{aligned} \hat{V}(\lambda,1) \quad &= \quad \hat{\alpha}(\lambda)\hat{Z}(\lambda,1) \\ \Rightarrow \quad \omega\left(\tfrac{d}{ds}\hat{V}\Big(\lambda(s)\Big)\right) \quad &= \quad \tfrac{d}{ds}\hat{\alpha}\Big(\lambda(s)\Big)\overline{\hat{\alpha}}\Big(\lambda(s)\Big) + \omega\left(\tfrac{d}{ds}\hat{Z}\Big(\lambda(s)\Big)\right) \end{aligned} \tag{3.2.6}$$

$\square$

Corollary 3.2.13. *Given a choice of reference paths $X^\pm(\lambda)$ for $A_{\pm\infty}(\lambda)$, and $Z(\lambda,\tau)$ as defined above, the **relative phase** of $Z(\lambda,+1)$,*

$$GP\Big(Z(K,+1)\Big) - GP\Big(X^+(K)\Big), \tag{3.2.7}$$

equals the winding of the induced phase if and only if $X^\pm(\lambda)$ each have the same index of degeneracy. In particular, the relative phase is the winding of the induced phase when $X^\pm(\lambda)$ are non-degenerate.

Proof. This is a direct consequence of Lemmas 3.2.10, 3.2.11 and 3.2.12. $\square$

3.3 The trivializations and the transition map

The unstable bundle is a non-trivial complex line bundle contained in the ambient trivial $\mathbb{C}^2$ vector bundle over the parameter sphere; for fixed λ, as τ moves between ± 1, the parameters in the sphere are the values (λ, τ) which describe the motion of solutions $Z(\lambda, \tau)$. Recall, taking a trivialization of this line bundle amounts to finding a linear isomorphism

$$\phi_\alpha : \mathcal{U}_\alpha \times \mathbb{C} \hookrightarrow \mathcal{U}_\alpha \times \mathbb{C}^2$$

where $\mathcal{U}_\alpha$ is a neighborhood in M, and the image of ϕ_α is the unstable bundle over $\mathcal{U}_\alpha$.

Definition 3.3.1. *Define the following:*

- *Let H_- be the **lower hemisphere** of M, given by*

  $$K^\circ \times \{\tau = -1\} \cup K \times \Big\{\tau \in [-1, 1]\Big\} \cup V \times \{\tau = +1\}$$

  where V is an open neighborhood in K° homotopy equivalent to S^1 with K in the closure of V. Assume no eigenvalue of $\mathcal{L}$ is contained in V. Thus H_- is an open neighborhood of M.

- *Let H_+ be the **upper hemisphere** of M, given by*

  $$K^\circ \times \{\tau = +1\} \cup K \times \Big\{\tau \in (-1, 1]\Big\}$$

  so H^+ is an open neighborhood of M.

- *Let Z and $\hat{Z}$ be as given in §3.2; abusing notation, let Z and $\hat{Z}$ also denote their extensions into $V \times \{\tau = +1\}$ so that for $\lambda \in V$, $Z(\lambda, +1)$ is smoothly compatible with the values $Z(\lambda, +1)$, $\lambda \in K$.*

- *For some non-degenerate reference path $X^+(\lambda)$ for $A_{+\infty}(\lambda)$, let $Y(\lambda, \tau)$ be in the center stable manifold of $X^+(\lambda)$. Extend Y into $K^\circ \times \{\tau = +1\}$ so that for $\lambda \in K^\circ$, $Y(\lambda, +1)$*

is an eigenvector for the unstable direction of $A_{+\infty}(\lambda)$, smoothly compatible with the values on the boundary K. Define the spherical projection of Y to be $\hat{Y}$.

For fixed (λ, τ), where they are defined, $\hat{Z}, \hat{Y}$ each span the unstable bundle. $\hat{Z}$ is defined over H_- and $\hat{Y}$ is defined over H_+, so that for any point p in the unstable bundle one may choose a unique $z \in \mathbb{C}$ for which $p \equiv (\lambda, \tau, z\hat{Z})$ if p is is over H_-, or choose a unique $y \in \mathbb{C}$ for which $p \equiv (\lambda, \tau, y\hat{Y})$ if p is over H_+. Thus the projections $\hat{Z}, \hat{Y}$ give choices of trivializations for the unstable bundle over H_-, H_+ respectively.

Definition 3.3.2. *Given Z, Y as above, and a choice of hemispheres $H_\pm$, define the following maps:*

$$\begin{array}{rccc} \phi_- : & H_- \times \mathbb{C} & \hookrightarrow & H_- \times \mathbb{C}^2 \\ & (\lambda, \tau, z) & \mapsto & \big(\lambda, \tau, z\hat{Z}(\lambda, \tau)\big) \end{array}$$

$$\begin{array}{rccc} \phi_+ : & H_+ \times \mathbb{C} & \hookrightarrow & H_+ \times \mathbb{C}^2 \\ & (\lambda, \tau, y) & \mapsto & \big(\lambda, \tau, y\hat{Y}(\lambda, \tau)\big) \end{array}$$

These maps are the ***trivializations of the unstable bundle*** *with respect to $H_\pm$, $\hat{Z}$ and $\hat{Y}$. The maps $\phi_\pm$ are linear vector bundle isomorphisms, and their composition $\hat{\phi} \equiv \phi_+^{-1} \circ \phi_-$ defined on $H_- \cap H_+ \times \mathbb{C}$ will define the transition map of the unstable bundle.*

Fixing τ such that $(\lambda, \tau) \in H_- \cap H_+ \ \forall \lambda \in K$, the transition map can be seen as a mapping from S^1 to $GL(1, \mathbb{C})$, ie: take the restriction of the composition of trivializations to the λ parameter

$$\begin{array}{rccc} {\phi_+}^{-1} \circ \phi_-(\lambda, \tau, -) : & K \cong S^1 & \to & GL(1, \mathbb{C}) \\ & (\lambda, -) & \mapsto & \hat{\phi}(-) \\ \hat{\phi} : & \mathbb{C} & \to & \mathbb{C} \\ & z & \mapsto & y \end{array}$$

where

$$z(\lambda, \tau)\hat{Z}(\lambda, \tau) = y(\lambda, \tau)\hat{Y}(\lambda, \tau)$$

Viewed this way

$$\phi_+{}^{-1} \circ \phi_-(-,\tau,-) \equiv \hat{\phi}_\tau$$

is seen to have a representation in the fundamental group of $GL(1,\mathbb{C}) \cong \mathbb{C}^*$. The fundamental group $\pi_1(\mathbb{C}^*) \cong \pi_1(S^1) \cong \mathbb{Z}$, so one may identify $\left[\hat{\phi}_\tau\right] \cong d$ where $d \in \mathbb{Z}$ is the winding of $\hat{\phi}_\tau$ about K.

Lemma 3.3.3. *The winding of the map $\hat{\phi}_\tau(\lambda)$ is equal to the* ***Chern number*** *of the unstable bundle, and therefore the total multiplicity of the eigenvalues contained in K°.*

Proof. See Alexander, Gardner & Jones [6] §6. □

Lemma 3.3.4. *For a choice of non-degenerate reference paths $X^\pm(\lambda)$ for $A_{\pm\infty}(\lambda)$, the winding of the induced phase equals the Chern number of the unstable bundle.*

Proof. Notice that for $(\lambda,+1) \in H_- \cap H_+$ the transition map can be described through the induced phase:

$$z \;\mapsto\; z\hat{Z}\Big(\lambda(s),+1\Big) \;\equiv\; z\zeta\Big(\lambda(s)\Big)\hat{X}^+\Big(\lambda(s)\Big) \;\equiv\; z\zeta\Big(\lambda(s)\Big)\hat{Y}\Big(\lambda(s),+1\Big) \;\mapsto\; z\zeta\Big(\lambda(s)\Big)$$

so that the transition map $\hat{\phi}$ is exactly given by $z \mapsto \zeta\Big(\lambda(s)\Big)z$. But the number of windings $\zeta\Big(\lambda(s)\Big)$ takes around the K is given by

$$d = \frac{1}{2\pi i}\int_{\zeta(K)} \frac{1}{z}dz \tag{3.3.1}$$

$$= \frac{1}{2\pi i}\int_0^1 \frac{\zeta'\Big(\lambda(s)\Big)}{\zeta\Big(\lambda(s)\Big)}\lambda'(s)ds \tag{3.3.2}$$

so that the Chern number of the unstable bundle is given by the equation (3.3.2) for the winding of the induced phase. □

3.4 The geometric phase and the transition map

The previous section establishes the relationship between the induced phase $\zeta(s)$, for non-degenerate reference paths, and the Chern number of the unstable bundle over M.

However, this must be related to the geometric phase in the Hopf bundle for a solution in the unstable manifold. Let $Z, \hat{Z}$ be defined as in §3.2. Then each $Z, \hat{Z} \in W^u$ for all ξ and

$$Z^\sharp \;\; := \;\; e^{i\mu_2\left(\lambda(s)\right)\xi} Z$$

is the corresponding solution to the A system at ξ. It remains to show that the geometric phase of the two solutions agree for each ξ, and to relate the phase to the winding of the transition map for the unstable bundle as $\xi \to +\infty$.

Theorem 3.4.1 (The method of geometric phase—case I). *Given a choice of reference paths $X^\pm(\lambda)$ for $A_{\pm\infty}(\lambda)$ and Z defined as in §3.2, the asymptotic relative phase of $Z\Big(\lambda, \tau(\xi)\Big)$,*

$$\lim_{\xi \to \infty} GP\Big(Z\Big(K, \tau(\xi)\Big)\Big) - GP\Big(X^+(K)\Big), \tag{3.4.1}$$

equals the total multiplicity of the eigenvalues enclosed by K if $X^\pm(\lambda)$ are non-degenerate.

Proof. This theorem is a direct consequence of Lemmas 3.3.3 and 3.3.4, and Corollary 3.2.13. □

Finally, the relationship between the solutions to the B, system defined for the proof, and the solutions to the A system will be established.

Proposition 3.4.2. *Let $X^-(\lambda)$ be a reference path for $A_{\pm\infty}$ and suppose Z and $Z^\sharp$ are solutions to the B and A system respectively, and that they agree at ξ_0; then for arbitrary finite ξ the geometric phase of $Z\Big(\lambda, \tau(\xi)\Big)$ and $Z^\sharp\Big(\lambda, \tau(\xi)\Big)$ agree.*

Proof. Suppose $\mu_2(\lambda) \equiv \alpha(\lambda) + i\beta(\lambda)$, and recall the solution to the A system given by

$$Z^\sharp\Big(\lambda, \tau(\xi)\Big) = e^{\mu_2(\lambda)(\xi - \xi_0)} Z\Big(\lambda, \tau(\xi)\Big).$$

Without loss of generality, suppose $\xi_0 = 0$ so that $Z^\sharp$ is the unique solution to the A system that agrees with Z at $\xi = 0$; the proof will not depend on the constant. The projection of $Z^\sharp$

onto the Hopf bundle is given by

$$\hat{Z}^{\sharp}\Big(\lambda, \tau(\xi)\Big) \equiv e^{i\beta(\lambda)\xi}\hat{Z}\Big(\lambda, \tau(\xi)\Big),$$

so that calculating the phase:

$$\begin{array}{rrcl} & \hat{Z}^{\sharp}\Big(\lambda(s), \tau(\xi)\Big) & = & e^{i\beta\big(\lambda(s)\big)\xi}\hat{Z}\Big(\lambda(s), \tau(\xi)\Big) \\ \Rightarrow & \frac{d}{ds}\hat{Z}^{\sharp}\Big(\lambda(s), \tau(\xi)\Big) & = & i\beta'\Big(\lambda(s)\Big)\lambda'(s)\xi e^{i\beta\big(\lambda(s)\big)\xi}\hat{Z}\Big(\lambda(s), \tau(\xi)\Big) \ + \\ & & & e^{i\beta\big(\lambda(s)\big)\xi}\frac{d}{ds}\hat{Z}\Big(\lambda(s), \tau(\xi)\Big) \\ \Rightarrow & \omega\left(\frac{d}{ds}\hat{Z}^{\sharp}\Big(\lambda(s), \tau(\xi)\Big)\right) & = & i\beta'\Big(\lambda(s)\Big)\lambda'(s)\xi + \omega\left(\frac{d}{ds}\hat{Z}\Big(\lambda(s), \tau(\xi)\Big)\right) \end{array} \quad (3.4.2)$$

But $\mu_2(\lambda), \mu_2'(\lambda)$ are each holomorphic by construction so that

$$\int_K \mu_2'(\lambda) = \int_K \alpha'(\lambda) + i\int_K \beta'(\lambda) \equiv 0$$

and the real and imaginary parts both must equal zero. The $i\beta'\Big(\lambda(s)\Big)\lambda'(s)\xi$ term thus vanishes in equation (3.4.2) when integrated for $s \in [0,1]$. This proves the geometric phase of $Z^{\sharp}(\lambda, \xi)$ of the A system corresponds to the phase of the solution $Z(\lambda, \xi)$ for the B system for arbitrary ξ. For systems defined on $\mathbb{C}^2$, one may thus obtain the total multiplicity of the eigenvalues contained in K° with a solution to either the A or B system utilizing the method of geometric phase. □

3.5 The method of geometric phase on the bistable equation

This section presents an example exploring Way's numerical method for computing the geometric phase on the Hopf bundle. This example illustrates some of the properties of the phase and its variation along paths, and it demonstrates a clear dependence on the length of the integration in the ξ direction, where the **relative phase** changes continuously from zero to the value of the multiplicity of the eigenvalue. The geometric phase of a differentiable path in the unstable manifold is not generically zero, as demonstrated in the examples. However, for symmetric systems, the relative phase will always transition from zero to the eigenvalue count, by the construction of the relative phase.

Returning to the bi-stable example, equation

$$u_t = u_{xx} + f(u) \qquad\qquad f(u) = u(u+1)(u-1)$$

consider the case when $c = 0$. Then $\xi = x$ and $u(\xi) = \sqrt{2}sech(\xi)$ is a time independent solution to the equation $u_t = u_{\xi\xi} - u + u^3$, with $-\infty < \xi < +\infty$. Consider the linearization $\mathcal{L}$ about the basic state. Trivially, 0 is an eigenvalue of multiplicity one for the linear operator $\mathcal{L}$. The dynamical systems formulation of $\mathcal{L}$ is

$$Y' = A(\lambda,\xi)Y \qquad\qquad \xi \in (-\infty,\infty)\,,\ \lambda \in \Omega \subset \mathbb{C}$$

$$A = \begin{pmatrix} 0 & 1 \\ \lambda + 1 - 6sech^2(\xi) & 0 \end{pmatrix} \qquad A_\infty(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda+1 & 0 \end{pmatrix} \tag{3.5.1}$$

which is equivalent to the operator $\mathcal{L}_\xi(p) = p_{\xi\xi} + f'(u(\xi))p$. The eigenvalues/vectors for the asymptotic system are of the form

$$+\sqrt{\lambda+1}, \qquad \begin{pmatrix} 1 \\ \\ \sqrt{\lambda+1} \end{pmatrix} \tag{3.5.2}$$

$$-\sqrt{\lambda+1}, \qquad \begin{pmatrix} 1 \\ \\ -\sqrt{\lambda+1} \end{pmatrix} \tag{3.5.3}$$

Take the contour K to be a circle of radius 0.1 about the origin in the complex plane. In the figures below plot the building of the geometric phase versus the integration interval in ξ. By discretizing the contour K into $10,000$ even steps, the geometric phase of the forward integrated loop of eigenvectors in equation (3.5.2) is plotted from $\xi = -11$ to $\xi = 11$ with two different scalings of initial conditions. For each point $\lambda \in K$, the initial condition is integrated forward in ξ and the equation (2.3.14) is computed, with the derivative approximated with the difference

$$\partial_s \,|_{s=s_0} \, u(s,\xi) \approx \frac{u(s_0+\delta s,\xi) - u(s_0-\delta s,\xi)}{2\delta s}.$$

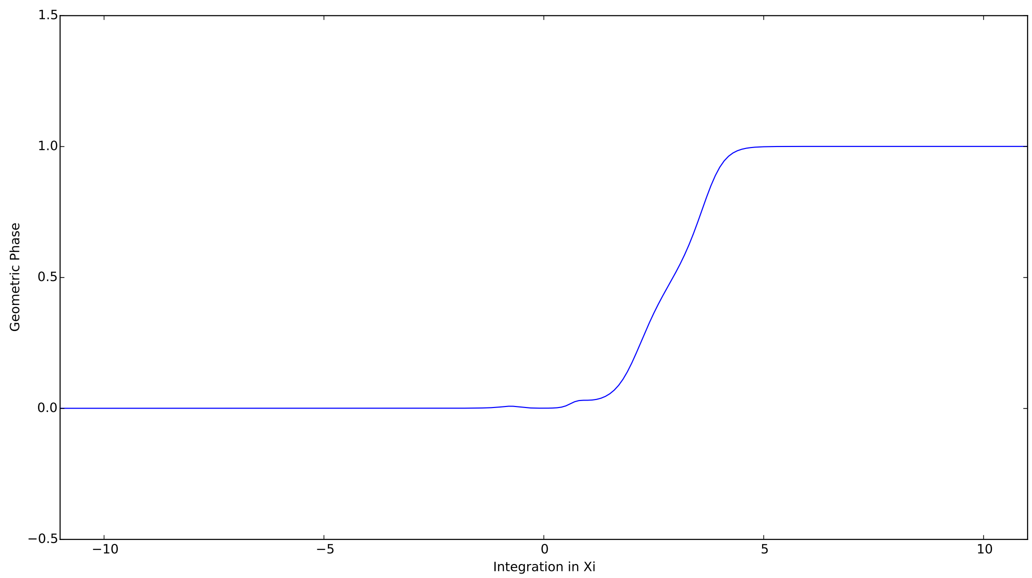

Figure 3.1: The phase profile for non-degenerate initial conditions.

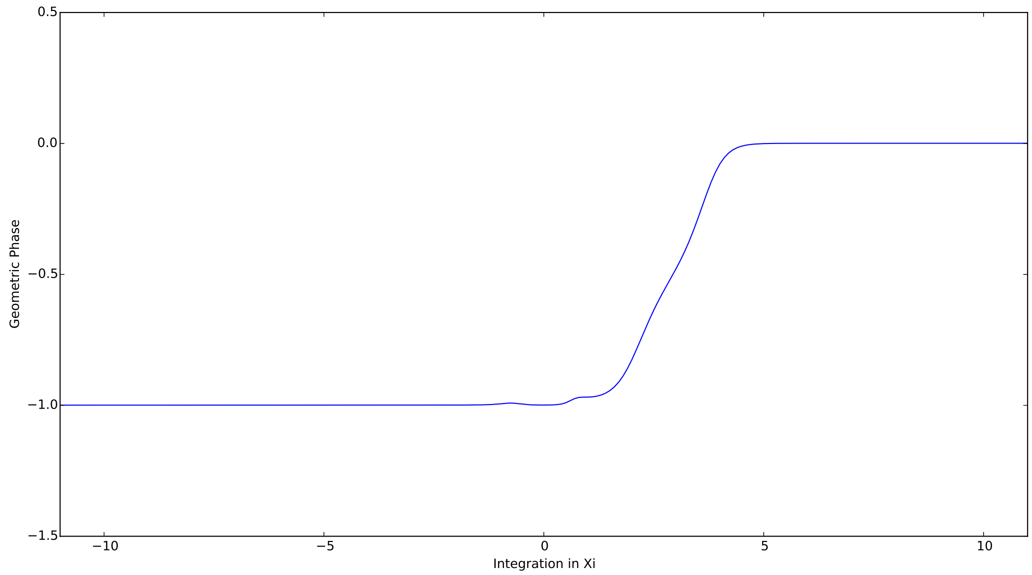

Figure 3.2: The phase profile with initial conditions scaled with simple pole.

The geometric phase experiences a transition near $\xi = 0$ in these two examples. Hence, the phase calculation need not be performed for ξ "close" to $+\infty$, but simply past a threshold where the change of phase occurs. The first figure plots the geometric phase of the eigenvectors in equation (3.5.2) exactly as the initial condition, but the second figure instead plots the initial condition scaled by the factor $\frac{1}{\lambda}$, so there is a pole enclosed at 0. In the degenerate case the geometric phase of the initial condition is -1, and thus the phase profile is translated by the index of the degeneracy.

Although in the above **non-degenerate** example, the initial geometric phase is zero, it need not be so in general. The contour K defined as the circle with center at 0.1 and radius 1 nears $\lambda = -1$, where $A_\infty(\lambda)$ is singular. In the plots below, the contour is discretized into $20,000$ even steps and the geometric phase of of the non-degenerate initial condition is evolved as in the previous example. For this contour, the geometric phase of the non-degenerate initial conditions in equation (3.5.2) has a different profile, beginning with phase greater zero and terminating with phase greater than the eigenvalue count.

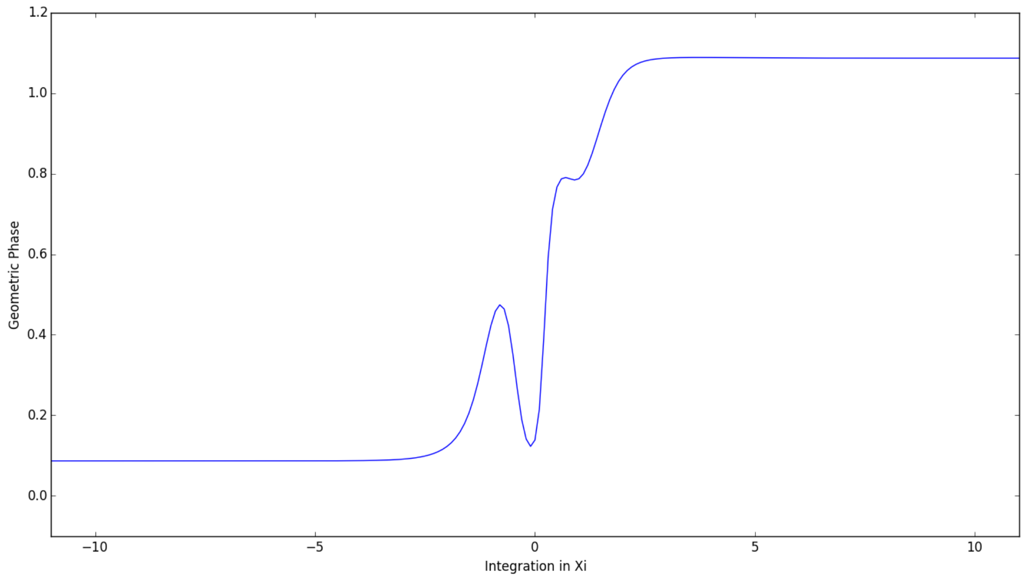

Figure 3.3: The geometric phase profile of the evolved solution.

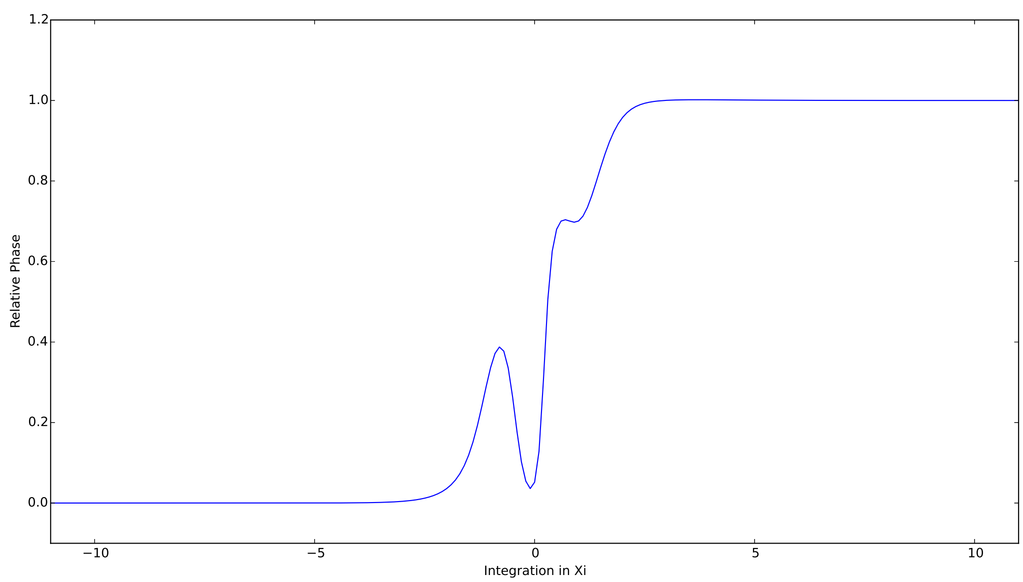

Figure 3.4: The relative phase profile of the evolved solution.

This specific example demonstrates the necessity of the **relative phase** formulation; in this system with symmetric asymptotic conditions, the relative phase may be formulated as

$$GP\Big(Z(K,\tau)\Big) - GP\Big(Z(K,-1)\Big) \tag{3.5.4}$$

because the reference paths may be chosen $X^{-}(\lambda) = X^{+}(\lambda)$. The relative phase is plotted as the terminal geometric phase minus the initial geometric phase; here the relative phase transitions between zero and the eigenvalue count as expected. This second example moreover demonstrates the non-uniform nature of the phase transition, which is also exhibited in the phase transition for the system defined by the Hocking-Stewartson pulse solution of the complex Ginzburg-Landau equation. The general method for systems defined on $\mathbb{C}^n$ is fully developed in the subsequent chapter, concluding with a numerical treatment of the Hocking-Stewartson pulse of the complex Ginzburg-Landau equation.

CHAPTER 4

The method for general equations

In considering general systems of reaction diffusion equations equations (2.1.1), this chapter adapts the techniques developed for scalar equations, taking advantage of the full generality in which the unstable bundle and Evans function can be constructed. Firstly, the case when $n > 2$ and there are $k > 1$ unstable directions will be considered—once a multi-dimensional formulation is established, the full generalization for systems with non-symmetric asymptotic limits will be presented. The statements of the method of geometric phase in this chapter reduce to the case in the previous chapter, so the theorems of this chapter may be considered the fully general statements for reaction diffusion systems defined on unbounded domains.

4.1 The determinant bundle of the unstable manifold

Suppose now the operator $\mathcal{L}$ defines an A system and B system, as in Chapter 3, but these systems are on $\mathbb{C}^n$ for $n > 2$. If for all $\lambda \in \Omega$, $A_\infty = A_{\pm\infty}$ has one unstable direction and $n-1$ stable directions, the proof in two dimensions holds; although the ambient complex dimension has increased, the unstable bundle is still one-dimensional. Likewise if the stable manifold is 1-dimensional, one may calculate the Chern number of the analogous stable bundle without any serious modification of the method.

Suppose more generally there are $1 < k < n-1$ unstable directions for the system A_∞. The k dimensional unstable bundle is again formed from the unstable manifold $W^u(\lambda, \tau)$ of the critical point $(0, -1)$, and the Chern number of this vector bundle equals the total multiplicity of the eigenvalues contained in K°. However, it is no longer sufficient to only consider solutions corresponding to a single eigenvector, as this will not capture the information of the full unstable bundle. To map the transition map of the unstable bundle, E, to a value in

S^1 this section introduces the determinant bundle constructed from a k dimensional vector bundle. This technique uses subspace coordinates, reducing the dimension of the unstable bundle to one, while raising the ambient complex dimension of the system. With respect to this coordinatization, the unstable manifold is a trajectory on which one can again calculate the geometric phase, and the goal is thus to apply the same method used in $\mathbb{C}^2$ to the determinant bundle of the k dimensional unstable space.

Definition 4.1.1. *The* $\mathbf{k}^{th}$ ***exterior power of*** $\mathbb{C}^{\mathbf{n}}$, $\Lambda^k(\mathbb{C}^n) \equiv \mathbb{C}^{\binom{n}{k}}$, *is the complex vector space of non-degenerate k forms on $\mathbb{C}^n$. $\Lambda^k(\mathbb{C}^n)$ is spanned by*

$$v = v_1 \wedge \cdots \wedge v_k \qquad v_i \in \mathbb{C}^n \;\; \forall i$$

and v is non-degenerate provided $\{v_i\}_1^k$ are linearly independent in $\mathbb{C}^n$.

Definition 4.1.2. *Given a dynamical system*

$$X' = AX \qquad ' = \tfrac{d}{d\xi} \qquad X \in \mathbb{C}^n$$

let $Y = Y_1 \wedge \cdots \wedge Y_k \in \Lambda^k(\mathbb{C}^n)$. The associated $\mathbf{A}^{(\mathbf{k})}$ ***system*** *on $\Lambda^k(\mathbb{C}^n)$ is generated by*

$$Y' = A^{(k)}Y \tag{4.1.1}$$

$$:= AY_1 \wedge \cdots \wedge Y_k + \cdots + Y_1 \wedge \cdots \wedge AY_k \tag{4.1.2}$$

Remark 4.1.3. *By equation (4.1.2) it is clear that the eigenvalues for the $A^{(k)}$ system are the sums of all k-tuples of eigenvalues for A. Thus for $A^{(k)}$, there is a unique eigenvalue of largest positive real part given by the sum of all eigenvalues with positive real part, including multiplicity.*

Definition 4.1.4. *Suppose $\mathcal{L}$ defines a system of the form (3.1.1) on $\mathbb{C}^n$. Denote $\{\mu_i^{\pm}\}_{i=1}^k$ the eigenvalues of positive real part for $A_{\pm\infty}(\lambda)$ respectively, and define $\mu^{\pm} := \sum_{i=1}^k \mu_i^{\pm}$. The* *corresponding* $\mathbf{A^{(k)}}$ ***and*** $\mathbf{B^{(k)}}$ ***systems*** *on* $\Lambda^k(\mathbb{C}^n) \equiv \mathbb{C}^{\binom{n}{k}}$ *are defined:*

$$\begin{aligned} &Y' = A^{(k)}(\lambda,\tau)Y \quad A^{(k)}_{\pm\infty}(\lambda) = \lim_{\xi\to\pm\infty} A^{(k)}(\lambda,\tau) \\ &\tau' = \kappa(1-\tau^2) \end{aligned} \tag{4.1.3}$$

$$\begin{aligned} B^{(k)}(\lambda,\tau) &:= \left(A^{(k)}(\lambda,\tau) - \mu^-(\lambda)\right) \qquad X' = B^{(k)}X \\ B^{(k)}_{\pm\infty}(\lambda) &:= \lim_{\xi\to\pm\infty} B^{(k)}(\lambda,\tau) \qquad \tau' = \kappa(1-\tau^2) \end{aligned} \tag{4.1.4}$$

Allen & Bridges [16] demonstrate that there is an explicit algorithm to compute the $A^{(k)}$ system (4.1.2) on the exterior power $\Lambda^k(\mathbb{C}^n)$ where the coefficients of $A^{(k)}$ are calculated through the inner product on $\mathbb{C}^n$.

Definition 4.1.5. *Let* $e_1, \cdots, e_n$ *denote the standard basis on* $\mathbb{C}^n$ *and* $\omega_1, \cdots, \omega_d$ *be the orthonormal basis for* $\Lambda^k(\mathbb{C}^n)$ *generated from the* $\{e_i\}_{i=1}^n$*, ie: all k-forms*

$$e_{i_1} \wedge \cdots \wedge e_{i_k} \qquad i_j < i_{j+1} \ \forall j.$$

If $x := x_1 \wedge \cdots \wedge x_k$ *and* $y := y_1 \wedge \cdots \wedge y_k$ *are k-forms in* $\Lambda^k(\mathbb{C}^n)$*, their* ***inner product*** *is defined*

$$\ll x, y \gg_k \;=\; det \begin{pmatrix} \langle x_1, y_1\rangle_{\mathbb{C}^n} & \cdots & \langle x_1, y_k\rangle_{\mathbb{C}^n} \\ \vdots & \ddots & \vdots \\ \langle x_k, y_1\rangle_{\mathbb{C}^n} & \cdots & \langle x_k, y_k\rangle_{\mathbb{C}^n} \end{pmatrix} \tag{4.1.5}$$

For $A^{(k)}x = \sum_{j=1}^k x_1 \wedge \cdots \wedge Ax_j \wedge \cdots \wedge x_k$, Allen and Bridges show the coefficients of $A^{(k)}$ can be computed following

$$A^{(k)}_{i,j} = \ll \omega_i, A\omega_j \gg_k \qquad i,j = 1,\cdots,d = \binom{n}{k}$$

An explicit calculation of the $A^{(2)}$ system on $\Lambda^2(\mathbb{C}^4)$ is given by Afendikov and Bridges [17] for the linearization of the complex Ginzburg-Landau equation about the Hocking-Stewartson pulse; this example was developed in the context of the asymptotic relative phase calculation in *The instability of the Hocking-Stewartson pulse and its geometric phase in the Hopf bundle* [11], and will be used to demonstrate the method of geometric phase for general unbounded domains at the end of this chapter. Generally for a 4×4 matrix A this algorithm generates the following $A^{(2)}$ system:

$$A^{(2)} = \begin{pmatrix} a_{11}+a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\ a_{32} & a_{11}+a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\ a_{42} & a_{43} & a_{11}+a_{44} & 0 & a_{12} & a_{13} \\ -a_{31} & a_{21} & 0 & a_{22}+a_{33} & a_{34} & -a_{24} \\ -a_{41} & 0 & a_{21} & a_{43} & a_{22}+a_{44} & a_{23} \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33}+a_{44} \end{pmatrix} \tag{4.1.6}$$

For the symmetric form of system (4.1.4), $B_\infty^{(k)}$ has a center direction of critical points, an unstable real direction, and all other directions are stable; the line of critical points is given by the span of the wedge of linearly independent eigenvectors corresponding to $\{\mu_i^-\}_{i=1}^k$.

Definition 4.1.6. *For all $(\lambda,\tau) \in M$, let $\Big\{w_i(\lambda,\tau)\Big\}_{i=1}^k$ be a spanning set for the unstable manifold W^u at (λ,τ), and define*

$$\Lambda^k\Big(W^u(\lambda,\tau)\Big) \equiv span_{\mathbb{C}}\Big\{w_1(\lambda,\tau)\wedge\cdots\wedge w_k(\lambda,\tau)\Big\}.$$

Then $\Lambda^k\Big(W^u(\lambda,\tau)\Big)$ can be taken as the fiber for a non-trivial vector bundle $\Lambda^k(E)$ over M with projection $\pi_{E^k} : E^k \to M$,

$$\begin{array}{ccc} \Lambda^k\Big(W^u(\lambda,\tau)\Big) & \longrightarrow & \Lambda^k(E) \\ & & \Big\downarrow \; \pi_{E^k} \\ & & M \end{array} \tag{4.1.7}$$

$\Lambda^k(E)$ is called the ***determinant bundle of the unstable manifold*** *over M; henceforth*

$\Lambda^k(E)$ *is referred to simply as the* ***determinant bundle****. The determinant bundle is a* ***line bundle****.*

Let the transition map of the unstable bundle E be denoted $\hat{\phi}_E$. The determinant bundle acquires its namesake from the construction of its transition map $\hat{\phi}^k_E$. The transition map of the k-dimensional unstable bundle is a λ dependent, non-singular mapping of k-frames of n dimensional complex vectors. Restricting to the equator of M, one may thus interpret the transition map

$$\begin{array}{rccc} \hat{\phi}_E : & S^1 & \to & GL(\mathbb{C}, k) \\ & \lambda & \mapsto & \psi(\lambda) \end{array}$$

so that it defines an element of $\pi_1\Big(GL(\mathbb{C}, k)\Big)$. But notice, $\det\Big(\hat{\phi}_E(\lambda)\Big) \in GL(\mathbb{C}, 1)$ for all $\lambda \in K$, so that the determinant induces a homomorphism of fundamental groups

$$\begin{array}{rccc} \det_* : & \pi_1\Big(GL(\mathbb{C}, k)\Big) & \to & \pi\Big(GL(\mathbb{C}, 1)\Big) \\ & \Big[\hat{\phi}_E\Big] & \mapsto & \Big[\det \circ \hat{\phi}_E\Big] \end{array}$$

Definition 4.1.7. *The mapping,*

$$\det \circ \hat{\phi}_E(\lambda) \equiv \hat{\phi}^k_E, \tag{4.1.8}$$

is the ***transition map of the determinant bundle****.*

Lemma 4.1.8. *The Chern number of the determinant bundle of the unstable manifold over M equals the Chern number of the unstable bundle, and therefore the total multiplicity of eigenvalues for $\mathcal{L}$ contained in K.*

Proof. This is proven by Alexander, Gardner & Jones [6] §6. □

For systems (4.1.3) with symmetric asymptotic limits, one may utilize the method of geometric phase, calculating the geometric phase of the solution $Z(\lambda, \tau)$ corresponding to the eigenvalue of most positive real part, where $Z(\lambda, \tau)$ describes the determinant bundle. These modifications are presented in the following theorem.

Theorem 4.1.9 (The method of geometric phase—case II). *Let the $A^{(k)}$ and $B^{(k)}$ systems be defined as in equations (4.1.3) and (4.1.4) above. Let $X^{\pm}(\lambda)$ be reference paths for $A^{(k)}_{\pm\infty}(\lambda)$ and suppose $Z\Big(\lambda, \tau(\xi)\Big)$ is in the center-unstable manifold of $X^{-}(\lambda)$ with respect to $B^{(k)}$. Then the asymptotic relative phase of $Z\Big(\lambda, \tau(\xi)\Big)$,*

$$\lim_{\xi \to \infty} GP\Big(Z\Big(K, \tau(\xi)\Big)\Big) - GP\Big(X^{+}(K)\Big), \tag{4.1.9}$$

equals the total multiplicity of the eigenvalues enclosed by the contour K if $X^{\pm}(\lambda)$ are non-degenerate.

Proof. As in the two dimensional case, Z forms a $\mathbb{C}$ analytic section of the line bundle over M for $\tau \in [-1, 1]$. §4 of Alexander, Gardner & Jones [6] shows that this solution is analytic on $[-1, +1)$ and §6 shows that the limit as $\xi \to +\infty$ is non-zero and continuous. The proof of locally uniform convergence in Proposition 3.2.8 holds here as well, so that the extension of Z to $Z(\lambda, +1)$ is $\mathbb{C}$ analytic.

Therefore take the projection of Z, $\hat{Z}$, onto the sphere

$$S^{\left(2\binom{n}{k}-1\right)} \subset \mathbb{C}^{\binom{n}{k}} \cong \Lambda^{k}\left(\mathbb{C}^{n}\right),$$

then with respect to $X^{\pm}(\lambda)$ the induced phase $\zeta(\lambda)$ is recovered as a value in S^{1}.

Let $Y(\lambda, \tau)$ be a solution to the $B^{(k)}$ system that is in the center-stable manifold of a non-degenerate reference path $\hat{X}^{+}(\lambda)$ at $\tau = +1$, and let $\hat{Y}$ be the projection of this solution. The trivializations of the determinant bundle can be expressed in terms of $\hat{Z}$ and $\hat{Y}$, which yields transition map

$$\hat{Z}(\lambda, +1) \equiv \zeta(\lambda)\hat{X}^{+}(\lambda) \equiv \zeta(\lambda)\hat{Y}(\lambda, +1)$$

The winding of $\zeta(\lambda)$ is thus equal to the Chern number of the determinant bundle, and is related to the geometric phase of $Z(\lambda, +1)$ by the same formulation described in the two dimensional case. $\square$

Thus in the case of k unstable directions, one may calculate the total multiplicity of the eigenvalues contained in the region K° by an adaptation of the method of geometric phase applied to the determinant bundle of the unstable manifold. The same proof as in Lemma 3.4.2 will demonstrate that the geometric phase is equivalent in both the $A^{(k)}$ and $B^{(k)}$ systems.

4.2 The general method for systems of equations on unbounded domains

This section includes the statement of the general method of geometric phase for systems of reaction diffusion equations on unbounded domains, defining a dynamical system for any $n \geq 2$. The preceding sections developed a method for finding the total multiplicity of eigenvalues for $\mathcal{L}$ in the region K°, but the method was restricted to the case for which $\lim_{\xi \to -\infty} A(\lambda, \xi) \equiv \lim_{\xi \to +\infty} A(\lambda, \xi)$. The unstable bundle construction, however, is valid for general systems $A_{\pm\infty}$ that **split** in Ω, i.e., each have exactly k unstable, and $n-k$ stable directions for every $\lambda \in \Omega$. The final modification is to account for systems with non-symmetric asymptotic limits. The following construction will reduce to that in the previous sections if the system is symmetric or the dimension of the unstable manifold is $k = 1$, so this may be considered to be the fully general statement of the method of geometric phase for systems on unbounded domains.

Define the determinant bundle system

$$\begin{aligned} Y' &= A^{(k)}(\lambda, \tau) Y \quad A^{(k)}_{\pm\infty}(\lambda) = \lim_{\xi \to \pm\infty} A^{(k)}(\lambda, \tau) \\ \tau' &= \kappa(1 - \tau^2) \end{aligned} \tag{4.2.1}$$

derived from the flow $Y' = AY$ on $\mathbb{C}^n$.

Given a non-degenerate reference path for $A_{-\infty}(\lambda)$, $X^-(\lambda)$, one may construct the center-unstable manifold of the direction of critical points at $\tau = -1$ in the $B^{(k)}$ system as before. However, the behavior of such a solution will differ when $\tau \to +1$. The dominating unstable eigenvalue for the system at $\tau = +1$ does not in general equal the value at $\tau = -1$, and to calculate the asymptotic relative phase a solution that is non-singular as $\xi \to \infty$ will be needed.

Definition 4.2.1. *Let $\mu^{\pm}(\lambda)$ be the eigenvalue of most positive real part for $A^{(k)}_{\pm\infty}(\lambda)$. For a reference path $X^{-}(\lambda)$ for $A^{(k)}_{-\infty}(\lambda)$ define the center-unstable manifold of $X^{-}(\lambda)$ in the $B^{(k)}$ system to be $Z(\lambda, \tau)$ for $\tau \in [-1, 1)$. Define*

$$Z^{\sharp}(\lambda, \tau) := e^{\left(\mu^{-}(\lambda)\xi\right)} Z(\lambda, \tau) \qquad \tau \in (-1, +1)$$

so that

$$\Gamma(\lambda, \tau) \quad := \quad \begin{cases} e^{\left(-\mu^{-}\xi\right)} Z^{\sharp}(\lambda, \tau) & \textit{for } \tau \in [-1, 0) \\ e^{\left(-\mu^{+}\xi\right)} Z^{\sharp}(\lambda, \tau) & \textit{for } \tau \in [0, +1) \\ \lim_{\xi \to \infty} e^{\left(-\mu^{+}\xi\right)} Z^{\sharp}(\lambda, \tau) & \textit{for } \tau = +1 \end{cases} \tag{4.2.2}$$

Proposition 4.2.2. *$\Gamma(\lambda, \tau)$ satisfies the equation*

$$Y' = \Psi(\lambda, \xi) Y \qquad \Psi \quad = \quad \begin{cases} \left(A^{(k)}(\lambda, \xi) - \mu^{-}(\lambda) I\right) \textit{ for } \xi \in (-\infty, 0) \\ \left(A^{(k)}(\lambda, \xi) - \mu^{+}(\lambda) I\right) \textit{ for } \xi \in [0, +\infty) \end{cases} \tag{4.2.3}$$

Moreover, $\Gamma(\lambda, \tau)$ is non-zero and analytic in λ for fixed τ, and spans the determinant bundle $\forall (\lambda, \tau) \in H_{-}$.

Proof. Notice that $\Gamma(\lambda, \tau)$ is a solution to equation (4.2.3) by construction and, moreover, the analyticity of Γ for $\tau \in [-1, +1)$ is obvious from the analyticity of Z. Under the flow defined by

$$Y' = \left(A^{(k)}(\lambda, \tau) - \mu^{+}(\lambda) I\right) Y \tag{4.2.4}$$

the eigenvector corresponding to $\mu^{+}(\lambda)$ is once again a line of critical points. The solution

$\Gamma(\lambda, \tau)$ will converge uniformly in λ to a non-zero critical point defined by the flow in Ψ, as is demonstrated in Lemma 6.1 in Alexander, Gardner & Jones [6], utilized in its full generality. Likewise, following the proof of the Proposition 3.2.8 in Chapter 3, $\Gamma(\lambda, \tau)$ indeed defines a section of the determinant bundle over the lower hemisphere H_-. $\square$

Theorem 4.2.3 (**The method of geometric phase—general systems on unbounded domains**). *If $X^{\pm}(\lambda)$ are reference paths for $A^{(k)}_{\pm\infty}(\lambda)$, and $\Gamma\Big(\lambda, \tau(\xi)\Big)$ is defined as in Definition 4.2.1, then the asymptotic relative phase of $\Gamma\Big(\lambda, \tau(\xi)\Big)$,*

$$\lim_{\xi \to \infty} GP\Big(\Gamma\Big(K, \tau(\xi)\Big)\Big) - GP\Big(X^+(K)\Big), \tag{4.2.5}$$

equals the total multiplicity of the eigenvalues enclosed by the contour K if $X^{\pm}(\lambda)$ are non-degenerate.

Proof. To adapt the determinant bundle method from here, it remains only define $Y, \hat{Y}$ appropriately so they converge to a non-degenerate reference path for $A^{(k)}_{+\infty}$. The construction of the induced parallel translation will follow analogously, as will the lemmas of §3. $\square$

Remark 4.2.4. *The equivalence of the geometric phase for $\Gamma(\lambda, \tau)$ and $Z^\sharp(\lambda, \tau)$ for $\tau \in (-1, 1)$ follows from the proof of Proposition 3.4.2.*

4.3 The Hocking-Stewartson pulse of the CGL equation

This section demonstrates the method of geometric phase for an ODE system defined on $\mathbb{C}^n$ with $n > 2$ and an unstable manifold of dimension $k = 2$; this situation will give rise to a compound matrix system of the form in equation (4.1.6), defining the phase space $\mathbb{C}^6$ where the geometric phase is computed.

The scaled, complex Ginzburg-Landau equation is given by

$$\rho e^{i\psi} Y_t = Y_{xx} - (1 + i\omega)^2 Y + (1 + i\omega)(2 + i\omega) \mid Y \mid^2 Y \tag{4.3.1}$$

where $\rho > 0$, ψ and ω are specified real parameters for the system. The Hocking-Stewartson pulse is the steady state solution for the Complex Ginzburg-Landau equation, given by

$$Y(x,t) = (cosh(x))^{-1-i\omega} \tag{4.3.2}$$

Let $\mathcal{L}$ be the linearization of the operator defining equation (4.3.1) about the Hocking-Stewartson pulse (4.3.2). Considering solutions proportional to $e^{\lambda t}$, one can derive a non-autonomous system on $\mathbb{C}^4$, with asymptotic limits in x, as done by Afendikov & Bridges [17]. The system is of the form

$$\begin{array}{ll} v' = A(x,\lambda)v & v \in \mathbb{C}^4 \\ \lim_{x\to\pm\infty} A(x,\lambda) = A_{\pm\infty}(\lambda) & ' = \frac{d}{dx} \end{array} \tag{4.3.3}$$

Remark 4.3.1. *The linearization of the complex Ginzburg-Landau equation about the pulse, $\mathcal{L}$, has essential spectrum on the set*

$$S_{ess} = \left\{\rho^{-1}e^{\mp i\psi}(\omega - s^2 - 1) \mp 2i\rho^{-1}\omega e^{\mp i\psi}, s \in \mathbb{R}^+\right\} \tag{4.3.4}$$

and for the parameter values $\omega = 3$, $\rho = \frac{1}{\sqrt{5}}$, and $\psi = \arctan(2)$ there is a known double eigenvalue at $\lambda = 0$, and simple eigenvalues at approximately $\lambda = -6.6357$ and $\lambda = 15$ estimated by Afendikov & Bridges [17].

Lemma 4.3.2. *For the parameter values $\omega = 3$, $\rho = \frac{1}{\sqrt{5}}$, and $\psi = \arctan(2)$ system (4.3.3) splits on the domain $\{\lambda \in \mathbb{C} : Re(\lambda) > 0\}$. Moreover, $\Omega \subset \mathbb{C}$ can be chosen such that $\{\lambda \in \mathbb{C} : Re(\lambda) > 0\} \subset \Omega$ and $-6.6357 \in \Omega$.*

Proof. Afendikov & Bridges [17] demonstrate that the autonomous limits, $A_{\pm\infty}(\lambda)$, each have exactly 2 stable and unstable eigenvalues respectively, for each λ such that $Re(\lambda) > 0$, and in general for $\lambda \notin S_{ess}$. For $\omega = 3$, $\rho = \frac{1}{\sqrt{5}}$, and $\psi = \arctan(2)$, the essential spectrum is a curve in $\mathbb{C}$ that does not intersect -6.6357; therefore an open $\Omega \subset \mathbb{C}$ can be chosen containing

$\lambda = -6.6357$ without intersecting the essential spectrum, and for such an Ω, system (4.3.3) splits on the domain. $\square$

In order to capture the winding of the unstable manifold of the asymptotic system $A_{-\infty}(\lambda)$, we define the dynamical system on the exterior algebra $\Lambda^2(\mathbb{C}^4)$. Explicitly, Afendikov & Bridges derive the compound matrix system

$$u_x = A(\lambda, x)u \qquad x \in \mathbb{R} \qquad \lambda \in \mathbb{C} \quad u \in \mathbb{C}^6$$

$$A(\lambda) = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ a_{32} & 0 & 0 & 0 & 0 & 0 \\ a_{42} & 0 & 0 & 0 & 0 & 1 \\ -a_{31} & 0 & 0 & 0 & 0 & -1 \\ -a_{41} & 0 & 0 & 0 & 0 & 0 \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & 0 \end{pmatrix} \tag{4.3.5}$$

with components defined

$$\begin{aligned}
a_{31} =& \lambda\rho\cos(\psi) + 1 - \omega^2 - (2-\omega^2)(\hat{q}_2^2 + 3\hat{q}_1^2) + 6\omega\hat{q}_1\hat{q}_2 \\
a_{32} =& -\lambda\rho\sin(\psi) - 2\omega - 2(2-\omega)\hat{q}_1\hat{q}_2 + 3\omega(\hat{q}_1^2 + 3\hat{q}_2^2) \\
a_{41} =& \lambda\rho\sin(\psi) + 2\omega - 2(2-\omega)\hat{q}_1\hat{q}_2 - 3\omega(3\hat{q}_1^2 - \hat{q}_2^2) \\
a_{42} =& \lambda\rho cos(\psi) + 1 - \omega^2 - (2-\omega^2)(\hat{q}_1^2 + 3\hat{q}_2^2) - 6\omega\hat{q}_1\hat{q}_2
\end{aligned}$$

and $\hat{q}_1, \hat{q}_2$ derived from the expression for the pulse in $\mathbb{C}^4$, where

$$\begin{aligned}
\hat{q}_1 =& \frac{\cos\Big(\omega\log\Big(\cosh(x)\Big)\Big)}{\cosh(x)} \\
\hat{q}_2 =& \frac{-\sin\Big(\omega\log\Big(\cosh(x)\Big)\Big)}{\cosh(x)}
\end{aligned}$$

The non-autonomous system (4.3.5) has the symmetric asymptotic limits

$$A_\infty(\lambda) = \lim_{x\to\pm\infty} A(\lambda,x) = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ -p(\lambda) & 0 & 0 & 0 & 0 & 0 \\ \eta(\lambda) & 0 & 0 & 0 & 0 & 1 \\ -\eta(\lambda) & 0 & 0 & 0 & 0 & -1 \\ -p(\lambda) & 0 & 0 & 0 & 0 & 0 \\ 0 & -p(\lambda) & \eta(\lambda) & -\eta(\lambda) & -p(\lambda) & 0 \end{pmatrix} \tag{4.3.6}$$

where the parameters are defined

$$p(\lambda) = 2\omega + \lambda\rho\sin(\psi) \tag{4.3.7}$$

$$\eta(\lambda) = 1 - \omega^2 + \lambda\rho\cos(\psi) \tag{4.3.8}$$

For the asymptotic system (4.3.6), non-degenerate reference paths can be constructed explicitly. The unique eigenvalues of most positive and most negative real part for system (4.3.6) are given by σ^+, σ^- respectively, and have associated eigenvectors

$$\sigma^+ = \sqrt{2}\sqrt{\eta + \sqrt{\eta^2 + p^2}}, \qquad X^+(\lambda) = \begin{pmatrix} 2\sigma^+ \\ -2p \\ (\sigma^+)^2 \\ -(\sigma^+)^2 \\ -2p \\ \sigma^+\big((\sigma^+)^2 - 2\eta\big) \end{pmatrix} \tag{4.3.9}$$

$$\sigma^- = -\sqrt{2}\sqrt{\eta + \sqrt{\eta^2 + p^2}}, \qquad X^-(\lambda) = \begin{pmatrix} 2\sigma^- \\ -2p \\ (\sigma^-)^2 \\ -(\sigma^-)^2 \\ -2p \\ \sigma^-\Big((\sigma^-)^2 - 2\eta\Big) \end{pmatrix}. \tag{4.3.10}$$

These eigenvectors correspond to the Grassmann coordinates for the un/stable subspace of the asymptotic system on $\mathbb{C}^4$, and for $x < 0$ and $\mid x \mid$ taken sufficiently large, the λ dependent initial conditions defined by (4.3.9) will approximate the unstable manifold of the fixed point 0 for the asymptotic system.

In each example below the contour K is chosen to be the circle of radius .1 about λ_0 where $\lambda_0 \in \{0, 15, -6.6537\}$. The contour is discretized into $10,000$ even steps, and for each fixed λ in the discretization of K, the the unstable eigenvector (4.3.9) is integrated from $x_0 = -10$ forward to some x_1. The Matlab ODE45 solver is used to find the trajectory of the initial condition $X^+(\lambda, x_0)$ with respect to the system (4.3.5), and the trajectory is stored at step sizes of .04 in x. To compute the relative phase in equation, the geometric phase of the solution and the reference path is computed with the Euler method from the connection equation (2.3.13). The relative phase is computed for each stored value of x and plotted for each of the three contours below—because the system is symmetric, the **relative** geometric phase is described by subtracting the **initial** geometric phase from the **terminal** geometric phase.

The first figure demonstrates the phase transition for the simple eigenvalue at $\lambda_0 = 15$, where the transition is almost monotonic.

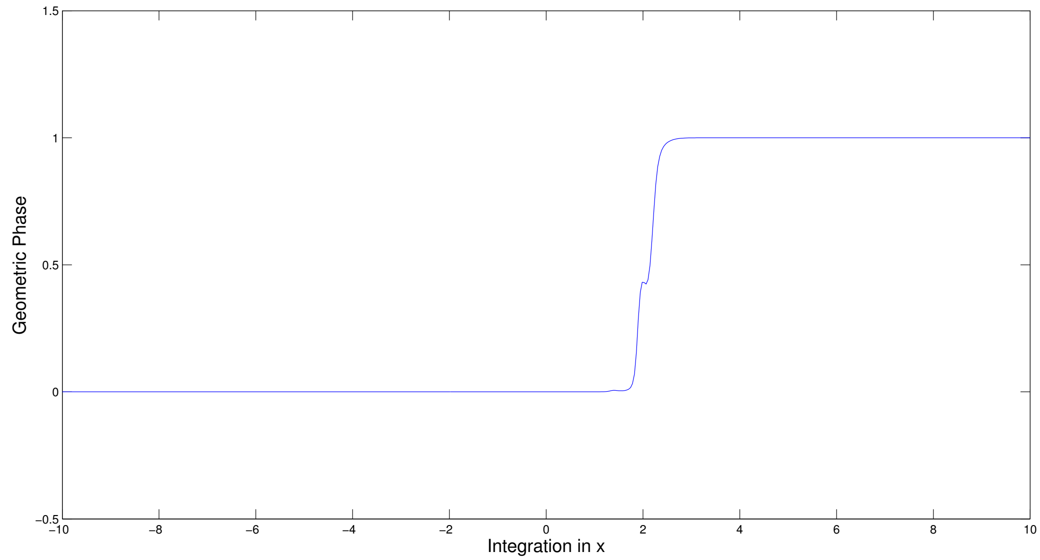

Figure 4.1: The phase transition plotted for the simple eigenvalue at $\lambda \approx 15$

However, the other two plots for $\lambda_0 \in \{0, -6.6357\}$ demonstrate a non-uniform transition both in terms of the monotonicity in the phase calculation, as well as the value of x for which the transition begins.

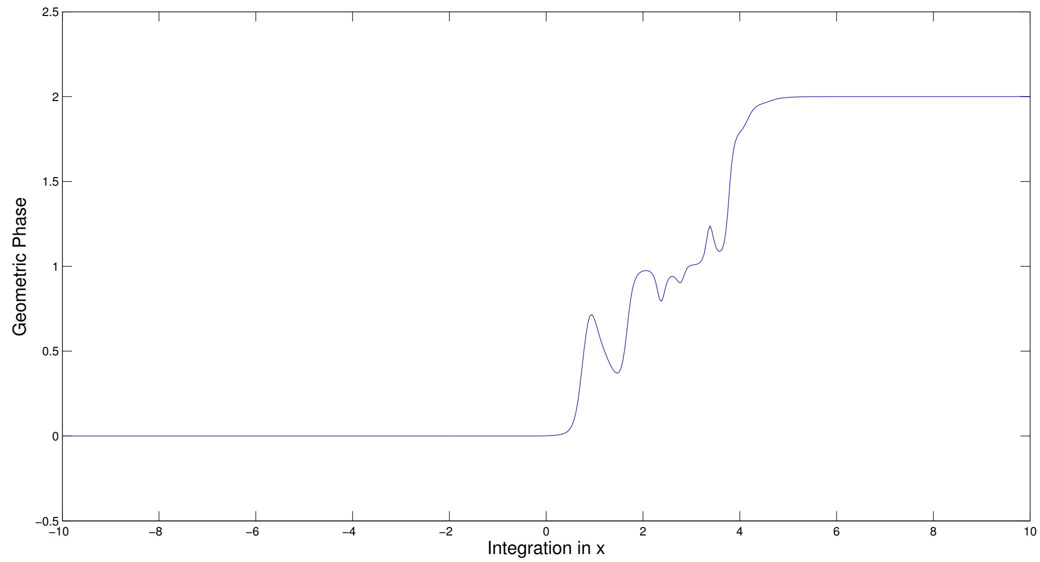

Figure 4.2: The phase transition plotted for the double eigenvalue at $\lambda = 0$.

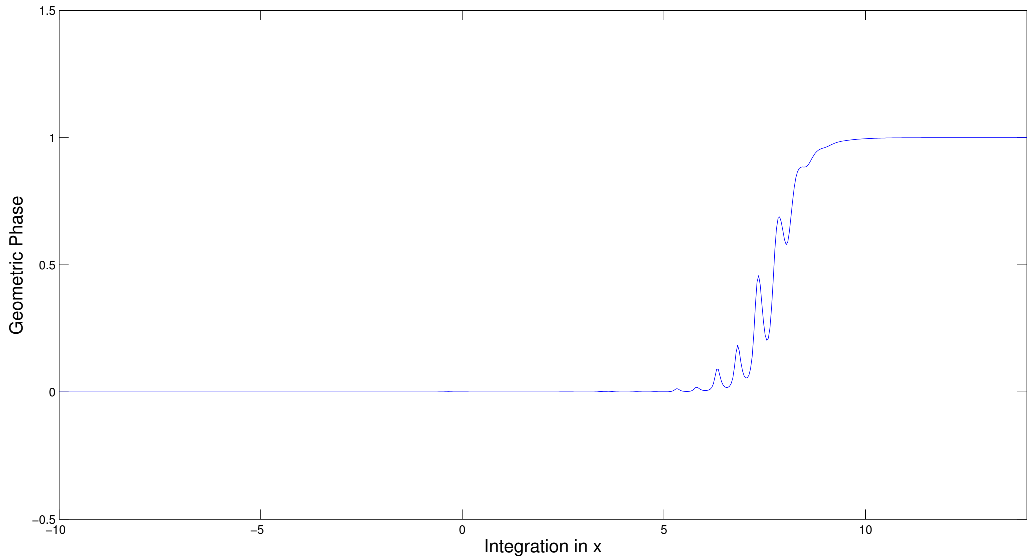

Figure 4.3: The phase transition plotted for the simple eigenvalue at $\lambda \approx -6.6357$

The scale in x direction for the plot of the phase transition at $\lambda_0 = -6.6357$ is longer, ending at $x_1 = 14$. Noticeably, the transition here begins later, and doesn't terminate until it is nearly at the end of the other plots, at $x_1 \approx 10$. This example in particular highlights the importance of understanding the phase transition for applications. It is still an open question as to what triggers the change of mode in the phase, how this is related to underlying wave, and the spectrum of the operator $\mathcal{L}$. Understanding the nature of the phase transition may itself provide additional means of analysing the the stability of the underlying wave.

CHAPTER 5

The method for systems on bounded domains

While the previous chapters dealt with ODE systems defined on unbounded domains, the Evans function may also be used to locate and count the spectrum of linear operators defined for boundary value problems. Gardner & Jones made further developments in the bundle construction for the Evans Function to study boundary value problems with parabolic boundary conditions [18], ie: problems of the form

$$u_t = Du_{xx} + f(x, u, u_x) \qquad (0 < x < 1)$$
$$u(x, 0) = u_0 \qquad B_0 u = 0 \qquad B_1 u = 0$$

where $u \in \mathbb{R}^n$, $f : \mathbb{R}^{2n+1} \to \mathbb{R}^n$ is C^2. The matrix D is a positive diagonal matrix and the boundary operators are defined

$$B_0 u = D^0 u(0, t) + N^0 u_x(0, t)$$
$$B_1 u = D^1 u(0, t) + N^1 u_x(0, t)$$

such that D^j, N^j are diagonal with entries α_i^j, β_i^j respectively that satisfy

$$\left(\alpha_i^j\right)^2 + \left(\beta_i^j\right)^2 = 1 \qquad 1 \leq i \leq n; \quad i = 1, 2$$

Austin & Bridges built upon and generalized these bundle methods into a vector bundle construction for boundary value problems for which the boundary conditions can depend on λ, and allow for general splitting of the boundary conditions [19]. This chapter will consider how the method of geometric phase can be adapted to boundary value problems, using the techniques Austin and Bridges developed for the general boundary conditions.

5.1 The boundary bundle for $\mathbb{C}^n$

For $n \geq 2$, consider a system of ODE's defining a flow on $\mathbb{C}^n$, derived from the linearization $\mathcal{L}$ of a reaction diffusion equation about a steady state. Assume the system is of the form

$$\begin{gathered} u_x = A(\lambda, x)u \qquad 0 < x < 1 \qquad \lambda \in \Omega \subset \mathbb{C} \\ a_j^*\left(\bar{\lambda}\right) : \mathbb{C} \to \mathbb{C}^n \quad j = 1, ..., n-k \quad b_j^*\left(\bar{\lambda}\right) : \mathbb{C} \to \mathbb{C}^n \quad j = 1, ..., k \end{gathered} \tag{5.1.1}$$

where $A(\lambda, x)$ depends analytically on λ, and the a_j^*, b_j^* are holomorphic functions of $\bar{\lambda}$ that describe the boundary conditions for the operator $\mathcal{L}$—the specific conditions are described with respect to the **section product** below.

The ambient trivial bundle is once again constructed from the product $M \times \mathbb{C}^n$. The vectors $\left(\lambda, x, a_j^*\right), \left(\lambda, x, b_j^*\right)$ for each $(\lambda, x) \in M$ are anti-holomorphic sections of the trivial bundle, motivating the above dual notation.

Definition 5.1.1. *For a pair $\nu(\lambda, x)$, $\eta(\lambda, x)$ where ν is a holomorphic section and η is an anti-holomorphic section of the trivial bundle $M \times \mathbb{C}^n$, their product is defined as:*

$$\langle \eta, \nu \rangle_\lambda = \sum_{j=1}^{n} \overline{\eta_j\left(\bar{\lambda}\right)} \nu_j(\lambda) \tag{5.1.2}$$

where η_j, ν_j are their respective components.

Remark 5.1.2. *This scalar product is holomorphic for all $\lambda \in \Omega$, and the boundary value problem is formulated as follows: $u(\lambda, x)$ is an eigen function of the operator $\mathcal{L}$ for the eigenvalue λ if and only if $u(\lambda, x)$ is a solution to $u_x = A(\lambda, x)u$ and*

$$\begin{aligned} \left\langle a_j^*\left(\bar{\lambda}\right), u(\lambda, 0)\right\rangle_\lambda &= 0 \qquad j = 1, ..., n-k \\ \left\langle b_j^*\left(\bar{\lambda}\right), u(\lambda, 1)\right\rangle_\lambda &= 0 \qquad j = 1, ..., k \end{aligned}$$

A significant difference in this construction from the unbounded systems is that there are no dynamics to consider on the caps of the parameter sphere, nor eigenvalues of a limiting system to describe eigenfunctions. What is needed then is an analogue to the unstable bundle that will trace the dynamics and pick up winding while traversing the parameter sphere between $\tau = \mp 1$. One choice is the orthogonal complement to the initial conditions, dimension k, and the manifold defined by their evolution. First is to show that these subspaces, and their evolution under the flow, vary holomoprhically with respect to $\lambda \in \Omega$. With a holomorphic basis, one may construct a non-trivial vector bundle over M through which the geometric phase can be computed as in previous chapters.

Theorem 5.1.3. *For a system of the form (5.1.1) derived from the operator $\mathcal{L}$ there exist analytic choices of orthogonal bases for $\mathbb{C}^n$ such that*

$$V_0 := \left\{\nu_j(\lambda) : \lambda \in \Omega\right\}_{j=1}^{n-k} \qquad U_0 := \left\{\xi_j(\lambda) : \lambda \in \Omega\right\}_{j=1}^{k} \qquad V_0 \oplus U_0 = \mathbb{C}^n \tag{5.1.3}$$

$$V_1 := \left\{\upsilon_j(\lambda) : \lambda \in \Omega\right\}_{j=1}^{n-k} \qquad U_1 := \left\{\eta_j(\lambda) : \lambda \in \Omega\right\}_{j=1}^{k} \qquad V_1 \oplus U_1 = \mathbb{C}^n \tag{5.1.4}$$

$$span_{\mathbb{C}}\{\nu_j\}_{j=1}^{n-k} = span_{\mathbb{C}}\{a_j^*\}_{j=1}^{n-k} \quad span_{\mathbb{C}}\{\eta_j\}_{j=1}^{k} = span_{\mathbb{C}}\{b_j^*\}_{j=1}^{k} \tag{5.1.5}$$

and with respect to the product of sections (5.1.2)

$$\left\langle a_i^*\left(\bar{\lambda}\right), \xi_j(\lambda)\right\rangle_\lambda = 0 \qquad 0 \le i \le n-k, \;\; 0 \le j \le k \tag{5.1.6}$$

$$\left\langle b_i^*\left(\bar{\lambda}\right), \upsilon_j(\lambda)\right\rangle_\lambda = 0 \qquad 0 \le i \le k, \;\; 0 \le j \le n-k \tag{5.1.7}$$

Proof. This is the content of Austin & Bridges' results in Lemmas 3.1 through 3.3 in [19] and the reader is referred there for a full discussion. □

Remark 5.1.4. *Reformulating the problem in this context, $u(\lambda, x)$ is an eigenfunction of $\mathcal{L}$ with eigenvalue λ if and only if*

$$u(\lambda, 0) \in span_{\mathbb{C}}\left\{U_0(\lambda)\right\}$$
$$u(\lambda, 1) \in span_{\mathbb{C}}\left\{V_1(\lambda)\right\}.$$

Foliating the subspaces $U_0(\lambda), U_1(\lambda) \subset \mathbb{C}^n$ on the caps of M and constructing subspaces that connect $U_0(\lambda), U_1(\lambda)$ will define a "boundary bundle" over M. Choosing $U_0(\lambda), U_1(\lambda)$ as fibers above the caps of the boundary bundle is analogous to the case of unbounded domains because, if λ is not an eigenvalue, a solution to the system $u' = Au$ cannot be in the span U_0 at $x = 0$ and in the span of V_1 at $x = 1$. Any collection of solutions $\left\{\gamma_j(\lambda, x)\right\}_{j=1}^{k}$ that satisfy the boundary conditions at $x = 0$, and are linearly independent for $(\lambda, 0)$, will be linearly independent for (λ, x) where $x \in [0, 1)$. In particular when λ is not an eigenvalue of $\mathcal{L}$, then $\left\{\gamma_j(\lambda, 1)\right\}_{j=1}^{k}$ are linearly independent and must span *some* compliment of $V_1(\lambda)$; in general this need not be the orthogonal complement, i.e., $U_1(\lambda)$, but it is possible to smoothly deform the solutions into $U_1(\lambda)$ with the projection operator.

Definition 5.1.5. *Define the λ dependent projection operator*

$$Q_\lambda : \mathbb{C}^n \to U_1(\lambda)$$

and define the orthogonal projection operator

$$P_\lambda = (I - Q_\lambda) : \mathbb{C}^n \to V_1(\lambda)$$

Proposition 5.1.6. *Let $u_j(\lambda, x)$ be solutions to the flow on $\mathbb{C}^n$ such that $u_j(\lambda, 0) = \xi_j(\lambda)$ for each $j = 1, ..., k$, and let $\left\{\eta_j(\lambda)\right\}_{j=1}^{k}$ be a holomorphic basis for $U_1(\lambda)$. Define*

$$\sigma_j(\lambda, x) \equiv \begin{cases} (I - xP_\lambda)\Big(u_j(\lambda, x)\Big) & (\lambda, x) \in K \times [0, 1] \\ \xi_j(\lambda) & (\lambda, 0) \in K^\circ \times \{0\} \end{cases}$$

Then $\left\{\sigma_j(\lambda, x)\right\}_{j=1}^{k}$ are linearly independent and holomorphic for all $(\lambda, x) \in M \setminus \Big(K^\circ \times \{1\}\Big)$.

Proof. This proposition follows immediately from the results of §4 in Austin & Bridges [19]. $\square$

Definition 5.1.7. *Define $\mathcal{E}_{\lambda,x} \subset \mathbb{C}^n$ to be the k dimensional subspace spanned by $\left\{\sigma_j(\lambda,x)\right\}_{j=1}^k$ for $(\lambda,x) \in K \times [0,1]$. Over $K^\circ \times \{0\}$ define $\mathcal{E}_{\lambda,0} = span_{\mathbb{C}}\left\{U_0(\lambda)\right\}$. Finally over $K^\circ \times \{1\}$, define $\mathcal{E}_{\lambda,1} = span_{\mathbb{C}}\left\{U_1(\lambda)\right\}$. For the fibers defined as above,*

$$E \equiv \left\{(\lambda, x, \mathcal{E}_{\lambda,x}) : (\lambda,x) \in M\right\}$$

*is defined to be the **boundary bundle** with respect to equation (5.1.1) over M.*

It follows from Proposition 5.1.6 that E is a holomorphic vector bundle for which $\left\{\sigma_j(\lambda,x)\right\}_{j=1}^k$ and $\left\{\eta_j(\lambda)\right\}_{j=1}^k$ define local trivializations over the upper and lower hemispheres of M, similarly to Chapter 3. For $(\lambda,x) \in K \times (0,1]$, define $\left\{\eta_j(\lambda,x)\right\}_{j=1}^k$ as the solutions to equation (5.1.1) which converge to $\left\{\eta_j(\lambda)\right\}_{j=1}^k$ at $x=1$, and extend the $\left\{\sigma_j(\lambda,1)\right\}_{j=1}^k$ holomorphically into an open set in K° homotopy equivalent to S^1.

Definition 5.1.8. *Define the trivializations of the boundary bundle E over open sets in M, by*

$$\begin{array}{rccc} \phi_- : & H_- \times \mathbb{C}^k & \hookrightarrow & H_- \times \mathbb{C}^n \\ & (\lambda, x, ze_j) & \mapsto & \Big(\lambda, x, z\sigma_j(\lambda,x)\Big) \end{array}$$

$$\begin{array}{rccc} \phi_+ : & H_+ \times \mathbb{C}^k & \hookrightarrow & H_+ \times \mathbb{C}^n \\ & (\lambda, x, ze_j) & \mapsto & \Big(\lambda, x, z\eta_j(\lambda,x)\Big) \end{array}$$

whereby the transition map at $\{1\} \times K$ is defined by the matrix $\hat{\phi}(\lambda,1,-) := \phi_+^{-1} \circ \phi_-(\lambda,1,-)$.

Lemma 5.1.9. *The winding of the determinant of the transition function,*

$$\det \circ \hat{\phi}_{(1,-)}(\lambda) : K \to GL(\mathbb{C}),$$

equals the total multiplicity of eigenvalues of $\mathcal{L}$ contained within K°.

Proof. This is Theorem 5.1 in Austin & Bridges [19]. □

5.2 The method of geometric phase for boundary value problems

With the bundle view of boundary value problems with $\lambda-$dependent boundary conditions, one is in the position to utilize the method geometric phase to relate the total multiplicity of eigenvalues contained within K° to the relative phase of paths in the bundle. Utilizing the determinant bundle, as in Chapter 4, the Chern number of the boundary bundle E will be recovered through the relative phase. In particular, the wedge product of the solutions $\left\{\sigma_j(\lambda, x)\right\}_{j=1}^{k}$ will form a solution to the associated system on $\Lambda^k\left(\mathbb{C}^n\right)$ $A^{(k)}$ for which one can compute the phase.

Definition 5.2.1. *Let $\mathcal{U}(\lambda, x) := \sigma_1(\lambda, x) \wedge ... \wedge \sigma_k(\lambda, x)$, and denote $\hat{\mathcal{U}}(\lambda, x)$ to be the spherical projection of $\mathcal{U}(\lambda, x)$. Similarly let $\eta(\lambda, x) := \eta_1(\lambda, x) \wedge ... \wedge \eta_k(\lambda, x)$, and $\hat{\eta}(\lambda, x)$ be the normalization of η in the exterior algebra. Then the line bundle over the parameter sphere with fibers defined by the span of $\mathcal{U}(\lambda, x)$ for $0 \leq x \leq 1$ and the span of $\eta(\lambda, x)$ for $x = 1$ is defined to be the* ***determinant bundle of the boundary bundle****.*

Note that $\hat{\mathcal{U}}$ may not be holomorphic, but as in previous sections it inherits infinite differentiability in the parameter s, where $\lambda(s) : [0, 1] \hookrightarrow K$. One may thus calculate the geometric phase of the vector $\hat{\mathcal{U}}\Big(\lambda(s), x\Big)$ on the Hopf bundle $S^{\left(2\binom{n}{k}-1\right)}$. From the above, one may define trivializations of the determinant bundle similarly to the previous sections via $\hat{\mathcal{U}}(\lambda, x)$ over H_- and $\hat{\eta}(\lambda, x)$ over H_+. The Chern number of this vector bundle is equal to the winding of the transition function, given exactly by the winding of $\det \circ \phi_{(1,-)}(\lambda)$.

Definition 5.2.2. *The relative phase of $\mathcal{U}(\lambda, x)$, as in Definition 5.2.1, is defined to be the quantity*

$$GP\Big(\mathcal{U}(K, x)\Big) - GP\Big(\eta(K, 1)\Big) \tag{5.2.1}$$

Theorem 5.2.3 (**The method of geometric phase—systems on bounded domains**).

Let $\mathcal{U}(\lambda, x), \eta(\lambda, x)$ be defined as in Definition 5.2.1; the relative phase of $\mathcal{U}(\lambda, 1)$,

$$GP\Big(\mathcal{U}(K, 1)\Big) - GP\Big(\eta(K, 1)\Big), \tag{5.2.2}$$

is equal to the total multiplicity of the eigenvalues enclosed by the contour K if $\mathcal{U}(\lambda, 0)$ and $\eta(\lambda, 1)$ are holomorphic and non-zero over K°.

Proof. The calculations of the winding of the transition function and the geometric phase are analogous to the calculations performed in Chapter 4; there is no difference in calculating the geometric phase and transition function with respect to these trivializations, and the proofs of the lemmas of Chapter 3 and Chapter 4 will also work for the boundary bundle setting. Therefore, the relative phase of $\mathcal{U}(\lambda, x)$ at $x = 1$ agrees with the total multiplicity of the eigenvalues contained in K° if the paths $\mathcal{U}(\lambda, 0)$ and $\eta(\lambda, 1)$ enclose no zeros or poles. $\square$

CHAPTER 6

The method in the Stiefel bundle

This chapter proves a reformulation of the Chern number calculation for the method of geometric phase, but formulating the calculation in the Stiefel bundle. The following propositions establish the relationship between the natural connection on the Hopf bundle and the universal connection on the Stiefel bundle, when considering a spanning frame for the unstable manifold. Although the exterior algebra and the determinant bundle is a natural setting to analytically study the Evans function, the *computation* of the Evans function in the exterior algebra becomes prohibitive for systems of large dimension. With the un/stable manifolds typically of dimension $k, n-k \approx \frac{n}{2}$, the state space dimension for the exterior algebra, $\mathbb{C}^{\binom{n}{k}}$, grows approximately exponentially in the dimension n [20], [21]. Several works have proposed methods to overcome the computational cost of the exterior power formulation in the Evans function and in particular, Humpherys & Zumbrun [20], and Ledoux, Malham & Thummler [21], develop shooting algorithms that grow in polynomial complexity, $\mathcal{O}(n^3)$, in the system dimension n.

These shooting formulations of the Evans function are similar to the method of geometric phase as presented in the above chapters but the work of Humpherys & Zumbruns computes *a spanning frame* of vectors, in $\mathbb{C}^n$, for the un/stable manifolds rather than the Grassmann coordinates in $\Lambda^k(\mathbb{C}^n)$; Avitabile & Bridges built on this, connecting the work directly to the integration of paths in the Stiefel manifold [22]. This chapter similarly considers the geometric phase of the unstable manifold in $\Lambda^k(\mathbb{C}^n)$, but in terms of an associated k frame of vectors and the evolution of this frame on the Stiefel manifold. After deriving original results describing the analytic formulation of the connection for a k frame of vectors, spanning the unstable manifold in $\Lambda^k(\mathbb{C}^n)$, the remaining sections in this chapter will discuss a possible numerical implementation of this analytic result, and compare it to similar formulations described above. The results of this section are to be understood as a novel formulation of the method of geometric phase, calculating the winding of the Evans function, but in the Stiefel bundle.

6.1 The geometric phase for a frame

The Stiefel manifold is a fundamental object in differential geometry and algebraic topology, and its geometric and topological properties are discussed, for instance, by Kobayashi & Nomizu [14] and Hatcher [23]. The Stiefel manifold also admits a principal fiber bundle structure—the Stiefel bundle and its canonical connection are defined as follows.

Definition 6.1.1. *Define $V(n,k)$ to be the set of matrices $V \in \mathbb{C}^{n\times k}$ such that $V^*V = I_{k\times k}$, $U(k)$ to be the unitary group over $\mathbb{C}^k$ and $Gr(n,k)$ to be the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$. The* ***Stiefel bundle*** *is the principal fiber bundle $V(n,k)$ over $Gr(n,k)$, induced by right multiplication by elements in the fiber $U(k)$. These spaces are related by the diagram*

$$\begin{array}{ccc} U(n) & \longrightarrow & V(n,k) \\ & & \Big\downarrow \pi \\ & & Gr(n,k) \end{array} \tag{6.1.1}$$

where π is the quotient map induced by the group action of $U(k)$.

Definition 6.1.2. *Let $V(s)$ be a differentiable path in the Stiefel manifold $V(n,k)$. The* ***canonical connection*** *of the Stiefel bundle is defined by the map ω such that*

$$\begin{array}{rcc} \omega: \; T\Big(V(n,k)\Big) & \to & \mathcal{U} \\ \omega\Big(\frac{d}{ds}V(s)\Big) & \mapsto & V(s)^*\frac{d}{ds}V(s) \end{array} \tag{6.1.2}$$

where $\mathcal{U}$ is the Lie algebra of the unitary group $U(k)$, ie: the $k\times k$ skew-Hermitian matrices.

Remark 6.1.3. *Narasimhan & Ramanan [24] demonstrate that the canonical connection of the Stiefel bundle represents a* ***universal connection*** *for principal fiber bundles with fiber* *given by the unitary group. In this sense, the canonical connection on the Stiefel bundle is a natural construction.*

Formulations of the Evans function utilizing the Stiefel manifold decompose the Grassmannian coordinates for un/stable manifolds into angular and radial parts; by utilizing only the angular part, and an associated $n \times k$ matrix, these methods describe the evolution of a loop in the Stiefel manifold. Likewise, for the method of geometric phase, one may consider this decomposition. Suppose as in the preceding chapters there is a system of ODE's defined on $\mathbb{C}^n$ derived from a reaction diffusion equation linearized about a steady state. Let this system be described by

$$\begin{aligned} Y' &= A(\lambda,\tau)Y & A_\infty(\lambda) &:= \lim_{\xi\to\pm\infty} A(\lambda,\tau) \\ \tau' &= \kappa(1-\tau^2) & Y &\in \mathbb{C}^n \\ A(\lambda,\tau) &= \begin{cases} A(\lambda,\xi(\tau)) & \text{for } \tau \neq \pm 1 \\ A_\infty(\lambda) & \text{for } \tau = \pm 1 \end{cases} \end{aligned} \tag{6.1.3}$$

Let K be a simple, closed contour in $\Omega \subset \mathbb{C}$ containing no eigenvalues for the operator associated to the system (6.1.3), and let $K^\circ \subset \mathbb{C}$ define the region enclosed by K. Suppose the asymptotic $A_{\pm\infty}(\lambda)$ systems split in Ω, and that they have exactly k and $n-k$ unstable and stable eigenvalues respectively. Define $W^u(\lambda,\xi)$ be the k dimensional unstable manifold for 0 in $\mathbb{C}^n$.

Definition 6.1.4. *For ξ_0 fixed such that $\Big(\lambda,\tau(\xi_0)\Big) \in K \times [-1,1]$, let $\Big\{\sigma_j(\lambda,\xi) \in \mathbb{C}^n\Big\}_{j=1}^k$ be an orthonormal set of vectors spanning the unstable manifold $W^u(\lambda,\xi_0)$. Let $\lambda(s) : [0,1] \to K$ be a parametrization of K and suppose the vectors $\Big\{\sigma_j\Big(\lambda(s),\xi_0\Big)\Big\}_{j=1}^k$ are each differentiable in s. The set of $\sigma_j(\lambda,\xi_0)$ is defined as a* ***frame for the unstable manifold at $\boldsymbol{\xi_0}$****, and* *likewise denote the matrix with columns given by the σ_j and the wedge product of the σ_j as*

$$\Sigma\Big(\lambda(s),\xi_0\Big) \equiv \Big(\sigma_1\Big(\lambda(s),\xi_0\Big),\cdots,\sigma_k\Big(\lambda(s),\xi_0\Big)\Big) \tag{6.1.4}$$

$$\sigma\Big(\lambda(s),\xi_0\Big) \equiv \sigma_1\Big(\lambda(s),\xi_0\Big)\wedge\cdots\wedge\sigma_k\Big(\lambda(s),\xi_0\Big) \tag{6.1.5}$$

respectively.

Lemma 6.1.5. *Given a frame for the unstable manifold $\Big\{\sigma_j\Big(\lambda(s),\xi_0\Big)\Big\}_{j=1}^{k}$ at ξ_0, the wedge product $\sigma(s)$ gives a path in the Hopf bundle $S^{\left(2\binom{n}{k}-1\right)} \subset \Lambda^k\left(\mathbb{C}^n\right)$.*

Proof. Consider the inner product on $\Lambda^k(\mathbb{C}^n)$, defined in equation (4.1.5); suppressing the dependence on the parameters, the norm squared of σ is given by

$$\begin{aligned} \ll \sigma,\sigma \gg_k &= \det\begin{pmatrix} \langle\sigma_1,\sigma_1\rangle_{\mathbb{C}^n} & \cdots & \langle\sigma_1,\sigma_k\rangle_{\mathbb{C}^n} \\ \vdots & \ddots & \vdots \\ \langle\sigma_k,\sigma_1\rangle_{\mathbb{C}^n} & \cdots & \langle\sigma_k,\sigma_k\rangle_{\mathbb{C}^n} \end{pmatrix} \\ &= \det\Big(I_{k\times k}\Big) \\ &= 1 \end{aligned}$$

and thus $\sigma\Big(\lambda(s),\xi_0\Big)$ is in Hopf bundle for all s. □

The above lemma motivates considering a solution in $\Lambda^k\left(\mathbb{C}^n\right)$ explicitly in terms of a frame for the unstable manifold and its radial component. Neglecting the radial component, one may calculate the geometric phase of a solution in the determinant bundle σ, but in terms of Σ and its component-wise ∂_s derivatives. This is formalized in the following lemma.

Lemma 6.1.6. *Let $\Gamma(\lambda,\tau)$ be a solution, as in Definition 4.2.1, spanning the determinant bundle for $\tau\in[-1,1]$. For fixed ξ_0 define*

$$\Gamma\Big(\lambda,\tau(\xi_0)\Big) \equiv \gamma(\lambda,\xi_0)\sigma_1\wedge\cdots\wedge\sigma_k(\lambda,\xi_0) \tag{6.1.6}$$

for a frame for the unstable bundle at ξ_0. Then the projection of $\Gamma(\lambda, \tau)$ on the Hopf bundle is given by

$$\hat{\Gamma}(\lambda, \tau) = \sigma(\lambda, \tau) \in S^{\left(2\binom{n}{k}-1\right)}, \tag{6.1.7}$$

and the connection of $\partial_s \hat{\Gamma}\Big(\lambda(s), \tau(\xi_0)\Big)$ in the Hopf bundle can be expressed in terms of the frame as

$$\sigma\Big(\partial_s \hat{\Gamma}\Big) = \sum_{i=1}^{k} \sum_{j=1}^{k} (-1)^{i+j} \langle \partial_s \sigma_i, \sigma_j \rangle_{\mathbb{C}^n} \tag{6.1.8}$$

Proof. Recall, one may distribute the s derivative over the wedge product via the Leibeniz rule, so that

$$\partial_s \sigma \quad = \quad \textstyle\sum_{i=1}^{k} \sigma_1 \wedge \cdots \wedge \partial_s \sigma_i \wedge \cdots \wedge \sigma_k.$$

Therefore natural connection of this tangent vector for the path $\hat{\Gamma}(\lambda, \tau)$ in the Hopf bundle is given by

$$\ll \partial_s \sigma, \sigma \gg_k,$$

which can be distributed over sum

$$\ll \partial_s \sigma, \sigma \gg_k = \sum_{i=1}^{k} \ll \sigma_1 \wedge \cdots \wedge \partial_s \sigma_i \wedge \cdots \wedge \sigma_k, \sigma \gg \tag{6.1.9}$$

$$= \sum_{i=1}^{k} \det \begin{pmatrix} \langle \sigma_1, \sigma_1 \rangle_{\mathbb{C}^n} & \cdots & \langle \sigma_1, \sigma_k \rangle_{\mathbb{C}^n} \\ \vdots & & \vdots \\ \langle \partial_s \sigma_i, \sigma_1 \rangle_{\mathbb{C}^n} & \cdots & \langle \partial_s \sigma_i, \sigma_k \rangle_{\mathbb{C}^n} \\ \vdots & & \vdots \\ \langle \sigma_k, \sigma_1 \rangle_{\mathbb{C}^n} & \cdots & \langle \sigma_k, \sigma_k \rangle_{\mathbb{C}^n} \end{pmatrix}. \tag{6.1.10}$$

Due to the orthonormality of the frame, this may be written as

$$\ll \partial_s\sigma, \sigma \gg_k = \sum_{i=1}^{k} \det \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ \langle \partial_s\sigma_i, \sigma_1\rangle_{\mathbb{C}^n} & \cdots & \langle \partial_s\sigma_i, \sigma_k\rangle_{\mathbb{C}^n} \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{pmatrix}, \tag{6.1.11}$$

and expanding cofactors along the row of derivatives in each term yields precisely the alternating sum in equation (6.1.8). □

Lemma 6.1.6 shows that the connection on the Hopf bundle $S^{\left(2\binom{n}{k}-1\right)} \subset \Lambda^k(\mathbb{C}^n)$ can be computed with respect to the frame for the unstable manifold. The following proposition will simplify this formulation in terms of a matrix equation for Σ.

Proposition 6.1.7. *Define*

$$\partial_s\Sigma = (\partial_s\sigma_1, \cdots, \partial_s\sigma_k) \tag{6.1.12}$$

$$j_k = \begin{pmatrix} (-1)^2 \\ (-1)^3 \\ \vdots \\ (-1)^{k+1} \end{pmatrix}. \tag{6.1.13}$$

Then for Γ *and* σ *as defined in Lemma 6.1.6*

$$\sigma\left(\partial_s\hat{\Gamma}\right) = j_k^T(\Sigma^*\partial_s\Sigma)j_k \tag{6.1.14}$$

Proof. It remains only to verify that the equations (6.1.8) and (6.1.14) agree. Notice, $\Sigma^*\partial_s\Sigma$

is the $k \times k$ matrix given by

$$\Sigma^*\partial_s\Sigma = \begin{pmatrix} \langle\partial_s\sigma_1, \sigma_1\rangle & \langle\partial_s\sigma_2\sigma_1\rangle & \cdots & \langle\partial_s\sigma_k, \sigma_1\rangle \\ \vdots & & & \vdots \\ \langle\partial_s\sigma_1, \sigma_k\rangle & \langle\partial_s\sigma_2\sigma_k\rangle & \cdots & \langle\partial_s\sigma_k, \sigma_k\rangle \end{pmatrix} \tag{6.1.15}$$

$$\Rightarrow \Sigma^*\partial_s\Sigma j_k = \begin{pmatrix} \sum_{i=1}^k (-1)^{i+1}\langle\partial_s\sigma_1, \sigma_i\rangle \\ \vdots \\ \sum_{i=1}^k (-1)^{i+1}\langle\partial_s\sigma_k, \sigma_i\rangle \end{pmatrix} \tag{6.1.16}$$

$$\Rightarrow j_k^T(\Sigma^*\partial_s\Sigma)j_k = \sum_{l=1}^{k}\sum_{i=1}^{k}(-1)^{2+l+i}\langle\partial_s\sigma_l, \sigma_i\rangle \tag{6.1.17}$$

□

The above formulation of the connection for the Hopf bundle $S^{\left(2\binom{n}{k}-1\right)}$ establishes a relationship with the canonical connection of the associated Stiefel manifold. The geometric implications of this relationship is currently unclear, but it is worth noting this fact for further research. Avitabile and Bridges make note that the trace of a skew symmetric matrix is pure imaginary, and thus the trace of the canonical connection of the Stiefel bundle generates a phase analogous to the geometric phase in the Hopf bundle [22]. The above formulation in Proposition 6.1.7 similarly yields an imaginary phase with the Stiefel bundle, and there may be important connections to explore geometrically. As of now, this establishes the analytic relationship between the universal connection for the Stiefel bundle and the natural connection of the Hopf bundle, allowing one to compute the relative phase. The next section will consider possible numerical implementations of this formulation.

6.2 Methods of continuous orthogonalization and future work

Given a frame for the unstable manifold $\left\{\sigma_j(\lambda,\tau)\right\}$ corresponding to a solution $\Gamma(\lambda,\tau) \in \Lambda^k(\mathbb{C}^n)$ spanning the unstable manifold of the associated $A^{(k)}(\lambda,\tau)$, one may compute the relative phase at τ strictly in terms of the frame for the unstable manifold. It may be useful, therefore, to evolve a frame for the unstable manifold as a point in the Stiefel bundle, with respect to the dynamics induced by $A(\lambda, \tau)$. Humphreys and Zumbrun [20] propose a formulation of this type for the traditional formulation of the Evans function, matching asymptotic conditions for the un/stable manifolds of the system, usually at $\tau = 0$; their method is based on the numerical methods of Drury [25] and Davey [26], which were proposed as an alternatives to the compound matrix method for numerical solutions boundary value problems.

Definition 6.2.1. *The **flow on the Stiefel bundle** with respect to a linear non-autonomous system,*

$$Y' = A(\lambda, \tau)Y \qquad A(\lambda, \tau) = \begin{cases} A_{\pm\infty}(\lambda) & \tau = \pm 1 \\ A\Big(\lambda, \tau(\xi)\Big) & \tau \in (-1, 1) \end{cases} \tag{6.2.1}$$

is generated by the matrix system

$$\begin{aligned} V' &= A(\lambda, \xi)V - Vg \\ g &= V^* AV - S(\xi, V) \end{aligned} \tag{6.2.2}$$

for $V \in V(n, k)$ and S a value in $\mathbb{C}^{k\times k}$ such that $S^* = -S$.

Lemma 6.2.2. *If $S(\xi, V)$ is **any** skew-Hermitian matrix valued function, depending smoothly on its arguments, then a solution to equation (6.2.2) with initial condition in $V(n, k)$ will remain in $V(n, k)$ for all $\xi \in (-\infty, \infty)$.*

Lemma 6.2.3. *If $S = 0$, so that g is set equal to V^*AV, then the flow on the Stiefel bundle is precisely the method of Drury and Davey, and moreover defines the evolution of a horizontal path in the Stiefel bundle with respect to the canonical connection.*

Proof. The above two lemmas are demonstrated by Avitabile and Bridges [22] in pages 1040-1041. □

Numerically integrating the flow on the Stiefel bundle, by continuously orthogonalizing the initial frame for the unstable manifold, may prove challenging. Dieci, Russell & Van

Vleck [27] demonstrated that implicit, Gauss-Legendre Runge-Kutta integration schemes preserve the orthonormality of the frame, but these implicit methods are computationally costly. Additionally they suggest that a solution may be integrated at each step in ξ and projected onto a unitary frame, but this may not preserve the differentiability in the path parameter for computing the connection. Avitabile and Bridges [22] have also suggested that the appropriate choice of g may be used as a stabilization term for explicit, two-step Runge-Kutta algorithms to balance the precision and computational costs, while maintaining the differentiability of the flow. The specific choice of numerical implementation will go beyond the scope of this thesis, and this final section is written to suggest future work on the method of geometric phase. Although the efficacy of numerically computing the connection on a frame for the unstable manifold is unclear, this chapter demonstrates that *analytically* one may compute the asymptotic relative phase and therefore the Chern class of the unstable bundle via a frame for the unstable bundle. A complete numerical study of the method of geometric phase applied to a frame in the Stiefel bundle is thus worthy of consideration.

CHAPTER 7

Conclusion

This final chapter will review the work of the thesis, highlighting the key results and the open questions arising from them.

7.1 Discussion of results and open questions

This thesis work formalizes the method of geometric phase for locating and counting eigenvalues of linear operators associated to reaction diffusion equations, as was originally proposed by Way [1]. The method demonstrated in this thesis differs significantly from Way's conjecture by realizing the necessity of computing the asymptotic relative phase, as was given in Definition 3.2.9, and passing to the exterior algebra system for the computation of the relative phase for systems of arbitrary dimension. The veracity of the method of geometric phase was demonstrated by equating this method to the winding number calculation in the Evans function for non-linear reaction diffusion equations on unbounded domains, as formulated by Alexander, Gardner & Jones [6]. Furthermore, this led to the natural extension to boundary problems for reaction diffusion equations of the form described by Austin & Bridges [19]. The main results describing this equivalence of these two winding number calculations are in Theorems 3.4.1, 4.1.9, 4.2.3 and 5.2.3.

While the above results are inspired from Way's work on the geometric phase in the Hopf bundle, this thesis makes an additional departure, demonstrating a novel formulation of the method of geometric phase in terms of a loop in the Stiefel bundle; the phase generated from the loop in the Stiefel bundle is shown to be related to the canonical connection on the Stiefel bundle and the result is described in Proposition 6.1.7. This calculation of the phase in the Stiefel bundle yields an analytic reformulation of the relative phase calculation in the Hopf bundle, and therefore the Chern number. Finally, building on Way's numerical study, this thesis includes demonstrations of the method of geometric phase with a new attention to the phase transition and dependence on the asymptotic conditions in the wave parameter.

More than a proof of concept, the numerical implementation of the analytic results leave important theoretical questions to be examined. The numerical examples exhibit a clear dependence on the length of the forward integration in the wave parameter direction and it suggests firstly that it is not in general necessary to integrate the λ dependent loop of eigenvectors to a value "close to $+\infty$", but rather, past some critical point at which there is a transition in the relative phase. The relative phase in the numerics initially equals zero, but there is a discernible transition of regimes terminating with the relative phase equal to the total multiplicity of the eigenvalues enclosed by the spectral path. Given that the proof of the method equates the relative phase to the Chern number of the unstable bundle over the parameter sphere, it seems intuitive that this should be the case. Indeed, the Chern number describes a gluing condition for the trivializations of the hemispheres, and the relative phase seems to "feel" the transition between these trivializations at some intermediate point, rather than at "$+\infty$". Understanding the nature of this transition is of critical importance to the computational method, and the relationship of the phase transition to the underlying wave is currently unclear. The numerical considerations also bring to light the apparently rich connections between the method of geometric phase in the Hopf bundle, and an equivalent formulation of a geometric phase in the Stiefel bundle. There appears to be room for improving the computational performance of the method by formulating the method of geometric phase as the evolution of the Stiefel bundle itself.

7.2 Concluding remarks

Evans function calculations are often useful as a stability index [6], describing the multiplicity of eigenvalues of positive real part by computing the winding of the Evans function along the imaginary axis, and bounding the integral of the winding along a semi-circle of radius r, as $r \to \infty$. In particular, in order to utilize the method of calculating the winding with the geometric phase in the Hopf bundle, it will be critical to understand the nature of the phase transition. As demonstrated in the numerical examples, the phase transition need not be uniform in the integration of the wave parameter, nor uniform across eigenvalues of the operator—indeed the calculation may fluctuate and the initiation and termination of the transition differs for each of the examples demonstrated in this work. For utilization as a stability index, one must understand the relationship between the transition and the underlying steady state to efficiently compute the eigenvalues. The geometric phase must eventually converge to the multiplicity of the eigenvalues enclosed by the contour, but a theoretical understanding of the transition of the phase will be an important development for both the numerical method and the understanding the eigenvalue problem itself—indeed the method of the geometric phase offers a unique insight into the continuous accumulation of the eigenvalue as driven by the system dynamics, a new insight not afforded by other Evans function methods.

Currently the general computational method of geometric phase is limited by the dependence on the exterior algebra formulation—for usual systems on $\mathbb{C}^n$, where the stable and unstable manifolds are of dimension approximately $\frac{n}{2}$, the dimension of phase space for the exterior algebra grows approximately exponentially in n, as discussed by Humpherys & Zumbrun [20]. However, the fact that the method of geometric phase relies only on *either* the unstable or stable manifold for the eigenvalue calculation highlights the potential for future reductions. The numerical calculation of the phase of a *frame of solutions* spanning the unstable manifold evolved via a continuous orthonogonalization scheme, as discussed in Section 6.2, is worthy of its own study. The method of geometric phase furthermore has the potential to be formulated entirely as a geometric phase of the Stiefel bundle itself, as suggested by Proposition 6.1.7.

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