Essays on the Theory of Conflict

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Abstract

JEREMY PETRANKA: Essays on the Theory of Conflict.
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This dissertation consists of two papers in the field of conflict theory. In the first paper, I offer a model of American Presidential politics in which voters utilize an information heuristic. Specifically, voters vote for the candidate who espouses their ideal policy platform. If both candidates advocate this platform, voters probabilistically choose a candidate based on a contest success function incorporating policy ambiguity and candidate personality. Using this framework, I find that the optimal level of policy "overlap" in any given election is based only on voters' sensitivity towards policy ambiguity. As such, the recent changes in American presidential election trends are also expected to rely on this sensitivity.

In the second paper, I propose a new microeconomic structure under which a ratio-form contest success function can be derived as a limit result using an underlying contest with multiple rounds and threshold success levels. This model generalizes the stochastic equivalence of ratio-form contest success functions and patent race games, allowing greater flexibility in the underlying structural interpretation. In addition, I formulate a spatial interpretation of the model in which the effectivity functions of a contest success function are related to a player's ability to increase his precision in hitting a target. Through the use of the threshold success level, I am able to relate a given effectivity function to a precision technology having desired productive characteristics.

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Chapter 1

Introduction

This document presents the two papers that form my dissertation in accordance with the Graduate School and Economics Department at UNC Chapel Hill.

The first paper is titled "A Conflict Theory of Voting", wherein I examine American presidential politics assuming voters follow an information heuristic. Variants of Downs's spatial model of voting have proven problematic in explaining trends in American presidential politics over the past five decades. Empirical studies have revealed increased voter independence, candidate ambiguity, and party partisanship, which are generally left unexplained using proximity models. In addition, voters have been found to be poorly informed, highly responsive to candidate personality, and to tend to follow a "fast and frugal" heuristic. I propose a new model based on the behavioral findings of Lau and Redlawsk (2006) which reflect these voter characteristics. In particular, voters vote for the candidate who espouses their ideal policy platform. If both candidates advocate this platform, voters probabilistically choose a candidate based on a contest success function incorporating policy ambiguity and candidate personality.

Using this framework, I find that the optimal level of policy "overlap" in any given election is based only on voters' sensitivity towards policy ambiguity. As such, the recent changes in American presidential election trends are also shown to rely on this sensitivity. In particular, I argue the primary system, campaign finance reform, and changing media climate have altered candidates' optimal strategies, leading to the current political environment.

The second paper is titled "A Threshold Interpretation of the Ratio-Form Contest Success Function". Here, I propose a new structure under which a ratio-form contest success function can be derived as a limit result using an underlying contest with multiple rounds and threshold success levels. This structure consists of a series of rounds in which players can exert effort to increase their likelihood of reaching a threshold success level. This threshold can be either absolute or relative to the other players. If any player reaches the threshold in a given round, he has the opportunity to win the contest. If not, a new round is played. As the probability of surpassing the threshold in any given round converges to zero, the probability of winning the full contest converges to the ratio-form contest success function.

This model generalizes the stochastic equivalence of ratio-form contest success functions and patent race games, allowing greater flexibility in the underlying structural interpretation. In addition, I formulate a spatial interpretation of this model in which the effectivity functions of a contest success function are related to a player's ability to increase his precision in hitting a target. Through the use of the threshold success level, I am able to relate a given effectivity function to a precision technology having desired productive characteristics.

Chapter 2

A Conflict Theory of Voting

2.1 Introduction

Since Anthony Downs (1957) employed the spatial framework of Hotelling (1929) and Smithies (1941) for use in analyzing voter behavior, proximity models have dominated the theoretical thinking of political theorists.[1] As with all spatial models, the proximity model views candidates and voters as points in an issue space. Voters are assumed to select the candidate which minimizes total "distance" between their ideal issue point and the candidate's issue point. Many of the spatial model's findings have been validated, especially in limited voting scenarios. However, its relevance as an explanatory tool for national elections has been called into question due to its inability to explain major trends in the American presidential political arena.

2.2 Trends in Modern American Presidential Politics

2.2.1 The Uninformed American Voter

> *The classic texts of democratic theory assume that for a democracy to function properly, citizens should be interested in, pay attention to, discuss, and actively participate in politics...Five decades of behavioral research in political science have left no doubt that only*

[1] While the research is too vast to relegate to a footnote, please refer to Enelow and Hinich (1984), Enelow and Hinich (1990), Coughlin (1992), Hinich and Munger (1997), and Samuel III and Grofman (1999) for recent summaries.

a tiny minority of the citizens in any democracy actually live up to these ideals. Interest in politics is generally weak, discussion is rare, political knowledge on the average is pitifully low, and few people actively participate in politics beyond voting. - Lau and Redlawsk (2006)

Considerable amounts of research by Converse (1964), Bennett (1996), Neuman (1986) and others have shown that the American voter's political knowledge is, on average, extremely low. Baum (2005), for instance, finds that according to the 2000 American National Election Study (ANES), 60% of respondents who indicated that they follow government and public affairs "hardly at all" or "only now and then" claimed to have voted. In an aggregation of 2000 survey questions asked over the last 50 years concerning questions one might expect an informed citizen to know, Carpini and Keeter (1996) find only 40% of the questions for which over half the population can answer correctly. Carpini (1999) notes that of the questions that cannot be answered by over half the population are "definitions of key terms such as liberal, conservative, primary elections, or the bill of rights; knowledge of many individual and collective rights guaranteed by the Constitution;...candidate and party stands on many important issues of the day; key social conditions such as the unemployment rate or the percentage of the public living in poverty or without health insurance; how much of the federal budget is spent on defense, foreign aid, or social welfare; and so on". As a stylized example, Carpini (1999) cites a 1992 report by the Center for the Study of Communication at the University of Massachusetts that found while 86% of a random sample of likely voters knew the Bush family dog was named Millie and 89% knew that Murphy Brown was the TV character criticized by Dan Quayle, only 15% knew that both candidates favored the death penalty and only 5% knew that both had proposed cuts in the capital gains tax.

These results are in stark contrast to the proximity model's theory of voter choice. Even if we assume voters have the capability of comparing seemingly incommensurate issues,[2] these findings imply the average voter does not have the necessary information to make the comparison. If voters are making issue-based choices, they are based on a much more limited amount

[2]Consider the voting difficulty faced by a staunchly pro-choice, pro-NRA individual.

of information than generally claimed using the proximity model.

2.2.2 The Independent American Voter and Ideological American Political Parties

Since the late 1960's, it has been well documented by Crotty and Jacobson (1984), Wattenberg (2000), Luttbeg and Gant (1995), and others that partisanship has had a declining effect on American voting decisions. Figure 2.1 demonstrates the rise of the self-proclaimed independent voter.[3] Spatial voting models offer two possible explanations for this phenomenon. First, underlying voter preferences have changed, making voters more moderate. While possible, it is hard to justify an underlying social shift with the speed and stability at which independent voters increased in the late 1960's and early 1970's. In addition, Norpoth and Rusk (2007) find that no political realignment occurred between 1932 (the New Deal Democratic realignment) and 1992.

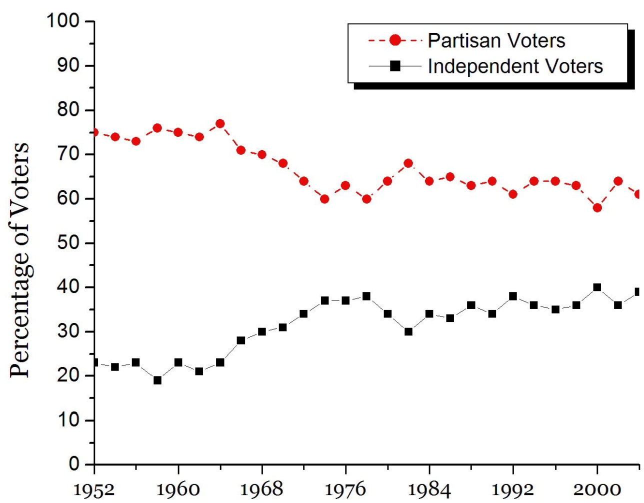

Figure 2.1: Rise of the Independent American Voter

Alternatively, spatial models can explain the rise in Independent voters by arguing the political parties have moved away from the moderate voter. This coincides with the findings

[3]Independents are classified as those labeling themselves "Independent Democrat", "Independent Republican", or "Independent Independent" on the ANES survey. On a 1-7 scale, it is those individuals labeling themselves a 3, 4, or 5. See Figure 2.2(a) for the full ANES scale.

of Groseclose, Levitt, and Snyder (1999), Jacobson (2000), Stonecash, Brewer, and Mariani (2003), and Brewer (2005) who have shown that regardless of the measure used, the ideological distance between the parties has been growing, with the Democrats becoming more liberal and the Republicans becoming more conservative. Unfortunately, this claim is in contrast to a hallmark of the spatial model, the Median Voter Theorem, which states that political parties' issue stances should converge to the stance of the median voter.[4]

2.2.3 The Ambiguous American Candidate

Coinciding with the increase in Independent voters is the common complaint that American presidential candidates are increasingly vague. Through the course of a presidential election, policy positions are rarely consistently unambiguous, causing Levine (1995) to claim "the major candidates rarely offer a clear choice of detailed, workable policy solutions on issues of importance to voters".

On the issue of NAFTA, John McCain, Barack Obama, and Hillary Clinton's websites offer the following proposals:[5]

- "The U.S. should engage in multilateral, regional and bilateral efforts to promote free trade, level the global playing field and build effective enforcement of global trading rules."
- "Obama believes that NAFTA and its potential were oversold to the American people. Obama will work with the leaders of Canada and Mexico to fix NAFTA so that it works for American workers. "
- "[Hillary] will also ensure that trade policies work for average Americans. Trade policy must raise our standard of living, and they must have strong protections for workers and the environment"

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[4]Models that assume candidates have preferences beyond simply winning the election do not display the Median Voter result. However, they also provide no justification as to why political parties have become more partisan.

[5]www.johnmccain.com, www.barackobama.com, and www.hillaryclinton.com as of 4/29/2008. Note Hillary Clinton and John McCain did not specifically mention NAFTA.

From these stances, it is extremely difficult to determine the exact measures each candidate will take on the issue. We seem far removed from the 1964 election when Barry Goldwater made his ideological position clear by proclaiming in his Republican National Convention acceptance speech, "Anyone who joins us in all sincerity, we welcome. Those, those who do not care for our cause, we don't expect to enter our ranks, in any case. And let our Republicanism so focused and so dedicated not be made fuzzy and futile by unthinking and stupid labels. I would remind you that extremism in the defense of liberty is no vice! And let me remind you also that moderation in the pursuit of justice is no virtue!".[6]

Spatial models have been able to offer extensive justification for candidate ambiguity. Early models such as Shepsle (1972) justified ambiguity by claiming voters were risk-loving.[7] Alesina and Cukierman (1990) and Aragones and Neeman (2000) have shown that candidates will choose to be ambiguous if they care about more than merely winning the election. Callander and Wilson (2008) have shown that candidates will respond to context-dependent voters by giving ambiguous policy stances. Aragones and Postlewaite (2002) give conditions under which voters with "intense" preferences will induce candidate ambiguity. However, these models have been silent as to why the level of candidate ambiguity has seemingly increased over the past five decades.[8]

In an effort to explain these recent trends in American politics, the rest of this chapter is organized as follows. In Section 2.3, I review recent findings in cognitive psychology, specifically related to voting behavior. In particular, I discuss the use of emotional cues and mental heuristics in the formation of voting decisions. In Section 2.4, I incorporate these findings into a modified spatial model of campaign strategy between two candidates. In Section 2.5, I solve for the optimal strategies of the candidates and in Section 2.6, I show how these

[6] On specific policy issues, Goldwater was equally uncompromising. In a campaign brochure found at http://www.4president.org/brochures/goldwater1964brochure.htm, Goldwater clearly states he is for an increase in State's rights, against the Civil Rights Act of 1964's public accommodations provision, for a decrease in Union power, and against expansion of government Welfare.

[7] Bartels (1986) and Berinsky and Lewis (2007) contradicted this claim by finding American voters tend to be risk averse.

[8] Consider the unambiguous issue stands of New Deal Democrats versus the current level of candidate ambiguity.

strategies can help explain the recent trends in American presidential elections. Section 2.7 offers a measurement of the efficacy of the proposed model and Section 2.8 discusses the multidimensional formulation of the game. Section 2.9 concludes.

2.3 Recent Findings in Voter Cognitive Psychology

While the trends in American politics have caused researchers to question the validity of Downs's proximity model, behavioral researchers starting with Herbert Simon (1955) have been simultaneously developing an alternate model of voter choice. Taking an agnostic view, these researchers study the process of making the voting decision, as opposed to postulating a specific voting strategy. Their findings generally contradict the choice mechanism proposed by Downs.

2.3.1 The Affective American Voter

A popular comic strip[9] describes the American Political Process as

1. Eligible candidates announce their desire to run for president.
2. Democratic/Republican parties hold lengthy, detailed meetings and voting sessions.
3. The top two nominees are entered in the general election in November.
4. The American public elects the cuter one.

While a mild exaggeration of the state of American politics, an extensive literature exists on the role of candidate personality on voter affect. Affect is defined as the emotional "feeling" an individual develops towards a specific choice. It is distinctly separate from rational intelligence, both conceptually and neurobiologically. In the political realm, familiarity, perceived truthworthiness, overall image, enthusiasm, and other "non-issue" traits have all been found to play an important role in voter behavior through their manipulation of voter emotions.

[9] http://www.toothpastefordinner.com/030408/american-political-process.gif

Kinder and Abelson (1981), Marcus, Newman, and Mackuen (2000), Rosenberg and McCafferty (1987), and Rosenberg, Hohan, McCafferty, and Harris (1986) all show that candidate image influences voter choice. In presidential elections, the effect is even more pronounced, with the ANES finding that the candidate who rates higher in the public's "Average Feeling Thermometer"[10] has not lost a presidential election since the question was created in 1968. Miller, Wattenburg, and Malanchuk (1986) even finds that voter perception of candidates relies more on personality than issue concerns. In addition, these perceptions are not superficial, but reflect performance-based criteria such as integrity and reliability.

Existing spatial models of voting have very little to say on candidate personality traits. Generally when personality is considered, it is included linearly in the voter's utility function, separating the role of issues and personality.[11] This has a two-fold effect. First, it removes any predictive power of the spatial model since any result inconsistent with the theory can be explained via the personality variable. Second, it allows for highly unrealistic scenarios in which a candidate with enough "personality" can win an election regardless of his issue stands. Alternatively, a grossly incompetent leader can win an election if his issue stands appeals to enough voters. Even without Miller et al.'s findings, it should be obvious that to a voter, candidate personality and issue stands play inseparable roles. In particular, *voters desire a candidate with the personality to enact their issue stands.*

2.3.2 Voter Heuristics

One of the generally accepted tenants of behavioral psychology is that individuals are systematically unable to perform exceedingly complex mental calculations. As such, in "hard" choice environments, such as the comparison of ideological issues present in national elections, humans use "fast and frugal" mental heuristics. As demonstrated by Gigerenzer, Todd, and

[10] Question VCF0201 in the Cumulative Data File. 1976 question text: "We'd also like to get your feelings about some groups in American society. When I read the name of a group, we'd like you to rate it with what we call a feeling thermometer. Ratings between 50 degrees-100 degrees mean that you feel favorably and warm toward the group; ratings between 0 and 50 degrees mean that you don't feel favorably towards the group and that you don't care too much for that group."

[11] Please refer to Azrieli (2009) for the necessary conditions this imposes on the voters' preferences for candidates.

Group (2000), these heuristics allow individuals to make a choice with minimal information requirements that can approach, or even exceed, the accuracy of modern computational techniques. Baron (1990), Hogarth (1987), and Payne, Bettman, and Johnson (1993) have all shown that when making a choice, individuals try to avoid trade-offs and instead focus on a single good reason to select an option. Despite being seemingly uninformed, research has shown that citizens generally make reasonable policy decisions when using heuristics. In a study of 45 policy issues, Althaus (1998) estimates that 80% of the sampled collective preferences were identical to those made by a highly-informed populace.

Despite their prevalence in behavioral research, heuristics are generally overlooked in economic analysis for two reasons. First, in many situations individuals act "as if" they are behaving without the use of a heuristic, making the heuristic itself irrelevant. Given the recent unexplained trends in American presidential elections, it should be clear this is not the case with voting behavior. Secondly, heuristics are highly environment-specific. A mental heuristic that works remarkably well in one scenario can be highly inaccurate in another. The recognition heuristic, in which humans select the choice whose name they recognize, can be highly detrimental if an individual is choosing a quiet city in which to retire.[12] Behavioral researchers define a heuristic's effectiveness in a specific environment as "ecological rationality".[13] Specifically, the study of ecological rationality involves analyzing the structure of environments, the structure of heuristics, and the match between them. The specificity inherent to ecological rationality is at odds with the economist's desire for generality.

It should be noted that heuristics do not necessarily imply individuals are behaving economically irrationally. While a behavioral psychologist would argue that individuals are unaware of their choice of heuristic, there is no inherent requirement that the choice of heuristic be subconscious. If choosing an ecologically rational heuristic reduces information search costs and mental conflict, then use of such a heuristic can be consistent with rational optimizing behavior.

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[12] With the recognition heuristic, they would end up in New York, Los Angeles, etc.

[13] Please refer to Chapter 1 of Gigerenzer, Todd, and Group (2000) for a more detailed description of ecological rationality.

2.3.3 Information Rationalization

In using noncompensatory decision strategies, it should be clear that voters will likely face contradictions through the course of the election. To imply voters are using a fast and frugal heuristic, it must be shown that voters do not treat all information equivalently. Information received after a choice is made must be ignored or rationalized such that mental conflict is avoided.

In a study published in the Journal of Cognitive Neuroscience, Westen, Blagov, Harenski, Kilts, and Hamann (2004) performed MRI brain-imaging on "committed" Republicans and Democrats to test this theory. During the study, each subject was exposed to a series of slides that demonstrated inconsistent statements made by their candidate of choice as well as the opposing candidate. As expected, when exposed to contradictory statements by their candidate, brain regions were activated which were involved in implicit emotion regulation and the elicitation of negative emotion. These areas were distinct from the areas of the brain involved in cold-reasoning and conscious emotion regulation. In other words, when exposed to contrary information, brain regions linked to "rational thought" were not used, preventing the information from entering the subject's decision calculus.

This effect can also be seen in the American National Election Studies data. Since 1976, all surveyed individuals who could correctly identify the Democratic candidate as weakly more liberal than the Republican candidate were asked to place each candidate on a Liberal/Conservative scale of 1-7. Individuals who voted for a candidate placed that candidate .05 closer to themselves than the average surveyed individual placed the candidate. More importantly, they placed the opposing candidate .91 points farther from themselves. This is in contract to nonvoters, who placed the Democratic candidate .32 points farther from themselves and the Republican candidate .28 points farther. As compared to nonvoters, voters appear to be processing information differently as predicted with the use of a fast and frugal heuristic.

2.3.4 Dynamic Information Processing

In an effort to determine what heuristics voters use, Richard R. Lau and David P. Redlawsk (Lau and Redlawsk, 2006) created an extensive experimental environment that mimicked the dynamic nature of presidential politics. In particular, individuals were faced with general information cues (such as "Candidate A's Stand on Taxes/Tax Reform") that they could select to learn the specific information contained within. As with a real election, spending the time to focus on a specific cue meant possibly ignoring another piece of information. Through this process and extensive pre/post-trial surveys, they were able to make conclusions as to how voters make their decisions. Among other findings, they determined:

Finding 1: In selecting the voter's preferred candidate, memory and affective perception both play a role. In other words, specific information cues as well as overall perception are used as a basis for choice. In selecting the voter's rejected candidate, however, affective memory does not play a role. The voter only uses specific information to reject the candidate, and does not retain an "overall feeling".

Finding 2: Voters recall more total information concerning their preferred candidate.

Finding 3: Of the information recalled, voters recall a higher percentage of positive information for their preferred candidate.

2.4 Conflict Model of Voting with Voter Heuristic

2.4.1 "Hear What You Want To Hear" Voter Heuristic

In an effort to explain the political trends in American presidential politics as well as incorporate the discoveries of behavioral researchers, I propose a voter heuristic inspired by Lau and Redlawsk's findings. Like Downs, I assume each voter has an ideal point at which they would like a candidate to locate.[14] Unlike Downs, I assume a voter uses the following heuristic in choosing his preferred candidate:

[14] Alternatively, I could assume each voter has an interval for which he would consider a candidate acceptable. The main results of the paper do not change.

1. A voter receives enough information to determine which candidate(s) put positive probability on his ideal issue point.

2. If neither candidate puts positive probability on the voter's ideal issue point, he does not vote.

3. If only one of the candidates puts positive probability on the voter's ideal issue point, he votes for that candidate.

4. If both candidates put positive probability on the voter's ideal issue point, he votes for each candidate as probabilistically determined by a function incorporating each candidate's ambiguity and personality characteristics.

Note that under this heuristic, the voter is displaying a form of lexicographic preferences over his ideal issue point. Particularly, regardless of any other criteria, a candidate will be selected only if his issue platform contains the voter's ideal point. Only if both candidates satisfy the first criteria will alternate criteria be considered. This coincides with Luce, Payne, and Bettman (2001), who find "the major form of emotion-focused coping relevant to decision processing is a desire to avoid particularly distressing explicit tradeoffs between attributes. That is, if tradeoff difficulty is elicited by the perception that valued goals must be given up, then the decision maker should try...to avoid these sacrifices altogether."

Note that under this heuristic, if a candidate is rejected in Step 2, a particular issue stance will cause the rejection. When asked to identify the reason for the rejection, the voter should be able to recall the exact reason. An emotional judgment will not be made, in that once the decision is reached, no additional information is required. If both candidates continue to Step 3, we expect a more affective decision to made, in that non-issue related cues are helping to form the vote. Note that memory will still play a role, as expected, since each candidate can only reach Step 3 if their memory-related issue cues are in line with the voter. These predictions are consistent with Lau and Redlawsk's Finding 1.

In addition, because this heuristic is noncompensatory, the voter will be expected to be exposed to contradictory information after his decision is made. As such, we would expect internal justifications to occur of the sort seen by Westen, Blagov, Harenski, Kilts, and Hamann (2004). In particular, "good" information will be retained, while "bad" information will be justified or neglected. This coincides with Lau and Redlawsk's Findings 2 and 3 which state that voters will recall more information concerning the preferred candidate, and the information will be of a more positive nature.

It is important to note that while Step 1 might seem informationally intensive, it is in fact a fast and frugal heuristic. Voters generally have a limited number of key issues on which they base their vote, restricting the number of issues on which they need information. In addition, voters do not need to analyze all policy positions by a candidate to determine if their ideal point is included. Instead, they need only look at the policy stances the candidate has emphatically rejected to determine if their ideal point is excluded. Given that the President's authority exists in his veto power, the only unambiguous statements a candidate can make concerning potential laws involves bills he will NOT enact. For instance, according to his website, John McCain's stance on abortion is:

> *John McCain believes Roe v. Wade is a flawed decision that must be overturned, and as president he will nominate judges who understand that courts should not be in the business of legislating from the bench. Constitutional balance would be restored by the reversal of Roe v. Wade, returning the abortion question to the individual states. The difficult issue of abortion should not be decided by judicial fiat.*

For pro-life voters, it is unclear whether he would support a federal ban on abortion, a federal ban on third-trimester abortions, etc. For pro-choice voters, however, is is unambiguous that he would support the overturn of Roe v. Wade, a critical issue.

It should be noted that this heuristic is essentially a formalization of the observation that "people hear what they want to hear". It is also important to realize that this heuristic is not proposed as a universal voting heuristic. It is specific to the environment of modern American presidential elections. I will later show that the proposed heuristic is ecologically rational in the American presidential election environment, but this might not be the case in a different election format, or even in a different period in American politics. Pre-1960, for instance, an even faster and more frugal heuristic of voting along party lines proved highly accurate. I will later discuss some of the changes that have occurred since 1960 that have led to the partial abandonment of the party heuristic.

2.4.2 Candidate Campaign Game

Having established the voter heuristic, I now turn to the actual game played by the candidates, D and R. These politicians, only concerned with winning, compete with each other in an election. As is common in spatial models, I assume a unidimensional issue space on the real line, $\Re$. As will be discussed later, unidimensionality is not required under the assumption that ambiguity in one issue does not relate to ambiguity in another. A uniformly distributed continuum of voters exists.

The sequence of the game is as follows.

1. Nature selects the most liberal possible position of the Democratic candidate, $\bar{D}_L$.[15] This can be viewed as an indication of past candidate voting bias, the current ideology of the Democratic party, or the natural checks-and-balances inherent in American politics. Simultaneously, nature selects the most conservative possible position of the Republican candidate, $\bar{R}_C > \bar{D}_L$. Both values are known by all players.

2. The Democratic candidate selects her most conservative campaign position, $D_C \in [\bar{D}_L, \bar{R}_C]$. This will establish her level of campaign ambiguity, $A_D \equiv D_C - \bar{D}_L$. Simultaneously, the Republican candidate selects his most liberal campaign position, R_L. This will establish his level of campaign ambiguity, $A_R \equiv \bar{R}_C - R_L$. Note I explicitly assume a candidate's strategy is a convex interval. This avoids dubious campaign strategies such as simultaneously claiming to be extremely pro-life, extremely pro-choice, but against abortions in the case of rape and incest. I explicitly focus on pure strategies.

3. Non-strategic voters select their candidate of choice using the proposed voter heuristic.

[15] I use the labels "Democratic" and "Republican" for ease of exposition. No connotations should be inferred.

In the scenario where both candidates offer voters their ideal points, support will be divided according to a voter contest success function.

4. The winner will be probabilistically selected according to an increasing, twice-continuously differentiable function mapping the percentage of voter support to the probability of winning, $\nu : [0,1] \mapsto [0,1]$, $\nu(\cdot)' > 0$. This function incorporates the uncertainty inherent to voter turnout, the ideosyncracies of the electoral college, etc.[16]

Since the function mapping voter support to probability of winning is increasing, I will use the terms "support" and "votes" interchangeably.

Voter Contest Success Function

To determine each candidate's percentage of voter support when policies overlap, I utilize the concept of a contest success function as detailed in Hirshleifer (2005) and is common in the rent-seeking literature. In particular, I assume that each candidate, i, with personal affect, α_i, and ambiguity, A_i, wins the following share of contested voters:

$$s_i(A_i) = \frac{\frac{\alpha_i}{(A_i)^m}}{\sum_i \frac{\alpha_i}{(A_i)^m}} \tag{2.1}$$

Making the bipartisan share of contested voter support in the model:

$$s_D \equiv \frac{\alpha_D}{\alpha_D + \alpha_R\left(\frac{D_C - \bar{D}_L}{\bar{R}_C - R_L}\right)^m} \tag{2.2}$$

$$s_R \equiv \frac{\alpha_R}{\alpha_R + \alpha_D\left(\frac{\bar{R}_C - R_L}{D_C - \bar{D}_L}\right)^m} \tag{2.3}$$

[16] ν is included in the model for two reasons. First, it removes a discontinuity in the candidates' best response functions. In particular, if candidates care only about winning and voter support has a one-to-one mapping with total votes, then a discontinuity will exist at the ambiguity level that ensures 50% of the vote. This will result in strategies in which the winning candidate is not concerned with maximizing her support, and the losing candidate will behave entirely arbitrarily. Neither seems to accurately represent American presidential politics. Second, the 2000 presidential elections in which the more popular candidate lost implies a probabilistic function is a reasonable assumption.

with

$\alpha_i > 0$: The affect potential of candidate i. This variable indicates the ability of the candidate to express confidence, competency, positive emotional appeal, and to garner a sense of trust in the voters.

$m > 0$: The mass effect parameter. As m decreases, the less detrimental ambiguity becomes.

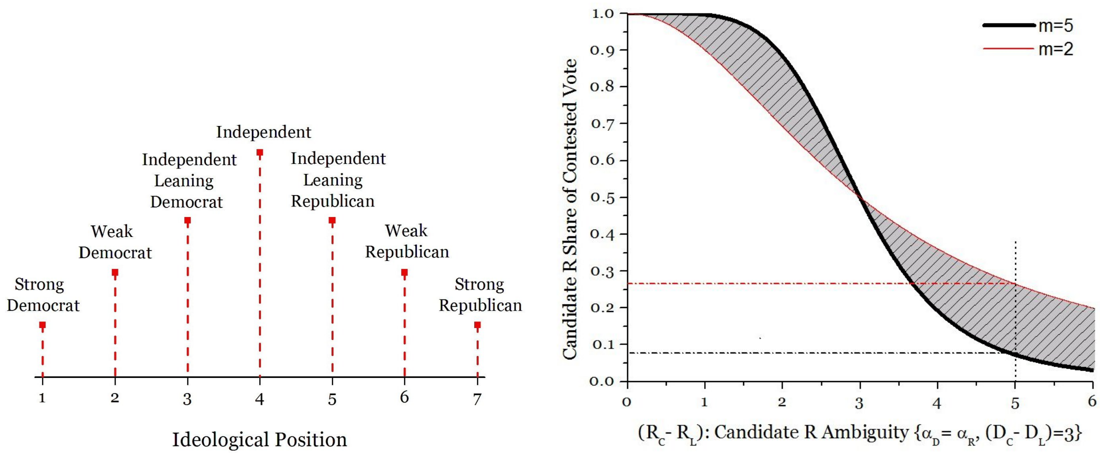
Intuitively, α_D and α_R can be described as follows. Assume each candidate expresses an identical level of ambiguity. $\frac{\alpha_D}{\alpha_D+\alpha_R}$ will represent the share of the contested votes won by Candidate D. Note that if $\alpha_D > \alpha_R$, Candidate D will win more than 50% of the contested votes on strength of personality. If $\alpha_D = 2$ and $\alpha_R = 1$, for instance, Candidate D will win 2/3 of the contested votes.

The mass effect parameter, m, can be interpreted as the cost of being ambiguous relative to the other candidate. To illustrate this intuition, assume the two candidates are competing for voters whose ideologies range from 1 to 7, as is standard in the American National Election Survey and is demonstrated in Figure 2.2(a). In addition, assume these candidates have identical affect potentials ($\alpha_D = \alpha_R$). Lastly, assume Candidate D has selected an ambiguity of 3. This would be the case, for instance, if she claimed to be a 2-4 on the ANES scale.[17] For values of m ranging from 2 to 5, Figure 2.2(b) shows the share of the contested votes candidate R will receive as his level of ambiguity changes. Note that lower values of m allow Candidate R to be more successful when he is more ambiguous than Candidate D. Likewise, lower values of m allow candidate D to be more successful when she is more ambiguous than Candidate R.

Note that the general contest success function expressed in Equation 2.1 displays the following desirable properties regarding voter behavior.

- As a candidate's ambiguity increases, his share of the contested votes will decrease.
- As a candidate's affect potential increases, his share of the contested votes will increase.

[17] This would be the case if she claimed to not be a Strong Democrat, but also was not a Republican. e.g. she campaigned on the platform of being a moderate Democrat.



(a) ANES Ideological Range

(b) Effect of m on Voter Share

Figure 2.2: Mass Effect Parameter

- If all candidates scale their level of ambiguity by the same amount,[18] each candidate's share of the contested vote will not change.
- If all candidates display an equal level of ambiguity, the more emotionally appealing candidate will win a higher share of the contested vote.
- The share of the contested voters sums to 1, and each candidate receives a share of the vote between 0 and 1.
- The choice between two alternatives is independent of a third candidate who receives no share of the vote. i.e. Independence of Irrelevant Alternatives.

Clark and Riis (1998) show that if $(\bar{D}_L, \bar{R}_C) = (-\infty, \infty)$, the contest success function *must* be in the form of Equation 2.1 for these properties to hold.

[18] If each candidate doubles his level of ambiguity, for instance.

Candidate Objective Functions

I assume each candidate achieves a utility of 1 from winning the election and 0 from losing. As such, each maximizes his expected utility by maximizing his probability of winning.[19] For the contest success function defined above and a given $\{\bar{D}_L, \bar{R}_C\}$, the expected utility for each candidate in the game represented in Figure 2.3 is therefore:

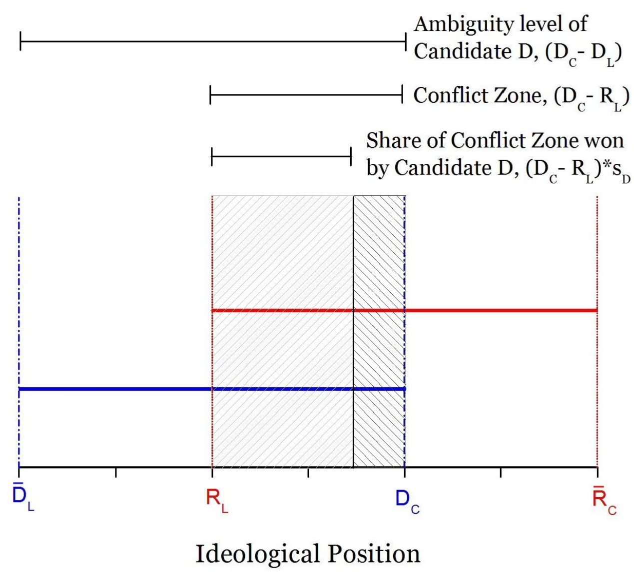


Figure 2.3: Election Game

$$V_D(D_C) = \nu\Bigg(\frac{1}{\bar{R}_C - \bar{D}_L}\Big(R_L - \bar{D}_L + s_D(D_C - R_L)\Big)\Bigg) \quad (2.4)$$

$$V_R(R_L) = 1 - V_D = 1 - \nu\Bigg(\frac{1}{\bar{R}_C - \bar{D}_L}\Big(R_L - \bar{D}_L + s_D(D_C - R_L)\Big)\Bigg) \quad (2.5)$$

[19] Instead of maximizing the probability of winning by maximizing vote share, we could also assume there exist n voters with iid uniform distributions over $[\bar{D}_L, \bar{R}_C]$ whose exact locations are unknown to the candidates. Per Ledyard (1984), the difference in probability a voter will vote for Candidate D and the probability a voter will vote for Candidate R is a good approximation for the probability Candidate D wins the election. If we interpret the share of contested voter support as the probability a voter in the contested range will vote for a particular candidate, this approximation can be used. Assuming candidates attempt to maximize their probability of winning under this scenario and using this approximation results in no fundamental changes.

2.5 Results

Given the complete information environment and the simultaneous moves of the candidates, I use the Nash equilibrium solution concept to determine the optimal strategies. Due to the symmetry of the problem, I assume w.l.o.g. that $\alpha_D > \alpha_R$.[20]

Proposition 1. *For* $m > \frac{\ln[\frac{\alpha_D}{\alpha_R}]}{W\left(\ln[\frac{\alpha_D}{\alpha_R}]\right)} + 1 \equiv m^*$,[21] *the optimal strategies for the candidates are:*

$$D_C^* = \frac{\exp\left[\frac{\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\right](m\bar{R}_C - \bar{D}_L) + \bar{D}_L(m-1)}{(m-1)\left[1+\exp\left[\frac{\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\right]\right]}$$

$$R_L^* = \bar{R}_C - \frac{m(\bar{R}_C - \bar{D}_L)}{(m-1)\left[1+\exp\left[\frac{\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\right]\right]}$$

where $W(y)$ is the real-valued branch of the Lambert W-function (also called the omega function), which is the function satisfying $W(y)e^{W(y)} = y$.[22]

Proof. I will solve for the Nash equilibrium using the necessary first order conditions for an interior solution. I will then verify the solution is, in fact, a maximum using the second order conditions. Lastly, for $m > m^*$, I will confirm the constraints $D_C \in [\bar{D}_L, \bar{R}_C]$ and $R_L \in [\bar{D}_L, \bar{R}_C]$ are not violated.

First Order Conditions

Rewriting the objective functions 2.4 and 2.5,

[20]For ease of exposition, the case where $\alpha_D = \alpha_R$ is ignored. The closed-form solutions are identical, with $m^* = 2$. It should be clear that the optimal solutions are symmetric, and no region will exist in which only one candidate will be fully ambiguous.

[21]Numerically, m^* is bounded below by 2.

[22]$W(y)$ is a multi-valued function over the complex reals. However, the real-valued branch of $W(y)$, when restricted to $y \in \Re_+$, is a positive, strictly increasing, concave function. Please refer to Corless, Gonnet, Hare, Jeffrey, and Knuth (1996) for further details.

$$
\begin{aligned}
V_D(D_C) &= \nu\Big(\frac{1}{\bar{R}_C - \bar{D}_L}\Big(R_L - \bar{D}_L + s_D(D_C - R_L)\Big)\Big) \\
V_R(R_L) &= 1 - V_D = 1 - \nu\Big(\frac{1}{\bar{R}_C - \bar{D}_L}\Big(R_L - \bar{D}_L + s_D(D_C - R_L)\Big)\Big).
\end{aligned}
$$

With Candidate R selecting his most liberal campaign position, R_L^*, the necessary condition for an interior optimum for candidate D is

$$
\begin{aligned}
\frac{\partial V_D}{\partial D_C} \quad &= \nu(\cdot)'\Bigg(\frac{1}{\bar{R}_C - \bar{D}_L}\Bigg[\frac{\alpha_D}{\alpha_D + \alpha_R\big(\frac{D_C^* - \bar{D}_L}{\bar{R}_C - R_L^*}\big)^m} \\
&\qquad - \frac{(D_C^* - R_L^*)\alpha_D\alpha_R m(D_C^* - \bar{D}_L)^{m-1}(\bar{R}_C - R_L^*)^{-m}}{\big(\alpha_D + \alpha_R\big(\frac{D_C^* - \bar{D}_L}{\bar{R}_C - R_L^*}\big)^m\big)^2}\Bigg]\Bigg) = 0, \qquad (2.6)
\end{aligned}
$$

or, simplified,

$$
\begin{aligned}
\alpha_D^2(\bar{R}_C - R_L^*)^m + \alpha_D\alpha_R(D_C^* - \bar{D}_L)^m - D_C^*\alpha_D\alpha_R m(D_C^* - \bar{D}_L)^{m-1} \\
+ R_L^*\alpha_D\alpha_R m(D_C^* - \bar{D}_L)^{m-1} = 0.
\end{aligned}
$$

Setting $\mu \equiv \frac{\alpha_R}{\alpha_D}$ and rearranging,

$$
\Big(\frac{(\bar{R}_C - R_L^*)}{(D_C^* - \bar{D}_L)}\Big)^m + \mu - \frac{\mu D_C^* m}{(D_C^* - \bar{D}_L)} + \frac{\mu R_L^* m}{(D_C^* - \bar{D}_L)} \quad = \quad 0. \qquad (2.7)
$$

With Candidate D selecting her most conservative campaign position, D_C^*, the necessary condition for an interior optimum for candidate R is

$$\frac{\partial V_R}{\partial R_L} = \nu(\cdot)'\Big(-1 + \frac{\alpha_D}{\alpha_D + \alpha_R\big(\frac{D_C^* - \bar{D}_L}{\bar{R}_C - R_L^*}\big)^m} + \frac{(D_C^* - R_L^*)\alpha_D\alpha_R(D_C^* - \bar{D}_L)^m(\bar{R}_C - R_L^*)^{(-m-1)}m}{\big(\alpha_D + \alpha_R\big(\frac{D_C^* - \bar{D}_L}{\bar{R}_C - R_L^*}\big)^m\big)^2}\Big) = 0, \quad (2.8)$$

or, simplified,

$$-\alpha_D\alpha_R\frac{(D_C^* - \bar{D}_L)^m}{(\bar{R}_C - R_L^*)^m} - \alpha_R^2\big(\frac{D_C^* - \bar{D}_L}{\bar{R}_C - R_L^*}\big)^{2m} + D_C^*\alpha_D\alpha_R m\frac{(D_C^* - \bar{D}_L)^m}{(\bar{R}_C - R_L^*)^{m+1}} - R_L^*\alpha_D\alpha_R m\frac{(D_C^* - \bar{D}_L)^m}{(\bar{R}_C - R_L^*)^{m+1}} = 0.$$

Setting $\phi \equiv \frac{\alpha_D}{\alpha_R}$ and simplifying,

$$-\phi(D_C^* - \bar{D}_L)^m(\bar{R}_C - R_L^*)^m - (D_C^* - \bar{D}_L)^{2m} + D_C^*\phi m(D_C^* - \bar{D}_L)^m(\bar{R}_C - R_L^*)^{m-1} - R_L^*\phi m(D_C^* - \bar{D}_L)^m(\bar{R}_C - R_L^*)^{(m-1)} = 0. \quad (2.9)$$

Setting $\eta \equiv (\bar{R}_C - R_L^*)$ and $\psi \equiv (D_C^* - \bar{D}_L)$, Eqn. (2.9) can be written

$$-\phi\psi^m\eta^m - \psi^{2m} + (\psi + \bar{D}_L)\phi m\psi^m\eta^{(m-1)} - (\bar{R}_C - \eta)\phi m\psi^m\eta^{(m-1)} = 0,$$

or, further simplified,

$$\big(\frac{\eta}{\psi}\big)^{(m-1)} = \frac{\psi}{\phi(-\eta + \psi m + \bar{D}_L m - \bar{R}_C m + \eta m)}. \quad (2.10)$$

Also note Eqn. (2.7) implies

$$(\frac{\eta}{\psi})^m \quad = \quad \mu\frac{-\psi+\psi m+\bar{D}_L m-\bar{R}_C m+\eta m}{\psi}, \tag{2.11}$$

which further implies

$$(\frac{\eta}{\psi})^m\frac{\psi}{\eta} \quad = \quad \mu\frac{-\psi+\psi m+\bar{D}_L m-\bar{R}_C m+\eta m}{\eta}. \tag{2.12}$$

Combined, Eqns. (2.10) and (2.12) tell us

$$\mu\frac{-\psi+\psi m+\bar{D}_L m-\bar{R}_C m+\eta m}{\eta} \quad = \quad \frac{\psi}{\phi(-\eta+\psi m+\bar{D}_L m-\bar{R}_C m+\eta m)},$$

which, after recognizing $\mu\phi = 1$, implies

$$\eta\psi = (-\psi+\psi m+\bar{D}_L m-\bar{R}_C m+\eta m)(-\eta+\psi m+\bar{D}_L m-\bar{R}_C m+\eta m),$$

or, after further simplification,

$$\psi \quad = \quad -\eta+\frac{m(\bar{R}_C-\bar{D}_L)}{m-1}. \tag{2.13}$$

Also note from Eqn. (2.10),

$$(\frac{\psi}{\eta})^m \quad = \quad \frac{\phi(-\eta+\psi m+\bar{D}_L m-\bar{R}_C m+\eta m)}{\eta}.$$

Substituting for ψ from Eqn. (2.13) and simplifying,

$$\left(\frac{\eta + \frac{m(\bar{R}_C - \bar{D}_L)}{m-1}}{\eta}\right)^m = \frac{\phi\big(-\eta - \eta m + \frac{m^2(\bar{R}_C - \bar{D}_L)}{m-1} + \bar{D}_L m - \bar{R}_C m + \eta m\big)}{\eta}$$
$$= \phi\Big(\frac{-\eta(m-1) + \bar{R}_C m - \bar{D}_L m}{(m-1)\eta}\Big).$$

Multiplying both sides by η,

$$\eta\Big(\frac{\eta + \frac{m(\bar{R}_C - \bar{D}_L)}{m-1}}{\eta}\Big)^m = \phi\Big(\eta + \frac{m(\bar{R}_C - \bar{D}_L)}{m-1}\Big),$$

which, after taking the natural log of both sides, implies,

$$\ln\Big(\frac{\eta + \frac{m(\bar{R}_C - \bar{D}_L)}{m-1}}{\eta}\Big) = \frac{\ln(\phi)}{m-1},$$

or, simplified,

$$\eta = \frac{m(\bar{R}_C - \bar{D}_L)}{(m-1)\big[1 + \exp[\frac{\ln[\phi]}{m-1}]\big]}.$$

Changing η back to $(\bar{R}_C - R_L^*)$ and ψ back to $(D_C^* - \bar{D}_L)$,

$$R_L^* = \bar{R}_C - \frac{m(\bar{R}_C - \bar{D}_L)}{(m-1)\big(1 + \exp\big[\frac{\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\big]\big)}. \tag{2.14}$$

Eqns. (2.13) and (2.14) imply

$$D_C^* - \bar{D}_L = \frac{m(\bar{D}_L - \bar{R}_C)}{(m-1)\big(1 + \exp\big[\frac{\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\big]\big)} + \frac{m(\bar{R}_C - \bar{D}_L)}{m-1},$$

which, when simplified, implies

$$D_C^* = \frac{\exp\big[\frac{\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\big](m\bar{R}_C - \bar{D}_L) + \bar{D}_L(m-1)}{(m-1)\big(1+\exp\big[\frac{\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\big]\big)}. \tag{2.15}$$

Second Order Conditions

With Candidate R again selecting his most liberal campaign position, R_L^*, a sufficient condition for an interior optimum for Candidate D is $\frac{\partial^2 V_D}{\partial D_C^2}(D_C^*) < 0$. To verify this condition holds,

$$\begin{aligned}\frac{\partial^2 V_D}{\partial D_C^2}(D_C^*) = \nu(\cdot)'\Bigg(& - \frac{2\alpha_D\alpha_R m(\bar{R}_C - R_L^*)^{-m}(D_C^* - \bar{D}_L)^{(m-1)}}{[\alpha_D + \alpha_R\big(\frac{D_C^*-\bar{D}_L}{\bar{R}_C - R_L^*}\big)^m]^2} \\ & - \frac{2\alpha_R\alpha_D m(\bar{R}_C - R_L^*)^{-m}(D_C^* - \bar{D}_L)^{(m-1)}}{[\alpha_D + \alpha_R\big(\frac{D_C^*-\bar{D}_L}{\bar{R}_C - R_L^*}\big)^m]^2} \\ & + \frac{4\alpha_D\alpha_R^2 m^2(D_C^* - R_L^*)(\bar{R}_C - R_L^*)^{-2m}(D_C^* - \bar{D}_L)^{2(m-1)}}{[\alpha_D + \alpha_R\big(\frac{D_C^*-\bar{D}_L}{\bar{R}_C - R_L^*}\big)^m]^3} \\ & - \frac{2(m-1)\alpha_D\alpha_R m(D_C^* - \bar{D}_L)^{(m-2)}(D_C^* - R_L^*)(\bar{R}_C - R_L^*)^{-m}}{[\alpha_D + \alpha_R\big(\frac{D_C^*-\bar{D}_L}{\bar{R}_C - R_L^*}\big)^m]^2}\Bigg) \\ & + \nu(\cdot)''\Bigg(\frac{1}{\bar{R}_C - \bar{D}_L}\Bigg[\frac{\alpha_D}{\alpha_D + \alpha_R\big(\frac{D_C^*-\bar{D}_L}{\bar{R}_C - R_L^*}\big)^m} \\ & \quad - \frac{(D_C^* - R_L^*)\alpha_D\alpha_R m(D_C^* - \bar{D}_L)^{m-1}(\bar{R}_C - R_L^*)^{-m}}{\big(\alpha_D + \alpha_R\big(\frac{D_C^*-\bar{D}_L}{\bar{R}_C - R_L^*}\big)^m\big)^2}\Bigg]\Bigg). \end{aligned} \tag{2.16}$$

Using the first order condition of Eqn. (2.6) and simplifying Eqn. (2.16) ,

$$\begin{aligned}\frac{\partial^2 V_D}{\partial D_C^2}(D_C^*) = \nu(\cdot)'\Big(& 2\alpha_D\alpha_R m(\bar{R}_C - R_L^*)^{-m}(D_C^* - \bar{D}_L)^{(m-1)}[\alpha_D + \alpha_R\big(\frac{D_C^* - \bar{D}_L}{\bar{R}_C - R_L^*}\big)^m]^{-2}\Big(-2 \\ & + 2\alpha_R m(D_C^* - R_L^*)(D_C^* - \bar{D}_L)^{(m-1)}(\bar{R}_C - R_L^*)^{-m}[\alpha_D + \alpha_R\big(\frac{D_C^* - \bar{D}_L}{\bar{R}_C - R_L^*}\big)^m]^{-1} \\ & - (m-1)(D_C^* - \bar{D}_L)^{-1}(D_C^* - R_L^*)\Big)\Big). \end{aligned} \tag{2.17}$$

To further evaluate $\frac{\partial^2 V_D}{\partial D_C^2}$ at the optimum, D_C^*, note Eqn. (2.6) implies

$$\frac{(D_C^* - R_L^*)\alpha_R m(D_C^* - \bar{D}_L)^{(m-1)}(\bar{R}_C - R_L^*)^{-m}}{\alpha_D + \alpha_R\big(\frac{D_C^* - \bar{D}_L}{\bar{R}_C - R_L^*}\big)^m} \quad = \quad 1.$$

In addition, Proposition 2 implies

$$-(m-1)(D_C^* - \bar{D}_L)^{-1}(D_C^* - R_L^*) \quad = \quad -\frac{\bar{R}_C - \bar{D}_L}{D_C^* - \bar{D}_L}.$$

Substituting these two equalities into Eqn. (2.17),

$$\frac{\partial^2 V_D}{\partial D_C^2} \quad = \quad \frac{\nu(\cdot)'\alpha_D\alpha_R m(\bar{R}_C - R_L^*)^{-m}(D_C^* - \bar{D}_L)^{(m-1)}}{[\alpha_D + \alpha_R\big(\frac{D_C^* - \bar{D}_L}{\bar{R}_C - R_L^*}\big)^m]^2}\Big(-\frac{\nu(\cdot)'(\bar{R}_C - \bar{D}_L)}{D_C^* - \bar{D}_L}\Big).$$

The first term in the above equation is greater than zero, while the second term is less than zero, implying

$$\frac{\partial^2 V_D}{\partial D_C^2}(D_C^*) < 0.$$

Symmetrically,

$$\frac{\partial^2 V_R}{\partial R_L^2}(R_L^*) < 0,$$

verifying we have solved for the Nash Equilibrium if the constraints are not violated.

Requirements to Avoid Constraint Violation

W.L.O.G. assume $\alpha_R > \alpha_D$, making $R_L^* > \bar{D}_L$ the relevant constraint. We will assume $R_L^* > \bar{D}_L$, and determine the required conditions on m, α_D, and α_R. All variables should be assumed to be at the optimum.

$R_L^* > \bar{D}_L$ and Eqn. (2.14) imply

$$\bar{D}_L \quad < \quad \bar{R}_C - \frac{m(\bar{R}_C - \bar{D}_L)}{(m-1)\big(1 + \exp\big[\frac{\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\big]\big)},$$

which, further simplified, requires

$$\exp\Big[\frac{\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\Big] \quad > \quad \frac{1}{m-1},$$

which, after taking the natural log of both sides and simplifying, tells us

$$\ln\Big[\frac{\alpha_R}{\alpha_D}\Big] \quad < \quad (m-1)\ln[m-1]. \tag{2.18}$$

Setting $y \equiv \ln[\frac{\alpha_R}{\alpha_D}]$ and $x \equiv (m-1)$, Eqn. (2.18) can be rewritten

$$y \quad < \quad x\ln[x],$$

which implies

$$\frac{\exp[\frac{y}{x}]}{x} \quad < \quad 1.$$

Multiplying both sides by y,

$$\frac{y}{x}\exp[\frac{y}{x}] < y,$$

which implies

$$\frac{y}{x} < W(y) \tag{2.19}$$

where $W(y)$ is the Lambert W-function (also called the omega function), which is defined as the inverse function of $f(W) = We^W$.

Changing y back to $\ln[\frac{\alpha_R}{\alpha_D}]$ and x back to $(m-1)$ in Eqn. (2.19),

$$R_L^* > \bar{D}_L \quad \Longleftrightarrow \quad m > \frac{\ln[\frac{\alpha_R}{\alpha_D}]}{W\big(\ln[\frac{\alpha_R}{\alpha_D}]\big)} + 1 \text{ when } \alpha_R > \alpha_D.$$

Symmetrically,

$$D_C^* < \bar{R}_C \quad \Longleftrightarrow \quad m > \frac{\ln[\frac{\alpha_D}{\alpha_R}]}{W\big(\ln[\frac{\alpha_D}{\alpha_R}]\big)} + 1 \text{ when } \alpha_D > \alpha_R.$$

□

Proposition 2. *For* $1 + \frac{\alpha_R}{\alpha_D} < m \leq m^*$*, the optimal strategies for the candidates are:*

$$\begin{aligned} D_C^* &= \bar{R}_C \\ R_L^* &= \bar{R}_C - \exp\left[\frac{\ln\left[\frac{(\bar{R}_C - \bar{D}_L)^m \alpha_R}{(m-1)\alpha_D}\right]}{m}\right] \end{aligned}$$

Proof. W.L.O.G., assume $\alpha_D > \alpha_R$. Defining $\phi \equiv \frac{\alpha_D}{\alpha_R}$, for $1 + \frac{1}{\phi} < m \leq m^*$, the above analysis shows a fully interior solution does not exist. To solve for the Nash equilibrium, I will instead assume Candidate D is fully ambiguous, show Candidate R will select an interior solution, then verify Candidate D's best response to Candidate R is, in fact, full ambiguity.

Step 1: Assume $D_C^* = \bar{R}_C$ and $\alpha_D > \alpha_R$. Find candidate R's optimal strategy, R_L^*, assuming an interior solution.

Eqn. (2.9), with the assumption that $D_C^* = \bar{R}_C$, implies at an interior optimum for candidate R,

$$\begin{aligned}-\phi(\bar{R}_C - \bar{D}_L)^m(\bar{R}_C - R_L^*)^m - (\bar{R}_C - \bar{D}_L)^{2m} + \bar{R}_C\phi m(\bar{R}_C - \bar{D}_L)^m(\bar{R}_C - R_L^*)^{m-1} \\ - R_L^*\phi m(\bar{R}_C - \bar{D}_L)^m(\bar{R}_C - R_L^*)^{m-1} &= 0,\end{aligned}$$

or, simplified,

$$(\bar{R}_C - \bar{D}_L)\Big(-1 - \frac{(\bar{R}_C - \bar{D}_L)^m}{(\bar{R}_C - R_L^*)^m\phi} + m\Big) = 0.$$

Further simplification and taking the natural log of both sides tells us

$$m\ln\left[\bar{R}_C - R_L^*\right] = \ln\left[\frac{(\bar{R}_C - \bar{D}_L)^m}{(m-1)\phi}\right],$$

or, simplified,

$$R_L^* = \bar{R}_C - \exp\left[\frac{\ln\left[\frac{(\bar{R}_C - \bar{D}_L)^m}{(m-1)\phi}\right]}{m}\right]. \tag{2.20}$$

Step 2: When $R_L = R_L^*$, show $\frac{\partial V_D}{\partial D_C} > 0$, implying $D_C = \bar{R}_C$ is an optimum for candidate D. Equations (2.6) and (2.20) imply $\frac{\partial V_D}{\partial D_C}$ evaluated at $D_C = \bar{R}_C$ equals

$$\frac{\partial V_D}{\partial D_C}(\bar{R}_C) = \nu(\cdot)'\left(\frac{\alpha_D\left(\alpha_D+\alpha_R\left(\frac{\bar{R}_C-\bar{D}_L}{\Omega}\right)^m\right)^{-1}}{\bar{R}_C-\bar{D}_L} - \frac{m\alpha_D\alpha_R(\bar{R}_C-R_L^*)(\Omega)^{-m}(\bar{R}_C-\bar{D}_L)^{m-1}\left(\alpha_D+\alpha_R\left(\frac{\bar{R}_C-\bar{D}_L}{\Omega}\right)^m\right)^{-2}}{\bar{R}_C-\bar{D}_L}\right)$$

with $\Omega \equiv \exp\left[\frac{\ln\left[\frac{(\bar{R}_C-\bar{D}_L)^m}{(m-1)\phi}\right]}{m}\right]$. Simplifying, $\frac{\partial V_D}{\partial D_C}(\bar{R}_C)$ equals

$$\nu(\cdot)'\left(\frac{\alpha_D}{\bar{R}_C-\bar{D}_L}\big(\alpha_D+\alpha_R(m-1)\phi\big)^{-1} - \frac{\alpha_D}{\bar{R}_C-\bar{D}_L}\exp\left[\frac{\ln\left[\frac{(\bar{R}_C-\bar{D}_L)^m}{(m-1)\phi}\right]}{m}\right]m\alpha_R\left(\frac{(m-1)\phi}{(\bar{R}_C-\bar{D}_L)^m}\right)(\bar{R}_C-\bar{D}_L)^{m-1}\big(\alpha_D+\alpha_R(m-1)\phi\big)^{-2}\right).$$

Recognizing $\phi = \frac{\alpha_D}{\alpha_R}$, this further reduces to

$$\frac{\partial V_D}{\partial D_C}(\bar{R}_C) = \nu(\cdot)'\left(\frac{\alpha_D}{\bar{R}_C-\bar{D}_L}\left(\frac{1}{m\alpha_D} - \exp\left[\frac{\ln\left[\frac{(\bar{R}_C-\bar{D}_L)^m}{(m-1)\phi}\right]}{m}\right]m\alpha_D\frac{(m-1)}{\bar{R}_C-\bar{D}_L}\frac{1}{(m\alpha_D)^2}\right)\right)$$

$$= \nu(\cdot)'\left(\frac{\alpha_D}{(\bar{R}_C-\bar{D}_L)m\alpha_D}\left(1-\frac{(m-1)}{\big((m-1)\phi\big)^{\frac{1}{m}}}\right)\right). \tag{2.21}$$

Using the same techniques as in Section (2.5), note when $\alpha_D > \alpha_R$, $m < m^*$ if and only if

$$\ln[\phi] > (m-1)\ln[m-1],$$

which implies

$$\phi > (m-1)^{m-1}.$$

Multiplying both sides by $(m-1)$, raising each side to the $\frac{1}{m}$ power, and simplifying,

$$1 - \frac{(m-1)}{\Big((m-1)\phi\Big)^{\frac{1}{m}}} > 0. \tag{2.22}$$

Combining Eqns. (2.22) and (2.21), we find $\frac{\partial V_D}{\partial D_C}(\bar{R}_C) > 0$, implying $D_C = \bar{R}_C$ is, in fact, candidate D's best response when candidate R is playing R_L^*.

Step 3: Determine the requirement on m that ensures $R_L = R_L^*$ does not violate the constraints.

Eqn. (2.20) implies

$$R_L^* = \bar{R}_C - \Big((\bar{R}_C - \bar{D}_L)^m (m-1)^{-1} \phi^{-1}\Big)^{\frac{1}{m}}.$$

To ensure the constraints are not violated, we need

$$\bar{D}_L \leq R_L^* = \bar{R}_C - \Big((\bar{R}_C - \bar{D}_L)^m (m-1)^{-1} \phi^{-1}\Big)^{\frac{1}{m}},$$

or, simplified,

$$1 \geq \Big((m-1)^{-1} \phi^{-1}\Big)^{\frac{1}{m}}.$$

Raising both sides to the $-m$ power (which reverses the inequality) and rearranging, we need

$$m \geq 1 + \frac{1}{\phi}. \tag{2.23}$$

$\square$

Proposition 3. *For* $0 \leq m \leq 1 + \frac{\alpha_R}{\alpha_D}$*, the optimal strategies for the candidates are:*

$$\begin{aligned} D_C^* &= \bar{R}_C \\ R_L^* &= \bar{D}_L \end{aligned}$$

Proof. W.L.O.G., assume $\alpha_D > \alpha_R$. Again defining $\phi \equiv \frac{\alpha_D}{\alpha_R}$, for $m < 1 + \frac{1}{\phi}$ the above analysis shows neither candidate will have an interior solution. To show both Candidates will be fully ambiguous, I will instead assume Candidate D is fully ambiguous, show Candidate R's best response to Candidate D is full ambiguity, then verify Candidate D's best response to Candidate R is, in fact, full ambiguity.

Step 1: When $D_C = \bar{R}_C$, show $\frac{\partial V_R}{\partial R_L} < 0$, implying $R_L = \bar{D}_L$ is an optimum for candidate R. Equation (2.8) implies when $D_C = \bar{R}_C$,

$$\begin{aligned} \frac{\partial V_R}{\partial R_L} &= \nu(\cdot)'\Bigg(-1 + \alpha_D\Bigg(\alpha_D + \alpha_R\Big(\frac{\bar{R}_C - \bar{D}_L}{\bar{R}_C - R_L}\Big)^m\Bigg)^{-1} \\ &\qquad + m\alpha_D\alpha_R\Bigg(\alpha_D + \alpha_R\Big(\frac{\bar{R}_C - \bar{D}_L}{\bar{R}_C - R_L}\Big)^m\Bigg)^{-2}(\bar{R}_C - \bar{D}_L)^m(\bar{R}_C - R_L)^{-m-1}\Bigg), \end{aligned}$$

$$\begin{aligned} = \ & \nu(\cdot)'\Bigg(\alpha_R\Bigg(\alpha_D + \alpha_R\Big(\frac{\bar{R}_C - \bar{D}_L}{\bar{R}_C - R_L}\Big)^m\Bigg)^{-2}\Big(\frac{\bar{R}_C - \bar{D}_L}{\bar{R}_C - R_L}\Big)^m\Bigg(- 2\alpha_D - \alpha_R\Big(\frac{\bar{R}_C - \bar{D}_L}{\bar{R}_C - R_L}\Big)^m\Bigg) \\ & + \alpha_R\Bigg(\alpha_D + \alpha_R\Big(\frac{\bar{R}_C - \bar{D}_L}{\bar{R}_C - R_L}\Big)^m\Bigg)^{-2}\Big(\frac{\bar{R}_C - \bar{D}_L}{\bar{R}_C - R_L}\Big)^m\Big(\alpha_D + m\alpha_D\Big)\Bigg) \end{aligned}$$

which, when simplified, implies $\frac{\partial V_R}{\partial R_L}$ equals

$$\nu(\cdot)'\Bigg(\bigg(\alpha_D+\alpha_R\Big(\frac{\bar{R}_C-\bar{D}_L}{\bar{R}_C-R_L}\Big)^m\bigg)^{-2}\Big(\frac{\bar{R}_C-\bar{D}_L}{\bar{R}_C-R_L}\Big)^m\bigg(-\Big(\frac{\bar{R}_C-\bar{D}_L}{\bar{R}_C-R_L}\Big)^m+(m-1)\phi\bigg)\Bigg). \quad (2.24)$$

Note that when $0 < m < 1+\frac{1}{\phi}$

$$\Big(\frac{\bar{R}_C-\bar{D}_L}{\bar{R}_C-R_L}\Big)^m \geq 1 > (m-1)\phi,$$

which implies

$$-\Big(\frac{\bar{R}_C-\bar{D}_L}{\bar{R}_C-R_L}\Big)^m+(m-1)\phi < 0. \quad (2.25)$$

Combining Eqns. (2.24) and (2.25) we find $\frac{\partial V_R}{\partial R_L} < 0$, implying $R_L = \bar{D}_L$ is Candidate R's best response when Candidate D is playing $\bar{R}_C$ and $0 < m < 1+\frac{1}{\phi}$.

Step 2: When $R_L = \bar{D}_L$, show $\frac{\partial V_D}{\partial D_C} > 0$, implying $D_C = \bar{R}_C$ is an optimum for Candidate D. Equation (2.6) implies when $R_L = \bar{D}_L$,

$$\begin{aligned}\frac{\partial V_D}{\partial D_C} &= \nu(\cdot)'\Bigg(\alpha_D\bigg(\alpha_D+\alpha_R\Big(\frac{D_L-\bar{D}_L}{\bar{R}_C-\bar{D}_L}\Big)^m\bigg)^{-1}\\ &\qquad -m\alpha_D\alpha_R\bigg(\alpha_D+\alpha_R\Big(\frac{D_L-\bar{D}_L}{\bar{R}_C-\bar{D}_L}\Big)^m\bigg)^{-2}(\bar{R}_C-\bar{D}_L)^{-m}(D_C-\bar{D}_L)^m\Bigg),\end{aligned}$$

which, when simplified, implies

$$\frac{\partial V_D}{\partial D_C} = \nu(\cdot)'\Bigg(\frac{\alpha_D}{\alpha_R}\bigg(\alpha_D+\alpha_R\Big(\frac{D_L-\bar{D}_L}{\bar{R}_C-\bar{D}_L}\Big)^m\bigg)^{-2}\bigg(\phi-(m-1)\Big(\frac{D_C-\bar{D}_L}{\bar{R}_C-\bar{D}_L}\Big)^m\bigg)\Bigg). \quad (2.26)$$

Note that since $\frac{D_C - \bar{D}_L}{\bar{R}_C - \bar{D}_L} \leq 1$, when $0 < m < 1 + \frac{1}{\phi}$

$$\phi - (m-1)\Big(\frac{D_C - \bar{D}_L}{\bar{R}_C - \bar{D}_L}\Big)^m > \phi - \frac{1}{\phi}\Big(\frac{D_C - \bar{D}_L}{\bar{R}_C - \bar{D}_L}\Big)^m \geq \phi - \frac{1}{\phi}. \tag{2.27}$$

Since we have assumed that $\alpha_D > \alpha_R$,

$$\phi - \frac{1}{\phi} = \frac{\alpha_D}{\alpha_R} - \frac{\alpha_R}{\alpha_D} > 0,$$

which, along with Eqns. (2.27) and (2.26), tell us $\frac{\partial V_D}{\partial D_C} > 0$, implying $D_C = \bar{R}_C$ is, in fact, candidate D's best response when candidate R is playing $\bar{D}_L$ and $0 < m < 1 + \frac{1}{\phi}$. □

For $\alpha_D = 2, \alpha_R = 1$, Figure 2.4 graphically shows Propositions 1 through 3 over a range of mass effect parameters.

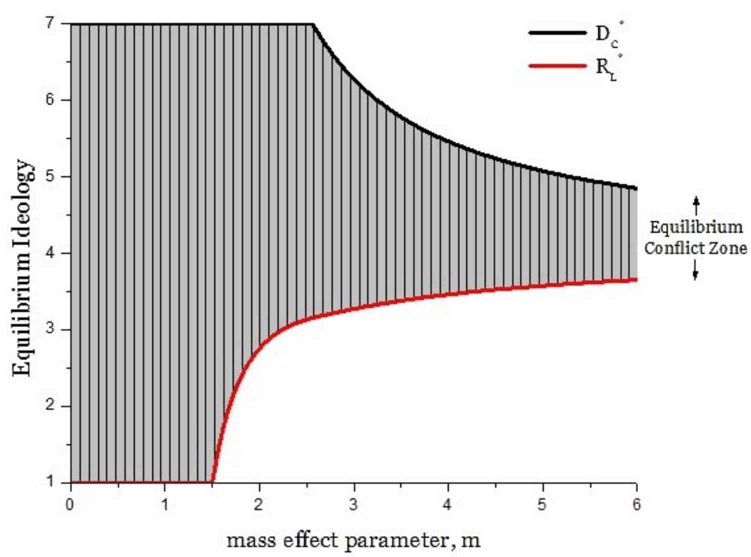


Figure 2.4: Nash Equilibria

For $m \leq m^*$, the above results show one or both candidates select full ambiguity as an optimal strategy. This appears in contrast to the current American political environment, in which presidential candidates tend to cater to voters ranging from their base to slightly beyond the median voter (Democrats try to appeal to Independents leaning Republican and vice versa). As such, I focus the remainder of the paper on scenarios in which $m > m^*$. For this range, Proposition 1 directly implies:

Proposition 4. *For $m > m^*$, the size of the conflict zone, $D_C^* - R_L^*$, is independent of the candidate affect potentials. Specifically,*

$$D_C^* - R_L^* = \frac{\bar{R}_C - \bar{D}_L}{m - 1}$$

This proposition is the key finding of the paper. Along with Proposition 1, it tells us that for a given $m > m^*$, while the optimal level of ambiguity might vary from election year to election year, the number of contested voters will not. Graphically, it indicates that different values of $\alpha_D > \alpha_R$ will vertically shift Figure 2.4, but the interior shape will remain the same. This result will prove central in explaining the recent trends in American presidential politics. Note that Proposition 1 also implies the candidate more likely to win will be more ambiguous than the losing candidate, a finding validated by Campbell (1983).

2.6 Explaining Trends in Modern American Presidential Elections

Before justifying the increase in self-proclaimed Independent American voters, increased partisanship of American political parties, and increased candidate ambiguity, it is necessary to offer definitions of political affiliation. It should be noted that party labels in America are meaningless outside the context of the prevailing political parties. Even in the last twenty years, we have seen the term "Republican" take on a significantly more morally conservative connotation. I therefore offer the following definitions:

Independent American Voter: *A voter whose ideal ideological stance is not consistently espoused by one and only one political party.*

Partisan American Voter: *A voter whose ideal ideological stance is consistently espoused by one and only one political party.*

Political Party: *A party comprised of partisan voters.*

In other words, an Independent American voter is one who cannot count on a specific party to be the sole party offering their ideal viewpoint. These definitions coincide with the trends found in Figure 2.5, which shows a strong relationship between partisanship and the voter's perception that one party is superior to the other on their most important issue.[23]

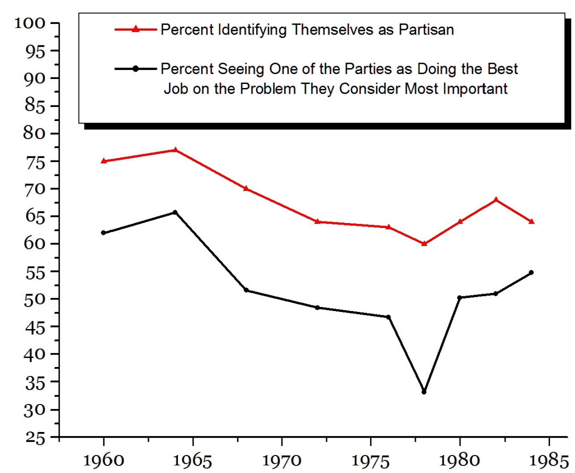


Figure 2.5: Independent Voters

With these definitions in mind, we return to the model. I assume that American presidential candidates have a bounded affect potential advantage over each other. This assumption is based on the inherent "political savvy" required to even be considered a presidential candidate. I model this assumption by assuming $\frac{\alpha_D}{\alpha_R} \in [A, B]$, where $\{A, B\} \in \{\Re^2_+ | A < B\}$. I also assume that neither party consistently offers more personable candidates. Defining $D^*_{C_{\max}}$ as the most conservative equilibrium position a Democratic candidate will adopt, $R^*_{L_{\min}}$ the most liberal

[23]As specified by ANES Cumulative Data variable VCF9012.

equilibrium position a Republican candidate will adopt, $[\bar{D}_L, D^*_{C_{\max}}]$ the voters targeted by Democratic candidates, $[R^*_{L_{\min}}, \bar{R}_C]$ the voters targeted by Republican candidates, and I the percentage of Independent voters, I find:

Proposition 5. *For $m > m^*$, changes due to specific candidate personalities will not affect either party's targeted base, the number of self-proclaimed Independent voters, or the level of party ambiguity. Analytically,*

$$\frac{\partial D^*_{C_{\max}}}{\partial \alpha_D} = 0, \quad \frac{\partial D^*_{C_{\max}}}{\partial \alpha_R} = 0, \quad \frac{\partial R^*_{L_{\min}}}{\partial \alpha_D} = 0,$$
$$\frac{\partial R^*_{L_{\min}}}{\partial \alpha_R} = 0, \quad \frac{\partial I}{\partial \alpha_D} = 0, \quad \frac{\partial I}{\partial \alpha_R} = 0$$

Proof. Within the affective potential bounds of $[A, B]$, note that Eqns. (2.14) and (2.15) imply the Democratic and Republican parties will, at most, cater to voters in the ranges:

$$\textbf{Targeted by Democratic Candidates: } \Big[\bar{D}_L, D^*_{C_{\max}} \equiv \frac{\exp[\frac{\ln[\frac{B}{A}]}{(m-1)}](m\bar{R}_C - \bar{D}_L) + \bar{D}_L(m-1)}{(m-1)\big(1+\exp[\frac{\ln[\frac{B}{A}]}{(m-1)}]\big)}\Big]$$
$$\textbf{Targeted by Republican Candidates: } \Big[R^*_{L_{\min}} \equiv \bar{R}_C - \frac{m(\bar{R}_C - \bar{D}_L)}{(m-1)\big(1+\exp[\frac{\ln[\frac{A}{B}]}{(m-1)}]\big)}, \bar{R}_C\Big].$$

The percentage of Independent voters equals

$$I = \frac{D^*_{C_{\max}} - R^*_{L_{\min}}}{\bar{R}_C - \bar{D}_L}.$$

It is straight-forward that $\frac{\partial D^*_{C_{\max}}}{\partial \alpha_D} = 0$, $\frac{\partial D^*_{C_{\max}}}{\partial \alpha_R} = 0$, $\frac{\partial R^*_{L_{\min}}}{\partial \alpha_D} = 0$, $\frac{\partial R^*_{L_{\min}}}{\partial \alpha_R} = 0$, $\frac{\partial I}{\partial \alpha_D} = 0$, and $\frac{\partial I}{\partial \alpha_R} = 0$. Intuitively, individual candidates do not affect a voter's stated ideology. Instead, only the recognized party bounds are critical. A single Democratic candidate running on a highly conservative platform would not systematically make Republicans declare themselves Democratic. □

Proposition 6. *For $m > m^*$, a decrease in m (corresponding to a decrease in the cost of being more ambiguous than the other candidate) will result in:*

- *The Democratic party will target a larger base.*
- *The Republican party will target a larger base.*
- *Candidates will become more ambiguous.*
- *The number of self-proclaimed Independent voters will increase*

Analytically,

$$\begin{aligned}\frac{\partial D^*_{C_{\max}}}{\partial m} &< 0\\ \frac{\partial R^*_{L_{\min}}}{\partial m} &> 0\\ \frac{\partial I}{\partial m} &< 0\end{aligned}$$

Proof. Using the definitions above,

$$\begin{aligned}\frac{\partial R^*_{L_{\min}}}{\partial m} &= -\frac{(\bar{R}_C - \bar{D}_L)}{(m-1)\big(1+\exp[\frac{\ln[\frac{A}{B}]}{(m-1)}]\big)} + \frac{m(\bar{R}_C - \bar{D}_L)}{(m-1)^2\big(1+\exp[\frac{\ln[\frac{A}{B}]}{(m-1)}]\big)}\\ &\quad - \frac{m(\bar{R}_C - \bar{D}_L)\ln[\frac{A}{B}]\exp[\frac{\ln[\frac{A}{B}]}{(m-1)}]}{(m-1)^3\big(1+\exp[\frac{\ln[\frac{A}{B}]}{(m-1)}]\big)^2}\\ &= -\frac{(\bar{R}_C - \bar{D}_L)}{(m-1)^2\big(1+\exp[\frac{\ln[\frac{A}{B}]}{(m-1)}]\big)^2}\Big(\exp\big[\frac{\ln[\frac{A}{B}]}{(m-1)}\big]\big(\frac{m}{m-1}\ln[\frac{A}{B}]-1\big)\Big).\end{aligned}$$

The first term in the above equation is negative. The second term is also negative, since $A < B$ and $m > 1$ by the assumption of an interior optimum and Proposition 1 . Therefore,

$$\frac{\partial R^*_{L_{\min}}}{\partial m} > 0. \qquad (2.28)$$

As m decreases, candidate R becomes more ambiguous. Symmetrically,

$$\frac{\partial D^*_{C_{\max}}}{\partial m} < 0. \tag{2.29}$$

To show $\frac{\partial I}{\partial m} < 0$,

$$\begin{aligned}\frac{\partial I}{\partial m} &= \frac{\partial \frac{D^*_{C_{\max}} - R^*_{L_{\min}}}{(\bar{R}_C - \bar{D}_L)}}{\partial m} \\ &= \frac{\frac{\partial D^*_{C_{\max}}}{\partial m} - \frac{\partial R^*_{L_{\min}}}{\partial m}}{(\bar{R}_C - \bar{D}_L)},\end{aligned}$$

which, using Eqns. (2.28) and (2.29), is less than 0. □

Intuitively, as ambiguity becomes less harmful to candidates, they will choose to be more vague in their campaign platforms. This will have the effect of causing fewer voters to be convinced that a single party can implement their ideal point. For instance, consider an environment in which ambiguity is harmful. Assume that each candidate caters to their own base. Democrats would offer policy platforms that include the ideal points of voters between Strongly Democratic and Independent (1-4 on the ANES 1-7 scale). Republicans would offer the opposite policy platforms (4-7). Only true Independents (a 4 on the ANES scale) would identify themselves as an Independent, since every other voter type recognizes a single party who identifies with their ideal point. As ambiguity becomes less harmful, the Democratic candidate would offer a more conservative policy stand, including the ideal points of Independent leaning Republican voters (a 5 on the ANES scale). Likewise, Republicans would offer a more liberal policy stand, including the ideal points of Independent leaning Democratic voters (a 3 on the ANES scale). With decreased ambiguity cost, all voters between 3 and 5 would now identify themselves as Independent, since both parties offer their ideal point.

Note Proposition 6 also directly implies:

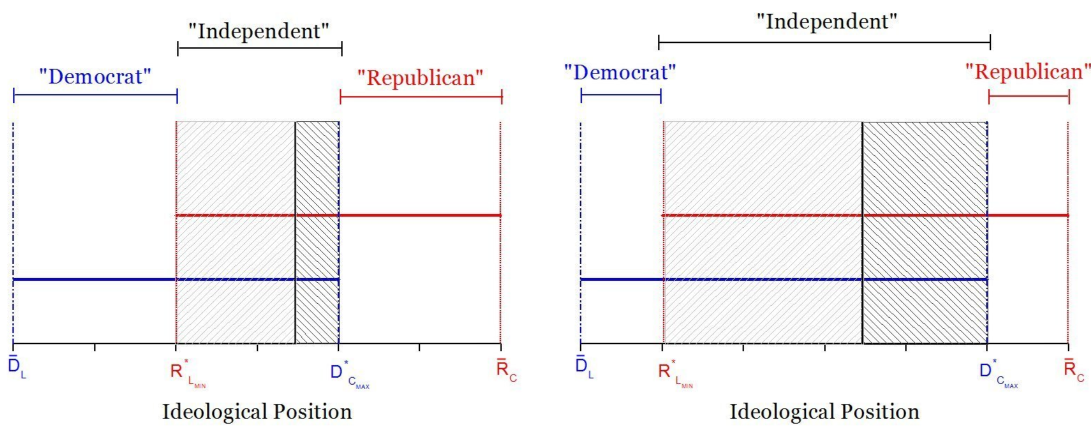
Proposition 7. *For $m > m^*$, a decrease in m (corresponding to a decrease in the cost of* *being more ambiguous than the other candidate) will result in:*

- *The Democratic party, on average, will consist of more liberal voters, resulting in increased partisanship.*
- *The Republican party, on average, will consist of more conservative voters, resulting in increased partisanship.*

Analytically,

$$\frac{\partial\left(\frac{R^*_{L_{\min}}+\bar{D}_L}{2}\right)}{\partial m} < 0$$
$$\frac{\partial\left(\frac{D^*_{C_{\max}}+\bar{R}_C}{2}\right)}{\partial m} > 0$$

Propositions 5 through 7 are demonstrated in Figure 2.6.



(a) m=4 (b) m=2.5

Figure 2.6: Effect of a Decrease in the Mass Effect Parameter, m

Given that the bounds of relative candidate political affect has not undergone a radical shift in the last fifty years,[24] the model predicts the rise in Independent voters and increase in

[24] John Kennedy, Ronald Reagan, and Barack Obama on the high side. George H. W. Bush, Michael Dukakis, and John Kerry on the low side.

party partisanship is due to a decrease in the value of the mass effect parameter, m. Or, put differently, over the past fifty years, ambiguity has become less detrimental for presidential candidates. Specifically, Figure 2.1 indicates ambiguity must have become significantly less detrimental in the late 1960's and early 1970's, and gradually less detrimental from 1980 - 2004.

2.6.1 Death of the Party Bosses 1968-1976

In 1968, the Democratic party nominated Hubert Humphrey for their presidential candidate. This is despite the fact that he had not won (or entered) a single primary. Politics of the day were dictated by party bosses, "who from the sanctity of the so-called smoke-filled rooms at nominating conventions handpicked 'their' candidate to be the party nominee....Party bosses, through a system that combined the disposition of jobs with political favors, support, and even protection, controlled the votes, the party, and thus the selection of all candidates" (Trent and Friedenberg, 1995).

Humphrey's nomination caused such disenfranchisement with Antiwar Democrats that the 1968 convention caused a riot in Chicago. Richard Nixon, the Republican nominee, ended up narrowly winning the general election. To avoid a recurrence, Democrats enacted party rule changes referred to as the McGovern-Fraser reforms. These reforms effectively took the convention out of the "back-rooms" and emphasized general primaries versus closed-room caucuses. Between 1968 and 1976, the percent of delegates selected via a Democratic primary (versus a caucus) increased from 38% to 73%.[25] In addition, in 1974, the Federal Election Campaign Act (FECA) was passed into law, limiting the amount of private money available to candidates receiving public support. This had the effect of limiting the influence of a small number of wealthy donors.

Both changes had substantial effects on the American political landscape. Significant

[25] In an effort to maintain the appearance of openness in the face of the Democratic changes, Republicans also enacted changes. Between 1968 and 1976, the percent of delegates selected via a Republican primary increased from 34% to 68%.

financial contributors could no longer require unambiguous promises from candidates. Party-bosses could not demand the candidate run on a specific party platform. There was no longer a small group of individuals to whom detailed promises must be made. As the role of closed-door caucuses became less pronounced in the nomination of presidential candidates, ambiguity became less harmful to individual nominees. Ambiguous messages could be delivered through the course of a campaign that could not occur in a single national convention during which the elected candidate was chosen internally.

In terms of the model, this had the effect of lowering the mass effect parameter, m, extremely quickly. In turn, this caused candidates to espouse more ambiguous campaign stances, increasing the number of self-proclaimed Independent voters. This downward trend in the caucus system lasted until the late 1970's, at which point the rise in the number of political primaries leveled off. This coincides with the period during which the number of Independent voters stabilized. As shown in Figure 2.1, since the effect of the campaign reform measures stabilized in 1976, the number of Independents has remained reasonably steady.

2.6.2 Rise of the Partisan Media 1980-2008

While the number of Independents in the last three decades has not shown the drastic rise of the late 1960's, Figure 2.1 does show a seemingly consistent increase. I argue this effect is in part due to cable and internet news, and the partisan bias consistent therein.

To explain this effect, note CNN, the first 24-hour cable news program, was founded in 1980. Since that time, the number of cable news outlets and internet news sources has grown exponentially, as well as the partisan slant of the media as a whole.[26] I argue this ability to receive biased news has increased the value of ambiguity (i.e. decreased m).

In particular, biased news outlets allow voters to self-select their information cues. In 2004, for instance, 52% of regular Fox viewers described themselves as politically conservative, while only 36% of CNN viewers and 33% of nightly network news viewers did the same.[27]

[26] Please see Goldberg (2003) and Alterman (2003) for details.

[27] http://people-press.org/reports/display.php3?ReportID=215

This corresponds to the findings of Groseclose and Milyo (2005), who show Fox News is more conservative than other news outlets. Using a biased news outlet that filters conflicting information allows voters to avoid the psychological cost of contradictory statements.[28] This has the effect of allowing candidates to be more ambiguous in their campaign platforms, in that there is a reduced risk of voters hearing contradictory statements.

As an example, when asked in 2003, "Is it your impression that the US has or has not found clear evidence in Iraq that Saddam Hussein was working closely with the al Qaeda terrorist organization?", a study by the Program on International Policy Attitudes[29] found that 67% of Fox News viewers believed the U.S. *had* found evidence linking the two. This is the highest of any other listed news organization, and is in considerable contrast to the 16% of NPR viewers who answered similarly.

2.7 Efficacy of the Heuristic

One of the requirements for a heuristic to be ecologically rational is that it performs "almost as well" as a more complicated decision process. In terms of our model, this implies for voters to be using the "hear what you want to hear" heuristic, it must perform reasonably well compared to a fully informed process. To claim a heuristic exists that systematically resulted in the "wrong" decision would be fundamentally suspect.

To judge the efficacy of the heuristic, it is impossible to use isolated voters. In particular, the assumption of any heuristic implies that voters will never become fully informed, and as such, never determine what the "right" decision should have been. A voter (at the time their vote is cast) always views his decision as correct. As such, to determine if the heuristic is valid, I will use an aggregate performance measure.

Specifically, I will assume that winning candidates enact policy stands at the mean of their issue intervals. Voters view their initial votes as "right" if the enacted policy stand is "closer"

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[28] Burke (2007) offers an alternative rationale for using biased news outlets that does not rely on the concept of psychological cost.

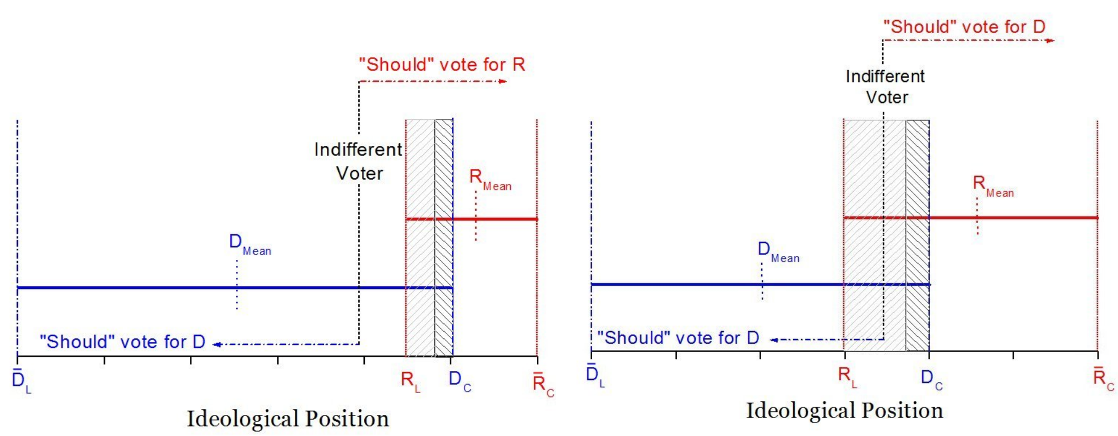
[29] http://65.109.167.118/pipa/pdf/oct03/IraqMedia_Oct03_rpt.pdf

(in a fully informed spatial sense) than their estimates of the losing candidate's policy stand, which we also assume is at the mean of their issue interval.[30] In addition, I assume that the share of voters voting for the Republican candidate are those on the more conservative side of the conflict zone (and vice versa). Call this metric the *efficacy metric*.

Proposition 8. *Using the efficacy metric, for scenarios in which* $\frac{\alpha_D}{\alpha_R} < 20$*, the percentage of voters voting "correctly" equals*

$$\begin{aligned}
&\textit{\% Voting ``Correctly'' (PVC)} \\
&= \quad 1 + \frac{1}{m-1}\min\left(\frac{2-m}{4} + \frac{m}{2\Big(1+\exp\big[\frac{m\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\big]\Big)}, \frac{\alpha_D}{\alpha_D + \alpha_R \exp\big[\frac{m\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\big]}\right) \\
&\qquad - \frac{1}{m-1}\max\left(\frac{2-m}{4} + \frac{m}{2\Big(1+\exp\big[\frac{m\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\big]\Big)}, \frac{\alpha_D}{\alpha_D + \alpha_R \exp\big[\frac{m\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\big]}\right),
\end{aligned}$$

Proof. W.L.O.G., assume $\alpha_D > \alpha_R$. In this scenario, there are two classes of problems as demonstrated in Figure 2.7:



(a) Case 1 (b) Case 2

Figure 2.7: Possible Interior Nash Equilibrium Outcomes

[30] As long as the assumed policy skew is identical for each party, the mean assumption is not critical. It is worth noting that a candidate who enacts policies considerably more extreme than expected will cause a significantly higher number of voters to be disappointed with their initial votes.

Case 1: Indifferent Voter (IV) $< R_L^*$

All variables should be assumed to be at the optimal values, as determined in Proposition 1.

$IV < R_L^*$ implies

$$\frac{\frac{\bar{D}_L + D_C^*}{2} + \frac{R_L^* + \bar{R}_C}{2}}{2} = \frac{\bar{D}_L + D_C^* + R_L^* + \bar{R}_C}{4} < R_L^*. \tag{2.30}$$

Simplified, this implies

$$2R_L^* \quad > \quad \bar{D}_L + (D_C^* - R_L^*) + \bar{R}_C,$$

which, along with Proposition 2, requires

$$2R_L^* > \bar{D}_L + \bar{R}_C + \frac{\bar{R}_C - \bar{D}_L}{m-1}.$$

Combined with Eqn. (2.14), this tells us

$$2\bar{R}_C - \frac{2m(\bar{R}_C - \bar{D}_L)}{(m-1)\big(1 + \exp\big[\frac{\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\big]\big)} > \bar{D}_L + \bar{R}_C + \frac{\bar{R}_C - \bar{D}_L}{m-1},$$

or, simplified,

$$\exp\Big[\frac{\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\Big] \quad > \quad \frac{m+2}{m-2}. \tag{2.31}$$

Proposition 1 implies $m > 2$. With Eqn. (2.31), this implies

$$\frac{\alpha_D}{\alpha_R} > \exp\left[(m-1)\ln[\frac{m+2}{m-2}]\right].$$

Figure 2.8 represents this inequality for values of m for which the equilibrium solutions would range from full ambiguity to 5% issue overlap, a seemingly reasonable range for American presidential politics. As shown, in order for Case 1 to occur, the affect potential ratio must exceed 20, implying that for an identical level of ambiguity, one candidate would win 95%+ of the contested votes.

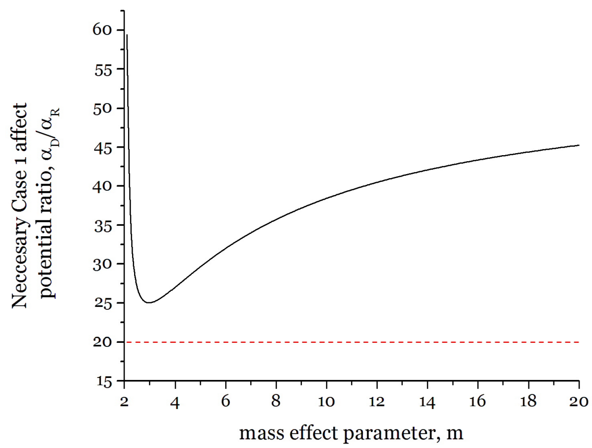


Figure 2.8: Case 1 parameter requirements

Case 2: Indifferent Voter (IV) $>= R_L^*$

Having established Case 1 does not occur if $\frac{\alpha_D}{\alpha_R} < 20$, I focus on Case 2. Note the assumption that $\alpha_D > \alpha_R$ ensures $IV < D_C^*$, allowing us to focus on the scenario shown in Figure 2.7(b). Again, all variables should be assumed to be at the optimal interior values, as determined in Proposition 1.

$$\begin{aligned}\%\text{ Voting "Correctly" (PVC)} \;\equiv\;& \frac{(R_L^* - \bar{D}_L) + (\bar{R}_C - D_C^*)}{\bar{R}_C - \bar{D}_L} \\ &+ \frac{(D_C^* - R_L^*)\min\Big(\frac{\frac{\bar{D}_L + \bar{R}_C + D_C^* + R_L^*}{4} - R_L^*}{D_C^* - R_L^*}, \Upsilon\Big)}{\bar{R}_C - \bar{D}_L} \\ &+ \frac{D_C^* - R_L^*}{\bar{R}_C - \bar{D}_L} \\ &- \frac{(D_C^* - R_L^*)\Big(\max\big(\frac{\frac{\bar{D}_L + \bar{R}_C + D_C^* + R_L^*}{4} - R_L^*}{D_C^* - R_L^*}, \Upsilon\big)\Big)}{\bar{R}_C - \bar{D}_L} \qquad (2.32)\end{aligned}$$

where $\Upsilon \equiv \frac{\alpha_D}{\alpha_D + \alpha_R\left(\frac{D_C^* - \bar{D}_L}{\bar{R}_C - R_L^*}\right)^m}$.

Note Eqn. (2.15) implies

$$D_C^* - \bar{D}_L = \frac{m \exp\left[\frac{\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\right](\bar{R}_C - R_L^*)}{(m-1)\left(1 + \exp\left[\frac{\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\right]\right)} \qquad (2.33)$$

and Eqn. (2.14) implies

$$\bar{R}_C - R_L^* = \frac{m(\bar{R}_C - \bar{D}_L)}{(m-1)\left(1 + \exp\left[\frac{\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\right]\right)}. \qquad (2.34)$$

Together, Eqns. (2.33) and (2.34) tell us

$$\frac{D_C^* - \bar{D}_L}{\bar{R}_C - R_L^*} = \exp\left[\frac{\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\right]. \qquad (2.35)$$

Plugging Eqn. (2.35) into Eqn. (2.32) and simplifying,

$$\begin{aligned}\text{PVC} &= 1+\frac{D_C^*-R_L^*}{\bar{R}_C-\bar{D}_L}\min\left(\frac{\bar{D}_L+\bar{R}_C+D_C^*-R_L^*-2R_L^*}{4(D_C^*-R_L^*)},\Xi\right)\\ &\quad -\frac{D_C^*-R_L^*}{\bar{R}_C-\bar{D}_L}\max\left(\frac{\bar{D}_L+\bar{R}_C+D_C^*-R_L^*-2R_L^*}{4(D_C^*-R_L^*)},\Xi\right) \qquad (2.36)\end{aligned}$$

where $\Xi \equiv \frac{\alpha_D}{\alpha_D+\alpha_R\exp\left[\frac{m\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\right]}$. Plugging Proposition 4 into Eqn. 2.36 and simplifying,

$$\begin{aligned}\text{PVC} &= 1+\frac{1}{m-1}\min\left(\frac{\bar{D}_L+\bar{R}_C+\frac{\bar{R}_C-\bar{D}_L}{m-1}-2R_L^*}{4\frac{\bar{R}_C-\bar{D}_L}{m-1}},\Xi\right)\\ &\quad -\frac{1}{m-1}\max\left(\frac{\bar{D}_L+\bar{R}_C+\frac{\bar{R}_C-\bar{D}_L}{m-1}-2R_L^*}{4\frac{\bar{R}_C-\bar{D}_L}{m-1}},\Xi\right).\end{aligned}$$

Plugging Eqn. (2.14) into Eqn. (2.37) and simplifying,

$$\begin{aligned}\text{PVC} &= 1+\frac{1}{m-1}\min\left(\frac{\bar{D}_L+\bar{R}_C+\frac{\bar{R}_C-\bar{D}_L}{m-1}-2\left(\bar{R}_C-\frac{m(\bar{R}_C-\bar{D}_L)}{(m-1)\left(1+\exp\left[\frac{m\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\right]\right)}\right)}{4\frac{\bar{R}_C-\bar{D}_L}{m-1}},\Xi\right)\\ &\quad -\frac{1}{m-1}\max\left(\frac{\bar{D}_L+\bar{R}_C+\frac{\bar{R}_C-\bar{D}_L}{m-1}-2\left(\bar{R}_C-\frac{m(\bar{R}_C-\bar{D}_L)}{(m-1)\left(1+\exp\left[\frac{m\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\right]\right)}\right)}{4\frac{\bar{R}_C-\bar{D}_L}{m-1}},\Xi\right),\end{aligned}$$

which, after simplification, implies

$$\begin{aligned}\text{PVC} &= 1+\frac{1}{m-1}\min\left(\frac{2-m}{4}+\frac{m}{2\left(1+\exp\left[\frac{m\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\right]\right)},\frac{\alpha_D}{\alpha_D+\alpha_R\exp\left[\frac{m\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\right]}\right)\\ &\quad -\frac{1}{m-1}\max\left(\frac{2-m}{4}+\frac{m}{2\left(1+\exp\left[\frac{m\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\right]\right)},\frac{\alpha_D}{\alpha_D+\alpha_R\exp\left[\frac{m\ln[\frac{\alpha_D}{\alpha_R}]}{m-1}\right]}\right).\end{aligned}$$

□

Figure 2.9 shows this percentage of voters for which the heuristic is effective. Affect potentials are shown such that a candidate facing another candidate with identical ambiguity will receive anywhere from 25% to 75% of the vote. Mass effect parameters are shown such that equilibrium conflict zones will range from 1/4 to 2/3 of the total possible political ideology. Assuming that American presidential politics exists within these ranges,[31] approximately 80%+ of voters will view their choice as correct, ex-post. This corresponds to the findings of Lau and Redlawsk (2006) who estimate that 77% of voters make the "correct" voting decision, given their self-proclaimed issue stances.

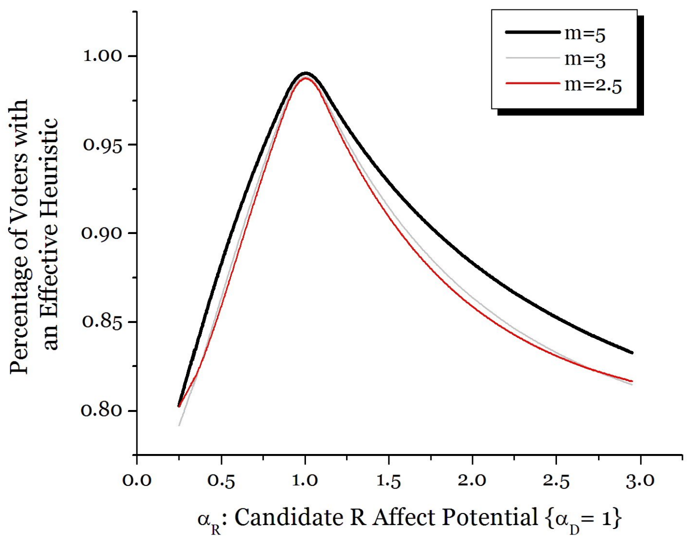


Figure 2.9: Efficacy of the "Hear What You Want To Hear" Heuristic

2.8 Multidimensional Issue Space

It is worth noting the effect of a multidimensional issue space on the model. If we assume that ambiguity in one issue does not affect the share of the contested vote in another issue,[32]

[31] Given the requirements necessary to win a party primary, it is reasonable to assume no two presidential candidates are significantly different in affect potential. Also, modern campaigns show a propensity for "appealing to the independent voter without alienating your base", implying a moderate level of conflict.

[32] For instance, a devote pro-life candidate's share of the pro-life vote will not be affected by an ambiguous stance on welfare reform.

then we can view each issue space as a separate game. A candidate's affect potential might vary from issue to issue, allowing one candidate to be specific on some issues and ambiguous on others. However, the best responses in each game will follow the form of Proposition 1.

Note the only significant change to the unidimensional model is the higher likelihood that voters will choose not to vote. If, for instance, a candidate offers the voter's ideal point in Issue 1, but does not in Issue 2, the heuristic will mandate the voter does not vote. This again coincides with the findings that individuals avoid making trade-offs between cues pointing in opposite directions. This result also offers an explanation for the highly correlated party stances on highly disparate issues (pro-social welfare, pro-union, pro-choice, etc.). Because a voter claiming to be pro-choice is likely to be pro-welfare, the optimal strategy for a Democrat is to offer both issue stances.

Lastly, note that the multidimensional model allows the concept of a voter satisficing strategy. In particular, because each game is separate, voters choosing to focus on only a few key issues do not affect any of the results.

2.9 Conclusion

This chapter has applied a voter heuristic to American presidential elections and applied the economic theory of conflict to help explain recent trends in American politics left unexplained by spatial models of voting. The model's key results are that individual candidate personality does not influence the trends of Independent voters, party partisanship, or long-term ambiguity. Instead, only the cost of being ambiguous is significant, with lower ambiguity costs related to increases in these trends. I have also made the case that the primary system, campaign finance reform, and changing media climate have all resulted in lower ambiguity costs. Lastly, I have shown that the proposed heuristic is ecologically rational, and offers similar efficacy rates to those observed in American politics.

While the model offers a starting point to understanding candidate and voter behavior, it has not addressed the rich environment inherent to campaign advertising strategy. I assumed the level of affect potential is fixed for each candidate, but this is an obvious simplification, given the effectiveness of negative campaigning, image consultants, etc. Further, I have not explored how the primary campaign game affects the general election. Exploring this subgame would likely offer further insight into optimal presidential campaign strategies. In particular, examining the tradeoff between more extreme and more personable candidates would likely prove valuable.

Chapter 3

A Threshold Interpretation of the Ratio-Form Contest Success Function

3.1 Introduction

Contest success functions have been widely applied to many economic applications involving uncertain conflict. Finding uses in standard military engagement (Hirshleifer, 1988; Baker, 2003), electoral campaigning (Snyder, 1989; Skaperdas and Grofman, 1995), litigation behavior (Katz, 1988; Farmer and Pecorino, 1999), and numerous other applications, contest success functions have allowed tractable predictions to inherently inconclusive scenarios.

Formally, consider a contest with n players in which each player, $i \in \{1 \ldots n\}$, exerts effort level, x_i.[1] A contest success function (CSF) is defined as

$$\mathbb{P} : \mathrm{dom}(\vec{x}) \to [0,1]^n \text{ such that } \sum_{i=1}^{n} \mathbb{P}_i = 1, \frac{\partial \mathbb{P}_i}{\partial x_i} > 0, \text{ and } \frac{\partial \mathbb{P}_i}{\partial x_j} < 0.$$

Thus, a contest success function maps a player's effort level into his probability of winning

[1] which can be interpreted as actual effort level, campaign expenditures, military expenditures, etc.

the contest.[2] Foremost of the class of CSF's is the "ratio" form,[3] which proposes

$$\mathbb{P}_i = \frac{f_i(x_i)}{\sum_{l=1}^{n} f_l(x_l)}, \tag{3.1}$$

where the *effectivity functions*, $f_i(x_i), i \in \{1, \ldots, n\}$, map individual efforts into the effective "output" entering the specification in Eqn. 3.1. Note the ratio form includes the seminal rent-seeking CSF of Tullock (1980) where $f_i(x) = f_j(x) \equiv x^r$ and the logistic-form proposed by Hirshleifer (1989) where $f_i(x) = f_j(x) \equiv \exp(kx)$.

Until very recently, the use of CSF's were justified in one of two ways.[4] The first approach leaves unexplained the microfoundations of the function, but instead appeals to its axiomatic properties. Skaperdas (1996), Kooreman and Schoonbeek (1997), Clark and Riis (1998), and Rai and Sarin (2009) have all shown that under various intuitive conditions, specific forms of the CSF are necessary. Being inherently normative, this approach leaves unanswered questions such as why a particular effectivity function should be used in the class of ratio-form CSF's. Clark and Riis (1998), for instance, show that under certain assumptions, the effectivity function must be of the form, $f_i(x_i) = \alpha_i x_i^r$ where $r > 0$ and $\alpha_i > 0$. However, as Perez-Castrillo and Verdier (1992) and Baye, Kovenock, and Vries (1994) demonstrate, the specific value of r can result in drastically different equilibria.

Alternatively, Katz (1988), Clark and Riis (1996), and Fu and Lu (2008) follow the random utility framework pioneered by McFadden (1973, 1974) in which a contest administrator is assumed to have preferences over noisy effort levels. The noise enters additively and is assumed to follow the extreme value distribution. As Fu and Lu (2008) recognize, this can imply an underlying ranking system in which the best outcome is submitted to the contest administrator.[5] In cases where the "best outcome" is the sole criteria for victory and the normalized

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[2]Instead of the probability of winning a winner-take-all contest, the CSF can also be interpreted as the share of a prize that is split among the players.

[3]Alternatively referred to as the logit or additive form.

[4]I am excluding a discussion of Tullock's (1975, 1980) "lottery" which is incapable of explaining irrational probabilities and was seemingly derived for purely illustrative purposes.

[5]Specifically, when the contest administrator is concerned with log-output. If the contest administrator is

error terms have units of log-output, this approach is highly warranted. However, in many scenarios such as military conflicts, a natural analogue is difficult to construct.

I offer an alternative microfoundation in which the limit of a player's probability of winning a "threshold" contest is stochastically equivalent to the ratio-form contest success function.[6] In particular, I propose a game in which players can exert effort to increase their probability of successfully "hitting a target", which is specific to the contest in question. When the number of targets hit in a specified period surpasses an exogenous threshold, the player can win the full contest. The threshold can be in two forms. First, it can be an exogenous absolute threshold in which the first player to surpass the threshold can win the contest. In this interpretation, the game is most like a patent race as specified by Baye and Hoppe (2003). Alternatively, the threshold can measure how much more successful a player must be than his opponents to be able to win the contest. In this interpretation, the game is most like competition during an evaluation period. In both cases, I derive a discrete and continuous form of the threshold game.

In a paper most similar to this chapter, Corchón and Dahm (2008) offer two alternate microfoundational approaches to modeling CSF's. The first approach requires a contest administrator about whom the players have incomplete information. In economic scenarios in which a contest administrator does not exist, an alternate justification is needed. In addition, when more than two players compete, the ratio form cannot be obtained when a single crossing property is assumed. Their second approach, which is extremely novel, finds that when players bargain over shares of a prize, each player has a bargaining weight of $f_i(x_i)$, and the disagreement point is $\vec{d} = \vec{0}$, then the ratio-form CSF specified in Eqn. 3.1 is the Nash-bargaining solution. Thus, in scenarios where the good is divisible,[7] both players are willing to bargain, and bargaining is necessary to receive utility, Corchón and Dahm's (2008) microfoundations seem highly appropriate from a positive perspective.

---

concerned with output, the equivalent interpretation implies a strictly logistic CSF.

[6]This structure does not address non-ratio-form CSF's, such as that used in Che and Gale (2000).

[7]Or, less likely, both parties can agree to bargain over probabilities of success.

I focus on scenarios, especially those having elements of conflict, that do not admit these assumptions. In a military conflict between countries, for instance, the disagreement point is not a zero share of the contested zone, but instead the status-quo land share. More generally, conflict scenarios tend to not involve players bargaining over the share of the prize, but instead involve players trying to surpass a threshold necessary for complete victory. In legal conflicts, lawyers try to convince the jury, at which point additional effort is unnecessary and total victory is achieved. In military conflicts, countries attempt to destroy tactical targets to induce surrender. In political conflicts, lobbyists try to convince politicians (with a sweet enough carrot or a big enough stick) to vote for their specific policy.

This chapter is structured as follows. In Section 1, I specify a discrete conflict game using an absolute threshold and show it induces the ratio-form CSF. In section 2, I extend this model to a continuous game. In Section 3, I specify a discrete game with a relative threshold and show it, too, induces the ratio-form CSF. In Section 4, I extend this model to a continuous game. In Section 5, I offer a spatial interpretation of the proposed models. In Section 6, I apply this model to specific examples in the literature. In Section 7, I offer a caveat to using this method. In Section 8, I conclude.

3.2 Discrete Model with Absolute Threshold

3.2.1 Model

In the following model, players compete in a series of rounds. Within each round, each player attempts to hit a target during each turn of the round. If, within the round, a player is the first to surpass an absolute threshold of successful hits, he has the ability to win the contest. If no player surpasses the threshold, the number of successful hits "resets", and a new round is played.

Formally, let $N = \{1, 2, \ldots, n\}$ be the set of players who may participate in a contest. The contest consists of (potentially) multiple rounds, with each round having $t \in \mathbb{Z}^+$ turns. The rules are as follows:

1. For every turn in a round, every player "makes an attempt". Player i makes a successful

attempt with probability

$$P_i \equiv g_i(x_i)\delta. \tag{3.2}$$

$g_i(x_i)$, Player i's *precision function*, captures the technology available to the player to increase his likelihood of making a successful attempt. This function should be specific to the game in question and must map all possible effort levels, x_i, into $[0, \frac{1}{\delta}]$ with $\frac{\partial g_i}{\partial x_i} > 0$. $\delta > 0$, the *technological boundary*, captures the technical/situational difficulties in making a successful attempt given the nature of the conflict. The results of the paper rely on the limit as δ approaches zero and are thus not applicable to conflict situations in which minimal effort can result in success with any reasonable certainty.

When developing an interpretation for the underlying structure, it should be noted the separability of P_i is crucial for the results.

2. A player is said to have reached the threshold potential if, during any turn in the round, he is the only player who has achieved his mth successful hit. If more than one player achieves his mth successful hit during the same turn, each successful player reaches the threshold potential with probability $\frac{1}{n}$. If no successful player reaches the threshold potential, the round ends and the contest continues.[8] The threshold potential represents the exogenous number of successful attempts required to have a possibility of winning the contest.

3. After the round is complete, if any player reaches the threshold potential, he wins the contest with probability $\alpha_i \in (0, 1]$. This probability incorporates personal characteristics of the individual players.

4. If no player wins at the end of the round, a new round is played.

[8] This specific tie-breaking rule is used only for ease of notation. The exact rule is inconsequential, as $\lim \delta \to 0$ implies the probability of two players each achieving his mth successful hit in a turn is dominated by the probability of only one player achieving the threshold.

Thus, the underlying contest is one in which the players attempt to reach a success threshold, and they can affect their probability of success through their choice of effort. While the fact that successful attempts in a given round do not transfer to future rounds will be crucial for the results, in many games this requirement is natural. In a military conflict, for instance, attacks normally come in coordinated surges. If, during a surge, enough tactical targets are destroyed, surrender becomes likely. If the opponent can "wait it out", he can regroup, rebuild, and the war effectively begins anew.[9]

It should be noted that not all elements of the model are needed for every economic application and some elements of the model can be interchanged with others depending on the scenario. For instance, the threshold potential, m, and the precision function, $g_i(x_i)$, can be adjusted to ensure the precision function has reasonable returns to scale. Also, note if $\alpha_i = 1$ for each player and $m = t = 1$, the contest reduces to a race in which the first player to have a successful attempt wins.

3.2.2 Results

I now show that under the discrete model with absolute threshold described in Section 3.2.1, the limit as the technical boundary, δ, approaches zero of the probability a given player wins the contest equals the ratio contest success function specified by Eqn. 3.1.

Lemma 1. *Define P_i^W as the probability that Player i wins a round. With t turns in a round and a threshold potential of m,*

[9] Consider the U.S. difficulty in dismantling Al Qaeda. According to the 9/11 Commission (2004), in response to the August 7, 1998 embassy attacks, U.S. forces launched Tomahawk missiles on August 20, 1998 at suspected Bin Ladin camps. The strikes are believe to have missed Bin Ladin by a few hours, at which point the surge in military activity ceased. Later, on December 20, 1998, intelligence reported that Bin Ladin was in a location heavily populated by civilians. On the decision to not strike, a lower-level official stated "We should have done it...We may well come to regret the decision not to go ahead". If either of these attempts at Al Qaeda's prime tactical target were successful, it could be argued that the war would have ended. By missing the target, the war continued tactically unchanged.

$$P_i^W = \alpha_i \sum_{h=m}^{t} \binom{h-1}{m-1} P_i^m (1-P_i)^{h-m} \Bigg[\prod_{j \in N \setminus i} \Big(1 - \sum_{k=m}^{h} \binom{k-1}{m-1} P_j^m (1-P_j)^{k-m}\Big) + \sum_{j \in N \setminus i} \frac{1}{n} \binom{h-1}{m-1} P_j^m (1-P_j)^{h-m} \Bigg] \quad (3.3)$$

Proof. Define $\rho \in N$ as the player who first reaches the threshold potential. We will first determine the probability that $\rho = i$, then determine Player i's probability of winning the round by noting that

$$\begin{aligned} P_i^W &= \text{Prob}(\text{Player } i \text{ wins a round} | \rho = i)\text{Prob}(\rho = i) \\ &= \alpha_i \text{Prob}(\rho = i) \end{aligned} \quad (3.4)$$

Define ρ_i^j as the event "player i reaches the threshold potential during turn j". Then

$$\text{Prob}(\rho = i) = \sum_{h=1}^{t} \text{Prob}(\rho_i^h)$$

Recognizing $\text{Prob}(\rho_i^h) = 0$ for $h < m$,

$$\text{Prob}(\rho = i) = \sum_{h=m}^{t} \text{Prob}(\rho_i^h). \quad (3.5)$$

Define $\Upsilon^i \in \{0,1\}^h$ as the outcome of Player i's attempts during the first h turns. $\Upsilon_r^i = 0$ if Player i was not successful in his rth attempt, and $\Upsilon_r^i = 1$ otherwise. Note $\#\big\{ \cup_{r=1}^{h} \{\Upsilon_r^i | \Upsilon_r^i = 1\}\big\}$ is the number of successful attempts by Player i in the first h turns. In addition, define m_i^h as the event "Player i has his mth successful attempt during turn h". Then,

$$
\begin{aligned}
&\text{Prob}(m_i^h) \\
&= \text{Prob}\Big(\Upsilon_i^h = 1 | \#\{\cup_{r=1}^{h-1}\{\Upsilon_r^i | \Upsilon_r^i = 1\}\} = m-1\Big)\text{Prob}\Big(\#\{\cup_{r=1}^{h-1}\{\Upsilon_r^i | \Upsilon_r^i = 1\}\} = m-1\Big) \\
&= P_i\Big[\binom{h-1}{m-1} P_i^{m-1}(1-P_i)^{h-m}\Big] \\
&= \binom{h-1}{m-1} P_i^m (1-P_i)^{h-m}.
\end{aligned} \tag{3.6}
$$

The probability that player i reaches the threshold potential in turn h is then

$$
\begin{aligned}
&\text{Prob}(\rho_i^h) \\
&= \text{Prob}\Big(m_i^h | \max_{j\in N\backslash i} \#\{\cup_{r=1}^{h}\{\Upsilon_r^j | \Upsilon_r^j = 1\}\} < m\Big)\text{Prob}\Big(\max_{j\in N\backslash i} \#\{\cup_{r=1}^{h}\{\Upsilon_r^j | \Upsilon_r^j = 1\}\} < m\Big) \\
&\qquad + \sum_{j\in N\backslash i} \frac{1}{n}\text{Prob}\big(m_i^h | m_j^h\big)\text{Prob}\big(m_j^h\big) \\
&= \text{Prob}(m_i^h) \prod_{j\in N\backslash i} \big(1 - \sum_{k=1}^{h} \text{Prob}(m_j^k)\big) + \sum_{j\in N\backslash i} \frac{1}{n}\text{Prob}\big(m_i^h\big)\text{Prob}\big(m_j^k\big) \\
&= \text{Prob}(m_i^h)\Big[\prod_{j\in N\backslash i} \big(1 - \sum_{k=1}^{h} \text{Prob}(m_j^k)\big) + \sum_{j\in N\backslash i} \frac{1}{n}\text{Prob}\big(m_j^k\big)\Big].
\end{aligned} \tag{3.7}
$$

Plugging Eqn. (3.6) into Eqn. (3.7) and recognizing Prob(m_j^k)=0 for $k < m$,

$$
\begin{aligned}
\text{Prob}(\rho_i^h) \;=\; & \binom{h-1}{m-1} P_i^m (1-P_i)^{h-m} \Bigg[\prod_{j\in N\backslash i} \Big(1 - \sum_{k=m}^{h} \binom{k-1}{m-1} P_j^m (1-P_j)^{k-m}\Big) \\
& + \sum_{j\in N\backslash i} \frac{1}{n} \binom{h-1}{m-1} P_j^m (1-P_j)^{h-m}\Bigg]
\end{aligned} \tag{3.8}
$$

Combining Eqns. (3.4), (3.5), and (3.8) gives the desired result. □

Using Lemma 1, we can now determine the overall probability that Player i wins the contest.

Lemma 2. *Define $\mathbb{P}_i$ as the probability that Player i wins the contest. With t turns in a round and a threshold potential of m,*

$$\mathbb{P}_i = \frac{\alpha_i g_i(x_i)^m \sum_{h=m}^{t} (1-g_i(x_i)\delta)^{h-m} \Bigg[\prod_{j \in N \setminus i} \Big(1 - \sum_{k=m}^{h} \binom{k-1}{m-1} g_j(x_j)^m \delta^m (1-g_j(x_j)\delta)^{k-m}\Big) + \sum_{j \in N \setminus i} \frac{1}{n} \binom{h-1}{m-1} g_j(x_j)^m \delta^m (1-g_j(x_j)\delta)^{h-m} \Bigg]}{\sum_{l=1}^{n} \alpha_l g_l(x_l)^m \sum_{h=m}^{t} (1-g_l(x_l)\delta)^{h-m} \Bigg[\prod_{j \in N \setminus l} \Big(1 - \sum_{k=m}^{h} \binom{k-1}{m-1} g_j(x_j)^m \delta^m (1-g_j(x_j)\delta)^{k-m}\Big) + \sum_{j \in N \setminus l} \frac{1}{n} \binom{h-1}{m-1} g_j(x_j)^m \delta^m (1-g_j(x_j)\delta)^{h-m} \Bigg]}$$

Proof. Note first that if $\sum_{l=1}^{n} P_l^W \neq 0$ then:[10]

$$\begin{aligned} \mathbb{P}_i &= P_i^W + \big(1 - \sum_{l=1}^{n} P_l^W\big) P_i^W + \big(1 - \sum_{l=1}^{n} P_l^W\big)^2 P_i^W + \dots \\ &= \frac{P_i^W}{1 - \big(1 - \sum_{l=1}^{n} P_l^W\big)} \\ &= \frac{P_i^W}{\sum_{l=1}^{n} P_l^W} \end{aligned} \tag{3.9}$$

In addition, by substituting Eqn. (3.3) into Eqn. (3.9), expanding the P_i's per Eqn. (3.2), and slightly rearranging,

[10]Note when $\delta = 0$, this condition is violated, causing a discontinuity in the probabilities. I thus focus solely on the limit result, with the reader being advised to consider the necessary interpretive caveats.

$$\mathbb{P}_i = \frac{\alpha_i g_i(x_i)^m \delta^m \sum_{h=m}^{t} (1-g_i(x_i)\delta)^{h-m} \Bigg[\prod_{j\in N\setminus i} \Big(1 - \sum_{k=m}^{h} \binom{k-1}{m-1} g_j(x_j)^m \delta^m (1-g_j(x_j)\delta)^{k-m}\Big) + \sum_{j\in N\setminus i} \frac{1}{n} \binom{h-1}{m-1} g_j(x_j)^m \delta^m (1-g_j(x_j)\delta)^{h-m} \Bigg]}{\sum_{l=1}^{n} \alpha_l g_l(x_l)^m \delta^m \sum_{h=m}^{t} (1-g_l(x_l)\delta)^{h-m} \Bigg[\prod_{j\in N\setminus l} \Big(1 - \sum_{k=m}^{h} \binom{k-1}{m-1} g_j(x_j)^m \delta^m (1-g_j(x_j)\delta)^{k-m}\Big) + \sum_{j\in N\setminus l} \frac{1}{n} \binom{h-1}{m-1} g_j(x_j)^m \delta^m (1-g_j(x_j)\delta)^{h-m} \Bigg]}$$

Cancelling the δ^m terms gives the desired result. □

Finally, we can use Lemma 2 to prove the first result of the paper.

Theorem 9. *Given the underlying model described in Section 3.2.1, the limit as δ approaches zero of the probability that Player i wins the contest is the ratio-form contest success function:*

$$\lim_{\delta\to 0} \mathbb{P}_i = \frac{\alpha_i g_i(x_i)^m}{\sum_{l=1}^{n} \alpha_l g_l(x_l)^m} \equiv \frac{f_i(x_i)}{\sum_{l=1}^{n} f_l(x_l)}$$

Proof. The Theorem is a direct result of taking the limit of $\mathbb{P}_i$ as defined in Lemma 2. In particular,

$$
\lim_{\delta\to\delta}\mathbb{P}_i =
$$

$$
\frac{\lim_{\delta\to\delta}\alpha_i g_i(x_i)^m \sum_{h=m}^{t}(1-g_i(x_i)\delta)^{h-m}\left[\prod_{j\in N\setminus i}\left(1-\sum_{k=m}^{h}\binom{k-1}{m-1}g_j(x_j)^m\delta^m(1-g_j(x_j)\delta)^{k-m}\right) + \sum_{j\in N\setminus i}\frac{1}{n}\binom{h-1}{m-1}g_j(x_j)^m\delta^m(1-g_j(x_j)\delta)^{h-m}\right]}{\lim_{\delta\to\delta}\sum_{l=1}^{n}\alpha_l g_l(x_l)^m \sum_{h=m}^{t}(1-g_l(x_l)\delta)^{h-m}\left[\prod_{j\in N\setminus l}\left(1-\sum_{k=m}^{h}\binom{k-1}{m-1}g_j(x_j)^m\delta^m(1-g_j(x_j)\delta)^{k-m}\right) + \sum_{j\in N\setminus l}\frac{1}{n}\binom{h-1}{m-1}g_j(x_j)^m\delta^m(1-g_j(x_j)\delta)^{h-m}\right]}
$$

$$
= \frac{\alpha_i g_i(x_i)^m\left[\prod_{j\in N\setminus i}(1)+(0)\right]}{\sum_{l=1}^{n}\alpha_l g_l(x_l)^m\left[\prod_{j\in N\setminus l}(1)+(0)\right]}
$$

$$
= \frac{\alpha_i g_i(x_i)^m}{\sum_{l=1}^{n}\alpha_l g_l(x_l)^m}
$$

□

Intuitively, this result is directly related to the fact that the game is repeated indefinitely. Thus, the probability that no player wins equals zero, even if δ is very small. The tractability is due to the fact that as δ approaches zero, the probability of having more than one player reach the threshold is dominated by the probability of having exactly one player reach the threshold.

Note that by setting $\alpha_i = 1$ for all players and $m = 1$, $\lim_{\delta\to 0}\mathbb{P}_i = \frac{g_i(x_i)}{\sum_{l=1}^{n} g_l(x_l)}$. Thus, when interpreting the model as a patent race in which the first player to innovate wins, the effectivity function, $f_i(x_i)$, is equal to the precision function, $g_i(x_i)$, and is directly interpreted as the mapping from effort into the probability of having a successful innovation. However, by adjusting the threshold level, m, a modeler is able to justify alternate returns to scale of the effectivity function.

3.2.3 Tullock's Rent-Seeking Function

As an example, consider the seminal rent-seeking function proposed by Tullock (1980),[11]

$$\mathbb{P}_i = \frac{x_i^m}{\sum_{l=1}^n x_l^m}.$$

Many papers, such as Garfinkel and Skaperdas (2000), explicitly assume $m = 1$. In the context of the discrete model with absolute threshold, this can be interpreted as a conflict in which the first player to make a successful attempt wins the contest. Specifically, players can linearly increase their probability of success for any given attempt via $P_i = x_i \delta$. Under this interpretation, the probability technology displays constant returns to scale, which may not accurately reflect the specifics of the economic application. In Garfinkel and Skaperdas (2000), for instance, this implies the military technology designed to increase precision displays constant returns to scale. If, instead, the game is interpreted as a series of military strikes in which each player requires two hits during any given strike, then the identical CSF implies an underlying technology displaying decreasing returns to scale:

$$\mathbb{P}_i = \frac{x_i}{\sum_{l=1}^n x_l} = \frac{(x_i^{1/2})^2}{\sum_{l=1}^n (x_l^{1/2})^2}.$$

Thus, by modifying the threshold potential, the effectivity functions can be modeled to display reasonable returns to scale. In addition, the α_i's can be interpreted as the exogenous probability of winning the game once the threshold is fulfilled, or a specification of the effectivity function.

[11] In the discrete model with absolute threshold, this assumes $\alpha_i = 1$ for all players.

3.3 Continuous Model with Absolute Threshold

3.3.1 Model

Given that the discrete model can be interpreted as a patent race requiring a threshold number of innovations, it is not surprising that a similar result to Baye and Hoppe (2003) holds, which shows a continuous patent game such as Loury (1979) and Dasgupta and Stiglitz (1980) is strategically equivalent to a contest game incorporating a ratio-form contest success function.

Specifically, players again compete in a series of rounds, each of time duration, T. As in the discrete case, players attempt to hit a target throughout the round. If a player is the first to surpass an absolute threshold of successful hits, he has the ability to win the contest. If no player surpasses the threshold, the number of successful hits "resets", and a new round is played.

Formally, let $N = \{1, 2, \ldots, n\}$ be the set of players who may participate in a contest. The contest consists of (potentially) multiple rounds, with each round having time duration, T. The rules are as follows:

1. Throughout each round, every player's number of successes follows a Poisson process. In particular, the time, t, at which each sequential independent success occurs is a random variable having exponential density,

$$P_i \equiv g_i(x_i)\delta \exp^{-g_i(x_i)\delta t}.$$

Thus, the probability that an individual is successful at or before time t equals

$$\int_0^t P_i dt = 1 - \exp^{-g_i(x_i)\delta t}.$$

As before, $g_i(x_i)$ is Player i's precision function and captures the technology available to the player to increase his likelihood of making a successful attempt in a given time period.

This function should be specific to the game in question and must map all possible effort levels, x_i, into $(0, M < \infty]$ with $\frac{\partial g_i}{\partial x_i} > 0$.[12] $\delta > 0$ again captures the technical/situational difficulties in making a successful attempt given the nature of the conflict.

2. A player is said to have reached the threshold potential if, at any time during a round, he is the only player who has achieved his mth successful hit. The threshold potential represents the exogenous number of successful attempts required to have a possibility of winning the contest. As the probability of two players each achieving his mth successful hit at the same instant is zero, any tie-breaking rule can be assumed.

3. After the round is complete, if any player reaches the threshold potential, he wins the contest with probability $\alpha_i \in (0, 1]$. This probability incorporates personal characteristics of the individual players.

4. If no player wins at the end of the round, a new round is played.

Thus, the underlying contest is essentially the same as the discrete model, but players' effort levels affect the time it takes to achieve a success.

3.3.2 Results

I now show that under the continuous model with absolute threshold described in Section 3.3.1, the limit as the technical boundary, δ, approaches zero of the probability a given player wins the contest equals the ratio contest function specified in Eqn. 3.1.

Lemma 3. *Define P_i^W as the probability that Player i wins a round. With each round having time duration, T, and a threshold potential of m,*

$$P_i^W = \frac{\alpha_i g_i(x_i)^m \delta^m}{(m-1)!} \int_0^T \exp^{-g_i(x_i)\delta t} t^{m-1} \prod_{j \in N \setminus i} \sum_{h=0}^{m-1} \frac{g_i(x_i)^h \delta^h t^h}{h!} dt \qquad (3.10)$$

[12]Note in the continuous case, $g_i(x_i)$'s upper bound need not relate to δ. In particular, while the hazard rate, $g_i(x_i)\delta$, has an interpretation of instantaneous conditional probability of success, it is not formally a probability measure. This fact implies Baye and Hoppe's (2003) restriction that the hazard rate in a patent game be defined over $[0, 1]$ is not necessary.

Proof. Define $\rho \in N$ as the player who first reaches the threshold potential. We will first determine the probability that $\rho = i$, then determine Player i's probability of winning the round by noting that

$$\begin{aligned} P_i^W &= \text{Prob}(\text{Player } i \text{ wins a round} | \rho = i)\text{Prob}(\rho = i) \\ &= \alpha_i \text{Prob}(\rho = i) \end{aligned} \tag{3.11}$$

Note with the defined underlying Poisson process, the probability density function of achieving m successes follows the $\Gamma(m, g_i(x_i)\delta)$ distribution,

$$\frac{g_i(x_i)^m \delta^m}{(m-1)!} t^{m-1} \exp^{-g_i(x_i)\delta t} \tag{3.12}$$

In addition, note if X is a $\Gamma\big(m, g_i(x_i)\delta\big)$ random variable, it has the property[13]

$$\text{Prob}(X \geq t) = \text{Prob}(Y < m) \tag{3.13}$$

where Y is a Poisson$\big(g_i(x_i)\delta t\big)$ random variable having

$$\text{Prob}(Y = h) = \frac{\exp^{-g_i(x_i)\delta t} g_i(x_i)^h \delta^h t^h}{h!} \tag{3.14}$$

Using Eqns. (3.12),(3.13), and (3.14), the probability that $n-1$ players do not achieve m successes by time t is

[13] Please refer to Casella and Berger (2002) for details.

$$\prod_{j \in N \setminus i} \int_t^{\infty} \frac{g_j(x_j)^m \delta^m}{(m-1)!} t^{m-1} \exp^{-g_j(x_j)\delta t} \, dt = \prod_{j \in N \setminus i} \sum_{h=0}^{m-1} \frac{g_j(x_j)^h \delta^h t^h}{h!}$$

Using Eqns. (3.12) and (3.15), the probability Player i wins a given round is

$$\text{Prob}(\rho = i) \quad = \quad \int_0^T \frac{g_i(x_i)^m \delta^m}{(m-1)!} \exp^{-g_i(x_i)\delta t} \, t^{m-1} \prod_{j \in N \setminus i} \sum_{h=0}^{m-1} \frac{g_i(x_i)^h \delta^h t^h}{h!} dt.$$

Combining this equation with Eqn. (3.11) and slightly rearranging gives the desired result. □

Using Lemma 3, we can now determine the overall probability that Player i wins the contest.

Lemma 4. *Define $\mathbb{P}_i$ as the probability that Player i wins the contest. With each round having time duration, T, and a threshold potential of m,*

$$\mathbb{P}_i \quad = \quad \frac{\alpha_i g_i(x_i)^m \int_0^T \exp^{-g_i(x_i)\delta t} \, t^{m-1} \prod_{j \in N \setminus i} \sum_{h=0}^{m-1} \frac{g_i(x_i)^h \delta^h t^h}{h!} dt}{\sum_{l \in N} \alpha_l g_l(x_l)^m \int_0^T \exp^{-g_l(x_l)\delta t} \, t^{m-1} \prod_{j \in N \setminus l} \sum_{h=0}^{m-1} \frac{g_l(x_l)^h \delta^h t^h}{h!} dt}$$

Proof. As in the proof of Lemma 2, note first that if $\sum_{l=1}^{n} P_l^W \neq 0$ then

$$\mathbb{P}_i \quad = \quad \frac{P_i^W}{\sum_{l=1}^{n} P_l^W} \tag{3.15}$$

Substituting Eqn. (3.10) into Eqn. (3.15) and canceling δ^m and $(m-1)!$ gives the desired result. □

Finally, we can use Lemma 4 to prove the following result of the paper.

Theorem 10. *Given the underlying model described in Section 3.3.1, the limit as δ approaches zero of the probability that Player i wins the contest is the ratio-form contest success function:*

$$\lim_{\delta \to 0} \mathbb{P}_i \quad = \quad \frac{\alpha_i g_i(x_i)^m}{\sum_{l=1}^{n} \alpha_l g_l(x_l)^m} \equiv \frac{f_i(e_i)}{\sum_{l=1}^{n} f_l(e_l)}$$

Proof. With $g_i(x_i)$ bounded,

$$\begin{aligned}
&\lim_{\delta \to 0} \frac{\alpha_i g_i(x_i)^m \int_0^T \exp^{-g_i(x_i)\delta t} t^{m-1} \prod_{j \in N \setminus i} \sum_{h=0}^{m-1} \frac{g_i(x_i)^h \delta^h t^h}{h!} dt}{\sum_{l \in N} \alpha_l g_l(x_l)^m \int_0^T \exp^{-g_l(x_l)\delta t} t^{m-1} \prod_{j \in N \setminus l} \sum_{h=0}^{m-1} \frac{g_l(x_l)^h \delta^h t^h}{h!} dt} \\
&= \frac{\lim_{\delta \to 0} \alpha_i g_i(x_i)^m \int_0^T \exp^{-g_i(x_i)\delta t} t^{m-1} \prod_{j \in N \setminus i} \sum_{h=0}^{m-1} \frac{g_i(x_i)^h \delta^h t^h}{h!} dt}{\lim_{\delta \to 0} \sum_{l \in N} \alpha_l g_l(x_l)^m \int_0^T \exp^{-g_l(x_l)\delta t} t^{m-1} \prod_{j \in N \setminus l} \sum_{h=0}^{m-1} \frac{g_l(x_l)^h \delta^h t^h}{h!} dt} \qquad (3.16)
\end{aligned}$$

Define $\frac{1}{z} \equiv \delta$ so that

$$\begin{aligned}
&\lim_{\delta \to 0} \alpha_i g_i(x_i)^m \int_0^T \exp^{-g_i(x_i)\delta t} t^{m-1} \prod_{j \in N \setminus i} \sum_{h=0}^{m-1} \frac{g_i(x_i)^h \delta^h t^h}{h!} dt \\
&= \lim_{z \to \infty} \alpha_i g_i(x_i)^m \int_0^T \exp^{-g_i(x_i)\frac{1}{z}t} t^{m-1} \prod_{j \in N \setminus i} \sum_{h=0}^{m-1} \frac{g_i(x_i)^h \frac{1}{z}^h t^h}{h!} dt
\end{aligned}$$

Note for all values of z,

$$\alpha_i g_i(x_i)^m \int_0^T \exp^{-g_i(x_i)\frac{1}{z}t} t^{m-1} \prod_{j \in N \setminus i} \sum_{h=0}^{m-1} \frac{g_i(x_i)^h \frac{1}{z}^h t^h}{h!}$$

is a real-valued measurable function uniformly bounded by

$$\alpha_i g_i(x_i)^m \int_0^T \exp^{-g_i(x_i)t} t^{m-1} \prod_{j \in N \setminus i} \sum_{h=0}^{m-1} \frac{g_i(x_i)^h t^h}{h!}.$$

Thus, the Bounded Convergence Theorem implies that

$$\begin{aligned}
&\lim_{z\to\infty} \alpha_i g_i(x_i)^m \int_0^T \exp^{-g_i(x_i)\frac{1}{z}t}\, t^{m-1} \prod_{j\in N\setminus i} \sum_{h=0}^{m-1} \frac{g_i(x_i)^h \frac{1}{z}^h t^h}{h!} dt \\
= \;& \alpha_i g_i(x_i)^m \int_0^T \lim_{z\to\infty} \exp^{-g_i(x_i)\frac{1}{z}t}\, t^{m-1} \prod_{j\in N\setminus i} \sum_{h=0}^{m-1} \frac{g_i(x_i)^h \frac{1}{z}^h t^h}{h!} dt \\
= \;& \frac{T^m \alpha_i g_i(x_i)^m}{m}
\end{aligned}$$

Using identical logic on the denominator of Eqn. (3.16), canceling the $\frac{T^m}{m}$ terms, and combining the above gives the desired result. □

Note that when $m = 1$, $T = \infty$, the continuous model with absolute threshold is equivalent to the patent race specified in Baye and Hoppe (2003).

3.4 Discrete Model with Relative Threshold

3.4.1 Model

In an alternate model, players again compete in a series of rounds. In this scenario, each player attempts to hit a target during each turn of the round. If, at the end of the round, the difference between one player's number of successful hits and the second most successful player's number of successful hits weakly exceeds the threshold potential, that player has the ability to win the contest. If no player surpasses the relative threshold, the number of successful hits "resets", and a new round is played.

Formally, let $N = \{1, 2, \ldots, n\}$ be the set of players who may participate in a contest. The contest consists of (potentially) multiple rounds, with each round having $t \in \mathbb{Z}^+$ turns. The rules are as follows:

1. For every turn in a round, every player "makes an attempt". Player i makes a successful attempt with probability

$$P_i \equiv g_i(x_i)\delta. \tag{3.17}$$

where $g_i(x_i)$ and δ are defined as in Section 1.

2. After the round is complete, a player is said to reach the threshold potential, m, if he has m or more successful attempts than the second most successful player during the round. Note this threshold is a relative benchmark representing how much better one player must perform compared to the other players.

3. After the round is complete, if any player reaches the threshold potential, he wins the contest with probability $\alpha_i \in (0, 1]$. This probability incorporates personal characteristics of the individual players.

4. If no player wins at the end of the round, a new round is played.

Thus, the underlying contest is one in which the players are competing against each other, attempting to demonstrate superiority in a given round. This model incorporates the concept of an "evaluation period" during which players are appraised.[14] When appropriate, the threshold level can be interpreted as an explicit satisficing requirement.

When considering potential applications of this model, it is again important to note that successful attempts in a given round can not transfer to future rounds. With a relative threshold, this implies a player who is extremely weak in a given round is not punished at any point in the future as long as two other players in the current round ensure neither exceeds the threshold. Thus, if there is not a natural separation during which previous successes can be "forgotten", this model should be used with caution.

[14] The results do not change if the "evaluation period" interpretation is discarded but the relative threshold is maintained. Specifically, the same results are achieved if the first player to have m more successful attempts than any other player is said to reach the threshold potential.

3.4.2 Results

I now show that under the contest described in Section 3.4.1, the limit as the technical boundary, δ, approaches zero of the probability a given player wins the contest equals the ratio contest success function specified by Eqn. 3.1.

Lemma 5. *Define P_i^W as the probability that Player i wins a round. With t turns in a round and a threshold potential of m,*

$$P_i^W = \alpha_i \sum_{h=m}^{t} \binom{t}{h} P_i^h (1-P_i)^{t-h} \Big[\prod_{j \in N \setminus i} \Big[\sum_{k=0}^{h-m} \binom{t}{k} P_j^k (1-P_j)^{t-k} \Big] \Big] \tag{3.18}$$

Proof. Define $\Upsilon^i \in \{0,1\}^t$ as the outcome of Player i's attempts in a given round. $\Upsilon_r^i = 1$ if Player i is successful in his rth attempt, and $\Upsilon_r^i = 0$ otherwise. Define the difference between the number of successful attempts made by Player i and the number of successful attempts made by the most successful remaining player as

$$\Delta_i \equiv \#\big\{ \cup_{r=1}^{t} \{\Upsilon_r^i | \Upsilon_r^i = 1\} \big\} - \max_{j \in N \setminus i} \#\big\{ \cup_{r=1}^{t} \{\Upsilon_r^j | \Upsilon_r^j = 1\} \big\} \tag{3.19}$$

To determine P_i^W, we will first determine the probability that Player i has m or more successful attempts than the second most successful player, then determine Player i's probability of winning the round by noting that

$$\begin{aligned} P_i^W &= \text{Prob}(\text{Player } i \text{ wins a round} | \Delta_i \geq m) \text{Prob}(\Delta_i \geq m) \\ &= \alpha_i \text{Prob}(\Delta_i \geq m) \end{aligned} \tag{3.20}$$

Disjointly partitioning the underlying set and using the definition of conditional probabilities, we find the probability that Player i has m or more successful attempts than the second most successful player equals

$$\text{Prob}(\Delta_i \geq m) = \sum_{h=1}^{t} \text{Prob}\big(\#\{ \cup_{r=1}^{t} \{\Upsilon_r^i | \Upsilon_r^i = 1\}\} = h\big)\text{Prob}\big(\Delta_i \geq m | \#\{ \cup_{r=1}^{t} \{\Upsilon_r^i | \Upsilon_r^i = 1\}\} = h\big)$$

or, recognizing $\text{Prob}\big(\Delta_i \geq m | \#\{ \cup_{r=1}^{t} \{\Upsilon_r^i | \Upsilon_r^i = 1\}\} = h\big) = 0$ when $h < m$

$$\text{Prob}(\Delta_i \geq m) = \sum_{h=m}^{t} \text{Prob}\big(\#\{ \cup_{r=1}^{t} \{\Upsilon_r^i | \Upsilon_r^i = 1\}\} = h\big)\text{Prob}\big(\Delta_i \geq m | \#\{ \cup_{r=1}^{t} \{\Upsilon_r^i | \Upsilon_r^i = 1\}\} = h\big) \quad (3.21)$$

Using Eqn. 3.19, further note

$$\text{Prob}(\Delta_i \geq m | \#\{ \cup_{r=1}^{t} \{\Upsilon_r^i | \Upsilon_r^i = 1\}\} = h\big) = \text{Prob}\Big(h - \max_{j \in N \setminus i} \#\{ \cup_{r=1}^{t} \{\Upsilon_r^j | \Upsilon_r^j = 1\}\} \geq m\Big)$$

which, after expanding the max term, implies

$$\text{Prob}(\Delta_i \geq m | \#\{ \cup_{r=1}^{t} \{\Upsilon_r^i | \Upsilon_r^i = 1\}\} = h\big) = \prod_{j \in N \setminus i} \text{Prob}\Big(h - \#\{ \cup_{r=1}^{t} \{\Upsilon_r^j | \Upsilon_r^j = 1\}\} \geq m\Big)$$

or, rearranging,

$$\text{Prob}(\Delta_i \geq m | \#\{ \cup_{r=1}^{t} \{\Upsilon_r^i | \Upsilon_r^i = 1\}\} = h\big) = \prod_{j \in N \setminus i} \text{Prob}\Big(\#\{ \cup_{r=1}^{t} \{\Upsilon_r^j | \Upsilon_r^j = 1\}\} \leq h - m\Big). \quad (3.22)$$

Disjointly partitioning the right-hand probability in Eqn. (3.22) further implies

$$\text{Prob}(\Delta_i \geq m | \#\{ \cup_{r=1}^{t} \{\Upsilon_r^i | \Upsilon_r^i = 1\}\} = h\big) = \prod_{j \in N \setminus i} \sum_{k=0}^{h-m} \text{Prob}\Big(\#\{ \cup_{r=1}^{t} \{\Upsilon_r^j | \Upsilon_r^j = 1\}\} = k\Big). \quad (3.23)$$

Combining Eqns. (3.20), (3.21), and (3.23), we find

$$P_i^W = \alpha_i \sum_{h=m}^{t} \text{Prob}\big(\#\{\cup_{r=1}^t \{\Upsilon_r^i | \Upsilon_r^i = 1\}\} = h\big) \prod_{j\in N\setminus i} \sum_{k=0}^{h-m} \text{Prob}\Big(\#\{\cup_{r=1}^t \{\Upsilon_r^j | \Upsilon_r^j = 1\}\} = k\Big).$$

Finally, recognizing $\text{Prob}\big(\#\{\cup_{r=1}^t \{\Upsilon_r^i | \Upsilon_r^i = 1\}\} = h\big)$ follows the Bernoulli distribution, we obtain the desired result,

$$P_i^W = \alpha_i \sum_{h=m}^{t} \binom{t}{h} P_i^h (1-P_i)^{t-h} \Bigg[\prod_{j\in N\setminus i} \Bigg[\sum_{k=0}^{h-m} \binom{t}{k} P_j^k (1-P_j)^{t-k} \Bigg] \Bigg]$$

□

Using Lemma 5, we can now determine the overall probability that Player i wins the contest.

Lemma 6. *Define $\mathbb{P}_i$ as the probability that Player i wins the contest. With t turns in a round and a threshold potential of m,*

$$\mathbb{P}_i = \frac{\alpha_i \sum_{h=m}^{t} \binom{t}{h} g_i(x_i)^h \delta^{h-m} (1-g_i(x_i)\delta)^{t-h} \Big[\prod_{j\in N\setminus i} \Big[\sum_{k=0}^{h-m} \binom{t}{k} g_j(x_j)^k \delta^k (1-g_j(x_j)\delta)^{t-k} \Big] \Big]}{\sum_{l=1}^{n} \alpha_l \sum_{h=m}^{t} \binom{t}{h} g_l(x_l)^h \delta^{h-m} (1-g_l(x_l)\delta)^{t-h} \Big[\prod_{j\in N\setminus l} \Big[\sum_{k=0}^{h-m} \binom{t}{k} g_j(x_j)^k \delta^k (1-g_j(x_j)\delta)^{t-k} \Big] \Big]}$$

Proof. As in the proof of Lemma 6, note first that if $\sum_{l=1}^{n} P_l^W \neq 0$ then

$$\mathbb{P}_i = \frac{P_i^W}{\sum_{l=1}^{n} P_l^W} \tag{3.24}$$

Substituting Eqn. (3.18) into Eqn. (3.24) and expanding the P_i's per Eqn. (3.17),

$$\mathbb{P}_i = \frac{\alpha_i \sum_{h=m}^{t} \binom{t}{h} g_i(x_i)^h \delta^h (1-g_i(x_i)\delta)^{t-h} \left[\prod_{j\in N\setminus i} \left[\sum_{k=0}^{h-m} \binom{t}{k} g_j(x_j)^k \delta^k (1-g_j(x_j)\delta)^{t-k} \right]\right]}{\sum_{l=1}^{n} \alpha_l \sum_{h=m}^{t} \binom{t}{h} g_l(x_l)^h \delta^h (1-g_l(x_l)\delta)^{t-h} \left[\prod_{j\in N\setminus l} \left[\sum_{k=0}^{h-m} \binom{t}{k} g_j(x_j)^k \delta^k (1-g_j(x_j)\delta)^{t-k} \right]\right]}$$

Factoring out δ^m and canceling gives the desired result. □

Finally, we can use Lemma 6 to prove the following result of the paper.

Theorem 11. *Given the underlying model described in Section 3.2.1, the limit as δ approaches zero of the probability that Player i wins the contest is the ratio-form contest success function:,*

$$\lim_{\delta\to 0} \mathbb{P}_i = \frac{\alpha_i g_i(x_i)^m}{\sum_{l=1}^{n} \alpha_l g_l(x_l)^m} \equiv \frac{f_i(x_i)}{\sum_{l=1}^{n} f_l(x_l)}$$

Proof. The Theorem is a direct result of taking the limit of $\mathbb{P}_i$ as defined in Lemma 6. In particular,

$$\lim_{\delta\to 0} \mathbb{P}_i = \frac{\lim_{\delta\to 0} \alpha_i \sum_{h=m}^{t} \binom{t}{h} g_i(x_i)^h \delta^{h-m} (1-g_i(x_i)\delta)^{t-h} \left[\prod_{j\in N\setminus i} \left[\sum_{k=0}^{h-m} \binom{t}{k} g_j(x_j)^k \delta^k (1-g_j(x_j)\delta)^{t-k} \right]\right]}{\lim_{\delta\to 0} \sum_{l=1}^{n} \alpha_l \sum_{h=m}^{t} \binom{t}{h} g_l(x_l)^h \delta^{h-m} (1-g_l(x_l)\delta)^{t-h} \left[\prod_{j\in N\setminus l} \left[\sum_{k=0}^{h-m} \binom{t}{k} g_j(x_j)^k \delta^k (1-g_j(x_j)\delta)^{t-k} \right]\right]} \tag{3.25}$$

The limit of the numerator equals

$$\begin{aligned}
&\lim_{\delta\to 0}\alpha_i\sum_{h=m}^{t}\binom{t}{h}g_i(x_i)^h\delta^{h-m}(1-g_i(x_i)\delta)^{t-h}\Big[\prod_{j\in N\setminus i}\Big[\sum_{k=0}^{h-m}\binom{t}{k}g_j(x_j)^k\delta^k(1-g_j(x_j)\delta)^{t-k}\Big]\Big]\\
&=\alpha_i\sum_{h=m}^{t}\lim_{\delta\to 0}\binom{t}{h}g_i(x_i)^h\delta^{h-m}(1-g_i(x_i)\delta)^{t-h}\Big[\prod_{j\in N\setminus i}\lim_{\delta\to 0}\Big[\sum_{k=0}^{h-m}\binom{t}{k}g_j(x_j)^k\delta^k(1-g_j(x_j)\delta)^{t-k}\Big]\Big]\\
&=\alpha_i\sum_{h=m}^{t}\lim_{\delta\to 0}\binom{t}{h}g_i(x_i)^h\delta^{h-m}(1-g_i(x_i)\delta)^{t-h}\Big[\prod_{j\in N\setminus i}\Big[\lim_{\delta\to 0}\binom{t}{0}g_j(x_j)^0\delta^0(1-g_j(x_j)\delta)^t\\
&\qquad\qquad+\lim_{\delta\to 0}\sum_{k=1}^{h-m}\binom{t}{k}g_j(x_j)^k\delta^k(1-g_j(x_j)\delta)^{t-k}\Big]\Big]\\
&=\alpha_i\sum_{h=m}^{t}\lim_{\delta\to 0}\binom{t}{h}g_i(x_i)^h\delta^{h-m}(1-g_i(x_i)\delta)^{t-h}\\
&=\alpha_i\lim_{\delta\to 0}\binom{t}{m}g_i(x_i)^m\delta^0(1-g_i(x_i)\delta)^{t-m}+\sum_{h=m+1}^{t}\lim_{\delta\to 0}\binom{t}{h}g_i(x_i)^h\delta^{h-m}(1-g_i(x_i)\delta)^{t-h}\\
&=\alpha_i\binom{t}{m}g_i(x_i)^m
\end{aligned}$$

where $\sum_{k=1}^{0}\equiv 0$. Using identical logic on the denominator and substituting into Eqn. (3.25),

$$\begin{aligned}
\lim_{\delta\to 0}\mathbb{P}_i &= \frac{\alpha_i\binom{t}{m}g_i(x_i)^m}{\sum_{l=1}^{n}\alpha_l\binom{t}{m}g_l(x_l)^m}\\
&= \frac{\alpha_i g_i(x_i)^m}{\sum_{l=1}^{n}\alpha_l g_l(x_l)^m}
\end{aligned}$$

□

3.5 Continuous Model with Relative Threshold

3.5.1 Model

In the continuous formulation of the relative threshold model, players again compete in a series of rounds, each of time duration, T. As in the discrete case, players attempt to hit targets throughout the round. If, at the end of the round, the difference between one player's number of successful hits and the second most successful player's number of successful hits weakly exceeds the threshold potential, that player has the ability to win the contest. If no player surpasses the relative threshold, the number of successful hits "resets", and a new round is played.

Formally, let $N = \{1, 2, \ldots, n\}$ be the set of players who may participate in a contest. The contest consists of (potentially) multiple rounds, with each round having time duration, T. The rules are as follows:

1. As in the continuous model with absolute threshold, every player's number of successes follows a Poisson process. In particular, the time, t, at which each sequential independent success occurs is a random variable having exponential density,

$$P_i \equiv g_i(x_i)\delta \exp^{-g_i(x_i)\delta t}.$$

   As before, the probability that an individual is successful at or before time t equals

$$\int_0^t P_i dt = 1 - \exp^{-g_i(x_i)\delta t},$$

   with $g_i(x_i)$ and δ having equivalent interpretations.

2. After the round is complete, a player is said to reach the threshold potential, m, if he has m or more successful attempts than the second most successful player during the round. As with the discrete case, this threshold is a relative benchmark representing how much better one player must perform compared to the other players.

3. After the round is complete, if any player reaches the threshold potential, he wins the contest with probability $\alpha_i \in (0, 1]$. This probability incorporates personal characteristics of the individual players.

4. If no player wins at the end of the round, a new round is played.

3.5.2 Results

I now show that under the continuous model with relative threshold described in Section 3.5.1, the limit as the technical boundary, δ, approaches zero of the probability a given player wins the contest equals the ratio contest function specified in Eqn. 3.1.

Lemma 7. *Define P_i^W as the probability that Player i wins a round. With each round having time duration, T, and a threshold potential of m,*

$$P_i^W = \alpha_i \sum_{h=m}^{\infty} \frac{\exp^{-g_i(x_i)\delta T} g_i(x_i)^h \delta^h T^h}{h!} \prod_{j\in N\setminus i} \sum_{k=0}^{h-m} \frac{\exp^{-g_j(x_j)\delta T} g_j(x_j)^k \delta^k T^k}{k!} \tag{3.26}$$

Proof. Note with the defined underlying Poisson process, the probability of achieving h successes in a time period of length, T, follows the Poisson distribution,

$$\text{Prob}(h \text{ successes}) = \frac{\exp^{-g_i(x_i)\delta T} g_i(x_i)^h \delta^h T^h}{h!}$$

Using this probability instead of the Bernoulli distribution and following identical logic to Lemma 5 gives the desired results. □

Using Lemma 7, we can now determine the overall probability that Player i wins the contest.

Lemma 8. *Define $\mathbb{P}_i$ as the probability that Player i wins the contest. With each round having time duration, T, and a threshold potential of m,*

$$\mathbb{P}_i = \frac{\alpha_i \sum_{h=m}^{\infty} \exp^{-g_i(x_i)\delta T} g_i(x_i)^h \frac{\delta^h T^h}{h!} \prod_{j\in N\setminus i} \sum_{k=0}^{h-m} \frac{\exp^{-g_j(x_j)\delta T} g_j(x_j)^k \delta^k T^k}{k!}}{\sum_{l=1}^{n} \alpha_l \sum_{h=m}^{\infty} \exp^{-g_l(x_l)\delta T} g_l(x_l)^h \frac{\delta^h T^h}{h!} \prod_{j\in N\setminus l} \sum_{k=0}^{h-m} \frac{\exp^{-g_j(x_j)\delta T} g_j(x_j)^k \delta^k T^k}{k!}}$$

Proof. As in the proof of Lemma 2, note first that if $\sum_{l=1}^{n} P_l^W \neq 0$ then

$$\mathbb{P}_i = \frac{P_i^W}{\sum_{l=1}^{n} P_l^W} \tag{3.27}$$

Substituting Eqn. (3.26) into Eqn. (3.27) gives the desired result. □

Finally, we can use Lemma 8 to prove the final result of the paper.

Theorem 12. *Given the underlying model described in Section 3.5.1, the limit as δ approaches zero of the probability that Player i wins the contest is the ratio-form contest success function:*

$$\lim_{\delta \to 0} \mathbb{P}_i = \frac{\alpha_i g_i(x_i)^m}{\sum_{l=1}^{n} \alpha_l g_l(x_l)^m} \equiv \frac{f_i(x_i)}{\sum_{l=1}^{n} f_l(x_l)}$$

As the proof follows Theorem 11 very closely, it is omitted.

3.6 Uniformly Distributed Spatial Game

Both discrete models are especially applicable to a class of contests which contains an underlying spatial component. In particular, let τ, a point on the real number line, $\Re$,[15] signify a strategically relevant "target" that must be "hit" in order for a player to have a chance of winning the contest. In military contests, for instance, this point could be a specific tactical target. Every attempt consists of "destroying" the interval, $[\delta_L, \delta_R]$, where $\delta_R - \delta_L = \delta$. If τ is within the interval, the attempt is successful. While the location of τ is known, the ability to hit τ is imperfect. Specifically, a player can only ensure δ_L is located within the *precision interval*, $\tau_\delta \equiv [\tau - \frac{1}{2g_i(x_i)}, \tau + \frac{1}{2g_i(x_i)}]$, where $g_i(x_i)$ is the precision function defined in Section 3.2.1. The random location of δ_L within τ_δ is uniformly distributed. Figure 3.1 represents this spatial contest.

[15] Expanding the model to multiple dimensions does not change the results.

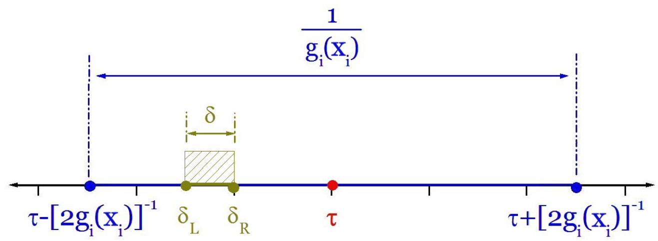

Figure 3.1: Spatial Contest

In this framework, $\frac{1}{g_i(x_i)}$ is the production function transforming effort into precision. In military contests, for example, this can be interpreted as the function mapping effort into weapon precision. Note the probability of hitting the target in this environment is $g_i(x_i)\delta$, as specified in the models presented in Sections 3.2.1 and 3.4.1.

3.7 Examples

3.7.1 Grossman's Model of Insurrections

Consider Grossman's (1991) model of insurrections in which he examines a model in which peasant families can divide their time between production, soldiering, and participating in an insurrection. Soldiers and insurrectionists engage in conflict to determine who receives government profits. Grossman specifically assumes the probability the insurrection is successful, β, is in the form

$$\beta = \frac{I^{1-\theta}}{S^{\sigma} + I^{1-\theta}}, 0 \leq \theta \leq 1, 0 < \sigma < 1$$

where I is the fraction of peasant time allocated to participating in the insurrection and S is the fraction of peasant time allocated to soldiering.[16]

Within the context of the discrete model with absolute threshold, this has a natural interpretation. In particular, assume the government and the insurrection each has a pivotal leader. Assume if the government leader is assassinated first, the insurrectionists place their

[16] Alternatively, I can be interpreted as the percentage of peasants who are insurgents and S as the percentage of peasants who are soldiers.

leader in power. Likewise, if the insurrection leader is assassinated first, the insurrection is defeated. Assume weapon technology is such that a solider "in range" of a target still has a small likelihood of successfully killing the target.[17] Also assume the government and the insurgents have access to different military technologies, so the probability of the insurgent getting in range of the government target is $I^{1-\theta}$ and the probability of the government getting in range of the insurgent target is S^{σ}. Theorem 9 implies the probability that the insurrection will be successful approaches $\beta = \frac{I^{1-\theta}}{S^{\sigma}+I^{1-\theta}}$.

Note this is not the only conflict interpretation for Grossman's CSF. An alternative interpretation is strictly spatial. Instead of assuming each side has a pivotal leader, assume each side has a pivotal physical target located at τ.[18] The strength of weapons is exogenous (and small), but the precision of the weapon can be increased via a technology of innovation. The insurgency can ensure their weapon lands within $\frac{I^{\theta-1}}{2}$ of the government's pivotal target, with the exact location of the strike a random variable uniformly distributed along the interval $[\tau - \frac{I^{\theta-1}}{2}, \tau + \frac{I^{\theta-1}}{2}]$. Likewise, the government's precision technology ensures their weapon lands within $\frac{S^{-\sigma}}{2}$ of the insurgent target. As before, the probability the insurrectionist weapon will destroy the government's pivotal target first approaches $\beta = \frac{I^{1-\theta}}{S^{\sigma}+I^{1-\theta}}$.

3.7.2 Model of Political Conflict

Utilizing the spatial interpretation of the model is not restricted to actual physical conflict. Consider the voting model of Chapter 2, in which voters probabilistically vote for each candidate who does not reject their ideal point according to the CSF:

$$s_i(A_i) = \frac{\frac{\alpha_i}{(A_i)^m}}{\sum_{l=1}^{n} \frac{\alpha_l}{(A_l)^m}}, i \in \{1, \ldots, n\}$$

where α_i represents each candidate's personal ability to sway voters, and A_i represents the

[17] In models of modern warfare, this assumption becomes suspect. However, the assumption of a continuous CSF also becomes suspect, since in many cases technology is such that victory is ensured via complete annihilation.

[18] Alternatively, assume each side has m pivotal targets that must be hit to ensure success.

ambiguity level of each candidate. Assuming that the two candidates are competing for voters whose ideologies range from 1 to 7, as is standard in the American National Election Survey, the ambiguity level is defined as the location of the candidate's most conservative stance minus the location of his most liberal stance.

As an interpretation of this CSF, assume that once a voter determines that neither candidate has rejected her ideal point, she then "pays attention" to determine which candidate is more likely to follow through. Voters are imperfectly informed, and each receives her information through the news media. Assume the news media broadcasts news segments, which randomly detail a specific policy stance for a specific candidate.[19] Once a given candidate is selected for discussion, the particular issue that is deemed newsworthy in any particular day is uniformly selected from the candidate's possible campaign stances. Even if a candidate is ambiguous about a certain issue, the media covers the fact that the particular issue is part of the candidate's campaign stance. Each news segment covers the issues within a policy interval of size δ.

In an effort to reduce information costs, voters use a partially-informative heuristic that economizes on the effort needed to reach a decision subject to the requirement that the heuristic is "generally" correct. This coincides with the findings of Lau and Redlawsk (2006: 138) who find voters are more inclined to use heuristics when candidate ideologies are similar. Assume the specific heuristic is a satisficing heuristic, in which each voter receives information until one of the candidates has passed a certain threshold, at which point the voter makes her decision. Specifically, assume each voter watches news segments until they have observed one of the candidates espouse her ideal point m more times than the other candidate times in a short enough time interval. As an illustrative (and entirely arbitrary) example, assume a pro-life voter watches the news every day, and if she hears a candidate state "I am against abortion" three more times than his opponent, she will believe the candidate with probability α_i and decide to vote for that candidate. If, however, neither candidate fulfills this requirement, the voter will watch the news the next day in the hopes one of the candidates satisfies her

[19]The random element of the news segments assumes a voter is watching an unbiased media report, an assumption based on the fact she is attempting to formulate a correct issue-based decision.

voting requirements. Note because the voter is only receiving additional information if both candidates have not rejected her ideal point, the voter will never hear a candidate reject her stance at this point. She instead is only trying to determine which is more sincere.

Under this heuristic, for a given ideal point and candidate, the probability Candidate i's stance on the point is discussed by the news media in a given broadcast is $\frac{\delta}{A_i}$. Assuming each candidate is equally likely to be discussed, Theorem 11 implies the probability of a voter voting for Candidate i approaches $\frac{\frac{\alpha_i}{(A_i)^m}}{\sum_{l=1}^{n} \frac{\alpha_l}{(A_l)^m}}$.[20]

3.8 Competition Costs

An important caveat to using this interpretation of the ratio-form CSF in economic models is that numerous rounds will likely be played before the contest concludes. Specifically, as $\delta \to 0$, the probability that the game ends after any $M \in \mathbb{Z}^+$ rounds approaches 1. Thus, if there is any nontrivial cost to participating in a round of conflict, the expected cost of the contest becomes unbounded as the technological boundary approaches zero. In some scenarios, this caveat will prove insurmountable. However, in many scenarios the cost of a round can naturally be assumed to relate to the size of δ. Defining c as the cost of a single round, a sufficient condition for the cost of the contest to remain finite is $\delta \to 0 \Rightarrow c \to 0$. For example, in the voting model discussed above, if the cost of watching a news item converges to zero as the length of time of the news item converges to zero, then as $\delta \to 0$, the expected cost of the conflict will converge to zero. In a military conflict, $\delta \to 0$ implies that weapons are not sufficiently destructive. If these nondestructive weapons are extremely cheap,[21] we can again avoid having costs explode.

It is useful to note this caveat and approach is similar in spirit to Binmore, Rubinstein, and

[20]It should be noted that in Chapter 2, I perform comparative statics on m, which is not appropriate in this context given the integer nature of m. If, instead, I assume the probability a particular issue is newsworthy is random, and each media outlet chooses to discuss Candidate i with probability $\frac{\delta}{A_i^m}$, I will find an equivalent result. This interpretation captures the concept that as politicians become more ambiguous, news stations are unwilling to attribute to them a particular issue stance.

[21]For instance, in contests of hunter-gather societies such as Baker (2003), it is natural that weapons are assumed to be extremely primitive.

Wolinsky's (1986) justification for the Nash Bargaining solution as a limit result of Rubinstein's (1982) alternating offers game. Specifically, the alternating offers game only approaches the Nash bargaining solution if the time necessary to make an offer is very small. Likewise, the model proposed here is only valid if the cost necessary to make an attempt is very small.

3.9 Conclusion

This chapter proposes a new interpretation for the ratio-form contest success function that models a contest as a game in which the players exert effort in an attempt to increase their probability of successfully performing some action. When enough successful attempts are made in a specific period, the player passes a threshold allowing them to win the full contest. As the limit of a the probability of a successful attempt converges to zero, the probability of winning the contest converges to the ratio-form CSF.

This approach contributes to the literature in three fundamental ways. First, if offers a justification for the ratio-form CSF in non-bargaining scenarios. While Corchón and Dahm's (2008) Nash-bargaining justification for ratio-form CSF's is extremely valid in bargaining scenarios, there exist economic scenarios that do not fall under their framework. In these cases, this alternate approach can be used when there is a small probability of success in any round, the contest cost converges to zero as the probability of success converges to zero, and a "race-to-threshold" interpretation of the underlying contest exists.

Second, my approach introduces a spatial interpretation of the ratio-form CSF. This can prove useful in modeling actual physical conflict, but also holds use for other economic applications. Using a similar framework, areas can be investigated which incorporate the concept of an ideological space with preferred ideal points. For instance, ratio-form CSF's in legal contexts such as Katz (1988) can be interpreted as lawyers competing to "hit" the ideal argument that will convince a judge/juror. If the space of possible arguments is spatial in nature, ratio-form CSF's can be justified using this approach.

Lastly, instead of viewing the effectivity function, $f_i(x_i)$, as a measure of utility as in the random utility formulation, I view it as related to an actual production function mapping effort to precision. Thus, its returns to scale has a productive meaning, and by appropriately defining the threshold limit, this production function can be given appropriate characteristics. This admits a productive interpretation of necessary axiomatic conditions imposed on effectivity functions such as those required by Rai and Sarin (2007) to induce “mixed homogeneity” in the CSF.

As a final note, this chapter should be viewed as complementary to Corchón and Dahm (2008) and Fu and Lu (2008), and should be viewed as only one approach to modeling CSF’s. Hopefully, additional foundations will arise justifying the use of ratio-form CSF’s in scenarios where Corchón and Dahm (2008), Fu and Lu (2008), and my assumptions are not appropriate.

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