Essays on Asset Pricing.

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A dissertation submitted to the faculty of the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Kenan-Flagler Business School.

Chapel Hill

2008

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Abstract

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Essays on Asset Pricing

(Under the direction of J. Conrad and M. Gultekin)

I am proposing a simple theory in which an investor distinguishes between positive and negative deviations in the portfolio value for risk estimation. The risk of the portfolio is defined as the average futile return on the portfolio. The investor tries to create such a portfolio that the unconditional average return is as high as possible while conditional (negative) return on the portfolio is as small (in absolute terms) as possible. I am not making any assumptions about the possible distribution of the stock prices and returns. However, assuming the normal (Gaussian) mutual distribution the solution reduces to the standard CAPM solution.

To my mother Svetlana and my father Georii

To my wife Anjelika

To my daughter Alesja and my son Yanka

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1 Introduction.

Recent papers in the literature have focused on the heterogeneity of investors. Although the heterogeneity of investors in terms of risk-aversion and discount factors may be small, personal wealth can differ significantly. And, whereas the amount invested by any poor investor might be small, there can be many such investors, so that their combined investment is not small and may affect average consumption and the allocation of wealth into risky and risk-free securities. Consequently, their behavior can affect prices in the market.

Merton's theory [1] -[3] predicts a linear relation between wealth and decision rules. In Merton's world an agent's consumption is linear in wealth and a fraction of the agent's wealth contributed to the risky asset is constant. However, if one relaxes some constraints, decisions are no longer linear in wealth. This non-linearity can have grave consequences for the notion of a representative agent. If one forms the representative agent and empirically finds his consumption/investment decisions, these decisions would appear to be not optimal. That is, it would seem as if an average investor behaves non-rationally. In this paper I show that the relaxation of several of these constraints has the effect that optimal decisions are no longer linear in wealth. Consequently, in order to test the theory, one needs to integrate weighted decision rules over wealth using the distribution of wealth values for all investors.

One of the constraints I relax is an introduction of an option to declare a bankruptcy. When does a declaration of a bankruptcy improve the well-being of an investor? Does a possibility of such a declaration affect the investment and consumption decisions? I find that, as with any option, the option to declare the bankruptcy is valuable to the investor and may affect his investment behavior. It is reasonable to expect that the value of the option is higher if the wealth of the investor is relatively small, and the option's value decreases with an increase in the investor's wealth. Because this option is not tradable, the value of the option is different for different investors and depends upon the personal characteristics of the investor (such as risk-aversion, discount factor, and wealth). In the present article, I argue that the effect of such an option causes the distribution of wealth to have an effect on the consumption/investment behavior of average investor.

One can look at this problem from a different point of view. The investor's wealth affects the investor's bankruptcy option value and, therefore, the investor's behavior. For a wealthy investor, the value of the option to receive subsistence is less valuable compared to a poorer investor. As a result it is difficult, and might be inappropriate, to combine investors and form a representative agent. Indeed, consider, for example, N relatively poor investors and one wealthy investor, whose wealth is equal to the combined wealth of the N poor investors. The poor investors have relatively valuable bankruptcy options. As opposed to that, the wealthy investor has just one option with a relatively small value (smaller than the value of any option of an investor from the first group). As a result the wealth of small investors are much better protected and the poor investors as a group might behave much more aggressively than the wealthier investor. E.g. the anecdotal evidence suggests that the majority of lottery players are poor individuals who contribute a significant fraction of their wealth to a very risky investment (lottery); a solo investor with the same level of wealth as combined wealth of poor investors is unlikely to contribute the same amount into a lottery. This suggests that the investment decisions and therefore consumption decisions are not linear in wealth and the formation of the representative agent is not warranted. Instead what one needs to do is to consider each investor individually, find the consumption and investment rules individually and only later aggregate (e.g. integrate with a given wealth distribution of investors). Because the rules are not linear in wealth, the averaged decision may not necessary satisfy the optimality conditions. As a result, it looks as if the average (or representative) investor behaves non-optimally.

1.1 Literature

One of the more important and interesting problems of finance is the portfolio selection and consumption decisions under uncertainty. The classical approach to the problem is that of Merton (see Merton [1] -[3]). This approach assumes that a time-separable von Neumann-Morgenstern utility function over the agent's consumption represents the agent's preferences. The agent tries to maximize a specific functional which, in particular, can be represented as a weighted sum (or integral) of utilities at particular moments of time. The utility function is an increasing concave function but for the simplification of calculations in many papers the utility function is assumed to be of a hyperbolic absolute risk aversion (HARA) family function. For example, in Merton's paper the utility is either $\ln(c(t))$ or $\frac{c(t)^{\gamma}-1}{\gamma}$, $\gamma < 1$ (if $\gamma = 0$ it reduces to the Bernoulli utility function $\ln(c(t))$). However there are other choices inside of the HARA family for the utility function (for example, see Rubinstein [4]). The important contribution of the classical solution is that the consumption rate at any time is directly proportional to current wealth, and the investment decision is simple: a fixed fraction of wealth α is invested into the risky asset and the remainder is invested in the risk-free asset. Unfortunately, that solution also suggests that, under plausible values of the risk aversion coefficient γ, the variance of per capita consumption should be much larger than the observed value; this problem is known as the equity premium puzzle (e.g. see Mehra and Prescott [6]). There have been many attempts to explain this puzzle. Some of these attempts question the assumption of time separability of the utility function (e.g. [8] , [9]); others consider models such as a habit-formation model, while several authors look at behavioral issues.

In this paper I will present a very simple approach which allows one to explain some intriguing problems with observed data. The most important difference between the classical Merton model and the present paper is a change of boundary conditions. I am relaxing the assumption that an investor experiences infinitely large negative utility if his wealth drops to zero or below. In the real world nobody would stop consuming or die instantly as his wealth falls below zero. In the developed economies, there are different mechanisms to prevent such an event (consider, for examples, food stamps to poor individuals, charities' contributions, etc.). I will introduce all these mechanisms into the model by introducing a "safety net" in the form of guaranteed consumption goods if the weighted sum of wealth and income falls to the pre-specified value. The second difference is a consideration of a more general HARA utility function (e.g. Rubinstein [4]) of the form $\ln(\frac{c-c_{\min}}{c_n})$ or $\frac{\left(\frac{c-c_{\min}}{c_n}\right)^{\gamma}-1}{\gamma}$. The third important consideration is the income of the agent; I will consider it similar to the consideration in Merton [2] . All three considerations contribute to the explanations described below, although for different combinations of parameters the degree of the effect on the final result may be different.

Using this approach one can show, first, that there are plausible values of the risk aversion coefficient γ, such that the variance of per capita consumption is close to the observed values; that is one can potentially explain the equity premium puzzle using this approach.

Second, the proposed solution allows for consumption/investment decisions to be non-linear functions of wealth. As wealth decreases, the proportion of wealth consumed increases and the investment in the risky asset also increases. Consumption is no longer independent of the characteristics of the risky asset (such as the instantaneous rate of return or variance) even for logarithmic utility function. The investors behave as if their risk aversion coefficient γ decreases (from classical theory point of view). The simple model (in discrete time) will show that even risk averse (rational) investors will play a lottery with negative NPV under some conditions.

In order to understand the intuition behind the first result let's consider the behavior of the agent who knows that in the bad state he will not be left alone and the government will subsidize him/her with consumption goods. Responding to this, the agent will consume a little bit more (in comparison to the classical Merton's world), specifically preferring to consume more right now (and increase utility dramatically) and later switch to the subsidized consumption goods (hence utility does not fall to $-\infty$). So, even if wealth is small, the agent will consume more than the same agent in Merton's world. Consequently, large oscillations in stock value will cause relatively large oscillations in wealth, but relatively small oscillations in consumption. Although it is not important in particular considerations, some agents will receive (constant) subsidized consumption at some moment in time and, therefore, their volatility of consumption is virtually zero. That effect also decreases the overall volatility of the consumption by all investors.

On the other hand if wealth is very low and close to the value at which the agent can switch to subsistence, he/she will find it beneficial to consume the rest of available wealth in a short burst to boost utility, knowing that there is guaranteed utility awaiting him/her after the wealth drops to specific level. As a result, the agent's decision to consume depends non-linearly on the wealth. As we will later see, consumption also depends on the financial parameters of the risky and risk-free assets. Also, as wealth approaches the lowest possible level, the agent may find it beneficial to risk more of his/her wealth in the hope of improving his/her situation dramatically if there is a positive jump in price of risky asset. If the price moves down, in the bad state, he/she does not lose a lot. So, the investor's investment decision (and the fraction of wealth devoted to the risky asset) is also non-linear with wealth.

I will consider an agent who has a (discounted) Bernoulli (logarithmic) utility of consumption. At time t the differential of the increase of total utility (marginal increase) due to consumption rate $c(t)$ is $e^{-\beta t}\ln(\frac{c(t)-c_{\min}}{c_n})dt$, where $c_{\min}$ and c_n are some constants, and β is the time-preference factor (personal discount rate). $c_{\min}$ can be thought of as a minimum consumption rate the agent still can bear. If the consumption rate is below that value, the agent dies and the utility is $-\infty$. c_n is a normalization factor which is useful for showing the invariance of the problem and for dimensional analysis.

There are two investment possibilities for the agent: either invest in a risky asset (with fixed parameters such as instantaneous expected return μ and variance σ) or a risk-free asset (with constant rate of return r). In addition to the investment opportunities, the agent increases his/her wealth by receiving income (from wages) with rate I. For simplicity I will assume that the income rate is constant $I - const$. In this world there is also a safety net in the form of consumption subsistence from the government if the weighted sum of wealth and income of the agent falls to a pre-specified value $\omega W_{sub} + \iota I$. Without losing any generality I will choose weights to be $\omega = r$ and $\iota = 1$ and denote $I_{sub} = rW_{sub} + I$. That value represents a deterministic rate of total income at the time the agent is eligible to switch to subsistence. If the total deterministic income is less than I_{sub} and the agent's wealth is at W_{sub} the agent may pay (legal, etc.) costs W_{sub} and forfeit his income I (so, his/her total wealth is zero and there is no income) in order to receive the consumption subsistence c_{sub} starting at this time (and lasting forever). In this model I allow subsistence only in the form of consumption goods and do not allow cash subsistence, therefore the switch to subsistence is irreversible: once the agent chooses to switch to subsistence consumption he/she will not be able to participate in market investments forever (the wealth/investable balance is zero and agent is not allowed to work and receive income). The irreversible switch to consumption subsistence is rational if $\beta I_{sub} < r c_{sub} \exp(\frac{\beta - r - \kappa}{\beta}) + c_{\min}(\beta - r\exp(\frac{\beta - r - \kappa}{\beta}))$, where $\kappa = \frac{(\mu - r)^2}{2\sigma^2}$. From this inequality one can see that if the threshold I_{sub} is large, the consumption subsistence must be sufficiently great to justify the switch (note that this condition is stronger than simple $I_{sub} < c_{sub}$ for all positive β and r). Intuitively, if the wealth and income of the agent is small, the expected (integrated discounted) utility is not very large and it is beneficial for the agent to consume everything quickly (increasing utility tremendously) and switch to the subsistence consumption. Indeed, from Merton's solution [1] compare the integrated utility of $J_{Merton}(W) = \frac{1}{\beta}\ln(\frac{\beta W + \frac{\beta}{r} I - \frac{\beta}{r} c_{\min}}{c_n}) + \frac{r + \kappa - \beta}{\beta^2}$ where c_n is some normalizing constant to the integrated utility of subsistence consumption here $J_{sub}(W) = \frac{1}{\beta}\ln\frac{c_{sub} - c_{\min}}{c_n}$. Note that at $\beta(rW_{sub} + I - c_{\min}) = r(c_{sub} - c_{\min})\exp(\frac{\beta - r - \kappa}{\beta})$ both integrated utilities are the same and at $rW_{sub} + I < I_{sub}$ the agent is better off he/she uses subsistence consumption option.

The paper is organized as follows. Section 2 deals with the very simple model which describes an agent in the deterministic world: the agent does not participate in risky markets. Section 3 analyzes the model under investigation and derives the Hamilton-Jacobi-Bellman (HJB) equation. In the fourth section, I describe the solution to the HJB equation. Section 5 shows the Monte Carlo simulations to support the theoretical results and examines the problems of interest: the equity premium puzzle and constancy/variability of the risky investment to wealth ratio and consumption rate to wealth ratio.

2 Simple Model. Zero fraction of investor's wealth in risky asset.

2.1 Definitions and mathematical representation.

Consider the agent with initial wealth W_0 who has income I and tries to maximize the value function

$$U = \int_0^{+\infty} e^{-\beta t} u_0 \ln(\frac{c(t) - c_{\min}}{c_0}) dt \tag{1}$$

The only investment opportunity is a risk-free asset with the rate of return r. Also, the agent receives steady income I.

The dynamics of wealth are

$$\frac{dW(t)}{dt} = rW(t) + I - c(t) \tag{2}$$

At $W = W_{sub}$ the agent may choose to switch to receive consumption subsistence in the form of consumption goods at rate c_{sub} (amount of consumption goods per unit of time).

2.2 Euler equation and solution to the Simple Model.

To solve the maximization problem one can express the consumption rate $c(t)$ from eq.2

$$c(t) = rW(t) + I - \frac{dW(t)}{dt} \tag{3}$$

substitute it into eq.1 and solve the standard variation problem using the Euler method with refraction boundaries. The corresponding ODE is

$$\frac{d^2W}{dt^2} + (\beta - 2r)\frac{dW}{dt} + r(r - \beta)W + (r - \beta)(I - c_{\min}) = 0 \tag{4}$$

with boundary conditions

$$W(t = 0) = W_0 \tag{5}$$

$$W(t = t_{sub}) = W_{sub} \tag{6}$$

and refraction condition

$$\left(\ln(rW - W' + I - c_{\min}) + \frac{W'}{rW - W' + I - c_{\min}}\right)|_{t=t_{sub}} = \ln(c_{sub} - c_{\min}) \tag{7}$$

where t_{sub} is the time at which the agent will choose to switch to the subsistence. If there is a solution to eq.4 with conditions 5-7 the optimal decision will involve switching.

The solution in terms of $W(t)$ is

$$W(t) = C_1 e^{rt} + C_2 e^{(r-\beta)t} + \frac{c_{\min} - I}{r} \tag{8}$$

where C_1 and C_2 are constants which are solutions to

$$C_1 + C_2 + \frac{c_{\min} - I}{r} = W_0 \tag{9}$$

$$C_1 e^{rt_{sub}} + C_2 e^{(r-\beta)t_{sub}} + \frac{c_{\min} - I}{r} = W_{sub} \tag{10}$$

$$r + r\frac{C_1}{C_2} e^{\beta t_{sub}} + \beta(r - \beta)t_{sub} = \beta \ln(\frac{e(c_{sub} - c_{\min})}{\beta C_2}) \tag{11}$$

The optimal consumption rate in this set-up can be found from $W(t)$ using eq.3

$$c(t) = \beta C_2 e^{(r-\beta)t} + c_{\min} \tag{12}$$

In general the system 9-11 can be solved only numerically. To obtain tractable analytical results one can consider specific values for some variables. Let $c_{\min} = I$, and $W_{sub} = 0$. Then

$$W(t) = (W_0 - C_2)e^{rt} + C_2 e^{(r-\beta)t} \tag{13}$$

$$c(t) = \beta C_2 e^{(r-\beta)t} + c_{\min} \tag{14}$$

where $C_2 = zW_0$ is solution to

$$z^{\rho} \times (z-1)^{1-\rho} = \frac{e(c_{sub} - c_{\min})}{\beta W_0} \tag{15}$$

where $\rho \equiv \frac{r}{\beta}$.

There is a real solution to eq.15 if $\rho \leq 1$ (or $r \leq \beta$) or $\rho > 1$ and $\rho^{\rho}(\rho-1)^{1-\rho} \leq \frac{e(c_{sub}-c_{\min})}{\beta W_0}$. Therefore, if $r \leq \beta$ it is always beneficial to switch to subsistence consumption at some moment of time; if $r > \beta$ the agent chooses to switch only if the initial wealth is smaller than the known value

$$W_0 \leq \frac{e(c_{sub} - c_{\min})}{\beta \rho^{\rho}(\rho-1)^{1-\rho}} \tag{16}$$

In case there is no real solution (only complex ones) to eq.15, the switch to subsistence consumption will yield smaller integrated utility and therefore is not beneficial; in that case $C_2 = W_0$, and

$$W(t) = W_0 e^{(r-\beta)t} \tag{17}$$

$$c(t) = \beta W_0 e^{(r-\beta)t} = \beta W(t) \tag{18}$$

The phase diagram below shows when the agent prefers to go bankrupt is given below on **Fig.1.**

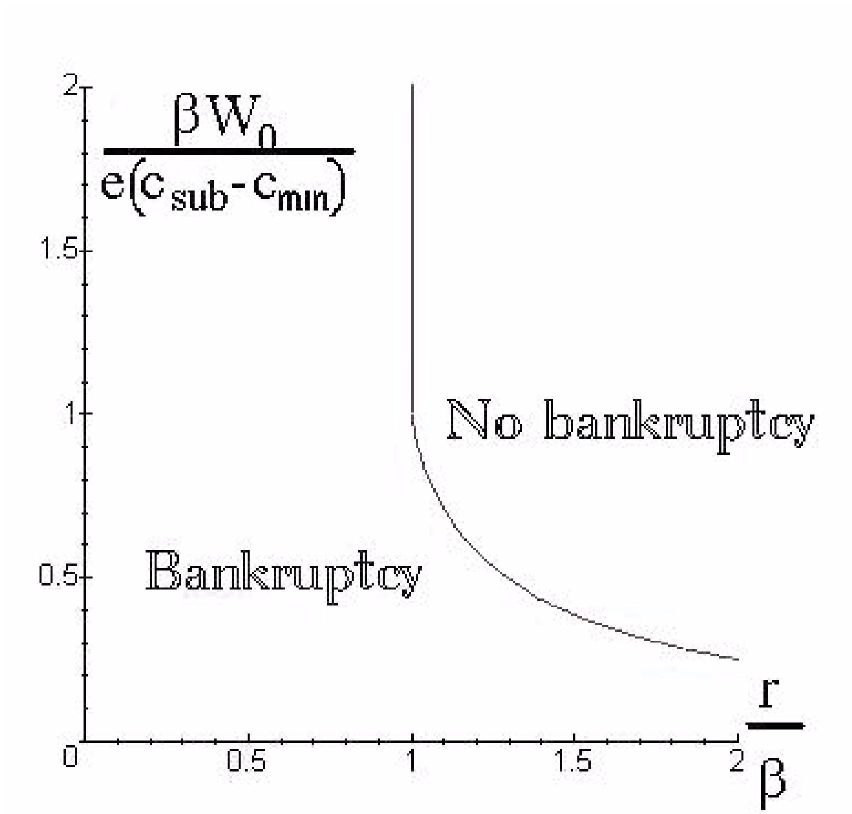

Fig.1. Phase diagram of bankruptcy decisions. *Decision rule to file or not to file bankruptcy as a function of variables* $\frac{\beta W_0}{e(c_{sub}-c_{\min})}$ *and ratio* $\frac{r}{\beta}$*. If* $\frac{r}{\beta} < 1$ *(e.g.* $r < \beta$*) it is always beneficial to declare the bankruptcy at some point of time. If* $r > \beta$ *the decision will also depend upon wealth. Is wealth is low enough (bottom right of the picture), the agent will declare the bankruptcy, if the wealth is large, the agent prefer to live on income from wages,* I*, and income from investment,* rW *(no bankruptcy region – upper right of the picture).*

People who do not participate in stock market, exhibit several interesting properties. First, they will deliberately consume more and declare bankruptcy if A) the risk-free interest rate in the economy is smaller then their personal discount factor or B) their initial wealth W_0 is small; in articular if W_0 is smaller than fraction of $\frac{e(c_{sub}-c_{\min})}{\beta}$ (the specific value of the fraction depends upon the ratio $\frac{r}{\beta}$, see eq.16).

The other interesting result is the consumption decision rule near the transition point. As wealth approaches the transition limit $W \to W_{sub}$, the optimal consumption approaches the value $c \to e \times c_{sub}$. That shows us that even in the deterministic case (i.e. without allowing the agent to participate in the stock market) one should not expect continuity of decision rules. Although the value function J is continuous at the transition point the first derivative of J with respect to W is not. Consequently, that is the first order phase transition.

The important result is that under some conditions, *but not always,* it is beneficial to consume more and faster and switch to subsistence rather than try to extend the life of one's "independent" consumption. Note, that this statement might not be true if one has a negative shock to his/her utility function as a result of transition to subsistence (e.g. if the switching to the subsistence instantaneously decreases utility by some value). That shock can be easily incorporated into the proposed solution but it will not contribute to the better understanding of the problem or intuition behind it, therefore I do not consider it in the present paper.

3 More realistic model.

Although a behavior of an agent discussed in the model in previous chapter is sub-optimal – the agent disregard risky component of the market – that behavior shows important features that are present in the optimal behavior, namely increased (relative) consumption at low wealth and optimality of declaration of bankruptcy. Below I consider the formal proof of that statements. First, I will define the problem of interest and introduce variable needed to solve the problem. Next, I will derive Hamilton-Jacobi-Bellman (HJB) equation, solve it, and discuss the results.

3.1 Definitions.

Consider an agent who may invest in either a risky asset with lognormal distribution of returns or a risk-free asset with a deterministic (and constant) rate of return. I will use words risky asset, share, stock, etc. or risk-free, bond, etc. interchangeably. I assume that there is only one risky asset the agent can buy. He/she will contribute $\alpha(t)$ of his/her wealth to the stock. The agent derives utility from a concave utility function. As an example, I will consider the Bernoulli (logarithmic) utility of consumption $u(c) = u_0 \ln(\frac{c(t)-c_{\min}}{c_n})$, where u_0 defines the level of utility and c_n is a normalization constant. The more general case of HARA utility function $\frac{\left(\frac{c-c_{\min}}{c_n}\right)^{\gamma}-1}{\gamma}$ is no harder and the choice of $\gamma = 0$ has no special meaning. One can easily generalize for any $\gamma < 1$. Also, I assume that utility is time-separable.

The agent is rational and risk-averse. He wants to maximize his expected (discounted) utility U

$$U = \int_0^{+\infty} e^{-\beta t} u_0 \ln(\frac{c(t) - c_{\min}}{c_n}) dt \tag{19}$$

where β is the agent's subjective discount factor for his/her utility, $c_{\min}$ is the minimum consumption (if the consumption falls below that the agent dies – utility is $-\infty$), u_0 is the level of utility function, and c_n is the normalization constant.

The two investments the agent may invest in are characterized by various parameters. The risky asset has instantaneous rate of return μ and volatility σ and the risk-free asset has deterministic rate of return r.

The dynamics of the risky asset's price are assumed to be

$$\frac{dS}{S} = \mu(t)dt + \sigma(t)dZ \tag{20}$$

where μ is the expected (instantaneous) rate or return and σ is the stock's volatility, both of which are assumed to be constant $\mu(t) = \mu = const$ and $\sigma(t) = \sigma = const$ and given exogenously.

The dynamics of the risk-free investment are

$$\frac{dB(t)}{B(t)} = r(t)dt \tag{21}$$

where we assume that the interest rate is deterministic and constant $r(t) = r = const$.

The agent invests a fraction α of his/her wealth in the risky asset and the fraction $1-\alpha$ in the risk-free asset. In this paper, I assume that the market is frictionless and the cost of changing the portfolio is zero. It is not difficult to introduce a flat fee for buying/selling securities; I will investigate that case in a later paper.

The agent has income rate $I - const$. Therefore, the change in the agent's wealth due to income is $dW(t) = Idt$. If the agent becomes very poor, there is a safety net in the form of consumption goods from the government. If the agent chooses, he/she may receive the goods if the deterministic rate of total income $rW + I$ falls to a pre-specified value I_{sub}. If the total deterministic income is less than I_{sub}, and the agent's wealth is at W_{sub}, the agent may pay (legal, etc.) costs W_{sub}, forfeit his/her income I and receive in perpetuity the consumption goods of value c_{sub} starting at this moment. As before, the switch to subsistence consumption is assumed to be irreversible.

The decision problem of the agent is to choose decision rules for consumption $c(W_0)$ and investment rule $\alpha(W_0)$ such as to maximize value function $J(W_0)$ given the initial wealth W_0. Because the utility function is time-separable, the decision rules depend only on the current value of the wealth and does not depend upon the history of decision rules or wealth levels.

3.2 Mathematical representation.

The agent wants to maximize his expected utility (see eq.19)

$$\max_{c_1(t),\ \alpha(t)} E[U \mid \Xi] \ = \tag{22}$$

$$\max_{c_1(t),\ \alpha(t)} E[u_0 \int\limits_0^{+\infty} e^{-\beta t} \ln(\frac{c(t) - c_{\min}}{c_n}) dt \mid \Xi]$$

where Ξ is the information available at time t, subject to dynamics equations eq.s 20 and 21.

The change of the total wealth can be written as the sum of income from wages, the change of the value of the stock, consumption, and risk-free interest

$$dW(t) = Idt + N(t)dS(t) - c(t)dt + r \times (1 - \alpha(t))W(t)dt \tag{23}$$

where α is a fraction of total wealth contributed to the stock $\alpha(t) = \frac{N(t)S(t)}{W(t)}$.

One can rewrite this in the following form (suppressing the time-dependence notations (t)).

$$dW = Idt + \alpha W \frac{dS}{S} - cdt + (1 - \alpha)rWdt \tag{24}$$

Substituting the dynamics of the stock from 20, and rearranging terms, one can find

$$dW = [\ \ W \times \{r + \alpha \times (\mu - r)\} + I - c]dt + \alpha W \sigma dZ \tag{25}$$

where the first term represents deterministic change of wealth due to investments $W \times \{r + \alpha \times (\mu - r)\}$, income I, and consumption $c(t)$; and the second term is stochastic contribution to the wealth form the risky investment.

The multiplication table for the increments of the variables is

$$dWdW = \alpha^2 W^2 \sigma^2 dt \tag{26}$$

3.3 Constraints.

The first important constraint in my model is that consumption $c(t)$ may not be less than $c_{\min}$ at any time t. Secondly, I don't consider any liquidity constraints (such as in Longstaff [5]), so the derivatives of α may be infinite. Due to instantaneous adjustment to the desirable ratio of investments, the state variables such as stock price $S(t)$ and the number of holding shares $N(t)$ will not be present in Hamilton-Jacobi-Bellman equation.

3.4 Hamilton-Jacobi-Bellman equation.

In this subsection, I consider the infinite time horizon. This restriction is not particularly important since it affects only the re-normalization of β. I will assume that the β parameter is normalized so as to take into account the probability of a death of the agent. Following Merton [1] we may introduce the characteristic function $J(t, W)$.

$$\begin{aligned} J(W(t_0)) &= \\ &= \max_{c_1(t),\ \alpha(t)} E_{t=0}[u_0 \int_{t=t_0}^{\tau_{sub}} e^{-\beta t} \ln(\frac{c(t) - c_{\min}}{c_n})dt\ + e^{-\beta \tau_{sub}} J_{sub}] \end{aligned} \tag{27}$$

where J_{sub} is the integrated (deterministic) utility the agent has at the time of switching to subsistence consumption. The characteristic function may be written in similar form which extends the integration to infinity

$$\begin{aligned} J(W(t_0)) &= \\ &= \max_{c_1(t),\ \alpha(t)} E_{t=t_0}[u_0 \int_{t=t_0}^{+\infty} e^{-\beta t} \ln(\frac{c^*(t) - c_{\min}}{c_n})dt\ ] \end{aligned} \tag{28}$$

where c^* is consumption $c(t)$ for $t \leq \tau_{sub}$ and c_{sub} for $t > \tau_{sub}$.

One can think about the equation 27 as the sum of the utilities of agent before he/she switches to subsistence consumption (first term of the equation) and after he/she switches to subsistence consumption (the second term). If the agent never chooses to switch to subsistence, that can be represented by $\tau_{sub} = +\infty$. For simplicity I will choose $t_0 = 0$ (although it does not change any results and one, if so wishes, may carry out all the transformations using general notation t_0)

The interpretation of the value function J is the total (integrated) expected utility of the agent under optimally chosen consumption $c(t)$ and investment decision rule $\alpha(t)$, given that all available information at time t is taken into account. For optimal value function $J(W) \geq J^*(W)$, where J^* is the integrated utility under any other decision rule $\alpha^*(W)$ or consumption level $c^*(W)$. In the text below, I suppress the function dependence notations and will write only J with the understanding that this is a function of the variable W.

From these considerations, we can derive that the optimal value function should satisfy the following partial differential equation, which is known as the Hamilton-Jacobi-Bellman (HJB) equation.

$$\frac{\partial J}{\partial t} + u_0 \ln \frac{c}{c_n} - \beta J + \widehat{\mathcal{A}} J = 0 \tag{29}$$

where the Dynkin operator $\widehat{\mathcal{A}}$ is

$$\widehat{\mathcal{A}} = \{[r + \alpha(\mu - r)]W + I - c\} \frac{\partial}{\partial W} + \frac{\alpha^2 W^2 \sigma^2}{2} \frac{\partial^2}{\partial W^2} \tag{30}$$

Because there is no direct dependence of J on t (integration of utility to infinity, see eq.28), the first term in eq.29 disappears.

Therefore

$$0 = \max_{\alpha,\ c_1}\{u_0 \ln\frac{c-c_{\min}}{c_n} - \beta J + ([r+\alpha(\mu-r)]W + I - c)\frac{\partial J}{\partial W} + \\ + \frac{\alpha^2 W^2 \sigma^2}{2}\frac{\partial^2 J}{\partial W^2}\} \tag{31}$$

Consider time $t \leq \tau_{sub}$. One can find the first order conditions with respect to c

$$\frac{u_0}{c(t)-c_{\min}} - \frac{\partial J}{\partial W} = 0 \tag{32}$$

and with respect to α

$$(\mu - r)W\frac{\partial J}{\partial W} + \alpha W^2\sigma^2\frac{\partial^2 J}{\partial W^2} = 0 \tag{33}$$

From eq.32 and eq.33 we can easily find the optimal consumption c and decision rule α

$$c = \frac{u_0}{\frac{\partial J}{\partial W}} + c_{\min} \tag{34}$$

and

$$\alpha = -\frac{(\mu-r)\frac{\partial J}{\partial W}}{W\sigma^2\frac{\partial^2 J}{\partial W^2}} \tag{35}$$

By substituting c and α from eq.s 34-35 into eq.31, we obtain the differential equation for optimal value function J

$$0 = -u_0\ln(\frac{c_n}{u_0}\frac{\partial J}{\partial W}) - \beta J + r(W + \frac{I-c_{\min}}{r})\frac{\partial J}{\partial W} - \frac{(\mu-r)^2(\frac{\partial J}{\partial W})^2}{2\sigma^2\frac{\partial^2 J}{\partial W^2}} - u_0 \tag{36}$$

Now we can consider time $t > \tau_{sub}$. In that case, the characteristic function J is a function of c_{sub} and does not depend on wealth W. Therefore from eq. 31 we can find that

$$J_{sub} = \frac{u_0}{\beta}\ln\frac{c_{sub}-c_{\min}}{c_n} \tag{37}$$

The equation 36 has many different solutions, one of which is Merton's solution [1] .

However, for different solutions (including Merton's solution) there is a difference in the agent behavior near very small or very large values of wealth. Mathematically that difference is described by boundary conditions. Because the eq.36 is the second order (ordinary) non-linear differential equation it needs *two* boundary conditions. The behavior of the agent near W_{sub} is clearly different from the behavior of Merton's agent; the first boundary condition is the condition at W_{sub}

$$J(W_{sub}) = J_{sub} = \frac{u_0}{\beta} \ln(\frac{c_{sub} - c_{\min}}{c_n}) \tag{38}$$

If the wealth of the agent is really large the influence of the safety net is insignificant and we expect the agent to behave according to Merton's decision rules. So the second boundary condition is

$$\lim_{W \to \infty} (J(W) - J_{Merton}(W)) = 0 \tag{39}$$

4 Solutions to the HJB equation.

4.1 Change of variables.

The equation 36 with boundary conditions 38 and 39 can be solved using the present notations, however, the introduction of new constants and variables helps to simplify the solution and to show the important features of it.

Let constants be

$$\alpha_0 = \frac{\mu - r}{\sigma^2} \tag{40}$$

$$\kappa = \frac{(\mu - r)^2}{2\sigma^2} = \frac{\mu - r}{2}\alpha_0 \tag{41}$$

$$h_1 = -\frac{\kappa + r - \beta + \sqrt{(\kappa + r - \beta)^2 + 4\kappa\beta}}{2} < 0 \tag{42}$$

$$h_2 = -\frac{\kappa + r - \beta - \sqrt{(\kappa + r - \beta)^2 + 4\kappa\beta}}{2} > 0 \tag{43}$$

Note that $h_1 h_2 = -\kappa\beta$. All units κ, r, β, h_1, and h_2 have the same dimensions $\frac{1}{s}$, so we can compare them. One of the properties we will later use is

$$0 < -h_1 < \kappa \tag{44}$$

for all positive values of r and β.

Introduce new variables

$$z = \ln(\frac{rW + I - c_{\min}}{I_{sub} - c_{\min}}) \tag{45}$$

$$\zeta^* = \frac{c - c_{\min}}{W + \frac{I - c_{\min}}{r}} \tag{46}$$

$$\alpha^* = \frac{W}{W + \frac{I - c_{\min}}{r}}\alpha \tag{47}$$

$$x = \frac{h_2}{h_2 - h_1}\frac{\alpha^* - \alpha_0^*}{\alpha_0^*} - \frac{\zeta^* - \beta}{h_2 - h_1} \tag{48}$$

$$y = \frac{h_1}{h_2 - h_1}\frac{\alpha^* - \alpha_0^*}{\alpha_0^*} - \frac{\zeta^* - \beta}{h_2 - h_1} \tag{49}$$

The meaning of ζ^* is easier to understand if we look at it assuming $I = 0 = c_{\min}$. In this case $\zeta = \frac{c}{W}$, and ζ^* is a rate (fraction) of wealth consumption. Note that dimensions of this new constant is also $\frac{1}{s}$. ζ^* has the same meaning but is normalized for I and $c_{\min}$. α^* is a re-normalized fraction of wealth invested into risky asset. x and y are new variables. The ODE eq.36 is transformed into a system of relatively simple differential equations on x and y.

$$\frac{dx}{dz} = -x + \frac{x(\kappa + h_2)}{\kappa(x - y + 1)} \tag{50}$$

$$\frac{dy}{dz} = -y + \frac{y(\kappa + h_1)}{\kappa(x - y + 1)} \tag{51}$$

It can be easily seen that if either x or y is zero they will stay zero at any z and, therefore, at any wealth. The case $x = 0 = y$ is the classical Merton's case. That solution satisfy the boundary condition eq.38 only for a very specific combination of parameters $c_{\min}$, I, β, κ, and

r. However the case $x = 0$ and $y \neq 0$ is the solution of interest. The point $(0; 0)$ is a stable point along y direction, so increasing z (in other words increasing wealth) the solution converges to $(0; 0)$ – to the Merton's solution. That is exactly what we are looking for.

So, the general solution of eq. 50-51 is

$$x(z) = 0 \tag{52}$$

$$h_1 z = \kappa \ln \left| \frac{y(z)}{y_0} \right| - (\kappa + h_1) \ln \left| \frac{y(z) + \frac{h_1}{\kappa}}{y_0 + \frac{h_1}{\kappa}} \right| \tag{53}$$

where y_0 can be found using boundary conditions eq.38 and eq.39. $y(z)$ is given in implicit form. Unfortunately there is no analytical expression for $y(z)$ in explicit form. However, general features of the solution can be easily found and are discussed below.

5 Monte Carlo simulation.

5.1 The Equity Premium Puzzle.

In that section I will show preliminary results of the possible explanation to the equity premium puzzle which this model provides.

The following values are used for the parameters

r	β	μ	σ	$I - c_{\min}$	I_{sub}	c_{sub}
0.04	0.05	0.2	0.2	400	500	800

The initial wealth was $W = 5000$, Monte Carlo step corresponds to 1 day, there are 1000 Monte Carlo steps.

The centered logarithms (deviations from the averages) of consumption and stock price are plotted below. The points are plotted with interval of 10 steps.

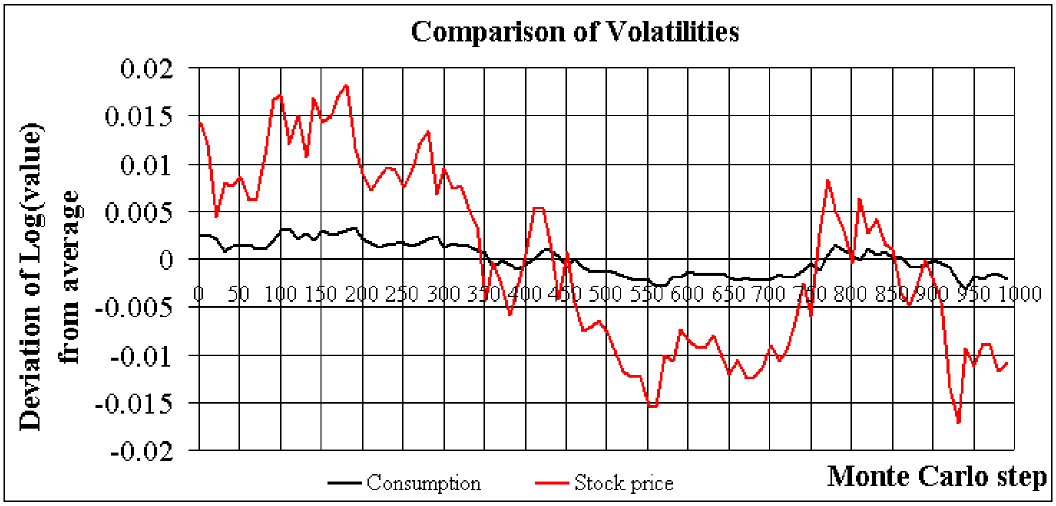

Fig.2. Simulation of volatilities. *Volatility of logarithm of stock is much larger than the volatility of logarithm of consumption. See text for more explanations.*

The volatility of the log of stock price is $\sigma = 0.2$, while the volatility of the log of consumption is $\sigma_{cons} \approx 0.015$. We see that the oscillation of logarithm of asset price is much larger than the oscillation of the logarithm of the consumption. The intuition behind the results is fairly simple. The agent knows that in the bad state he will receive subsistence from a government in the form of consumption goods. Therefore the agent prefers to consume more right now (and increase the utility dramatically) and, if bad state is realized, to switch to the subsistence (and keeping the utility at bearable level). So, even if wealth is small the agent will consume more than the same agent in the Merton's world. Such behavior smooths the consumption. Consequently, large oscillations in stock value will cause relatively small oscillations in consumption.

5.2 The constancy of ratios.

Another example of the application of the theory is the example that the fraction of wealth invested in the risky asset is not a constant fraction of total wealth; neither is the consumption proportional to wealth.

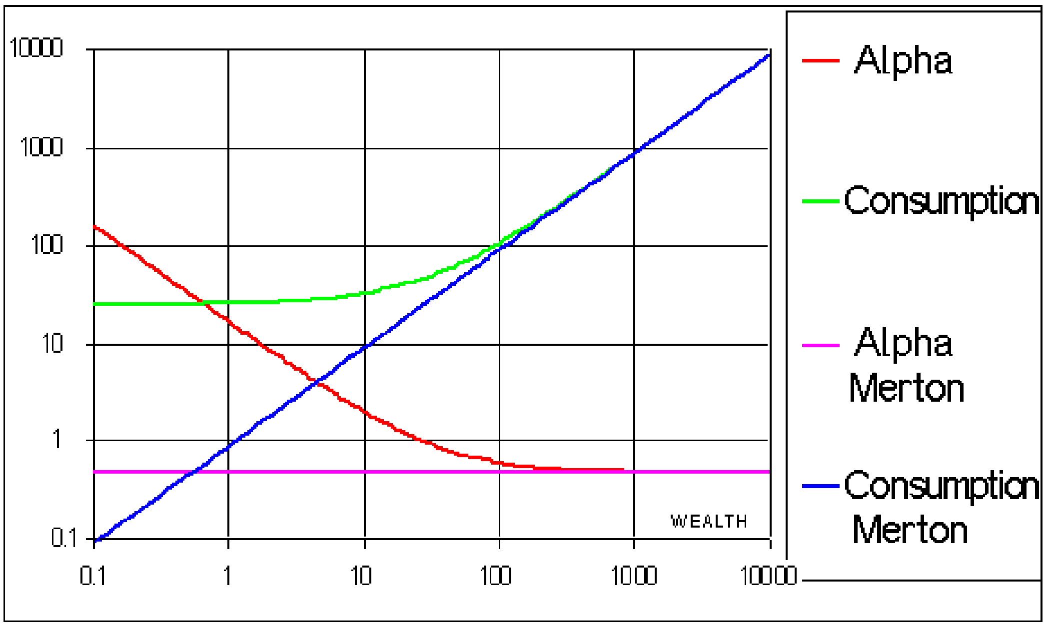

Fig.3. Comparison of decisions under Merton's model and my model. *The wealth is measured in thousands of dollars, consumption is measured in thousands of dolars per year, while* α *is measured in non-dimentional units. The subsistance level was choosen to be equivalent to* $\$700$ *per month.*

We see that for logarithmic utility function the behavior of people may differ from linear to level of wealth measured in tens of thousands of dollars. However, if the utility level of switching is larger than equivalent of $\$700/month$, the values of non-linearity will be significantly larger.

The extreme values of α at low wealths is the optimal behavior for poor agents. One of the assumptions of the model that there is no borrowing constrains. In real life there are borrowing limits, so the poor investor will be capped at α about 1. It is sub-optimal behavior for the agent, but the decision lies with lending authority. However, the poor people tends to borrow (relatively) more and invest almost all their wealth into risky projects, either buying stock, real estate, or starting risky business.

6 Conclusion.

In the present part I have considered the effect of the default possibility onto the decision making processes, such as allocation of money into risky asset and consumption decision.. Under some conditions an investor may prefer to intensify the consumption as wealth decreases and to declare a bankruptcy or default in order to increase the utility function.

If we assume that the investor's wealth is very low and close to the value at which the investor can switch to different regime (e.g. receive subsistance coupons or other benefits), he/she will find it beneficial to consume the rest of the available wealth in a short period of time, since the utility will increase dramatically (due to two reasons: first, larger consumption over short period of time increases the utility function, second, there is a guaranteed subsistence consumption, which has utility value, if the wealth drops to/below specific level). Consequently, agent's decision to consume depends non-linearly on the wealth. Also, as wealth approach the lowest possible level the agent may find it beneficial to risk more of his/her wealth in the hopes of improving his/her situation dramatically due to a positive jump in price of risky asset. In a bad state he/she does not lose a lot. So, agent's investment decisions (wealth contributed to the risky asset) are increasing with decreasing wealth and are also non-linear with wealth.

That analysis is true not only for subsidized consumption, but also in other cases such as strategic default (one needs just to use J_{sub} in 38, where J_{sub} is the value of utility if a firm declares default), irreversible fire-selling to another company, etc. The analysis also holds for different utility functions of HARA family. There are conditions then it is preferable to declare default at low wealth.

The decisions to allocate money to risky investment and for consumption is no longer linear in wealth, therefore a creation of the representative agent is no longer warranted. The total money value invested by the non-linear agents in the risky market may be substantial and it would seem that the representative agent behaves irrationally or as it has unusually high risk-aversion coefficient; however, each single agent might behave rationally and have reasonable risk-aversion coefficient.

Chapter II
Asset Allocation under New Definition of Risk.

7 Introduction.

The asset pricing problem is one of the most important problems in finance. The Markowitz and Sharpe-Lintner-Mossin theory is the classical theory of modern asset pricing ([28] -[33]). In 1990 Markowitz and Sharpe received a Nobel Prize in Economics for the work in Mean-Variance portfolio selection and CAPM. Many researchers and practitioners are using CAPM in day-to-day activities. However, there is evidence in the literature that the model may not be perfect.

One of the most important of many assumptions of CAPM is that it defines the risk of an asset in a well diversified portfolio as its covariance with an optimal portfolio (market portfolio). That risk stems from the non-diversifiable portion of variance. The variance includes both positive deviations and negative deviations from the average. However, if one's portfolio value may increase by 50% it is hardly a risk; in contrast, if portfolio value may decrease by 50%, it is a significant risk to the portfolio owner. CAPM considers both as a risk to the portfolio.

Researchers recognized long ago the limitations of the Mean-Variance Criterion and CAPM. Although the Mean-Variance Criterion approximately holds in many cases when the distribution of returns is not Normal (for discussion see [35]), there have been many attempts to improve this investment criterion. Perhaps, the simplest is the 'Geometric Mean return' maximization. The proponents of this criterion argue that a portfolio selected based on the maximum geometric mean return has the highest probability of achieving any given level of wealth/return. In addition, the theory is relatively simple. The criterion assumes the utility of potential investor in a very specific form: logarithmic. As a consequence, that theory may not be appropriate for every investor. Also, if the returns are log-normally distributed the geometric mean return maximization reduces to the Mean-Variance Criterion.

Another popular method is the 'Safety First' Criterion. The first model of this type was developed by Roy [36] . The criterion calls for minimizing the probability of the return falling below a specified value: $\min(\mathit{Prob}\,(R_{prtfolio} < R_{\lim}))$. For normally distributed returns the criterion is very similar to the Mean-Variance Criterion. In the general case, it cannot be reduced to Mean-Variance. However, the criterion does not take into account the distribution of return below the threshold value and therefore cannot capture risk accurately. More than that, the approach does not take into account the distribution of returns above the threshold. In other words the criterion looks only at the 'risk' of the portfolio disregarding possible rewards. The second 'Safety First' Criterion was developed by Telser [37] . His suggestion was to maximize the return on portfolio $R_{portfolio}$ given that the probability of return below $R_{\lim}$ does not exceed a pre-specified value $\alpha : \mathit{Prob}\,(R_{prtfolio} < R_{\lim}) \leq \alpha$. The two values $R_{\lim}$ and α are independent values and it might happen that for some combinations there are no feasible solutions/portfolios. The second problem with the criterion is that under some conditions the best portfolio requires unlimited borrowing. One more 'Safety First' Criterion was introduced by Kataoka [38] : for this criterion one maximizes the lower limit $R_{\lim}$ such that *Prob* $(R_{portfolio} < R_{\lim}) < \alpha$ for an α given in advance. Although the criterion takes into account the distribution of returns in the non-desirable region, it does not consider the rewards associated with portfolio. Again, as with first criterion, it looks only at 'the risk' of the portfolio.

There are also criteria based on higher moments of the return distribution. They better capture the behavior of returns and allow for a richer trade-off between reward and risk. However, the introduction of the third moment will not capture the whole dynamics of returns and the mathematical development of such a model is difficult.

In addition to the models mentioned above, there are models which consider semivariance as a measure of risk. For example, one of the models maximizes the Sortino ratio [39] . The ratio is defined as $\frac{\bar{r}_p - t}{\theta_{r_p}(t)}$, where $\bar{r}_p$ is the expected return on portfolio, $\theta_{r_p}(t) = (\int\limits_{-\infty}^{t} (t - \tilde{r}_p)^2 \; pdf(\tilde{r}_p) \; d\tilde{r}_p)^{1/2}$ is a second lower partial moment (or semi-standard deviation), and t is a cutoff value for undesirable returns. The semi-standard deviation serves as a measure of risk and the goal is to maximize the ratio.

A closely related set of popular models is a set of models with specific utility function over wealth/returns. Consider, for example, the power utility function (in Rubinstein [40] it is called constant proportional risk averse (CPRA) function) $U(C) \sim \frac{C^{-b}}{1-b}$. Based on the formula from Rubinstein [40] , Leland [41] derives the risk measure as $B_p = \frac{Cov(r_p, -(1+r_{mkt})^{-b})}{Cov(r_{mkt}, -(1+r_{mkt})^{-b})}$ where r_p is return on portfolio, while r_{mkt} is return on market portfolio.

In this paper I propose a simple single-period theory which allows one to distinguish between positive and negative deviations in the value of a portfolio. The theory is built on a mixture of CAPM (Sharpe's approach and semi-variance approach) and the "Safety First" approach. The risk is defined as the average non-desirable returns on the portfolio. An investor tries to create a portfolio so that the unconditional average return is as high as possible while conditional non-desirable returns on the portfolio are as small (in absolute values) as possible. I make no assumptions about the possible distribution of the stock prices. However, assuming the normal (Gaussian) distribution it would be possible to obtain a closed form solution. It can be shown that in the case of normal distribution the solution reduces to the CAPM solution. In other words CAPM is a particular case of the proposed solution when the distribution of returns is normal. If the Normal distribution is not assumed the result is stated via the integral of a distribution density. The preliminary empirical findings agree with the observed market returns. In addition the model allows one to explain the non-zero α of the stock-market distribution and allows for many risk factors (such as risk-free rate, inflation rate, PE ratio, book-to-market ratio, etc.).

8 Simple Models.

8.1 Linear and non-linear Models.

The CAPM and APT are linear-risk models, e.g. if one doubles the holdings in the risky component of portfolio by borrowing at risk free rate, the risk will double. Let us consider model with a non-linear change in risk.

The simplest possible non-linear model is to measure the risk as the probability of return dropping below a pre-specified value. Consider, for example, an investor who is concerned with her portfolio *value decrease,* e.g. she consider as the risk only a situation when the value of the portfolio drops below the initial value. Assume the portfolio has expected return μ and standard deviation $\sigma = \frac{1}{3}\mu$. The probability of a return below zero (decrease in value) is very small, about 0.15%. Now, assume there is a risk free asset with risk-free rate of return $r_f = \frac{2}{3}\mu$. The investor borrows at the risk free rate the same amount as she previously invested and invests everything into the risky asset. The probability of loss in value increases dramatically: 2.5%, more than 15 times. If the investor borrows three times the initial amount, the probability will jump more than 100 times, to about 16%. The increase in risk is definitely not linear with the holdings in risky component of the investor's portfolio.

Because investors do not consider the large positive return as risky, but do consider negative and low returns as undesirable or risky, one might be interested in developing a risk measure that includes low/negative return but does not treat positive returns as risky. The probability of bad returns is not a good measure for several reasons. However, one can consider the expected value of bad returns as a measure of portfolio's risk.

8.2 "Uniform" model comparison.

Consider two assets A and B. Asset A has a return that is uniformly distributed between 10% and 12%, while the return on asset B is uniformly distributed between 10% and 11%. Asset A has four times the variance of asset B (and therefore higher risk) and 0.5% higher expected return. According to the Mean-Variance criterion, a less risk-averse investor would prefer asset A, while a more risk-averse investor would prefer asset B. There is little suggestion from the model that the market may not be at equilibrium. However, one can see that asset A definitely dominates asset B (in the first order stochastic dominance sense). That is, the cumulative distribution function, CDF_B, of asset B is always larger than CDF_A of asset A, although it does not reaches 1 while CDF_A is zero (therefore one cannot arbitrage).

It is known that the Mean-Variance Criterion works only in the case of either quadratic utility function (which is not a case in the real world), or when the distribution of returns is normal (see, e.g. [34]). However, empirically there is evidence that the returns are not distributed normally or log-normally. Therefore there is a need for other Criteria which define risk differently. One can consider criteria which, instead of measuring risk by variance, take into account the undesirable results, where an investor considers the outcome as undesirable if the return on an asset (or portfolio) is below specific value γ. The investor considers the expected value of undesirable results for the asset (or portfolio) and the unconstrained expected value of the asset. Based on these two numbers the investor will make a decision.

Consider the same assets A and B. No matter what the investor's value of γ is, the undesirable expected return is the same for both assets, $\max(\frac{\gamma+10\%}{2}, 10\%)$, while the unconstrained expected return for asset A, 11%, is higher than for asset B, 10.5%. Therefore any investor with any utility function (which increases with expected return) will prefer asset A.

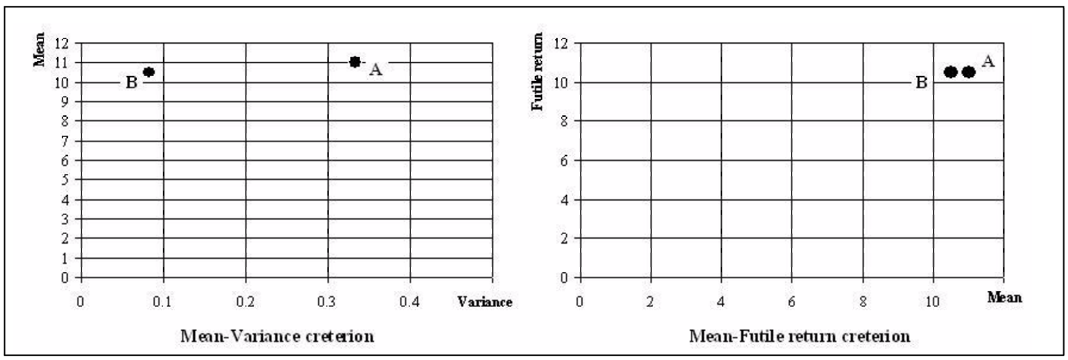

Fig. 4. Comparison of Mean-Variance criterion approach and Mean-Futile return criterion approach. *Using the MV approach it is impossible to tell wich asset is more preferable for investors with different utility finctions, however, the second approach shows clearly that asset A is better for* any *investor.*

9 Model.

Different investors think about risk differently. It may be "Investment products offered through XYZ are not guaranteed by XYZ, are not FDIC-insured, and may lose value", the other case might be "I don't want my portfolio to lose more than 15%", or "return compared to benchmark such as S&P 500", where the return less than the S&P 500 is 'bad' and if it is above, it is considered 'good'. Examples above show that even professional investors do not look at risk as simple variance of returns. Majority of less educated investors do not even understand the notion of variance and, therefore, look for something different as a measure of downside effects (risk).

In the model below, as a measure of risk I propose to use the expected value of "futile" returns (for formal definition see below). In order to decide whether return is not good, one can use different benchmarks, such as 0 (no loss in value), r_f (to make at least what one can make without a risk), or a market index, such as commonly used S&P 500 (beat/not beat the market).

An investor wants to maximize the unconstrained expected return of her portfolio, ${}_1E(r_p)$ consisting of a risk-free security with risk-free return rate r_{r_free} and weight w and risky asset with return r_{risky} and weight $(1-w)$

$$ {}_1E(r_p) = \int ((1-w) \times r_{risky} + w \times r_{r\_free}) \times f(r_{risky}) dr_{risky} \tag{54} $$

while keeping low returns under control, in other words striking the balance between unconstrained returns and expected futile returns ${}_1\phi(r_p)$

$$ {}_1\phi(r_p) = \int\limits_{((1-w)\times r_{risky}+w\times r_{r\_free})<\gamma} ((1-w) \times r_{risky} + w \times r_{r\_free}) \times f(r_{risky}) dr_{risky} \tag{55} $$

(where sub-index 1 signifies the number of risky securities).

For a portfolio consisting of two or more risky assets the definition of expected futile returns is very similar:

$$ {}_n\phi(r_p) = \int\limits_{r_{portfolio}<\gamma} (w_a \times r_a + w_b \times r_b + ... w_n \times r_n + w_{r\_free} \times r_{r\_free}) \times f(r_a, r_b, ...r_n) dr_a dr_b ... dr_n \tag{56} $$

where a, b, ... n are risky securities and $f(r_a, r_b, ...r_n)$ is the distribution function of returns.

The most general form for that type of risk function can be written as

$$\Phi_{most\_general}(r_p) = \frac{\int\limits_{r_{portfolio}<\gamma} u(r_{portfolio}, \gamma) \times f(r_i)\Pi dr_i}{\int\limits_{r_{portfolio}<\gamma} f(r_i)\Pi dr_i} \times \upsilon\Big(\int\limits_{r_{portfolio}<\gamma} f(r_i)\Pi dr_i \Big) \quad (57)$$

where $u(x, y)$ is the distaste function of return falling to x below acceptable threshold y, and $\upsilon(x)$ is the weight function which shows how investor weight the probability of falling short in returns in the risk function. The $u(x, y)$ function can be any non-negative function for $x < y$. Note, that $u(x, y) = (y - x)^2$ will give the semi-variance as a measure for risk; $u(x, y) = (x - E(x))^2$ with $y = +\infty$ generates the variance of the portfolio. Other functions such as $u(x, y) = (y - x)e^{-(y-x)/\chi}$ or $u(x, y) = (y - x) \times (1 - e^{-(y-x)/\chi})$, etc. may be considered. $u(x) = xe^{-x/\chi}$ places progressively smaller weight on extremely bad returns, while $u(x) = x(1 - e^{-x/\chi})$ almost disregards the small deviation (much less than χ) from acceptable returns. $\upsilon(x) = 1$ means the investor is interested in the conditional (normalized) expected return of undesirable states, while $\upsilon(x) = x$ means that the investor fully adjusts for the probability that the return will drop below threshold value γ. Although only empirical data can tell us the functions $u(x, y)$ and $\upsilon(x)$ I will assume the following forms: $u(x, y) = y - x$ and consider two possibilities for $\upsilon(x)$: either $\upsilon(x) = 1$ or $\upsilon(x) = x$. The chosen form $u(x, y) = y - x$ is linear in returns falling short by $y - x$ units, so it is placing much smaller weight on large bad deviations from the threshold compared to semi-standard deviation case, e.g. $u(x, y) = (y-x)^2$. This corresponds to the investor having less risk-averse utility function.

Therefore for a portfolio of several assets one may write the following risk measures:

based on conditional expected futile returns

$$\Phi_n(r_p) = \gamma - \frac{\int\limits_{r_{portfolio}<\gamma} r_{portfolio} \times f(r_i)\Pi dr_i}{\int\limits_{r_{portfolio}<\gamma} f(r_i)\Pi dr_i} \tag{58}$$

or based on absolute expected futile returns

$$\Phi_{nn}(r_p) = \int\limits_{r_{portfolio}<\gamma} (\gamma - r_{portfolio}) \times f(r_i)\Pi dr_i \tag{59}$$

where sub-index n signifies *normalized* returns (e.g. $v(x) = 1$), and sub-index nn signifies *non-normalized* (e.g. $v(x) = x$). Theoretically it is not possible to say what measure investors are using in real life and future studies are necessary in order to distinguish between the two. The decision can be made only by comparing them using empirical data.

9.1 Expected Futile Return Models.

The model assumes that the investors think about risk/loss only as the non-normalized expected value of futile returns. The investor estimates the expected value of possible low returns as they contribute to the total unconditional expected return. In other words the investor calculates the conditional expected value of possible low returns and multiplies it by the probability of the low return occurrence. The positive side of that definition is that the risk/loss function, defined by futile returns, is weighted according to the probability. If the expected futile return is very small (large negative) but the probability of it is very low, the investor will not be worried about that greatly.

9.1.1 Optimization.

One Deterministic Asset and One Stochastic Asset. Consider the portfolio which consists of a deterministic asset and a stochastic asset. An investor contributes w fraction of total wealth to the deterministic asset and $1 - w$ to the stochastic asset. The return on that portfolio, r_p, is

$$r_p = w r_{rf} + (1 - w) r_a \tag{60}$$

where r_{rf} is return on the deterministic asset and r_a is return on the other asset.

One can easily find the unconditional expected return on the portfolio

$$_1E_{nn}(r_p) = w r_{rf} + (1 - w)\mu_a \tag{61}$$

where μ_a is unconditional expected return on risky portfolio. The risk is determined by the low returns of the portfolio. Investors don't like returns below some value γ and consider all return below that value as non-acceptable. Investors want to minimize risk, $_1\Phi_{nn}(r_p)$, defined by

$$\begin{aligned} {}_1\Phi_{nn}(r_p) &= \int\limits_{((1-w)\times r_{risky}+w\times r_{rf})<\gamma} (\gamma - ((1-w)\times r_{risky} + w \times r_{rf})) \times f(r_{risky}) dr_{risky} = \\ &= (\gamma - w \times r_{rf}) \int\limits_{((1-w)\times r_{risky}+w\times r_{rf})<\gamma} f(r_{risky}) dr_{risky} - \\ &\quad (1-w) \int\limits_{((1-w)\times r_{risky}+w\times r_{rf})<\gamma} r_{risky} \times f(r_{risky}) dr_{risky} \end{aligned} \tag{62}$$

That is the simplest form for risk until we specify the return distribution of risky asset. Because later we will be interested in the risk of portfolio consisting of the Market and a risk-free asset, I will consider the risk associated with that portfolio. Since the market portfolio consists of many different assets, it is reasonable to assume that the distribution of return on the market is very close to a Gaussian distribution. In the paper I will assume that the market return distribution is indeed Gaussian $N(\mu_m, \sigma_m)$, while the distribution of returns on any risky asset is not necessarily Gaussian. For $w < 1$ the risk, or loss, function for portfolio consisting of the risk-free asset and the Market, is

$$
\begin{aligned}
{}_1\Phi_{nn}(r_p)_{w<1} &= ([\mu_m + w \times (r_f - \mu_m)] - \gamma) \times \frac{\operatorname{erf}(\kappa) - 1}{2} + \frac{1-w}{\sqrt{2\pi}} \sigma_m \exp(-\kappa^2) \\
&= \frac{1-w}{\sqrt{2\pi}} \sigma_m \times [\sqrt{\pi}\kappa \times (\operatorname{erf}(\kappa) - 1) + \exp(-\kappa^2)] \qquad (63)
\end{aligned}
$$

and for $w > 1$

$$
\begin{aligned}
{}_1\Phi_{nn}(r_p)_{w>1} &= -([\mu_m + w \times (r_f - \mu_m)] - \gamma) \times \frac{\operatorname{erf}(\kappa) + 1}{2} - \frac{1-w}{\sqrt{2\pi}} \sigma_m \exp(-\kappa^2) \\
&= -\frac{1-w}{\sqrt{2\pi}} \sigma_m \times [\sqrt{\pi}\kappa \times (\operatorname{erf}(\kappa) + 1) + \exp(-\kappa^2)] \qquad (64)
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa &= \frac{\mu_m - \frac{\gamma - w r_f}{1-w}}{\sqrt{2}\sigma_m} = \frac{1}{\sqrt{2}(1-w)} \frac{{}_1E_{nn}(r_p) - \gamma}{\sigma_m} \qquad (65) \\
&= \frac{1}{\sqrt{2}} \times \frac{{}_1E_{nn}(r_p) - \gamma}{{}_1E_{nn}(r_p) - r_f} \times \frac{\mu_m - r_f}{\sigma_m} \qquad (66)
\end{aligned}
$$

and $\operatorname{erf}(x)$ is the error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int\limits_0^x \exp(-t^2) dt$. From equation 66 we see the meaning of κ. It is the leveraged Sharpe ratio $\frac{\mu_m - r_f}{\sigma_m}$. The leverage coefficient is the ratio of ${}_1E_{nn}(r_p) - \gamma$, or how well the portfolio is expected to perform above the minimum acceptable level γ, to the ${}_1E_{nn}(r_p) - r_{f,}$ how well the portfolio is expected to perform relative to the risk-free asset r_f.

For $\gamma < r_f$ the typical behavior of risk function Φ as a function of expected return ${}_1E_{nn}(r_p)$ is given on the following graph ($r_f = 0.03$, $\mu_m = 0.1$, $\sigma_m = 0.1$ and $\gamma = 0.01$)

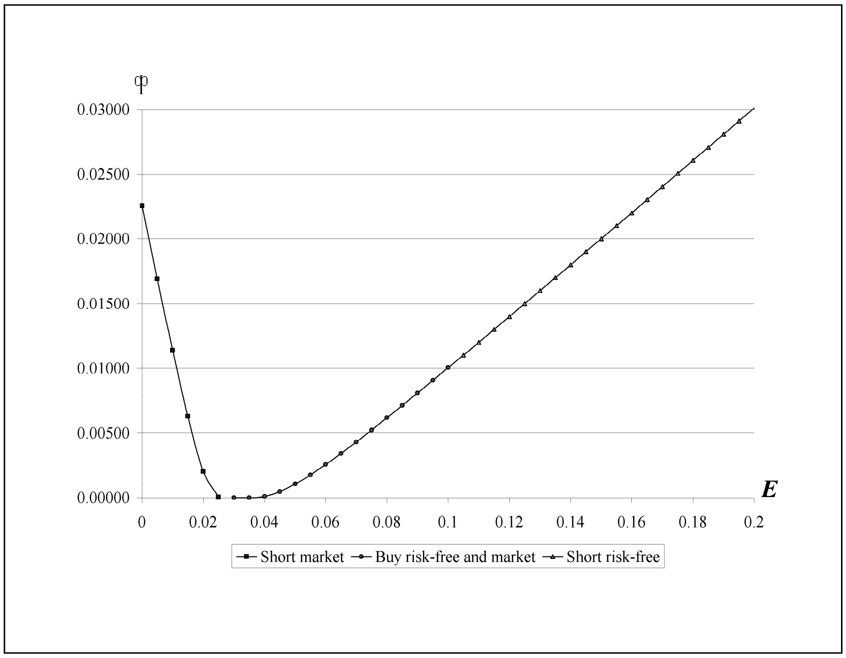

Fig. 5. Risk - Expected return preferences. *Vertical axis is risk function Φ and horizontal axis is expected return $E(R_{nn\_rf})$.*

As in CAPM we see that there is no reason to short the market, because in that case an investor will have much smaller return at the same level of risk/expected loss. Also, similar to CAPM, if the expected return on the portfolio increases, asymptotically the risk function increases linearly with the expected return

$$-\left[\gamma\frac{\mathrm{erf}(\frac{\mu_m-r_f}{\sqrt{2}\sigma_m})-1}{2}+\frac{r_f}{\mu_m-r_f}\frac{\sigma_m}{\sqrt{2\pi}}e^{-\frac{(\mu_m-r_f)^2}{2\sigma_m^2}}\right]+$$
$$+\left[\frac{\mathrm{erf}(\frac{\mu_m-r_f}{\sqrt{2}\sigma_m})-1}{2}+\frac{1}{\mu_m-r_f}\frac{\sigma_m}{\sqrt{2\pi}}e^{-\frac{(\mu_m-r_f)^2}{2\sigma_m^2}}\right]\times E \tag{67}$$

As the expected return decreases to negative infinity, the risk function also asymptotically approaches the linear function, but it is not of interest, because such behavior is sub-optimal.

Two Stochastic Assets. Now we will turn our attention to the case of two stochastic assets. The Mean-Variance Criterion considers only the first and second moments of the distribution. That information is sufficient if the distribution is jointly-normal (or the investors utility function is quadratic), which is not observed in real world. That is why I will consider the joint distribution $f(r_a, r_b)$ which is not Gaussian. The investor looks for a trade-off between higher unconstrained expected return

$$ {}_2E_{nn}(r_p) = w\mu_a + (1-w)\mu_b \tag{68}$$

and lower expected futile return

$$ {}_2\Phi_{nn}(r_p) = \iint\limits_{w\times r_a+(1-w)\times r_b<\gamma} (\gamma - (w \times r_a + (1-w) \times r_b)) \times f(r_a, r_b) dr_a dr_b \tag{69}$$

Unless one know the analytical formula for the distribution function, eq. 69 cannot be simplified.

Now we would like to compare the decisions of choosing the optimal portfolio based on the Mean-Variance Criterion and the proposed criterion. I assume there is one risk-free asset and only two risky assets to choose from. The return on the risk-free asset is $r_f = 0.03$. The actual joint distribution of probabilities for two risky assets is a non-symmetric distribution pictured on figures below:

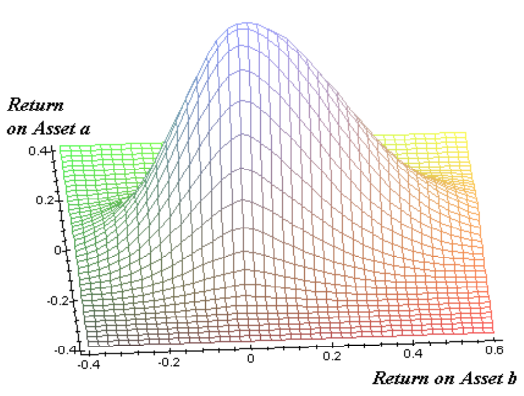

Fig. 6. Three-dimensional representation of non-Gaussian distribution of returns on assets A and B.

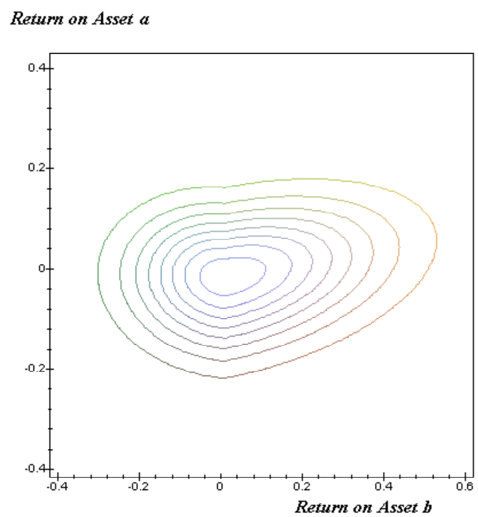

Fig. 7. Contour representation of non-Gaussian distribution of assets' A and B returns.

The first picture gives the three dimensional view of the distribution function, while the second gives the contours of equal values in *pdf*. An investor who bases her decisions on Mean-Variance approach needs to know first and second moments of that distribution. It is very easy to find them. The corresponding (rounded) moments are: expected returns $\mu_a = 0.054$, $\mu_b = 0.09$, standard deviations $\sigma_a = 0.2$, $\sigma_b = 0.4$, and correlation coefficient $\rho_{ab} = -0.1$. She does not pay attention to the non-symmetry of the distribution and is not interested in higher moments. Now she can use a standard technique to find the dependence of risk, as measured by the standard deviation of portfolio, on the expected return of portfolio. Taking into account that she can borrow/lend at risk-free rate $r_f = 0.03$, she decides to contribute $w = 0.64$ or 64% of her risky holdings into asset a and $1 - w = 0.36$ or 36% of her risky holdings into asset b. She barely decreases her risk (by 14%) as compared to risk of asset a, and more than twice reduces her risk as compared to asset b, however, she increases her expected return by roughly 24% compared to asset a. What proportion of her wealth she puts in risk-free asset and what proportion into the risky asset cannot be found without further assumptions about utility function over benefits (expected return) and risks (standard deviation) of the portfolio.

The second investor, who bases his decision on the proposed model, takes into account all points of the distribution, therefore accounting for all moments. Unfortunately, he pays the price for that. If the distribution is not known analytically, the only way to calculate the needed values is to calculate them numerically, which is generally resource- and time-consuming. Using eq.s 68 and 69 the investor can calculate expected return and corresponding risk for portfolios with different composition of assets a and b. Combined with ability to borrow/lend at risk-free rate, the investor finds the optimal portfolio to consist of $w = 0.61$ or 61% of asset a and $1-w = 0.39$ or 39% of asset b. The investor decreases his risk by 28% as compared to risk of asset a, and more than three times reduces the risk as compared to asset b. The increase in expected return is approximately 25% compared to asset a.

The picture below plots both decision making graphs on the same picture. One can see that the second investor behaves as if he had access to asset with higher than 3% risk-free rate.

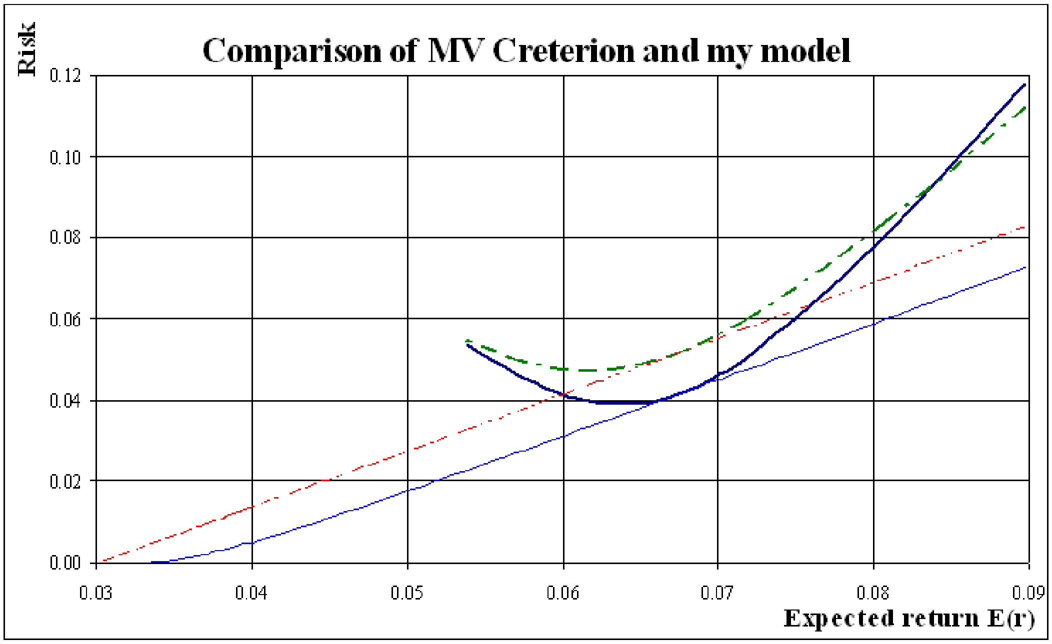

Fig. 8. Comparison of Mean-Variance Criterion and my model. *Dashed lines – 2-stock Efficient frontier (Capital Market line) and preference function for investor who behaves according to Mean-Variance Creterion; solid lines for investor who behaves accoring to my model. Investor who behaves according to my model realizes higher expected return and percieves lower risk.*

Although the risks are not directly comparable, the second investor will feel better, as his risk is reduced more in relative terms. In addition he will enjoy a higher expected rate of return.

9.1.2 Partial Equilibrium in the Model.

Expected Return on Stochastic Asset. In equilibrium, all rational homogenous investors hold the same portfolio. Everybody holds the same optimal proportion of all stochastic market assets. Because the market capitalization of any particular stock is much smaller than the total market capitalization, the proportion of each stock holdings in the optimal portfolio is very small and for calculation purposes can be assumed to be zero.

If one adds the newly issued stock to his portfolio in zero weight, the portfolio's return still be the same, market return, and the expected probable loss (expected value of futile returns) will not change either. Therefore on the plane $\Phi(r_p)$ vs. $E(r_p)$ the line of new asset–market different combinations should go through the market point, $(E_{market}, \Phi_{market})$. The new holding should not create room for improving both expected return and expected probable loss. Otherwise the asset will be highly desirable and every investor starts buying it, bidding the price up and the return down until the market finds a new equilibrium and the proportion of the new asset in everybody's portfolio is close to zero. Mathematically it can be expressed as that the derivatives $\frac{d\Phi}{dE}$ of portfolio consisting of the market and new asset portfolio and the portfolio consisting of the market and deterministic asset must be the same at the point where portfolios consist of market only. Indeed, assume that that is not the case. Consider that the derivative of the market and new asset portfolio is less than derivative of the market and deterministic asset portfolio. Because marginally the first portfolio provide smaller increase in the probable expected loss, one can short (buy) the new asset (if $\mu_a < \mu_m$ ($\mu_a > \mu_m$)), buy the market portfolio and invest to (borrow from) the deterministic asset. The created portfolio will either have higher expected return (at the same level of expected probable loss), or lower expected probable loss (at the same level of expected return), or both. Therefore we conclude that

$$\frac{d_1\Phi_{nn}}{d_1E_{nn}} = \frac{d_2\Phi_{nn}}{d_2E_{nn}} \tag{70}$$

From the equation 70 and the equations 63 and 69 on risk functions Φ, we can derive the relationship between μ_a and other variables

$$\begin{aligned}
\mu_a - r_f &= \left(1 - \frac{\iint\limits_{r_m<\gamma} (r_a - r_m) \times f(r_a, r_m) dr_a dr_m}{(r_f - \mu_m)\frac{1+\mathrm{erf}(\kappa_m)}{2} + \frac{\sigma_m}{\sqrt{2\pi}}\exp(-\kappa_m^2)}\right) \times (\mu_m - r_f) = \\
&= \chi_{nn} \times (\mu_m - r_f) \qquad (71)
\end{aligned}$$

where $\kappa_m = \frac{\mu_m - \gamma}{\sqrt{2}\sigma_m}$ and $f(r_a, r_m)$ is the joint distribution function for the new asset's and market's returns. If the joint distribution function is Gaussian, eq.71 reduces to the classical CAPM result (see Appendix for a proof)

$$\mu_a - r_f = \rho\frac{\sigma_a}{\sigma_m} \times (\mu_m - r_f) = \beta \times (\mu_m - r_f) \qquad (72)$$

9.2 Conditional Expected Futile Return Models.

The most important distinction of the model below is that it normalizes the expected value of bad returns by the probability of bad return occurrence. In terms of eq. 57 the function $\upsilon(x) = 1$. That definition is a more accurate definition of conditional expected futile returns, but lacks the ability to capture the weight of occurrence of the bad returns. The good side is that the investor sees the expected bad return conditional on its occurrence. It is reasonable to assume that the investor wants to know how much he is going to lose if something bad happens (in some sense it is similar to Value-at-Risk). The intuition and consideration is the same as in the first model case. The only difference is numerical equations and formulas.

9.2.1 Optimality.

One Deterministic Asset and One Stochastic Asset (Market). The portfolio consists of a deterministic asset and a stochastic asset. w is a fraction of total wealth contributed to the deterministic asset and $1-w$ is contributed to the stochastic asset. The return on the portfolio, r_p, is

$$r_p = wr_{rf} + (1-w)r_m \tag{73}$$

where r_{rf} is return on the deterministic asset and r_m is return on the other asset.

One can easily find unconditional expected return on the portfolio

$$_1E_n(r_p) = wr_{rf} + (1-w)\mu_m \tag{74}$$

where μ_m is unconditional expected return on risky portfolio. Again, we will introduce the expected probable low return function, $_1\Phi_n(r_p)$, defined by

$$\begin{aligned}
{}_1\Phi_n(r_p) &= \frac{\displaystyle\int\limits_{((1-w)\times r_m + w\times r_{rf})<\gamma} (\gamma - ((1-w)\times r_m + w\times r_{rf}))\times f(r_m)dr_m}{\displaystyle\int\limits_{((1-w)\times r_m + w\times r_{rf})<\gamma} f(r_m)dr_m} = \\
&= \gamma - w\times r_f - \frac{(1-w)\times \displaystyle\int\limits_{((1-w)\times r_m + w\times r_{rf})<\gamma} r_m \times f(r_m)dr_m}{\displaystyle\int\limits_{((1-w)\times r_m + w\times r_{rf})<\gamma} f(r_m)dr_m} = \\
&= \gamma - w\times r_f - (1-w)\times\mu_m + \\
&\quad + \sqrt{\frac{2}{\pi}}\times(1-w)\times\sigma_m\times\frac{e^{-\kappa^2}}{1-\mathrm{erf}(\kappa)}
\end{aligned} \tag{75}$$

where $w<1$,

$$\kappa = \frac{\mu_m - \frac{\gamma - wr_f}{1-w}}{\sqrt{2}\sigma_m} = \frac{(wr_f - (1-w)\mu_m) - \gamma}{\sqrt{2}\sigma_m(1-w)} \tag{76}$$

$$= \frac{1}{\sqrt{2}}\times\frac{_1E_n(r_p) - \gamma}{_1E_n(r_p) - r_f}\times\frac{\mu_m - r_f}{\sigma_m} \tag{77}$$

and $\operatorname{erf}(x)$ is the error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int\limits_0^x \exp(-t^2)dt$.

Two Stochastic Assets. Consider the case of two stochastic assets. I will consider the case where one of the stochastic assets is the asset with normally distributed unconditional returns. The joint distribution $f(r_a, r_m)$, however, is not assumed to be Gaussian. The investor needs to choose between higher unconstrained expected return

$$ {}_2E_n(r_p) = w\mu_a + (1-w)\mu_m \tag{78} $$

and lower expected futile return

$$ {}_2\Phi_n(r_p) = \gamma - \frac{\iint\limits_{w\times r_a+(1-w)\times r_m<\gamma} (w \times r_a + (1-w) \times r_m) \times f(r_a, r_m) dr_a dr_m}{\iint\limits_{w\times r_a+(1-w)\times r_m<\gamma} f(r_a, r_m) dr_a dr_m} \tag{79} $$

As in the first model if one makes no additional assumptions about the distribution function, that equation cannot be simplified.

9.2.2 Partial Equilibrium.

Expected Return on Stochastic Asset. In equilibrium, all rational homogeneous investors are expected to hold the same portfolio. Therefore the proportion of each stock holdings in the optimal portfolio is very small and can be assumed to be zero. As in first model, the derivatives $\frac{d\Phi}{dE}$ of the market and stochastic asset portfolio, and the market and deterministic asset portfolio must be the same at point $w = 0$.

$$ \frac{d_1\Phi_n}{d_1E_n} = \frac{d_2\Phi_n}{d_2E_n} \tag{80} $$

From the equation 80 and the equations 78 and 79 we can derive the relationship between μ_a and other variables.

$$ \mu_a - r_f = $$

$$= \left(1 - \frac{\frac{1}{\zeta_0} \times \left(\zeta_a - \zeta_m + (r_f - \mu_m + \sqrt{\frac{2}{\pi}}\sigma_m \frac{\exp(-\kappa_m^2)}{1-\mathrm{erf}(\kappa_m)}) \times \int\limits_{-\infty}^{+\infty} (\gamma - r_a) \times f(r_a, \gamma) dr_a\right)}{r_f - \mu_m + \sqrt{\frac{2}{\pi}}\sigma_m \frac{\exp(-\kappa_m^2)}{1-\mathrm{erf}(\kappa_m)} \times (1 + 2\kappa_m \kappa_{rf} - \frac{2\kappa_{rf}}{\sqrt{\pi}} \frac{\exp(-\kappa_m^2)}{1-\mathrm{erf}(\kappa_m)})}\right) \times$$

$$\times(\mu_m - r_f) \qquad \text{or}$$

$$= \chi_n \times (\mu_m - r_f) \tag{81}$$

where

$$\zeta_0 = \iint\limits_{w\times r_a + (1-w)\times r_m < \gamma} f(r_a, r_m) dr_a dr_m \tag{82}$$

$$\zeta_a = \iint\limits_{w\times r_a + (1-w)\times r_m < \gamma} r_a \times f(r_a, r_m) dr_a dr_m \tag{83}$$

$$\zeta_m = \iint\limits_{w\times r_a + (1-w)\times r_m < \gamma} r_m \times f(r_a, r_m) dr_a dr_m \tag{84}$$

$$\kappa_m = \frac{r_f - \gamma}{\sqrt{2}\sigma_m} \tag{85}$$

$$\kappa_m = \frac{\mu_m - \gamma}{\sqrt{2}\sigma_m} \tag{86}$$

and $f(r_a, r_m)$ is the joint distribution function for the new asset's and market's returns. If the joint distribution function is Gaussian, eq.81 reduces to the classical CAPM result as in the first model.

10 Comment about α.

The proposed approach allows us to explain the existence of non-zero α in regression of asset returns on market return. Indeed, if the distribution of returns is not mutually Gaussian (normal), the coefficient of proportionality in eq. 71 or eq. 81 is not β. Therefore, the regression line of $r_a - r_f$ on $r_m - r_f$ should not go through zero or, in other words, $\alpha \neq 0$

$$\begin{aligned}\alpha &= \left(1 - \frac{\iint\limits_{r_m<\gamma} (r_a - r_m) \times f(r_a, r_m) dr_a dr_m}{(r_f - \mu_m)\frac{1+\mathrm{erf}(\kappa_m)}{2} + \frac{\sigma_m}{\sqrt{2\pi}}\exp(-\kappa_m^2)} - \beta\right) \times (\mu_m - r_f) \\ &= (\chi - \beta) \times (\mu_m - r_f) \neq 0 \qquad (87)\end{aligned}$$

where

$$\chi = 1 - \frac{\iint\limits_{r_m<\gamma} (r_a - r_m) \times f(r_a, r_m) dr_a dr_m}{(r_f - \mu_m)\frac{1+\mathrm{erf}(\kappa_m)}{2} + \frac{\sigma_m}{\sqrt{2\pi}}\exp(-\kappa_m^2)} \neq \beta \qquad (88)$$

The following picture illustrate the possible returns distribution (such that $\alpha \neq 0$)

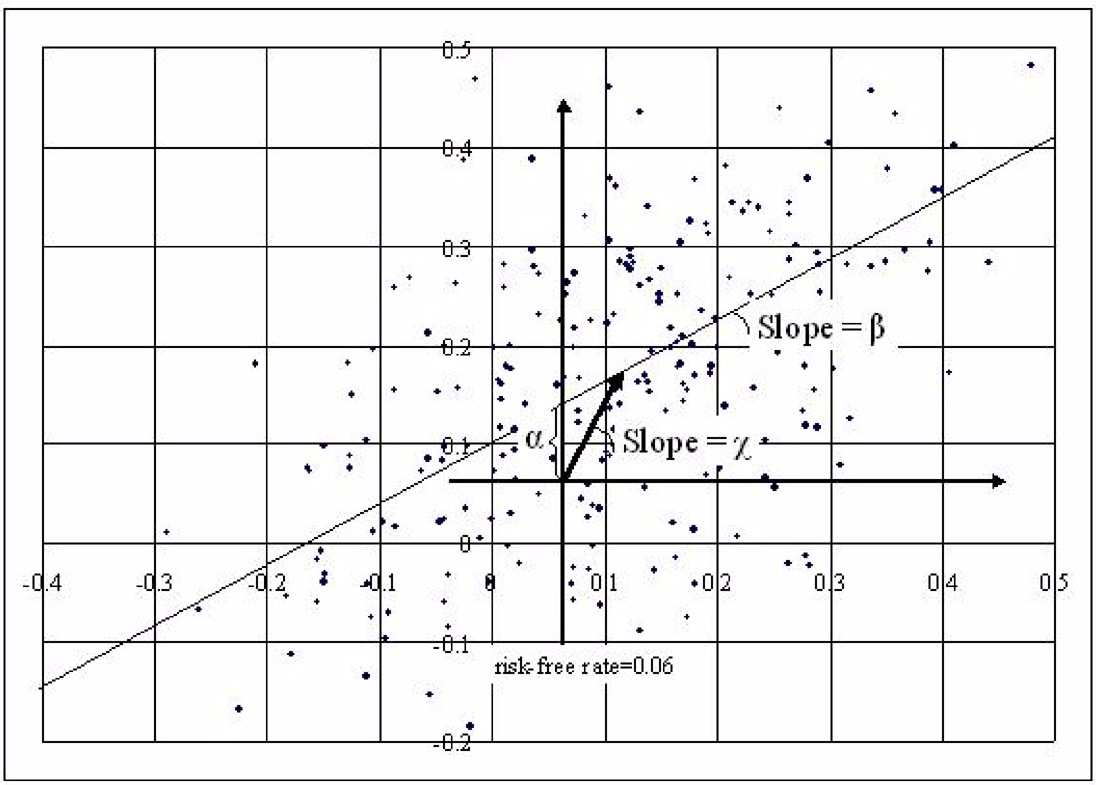

Fig. 9. Illustration of non-zero value of α. *Slope* β *as the covariance between returns and slope* χ *as coefficient of proportionality between excess return on stock and excess return on market in my model are not necesserely the same. That gives rise to non-zero value of* α.

11 Conclusions.

In this paper I have considered a model in which risk is defined differently from previous models. Specifically the risk is defined as the expected value of the returns below a specified threshold. The specific analytical form (e.g. functions and coefficients in eq.57) depends upon the specific form of a representative investor's utility function over returns. The question of what utility function better describes the investor's behavior is an empirical question and cannot be solved analytically. Therefore the question what model is right (e.g. Sortino model [39] , Rubinstein-Leland model [40] , [41] , or model presented above) can be solved only after careful empirical investigation.

I have defined the risk as $\Phi_n(r_p) = \gamma - \frac{\int\limits_{r_{portfolio}<\gamma} r_{portfolio} \times f(r_i)\Pi dr_i}{\int\limits_{r_{portfolio}<\gamma} f(r_i)\Pi dr_i}$ for the portfolio. Based on that definition I am able to derive the partial equilibrium for the asset and have found the expected return on the asset (eq.s 71 and 81 for two similar models). The results are simple to interpret. The expected return on the asset is always proportional to the market premium, but the coefficient of proportionality is no longer the (normalized) covariance of returns between the asset and the market. More than that the coefficient may be presented as a sum of terms each depending upon other specific risks

$$\mu_a - r_f = (\beta + \beta_{risk\_1} \times F_{risk\_1} + ... + \beta_{risk\_k} \times F_{risk\_k}) \times (\mu_m - r_f) \tag{89}$$

where $\beta + \beta_{risk\_1} \times F_{risk\_1} + ... + \beta_{risk\_k} \times F_{risk\_k}$ is the Taylor series of the coefficient in 71 or 81. From that equation we see that all combined risks $F_{risk\_z} \times (\mu_m - r_f)$ go to zero as the market premium goes to zero. One can orthogonalize the risks, but still all of them will depend on $(\mu_m - r_f)$ and will not be independent.

Several test may be designed to check the theory. The first one is to look at the equations 71 or 81 directly. One can measure distributions $f(r_a, r_m)$ and find, at least in principle, values χ. Knowing χ it is possible to calculate the values of $r_a - r_m$ and compare it to the empirical values.

The second approach is to check predictions for α (eq.88). Knowing χ one can calculate predicted values of α and compare it to the observed values.

The third one is to check eq.89.The theory predicts that the excess return on stock $r_a - r_m$ must depend not on separate risk factors as *Price/Earnings ratio* PE, *Market-to-Book ratio* MtB, etc., but rather on the product of these ratios to the excess market return. E.g. one should regress $r_a - r_f$ on $PE \times (r_m - r_f)$ (or $(PE - averagePE) \times (r_m - r_f)$, which will re-normalize coefficient before $(r_m - r_f)$ without changing the meaning of the equation), $MtB \times (r_m - r_f)$, *Size* $\times (r_m - r_f)$, etc. That regression should give much better results compared to regression on separate risk factors.

Appendix 1.

Hyperbolic Absolute Risk Aversion (HARA) function.

HARA family of utility functions is defined as follows.

$$-\frac{u(x)''}{u(x)'} = \frac{1-\gamma}{ax+b} > 0 \tag{90}$$

where a, b, and γ are some constants. Assuming $ax + b > 0$ one can see that $\gamma < 1$. There are three subfamilies of solutions.

If $\gamma = 0$, $a \neq 0$ the solution is

$$u(x) = u_0 + u_n \ln(ax + b) \tag{91}$$

If $\gamma \neq 0$, $\gamma < 1$, $a \neq 0$ the solution is

$$u(x) = u_0 + u_n \frac{(ax+b)^\gamma - 1}{\gamma} \tag{92}$$

If $\gamma < 1$, $a = 0$ the solution is

$$u(x) = u_0 - u_n \exp(-\frac{1-\gamma}{b}x) \tag{93}$$

where u_0 and $u_n > 0$ are some constants.

Appendix 2.

Proof of reducing the proposed theory to CAPM under the assumption of the mutually normal distribution of returns.

Let the mutual distribution of a stock and market returns be Gaussian

$$
\begin{aligned}
f(r_a, r_m) &= \\
&= \frac{1}{2\pi\sigma_a\sigma_m\sqrt{1-\rho_{a,m}^2}} \times \\
&\times \exp\left\{-\frac{1}{2(1-\rho_{a,m}^2)}\left(\frac{(r_a-\mu_a)^2}{\sigma_a^2} - 2\rho\frac{(r_a-\mu_a)(r_m-\mu_m)}{\sigma_a\sigma_m} + \frac{(r_m-\mu_m)^2}{\sigma_m^2}\right)\right\}
\end{aligned} \tag{94}
$$

Then, if the holdings of particular asset is very low, e.g. $w = 0$

$$
\iint\limits_{w=0, r_m<\gamma} (r_a - r_m) \times f(r_a, r_m) dr_a dr_m = (\mu_a - \mu_m)\frac{1+\mathrm{erf}(\kappa_m)}{2} + \frac{\sigma_m - \rho_{a,m}\sigma_a}{\sqrt{2\pi}}\exp(-\kappa_m^2) \tag{95}
$$

Substituting it into 71

$$
\mu_a - r_f = \left(1 - \frac{(\mu_a-\mu_m)\frac{1+\mathrm{erf}(\kappa_m)}{2} + \frac{\sigma_m-\rho_{a,m}\sigma_a}{\sqrt{2\pi}}\exp(-\kappa_m^2)}{(r_f-\mu_m)\frac{1+\mathrm{erf}(\kappa_m)}{2} + \frac{\sigma_m}{\sqrt{2\pi}}\exp(-\kappa_m^2)}\right) \times (\mu_m - r_f) \tag{96}
$$

and solving for μ_a one receives

$$
\mu_a - r_f = (\mu_m - r_f) \times \rho_{a,m} \times \frac{\sigma_a}{\sigma_m} = \beta_a \times (\mu_m - r_f) \tag{97}
$$

where

$$
\beta_a = \rho_{a,m} \times \frac{\sigma_a}{\sigma_m} \tag{98}
$$

That is CAPM statement (eq.72).

One should proceed similarly in order to proof that 81 reduces to CAPM in case of mutual Gaussian distribution.

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