A Topological Classification of D-Dimensional Cellular Automata

by
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ABSTRACT

EMILY GAMBER: A Topological Classification of D-Dimensional Cellular Automata

(Under the direction of Professor Jane Hawkins)

We give a classification of cellular automata in arbitrary dimensions and on arbitrary subshift spaces from the point of view of symbolic and topological dynamics. A cellular automaton is a continuous, shift-commuting map on a subshift space; these objects were first investigated from a purely mathematical point of view by Hedlund in 1969. In the 1980's, Wolfram categorized one-dimensional cellular automata based on features of their asymptotic behavior which could be seen on a computer screen. Gilman's work in 1987 and 1988 was the first attempt to mathematically formalize these characterizations of Wolfram's, using notions of equicontinuity, expansiveness, and measure-theoretic analogs of each. We introduce a topological classification of cellular automata in dimensions two and higher based on the one-dimensional classification given by Kůrka. We characterize equicontinuous cellular automata in terms of periodicity, investigate the occurrence of blocking patterns as related to points of equicontinuity, demonstrate that topologically transitive cellular automata are both surjective and have sensitive dependence on initial conditions, and construct subshift spaces in all dimensions on which there exists an expansive cellular automaton. We provide numerous examples throughout and conclude with two diagrams illustrating the interaction of topological properties in all dimensions

for the cases of an underlying full shift space and of an underlying subshift space with dense shift-periodic points.

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CHAPTER 1

Introduction

A cellular automaton is a tool used to model complex systems, making discrete simulations of intricate processes. Originally introduced by John von Neumann, following a suggestion of Stanislaw Ulam in the early 1950's, the purpose of this new tool was to construct a simple mathematical model capable of both universal computation and self-reproduction [**1**]. High performance computer systems and parallel processing have contributed to the popularity of cellular automata; computer implementation is quite easy due to the local and parallel nature of these objects. Various types of processes are simulated with cellular automata, cutting across many academic disciplines. Spin glass systems, reaction/diffusion processes in physics, tumor growth and excitement of muscle tissue in biology, and simulation of Turing machines in computer science are just a few of the existing applications [**15**].

Cellular automata were first investigated from a purely mathematical point of view in 1969 with Hedlund's formative paper [**12**]. This work was motivated by then-current problems in symbolic dynamics, possibly those of a cryptographic nature. When Wolfram turned his attention to cellular automata via computer simulation in the early 1980's, the subject gained momentum. Wolfram categorized one-dimensional cellular automata based on features of their asymptotic behavior which could be seen on a computer screen

[**34, 35**]. Gilman's work in 1987 and 1988 was the first attempt to mathematically formalize these characterizations of Wolfram's [**9, 10**]. He utilized the notions of equicontinuity and expansiveness, as well as measure theoretic analogs of each. There are other classifications of one-dimensional cellular automata based on different types of properties, see e.g., [**20**] and the references therein. While measure is intrinsic to Gilman's partition, Kůrka has a purely topological classification centered on equicontinuity, expansiveness, and sensitivity [**19**], and Hurley has categorized cellular automata by their attractors [**13**].

Although Ishii has developed a measure theoretic version of Wolfram's classification in dimension two [**14**], much of the literature devoted to higher dimensional cellular automata pertains to the computational complexity and decidability of various properties. Manzini, Margara, and others have examined a variety of properties of linear cellular automata, that is those whose local rule is only a linear combination of the neighbors' values, in higher dimensions [**6, 22**].

Here, we extend the one-dimensional topological classification of Kůrka for cellular automata on the full shift space, to higher dimensional subshift spaces. Our classification centers on equicontinuity, the topological property of almost equicontinuity, sensitive dependence on initial conditions, and expansivity. Some of the results from the one-dimensional case extend to full shift spaces in higher dimensions and to subshift spaces having dense shift-periodic points. However, the classification as a whole does not move up to all dimensions. In particular, there is a notion of a blocking word in dimension one that characterizes almost equicontinuous cellular automata as exactly those which do not have sensitive dependence on initial conditions [**19**]. This is due to the fact that, in one dimension, discrepancies in initial points can propagate towards the center from only the right or the left. In higher dimensions, however, there are many more directions in which an initial difference can alter a value, and so the dichotomy result does not extend as is to all dimensions. We introduce other notions of blocking, those of fully blocking and of blocking a cross, in order to obtain sufficient conditions for a cellular automaton to be almost equicontinuous. The dimension of the shift space also has an impact on the sheer existence of expansive cellular automata. While there are many examples of expansive cellular automata on one-dimensional full shift spaces, Shereshevsky has shown that an expansive cellular automaton can not exist on a full shift space in dimension higher than 1 [**31**]. To counter this, we construct subshift spaces in all dimensions on which there is an expansive cellular automaton, and investigate a class of subshifts on which expansive cellular automata can exist.

We begin with the basic definitions for symbolic dynamics and cellular automata in Section 2.1, and give three examples of cellular automata, one on a one-dimensional full shift space, one on a two-dimensional full shift space, and one on a two-dimensional subshift space, in Section 2.3. The remainder of Chapter 2 gives the basic definitions for more general topological dynamical systems, and concludes with a dichotomy result for cellular automata: one must either have sensitive dependence on initial conditions, or there exists a point of equicontinuity.

In Chapter 3, we address the property of equicontinuity. We first give an equivalent definition for this property particular to cellular automata, and extend the following two one-dimensional results from [**20**] to the setting where the underlying shift space is a subshift on which the shift-periodic points are dense: a cellular automaton is equicontinuous if and only if it is eventually periodic, and a cellular automaton is both surjective and equicontinuous if and only if it is periodic. In Section 3.2, we give a number of examples of cellular automata which are equicontinuous. These include the identity, the zero map, and in fact any cellular automaton with radius 0. Beyond these somewhat trivial examples, we give a construction to build an equicontinuous $(D+1)$-dimensional cellular automaton from a D-dimensional one. In Section 3.3, we investigate periodic points under a cellular automaton which may or may not be points of equicontinuity. First, we show that a periodic point under the shift must be eventually periodic under a cellular automaton, and second, an attracting periodic point for a cellular automaton must be fixed under both the cellular automaton and the shift, generalizing the one-dimensional result in [**13, 20**].

Since equicontinuity is typically too strong a property to expect in general, we next turn our attention to the property of almost equicontinuity, that is, that the set of equicontinuity points is residual. In dimension one, there are three equivalent properties for a CA: being almost equicontinuous, having sensitive dependence on initial conditions, and having so-called blocking words [**19**]. It is an extension of this theorem that we approach in Chapter 4. In order to do so, we introduce the notion of blocking and of fully blocking in all dimensions based on the one-dimensional definition given by Blanchard and Tisseur [**2**]. We discuss the one-dimensional definition and the key idea in this equivalence first in Section 4.1. Then, we move into higher dimensions in Section 4.2. We prove that being almost equicontinuous implies not having sensitive dependence on initial conditions, that not having sensitive dependence on initial conditions implies the existence of blocking patterns, and that the existence of fully blocking patterns implies being almost equicontinuous. However, having fully blocking patterns is not a necessary condition for being almost equicontinuous, and we give another sufficient condition for this property. Section 4.3 provides four examples of cellular automata exhibiting almost equicontinuity, one of which has the two-dimensional Golden Mean subshift as its underlying shift space.

We examine sensitive dependence on initial conditions further in Chapter 5. We begin by returning to our construction of a $(D+1)$-dimensional cellular automaton from a D-dimensional one from Section 3.2. We show that such a cellular automaton has sensitive dependence on initial conditions if and only if the D-dimensional one from which it is built has sensitive dependence on initial conditions. We also discuss topological transitivity, and extend the one-dimensional results that a topologically transitive cellular automaton is surjective, and either has sensitive dependence on initial conditions or consists of a single periodic orbit, as in [**20**]. In Section 5.3, we give some examples of cellular automata having these properties: the directional shifts are all topologically transitive on full shift spaces, and we give two product cellular automata which are sensitive but not transitive.

In Chapter 6, we address expansive cellular automata. By a result of Shereshevsky, there can be no expansive cellular automata on any full shift space in dimension $D \geq 2$ [**31**]. However, we build a subshift space in every dimension on which there is an expansive cellular automaton. To this end, we turn to the work of Boyle and Lind on the subdynamics of an expansive D-dimensional action [**3**]. As the D-dimensional shift action is an expansive action, this gives us information about the directional shifts. In fact, the first expansive cellular automata we use in our construction are directional shifts.

The shift spaces are derived as complete history spaces of a cellular automaton acting on a shift space one dimension lower. Shereshevsky has further shown that if $F : X \to X$ is an expansive cellular automaton, where $X \subseteq A^{\mathbb{Z}^D}$ with $D \geq 2$, then the underlying shift action on X must have entropy zero [**31**]. We show that the complete history spaces have zero entropy with respect to the shift, and as such, these shift spaces can support expansive cellular automata. Finally, we address some examples of subshift spaces with expansive cellular automata in Section 6.3. A large class of these in dimension two is given by Kitchens and Schmidt [**17**], and we discuss a possibility to extend this to a class in every dimension.

We conclude in Chapter 7 with diagrams illustrating the interaction of all the properties discussed in earlier chapters. One diagram holds for cellular automata on a full shift space, and the other holds for cellular automata on a subshift space. The main differences between the two are that first, no expansive cellular automata can exist on a full shift space, and second, our proofs regarding fully blocking patterns rely on the fact that on a full shift space, patterns can always be pieced together in a particular way. In Chapter 8, we give a variety of possibilities to extend the work. These include not only a refinement of the current classification, but moving in entirely new directions as well. We have not yet put any measures on the shift spaces, and certainly investigating measure-theoretic properties will give interesting insight to the nature of cellular automata. Also, the wide range of physical phenomena which can be modeled with cellular automata leave open numerous possibilities for future work.

CHAPTER 2

Preliminaries

Cellular automata are studied and used for modeling in a variety of academic disciplines, and our approach comes from symbolic and topological dynamics. We begin, then, with the basic definitions in symbolic dynamics, fixing a definition for cellular automata in this setting. We illustrate these notions with three examples of cellular automata, one on a one-dimensional full shift space, one on a two-dimensional full shift space, and one on a two-dimensional subshift space. Then we give the basic definitions for more general topological dynamical systems, as our classification is based on topological properties. We conclude the chapter with the first dichotomy result for cellular automata: one must either have sensitive dependence on initial conditions, or there exists a point of equicontinuity.

2.1. Symbolic Systems and Cellular Automata

Many different presentations and notations abound in the literature for symbolic systems, even among papers by the same author; the presentation which follows is a unified conglomeration. A detailed look at this material can be found in [**18, 21, 28**].

Let A be a finite set and $|A|$ its cardinality. For $|A| \geq 2$, A is an *alphabet.* A *word* in A is any finite sequence from A, $u = u_0 \cdots u_{n-1}$. The length of u, $|u|$, is n. A D-dimensional generalization of a word is a *pattern* in A, a set of values from A on a finite path-connected (in $\mathbb{Z}^D$) subset of coordinates $E \subseteq \mathbb{Z}^D$. For instance, the following is a two-dimensional pattern of size $(r+1)\times(s+1)$.

$$u = \begin{array}{ccccc} u_{0,s} & u_{1,s} & u_{2,s} & \cdots & u_{r,s} \\ & & \vdots & & \\ u_{0,1} & u_{1,1} & u_{2,1} & \cdots & u_{r,1} \\ u_{0,0} & u_{1,0} & u_{2,0} & \cdots & u_{r,0} \end{array} . \tag{2.1}$$

Now we form the D-dimensional *full shift spaces*, $A^{\mathbb{Z}^D}$. A point $x \in A^{\mathbb{Z}}$ is a doubly infinite sequence of letters from A,

$$x = \cdots x_{-2}x_{-1}.x_0x_1x_2\cdots , \tag{2.2}$$

where we use a decimal point to denote the 0^{th} position of x. Points in $A^{\mathbb{Z}^2}$ are doubly infinite sequences of points in $A^{\mathbb{Z}}$, arranged vertically:

$$x = \begin{array}{ccccccc} & & & \vdots & & & \\ \cdots & x_{(-2,2)} & x_{(-1,2)} & x_{(0,2)} & x_{(1,2)} & x_{(2,2)} & \cdots \\ \cdots & x_{(-2,1)} & x_{(-1,1)} & x_{(0,1)} & x_{(1,1)} & x_{(2,1)} & \cdots \\ \cdots & x_{(-2,0)} & x_{(-1,0)} & .x_{(0,0)} & x_{(1,0)} & x_{(2,0)} & \cdots \\ \cdots & x_{(-2,-1)} & x_{(-1,-1)} & x_{(0,-1)} & x_{(1,-1)} & x_{(2,-1)} & \cdots \\ \cdots & x_{(-2,-2)} & x_{(-1,-2)} & x_{(0,-2)} & x_{(1,-2)} & x_{(2,-2)} & \cdots \\ & & & \vdots & & & \end{array} , \tag{2.3}$$

where the decimal point denotes the $(0,0)^{th}$ position of x. Shift spaces in higher dimensions are defined similarly; points in $A^{\mathbb{Z}^D}$ are indexed by D-vectors of integers and have values from A at each coordinate. For a point $x \in A^{\mathbb{Z}^D}$ and a subset $E \subseteq \mathbb{Z}^D$, $x|_E$ is the pattern which results from restricting x to the coordinates given by E. If E is infinite, we call $x|_E$ an *infinite pattern*. We say that a word (in one-dimension) or a pattern (in higher dimensions) u *occurs* in a point $x \in A^{\mathbb{Z}^D}$ if there exists a finite subset, $E \subseteq \mathbb{Z}^D$, so that $x|_E = u$. For $n < m$, let $\langle n, m \rangle = \{i \in \mathbb{Z} : n \leq i \leq m\}$ be a closed interval of integers.

For a vector of integers, $\vec{\imath} = (i_1, i_2, \cdots, i_D) \in \mathbb{Z}^D$, denote by $||\vec{\imath}||$ the maximum of the components, $\max\{i_1, i_2, \cdots, i_D\}$. We define a metric d on $A^{\mathbb{Z}^D}$ by setting $d(x, y) = 0$ if $x = y$ and for $x \neq y \in A^{\mathbb{Z}^D}$,

$$d(x,y) = 2^{-k}, \text{ where } k = \inf\{||\vec{\imath}|| : x_{\vec{\imath}} \neq y_{\vec{\imath}}\}. \tag{2.4}$$

Under this metric, points in $A^{\mathbb{Z}}$ are close if they agree on a large central word, $x|_{\langle -k,k \rangle} = y|_{\langle -k,k \rangle}$, points in $A^{\mathbb{Z}^2}$ are close if they agree on a large central square,

$$\begin{array}{ccccc} x_{-k,\,k} & \cdots & x_{0,k} & \cdots & x_{k,\,k} \\ \vdots & & & & \vdots \\ x_{-k,\,0} & \cdots & x_{0,0} & \cdots & x_{k,\,0} \\ \vdots & & & & \vdots \\ x_{-k,\,-k} & \cdots & x_{0,-k} & \cdots & x_{k,\,-k} \end{array}, \tag{2.5}$$

and points in $A^{\mathbb{Z}^D}$ are close if they agree on a large central hypercube.

A basis for the topology determined by this metric is given by the cylinder sets, $[u]_{\vec{\imath}} = \{x \in A^{\mathbb{Z}^D}$ containing the pattern u beginning at the coordinates given by $\vec{\imath}\}$. These sets are both open and closed. As $A^{\mathbb{Z}^D}$ is a countable product of finite discrete spaces for each $D > 0$, the full shift spaces are compact.

To define a map on $A^{\mathbb{Z}^D}$, we simply describe the element in position $\vec{\imath}$ of the image for arbitrary $\vec{\imath} \in \mathbb{Z}^D$. On each full shift space, we have a $\mathbb{Z}^D$ action given by the *shift* *transformations*: for each $n \in \mathbb{Z}^D$, define

$$(2.6) \qquad (\sigma_{\vec{n}} x)_{\vec{\imath}} = x_{\vec{\imath}+\vec{n}}.$$

In one dimension, $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by $\sigma(x)_i = x_{i+1}$; the action of σ shifts terms one position to the left,

$$(2.7) \qquad \sigma(\cdots x_{-1}.x_0 x_1 \cdots) \;=\; \cdots x_{-1} x_0 . x_1 x_2 \cdots .$$

Let $\{\overrightarrow{e_j} : j = 1, \cdots, D\}$ be the standard orthonormal basis vectors of $\mathbb{R}^D$ with a 1 in position j and 0's elsewhere. Generators for the $\mathbb{Z}^D$ shift action are $\{\sigma_{\overrightarrow{e_j}} : j = 1, \cdots, D\}$, the shifts in the directions of each $\overrightarrow{e_j}$. It is straightforward to check that the shift is a continuous action on each of the full shift spaces. A closed, shift-invariant subset $X \subseteq A^{\mathbb{Z}^D}$ is a *subshift space*; every subshift space is a compact metric space using the metric d defined in (2.4).

A *cellular automaton, CA*, on a shift space $X \subseteq A^{\mathbb{Z}^D}$ is a continuous function $F : X \to X$ which commutes with the action of the shift. A trivial example of a CA is $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ itself. A theorem of Curtis, Hedlund, and Lyndon (Theorem 3.4, [**12**]) states that $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a CA if and only if there is a positive integer r, called the *radius*, and a function $f : A^{2r+1} \to A$, called the *local rule*, such that $F(x)_i = f(x_{i-r}, \cdots, x_{i+r})$. The radius of the shift, σ, is 1 and the local rule is $f(a, b, c) = c$. It is also the case in higher dimensions that a self-map of a shift space is a CA if and only if there is a radius and local rule as above; the proof in [**12**] extends. By a relabeling of the alphabet, every CA with radius r is conjugate to a CA with radius 1, so we will often assume radius 1. The definition of a conjugacy is given in Section 2.4.

2.2. Surjectivity of Cellular Automata

One of the earliest discoveries regarding properties of cellular automata were the "Garden of Eden" theorems of Moore and Myhill in 1962 and 1963, respectively [**16**]. A *Garden of Eden* for a CA is a point which is not in the image; it is so-named since a point unobtainable via iteration of the CA can only occur at the beginning of time. These theorems relate the properties of injectivity and surjectivity for two-dimensional cellular automata by passing to the set of points which have only finitely many non-zero entries, called the set of *finite configurations*. Let F_f denote the restriction of F to the set of finite configurations.

THEOREM 2.2.1 (Moore [**25**], Myhill [**26**]). *A cellular automaton* $F : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ *is surjective if and only if* F_f *is injective.*

An easy corollary to this theorem is that an injective CA must also be surjective, since an injective CA is certainly still injective when restricted to the set of finite configurations. A direct proof of this result is given for one-dimensional CA's in [**12**], and an extension of the Garden of Eden theorems of Moore and Myhill to all dimensions is given in [**29**].

THEOREM 2.2.2 (Hedlund [**12**], Richardson [**29**]). *Let* $F : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ *be an injective cellular automaton. Then* F *is also surjective.*

However, a CA which is injective on the set of finite configurations need not be injective on the full shift space, and in fact, the statement that a surjective CA must also be injective is not true in any dimension [**16**]. There is a sense that an onto CA is finite-to-one though, by considering the pre-images of cylinder sets. Maruoka and Kimura introduce the following definitions and notation in order to give this depiction. Denote by

$$B_{k,\ell} = \{\vec{\imath} \in \mathbb{Z}^D : k \leq ||\vec{\imath}|| \leq l\} \tag{2.8}$$

the annular ring of coordinates, and by

$$P_{k,\ell} = \{x|_{B_{k,\ell}} : x \in A^{\mathbb{Z}^D}\} \tag{2.9}$$

the patterns occurring in these coordinates. We need to discuss the image of a pattern, and so we introduce the following notation. Let $p \in A^{s_1 \times \cdots \times s_D}$ be a pattern given by $\{p_{(i_1,\cdots,i_D)} : 0 \leq i_j \leq s_j - 1, j = 1, \cdots, D\}$. For a CA $F : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ of radius r and local rule f, denote by $F(p)$ the pattern of size $(s_1 - 2r) \times \cdots \times (s_D - 2r)$ given by $(Fp)_{\vec{\imath}} = f\left(\{p_{\vec{\jmath}} : ||\vec{\jmath} - \vec{\imath}|| \leq r\}\right)$. A cellular automaton $F : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ is said to be *k-balanced* if for all patterns $p \in P_{0,k}$, we have

$$|\{p' \in P_{0,k+1} : F(p') = p\}| = |A|^{|B_{k+1,k+1}|} = |A|^{(2k+3)^D - (2k+1)^D}. \tag{2.10}$$

That is, all cylinder sets of $\left(\prod_{j=1}^{D} \langle -k, k \rangle\right)$-blocks have the same number of pre-images. We say that F is *balanced* if F is k-balanced for all $k \geq 1$. The concept of balanced lets us approach surjectivity by determining whether all cylinder sets of the same size have an equal number of pre-images. Moreover, there is a seemingly weaker statement, that each cylinder set have a non-empty pre-image, that guarantees a CA is balanced.

THEOREM 2.2.3 (Maruoka, Kimura [**23**]). *Let $F : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be a cellular automaton. The following are equivalent:*

(1) *F is surjective.*

(2) *F is balanced.*

(3) *For all* $k \geq 1$ *and every pattern* $p \in P_{0,k}$, *there exists a pattern* $p' \in P_{0,k+1}$ *such that* $F(p') = p$.

Although Theorem 2.2.3 gives a local characterization of the global property of surjectivity, it is still a challenging task in general to determine whether a given CA is surjective. In fact, Kari has shown that detecting the answer is computationally undecidable for two-dimensional CA's [**15**]. This is in stark contrast to the one-dimensional case, where Amoroso and Patt have given an explicit algorithm to decide whether a CA $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is surjective [**15**].

2.3. First Examples of Cellular Automata

In dynamics, there is great interest in the asymptotic behavior of a system, and this is certainly the case in studying CA's. A useful way to track orbits under a one-dimensional CA consists of writing the iterates of a point underneath one another as follows:

$$\begin{array}{rccccccc}
x = & \cdots & x_{-2} & x_{-1} & .\, x_0 & x_1 & x_2 & \cdots \\
Fx = & \cdots & (Fx)_{-2} & (Fx)_{-1} & .\, (Fx)_0 & (Fx)_1 & (Fx)_2 & \cdots \\
F^2x = & \cdots & (F^2x)_{-2} & (F^2x)_{-1} & .\, (F^2x)_0 & (F^2x)_1 & (F^2x)_2 & \cdots \\
 & & & & \vdots & & &
\end{array}$$

Example 2.3.1. Let $A = \{0, 1\}$ and consider the CA $S : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined by $(Sx)_i = x_{i-1} + x_{i+1} \pmod 2$. We will show that S satisfies condition (3) in Theorem 2.2.3, and thus, S is *surjective*. Given a word $w = w_0 \cdots w_k$, let $w' = w_0\ w_1\ 0\ 0\ w_2\ w_3\ w_4 - w_2\ w_5 - w_3\ w_6 - w_4 - w_2\ w_7 - w_5 - w_3\ \cdots w_k - w_{k-2} - w_{k-4} - \cdots - w_2$. Then $S(w') = w$, as desired. S is *not injective* however; it is a four-to-one mapping. To see the dynamics of S, we show the orbit of the point $x = \cdots 0\ 0\ .1\ 0\ 0 \cdots$ in Figure 2.1.

$$
\begin{array}{rcccccccccccccccc}
x = & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & .1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
Fx = & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & .0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
F^2x = & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & .0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
F^3x = & \cdots & 0 & 0 & 0 & 1 & 0 & 1 & .0 & 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\
F^4x = & \cdots & 0 & 0 & 1 & 0 & 0 & 0 & .0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
F^5x = & \cdots & 0 & 1 & 0 & 1 & 0 & 0 & .0 & 0 & 0 & 1 & 0 & 1 & 0 & \cdots \\
 & & & & & & & & \vdots & & & & & & &
\end{array}
$$

FIGURE 2.1. Orbit of $\cdots 0.10 \cdots$ under S

A more illustrative way to view points in $\{0,1\}^{\mathbb{Z}}$ is to let 0's be represented by white space and let 1's be represented by black space. Figure 2.2 shows the same orbit after more iterations in this fashion.

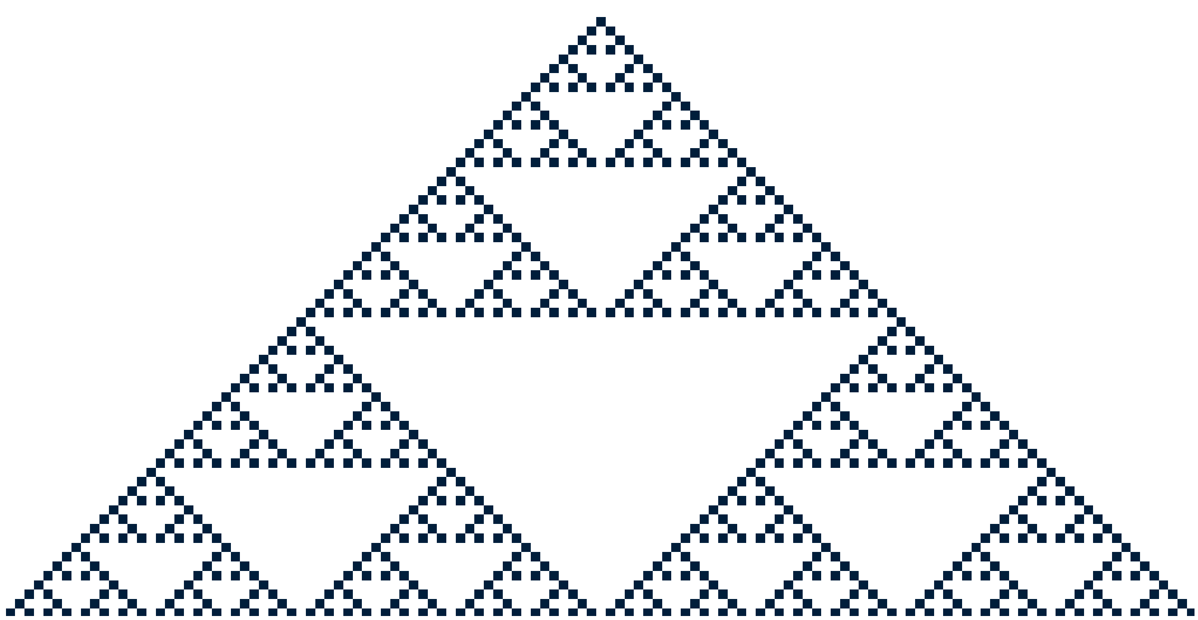
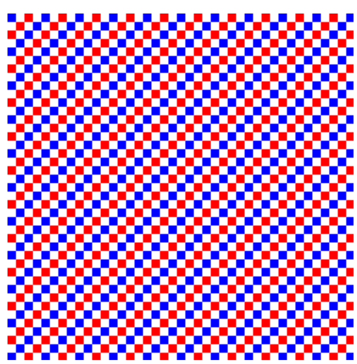


FIGURE 2.2. Color representation of the orbit of $\cdots 00.100 \cdots$ under S

An obvious benefit to visualizing orbits under a one-dimensional CA is that the space-time diagram only requires two dimensions. However, a single point in a two-dimensional space fills up the entire plane, so the space-time diagram of an orbit under a CA on a two-dimensional space would require three dimensions. Thus, in order to visualize orbits under a two-dimensional CA, we must show a series of iterates.

EXAMPLE 2.3.2. Let $A = \{\textcolor{red}{\blacksquare}, \textcolor{blue}{\blacksquare}, \textcolor{yellow}{\blacksquare}, \square\}$, and define the CA $P : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ to describe the movement of three different colored particles in white space as follows. A $\textcolor{red}{\blacksquare}$ particle moves both northeast and southwest leaving a trail, a $\textcolor{blue}{\blacksquare}$ particle moves both northwest and southeast leaving a trail, and a $\textcolor{yellow}{\blacksquare}$ particle is a wall that annihilates any other particle which runs into it. When a $\textcolor{red}{\blacksquare}$ and a $\textcolor{blue}{\blacksquare}$ particle try to occupy the same space, they annihilate each other. This is indeed a CA; we give the radius one local rule definition in Figure 2.3, where we represent the neighborhood by $\begin{array}{|c|c|c|}\hline \text{NW} & \text{N} & \text{NE} \\ \hline \text{W} & & \text{E} \\ \hline \text{SW} & \text{S} & \text{SE} \\ \hline \end{array}$. The dynamics of P is illustrated in Figures 2.4 through 2.7.

P is *not* a *surjective* CA. For, we show that the point

$y =$ 

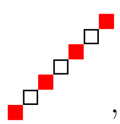
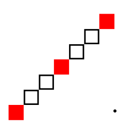
$$\color{yellow}{\blacksquare} \quad \mapsto \quad \color{yellow}{\blacksquare}$$

$$\color{red}{\blacksquare} \quad \mapsto \quad \begin{cases} \square & \text{if NW} = \color{blue}{\blacksquare} \text{ or SE} = \color{blue}{\blacksquare}\,, \\ \color{red}{\blacksquare} & \text{otherwise.} \end{cases}$$

$$\color{blue}{\blacksquare} \quad \mapsto \quad \begin{cases} \square & \text{if NE} = \color{red}{\blacksquare} \text{ or SW} = \color{red}{\blacksquare}\,, \\ \color{blue}{\blacksquare} & \text{otherwise.} \end{cases}$$

$$\square \quad \mapsto \quad \begin{cases} \color{red}{\blacksquare} & \text{if either NE} = \color{red}{\blacksquare} \text{ or SW} = \color{red}{\blacksquare}\,\text{, and both NW} \neq \color{blue}{\blacksquare} \text{ and SE} \neq \color{blue}{\blacksquare}\,, \\ \color{blue}{\blacksquare} & \text{if either NW} = \color{blue}{\blacksquare} \text{ or SE} = \color{blue}{\blacksquare}\,\text{, and both NE} \neq \color{red}{\blacksquare} \text{ and SW} \neq \color{red}{\blacksquare}\,, \\ \square & \text{otherwise.} \end{cases}$$

FIGURE 2.3. Local Rule for P

is not in the image of P. Suppose $x \in A^{\mathbb{Z}^2}$ is a point mapping to y, and that $y_{(0,0)} = \color{red}{\blacksquare}$. First consider the values $x_{(i,i)}$, where $i \in \mathbb{Z}$. Since P "moves" $\color{red}{\blacksquare}$'s in both directions along a diagonal, these values of x would need to be a combination of $\color{red}{\blacksquare}$'s and $\square$'s such as  ,  , or  . As y also has diagonal lines of $\color{red}{\blacksquare}$'s at coordinates $(i, i + 4c)$ for $i, c \in \mathbb{Z}$, then for each $c \in \mathbb{Z}$, the values $x_{(i,i+4c)}$ must follow one of these types of infinite patterns also. One way for $y_{(0,2)} = \color{blue}{\blacksquare}$ would be to have $x_{(0,2)} = \color{blue}{\blacksquare}$; however, this would prohibit $y_{(1,1)} = \color{red}{\blacksquare}$. The only other way to have $y_{(0,2)} = \color{blue}{\blacksquare}$ would be for $x_{(0,2)} = \square$ with either $x_{(-1,3)} = \color{blue}{\blacksquare}$ or $x_{(1,1)} = \color{blue}{\blacksquare}$. Respectively, this prohibits either $y_{(-1,3)} = \color{red}{\blacksquare}$ or $y_{(1,1)} = \color{red}{\blacksquare}$. Thus, there is no $x \in A^{\mathbb{Z}^2}$ having $P(x) = y$. As P is not surjective, it *cannot be injective* either by Theorem 2.2.2. We see this explicitly in Figure 2.8; the two points y_1 and y_2 both map to y_2 under P.

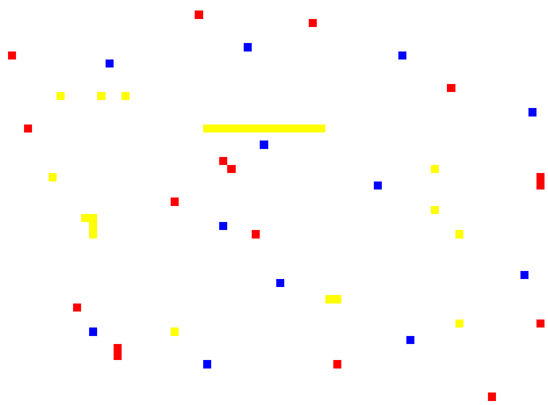


FIGURE 2.4. Initial point for P

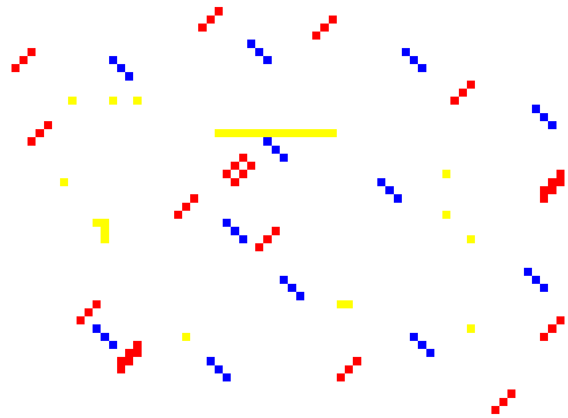


FIGURE 2.5. One iteration of P

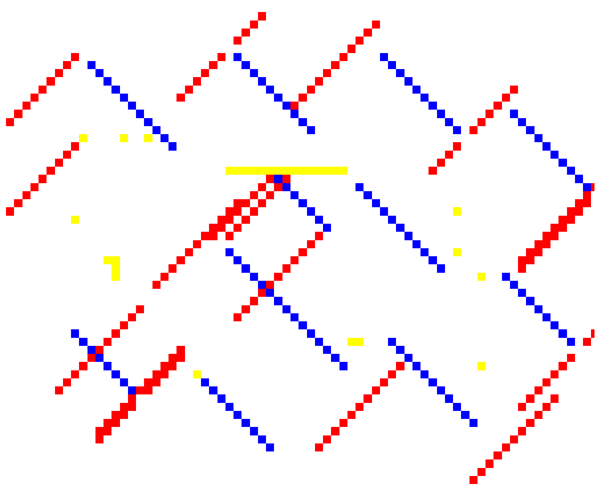


FIGURE 2.6. Five iterations of P

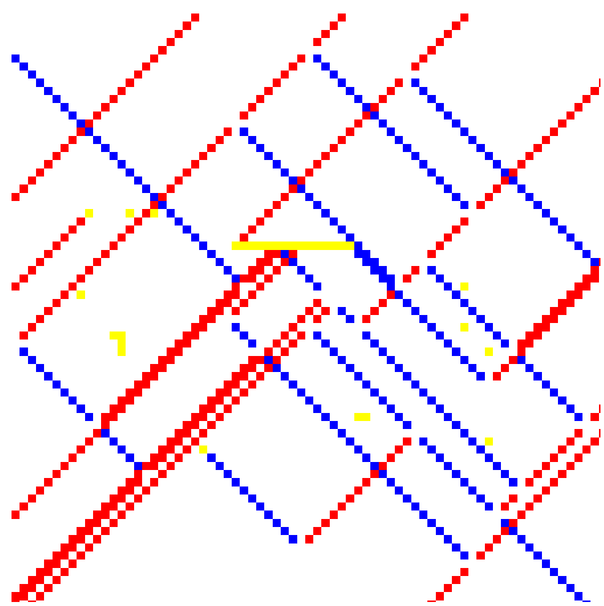


FIGURE 2.7. 100 iterations of P

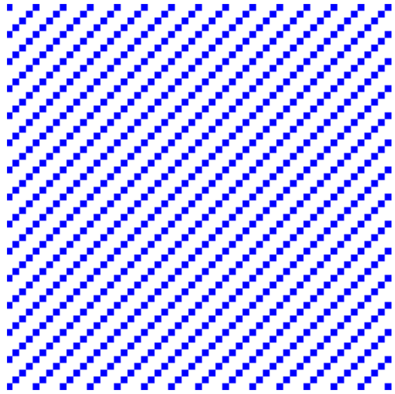
$y_1 =$  $y_2 =$ 

FIGURE 2.8. P is not injective: $P(y_1) = P(y_2) = y_2$

EXAMPLE 2.3.3. Consider the two-dimensional *Golden Mean Shift Space*, given by

$$X_{\left\{11,\ \begin{smallmatrix}1\\1\end{smallmatrix}\right\}} = \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : x|_{\{(i,j),(i+1,j)\}} \neq 1\ 1 \text{ and } x|_{\{(i,j),(i,j+1)\}} \neq \begin{matrix}1\\ \\1\end{matrix} \right\}. \tag{2.11}$$

This space is referred to as such because the shift, σ, has entropy $\frac{1+\sqrt{5}}{2}$ on the corresponding one-dimensional subshift space, $\left\{x \in \{0,1\}^{\mathbb{Z}} : x_i x_{i+1} \neq 11\right\}$. (We will discuss entropy more thoroughly in Chapter 6.) Returning to a CA on this space however, define $G : X_{\left\{11,\ \begin{smallmatrix}1\\1\end{smallmatrix}\right\}} \to X_{\left\{11,\ \begin{smallmatrix}1\\1\end{smallmatrix}\right\}}$ by the radius one local rule which sends the pattern $\begin{matrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{matrix}$ to 1 and all other 3×3 patterns to 0. First, we note that the image of G is, in fact, contained in $X_{\left\{11,\ \begin{smallmatrix}1\\1\end{smallmatrix}\right\}}$. For,

$$(2.12) \qquad (Gx)_{(i,j)} = 1 \quad \Leftrightarrow \quad \begin{array}{ccc} x_{(i-1,j+1)} & x_{(i,j+1)} & x_{(i+1,j+1)} \\ x_{(i-1,j)} & x_{(i,j)} & x_{(i+1,j)} \\ x_{(i-1,j-1)} & x_{(i,j-1)} & x_{(i+1,j-1)} \end{array} = \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{array},$$

$$(2.13) \qquad (Gx)_{(i+1,j)} = 1 \quad \Leftrightarrow \quad \begin{array}{ccc} x_{(i,j+1)} & x_{(i+1,j+1)} & x_{(i+2,j+1)} \\ x_{(i,j)} & x_{(i+1,j)} & x_{(i+2,j)} \\ x_{(i,j-1)} & x_{(i+1,j-1)} & x_{(i+2,j-1)} \end{array} = \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}, \text{ and}$$

$$(2.14) \qquad (Gx)_{(i,j+1)} = 1 \quad \Leftrightarrow \quad \begin{array}{ccc} x_{(i-1,j+2)} & x_{(i,j+2)} & x_{(i+1,j+2)} \\ x_{(i-1,j+1)} & x_{(i,j+1)} & x_{(i+1,j+1)} \\ x_{(i-1,j)} & x_{(i,j)} & x_{(i+1,j)} \end{array} = \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}.$$

But (2.12) and (2.13) contradict one another, so they cannot both hold at the same time; neither can both (2.12) and (2.14) hold at the same time. Thus $G\left(X_{\left\{11,\ \begin{smallmatrix}1\\1\end{smallmatrix}\right\}}\right) \subseteq X_{\left\{11,\ \begin{smallmatrix}1\\1\end{smallmatrix}\right\}}$. The dynamics of this CA is illustrated in Figures 2.9 through 2.12.

G is *not* a *surjective* CA, for the pattern $u = 1\ 0\ 0\ 1$ is not in the image of G. In order for a pattern, v, to map to u, we would need to have $\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}$ centered at each 1 in u. But this means v would look like $\begin{smallmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{smallmatrix}$, and of course this pattern is forbidden in all points of $X_{\left\{11,\ \begin{smallmatrix}1\\1\end{smallmatrix}\right\}}$. Thus by Theorem 2.2.2, G is *not injective* either.

2.4. Topological Dynamical Systems

Here we present the standard introductory definitions from topological dynamics. For further reference on these notions, see [**4, 5, 27, 32**].

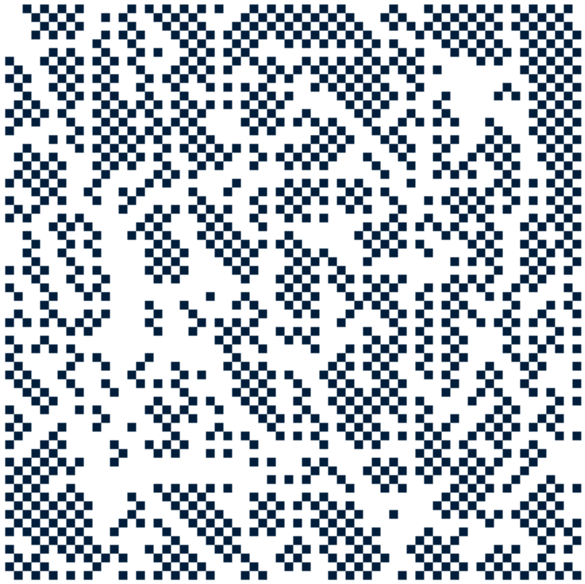


Figure 2.9. Initial point for G

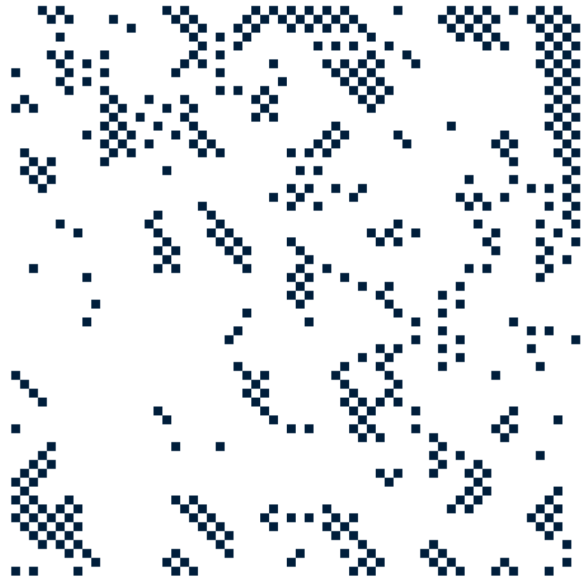


Figure 2.10. One iteration of G

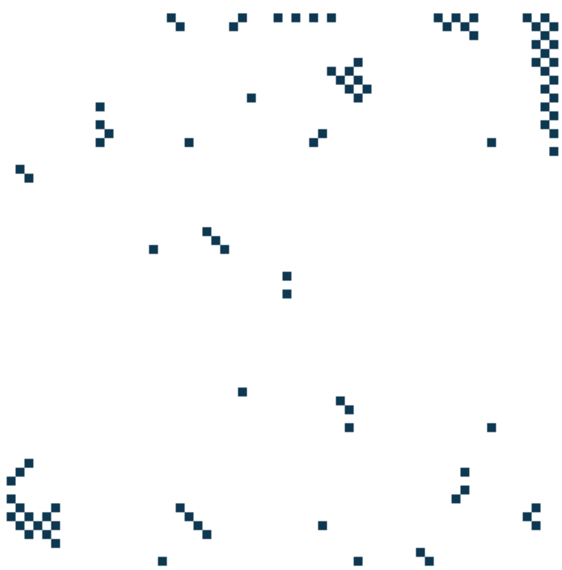


Figure 2.11. Two iterations of G

Figure 2.12. Three iterations of G

By a *dynamical system*, we will mean a pair (Y, T) consisting of a compact metric space Y and a continuous map $T : Y \to Y$. A subset $W \subseteq Y$ is *invariant* if $T(W) \subseteq W$. A *homomorphism* of dynamical systems, $\phi : (Y, T) \to (Z, S)$, is a continuous map $\phi : Y \to Z$ such that $\phi \circ T = S \circ \phi$. If ϕ is surjective, we say that it is a *factor map*, and if it is bijective, ϕ is called a *conjugacy*. Denote the n^{th} *iterate* of T by $T^n = T \circ T \circ \cdots \circ T$ (n times); by convention, $T^0 = Id$. The *orbit* of a point y is the set $\mathcal{O}(y) = \{T^n y : n \geq 0\}$. A point $y \in Y$ is *periodic* if $\exists\ p \geq 0$ such that $T^p y = y$. The *period* of y is $p = \min\{k : T^k y = y\}$. If $Ty = y$, we say that y is *fixed*. A point y is *eventually periodic* (*pre-periodic*, respectively) if $\exists\ m \geq 0$ (> 0, respectively), called the pre-period, such that $T^m y$ is periodic. We say that T is *periodic* if there is $p \geq 0$ such that $T^p = T$ as functions. That is, $T^p y = Ty$ for all $y \in Y$. T is *eventually periodic* (*pre-periodic*, respectively) if $\exists\ m \geq 0$ (> 0, respectively), called the pre-period, such that T^m is periodic in the above sense.

In contrast to periodic dynamical systems are those which jumble the space to some extent over time. A first notion of this type of behavior is captured by the property of transitivity. A dynamical system (Y, T) is *(topologically) transitive* if there is a point $y \in Y$ with a dense forward orbit, $Y = \overline{\{T^n y : n \geq 0\}}$. That is to say, from the initial point y, we can get arbitrarily close to any other point in the space Y via iteration by T. The shift $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a transitive mapping: any point containing all finite words in the positive indices has a dense forward orbit under σ. We say that a dynamical system (Y, T) is *(topologically) mixing* if for every pair of non-empty open sets U, V, there exists a $N \geq 0$ such that $T^n U \cap V \neq \emptyset$ for all $n \geq N$. This is a stronger property than transitivity, as not only must there be a single point whose orbit occasionally gets near other points, but a part of every neighborhood must stay near all other neighborhoods beyond some time. The shift $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is also mixing, because given any two cylinder sets $U = [B]_i$ and $V = [C]_j$, we can take $N > i - j + |B| - 1$. Then for $n \geq N$, there exist points in $A^{\mathbb{Z}}$ having the word B beginning at index $i - n$ and ending at index $i - n + |B| - 1 < j$ which also have the word C beginning at index j, thus $\sigma^n U \cap V \neq \emptyset$ for $n \geq N$, as desired.

A point y is an *equicontinuity point* of a dynamical system (Y, T) if $\forall\, \varepsilon > 0, \exists\, \delta > 0$ such that $d(x, y) < \delta \Rightarrow d(T^n x, T^n y) < \varepsilon\ \forall\, n \geq 0$. A dynamical system is *equicontinuous* if each of its points is an equicontinuity point. Essentially, an equicontinuous system is one for which points initially close have orbits which stay close for all time. We say that a dynamical system is *almost equicontinuous* if the set of equicontinuity points contains an intersection of dense open sets.

A dynamical system (Y, T) is said to be *expansive* if $\exists\, \varepsilon > 0$ such that $\forall\, x \neq y \in Y$ $d(T^n x, T^n y) \geq \varepsilon$ for some $n \in \mathbb{N}$, or $n \in \mathbb{Z}$ if T is invertible. In such a case, ε is an *expansive constant* for T. An expansive system is one in which distinct points, no matter how close initially, will eventually be pushed apart by the action of the transformation.

A dynamical system (Y, T) has *sensitive dependence on initial conditions* if $\exists\, \varepsilon > 0$ such that $\forall\, y \in Y$, and $\delta > 0, \exists\, x$ with $d(x, y) < \delta$ and $d(T^n x, T^n y) \geq \varepsilon$ for some $n \geq 0$. We will refer to this property simply as *sensitive*. In this case, ε is called a *sensitive constant*. Sensitivity differs from expansivity by not requiring that every pair of distinct points necessarily get pushed apart, but that for each $y \in Y$, we can find points arbitrarily close to y which eventually do get pushed away. We say that (Y, T) is *sensitive at* y, or that $y \in Y$ is a *point of sensitivity*, if $\exists\, \varepsilon_y > 0$ such that $\forall\, \delta > 0, \exists\, x$ with $d(x, y) < \delta$ and $d(T^n x, T^n y) \geq \varepsilon_y$ for some $n \geq 0$.

Although at first glance, the definitions of equicontinuity and sensitivity look as though the properties cannot hold simultaneously, there are some subtleties to notice here. An equicontinuous transformation is defined so that every point is a point of equicontinuity. Moreover, such a map is *uniformly equicontinuous*; that is, for every $\varepsilon > 0$, there is a $\delta > 0$ which works for every point in the space: for all x, y with $d(x, y) < \delta$, $d(T^n x, T^n y) < \varepsilon$ for all $n \geq 0$. If there are no points of equicontinuity for a system, then every point is a point of sensitivity. However, this does not guarantee that the system is sensitive. For sensitivity is defined in a uniform way; there is an $\varepsilon > 0$ that works for every point in the space. An example of a system which is not sensitive, but for which every point is a point of sensitivity is given in [**19**]. In contrast though, any system having a point of equicontinuity cannot also be sensitive.

PROPOSITION 2.4.1. *Let (Y, T) be a dynamical system If T has a point of equicontinuity, then T is not sensitive.*

Proof. Let $y \in Y$ be a point of equicontinuity for T. Suppose that T is sensitive. Then $\exists\ \varepsilon > 0$ such that for all $z \in Y$ and $\delta > 0$, there exists $x \in Y$ with $d(x, z) < \delta$ and $d(T^n x, T^n z) \geq \varepsilon$ for some $n \geq 0$. But by the definition of an equicontinuity point, the above property does not apply to y, and hence T is not sensitive. $\square$

In Section 4.2, we give a complete characterization of CA's which are not sensitive, Theorems 4.2.1 through 4.2.3.

CHAPTER 3

Equicontinuity Properties

The property of equicontinuity captures the notion of predictable behavior. For cellular automata, we see that this is incredibly rigid, as equicontinuous CA's are exactly those which are eventually periodic. We also investigate eventually periodic points for non-equicontinuous cellular automata, and provide numerous examples.

3.1. Equicontinuous Cellular Automata

We will first address equicontinuous cellular automata, that is, those for which every point is a point of equicontinuity. We give an equivalent definition for this property, and then extend the following two one-dimensional results from [**20**] to the setting where the underlying shift space is a subshift on which the shift-periodic points are dense: a cellular automaton is equicontinuous if and only if it is eventually periodic, and a cellular automaton is both surjective and equicontinuous if and only if it is periodic.

THEOREM 3.1.1. *Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift and let $F : X \to X$ be a cellular automaton. The following statements are equivalent:*

(1) *F is equicontinuous,*

(2) *$\exists\ M \geq 0$ such that for $x, y \in X$ with $d(x,y) < 2^{-M}$, $d(F^n x, F^n y) < 1\ \forall\ n \geq 0$.*

The proof is straightforward, though it does not appear to be in the literature.

Proof. (*1* $\Rightarrow$ *2*) The equicontinuity of F implies that for $\varepsilon = 1$, there exists a $\delta = 2^{-M}$ satisfying the property given in (2).

(*2* $\Rightarrow$ *1*) Let $\varepsilon = 2^{-k} > 0$, and take $\delta = 2^{-(k+M)}$. Then for a pair $x, y \in X$ with $d(x, y) < \delta \leq 2^{-M}$, the distance between their iterates being smaller than 1 means that $(F^n x)_{\vec{0}} = (F^n y)_{\vec{0}}$ $\forall$ $n \geq 0$. By our choice of δ, we also have $d(\sigma_{\vec{\imath}} x, \sigma_{\vec{\imath}} y) < 2^{-M}$ for $\vec{\imath} = (i_1, \cdots, i_D)$ with $|i_1|, |i_2|, \cdots, |i_D| \leq k$, and so $(F^n(\sigma_{\vec{\imath}} x))_{\vec{0}} = (F^n(\sigma_{\vec{\imath}} y))_{\vec{0}}$. Then for $|i_1|, |i_2|, \cdots, |i_D| \leq k$ and for all $n \geq 0$,

$$(F^n x)_{\vec{\imath}} = (\sigma_{\vec{\imath}}(F^n x))_{\vec{0}} = (F^n(\sigma_{\vec{\imath}} x))_{\vec{0}} = \\ (F^n(\sigma_{\vec{\imath}} y))_{\vec{0}} = (\sigma_{\vec{\imath}}(F^n y))_{\vec{0}} = (F^n y)_{\vec{\imath}}. \tag{3.1}$$

Thus $d(x, y) < \delta$ implies $d(F^n x, F^n y) < \varepsilon$ $\forall$ $n \geq 0$, and hence F is equicontinuous. $\square$

The next theorem characterizes equicontinuous CA's as those which are eventually periodic, extending Theorem 5.2 in [**20**], which is in the setting of one-dimensional CA's on the full shift space. We give the result in the more general setting of a CA on any subshift which has a dense set of shift periodic points. When σ is a $\mathbb{Z}^D$ action, $x \in X$ is σ-periodic if the set $\{\sigma_{\vec{\imath}}(x) : \vec{\imath} \in \mathbb{Z}^D\}$ is finite. A point is σ-periodic if and only if it has a period for each $\sigma_{\vec{e_j}}$ in the traditional sense. The full shift spaces in all dimensions and transitive one-dimensional subshifts of finite type (for every pair of allowable words u and v, there is an allowable word w so that uwv is an allowable word) each have a dense set of shift periodic points. Further, two-dimensional SFT's with strong specification (see Ward, [**33**]) and two-dimensional SFT's with the uniform filling property (see Robinson and Şahin, [**30**]) are also shown to have a dense set of shift periodic points. However, a general characterization of higher dimensional subshifts with this property is unknown.

THEOREM 3.1.2. *Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift with dense σ-periodic points, and let $F : X \to X$ be a cellular automaton. F is equicontinuous if and only if F is eventually periodic.*

Proof. ($\Leftarrow$) Let r be the radius of F, and assume $\exists\, m \geq 0,\ p > 0$ such that $F^{m+p} = F^m$. Take $M = r(m+p)$. For $x, y \in X$ with $d(x, y) < 2^{-M}$, we have

$$d(Fx, Fy) < 2^{-M} \cdot 2^r = 2^{-r(m+p-1)} < 1. \tag{3.2}$$

Further, $d(x, y) < 2^{-M}$ implies that for each $n < m + p$,

$$d(F^n x, F^n y) < 2^{-M} \cdot 2^{nr} = 2^{-r(m+p-n)} < 1. \tag{3.3}$$

Now as $F^{m+p} = F^p$, the two sequences of patterns consisting of the central symbols of the iterates, $(F^n x)_{\vec{0}}$ and $(F^n y)_{\vec{0}}$, each form an eventually periodic sequence with pre-period m and period p. But since the first $m + p > m$ elements are equal, we have the equality $(F^n x)_{\vec{0}} = (F^n y)_{\vec{0}}$ for all $n \geq 0$. Therefore,

$$d(x, y) < 2^{-M} \Rightarrow d(F^n x, F^n y) < 1\ \forall\ n \geq 0, \tag{3.4}$$

and by Theorem 3.1.1, F is equicontinuous.

($\Rightarrow$) Assume F is equicontinuous, and let M be the constant resulting from Theorem 3.1.1 (2). Let $x \in X$, and consider the central $\prod_{j=1}^{D}(2M + 1)$ pattern of x, $u_x = x|_{\prod_{j=1}^{D}\langle -M, M\rangle}$. Since the shift periodic points are dense in X, there exists a $z \in [u_x]_{-M\vec{e}}$ which is periodic under the shift, where $\vec{e} = \sum_{j=1}^{D} \overrightarrow{e_j}$ is the sum of all basis vectors. Denote by $\vec{p} = (p_1, \cdots, p_D)$ the period vector of z; that is, $\sigma_{\overrightarrow{e_j}}^{p_j} z = z$ for each j.

Now as F commutes with the action of the shift, for each $n \geq 0$ and basis vector $\overrightarrow{e_j}$, we have

$$\sigma_{\overrightarrow{e_j}}^{p_j}(F^n z) = F^n(\sigma_{\overrightarrow{e_j}}^{p_j} z) = F^n z. \tag{3.5}$$

That is, each iterate $F^n z$ is also periodic for the shift with the same period vector $\vec{p}$. This puts an upper bound on the cardinality of the set of iterates of z,

$$|\{F^n z : n \geq 0\}| \leq |A|^{p_1 p_2 \cdots p_D}, \tag{3.6}$$

which is finite. Hence there must be a repetition in the set of iterates; let the first one be $F^{m_{u_x}+p_{u_x}} z = F^{m_{u_x}} z$. Thus the set of iterates $\{F^n z : n \geq 0\}$ forms an eventually periodic sequence with pre-period $m_{u_x} \geq 0$ and period $p_{u_x} > 0$. We use the subscript u_x on both the pre-period and period of this sequence as these quantities depend only on the pattern u_x. Now for all y in the cylinder $[u_x]_{-M\vec{e}}$, $d(y, z) < 2^{-M}$ and hence $(F^n y)_{\vec{0}} = (F^n z)_{\vec{0}}$ $\forall$ $n \geq 0$. Therefore $(F^n y)_{\vec{0}}$ is also an eventually periodic sequence with pre-period m_{u_x} and period p_{u_x}. Let

$$m = \max\{m_u\} \quad \text{and let} \quad p = \prod p_u, \tag{3.7}$$

where the maximum and the product are each taken over all patterns $u \in A^{\prod_{j=1}^{D}(2M+1)}$. Since $\forall$ $x \in X$, the pattern $u_x = x|_{\prod_{j=1}^{D}\langle -M, M\rangle}$ is one of those that the maximum and product are taken over, we have $(F^{m+p}x)_{\vec{0}} = (F^m x)_{\vec{0}}$. Using the commutativity of F and the shift maps gives the equality

$$\begin{aligned}(F^{m+p}x)_{\vec{\imath}} = \big(\sigma_{\vec{\imath}}(F^{m+p}x)\big)_{\vec{0}} = \big(F^{m+p}(\sigma_{\vec{\imath}}x)\big)_{\vec{0}} = \\ (F^m(\sigma_{\vec{\imath}}x))_{\vec{0}} = (\sigma_{\vec{\imath}}(F^m x))_{\vec{0}} = (F^m x)_{\vec{\imath}}\end{aligned} \tag{3.8}$$

for each $\vec{\imath} \in \mathbb{Z}^D$. Hence, $F^{m+p} = F^m$, and so F is eventually periodic. $\square$

Further, an equicontinuous cellular automaton which is also surjective must be periodic. This seems to be well known, but a proof is not available in the literature.

THEOREM 3.1.3. *Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift with dense σ-periodic points, and let $F : X \to X$ be a cellular automaton. F is both equicontinuous and surjective if and only if F is periodic.*

Proof. ($\Rightarrow$) Suppose F is both equicontinuous and surjective. By the previous theorem, there are minimal integers $m \geq 0$ and $p > 0$ so that $F^{m+p} = F^m$. Assume to the contrary that $m > 0$, i.e., that F is only eventually periodic and not periodic. For an arbitrary $x \in X$, there must be a point $y \in X$ with $Fy = x$. Then we have both of the following:

$$F^m y = F^{m+p} y = F^{m+p-1}(Fy) = F^{m+p-1} x \tag{3.9}$$

$$F^m y = F^{m-1}(Fy) = F^{m-1} x, \tag{3.10}$$

so that $F^{m+p-1}x = F^{m-1}x$. As x was arbitrary, $F^{m-1+p} = F^{m-1}$, and so m is not the pre-period of F. Therefore, $m = 0$, and $F^p = F^0 = Id$ is periodic.

($\Leftarrow$) This direction is trivial, as F periodic of period p implies that for every $x \in X$, $F(F^{p-1}x) = x$; therefore F is surjective. Equicontinuity of F is then given by Theorem 3.1.2. $\square$

3.2. Examples of Equicontinuous Cellular Automata

EXAMPLE 3.2.1. Let $X \subseteq A^{\mathbb{Z}^D}$ be any subshift space and let $I : X \to X$ be the identity map. For $\varepsilon > 0$, simply set $\delta = \varepsilon$. Then for all $x, y \in X$ with $d(x, y) < \delta$, we have $d(I^n x, I^n y) = d(x, y) < \varepsilon$ for all $n \geq 0$ and hence I is *equicontinuous*. Clearly, I is *surjective* also and has period 1.

EXAMPLE 3.2.2. Let A be any finite set and let $O : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be the zero map, i.e. $O(x)_{\vec{\imath}} = 0$ for all $\vec{\imath} \in \mathbb{Z}^D$ and $x \in A^{\mathbb{Z}^D}$. For $\varepsilon > 0$, again let $\delta = \varepsilon$. For $x, y \in A^{\mathbb{Z}^D}$ with $d(x, y) < \delta$, clearly the 0^{th} iterates of x and y are within epsilon, and since for any $n > 0$, $(O^n x)_{\vec{\imath}} = (O^n y)_{\vec{\imath}} = 0$ for all $\vec{\imath} \in \mathbb{Z}^D$, we have $d(O^n x, O^n y) = 0 < \varepsilon$ for $n > 0$ also. Hence O is *equicontinuous*. Since there is only one point in the image of O, it is certainly *not surjective*, and we see that $m = 1,\ p = 1$.

EXAMPLE 3.2.3. Let A be any finite set and let $F : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be a CA with radius 0 and local rule $f : A \to A$. For $\varepsilon = 2^{-k} > 0$, again let $\delta = \varepsilon$. Now for $x, y \in A^{\mathbb{Z}^D}$ with $d(x, y) < \delta$, we have $x_{\vec{\imath}} = y_{\vec{\imath}}$ for $|i_1|, |i_2|, \cdots, |i_D| \leq k$. Then for $|i_1|, |i_2|, \cdots, |i_D| \leq k$ and $n \geq 0$, $(F^n x)_{\vec{\imath}} = f^n (x_{\vec{\imath}}) = f^n (y_{\vec{\imath}}) = (F^n y)_{\vec{\imath}}$, and so $d(F^n x, F^n y) < \varepsilon$ for all $n \geq 0$. Thus any radius 0 CA is *equicontinuous*.

EXAMPLE 3.2.4 ([**20**]). Let $E : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ be given by $(Ex)_i = x_i + x_{i-1} \cdot x_{i+1}$ (mod 2). E is *not surjective*, as we will show that the point $\cdots 1\ 0\ 1\ .0\ 1\ 0 \cdots$ is not in the image. By inspection, we see that $E^{-1}([010]_i) \subseteq [00100]_{i-2} \cup [00111]_{i-2} \cup [11100]_{i-2}$. However, as $E([00]_i) \subseteq [00]_i$, we see $E^{-1}([10101]_{i-1}) = \emptyset$. E is *eventually periodic* though, having pre-period 2 and period 2; thus it is an *equicontinuous* CA. The dynamics for a typical orbit are shown in Figure 3.1.

EXAMPLE 3.2.5. A class of two-dimensional examples can be obtained from equicontinuous one-dimensional cellular automata. We define the two-dimensional action by letting the one-dimensional CA act on the rows of points in $A^{\mathbb{Z}^2}$. Precisely, let A be a

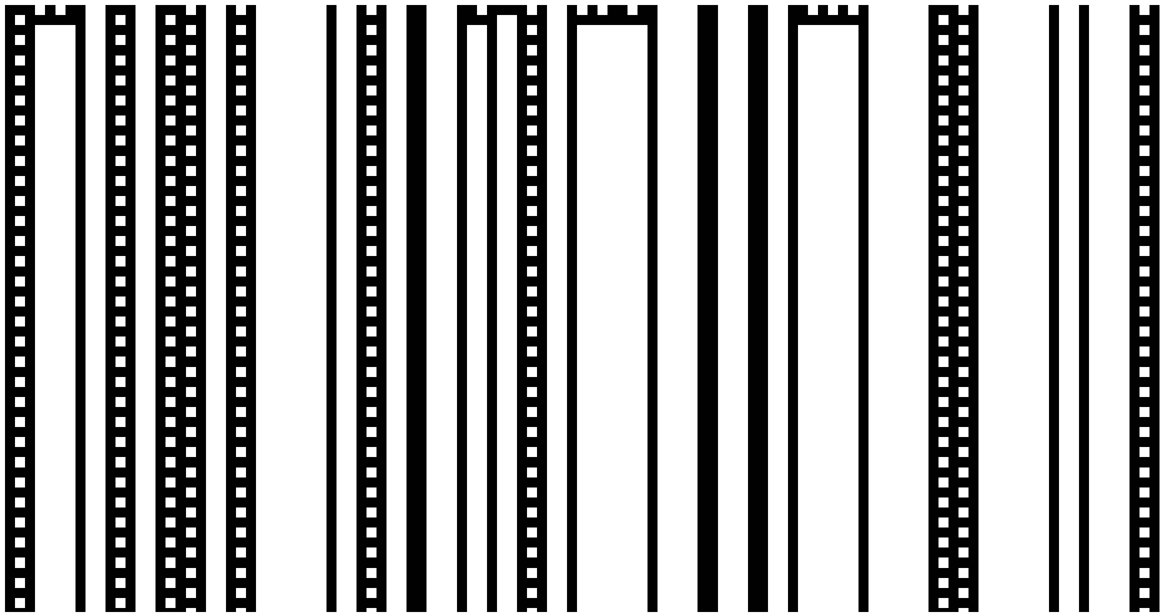


Figure 3.1. Orbit under E

finite set and $G : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be an equicontinuous one-dimensional CA. For $j \in \mathbb{Z}$, let $H_j : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}}$ be the restriction map to the j^{th} row, given by $(H_j\, x)_i = x_{(i,j)}$. Now define the map $F : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ by $(Fx)_{(i,j)} = (G \circ H_j\, x)_i$. We notice that as $(G \circ H_j\, x)$ and $(H_j \circ Fx)$ both represent the j^{th} row of Fx, we have $(F^n x)_{(i,j)} = (G^n \circ H_j\, x)_i$. To see that F is equicontinuous, let $\varepsilon > 0$. As G is equicontinuous, $\exists\ \delta_G = 2^{-k} > 0$ such that for $x, y \in A^{\mathbb{Z}}$ with $d(x,y) < \delta_G$, $d(G^n x, G^n y) < \varepsilon\ \forall\ n \geq 0$. Let $\delta = \delta_G$. Then for $x, y \in A^{\mathbb{Z}^2}$ with $(x,y) < \delta$, we have $d(H_j\, x, H_j\, y) < \delta = \delta_G$ for $|j| \leq k$. Now $d(F^n x, F^n y) = d(G^n \circ H_j\, x, G^n \circ H_j\, y) < \varepsilon\ \forall\ n \geq 0$.

This construction extends to higher dimensions so that from a D-dimensional equicontinuous CA, we can create a $(D+1)$-dimensional equicontinuous CA on the same alphabet. We view an arbitrary point in $A^{\mathbb{Z}^{D+1}}$ as an infinite number of points in $A^{\mathbb{Z}^D}$ by fixing the last coordinate. Specifically, for each $j \in \mathbb{Z}$, let $H_j : A^{\mathbb{Z}^{D+1}} \to A^{\mathbb{Z}^D}$ be the restriction map given by $(H_j x)_{(i_1,\cdots,i_D)} = x_{(i_1,\cdots,i_D,j)}$. Then we let the $(D+1)$-dimensional CA act by applying the D-dimensional rule independently to each of these lower dimensional points.

To illustrate this construction, recall the CA $E : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ from Example 3.2.4. Let the two-dimensional CA $E_2 : \{0,1\}^{\mathbb{Z}^2} \to \{0,1\}^{\mathbb{Z}^2}$ be derived from E as outlined above. We illustrate the dynamics of E_2 in Figures 3.2 through 3.5. For the initial point, we choose a point in $\{0,1\}^{\mathbb{Z}^2}$ which has as its center row the initial sequence, $x_0 \in \{0,1\}^{\mathbb{Z}}$, which is seen in Figure 3.1. As we iterate E_2 then, in the center row we see the iterates Ex_0, E^2x_0, and E^3x_0.

3.3. Equicontinuity Points for Cellular Automata

We now turn our attention to cellular automata which have points of equicontinuity, but are not equicontinuous mappings. Since equicontinuity is linked with periodicity, we investigate periodic points. We show that a periodic point under the shift must be eventually periodic under a cellular automaton, and that an attracting periodic point for a cellular automaton must be fixed under both the cellular automaton and the shift, as in [**13, 20**].

Theorem 3.1.2 is not only useful for characterizing equicontinuous cellular automata, but its proof suggests a more general result. If we have any CA, F, on a subshift, $X \subseteq A^{\mathbb{Z}^D}$, then even if F is not equicontinuous, we can obtain F-eventually periodic points. These are the points which are periodic under the shift action.

Proposition 3.3.1. *Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift space and let $F : X \to X$ be a cellular automaton. If $x \in X$ satisfies $\sigma_{\vec{\imath}}x = x$ for some $\vec{\imath} \in \mathbb{Z}^D$, then there exist integers $m \geq 0$ and $p > 0$ so that $F^{m+p}x = F^m x$.*

Figure 3.2. An Initial point for E_2

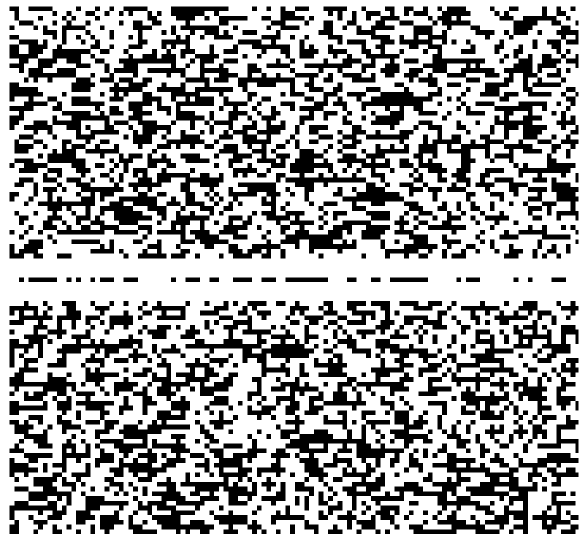


Figure 3.3. One Iteration of E_2

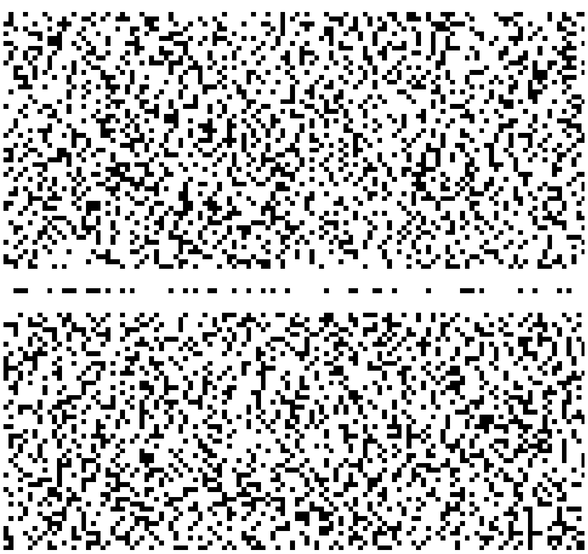


Figure 3.4. Two Iterations of E_2

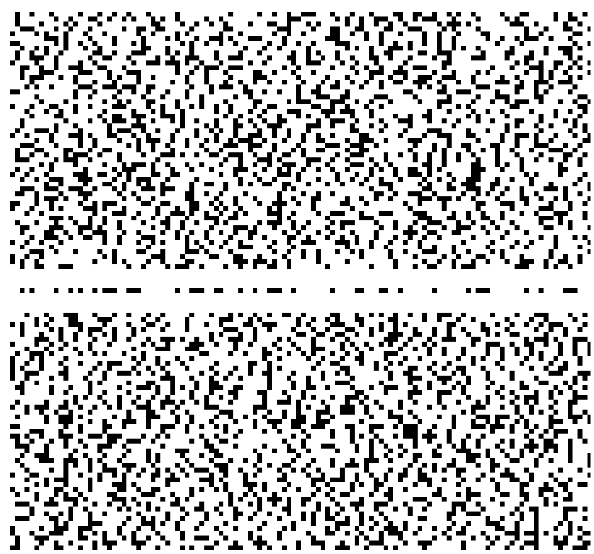


Figure 3.5. Three Iterations of E_2

Proof. Suppose that $\sigma_{\vec{\imath}} x = x$ for some point $x \in X$ and vector $\vec{\imath} = (i_1, \cdots, i_D) \in \mathbb{Z}^D$. Then for each $n \geq 0$, we have $\sigma_{\vec{\imath}}(F^n x) = F^n(\sigma_{\vec{\imath}} x) = F^n x$, and so each iterate $F^n x$ is also fixed under $\sigma_{\vec{\imath}}$. As in the proof of Theorem 3.1.2, we have a bound on the set of iterates, $|\{F^n x : n \geq 0\}| \leq |A|^{\Pi_{j=1}^{D} i_j}$; the set of iterates of x is a finite set. Therefore, there must be a repetition, $F^m x = F^{m+p} x$, for some $m \geq 0$ and $p > 0$. Thus x is F-eventually periodic. □

From Hurley's work investigating the attractor classification of CA, [**13**], Kůrka has extracted the one-dimensional result that any attracting periodic point for a CA must be a fixed point, with respect to both the CA and the shift, [**20**]. Hurley gave a result in arbitrary dimensions, and while his proof used the existence of a minimal quasi-attractor of full measure for a CA, Kůrka uses only topological notions. Our Theorem 3.3.2 is a strengthening of this result to the case of a subshift space, in any dimension, on which the generators for the shift action are each mixing. We use Kůrka's approach in our proof, keeping our result topological.

Let (Y, T) be a dynamical system. A subset $V \subseteq Y$ is an *attractor* if there exists a non-empty open set U such that $T(\overline{U}) \subseteq U$ and $V = \bigcap_{n \geq 0} T^n(U)$. A periodic point $y \in Y$ is *attracting* if its forward orbit $\mathcal{O}(y) = \{T^n y : n \geq 0\}$ is an attractor. Any attracting periodic point is a point of equicontinuity.

THEOREM 3.3.2. *Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift space on which each of the generators for the shift action, $\sigma_{\vec{e_j}}$, is topologically mixing, and let $F : X \to X$ be a cellular automaton. If $x \in X$ is an attracting periodic point for F, then $Fx = x$ and $\sigma_{\vec{\imath}}\, x = x$ for all $\vec{\imath} \in \mathbb{Z}^D$. That is, x is a fixed point with respect to both the CA and the shift action.*

Proof. Let p be the period of F, i.e., $F^p x = x$. By definition of attracting, there exists a non-empty open set U such that $F(\overline{U}) \subseteq U$ and $\bigcap_{n\geq 0} F^n(U) = \mathcal{O}(x) = \{x, Fx, \cdots, F^{p-1}x\}$. Thus, there exists a non-empty $U_0 \subseteq U$ so that $y \in U_0 \Rightarrow \lim_{n\to\infty} F^{np}y = x$. Since $\sigma_{\overrightarrow{e_j}}$ is topologically mixing on X for each standard basis vector $\overrightarrow{e_j}$, then there exist $K_1, K_2, \cdots, K_D \geq 0$ so that for each $1 \leq j \leq D$ and all $k \geq K_j$, $\sigma^k_{\overrightarrow{e_j}}(U_0) \cap U_0 \neq \emptyset$. In particular,

$$\sigma^{K_j}_{\overrightarrow{e_j}}(U_0) \cap U_0 \neq \emptyset \text{ and } \sigma^{K_j+1}_{\overrightarrow{e_j}}(U_0) \cap U_0 \neq \emptyset. \tag{3.11}$$

Then for $y_j \in \sigma^{K_j}_{\overrightarrow{e_j}}(U_0) \cap U_0$, we have points $y'_j \in U_0$ with $\sigma^{K_j}_{\overrightarrow{e_j}}(y'_j) = y_j$, so that $\lim_{n\to\infty} F^{np}(y'_j) = x$. Then by applying $\sigma^{K_j}_{\overrightarrow{e_j}}$, using continuity of each $\sigma_{\overrightarrow{e_j}}$, and using commutativity of F with each $\sigma_{\overrightarrow{e_j}}$, we have:

$$\sigma^{K_j}_{\overrightarrow{e_j}} x = \sigma^{K_j}_{\overrightarrow{e_j}}\left(\lim_{n\to\infty} F^{np}(y'_j)\right) = \lim_{n\to\infty} F^{np}\left(\sigma^{K_j}_{\overrightarrow{e_j}}(y'_j)\right) = \lim_{n\to\infty} F^{np}(y_j). \tag{3.12}$$

Similarly, for $z_j \in \sigma^{K_j+1}_{\overrightarrow{e_j}}(U_0) \cap U_0$, we have

$$\sigma^{K_j+1}_{\overrightarrow{e_j}} x = \lim_{n\to\infty} F^{np}(z_j). \tag{3.13}$$

But as $y_j,\ z_j \in U_0$, we also have

$$\lim_{n\to\infty} F^{np}(y_j) = \lim_{n\to\infty} F^{np}(z_j) = x. \tag{3.14}$$

Thus $\sigma^{K_j}_{\overrightarrow{e_j}} x = \sigma^{K_j+1}_{\overrightarrow{e_j}} x$, and as $\sigma_{\overrightarrow{e_j}}$ is a homeomorphism, $\sigma_{\overrightarrow{e_j}}\ x = x$. Since this is true for each $1 \leq j \leq D$, x is a fixed point for the shift action. Since x is fixed, we have $x_{\vec{\imath}} = a$ for all $\vec{\imath} \in \mathbb{Z}^D$.

But for each $\overrightarrow{e_j}$, $\sigma_{\overrightarrow{e_j}}(Fx) = F(\sigma_{\overrightarrow{e_j}}\ x) = Fx$, and so Fx must also be a fixed point for the shift; let $(Fx)_{\vec{\imath}} = b$ for all $\vec{\imath} \in \mathbb{Z}^D$. We will now show that $Fx = x$. We first note that as x is attracting, there also exists a non-empty, open $U_1 \subseteq U$ so that

$y \in U_1 \Rightarrow \lim_{n\to\infty} F^{np} y = Fx$. Since U_0 and U_1 are open and $x \in U_0,\ Fx \in U_1$, there is a positive number r so that the open ball about x with radius 2^{-r} and the open ball about Fx with radius 2^{-r} are contained in the sets U_0 and U_1, respectively. As the generators for the shift action are mixing on X, there is a point $z \in X$ such that the patterns $x|_{\prod_{j=1}^{D}\langle -r,r\rangle}$ and $(Fx)|_{\prod_{j=1}^{D}\langle -r,r\rangle}$ both appear in z. Let $\overrightarrow{v_1}, \overrightarrow{v_2} \in \mathbb{Z}^D$ be such that $d(\sigma_{\overrightarrow{v_1}} z, x) < 2^{-r}$ and $d(\sigma_{\overrightarrow{v_1}} z, Fx) < 2^{-r}$. Thus, $\sigma_{\overrightarrow{v_1}} z \in U_0$ and $\sigma_{\overrightarrow{v_2}} z \in U_1$. Now we have

$$\lim_{n\to\infty} F^{np}(\sigma_{\overrightarrow{v_1}} z) = x \tag{3.15}$$

$$\sigma_{\overrightarrow{v_1}}\left(\lim_{n\to\infty} F^{np} z\right) = x \tag{3.16}$$

$$\lim_{n\to\infty} F^{np} z = \sigma_{\overrightarrow{-v_1}} x = x. \tag{3.17}$$

Similarly, we have

$$\lim_{n\to\infty} F^{np}(\sigma_{\overrightarrow{v_2}} z) = Fx \tag{3.18}$$

$$\sigma_{\overrightarrow{v_2}}\left(\lim_{n\to\infty} F^{np} z\right) = Fx \tag{3.19}$$

$$\lim_{n\to\infty} F^{np} z = \sigma_{\overrightarrow{-v_2}} Fx = Fx. \tag{3.20}$$

Therefore, by comparing (3.17) and (3.20), we see that x is also fixed under the CA, F.

□

Now, we can realize Hurley's result as a corollary to Theorem 3.3.3, since each generator for the shift action on a full shift space is mixing.

Corollary 3.3.3. *Let $F : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be a cellular automaton. If $x \in A^{\mathbb{Z}^D}$ is an attracting periodic point for F, then $Fx = x$ and $\sigma_{\vec{\imath}}\ x = x$ for all $\vec{\imath} \in \mathbb{Z}^D$. That is, x is a fixed point with respect to both the CA and the shift action.*

CHAPTER 4

Almost Equicontinuity Properties

The property of being equicontinuous at every point is too restrictive for classifying CA's, for a number of cellular automata are not equicontinuous but still have points of equicontinuity. Recall that *almost equicontinuous* refers to a topological property; it means that the set of equicontinuity points is residual. In dimension one, Kůrka has shown that every CA is either almost equicontinuous or has sensitive dependence on initial conditions [**19**]. Therefore, we study the property of almost equicontinuous in higher dimensions as well. It is in this chapter that the classification separates from the one-dimensional case.

4.1. History in Dimension One

We will begin in dimension one with a brief review of the literature, so that when we give our higher dimensional results in Section 4.2, the role of the dimension will be clearer. In one dimension, almost equicontinuous CA's are characterized by the existence of blocking words, first introduced by Blanchard and Tisseur [**2**].

Let $X \subseteq A^{\mathbb{Z}}$ be a subshift space. A word u is called *s-blocking* for a CA $F : X \to X$ if $s \leq |u|$ and there exists a non-negative integer $p \leq |u| - s$, called the *offset*, so that for all $x, y \in [u]_0$, $F^n x_{\langle p,p+s)} = F^n y_{\langle p,p+s)}$ $\forall$ $n \geq 0$. Illustrated in Figure 4.1, this means that each occurrence of the pattern u in x determines a length s word in all iterates of x; p is the offset.

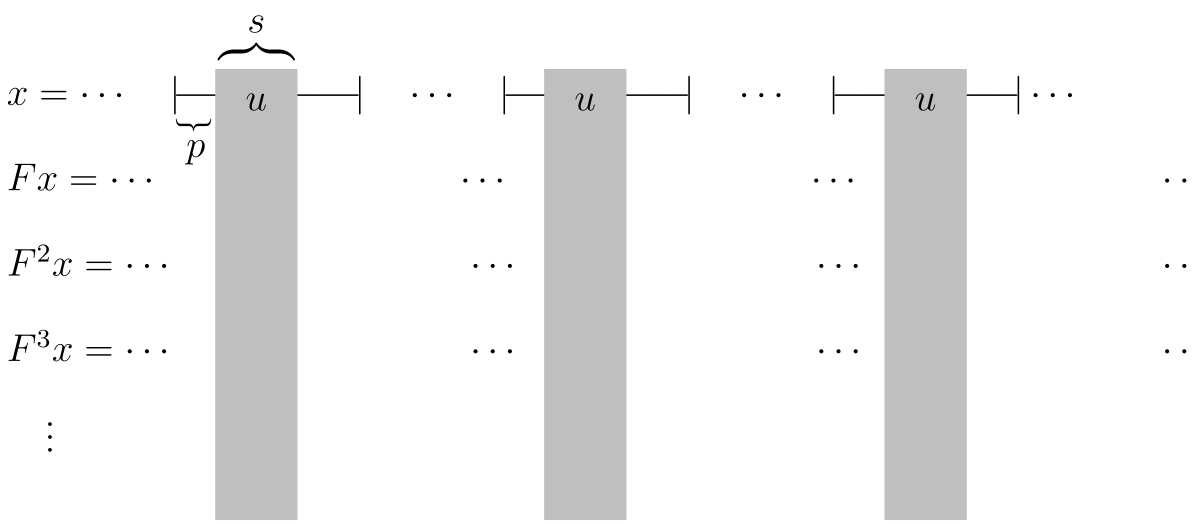


Figure 4.1. A Blocking Word for a 1D Cellular Automaton

As we have mentioned, any almost equicontinuous cellular automaton has a blocking word. This comes directly from the definition of a point of equicontinuity, and the fact that the metric is discrete on shift spaces. Further, the existence of a blocking word is a sufficient condition for almost equicontinuity, as was proven by Kůrka.

Theorem 4.1.1 (Kůrka,[**19**]). *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton with radius r. The following conditions are equivalent:*

(1) *F is not sensitive.*

(2) *There exists an r-blocking word for F.*

(3) *F is almost equicontinuous.*

While we will not reproduce the proof of Theorem 4.1.1 completely, it is instructive for moving the result into higher dimensions for us to provide an illustration of the key idea in the equivalence. Clearly, (3) $\Rightarrow$ (1); again (1) $\Rightarrow$ (2) is easily seen by combining the definition of an equicontinuity point and the nature of the metric on a shift space. So, we regard the remaining implication, (2) $\Rightarrow$ (1). Let u be the r-blocking word for F

guaranteed in the hypothesis. The set of points containing infinitely many occurrences of u in both the positive and negative coordinates is a residual set, and we will illustrate why a point in this set is one of equicontinuity in Figure 4.2. For such a point x, suppose $\varepsilon > 0$ is given; then choose $\delta \leq \varepsilon$ so that an occurrence of u is contained on both the positive and the negative side within the δ region. Since u is blocking and has length at least r, then for all time n, the values in the ε region cannot depend on values outside the δ region.

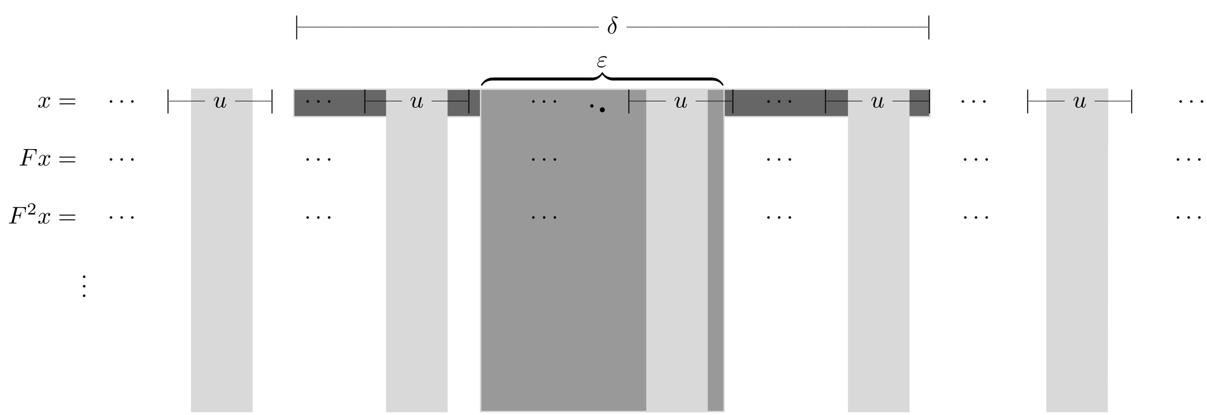


FIGURE 4.2. A point with infinitely many occurrences of u is one of equicontinuity

4.2. Almost Equicontinuity in Higher Dimensions

In order to develop our extension of the preceding theorem in higher dimensions, we prove versions of each implication in Theorems 4.2.1 through 4.2.3, rather than proving a single equivalence. We begin by noting the well-known fact that any dynamical system without isolated points which is almost equicontinuous cannot be sensitive, corresponding to (3) $\Rightarrow$ (1) in Theorem 4.1.1.

THEOREM 4.2.1. *Assume that X has no isolated points, and let (X, T) be an almost equicontinuous dynamical system. Then T is not sensitive.*

Proof. Since T is almost equicontinuous, T has a point of equicontinuity, x. Suppose that T is also sensitive. Then $\exists\ \varepsilon > 0$ such that for all $z \in X$ and $\delta > 0$, there exists $y \in X$ with $d(y, z) < \delta$ and $d(T^n y, T^n z) \geq \varepsilon$ for some $n \geq 0$. But by the definition of an equicontinuity point, the above property does not apply to x, and hence T is not sensitive. $\square$

We next extend the notion of blocking to patterns in dimensions two and higher, and examine the existence of such patterns and their relation to almost equicontinuity. We describe blocking first in two dimensions, where the definition is easier to state. A rectangular pattern u of size $k \times l$ is said to be (r, s)- *blocking* if there exist non-negative integers $p \leq k - r$ and $q \leq l - s$ such that for all $x, y \in [u]_{0,0}$ and $n \geq 0$, we have

$$(F^n x)|_{\langle p,\ p+r) \times \langle q,\ q+s)} = (F^n y)|_{\langle p,\ p+r) \times \langle q,\ q+s)}. \tag{4.1}$$

The pair (p, q) is called the *offset*. That is to say, if the pattern u occurs in a point x at the coordinates of E, the values of $F^n x$ at a subset of the coordinates of E are determined for all time n. See Figure 4.3 for an illustration; the entire shaded region is an occurrence of u, and the lighter shaded region is the part which is determined for all time.

If $r = k$, $s = l$ and $p = q = 0$, we say that u is *fully blocking.* A fully blocking pattern is one whose occurrence in x at E determines the values of $F^n x$ in all coordinates of E for all time n. In higher dimensions, a pattern u of size $k_1 \times k_2 \times \cdots \times k_D$ is $(r_1, r_2, \cdots, r_D)$-*blocking* if the occurrence of u in a point x determines the values of $F^n x$ in a $r_1 \times r_2 \times \cdots \times r_D$ hypercube of coordinates for all n. If each $k_i = r_i$, then u is said

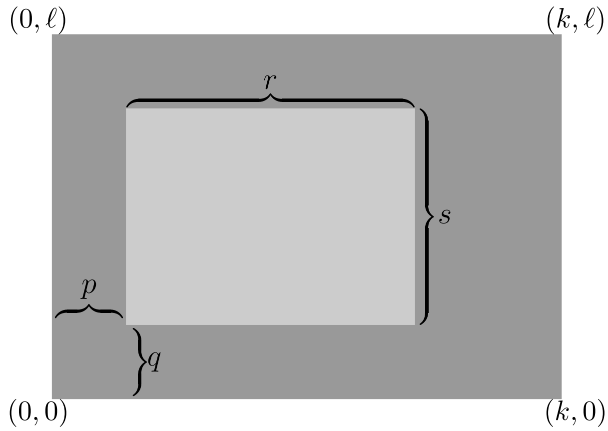


FIGURE 4.3. A Blocking Pattern for a 2D Cellular Automaton

to be *fully blocking*; that is, the values of $F^n x$ are determined in all coordinates where u occurs in x for all time n. For Example 2.3.3, the cellular automaton acting on the two-dimensional Golden Mean subshift space, any square pattern of 0's with side length at least 2 is fully blocking; once such a pattern occurs, there will always be 0's in those positions under iteration of G.

We now address the existence of blocking patterns for a CA. Theorem 4.2.2 is our version of (1) $\Rightarrow$ (2) in Theorem 4.1.1, showing that if a CA is not sensitive, then a blocking pattern exists.

THEOREM 4.2.2. *Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift and $F : X \to X$ be a cellular automaton with radius r which is not sensitive. Then there exists an $(r, r, \cdots, r)$-blocking pattern for F.*

Proof. Let $m \in \mathbb{Z}$ be such that $2m+1 \geq r$. Since F is not sensitive, for $\varepsilon = 2^{-m}$, $\exists\, x \in X$ and $\delta = 2^{-m-p}$, $p \geq 0$ such that for all $y \in X$, $d(x,y) < \delta \Rightarrow d(F^n x, F^n y) < \varepsilon\ \forall\ n \geq 0$. Let the pattern u be a central hypercube of x,

$$u = x|_{\prod_{j=1}^{D} \langle -(m+p), m+p \rangle} \in A^{\prod_{j=1}^{D}(2m+2p+1)}.$$

Now for $y, z \in [u]_{-(m+p)\vec{e}}$, we have $d(F^n y, F^n z) < \varepsilon \ \forall \ n \geq 0$, since $d(x, y) < \delta$ and $d(x, z) < \delta$ imply that both $d(F^n x, F^n y) < \varepsilon$ and $d(F^n x, F^n z) < \varepsilon$ for all $n \geq 0$. Thus u blocks a hypercube of size $(2m+1) \times (2m+1) \times \cdots \times (2m+1)$, and as m was chosen so that $2m + 1 \geq r$, u is an $(r, r, \cdots, r)$-blocking pattern. $\square$

If we assume that the existing blocking pattern is actually fully blocking on a full shift space, then we obtain almost equicontinuity of the CA. This provides a version of $(2 \Rightarrow 3)$ in Theorem 4.1.1. The proof we give relies on the fact that in a full shift space, patterns can always be fit together.

Theorem 4.2.3. *Let $F : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be a CA with radius r. If there exists a fully blocking pattern of size $k \times k \times \cdots \times k$ for F, where $k \geq r$, then F is almost equicontinuous.*

Proof. Let $u \in A^{\prod_{j=1}^{D} k}$ be a fully blocking pattern, where $k \geq r$. That is, u is a size $k \times k \times \cdots \times k$ pattern and when u occurs in a point of $A^{\mathbb{Z}^D}$, the values in the $k \times k \times \cdots \times k$ frame of the coordinates where u is are determined for all iterates of that point.

Let the sets G_n be the following:

$$G_n = \{x \in A^{\mathbb{Z}^D} : \exists\ \delta = \delta(n, x) \text{ such that} \quad d(x, y) < \delta \Rightarrow d(F^i x, F^i y) < 2^{-n}\ \forall\ i \geq 0\}. \tag{4.2}$$

Clearly, $\bigcap_{n \geq 0} G_n$ is the set of equicontinuity points for F. We will show that G_n is open, and using u, we will also show that G_n is dense for each n. This will prove that the set of equicontinuity points contains a residual set; that is, that F is almost equicontinuous.

Claim 1: G_n is open for each $n \geq 0$.

Fix $n \geq 0$, and let $x \in G_n$. Let δ be the $\delta(n,x)$ guaranteed by (4.2), the definition of G_n. We claim that $B_\delta(x) \subseteq G_n$. For, let $y \in B_\delta(x)$. In order for y to be in G_n, we need an $\alpha = \alpha(n,y)$ so that $d(y,z) < \alpha \Rightarrow d(F^i y, F^i z) < 2^{-n}\ \forall\ i \geq 0$. Let

$$\alpha = \min\left\{\frac{d(x,y)}{2}, \frac{\delta - d(x,y)}{2}\right\}. \tag{4.3}$$

Then for $d(y,z) < \alpha$, we have

$$\begin{aligned} d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + \frac{\delta - d(x,y)}{2} &= \\ \frac{\delta}{2} + \frac{d(x,y)}{2} < \frac{\delta}{2} + \frac{\delta}{2} &= \delta. \end{aligned} \tag{4.4}$$

Thus for each $i \geq 0$, $d(F^i x, F^i z) < 2^{-n}$. That is, $(F^i x)_{\vec{\jmath}} = (F^i z)_{\vec{\jmath}}$ for each $i \geq 0$ and $\vec{\jmath} = (j_1, j_2, \cdots, j_D)$ with $|j_1|, |j_2|, \cdots, |j_D| \leq n$. Now by the choice of δ, we also have

$$y \in B_\delta(x) \Rightarrow d(F^i x, F^i y) < 2^{-n}\ \forall\ i \geq 0, \tag{4.5}$$

and thus $d(F^i y, F^i z) < 2^{-n}\ \forall\ i \geq 0$. Therefore,

$$d(y,z) < \alpha \Rightarrow d(F^i y, F^i z) < 2^{-n}\ \forall\ i \geq 0, \tag{4.6}$$

and hence $y \in G_n$. Thus $B_\delta(x) \subseteq G_n$, and so G_n is open.

Claim 2: G_n is dense for each $n \geq 0$.

Let B be a hypercube pattern with side length a multiple of k. We will show that for all $\vec{p} \in \mathbb{Z}^D$, $G_n \cap [B]_{\vec{p}} \neq \emptyset$; thus G_n is dense. We build a point $x \in [B]_{\vec{p}}$ by placing the pattern B in the proper place, then filling out the rest of the coordinates with the pattern u. We will get the picture of Figure 4.4 for x in $A^{\mathbb{Z}^2}$.

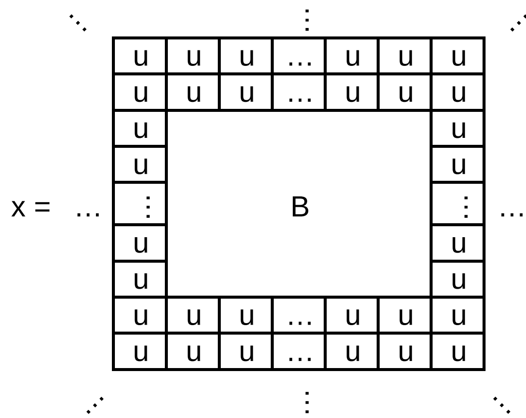


FIGURE 4.4. Creation of the point $x \in [B]_{\vec{p}} \cap G_n$

We have $x \in [B]_{\vec{p}}$, and we will now show that $x \in G_n$ also. Let $\delta = 2^{-(k+m)}$, where $m = \max\{n, \text{ side length of B}\}$. Suppose $y \in A^{\mathbb{Z}^D}$ has $d(x, y) < \delta$; we will show that all of the iterates $F^i x$ and $F^i y$, $i \geq 0$, are within 2^{-n}. By construction of δ, since y is within δ of x, y must contain the pattern B and at least one border of u patterns around B in the same location as x. Further, y will contain a border of u patterns inside the central $n \times n \times \cdots \times n$ region, again by construction of δ. Now as u is a $(k, k, \cdots, k)$-blocking pattern, the values at the coordinates of y which contain u (the frame around B) will be determined for all iterates $F^i y$. Since F has radius r and u has side length $k \geq r$, for each iterate $F^i y$, the values at the coordinates where y has the pattern B can only depend on the values at those same coordinates as well as the coordinates where y has the pattern u. That is, for all $i \geq 0$, the values in the central $n \times n \times \cdots \times n$ region of $F^i y$ are determined by the values of the coordinates in the central $(k+m) \times (k+m) \times \cdots \times (k+m)$ region of y, which equal the values of the coordinates in the central $(k+m) \times (k+m) \times \cdots \times (k+m)$ region of x. Therefore, $d(x, y) < \delta \Rightarrow d(F^i x, F^i y) < 2^{-n} \ \forall \ i \geq 0$, and thus $x \in G_n$.

Therefore, we have that G_n is an open, dense set for each $n \geq 0$, and that the intersection of all G_n's is contained in the set of equicontinuity points of F. Hence, F is almost equicontinuous. $\square$

What we have now shown in higher dimensions, with Theorems 4.2.1 through 4.2.3, is that almost equicontinuous implies not sensitive, not sensitive implies the existence of blocking patterns, and the existence of fully blocking patterns implies almost equicontinuity. However, not all almost equicontinuous CA's have fully blocking patterns; one such example is given in the next section. It has not yet been determined if there are any two-dimensional CA's which have a non-fully blocking pattern but are not almost equicontinuous.

The difficulty in extending Theorem 4.1.1 straightaway lies in the observation that in one dimension, the only directions from which values can propagate toward the center coordinate are the left and the right. In dimension 2, however, values can propagate toward the center coordinate from the left, the right, the top, the bottom, or any angle, depending on the rule of the CA. Recall that having a blocking pattern means that once we see a particular pattern, we will see a determined sequence of patterns in a subset of the positions where the blocking pattern occurred. What this accomplishes in one dimension is that a blocking pattern on either side of our coordinates of interest (in the epsilon region) seals off these central coordinates from outside influence, a sufficient condition for having the property of almost equicontinuity. However, existence of a blocking pattern in two dimensions does not appear to guarantee this. A first approach to guarantee almost equicontinuity is to use a blocking pattern in a ring around the coordinates of interest. However, as the patterns determine only a subset of the coordinates values for all time, it is conceivable that some far away value could affect the central coordinates by seeping in through the cracks in the patterns, that is, through the coordinates which are part of the blocking pattern which are not determined by the pattern's presence. This is illustrated in Figure 4.5; u is a blocking pattern which determines only the values in the shaded coordinates. The red coordinates and diagonal line represent the region through which an unwanted value could propagate toward the center.

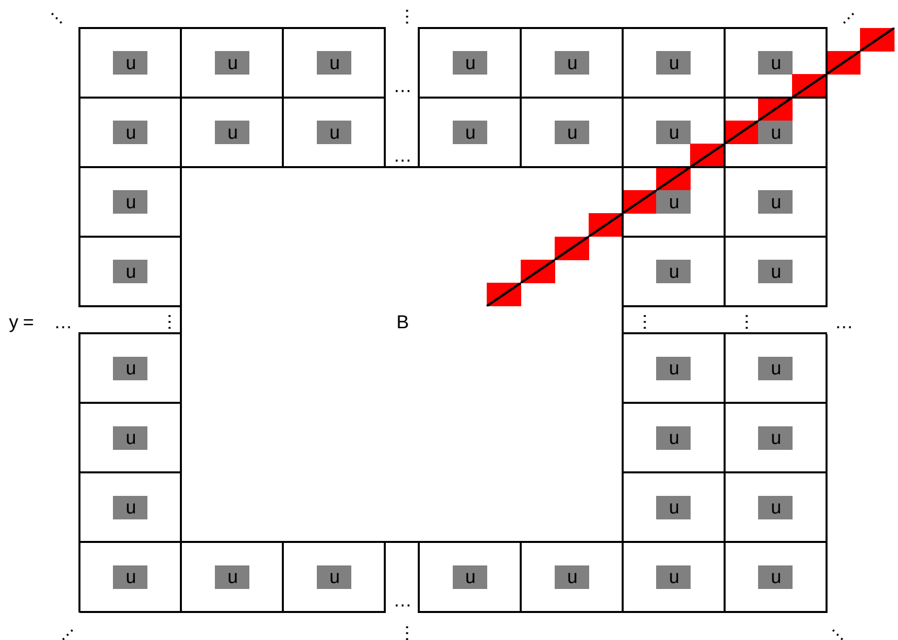


FIGURE 4.5. Potential Problem with Non-Fully Blocking Patterns

Our next result guarantees almost equicontinuity when there is a blocking pattern, not necessarily fully-blocking, which allows for no cracks when the pattern is used to form a boundary around the coordinates in the epsilon region. Let $F : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ be a CA with radius r. We say that a pattern u of size $k \times \ell$ *blocks a cross containing an* $r \times r$ *square for* F if, in addition to being (τ, s)-blocking with offset (p, q) for some $\tau, s \geq r$,

$p \leq k - \tau$ and $q \leq l - s$, u is also (τ, q)-blocking with offset $(p, 0)$, $(\tau, \ell - s - q)$-blocking with offset $(p, q + s)$, (p, s)-blocking with offset $(0, q)$, and $(k - p - \tau, s)$-blocking with offset $(p + \tau, q)$. As illustrated in Figure 4.6, the entire shaded region is an occurrence of u and the lighter shaded regions are coordinates whose values are determined for all time. Theorem 4.2.4 shows that fully blocking is not a necessary condition for almost equicontinuity.

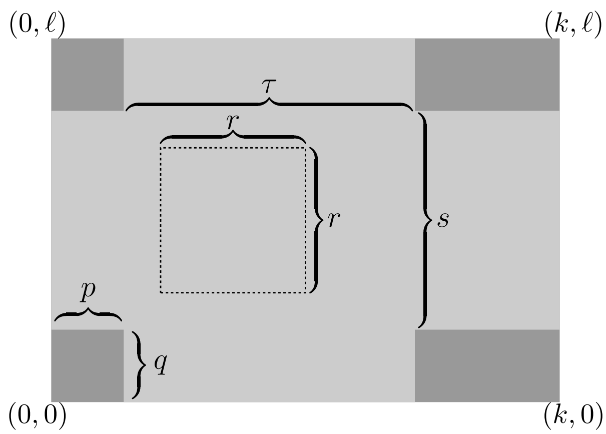


FIGURE 4.6. A Pattern Blocking a Cross for a 2D Cellular Automaton

THEOREM 4.2.4. *Let* $F : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ *be a cellular automaton with radius* r*. If there exists a pattern* u *which blocks a cross containing an* $r \times r$ *square for* F*, then* F *is almost equicontinuous.*

Proof. This proof is essentially identical to the proof of Theorem 4.2.3. The only difference is that the iterates of the point formed using the pattern u, $x \in [B]_{p,q} \cap G_n$, do not have all values determined in coordinates outside of those containing the pattern B. Figure 4.7 shows what x will look like in this situation, where the shaded coordinates represent the cross determined by the occurrence of u which contains an $r \times r$ square. Since the shaded region of x, the coordinates whose values are determined for all time,

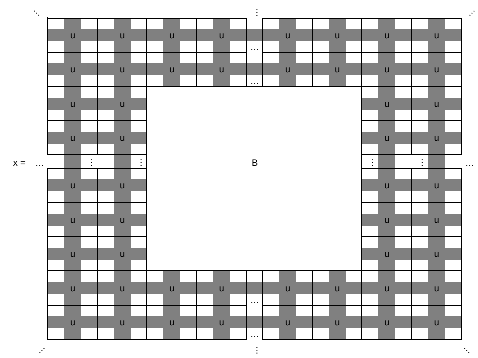


FIGURE 4.7. Creation of the point $x \in [B]_{p,q} \cap G_n$

has no gaps, x will be in each set G_n, and the proof of Theorem 4.2.3 holds here as well.

□

One final note on fully blocking patterns is that if such patterns are prevalent in the space, we can guarantee not just almost equicontinuity, but equicontinuity of the CA. In dimension one, Blanchard and Tisseur have shown that if there is an $s > 0$ such that every word of length at least s is blocking for a CA, then the CA is equicontinuous [**2**]. What we show is that if every big enough pattern is fully blocking for a CA, then the CA is equicontinuous.

THEOREM 4.2.5. *Let $F : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be a cellular automaton with radius r. If there exists $\ell \geq r$ such that for all patterns u of size $\|u\| \geq \ell$, u is fully blocking, then F is equicontinuous.*

Proof. Let $x \in A^{\mathbb{Z}^D}$; we will show that x is a point of equicontinuity. For $\varepsilon = 2^{-k} > 0$, set $\delta = 2^{-(k+\ell)}$. Now for $y \in A^{\mathbb{Z}^D}$ with $d(x, y) < \delta$, y contains the pattern $x|_{\prod_{j=1}^{D}\langle -k,k\rangle}$ in the same location as x, and by the choice of δ, y also contains the same values as x in the central annular region between the coordinates $\prod_{j=1}^{D}\langle -k, k\rangle$ and $\prod_{j=1}^{D}\langle -\ell, \ell\rangle$. We can cover this annular region by patterns of side length at least ℓ, which by assumption are all fully blocking. Thus, the values in this annular region for all iterates $F^i y$, $i \geq 0$, will be determined just by the values in these positions. Now since F has radius r and $\ell \geq r$, for each iterate $F^i y$, the values at the central $k \times \cdots \times k$ coordinates can only depend on the values of the coordinates in the central $(k + r) \times \cdots \times (k + r)$ region of y, which equal the values of the coordinates in the central $(k + r) \times \cdots \times (k + r)$ region of x, since $k + r \leq k + \ell$. Therefore, these values will be equal for all time; hence $d(x, y) < \delta \Rightarrow d(F^i x, F^i y) < 2^{-k} = \varepsilon \ \forall\ i \geq 0$.

□

4.3. Examples of Almost Equicontinuous Cellular Automata

Example 4.3.1. Recall the Golden Mean subshift space, $X_{\left\{11,\ \begin{smallmatrix}1\\1\end{smallmatrix}\right\}}$, and the CA $G : X_{\left\{11,\ \begin{smallmatrix}1\\1\end{smallmatrix}\right\}} \to X_{\left\{11,\ \begin{smallmatrix}1\\1\end{smallmatrix}\right\}}$ given in Example 2.3.3. We first show that the point x_0 consisting entirely of 0's is a *point of equicontinuity* for G. Note that x_0 is a fixed point of G. Fix an $\varepsilon = 2^{-k}$ and suppose that $d(x, x_0) < 2^{-(k+1)}$. Since every 0 in x in this central region has 0's in at least two of its left, right, top, and bottom neighbors, none of them will get mapped to a 1 in Gx. Similarly, all iterates $G^n x$ will retain the central $(2(k + 1) + 1) \times (2(k + 1) + 1)$ pattern of 0's, and so $d(G^n x, G^n x_0) < \varepsilon$ for all $n \geq 0$.

However, G is *not* an *equicontinuous* CA, as we next show. Consider the "checkerboard" period 2 points,

$$y_0 = \begin{array}{ccccccc} & & & \vdots & & & \\ & 1 & 0 & 1 & 0 & 1 & \\ & 0 & 1 & 0 & 1 & 0 & \\ \cdots & 1 & 0 & 1 & 0 & 1 & \cdots \\ & 0 & 1 & 0 & 1 & 0 & \\ & 1 & 0 & 1 & 0 & 1 & \\ & & & \vdots & & & \end{array} \quad \mapsto \quad y_1 = \begin{array}{ccccccc} & & & \vdots & & & \\ & 0 & 1 & 0 & 1 & 0 & \\ & 1 & 0 & 1 & 0 & 1 & \\ \cdots & 0 & 1 & 0 & 1 & 0 & \cdots \\ & 1 & 0 & 1 & 0 & 1 & \\ & 0 & 1 & 0 & 1 & 0 & \\ & & & \vdots & & & \end{array}$$

Fix $\varepsilon = 1$, and suppose there is a $\delta = 2^{-M}$ so that $d(x, y_0) < \delta \Rightarrow d(G^n x, G^n y_0) < 1$ for all $n \geq 0$. Take x^* to agree with y_0 in the central $(2M+1) \times (2M+1)$ pattern, and to be filled with 0's in all other coordinates. Now the 0's in the $\pm M^{th}$ rows and columns have only two 1's in their left, right, top, and bottom neighbors, and so they will not change to 1's in Gx^*. Under the next iteration of G, we see that the 0's in the $\pm(M-1)^{st}$ rows and columns do not change to 1's, and as we continue to apply L, the 1's eventually disappear completely. In particular, $G^n x^* = x_0$ for $n \geq M+1$. But as $(G^n y_0)_{0,0}$ alternates between 0 and 1 as n increases, $d(G^n x^*, G^n y_0) \not< 1$ for all n.

In fact, G is *almost equicontinuous*. As in Theorem 4.2.3, we use the fully blocking pattern $\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$ in order to construct the open dense sets G_n, whose intersection is the set of equicontinuity points.

Example 4.3.2. Let $A = \{0, 1\}$, and let $R : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ be given by the radius 1 local rule,

$$r \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} = \begin{cases} 0 & \text{if } \sum_{i=1}^{9} x_i \leq 3 \\ 1 & \text{if } \sum_{i=1}^{9} x_i \geq 4 \end{cases} . \tag{4.7}$$

Neither 0 nor 1 are (1,1)-blocking patterns for R, but consider the pattern $u = \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}$. Since each 1 neighbors at least three other 1's, $u \mapsto u$, and hence u is a fully blocking pattern. Then by Theorem 4.2.3, R is *almost equicontinuous*. The dynamics of R are illustrated in Figures 4.8 and 4.9.

By a pre-image of cylinder count, we will show that R is *not surjective*. Since a pattern $u \in \{0, 1\}^{3 \times 3}$ maps to 0 when there are no more than three 1's present in u, we

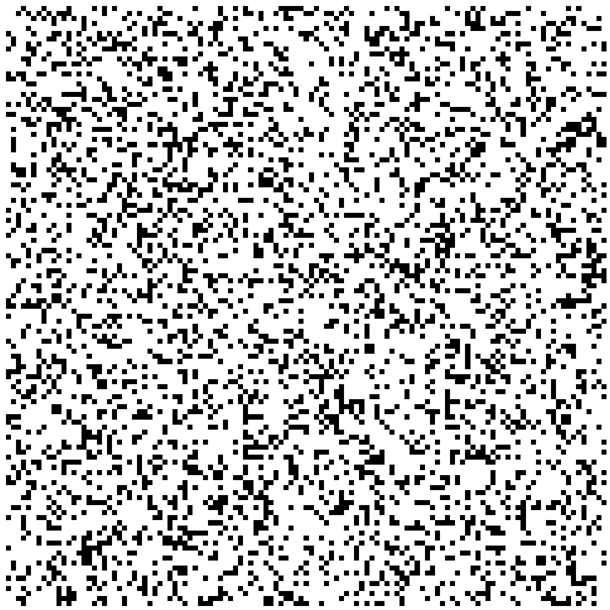


Figure 4.8. An initial point for R

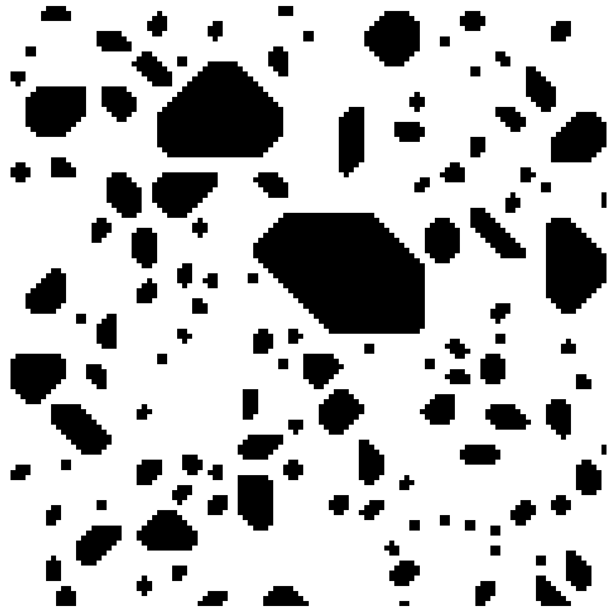


Figure 4.9. Sixty-four iterations of R

see that there are $1 + \binom{9}{1} + \binom{9}{2} = 46$ such patterns. On the other hand, the remaining $2^9 - 46 = 466$ patterns of size 3×3 map to 1, and so R is not 1-balanced. Thus by Theorem 2.2.3, R is not surjective. By Theorem 2.2.2, R is *not injective* either.

Example 4.3.3. Let $A = \{0, 1\}$, and let $M : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ be given by the radius 1 local rule,

$$m\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} = \begin{cases} 0 & \text{if } \displaystyle\sum_{i=1}^{9} x_i \leq 4 \\ 1 & \text{if } \displaystyle\sum_{i=1}^{9} x_i \geq 5 \end{cases}. \tag{4.8}$$

M successively homogenizes the space, illustrated in Figures 4.10 and 4.11.

There are *no fully blocking $k \times k$ patterns* for this CA, since the values in the corners of a square neighbor only three other known values, and at least four neighboring values must be known in order to determine the value in the next iteration. However, consider

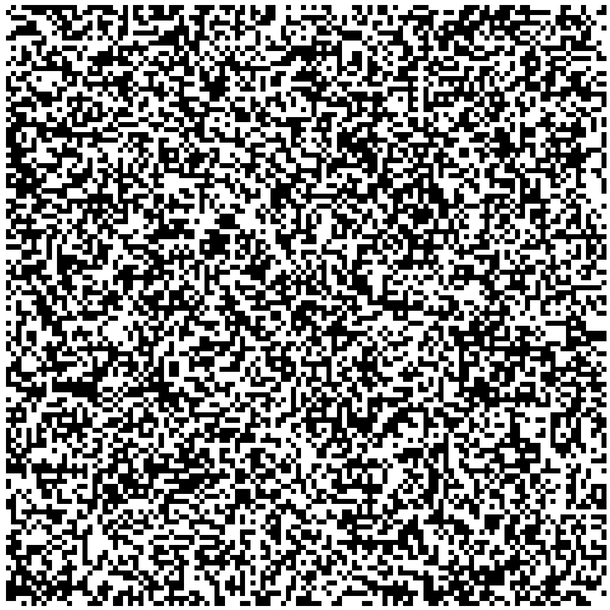


FIGURE 4.10. An initial point for M

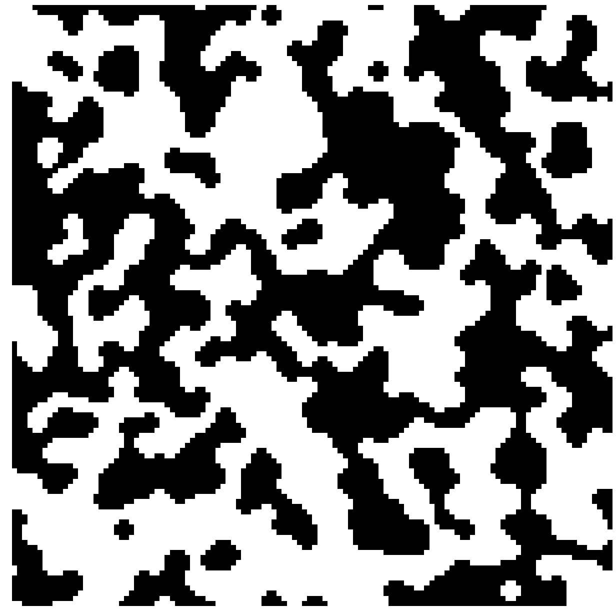


FIGURE 4.11. Twenty-seven iterations of M

the pattern $u = \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}$. After one iteration, $u \mapsto \begin{array}{cccc} & 1 & 1 & \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ & 1 & 1 & \end{array}$, which is invariant, since each of the 1's neighbor at least four other 1's. Thus u is a blocking pattern; it determines a cross of 1's containing a 2×2 square for all time. Now as M has radius 1, M is *almost equicontinuous* by Theorem 4.2.4. Note that this CA is *not equicontinuous* though, for consider the point

$$x = \begin{array}{ccccccc} & & & \vdots & & & \\ & 1 & 0 & 1 & 0 & 1 & \\ & 0 & 1 & 0 & 1 & 0 & \\ \cdots & 1 & 0 & 1 & 0 & 1 & \cdots \\ & 0 & 1 & 0 & 1 & 0 & \\ & 1 & 0 & 1 & 0 & 1 & \\ & & & \vdots & & & \end{array}$$

Since each value is surrounded by four 0's and four 1's, this is a fixed point under M. It is *not an equicontinuity point* though: for $\varepsilon = 2^{-k}$ and $\delta = 2^{-m}$, let $y \in B_\delta(x)$ so that

$y_{\vec{\imath}} = 0$ for each $||\vec{\imath}|| \geq m+1$. That is, y agrees with x in a large central square and has 0's elsewhere. Under iteration of M, the 0's in y propagate towards the center until $M^{m+1}y$ consists solely of 0's.

M is *not* a *surjective* CA either; we will appeal to Theorem 2.2.3 by counting the pre-image patterns of the four 1×2 patterns. Some careful counting is required, but we find that $|M^{-1}(00)| = |M^{-1}(11)| = 1518$ and $|M^{-1}(01)| = |M^{-1}(10)| = 530$, totalling the $2^{12} = 4096$ possible 3×4 patterns. Hence M is not 2-balanced, and thus neither surjective *nor injective*.

EXAMPLE 4.3.4. Let $A = \{$🟥 , 🟦 , 🟨 , ◻$\}$, and let $P : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ be the CA given in Example 2.3.2, describing the movement of three different colored particles through space.

Clearly, 🟨 is a (1,1)-blocking pattern, and since P has radius 1, Theorem 4.2.3 implies that $(A^{\mathbb{Z}^2}, P)$ is *almost equicontinuous*. As the point consisting of a single 🟥 particle in all white space evolves under iteration to a longer and longer diagonal red line through white space, P cannot be eventually periodic, and hence is *not equicontinuous*.

CHAPTER 5

Sensitive Dependence Properties

We now turn our attention away from predictable cellular automata and investigate those CA's which do not have points of equicontinuity. Recall Proposition **??**, that any CA without a point of equicontinuity must have sensitive dependence on initial conditions. In keeping with our goal of lifting one-dimensional CA phenomena to higher dimensions, we first build a class of examples in dimension $D+1$ from CA's in dimension D, showing that the new CA is sensitive if and only if the D-dimensional CA is. We give another way to construct sensitive CA's by taking the product of two CA's, at least one of which is sensitive. We then investigate the property of topological transitivity, as a transitive CA on a one-dimensional full shift space must have sensitive dependence on initial conditions [**19**]. Finally, we discuss some concrete examples of sensitive CA's.

5.1. Constructions of Sensitive Cellular Automata

Recall the construction of Example 3.2.5, in which we used a D-dimensional equicontinuous CA to build a $(D+1)$-dimensional equicontinuous CA by letting the lower dimensional rule act on restrictions of a higher dimensional point. As we show in Theorem 5.1.1, the properties of sensitive and of not sensitive are also preserved in this construction. We give the proof for dimension $D = 1$ to $D + 1 = 2$, where the terminology is easier; however, it is easily applicable to arbitrary D.

THEOREM 5.1.1. *Let $G : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton. For each $j \in \mathbb{Z}$, let $H_j : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}}$ be the restriction $(H_j x)_i = x_{(i,j)}$. Now define the cellular automaton $F : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ by $(Fx)_{(i,j)} = (G \circ H_j x)_i$. Then F is sensitive if and only if G is sensitive.*

Proof. ($\Leftarrow$) Assume that G is sensitive. Then there is an $\varepsilon_G = 2^{-k}$ such that for all $x \in A^{\mathbb{Z}}$ and $\delta > 0$, there is some $y \in B_\delta(x)$ and $n > 0$ such that $d(G^n x, G^n y) \geq \varepsilon_G$. We claim that $\varepsilon = \varepsilon_G$ is a sensitive constant for F also. For, let $x \in A^{\mathbb{Z}^2}$ and let $\delta = 2^{-m} > 0$. Let $y_0 \in B_\delta(H_0 x)$ be the one-dimensional sequence and $n > 0$ be the iterate guaranteed by the sensitivity of G to have $d(G^n \circ H_0 x, G^n y_0) \geq \varepsilon$. Now there is a point $y \in A^{\mathbb{Z}^2}$ having both $d(x, y) < \delta$ and $H_0 y = y_0$ since $d(H_0 x, y_0) < \delta$. Then we have $d(G^n \circ H_0 x, G^n \circ H_0 y) = d(G^n \circ H_0 x, G^n y_0) \geq \varepsilon$. That is, after n iterations of F, there are differences in the ε-regions of iterates of x and y in the central row, so that $d(F^n x, F^n y) \geq \varepsilon$. Therefore, F is sensitive.

($\Rightarrow$) Assume G is not sensitive. Let $\varepsilon = 2^{-k} > 0$; then $\exists\ x \in A^{\mathbb{Z}}$ and $\delta_G = 2^{-m} > 0$ such that for all $y \in B_{\delta_G}(x) \subseteq A^{\mathbb{Z}}$, the iterates of x and y under G stay close; $d(G^n x, G^n y) < \varepsilon$ for all $n \geq 0$. Take $z \in A^{\mathbb{Z}^2}$ such that $H_j z = x$ for all $|j| \leq \max\{k, m\}$. That is, z is a point in the infinitely wide cylinder set

$$\begin{bmatrix} x \\ x \\ \vdots \\ x \\ x \end{bmatrix},$$

having the central $2k+1$ and $2m+1$ rows all equal to the one-dimensional sequence x. Now take $\delta = \min\{\delta_G, \varepsilon\}$.

Claim: $\forall\, y \in B_\delta(z) \subseteq A^{\mathbb{Z}^2},\ d(F^n y, F^n z) < \varepsilon$ for all $n \geq 0$.

Let $y \in A^{\mathbb{Z}^2}$ such that $d(y,z) < \delta$. Then for $|j| \leq m,\ k$, we have $d(H_j y, H_j z) = d(H_j y, x) < \delta$. So for these j,

$$d(G^n \circ H_j y, G^n \circ H_j z) < \varepsilon\ \forall\, n \geq 0. \tag{5.1}$$

That is, for $|i| \leq k$, $(G^n \circ H_j y)_i = (G^n \circ H_j z)_i$ for all $n \geq 0$. Thus, for $|i|, |j| \leq k$, $(F^n y)_{i,j} = (F^n z)_{i,j}$ for all $n \geq 0$, or $d(F^n y, F^n z) < \varepsilon$ for all $n \geq 0$. Therefore, F is not sensitive. □

Another way to construct new sensitive CA's from old ones is by taking products of CA's, at least one of which is sensitive. Let A be a finite alphabet, let $D \geq 1$, and let $F_1,\ F_2 : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be two cellular automata. The aim is to define a CA $F := F_1 \times F_2$ on the space $(A \times A)^{\mathbb{Z}^D}$, where a point in $(A \times A)^{\mathbb{Z}^D}$ has a pair of values from A, (a_1, a_2), at each coordinate of $\mathbb{Z}^D$. In order to do so, let $P_1 : (A \times A)^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be the projection map onto the first component, and similarly let $P_2 : (A \times A)^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be projection onto the second component. Now we define $F : (A \times A)^{\mathbb{Z}^D} \to (A \times A)^{\mathbb{Z}^D}$ by

$$(Fx)_{\vec{\imath}} = \left([F_1 \circ P_1 x]_{\vec{\imath}},\ [F_2 \circ P_2 x]_{\vec{\imath}}\right). \tag{5.2}$$

Using the local rules of F_1 and F_2, we see that F is actually a cellular automaton. Notice that, for $j \in \{1, 2\}$, as $(F_j \circ P_j\, x)$ and $(P_j \circ Fx)$ both represent the j^{th} component of Fx, we have $(P_j \circ F^n x) = (F_j^n \circ P_j\, x)$.

Proposition 5.1.2. *Let $F_1,\ F_2 : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be any cellular automata, and let $F := F_1 \times F_2 : (A \times A)^{\mathbb{Z}^D} \to (A \times A)^{\mathbb{Z}^D}$ be the cellular automaton defined in* (5.2)*. Then if either F_1 or F_2 is sensitive, F is sensitive as well.*

Proof. Suppose, without loss of generality, that F_1 is a sensitive CA. Then by definition, there exists $\varepsilon_0 > 0$ such that for all $z \in A^{\mathbb{Z}^D}$ and all $\delta > 0$, there exists $y \in A^{\mathbb{Z}^D}$ with $d(z, y) < \delta$ such that $d(F_1^n z, F_1^n y) \geq \varepsilon_0$ for some $n \geq 0$. We show that ε_0 is a sensitive constant for F also. Let $x \in (A \times A)^{\mathbb{Z}^D}$, and fix $\delta > 0$. Then there exists $y \in A^{\mathbb{Z}^D}$ and $n_0 \geq 0$ such that $d(P_1 x, y) < \delta$, but $d(F_1^{n_0} \circ P_1 x, F_1^{n_0} y) \geq \varepsilon_0$. Let $w \in (A \times A)^{\mathbb{Z}^D}$ be given by $P_1 w = y$ and $P_2 w = P_2 x$. Thus we have,

$$d(x, w) = \max\{d(P_1 x, P_1 w), d(P_2 x, P_2 w)\} = \{d(P_1 x, y), 0\} = d(P_1 x, y) < \delta. \tag{5.3}$$

But also,

$$\begin{aligned} d(F^{n_0} x, F^{n_0} w) &= \max\{d(P_1 \circ F^{n_0} x, P_1 \circ F^{n_0} w), d(P_2 \circ F^{n_0} x, P_2 \circ F^{n_0} w)\} \\ &= \max\{d(F_1^{n_0} \circ P_1 x, F_1^{n_0} \circ P_1 w), d(F_2^{n_0} \circ P_2 x, F_2^{n_0} \circ P_2 w)\} \\ &= \max\{d(F_1^{n_0} \circ P_1 x, F_1^{n_0} y), d(F_2^{n_0} \circ P_2 x, F_2^{n_0} \circ P_2 x)\} \qquad (5.4) \\ &= \max\{d(F_1^{n_0} \circ P_1 x, F_1^{n_0} y), 0\} \\ &= d(F_1^{n_0} \circ P_1 x, F_1^{n_0} y) \geq \varepsilon_0. \end{aligned}$$

Therefore, F is sensitive with constant ε_0 also. ∎

We defer examples of this type to Section 5.3.

5.2. Topologically Transitive Cellular Automata

Recall that a dynamical system (Y, T) is *transitive* if there is a point $y \in Y$ with a dense forward orbit. We discuss transitivity here since transitive CA's on a one-dimensional full shift space are sensitive [**19**]. We generalize the proof of this result so that in higher dimensions as well, a transitive CA must be sensitive if the underlying subshift space is infinite.

THEOREM 5.2.1. *Let $F : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be a cellular automaton and suppose $X \subseteq A^{\mathbb{Z}^D}$ is an F-invariant subshift. If (X, F) is transitive, then it is either sensitive or consists of a single periodic orbit.*

Proof. Suppose that (X, F) is transitive but not sensitive. By a result of Glasner and Weiss, (Lemma 1.2, [**11**]), (X, F) is uniformly rigid. That is, there exists a sequence $n_k \nearrow$ such that $\{F^{n_k}\}$ tends uniformly to the identity on X. So for $\varepsilon = 1$, there is an $n \geq 0$ so that for all $x \in X$, $d(F^n x, x) < 1$. That is, $(F^n x)_{\vec{0}} = x_{\vec{0}}$ for all x and this n. Now for any integer vector $\vec{\imath} \in \mathbb{Z}^D$ and all $x \in X$ we have

$$(F^n x)_{\vec{\imath}} = (\sigma_{\vec{\imath}} \circ F^n x)_{\vec{0}} = (F^n \circ \sigma_{\vec{\imath}} x)_{\vec{0}} = (\sigma_{\vec{\imath}} x)_{\vec{0}} = x_{\vec{\imath}}. \tag{5.5}$$

Therefore, $F^n = Id$, and so X must consist of a single periodic orbit. □

COROLLARY 5.2.2. *Every cellular automaton on an infinite subshift space which is transitive is also sensitive.*

Proof. This is clear, since the only alternative to a transitive CA being sensitive is that the subshift must consist of a single periodic orbit. However, this would then make the subshift space finite. □

We can say even more about transitive CA's, following a result for a general dynamical system. Theorem 5.2.3 appears regularly in the literature without proof, so we include one.

THEOREM 5.2.3. *Let (X, T) be a dynamical system with no isolated points. If T is transitive, T is also surjective.*

Proof. Let $x \in X$ be the point with a dense forward orbit and let $y \in X$ be arbitrary. Then we have a sequence of iterates, $\{T^{n_k}(x)\}$, $n_k > 0$, which converges to y. Consider the sequence $\{T^{n_k-1}(x)\}$. Since X is compact, there is a convergent subsequence, $\{T^{n_{k_i}-1}(x)\} \to x_0$. Continuity of T implies that

$$\{T(T^{n_{k_i}-1}x)\} = \{T^{n_{k_i}}(x)\} \to T(x_0). \tag{5.6}$$

Then as X is a metric space, the limit of this sequence is unique, and so $T(x_0) = y$. Thus T is surjective. $\square$

COROLLARY 5.2.4. *Every cellular automaton on an infinite subshift space which is transitive is surjective.*

5.3. Examples of Sensitive Cellular Automata

We illustrate the ideas brought out in the preceding sections with examples.

EXAMPLE 5.3.1. Let A be any finite alphabet, $D \geq 1$, and $\vec{v} \in \mathbb{Z}^D$ any vector. Then the directional shift $\sigma_{\vec{v}}$ is *sensitive* on the full shift space $A^{\mathbb{Z}^D}$, with sensitive constant $\varepsilon = 1$. For, given any x and any $\delta = 2^{-k} > 0$, take a point $y \in A^{\mathbb{Z}^D}$ such that the central pattern $y|_{\prod_{j=1}^{D}\langle -k,k\rangle}$ agrees with x, but having $y_{\vec{\imath}} \neq x_{\vec{\imath}}$ for all $||\vec{\imath}|| \geq k+1$. Then, as y is shifted in the direction of $\vec{v}$, these discrepancies will be closer to the origin, resulting in relatively large distances between the iterates of x and of y. In fact, we have $d\left(\sigma_{\vec{v}}^{k+1}x, \sigma_{\vec{v}}^{k+1}y\right) = 1$, since $\left(\sigma_{\vec{v}}^{k+1}x\right)_{\vec{0}} = x_{(k+1)\cdot\vec{v}}$ and $||\vec{v}|| \geq 1 \Rightarrow ||(k+1)\cdot\vec{v}|| \geq k+1$.

EXAMPLE 5.3.2. Let A be any finite alphabet, $D \geq 1$, and $\vec{v} \in \mathbb{Z}^D$ any vector. By Proposition 5.1.2, the CA $\sigma_{\vec{v}} \times I$ is *sensitive* on $(A \times A)^{\mathbb{Z}^D}$, where I is the identity map. This CA is easily seen to be *surjective*, since both $\sigma_{\vec{v}}$ and I are surjective on $A^{\mathbb{Z}^D}$. However, I is not transitive, and so $\sigma_{\vec{v}} \times I$ is *not transitive* either.

EXAMPLE 5.3.3. Let $X_{\left\{11,\ \begin{smallmatrix}1\\1\end{smallmatrix}\right\}} \subseteq \{0,1\}^{\mathbb{Z}^2}$ be the two-dimensional Golden Mean subshift space as in Example 2.3.3. We claim that for any vector $\vec{v} \in \mathbb{Z}^D$, $\sigma_{\vec{v}}$ is *sensitive.* We cannot simply refer to the discussion in Example 5.3.1 on this subshift space, since we could potentially be creating points $y \in \{0,1\}^{\mathbb{Z}^2} \setminus X_{\left\{11,\ \begin{smallmatrix}1\\1\end{smallmatrix}\right\}}$ by changing all the values of a given x outside the central $k \times k$ region. However, given an $x \in X_{\left\{11,\ \begin{smallmatrix}1\\1\end{smallmatrix}\right\}}$ and $\delta = 2^{-k}$, choose $y \neq x \in A^{\mathbb{Z}^2}$ such that $y|_{\prod_{j=1}^{D}\langle -k,k\rangle} = x|_{\prod_{j=1}^{D}\langle -k,k\rangle}$ as follows. If x has a 1 outside this central δ region in the $\vec{v}$ direction, that is, if $x_{j\vec{v}} = 1$ for some $j \in \mathbb{Z}$, then take $y_{j\vec{v}} = 0$ and $y_{\vec{\imath}} = x_{\vec{\imath}}$ for all other $\vec{\imath} \neq j\vec{v}$. For such $x,\ y \in X_{\left\{11,\ \begin{smallmatrix}1\\1\end{smallmatrix}\right\}}$, we have $d(x,y) < \delta$ and $d(\sigma_{\vec{v}}^{j}x, \sigma_{\vec{v}}^{j}y) = 1$. If x does not have a 1 outside the central $k \times k$ region in the $\vec{v}$ direction, that is if $x_{j\vec{v}} = 0$ for all $j \in \mathbb{Z}$, then take $y_{j\vec{v}} = 1$ for all $||j\vec{v}|| \geq k+2$ and $y_{\vec{\imath}} = 0$ for all other $\vec{\imath}$. Then $y \in X_{\left\{11,\ \begin{smallmatrix}1\\1\end{smallmatrix}\right\}}$, $d(x,y) < \delta$, and $d(\sigma_{\vec{v}}^{k+2}x, \sigma_{\vec{v}}^{k+2}y) = 1$. Therefore, $\sigma_{\vec{v}}$ is sensitive on $X_{\left\{11,\ \begin{smallmatrix}1\\1\end{smallmatrix}\right\}}$ with constant 1.

EXAMPLE 5.3.4. Let $A = \{0,1\}$, and consider the space $\mathcal{X} \subseteq (A \times A)^{\mathbb{Z}^2}$ defined by

$$\mathcal{X} = \left\{ x \in (A \times A)^{\mathbb{Z}^2} : P_1 x,\ P_2 x \in X_{\left\{11,\ \begin{smallmatrix}1\\1\end{smallmatrix}\right\}} \right\}. \tag{5.7}$$

Now on $\mathcal{X}$, we have the product CA $\sigma_{\vec{v}} \times G$, where $G : X_{\left\{11,\ \substack{1\\1}\right\}} \to X_{\left\{11,\ \substack{1\\1}\right\}}$ is the CA given in Example 2.3.3 and $\vec{v} \in \mathbb{Z}^D$ is any integer vector. Since $\sigma_{\vec{v}}$ was seen to be sensitive in Example 5.3.3, then $\sigma_{\vec{v}} \times G$ is also *sensitive* by Proposition 5.1.2. As discussed in Example 5.3.3, G is not surjective; let $y \in X_{\left\{11,\ \substack{1\\1}\right\}}$ be a point which is not in the image of G. Then for any $x \in X_{\left\{11,\ \substack{1\\1}\right\}}$, the point $w \in \mathcal{X}$ having $P_1 w = x$ and $P_2 w = y$ will not be in the image of $\sigma_{\vec{v}} \times G$. Therefore, $\sigma_{\vec{v}} \times G$ is *not surjective* and thus *not transitive.*

CHAPTER 6

Expansive Properties

We now turn to a discussion of expansive cellular automata, where the dimension of the underlying shift space plays a much larger role. By a result of Shereshevsky, there are no expansive cellular automata on any full shift space in dimension $D \geq 2$. Further, if $F : X \to X$ is an expansive cellular automaton, where $X \subseteq A^{\mathbb{Z}^D}$ with $D \geq 2$, then the underlying shift action on X must have entropy zero [**31**]. Using complete history spaces (in the language of Milnor, [**24**]) we show that there are subshift spaces in all dimensions which have expansive CA's. To get such spaces, we use complete histories, first under directional shifts, and then under any bijective CA. We raise the question of whether any complete history space coming from a CA has expansive directional shifts, and give some partial answers. To this end, we show that the shift action has entropy 0 on such spaces, and so an expansive CA could exist on this space. We conclude with some concrete examples of spaces which do have expansive directions. This results in a construction of an expansive CA on some subshift $X \subseteq A^{\mathbb{Z}^D}$ for each D, in contrast to the result in [**31**].

6.1. Expansive Cellular Automata on Subshift Spaces

We first introduce some notation. For $j \in \mathbb{Z}$, let $\pi_j : A^{\mathbb{Z}^{D+1}} \to A^{\mathbb{Z}^D}$ be the restriction map onto the j^{th} D-dimensional Euclidean subspace, given by $(\pi_j\, x)_{(i_1,\cdots,i_D)} = x_{(i_1,\cdots,i_D,j)}$.

When $D = 1$, π_j gives the j^{th} row of a point of $A^{\mathbb{Z}^2}$, when $D = 2$, π_j gives the j^{th} plane of a point of $A^{\mathbb{Z}^3}$, and so on.

Let $F : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be a cellular automaton. We will define a $(D+1)$-dimensional subshift space of complete histories,

$$X_F = \{x \in A^{\mathbb{Z}^{D+1}} : \forall\ j \in \mathbb{Z},\ \pi_j x = F \circ \pi_{j+1} x\}. \tag{6.1}$$

Thus, a point in X_F consists of an orbit of a D-dimensional point under F. If F is not invertible, then there will be multiple points in X_F with equal "half histories;" that is, for $k \in \mathbb{Z}$, there are points $x \neq y \in X_F$ with $\pi_j x = \pi_j y$ for all $j \leq k$.

We will use complete history spaces to inductively build up subshift spaces in all dimensions which have expansive CA's. The maps will be simply directional shifts. Since these are components of the $\mathbb{Z}^D$ shift action, σ, we first introduce expansivity for higher dimensional actions, as given by Boyle and Lind [**3**], for our setting.

Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^D$, and for a vector $\vec{w} \in \mathbb{R}^D$ and a linear subspace $V \subseteq \mathbb{R}^D$, define the distance from $\vec{w}$ to V to be

$$\operatorname{dist}(\vec{w}, V) = \inf\{\|\vec{v} - \vec{w}\| : \vec{v} \in V\}. \tag{6.2}$$

For each linear subspace V and $t > 0$, define $V^t = \{\vec{w} \in \mathbb{R}^D :\ \operatorname{dist}(\vec{w}, V) \leq t\}$ to be a thickening of V by t. If W is any closed subset of $\mathbb{R}^D$, consider the following new function on $A^{\mathbb{Z}^D} \times A^{\mathbb{Z}^D}$:

$$d_\sigma^W(x, y) = \sup\{d(\sigma_{\vec{\imath}}\ x, \sigma_{\vec{\imath}}\ y) : i \in W \cap \mathbb{Z}^D\}. \tag{6.3}$$

If $W \cap \mathbb{Z}^D = \emptyset$, then set $d_\sigma^W(x, y) = 0$.

A subspace $V \subseteq \mathbb{R}^D$ is said to be *expansive* for σ if there exist $\varepsilon,\ t > 0$ such that $d_\sigma^{Vt}(x, y) \leq \varepsilon$ implies $x = y$. We say that the $\mathbb{Z}^D$ action σ is *expansive* if $\mathbb{R}^D$ is an expansive subspace for σ; that is, if there exists an expansive constant, $\varepsilon > 0$, such that $d(\sigma_{\vec{\imath}}\ x, \sigma_{\vec{\imath}}\ y) \leq \varepsilon$ for all $\vec{\imath} \in \mathbb{Z}^D$ implies $x = y$.

Clearly, the $\mathbb{Z}^D$ shift action, σ, is expansive on any subshift space $X \subseteq A^{\mathbb{Z}^D}$, using $\varepsilon = 1$. Combining this new definition with our earlier terminology, a directional shift, $\sigma_{\vec{\imath}}$, is expansive if the one-dimensional subspace of $\mathbb{R}^D$ generated by the directional vector $\vec{\imath}$ is an expansive subspace for the $\mathbb{Z}^D$ action σ.

The main result that we use to find expansive directions for the shift is that these directions are all determined by the expansive subspaces of co-dimension 1.

THEOREM 6.1.1 (Boyle, Lind [**3**]). *A directional shift, $\sigma_{\vec{\imath}}$, is expansive on any infinite subshift space, $X \subseteq A^{\mathbb{Z}^D}$, if and only if the direction vector, $\vec{\imath}$, lies in an expansive subspace $V \subseteq \mathbb{R}^D$ which has dimension $D - 1$.*

The following result tells us that we always obtain expansive subspaces of co-dimension 1 on complete history subshift spaces coming from a bijective CA.

THEOREM 6.1.2. *Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift, let $F : X \to X$ be a bijective cellular automaton, and let $X_F \subseteq A^{\mathbb{Z}^{D+1}}$ be the subshift space of complete histories under F. Then the following subspace is expansive for the $\mathbb{Z}^{D+1}$ shift action:*

$$V = \{(i_1, \cdots, i_D, 0) : i_j \in \mathbb{Z}\ \forall\ j = 1, \cdots, D\} \subseteq \mathbb{Z}^{D+1}.$$

Proof. Suppose $x, y \in X_F$ are such that $d(\sigma_{\vec{\imath}}\ x, \sigma_{\vec{\imath}}\ y) < 1$ for all $\vec{\imath} \in V$. That is, x and y agree in all coordinates of the form $(i_1, \cdots, i_D, 0)$, for any integers i_j. Since $\pi_j x = F \circ \pi_{j+1} x$ and $\pi_j y = F \circ \pi_{j+1} y$ by the definition of the space X_F, we further have that x and y agree on all coordinates of the form $(i_1, \cdots, i_D, i_{D+1})$, where $i_{D+1} < 0$. But as F is bijective, F^{-1} is uniquely defined and so we have both $F^{-1} \circ \pi_j x = \pi_{j+1} x$ and $F^{-1} \circ \pi_j y = \pi_{j+1} y$. Thus x and y also must agree on coordinates of the form $(i_1, \cdots, i_D, i_{D+1})$ where $i_{D+1} > 0$. Hence $x = y$, and so the subspace V is expansive for the $\mathbb{Z}^{D+1}$ shift. □

Combining Theorems 6.1.1 and 6.1.2, we have the following result regarding expansive directional shifts.

Corollary 6.1.3. *Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift, let $F : X \to X$ be a bijective cellular automaton, and let $X_F \subseteq A^{\mathbb{Z}^{D+1}}$ be the subshift space of complete histories under F. Then for any $\vec{\imath} = (i_1, \cdots, i_D, 0) \in \mathbb{Z}^{D+1}$, $\sigma_{\vec{\imath}}$ is an expansive CA on X_F.*

We remark that Theorem 6.1.2 is false if F is not bijective. Consider the one-dimensional CA $L : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ by $(Lx)_i = x_i + x_{i+1} \pmod 2$. This gives rise to the two-dimensional complete history space

$$X_L = \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : x_{i,j} + x_{i+1,j} + x_{i,j-1} \equiv_2 0 \right\}. \tag{6.4}$$

The subspace $V = \{(i,0) : i \in \mathbb{Z}\}$ is not expansive for the two-dimensional shift, however. For, suppose $x, y \in X_L$ have $d(\sigma_{(i,0)} x, \sigma_{(i,0)} y) < 1$ for all $i \in \mathbb{Z}$. In particular, $x_{(i,0)} = y_{(i,0)}$ for all $i \in \mathbb{Z}$, or $\pi_0 x = \pi_0 y$. Then as X_L is a complete history space, we have $\pi_j x = \pi_j y$ for all $j \le 0$, or $x_{(i,j)} = y_{(i,j)}$ for all $i \in \mathbb{Z}$ and $j \le 0$. Now since L is a two-to-one map, there exists a $z \ne \pi_1 y \in \{0,1\}^{\mathbb{Z}}$ with $Lz = \pi_0 x$. Further, there are two points in $\{0,1\}^{\mathbb{Z}}$ which L maps to z and two which are mapped to $\pi_1 y$, and so on, so that there are infinitely many different points $y' \in X_L$ having $\sigma_{(i,0)} x = \sigma_{(i,0)} y'$ for all $i \in \mathbb{Z}$.

Now we establish our main theorem, that there are subshift spaces in all dimensions on which there are expansive CA.

THEOREM 6.1.4. *For any $D \geq 1$, there exists a subshift space $X \subseteq A^{\mathbb{Z}^D}$ and a cellular automaton $F : X \to X$ such that F is expansive.*

Proof. We will prove this by induction on D. For $D = 1$, the shift map, σ, is expansive on any full shift space. For if all iterates of x and y agree on the central coordinate ($\varepsilon = 1$), then all coordinates of x and y are equal, and $x = y$.

Assume that for $D \geq 1$, $X \subseteq A^{\mathbb{Z}^D}$ is a subshift space and $\vec{v}$ is an expansive direction for the $\mathbb{Z}^D$ shift map on X. That is, the CA $\sigma_{\vec{v}}$ is expansive on X. Now let $X_{\sigma_{\vec{v}}} \subseteq A^{\mathbb{Z}^{D+1}}$ be the complete history space under $\sigma_{\vec{v}}$. Since $\sigma_{\vec{v}}$ is bijective for any $\vec{v} \in \mathbb{Z}^D$, then by Corollary 6.1.3, we have many expansive CA's $\sigma_{\vec{\imath}}$ on $X_{\sigma_{\vec{v}}} \subseteq A^{\mathbb{Z}^{D+1}}$, where $\vec{\imath} = (i_1, \cdots, i_D, 0) \in \mathbb{Z}^{D+1}$. Therefore, the statement holds for all dimensions $D \geq 1$. $\square$

As in Theorem 6.1.4 we can construct a subshift space in any dimension with an expansive shift direction by starting with a shift on a one-dimensional full shift space, choosing an expansive shift direction in the two-dimensional subshift space of complete histories, and continuing to choose expansive directions as we move up one dimension at a time. Alternatively, we do not always have to begin in dimension one; if we begin with a bijective CA on a subshift space in dimension D, we obtain many expansive directions on the complete history space in dimension $D + 1$ coming from this CA by Corollary 6.1.3. For example, any one of the *cyclic* CA's in dimension D could be a starting point for this construction. Let $A = \mathbb{Z}_m$ be the finite group of integers mod m, and let $C : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be given by $(Cx)_{\vec{\imath}} = x_{\vec{\imath}} + 1 \pmod{m}$. C has radius 0, and thus is *equicontinuous* as in Example 3.2.3. C is *injective*, since the map $B : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ defined by $(By)_{\vec{\imath}} = y_{\vec{\imath}} - 1 \pmod{m}$ is clearly an inverse for C. Then by Theorem 2.2.3, C is also *surjective*; C is thus a *bijection*.

6.2. Entropy of Complete History Spaces

We now look beyond the case of having a bijective CA on a subshift space in dimension D, and consider the subshift space of complete histories in dimension $D+1$ coming from an arbitrary CA. Does such a space always have expansive directions for the shift? Certainly, there are some spaces which do; the complete history space X_L defined by Equation (6.4) has many, these are addressed in the following section. To begin, we show that such any complete history space has entropy 0 with respect to the D-dimensional shift action, so that it is at least possible for an expansive CA to exist on such a space.

We use the following definition of (topological) entropy of σ, given in [**21**]. Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift space, and denote by

$$(X)|_{\prod_{j=1}^{D+1}\langle 0,n-1\rangle} = \{x|_{\prod_{j=1}^{D+1}\langle 0,n-1\rangle} : x \in X\} \tag{6.5}$$

the set of all hypercube patterns of side length n occurring in points of X. The *entropy of the shift action on* X, or the *entropy of* X is given by

$$h(\sigma, X) = h(X) = \lim_{n\to\infty} \frac{1}{n^{D+1}} \log \left| (X)|_{\prod_{j=1}^{D+1}\langle 0,n-1\rangle} \right|. \tag{6.6}$$

That is, the entropy of a shift space is the exponential asymptotic growth rate of the number of patterns occurring in points of the space. By building a complete history space from any CA in any dimension, we substantially limit the number of patterns which can occur in the higher dimensional space, and it turns out that the $\mathbb{Z}^{D+1}$ shift action has entropy 0 on a complete history space.

THEOREM 6.2.1. *Let $F : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be a cellular automaton and let $X_F \subseteq A^{\mathbb{Z}^{D+1}}$ be the subshift space of complete histories under F. Then the $\mathbb{Z}^{D+1}$ shift action, σ, has entropy 0 on X_F.*

Proof. Let $n \geq 1$, and let $x \in X_F$. In order to compute the entropy of the shift action on X_F, we will consider the possibilities for $x_{(i_1, i_2, \cdots, i_{D+1})}$, where $0 \leq i_1, i_2, \cdots, i_{D+1} \leq n-1$.

First, consider $i_{D+1} = n-1$, the "top" level of this hypercube. We have no restrictions on what values can occur in these n^D positions, since the full shift space is the domain of F. Since every CA is conjugate to a CA with radius 1 by a relabeling of the alphabet, we will assume that $r = 1$. So, this choice of values will determine the values at the level where $i_{D+1} = n-2$ in a hypercube with side length $n-2$, the values at the level where $i_{D+1} = n-3$ in a hypercube with side length $n-4$, and so on, so that a hyperpyramid of values is determined from this initial choice. Illustrated in Figure 6.1 for $D = 1$, the choice of values in a line determines a triangle, and illustrated in Figure 6.2 for $D = 2$, the choice of values in a square determines a pyramid.

Next we consider $i_{D+1} = n-2$. Most of the values have been determined by the image of F, but a "ring" of $n^D - (n-2)^D$ positions remains independent of our previous choice. Notice that for $D = 1$, there are just 2 such positions, the leftmost and rightmost coordinates in the line. These values, along with the ones chosen previously, are all F needs to determine another layer of the hyperpyramid. See Figures 6.3 and 6.4 for illustrations in two and three dimensions.

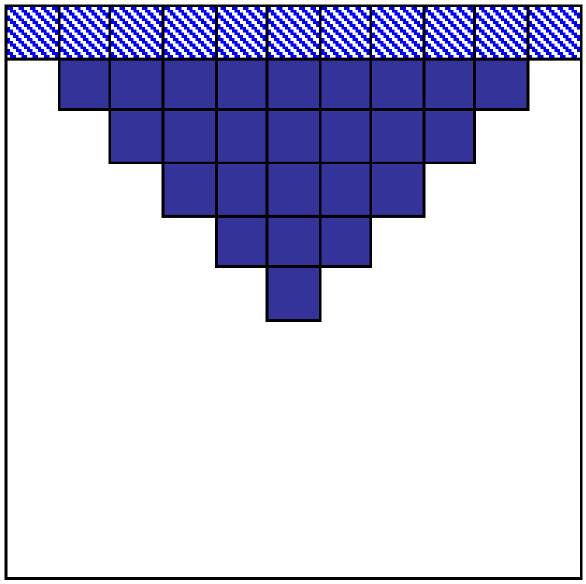


FIGURE 6.1. $D = 1$: Values in a line determine a triangle

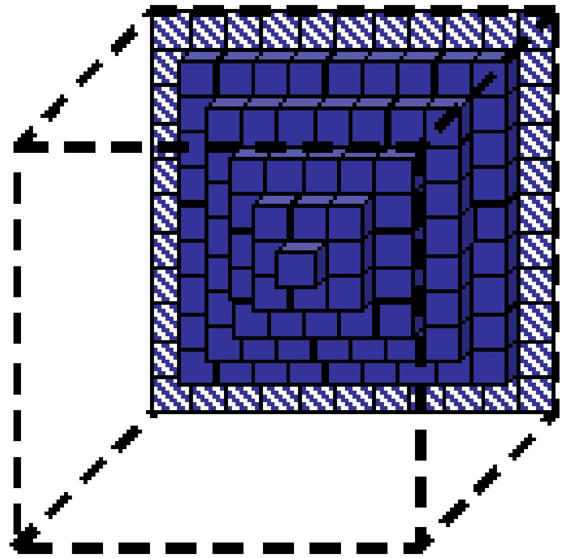


FIGURE 6.2. $D = 2$: Values in a square determine a pyramid

When $i_{D+1} = n - 3$, we have the same sized ring of $n^D - (n-2)^D$ positions whose values are independent of our previous choices. Again, from these values, F will determine another layer on the hyperpyramid. We see that this will occur for each i_{D+1} between 0 and $n - 2$. Thus we have choice of n^D values when $i_{D+1} = n - 1$, and choice of $n^D - (n-2)^D$ values when $0 \le i_{D+1} \le n - 2$. As we are working on the full shift space, all patterns are allowed, and so we have:

$$\left| \{ x|_{\prod_{j=1}^{D+1} \langle 0, n-1 \rangle} : x \in X_F \} \right| = |A|^{n^D + (n-1) \cdot (n^D - (n-2)^D)}. \tag{6.7}$$

Substituting (6.7) into the entropy definition in (6.6) and simplifying, we have the following computation for entropy:

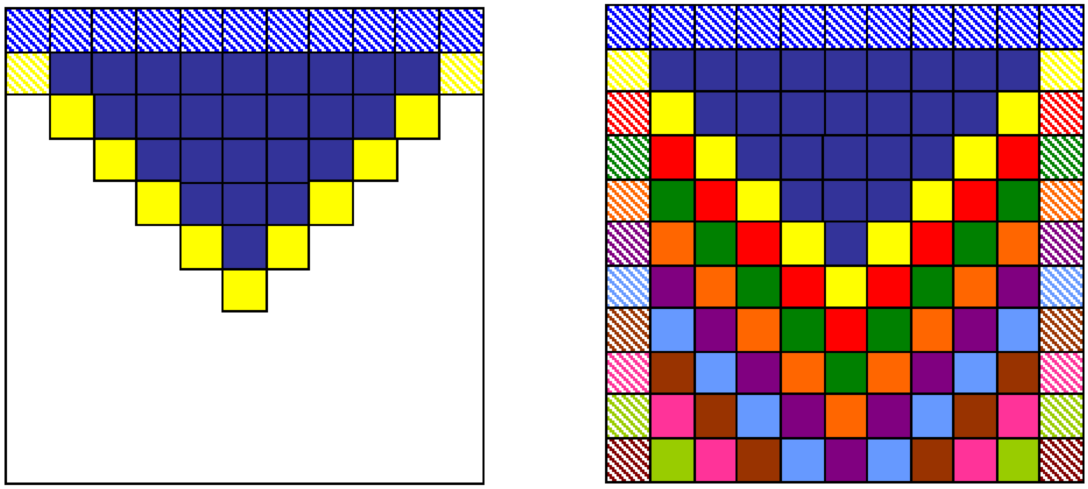


FIGURE 6.3. $D = 1$: Values in outer coordinates determine layers of a triangle

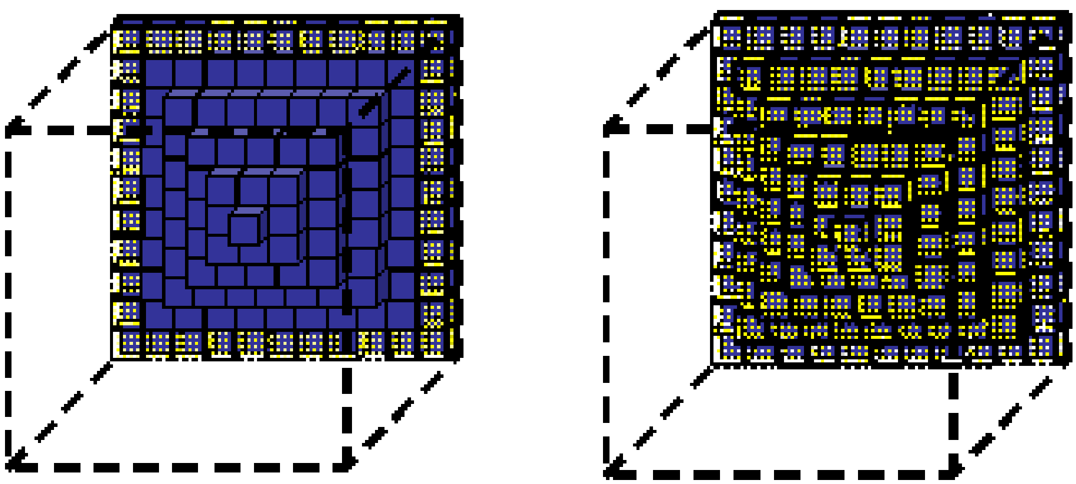


FIGURE 6.4. $D = 2$: Values in a square ring determine layers of a pyramid

$$h(X_F) = \lim_{n\to\infty} \frac{1}{n^{D+1}} \log \left| (X_F)|_{\prod_{j=1}^{D+1}\langle 0,n-1\rangle} \right| \tag{6.8}$$

$$= \lim_{n\to\infty} \frac{1}{n^{D+1}} \log |A|^{n^D+(n-1)\cdot(n^D-(n-2)^D)} \tag{6.9}$$

$$= \lim_{n\to\infty} \frac{n^D+(n-1)(n^D-(n-2)^D)}{n^{D+1}} \log |A| \tag{6.10}$$

$$= \log |A| \cdot \lim_{n\to\infty} \frac{n^D+(n-1)\left(n^D-\sum_{k=0}^{D}\binom{D}{k}n^k(-2)^{D-k}\right)}{n^{D+1}} \tag{6.11}$$

$$= \log |A| \cdot \lim_{n\to\infty} \frac{n^D+(n-1)\sum_{k=0}^{D-1}\binom{D}{k}n^k(-2)^{D-k}}{n^{D+1}} \tag{6.12}$$

$$= \log |A| \cdot 0 = 0, \tag{6.13}$$

since the polynomial in the numerator of (6.12) has degree D and the polynomial in the denominator of (6.12) has degree $D+1$. $\square$

6.3. A Class of Examples Having Expansive Directions

We next consider a class of two-dimensional examples of Kitchens and Schmidt, described here in the setting of complete history spaces. In [**17**], Kitchens and Schmidt determine all of the expansive directions on particular subshifts of $\{0,1\}^{\mathbb{Z}^2}$ defined by a local condition; we state their result in a more restricted setting, where a one-dimensional additive cellular automaton gives a two-dimensional complete history space which satisfies their condition.

By a one-dimensional *additive* CA of radius r, we mean that $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ has the form $(Fx)_i = \sum_{j=-r}^{r} c_j \cdot x_j$, where each c_j is either 0 or 1. Let $m = \min\{j \in [-r,r] : c_j \neq 0\}$ denote the *memory* of F and $a = \max\{j \in [-r,r] : c_j \neq 0\}$ the *anticipation* of F. The next result states that for an additive CA $F : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$, almost every direction $\vec{\imath} \in \mathbb{Z}^2$ is an expansive direction on the complete history space $X_F \subseteq \{0,1\}^{\mathbb{Z}^2}$.

Theorem 6.3.1 (Kitchens, Schmidt [**17**]). *Let $F : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ be an additive cellular automata with memory* m *and anticipation* a*, and let $X_F \subseteq \{0,1\}^{\mathbb{Z}^2}$ be the complete history space under F. For any $\vec{\imath} \in \mathbb{Z}^2 \setminus \{\pm(1,0), \pm(-m,1), \pm(a,1)\}$, the directional shift $\sigma_{\vec{\imath}}$ is expansive on X_F.*

Example 6.3.1. Recall the two-to-one CA $L : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ described in the discussion following Corollary 6.1.3; $(Lx)_i = x_i + x_{i+1} \pmod 2$. First, note that L is *not expansive*, as given any $\varepsilon = 2^{-k} > 0$, we can choose $x \neq y$ that agree on the right tails, $x_{\langle -k,\infty)} = y_{\langle -k,\infty)}$; these points have all of their iterates within ε. L is *sensitive*, however, since a difference on the right will propagate to the central coordinate. We see that L is an additive CA with memory 0 and anticipation 1, so that there are many directional shifts on the complete history space $X_L \subseteq \{0,1\}^{\mathbb{Z}^2}$ defined in Equation (6.4). The directional shift $\sigma_{\vec{\imath}}$ is *expansive* on X_L for any $\vec{\imath} \notin \{\pm(1,0), \pm(0,1), \pm(1,1)\}$, by Theorem 6.3.1. Now by choosing any such i, the complete history subshift space under $\sigma_{\vec{\imath}}$ in $\{0,1\}^{\mathbb{Z}^3}$ will have expansive directions, and we can continually apply Corollary 6.1.3, ultimately giving an expansive CA on a D-dimensional subshift space.

Theorem 6.3.1 provides another way to begin the induction in the proof of Theorem 6.1.4 without using the one-dimensional shift to get expansive directions on two-dimensional subshift spaces. In the proof of Theorem 6.1.4, we obtained a subshift space in dimension D with expansive directions by starting on a one-dimensional space and taking the complete history space under an expansive shift direction at each step until we arrived in dimension D. In Example 6.3.1, we begin with an additive CA in dimension one to get many expansive directions on the two-dimensional subshift space from which to choose how to inductively increase the dimension of the CA. Using Theorem 6.3.1, any additive CA gives rise to an expansive direction on a D-dimensional subshift space in the same way. Both of these approaches to a construction of a D-dimensional subshift space with an expansive direction start in dimension one.

CHAPTER 7

Conclusion

In [**20**], Kůrka gives a diagram showing the interaction of topological properties for one-dimensional cellular automata. We give analogous diagrams illustrating the relationships among the topological properties for higher dimensional cellular automata; Figure 7.1 is for the case where the base space is a full shift space and Figure 7.2 is for the case where the base space is a subshift spaces with dense σ-periodic points.

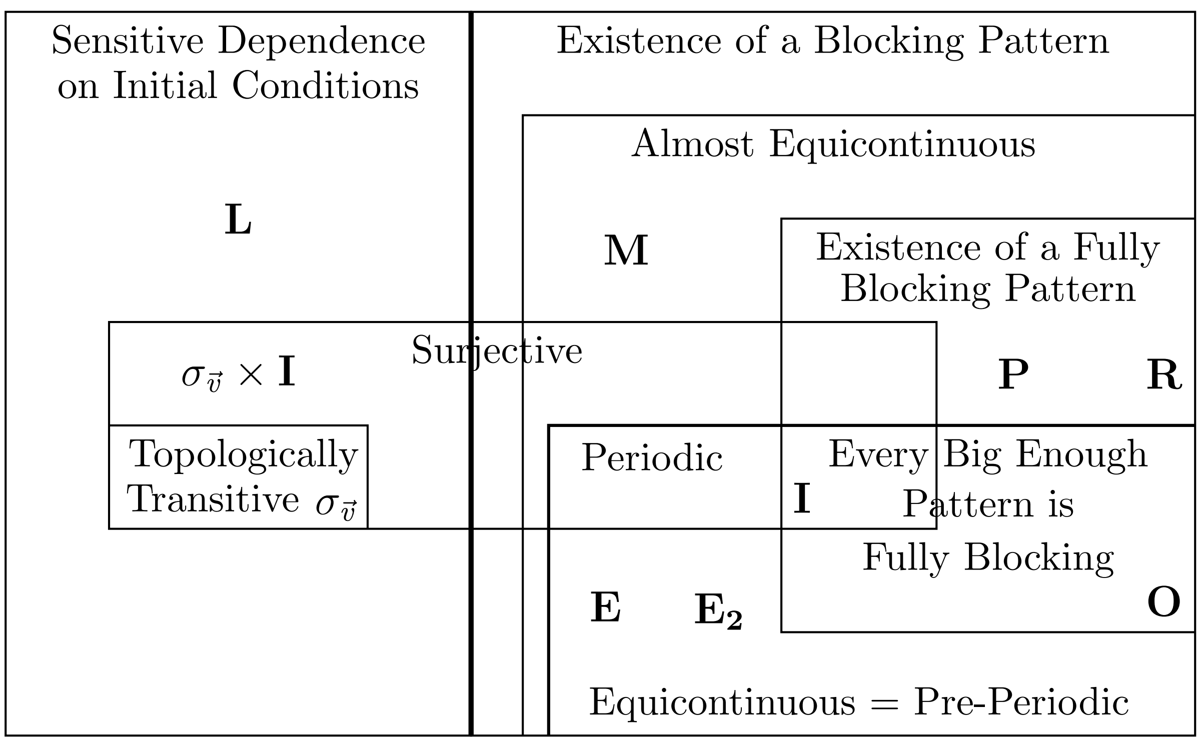


Figure 7.1. Classification of CA's on $A^{\mathbb{Z}^D}$

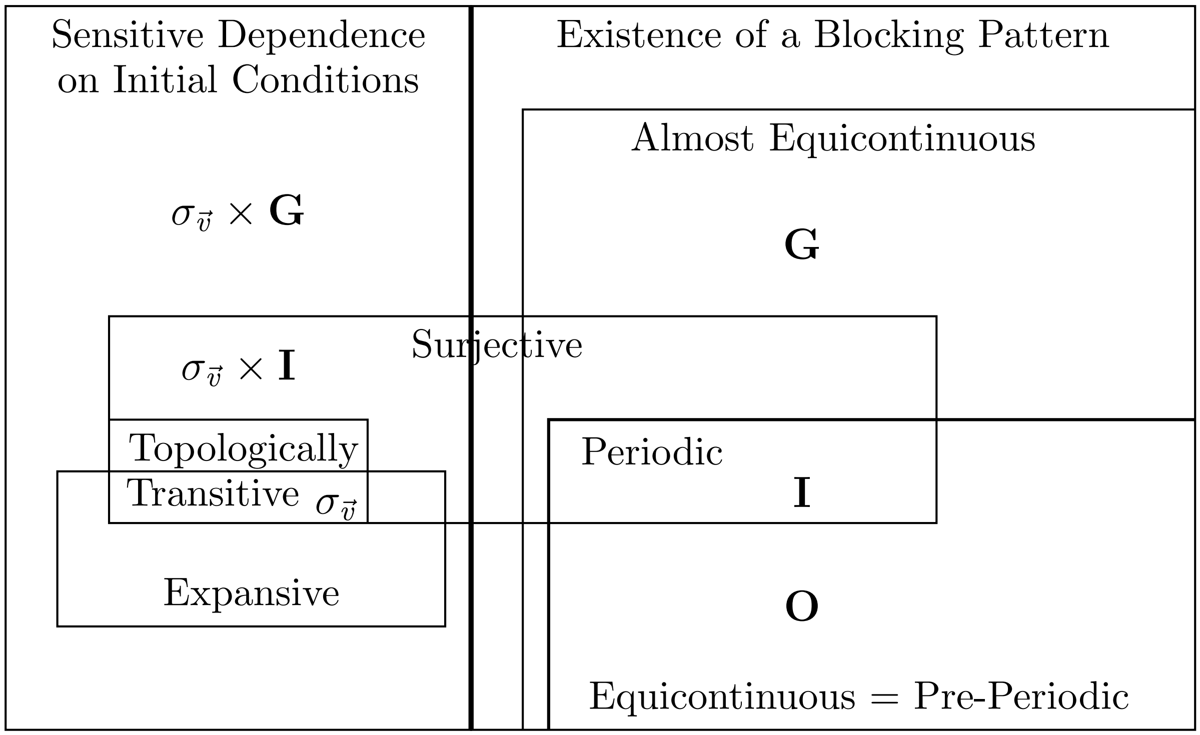


FIGURE 7.2. Classification of CA's on $X \subseteq A^{\mathbb{Z}^D}$

COROLLARY 7.1. *The diagrams in Figures 7.1 and 7.2 summarize the following results from previous chapters:*

(1) *Every almost equicontinuous CA is not sensitive (Theorem 4.2.1).*

(2) *Every CA which is not sensitive has a r^D-blocking pattern (Theorem 4.2.2).*

(3) *Every CA which has a fully blocking r^D pattern is almost equicontinuous (Theorem 4.2.3).*

(4) *Every CA in which every k^D pattern, $k \geq r$, is fully blocking is equicontinuous (Theorem 4.2.5).*

(5) *Every equicontinuous CA is eventually periodic (Theorem 3.1.2).*

(6) *Every surjective, equicontinuous CA is periodic (Theorem 3.1.3).*

(7) *Every topologically transitive CA is both sensitive and surjective (Corollaries 5.2.2, 5.2.3).*

(8) *In every dimension, there exists a subshift space having an expansive CA (Theorem 6.1.4).*

Additionally, we have examined these properties in the context of numerous concrete examples. The letters in Figures 7.1 and 7.2 correspond to examples treated, and prove the non-empty overlap of some of the topological classes.

COROLLARY 7.2. *The non-emptiness of various topological classes is provided by the following examples:*

I: *The identity CA of Example 3.2.1 is equicontinuous, surjective, and periodic, and every* 1^D *pattern is fully blocking.*

O: *The zero CA of Example 3.2.2 is equicontinuous and pre-periodic, and every* 1^D *pattern is fully blocking, but it is neither surjective nor periodic.*

E: *Example 3.2.4 is equicontinuous and pre-periodic, but it is neither surjective nor periodic, and while every long enough word is blocking, there are arbitrarily large words which are not fully blocking.*

E_2: *The two-dimensional version of* E *given in Example 3.2.5 is also equicontinuous and pre-periodic, neither surjective nor periodic, and there are arbitrarily large patterns which are not fully blocking.*

G: *On the two-dimensional Golden Mean shift space, the CA of Example 4.3.1 has a fully blocking pattern and is almost equicontinuous, but is neither equicontinuous nor surjective.*

R: *Example 4.3.2 has a fully blocking pattern and is almost equicontinuous, but is not surjective.*

M: *The majority CA of Example 4.3.3 has no fully blocking* $k \times k$ *patterns, but has a pattern which blocks a cross containing a* 1×1 *square and is almost equicontinuous; it is neither equicontinuous nor surjective.*

P: *The moving particles CA of Example 4.3.4 has a fully blocking pattern and is almost equicontinuous, but is neither equicontinuous nor surjective.*

$\sigma_{\vec{v}}$: *The directional shift CA's, Example 5.3.1, are topologically transitive, surjective and sensitive. Additionally,* $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ *is expansive. Further, these CA are sensitive and surjective on the Golden Mean subshift space, Example 5.3.3.*

S: *The sum CA of Example 2.3.1 is expansive, transitive, sensitive and surjective.*

$\sigma_{\vec{v}} \times I$: *The product CA of Example 5.3.2, product of a directional shift and the identity, is surjective and sensitive, but is not transitive.*

$\sigma_{\vec{v}} \times G$: *The product CA of Example 5.3.3 is sensitive, but is neither surjective nor transitive on the subshift space* $\mathcal{X}$*.*

L: *The additive CA of Example 6.3.1 is sensitive but not expansive, and is surjective.*

We have given a topological classification of cellular automata both on full shift spaces, $A^{\mathbb{Z}^D}$ in dimension $D \geq 2$, and on subshift spaces, $X \subseteq A^{\mathbb{Z}^D}$, based on the topological classification of cellular automata $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ given in [**20**]. Some facts generalize from the one-dimensional results: an equicontinuous CA is eventually periodic, an equicontinuous and surjective CA is periodic, an attracting periodic point must be fixed for the shift action and the CA, a periodic point for the shift is eventually periodic under the CA, and a transitive CA is both sensitive and surjective. What differs in higher dimensions is the notion of blocking and the existence of expansive CA's.

In dimension one, a CA is either almost equicontinuous or it is sensitive; this dichotomy is proven via the existence of a blocking word for a CA which is not sensitive. In higher dimensions, we do not know that the existence of a blocking pattern which is not fully blocking will guarantee that the CA is almost equicontinuous. Certainly, such a pattern will allow the construction of a point of equicontinuity for the CA. However, we obtained a sufficient condition, that there exists a fully blocking pattern, for a CA in higher dimensions to be almost equicontinuous. This condition is not necessary, though, as we have given examples of CA's on $\{0,1\}^{\mathbb{Z}^2}$ which are almost equicontinuous but do not have any fully blocking patterns. In dimension one, a CA is equicontinuous if and only if every long enough word is blocking. In higher dimensions, we have shown that if every big enough pattern is fully blocking, then the CA is equicontinuous. Again, we have established that this is not a necessary condition for higher dimensions, by providing an example of an equicontinuous two-dimensional CA that has arbitrarily large patterns which are not fully blocking.

In dimension one, the shift map on any full shift space is one example of an expansive CA. There can be no expansive CA's on any full shift space in dimension $D \geq 2$, as Shereshevsky has shown. However, we have constructed a subshift space in every dimension on which there is an expansive CA. We accomplished this by taking an expansive direction for the shift action in dimension D and building the complete history space as a subshift space in dimension $D+1$. Shereshevsky has also determined that if $F : X \to X$ is an expansive CA, where $X \subseteq A^{\mathbb{Z}^D}$, then the entropy of the $\mathbb{Z}^D$ shift action must be 0 on

X. We have identified a class of subshift spaces on which expansive CA's may exist, by showing that a complete history space arising from any surjective CA will have entropy 0 with respect to the shift action.

CHAPTER 8

Future Directions

We are interested in continuing to refine our topological classification. We have shown that there is a subshift space in every dimension on which there exists an expansive direction for the shift action. Further, we have shown that from a cellular automaton in any dimension D, the complete history space in dimension $D+1$ has entropy 0, so that expansive directions can exist on this subshift space; we would like to determine whether all such subspaces have an expansive direction for the shift. Recall that Kitchens and Schmidt gave a class of subshift spaces $X \subseteq \{0,1\}^{\mathbb{Z}^2}$ with many expansive directions [**17**]; we plan to apply their techniques to higher dimensions in order to determine the expansive directions for subshift spaces of $\{0,1\}^{\mathbb{Z}^{D+1}}$ defined in a similar manner. That is, if $F : \{0,1\}^{\mathbb{Z}^D} \to \{0,1\}^{\mathbb{Z}^D}$ is defined by adding (mod 2) the coordinates in a shape, then we would like to show that the complete history space $X_F \subseteq \{0,1\}^{\mathbb{Z}^{D+1}}$ has many expansive directions.

We have shown implications among the properties of the existence of a fully-blocking pattern, almost equicontinuity, not having sensitive dependence on initial conditions, the existence of an equicontinuity point, and the existence of a blocking pattern which is not necessarily fully-blocking. We would like either to prove the final implication, making all of the statements equivalent, or to produce an example which has both a point of equicontinuity and a blocking pattern, but does not have any fully-blocking patterns. A possibility to address this question is to apply some techniques from percolation theory.

Although many topological properties were investigated in the thesis, we aim to consider the role of both open and closing cellular automata in higher dimensions. An *open* dynamical system, (Y, T), is one for which $T(U)$ is open for all open sets $U \subseteq Y$. Hedlund showed that $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is open if and only if there is a constant $m > 0$ such that $|F^{-1}x| = m$ for all $x \in A^{\mathbb{Z}}$ [**12**]. A CA $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is *right closing (respectively, left closing)* if for every pair of points $x \neq y \in A^{\mathbb{Z}}$ such that $x_n = y_n$ for all $n \leq m$ for some $m \in \mathbb{Z}$ (respectively, for all pairs $x \neq y \in A^{\mathbb{Z}}$ such that $x_n = y_n$ for all $n \geq m$ for some $m \in \mathbb{Z}$), $Fx \neq Fy$. That is, F is a right (left) closing CA if all points x and y which agree on a left (right) tail and have $Fx = Fy$ also have $x = y$. A CA is *closing* if it is either left closing or right closing. Kůrka has shown that for every CA $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, F being open implies F is closing, F being closing implies F is surjective, and F being open and almost equicontinuous implies F is bijective [**20**]. We are interested to see if these implications hold in higher dimensions as well.

In addition to the investigation of topological properties, a lot of measure-theoretic work has been done in one dimension. Gilman has shown that every CA $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is either essentially equicontinuous or almost expansive. That is, it is equicontinuous on closed, invariant sets of measure arbitrarily close to 1; or, there is an expansive constant $\varepsilon > 0$ so that for all $x \in A^{\mathbb{Z}}$, the set of points $y \in A^{\mathbb{Z}}$ having $d(F^n x, F^n y) < \varepsilon$ for all $n \geq 0$ has measure 0. We extended this result to the case of irreducible shifts of finite type in [**8**], and refer the reader there for further definitions, results, and proofs. In order to state the one-dimensional measure-theoretic classification, we recall the following definitions and notation. Let $F : X \to X$ be a CA on a subshift $X \subseteq A^{\mathbb{Z}}$, and for any $x \in X$ and any interval $I = [a, b]$, define $\theta_I x \subseteq (A^{|I|})^{\mathbb{N}}$ by $(\theta_I x)_i = (F^i x)_a \cdots (F^i x)_b$. That is, $\theta_I x$ is a column of width I in the "forward history" subshift space in $A^{\mathbb{Z}^2}$. Now define $D_I(x) = \{y \in X : \theta_I(y) = \theta_I(x)\}$, and consider the following hypothesis: $F : X \to X$ is a CA with radius r on an irreducible subshift of finite type, $X \subseteq A^{\mathbb{Z}}$, of order l with a stationary, fully-supported Markov measure μ. Once we fix such a measure on X, the class to which a CA belongs is determined by the measure of its sets $D_I(x)$. In fact, different μ can change the class to which a CA belongs. Define Condition (8.1) by:

$$\mu(D_I(x)) > 0 \text{ for some } x \in X \text{ and } I \text{ of width at least } \max\{l,\ 2r\}. \tag{8.1}$$

Theorem 8.0.1 (Gilman [**10**], Gamber [**8**]). *(a) If F satisfies Condition* (8.1)*, then $\forall\ \varepsilon > 0$, $\exists$ a closed, F-invariant subset $Y \subseteq X$ such that $\mu(Y) > 1 - \varepsilon$ and $F|_Y$ is equicontinuous.*

(b) If F does not satisfy Condition (8.1)*, then for each $x \in X$, for μ-a.e. $y \in X$, $\exists\ i > 0$ such that $d(F^i x, F^i y) \geq 2^{-r}$.*

As yet, we have not considered the possible measures on shift spaces in dimensions two and higher; we anticipate that doing so will yield interesting results regarding the properties of a CA. In addition to considering a measure-theoretic classification for cellular automata in higher dimensions, there are topological properties which can be cast in terms of a measurable property. For instance, it is known that a CA $F : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ is surjective if and only if F preserves the uniform Bernoulli measure, μ, on $A^{\mathbb{Z}^D}$, that is, the product measure which arises from assigning probability $\frac{1}{|A|}$ to each $a \in A$. This can be seen via the notion of balanced, as in Theorem 2.2.3. Since F is surjective if and only if $|F^{-1}p| = |A|^{(2k+3)^D-(2k+1)^D}$ for all patterns $p \in P_{0,k}$, we see that F is surjective if and only if $\mu(F^{-1}p) = |A|^{(2k+3)^D-(2k+1)^D} \cdot |A|^{-(2k+3)^D} = |A|^{-(2k+1)^D}$, since all pre-image patterns of p of size $(2k+1)^D$ have the same μ measure. Then since $\mu(p) = |A|^{-(2k+1)^D}$, F is surjective if and only if F preserves μ.

Finally, many physical applications of cellular automata were mentioned in the introduction, and we anticipate exploring some of these in detail. In particular, there is a specific model of HIV evolution in lymph nodes that appears in the literature [**7**]. The purported advantage of using such a model as opposed to a traditional ODE approach is that this CA captures the three phases of the illness, the initial infection which lasts a few weeks, the latency stage which can last 5 to 10 years, and finally the onset of AIDS. This model is the subject of Jessica Hubbs' master's project at UNC, Chapel Hill. We are interested in exploring this model and some variations, as well as looking at some other physical applications.

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