Exploring Interpolation Techniques in Enumerative Geometry

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I Introduction

In this paper we examine the invariant subsets A of a vector space V, when we act by a group, G. Gathering some of the information about A in the equivariant cohomology ring $H^*_G(V)$ is an important area of study in enumerative geometry. The descriptions we seek to find for an invariant subspace satisfies certain universal properties. A notable example of this is given a vector bundle with fiber V and structure group G over a compact manifold M looking at some cohomological data for A-points of a section can be done in the equivariant cohomology ring instead of the cohomology ring of M. The fundamental cohomology class and Segre-Schwartz-MacPherson class are two such objects that are the same in both cohomology rings.

Since these classes are universal in the above sense, we expect that it is very difficult to determine the fundamental class and the Chern-Schwartz-MacPherson class. However, it is possible in certain situations.

In this paper we find the fundamental class and the CSM/SSM classes in the equivariant cohomology rings for two separate groups actions. Before we make our way through calculations, we give some intuitive definition of what these classes are. The basis for the computations we do is from interpolation characterizations of the CSM class and the fundamental classes, which are stated in a way applicable to our situation. We then work through the computation one of these for two seperate group actions.

The culmination of the first section is the statement and the proof of a well-known result called the Porteous identity. The second section ends with some conjectures based on the lower-dimensional results.

II Fundamental Class

Let M^n be a compact orientable manifold. For certain subsets A^a, where the upper index indicates its real dimension, there is an object, $\mu_{A\subset M}$ in the a^{th} homology group $H_a(M)$ called the fundamental homology class of A in M. See Appendix A in [MnS] for details. To obtain an element in the cohomology of M we can then utilize Poincaré duality, which is an isomorphism between $H_a(M)$ and $H^{n-a}(M)$, the a^{th} homology group and the $(n-a)^{\text{th}}$ cohomolgy group. The image of $\mu_{A\subset M}$ under this isomorphism is then called the fundamental cohomology class of A in M and is denoted $[A]$, the M is omitted from this notation.

Let G be an algebraic group which acts algebraically on a vector space V^n. Let A^a be an orbit closure. It is a fact that there is an analogous notion to the fundamental class, $[A]$, for A in V, which is a member of the equivariant cohomology of V, $H_G^{n-a}(V)$. This construction can be found in [FeRR] and Chapter 8 of [MS].

Suppose ξ is an orbit of G acting on V and $x \in \xi$. Let $G_\xi < G$ be the stabilizer of x. We use G_ξ instead of G_x to denote the stabilizer because the stabilizer subgroups of points in an orbit are conjugate to each other, and hence are isomorphic. Then G_ξ acts on T_xV and keeps $T_x\xi$, which we will denote T_ξ, invariant, and so G_ξ can act on $\nu_\xi = T_xV/T_\xi$, which we will call the normal space of ξ. Not only is G_ξ independent of the choice of x, but its representations on T_ξ and ν_ξ are as well. We define the Euler class, denoted $e(\xi)$, as the product of the weights of the action on ν_ξ, which belongs to $H^*_{G_\xi}(pt)$.

If $\xi \hookrightarrow V$ then there is an induced pullback map $H^*_G(\xi) \leftarrow H^*_G(V)$ of the equivariant cohomology rings. It is a fact that $H^*_{G_\xi}(pt) \cong H^*_G(\xi)$ and hence there is an induced map $\varphi_\xi : H^*_G(V) \to H^*_{G_\xi}(pt)$, called the restriction homomorphism.

The next two theorems, whose proofs can be found in [FeRR], gives us a method of computing the fundamental cohomology classes.

Theorem II.1. *Suppose G acts on V as above. Assume that there are only finitely many orbits and the Euler class of ξ is not a zero divisor in $H^*_{G_\xi}(pt)$ for all $\xi \in V/G$. Let η be an orbit. The class $[\eta]$ satisfies the following relationships*

1. $\varphi_\xi[\bar{\eta}] = e(\nu_\eta)$, *if* $\xi = \eta$.
2. $\varphi_\xi[\bar{\eta}] = 0$, *if* $\operatorname{codim}\xi \leq \operatorname{codim}\eta$, *and* $\xi \neq \eta$ *is an orbit.*

In practice we will see that φ_ξ are substitutions in a polynomial ring. Hence, we call 1 and 2 interpolation conditions.

Theorem II.2. *Under the same hypotheses stated above, the interpolation conditions listed above uniquely determine the fundamental class $[\bar{\eta}]$.*

II.1 Example

We will work through an example of how one could use the interpolation techniques, theorems (II.1) and (II.2), in the calculation of a fundamental class. Here, let G be the group $GL_n(\mathbb{C}) \times GL_m(\mathbb{C})$ and have this act on $V = \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^m)$ by

$$(A, B).M = BMA^{-1}. \tag{1}$$

A notable requirement is that V have finitely many orbits. Two elements of V are in the same orbit if and only if they have the same rank (and so have the same corank, cork $= n - \mathrm{rk}$). Therefore there are only $\min(m,n)+1$ orbits and for $0 \le r \le \min(n,m)$ they will be denoted Σ^r and

$$\Sigma^r = \{M \in V \,|\, \mathrm{cork}(M) = r\}. \tag{2}$$

Now we temporarily limit ourselves to the case when $n \le m$. We also define $\ell := m - n \ge 0$.

An element of Σ^r is

$$x = \begin{array}{c} \overbrace{\hphantom{1 \quad 0 \quad \cdots \quad 1}}^{n-r} \hphantom{\quad 0 \quad \cdots \quad 0} \\ \left(\begin{array}{cccc|ccc} 1 & 0 & \cdots & & & \cdots & 0 \\ 0 & 1 & \cdots & & & \cdots & 0 \\ & & \ddots & & & & \\ & & & 1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & & & 0 & \cdots & 0 \\ \vdots & \cdots & & & \vdots & \ddots & \vdots \\ 0 & \cdots & & & 0 & \cdots & 0 \end{array}\right) \end{array}, \tag{3}$$

or written in block form as

$$\left(\begin{array}{c|c} I_{n-r\times n-r} & 0_{n-r\times \ell+r} \\ \hline 0_{r\times n-r} & 0_{r\times \ell+r} \end{array}\right). \tag{4}$$

We should distinguish between the normal and tangent spaces of the representative of Σ^r. The normal space, ν_{Σ^r}, corresponds to the $r \times (\ell - r)$ block in the bottom right of the matrix. The tangent space, T_{Σ^r} consists of the remaining three blocks in the matrix.

If we are trying to utilize the interpolation equations, (II.1), we must determine the orbits ξ such that $\operatorname{codim} \xi \le \operatorname{codim} \Sigma^r$. Since ξ is an orbit of G, $\xi = \Sigma^k$ for some k. Since the codimension of Σ^k has to be less than the codimension of Σ^r, we must have $k \le r$.

Let T^n be the torus of diagonal matrices in GL_n. Then $H^*_{T^n\times T^m}(pt) \supset H^*_{GL_n\times GL_m}(pt)$ because of the splitting lemma. Moreover $H^*_{T^n\times T^m}(pt) = \mathbb{Z}[\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n]$ and $H^*_{GL_n\times GL_m}(pt) = \mathbb{Z}[\alpha_1,\times,\alpha_n,\beta_1,\ldots,\beta_m]^{S_n\times S_m} = \mathbb{Z}[a_1,\ldots,a_n,b_1,\ldots,b_m]$, where a_i (resp. b_i) is the i^{th} symmetric polynomial in the α (resp. β) variables. The a_i are called the universal Chern classes for GL_n and b_i are the the universal Chern classes for GL_m. So we can consider a polynomial in the a_i's and b_j's in $H^*_{T^n\times T^m}(pt)$ or $H^*_{GL_n\times GL_m}(pt)$. We'll restrict our attention to the action of $T^n \times T^m \leq GL_n \times GL_m$. So instead of a general $(A,B) \in GL_n \times GL_m$, we can consider A and B to diagonal matrices.

First we will work through a more explicit example by calculating the value of $[\bar{\Sigma^2}]$ for $\Sigma^2 \subset \mathrm{Hom}(\mathbb{C}^3, \mathbb{C}^3)$– that is when $r = 2$ and $n = m = 3$. We have to calculate how G acts on this representative of Σ^r.

Let $e_{i,j}$ be the homomorphism which sends the i^{th} basis vector of $\mathbb{C}^3$ to the j^{th} basis vector of $\mathbb{C}^3$– that is $e_{i,j}$ is the matrix with a 1 in the i^{th} row and the j^{th} column and zeros everywhere else. So we can see that the representative, x as above, of Σ^r is just $x = \sum_1^{n-r} e_{j,j}$. In this explicit example, it is just $e_{1,1}$.

Next, we see that the way that an element of $T^3 \times T^3$ acts on $e_{i,j}$ is by

$$\left(\begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix} \right) .e_{i,j} = \beta_j \alpha_i^{-1} e_{i,j}. \tag{5}$$

The conditions which stabilize the $x \in \Sigma^r$ is requiring $\beta_j \alpha_j^{-1} = 1$ for $j = 1, \ldots, n-r$. By abuse of notation, we will use the same letter α, β for the entry on the diagonal and the corresponding weights. For the i^{th} row and j^{th} column position in the normal space, the weight of the action of $T^3 \times T^3$ is $\beta_j - \alpha_i$. Therefore, restriction homomorphisms, φ_{Σ^k}, are going to map α_j to β_j for $j = 1, \ldots, n-k$.

From (II.1), we know that the properties which uniquely determine the fundamental class are: (1) When we restrict to what stabilizes Σ^k, $0 \leq k < 2$, that $[\bar{\Sigma^2}]$ is 0, and (2) while restricting to what stabilizes Σ^2, the fundamental class is equal to the Euler class of ν_{Σ^r}.

The Euler class of ν_{Σ^r} is the product of the weights of the action in the normal space. Therefore,

$$e(\nu_{\Sigma^2}) = (\beta_2 - \alpha_2)(\beta_3 - \alpha_2)(\beta_2 - \alpha_3)(\beta_3 - \alpha_3).$$

The fundamental class of Σ^r, $[\bar{\Sigma}^r] \in H^{2\,\mathrm{codim}_{\mathbb{C}} \Sigma^2}_{GL_3 \times GL_3}(pt)$, and hence it is a symmetric polynomial in the α and β variables. Because of this we know that the fundamental class can be written as an element of $\mathbb{C}[a_1, a_2, a_3, b_1, b_2, b_3]$ where a_j and b_j are the j^{th} elementary symmetric polynomial in α_- and β_- and $\deg b_j = \deg a_j = 2j$. Since $[\bar{\Sigma}^r] \in H^8_G(pt)$, $[\bar{\Sigma}^r]$ is a degree eight polynomial. Equivalently, $[\bar{\Sigma}^r]$ is a degree four polynomial with the convention $\deg a_j = \deg b_j = j$. We will use the second convention.

In this scenario, for the restriction of Σ^0 we need

$$[\bar{\Sigma}^2] \in \ker \varphi_{\Sigma^0}, \qquad \text{where } \varphi_{\Sigma^0} : \alpha_j \mapsto \beta_j,\ j = 1, 2, 3. \tag{6}$$

The restriction of Σ^1 is

$$[\bar{\Sigma}^2] \in \ker \varphi_{\Sigma^1}, \qquad \text{where } \varphi_{\Sigma^1} : \alpha_j \mapsto \beta_j,\ j = 1, 2. \tag{7}$$

Finally, the restriction to Σ^2 is

$$\varphi_{\Sigma^2}([\bar{\Sigma}^2]) = (\beta_2 - \alpha_2)(\beta_3 - \alpha_2)(\beta_2 - \alpha_3)(\beta_3 - \alpha_3), \qquad \text{where } \varphi_{\Sigma^2} : \alpha_1 \mapsto \beta_1. \tag{8}$$

These three restrictions on polynomial $[\bar{\Sigma}^2]$ uniquely determine it.

By using a computer algebra program, we find out that the only degree 4 polynomial in $\mathbb{C}[a_1, a_2, a_3, b_1, b_2, b_3]$ which satisfies the restriction conditions (and is therefore the fundamental class of Σ^2) is

$$[\bar{\Sigma}^2] = a_1^2 b_2 - a_1 a_2 b_1 - a_1 b_1 b_2 + a_2 b_1^2 - a_1 a_3 + a_1 b_3 + a_2^2 - 2 a_2 b_2 + a_3 b_1 - b_1 b_3 + b_2^2. \tag{9}$$

Let us define c_j, $j \in \mathbb{N}$ so that

$$1 + c_1 t + c_2 t^2 + \cdots = \frac{1 + b_1 t + b_2 t^2 + b_3 t^3}{1 + a_1 t + a_2 t^2 + a_3 t^3} \tag{10}$$

where the right hand side is understood to be the formal power series of the quotient. Then the first few terms, and the ones that we will utilize, are

$$\begin{aligned} c_1 &= b_1 - a_1 \\ c_2 &= b_2 - a_1 b_1 - a_2 + a_1^2 \\ c_3 &= b_3 - a_3 - a_2 b_1 - a_1 b_2 + a_1^2 b_1 + 2 a_1 a_2 - a_1^3 \end{aligned}$$

We can now rewrite the fundamental class, in a much cleaner and succinct form, as

$$[\bar{\Sigma}^2] = c_2^2 - c_1 c_3 = \det \begin{pmatrix} c_2 & c_3 \\ c_1 & c_2 \end{pmatrix}. \tag{11}$$

Writing the fundamental class in this determinant form generalizes to theorem (II.5).

II.2 Porteous Formula for $[\bar{\Sigma}^r]$

For two polynomials p and q we define the resultant of p and q as

$$R(p,q) = \prod_{(x,y):p(x)=q(y)=0} (x-y).$$

Claim II.3 (Sylvester's Theorem, [Sylv])**.** Suppose $p = t^n + a_1 t^{n-1} + \cdots + a_n$ and $q = t^m + b_1 t^{m-1} + \cdots + b_m$, where b_i's and a_j's are constants. Define

$$M = \left(\begin{array}{ccccccccc}
1 & b_1 & b_2 & \dots & b_m & 0 & \dots & & 0 \\
0 & 1 & b_1 & b_2 & \dots & b_m & 0 & \dots & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & & 0 \\
\vdots & \vdots & \ddots & \ddots & & & & & 0 \\
0 & \dots & & 0 & 1 & b_1 & b_2 & \dots & b_m \\
\hline
1 & a_1 & a_2 & \dots & a_n & 0 & \dots & & 0 \\
0 & 1 & a_1 & a_2 & \dots & a_n & 0 & \dots & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & & 0 \\
\vdots & \vdots & \ddots & \ddots & & & & & 0 \\
0 & \dots & & 0 & 1 & a_1 & a_2 & \dots & a_n
\end{array}\right),$$

where there are m rows involving the $a_{\_}$ terms and n rows involving $b_{\_}$ terms. Then

$$R(p,q) = \det M.$$

Proof. We note that $R(p,q) = 0$ if and only if p and q have a common root. This occurs if and only if p and q have a common divisor, which happens if and only if there are polynomials A and B such that $\deg A \le \deg q - 1$ and $\deg B \le \deg p - 1$ with $Ap + Bq = 0$.

If $Ap + Bq = 0$ then there is some non-trivial linear relation between the rows of M. Conversely, if there is a non-trivial relation between the rows of M then we can find A and B as above so that $Ap + Bq = 0$. This implies that $\det M$ is a constant multiple of $R(p,q)$, say $\det M = cR(p,q)$. Therefore determining the coefficients of b_m^n in the expansion of $\det M$ and $R(p,q)$ will determine the constant c.

The coefficient of b_m^n in $\det M$ is $\operatorname{sgn} \pi$ where π is the permutation

$$\pi : j \mapsto \begin{cases} j+m & : 1 \le j \le n \\ j-n & : n < j \le n+m \end{cases}.$$

We can write π in two-row notation as

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n & n+1 & n+2 & \dots & m+n \\ m+1 & m+2 & \dots & m+n & 1 & 2 & \dots & m \end{pmatrix}.$$

It takes n transpositions to move 1 to the first position and not change the order of $m+1, \dots, m+n$. It takes n transpositions to move each of the remaining $2, \dots, m$ to get the identity permutation. Therefore π is the composition of mn transpositions and so $\operatorname{sgn} \pi = (-1)^{mn}$.

If x_j, $1 \le j \le m$ are the roots of p then $b_m = (-1)^m x_1 x_2 \dots x_m$. If the roots of q are y_j, $1 \le j \le n$, then expanding $\prod(x_j - y_i)$ we see that the coefficient of b_m^n is also $(-1)^{mn}$. Therefore, $\det M = R(p, q)$.

□

Let us set c_j, $j \in \mathbb{N}$, so

$$1 + c_1 t + c_2 t^2 + c_3 t^3 + \dots = \frac{1 + b_1 t + b_2 t^2 + \dots + b_m t^m}{1 + a_1 t + a_2 t^2 + \dots + a_n t^n} \tag{12}$$

as a formal sum.

Claim II.4. For M and c_j defined as above, we have

$$\det M = (-1)^{mn} \det \begin{pmatrix} c_m & c_{m+1} & c_{m+2} & \dots & c_{m+n-1} \\ c_{m-1} & c_m & c_{m+1} & \iddots & \\ \vdots & c_{m-1} & c_m & \ddots & \\ \vdots & \vdots & \ddots & \ddots & \\ c_{m-n+1} & & & & c_m \end{pmatrix}_{n \times n}$$

Proof. Let C be the $n \times n$ matrix on the right and side.

Define for $j \in \mathbb{N}$, d_j so that

$$d_0 + d_1 t + d_2 t^2 + \dots = \frac{1}{1 + a_1 t + \dots + a_n t^n}.$$

Since

$$(1 + d_1 t + d_2 t^2 + \dots)(1 + a_1 t + \dots + a_r t^r) = 1 \tag{13}$$

holds, the coefficients of t^k on the left-hand side are all 0. Therefore, setting $a_k = 0$ for $k > n$ we arrive at

$$d_1 + a_1 = d_2 + a_1 d_1 + d_0 = \dots = \sum_{j=0}^{k} a_j d_{k-j} = 0, \qquad \forall k \in \mathbb{N}. \tag{14}$$

If we set b_k to be 0 for $k > m$ then we can see that

$$c_k = \sum_{j=0}^{k} b_j d_{k-j}. \tag{15}$$

Next, set

$$B = \begin{pmatrix} 1 & b_1 & b_2 & \dots & b_m & 0 & \dots & & 0 \\ 0 & 1 & b_1 & b_2 & \dots & b_m & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & & 0 \\ \vdots & \vdots & \ddots & \ddots & & & & & 0 \\ 0 & \dots & & 0 & 1 & b_1 & b_2 & \dots & b_m \end{pmatrix}$$

and

$$A = \begin{pmatrix} 1 & a_1 & a_2 & \dots & a_n & 0 & \dots & & 0 \\ 0 & 1 & a_1 & a_2 & \dots & a_n & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & & 0 \\ \vdots & \vdots & \ddots & 1 & a_1 & \dots & & & a_n \end{pmatrix}.$$

Then M in block form is simply $\left(\frac{B}{A}\right)$. Set

$$D = \begin{pmatrix} 1 & d_1 & d_2 & \dots & d_{n+m-1} \\ 0 & 1 & d_1 & \dots & d_{n+m-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & & \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \in \mathrm{Hom}(\mathbb{C}^{n+m}, \mathbb{C}^{n+m}).$$

Since D is upper-triangular with diagonal entries 1, $\det D = 1$, and so $\det(MD) = \det M$. Because of the relationships in (14) and (15),

$$MD = \left(\begin{array}{cccc|cccc} 1 & c_1 & c_2 & \dots & c_m & c_{m+1} & \dots & c_{m+n-1} \\ 0 & 1 & c_1 & \dots & c_{m-1} & c_m & \dots & c_{m+n-2} \\ & & & & \vdots & & & \\ 0 & & \dots & & c_{m-n+1} & c_{m-n+2} & \dots & c_m \\ \hline 1 & 0 & \dots & 0 & 0 & \dots & & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & & 0 \\ \vdots & & \ddots & \vdots & 0 & \dots & & 0 \\ 0 & \dots & 0 & 1 & 0 & \dots & & 0 \end{array}\right). \tag{16}$$

Expanding from the bottom left block, we see that $\det MD = (-1)^{mn} \det C$. $\square$

Theorem II.5 (Porteous Formula, [Port]). *Consider the group action described above on* $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$*, with* $n \leq m$*. If* $\ell = m - n$*, and* c_j *are defined as in (12), then*

$$[\bar{\Sigma}^r] = (-1)^{r(r+\ell)} \det \begin{pmatrix} c_{r+\ell} & c_{r+\ell+1} & c_{r+\ell+2} & \cdots & c_{r+\ell+r-1} \\ c_{r+\ell-1} & c_{r+\ell} & c_{r+\ell+1} & \ddots & \\ \vdots & c_{r+\ell-1} & c_{r+\ell} & \ddots & \\ \vdots & \vdots & \ddots & \ddots & \\ c_{\ell+1} & & & & c_{r+\ell} \end{pmatrix}_{r\times r}. \tag{17}$$

Proof. Let $C = C_r$ be the matrix appearing in the theorem. The interpolation theorems, (II.1) and (II.2), imply we need to show that when we restrict to $\beta_j = \alpha_j$ for $j = 1, 2, \ldots, k$ for $k \geq n - r + 1$ that $\det C = 0$ and when $k = r$, $\det C = e(\nu_{\Sigma^r})$.

We start by showing that $\det C$ vanishes when we set $\beta_j = \alpha_j$ for $j = 1, 2, \ldots n - r + 1$. Doing this we get

$$(1 + c_1 t + c_2 t^2 + \ldots) \left(\prod_{j=n-r+2}^{n} (1 + \alpha_j t) \right) = \prod_{j=m-r+2}^{m} (1 + \beta_j t). \tag{18}$$

To ease with notation we will relabel $\alpha_{j+n-r+1}$ as simply α_j, and do a similar process to the β's. We will also write a_j as the j^{th} elementary symmetric polynomial in the $r-1$ remaining α variables and b_j as the j^{th} elementary symmetric polynomial in the $r + \ell - 1$ remaining β variables, e.g. a_1 is now $\alpha_1 + \cdots + \alpha_{r-1}$. With these above simplifications in notation, the above equation becomes

$$(1 + c_1 t + c_2 t^2 + \ldots)(1 + a_1 t + \cdots + a_{r-1} t^{r-1}) = (1 + b_1 t + \cdots + b_{r+\ell+1} t^{r+\ell-1}) \tag{19}$$

Expanding and collecting the coefficients of t^j, we see that

$$\begin{aligned}
b_1 &= c_1 + a_1 \\
b_2 &= c_2 + a_1 c_1 + a_2 \\
&\vdots \\
b_r &= c_r + a_1 c_{r-1} + \cdots + a_{r-1} c_1 + 0 \\
b_{r+1} &= c_{r+1} + a_1 c_r + \cdots + a_{r-1} c_2 \\
&\vdots \\
b_{r+\ell-1} &= c_{r+\ell-1} + a_1 c_{r+\ell-2} + \cdots + a_{r-1} c_\ell \\
0 &= c_{r+\ell} + a_1 c_{r+\ell-1} + \cdots + a_{r-1} c_{\ell+1} \\
&\vdots \\
0 &= c_{r+\ell+r-1} + a_1 c_{r+\ell+r-2} + \cdots + a_{r-1} c_{r+\ell-1}
\end{aligned}$$

which implies

$$\begin{pmatrix} c_{r+\ell} \\ c_{r+\ell+1} \\ \vdots \\ c_{r+\ell+r+1} \end{pmatrix} = a_1 \begin{pmatrix} c_{r+\ell-1} \\ c_{r+\ell} \\ \vdots \\ c_{r+\ell+r-2} \end{pmatrix} + \cdots + a_{r-1} \begin{pmatrix} c_{\ell+1} \\ c_{\ell+2} \\ \vdots \\ c_{r+\ell} \end{pmatrix}. \tag{20}$$

This is a sufficient condition to show that $\det C$ is 0, as this shows that the last row is a linear combination of the first $r-1$ rows.

Restricting $\beta_j = \alpha_j$ for $j = 1, 2, \ldots, n-r+k$ for $k \geq 2$ is more restrictive than the case when $k = 1$. We could go through the same relabeling process and see that we would arrive at the existence linear relationship between the rows that involve $r-k$ of the rows. In particular, $\det C \in \ker \varphi_{\Sigma^k}$ for $k \geq n-r+1$.

Suppose we set $\beta_j = \alpha_j$ for $j = 1, \ldots, n-r$ and re-label and re-index as before. Let us set $p = 1 + b_1 t + \cdots + b_{r+\ell} t^{r+\ell}$ and $q = 1 + a_1 t + \cdots + a_r t^r$. Set M as the matrix that appears in (II.3) for the current p and q. We examine the resultant of p and q and see that

$$R(p,q) = \prod_{j=1}^{r+\ell} \prod_{k=1}^{r} (\beta_j - \alpha_k) = e(\nu_{\Sigma^r}).$$

By (II.3) and (II.4), this proves the theorem since the polynomials are of degrees r and $r+\ell$ instead of m and n. $\square$

If we consider the case when $n > m$ we can see that switching the roles of α's and β's give similar restriction cancelations. Therefore, for $\ell' = n - m$,

$$[\bar{\Sigma}^r] = \det \begin{pmatrix} c'_{r+\ell'} & c'_{r+\ell'+1} & c'_{r+\ell'+2} & \dots & c'_{r+\ell'+r-1} \\ c'_{r+\ell'-1} & c'_{r+\ell'} & c'_{r+\ell'+1} & \ddots & \\ \vdots & c'_{r+\ell'-1} & c'_{r+\ell'} & \ddots & \\ \vdots & \vdots & \ddots & \ddots & \\ c'_{\ell'+1} & & & & c'_{r+\ell'} \end{pmatrix}_{r\times r}$$

where

$$1 + c'_1 t + c'_2 t^2 + \cdots = \frac{1 + a_1 t + a_2 t^2 + \cdots + a_n t^n}{1 + b_1 t + b_2 t^2 + \cdots + b_m t^m}.$$

Going through the same steps we see that this formula holds, up to sign. In order to determine the sign, we note that

$$\det \left(\begin{array}{ccccccccc} 1 & a_1 & a_2 & \dots & a_{r+\ell'} & 0 & \dots & & 0 \\ 0 & 1 & a_1 & a_2 & \dots & a_{r+\ell'} & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & & 0 \\ \vdots & \vdots & \ddots & \ddots & & & & & 0 \\ 0 & \dots & & 0 & 1 & a_1 & a_2 & \dots & a_{r+\ell'} \\ \hline 1 & b_1 & b_2 & \dots & b_r & 0 & \dots & & 0 \\ 0 & 1 & b_1 & b_2 & \dots & b_r & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & & 0 \\ \vdots & \vdots & \ddots & \ddots & & & & & 0 \\ 0 & \dots & & 0 & 1 & b_1 & b_2 & \dots & b_r \end{array}\right) = (-1)^{r(r+\ell')} \prod_{j=1}^{r+\ell'} \prod_{k=1}^{r} (\alpha_j - \beta_k)$$

$$= \prod_{j=1}^{r+\ell'} \prod_{k=1}^{r} (\beta_k - \alpha_j)$$

and so the sign is indeed $(-1)^{r(r+\ell')}$.

III CSM and SSM Classes

We will follow the formal definition of the Chern-Schwartz-MacPherson class as laid out in Section 3.2 of [Ohm]. See also the references therein.

For a complex variety X of dimension n and a subvariety $W \subset X$, we denote the the characteristic function which takes value 1 on W and 0 otherwise by $\mathbb{1}_W$. We say α is a constructible function if $\alpha = \sum a_i \mathbb{1}_{W_i}$, for $a_i \in \mathbb{Z}$. Let $\mathcal{F}(X)$ be the abelian group of constructible functions over a space X.

First, the restriction of α to an algebraic subset $Z \subset X$ is usual restriction. Then $\alpha|_Z = \sum b_i \mathbb{1}_{V_i}$ and we define the integral of α over the algebraic set Z as

$$\int_Z \alpha = \sum b_i \chi(V_i),$$

where χ means the Euler characteristic. Moreover, for morphisms $f : X \to Y$ the pushforward is defined as

$$f_* : \mathcal{F}(X) \to \mathcal{F}(Y), \qquad f_*(\alpha)(y) = \int_{f^{-1}(y)} \alpha \quad (y \in Y).$$

For $pt : X \to pt$, $\int_X \alpha = pt_* a$. It holds that

$$(f \circ g)_* = f_* \circ g_*.$$

The pullback $f^* \mathcal{F}(Y) \to \mathcal{F}(X)$ is defined as $f_* \alpha = \alpha \circ f$.

The group of constructible functions $\mathcal{F}(X)$ and the Borel-Moore homology H_* define covariant functors $Var \to Ab$, from the category of complex algebraic varieties and proper morphisms to the category of abelian groups.

The total Chern class of the tangent bundle of a smooth variety is $c(TX) = 1 + c_1(TX) + \cdots + c_n(TX)$.

Theorem III.1 (Schwartz-MacPherson)**.** *There is a unique transformation*

$$C_* : \mathcal{F} \to H_*$$

between these functors so that $C_*(\mathbb{1}_X) = c(TX) \frown \mu_X$ if X is smooth, where μ_X is the fundamental homology class of X. This transformation satisfies

- *$C_*(\alpha + \beta) = C_*(\alpha) + C_*(\beta)$, i.e. C_* is an additive homomorphism.*
- *$C_* f_*(\alpha) = f_* C_*(\alpha)$ for proper morphisms $f : X \to Y$.*

We then define the Chern-Schwartz-MacPherson class of X in the homology ring as $c^{SM}(X) := C_*(\mathbb{1}_X) \in H_*^{BM}(X)$. In our settings, all varieties considered will be a subset of a compact, complex variety V. There is then a map $H_*(X) \to H_*(V)$

induced by the containment $X \subset V$ and because of Poincaré duality there is an isomorphism $H_*(V) \overset{\cong}{\longleftrightarrow} H^{\dim V - *}(V)$. With these maps we can view $c^{SM}(X) \in H^*(V)$ instead of $H_*(X)$. So $c^{SM}(X) \in H_*^{BM}(X)$ is not used. We only consider $c^{SM}(X) \in H^*(V)$.

Now consider the ambient space V to be acted upon by an algebraic group G, with only one point fixed by all of G. Ohmoto proved in [Ohm] that (III.1) extends to G-equivariant cohomology. If $X \subset V$ is a G-invariant subvariety then $c^{SM}(X) \in H_G^*(V)$, where the G is suppressed from the notation of the CSM class. We can now define the Segre-Schwartz-MacPherson, SSM, class to be

$$s^{SM}(X) = \frac{c^{SM}(X)}{c(V)} \in H_G^{\Pi}(V).$$

Here $c(V)$ is the total Chern class defined earlier.

In this situation there is, again, a useful interpolation theorem that allows us to determine the CSM class. Recall the restriction homomorphisms, φ_ξ, and the tangent spaces, T_ξ. The number of zero weights of the action of G_ξ has on T_ξ will be denoted k_ξ.

Theorem III.2 ([RR]-unpublished)**.** *Suppose G acts on V, a vector space, as above and suppose that there are only finitely many orbits. If η and ζ are orbits then the following statements hold:*

1. $\varphi_\zeta\big(c^{SM}(\eta)\big) = c(T_\eta)e(\nu_\eta)$ *if* $\zeta = \eta$.
2. $\varphi_\zeta\big(c^{SM}(\eta)\big)$ *is divisible by* $c(T_\zeta)$.
3. $\varphi_\zeta\big(c^{SM}(\eta)\big)$ *has degree strictly less than* $2(\dim_{\mathbb{C}} V - k_\zeta)$.

Note that in the above theorem we deal with $c^{SM}(\eta)$ and not $c^{SM}(\bar{\eta})$.

Theorem III.3 ([RR]-unpublished)**.** *The previous set of conditions uniquely determine the Chern-Schwartz-MacPherson class of the orbit η.*

The lowest degree term of $c^{SM}(\eta)$ is $[\bar{\eta}]$, the fundamental class of the closure of η.

Before working through an example to use the theorem, we are going to take a brief aside in order to describe the Schur functions which play an important role in the conjectures we make.

III.1 Schur Functions

Given a partition, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, of an integer n we can form the Schur polynomial (or function), denoted $s_\lambda(x)$, defined by

$$s_\lambda(x) = \det[h_{\lambda_i - i + j}(x)]_{k \times k}, \tag{21}$$

where $h_\ell(x)$ is the complete homogeneous symmetric polynomial of degree ℓ in n variables $x_1, \ldots, x_n$. We can also form the Young diagram of λ. This consists of n boxes, left justified with λ_1 boxes in the top row, λ_2 boxes in the row below that, and so on. For example,

$$(5,3,2) \qquad \longleftrightarrow \qquad \ydiagram{5,3,2} \quad .$$

Many times we will write the associated Young diagram instead of $(\lambda_1, \ldots, \lambda_n)$. The conjugate partition of λ is the partition obtained by reflecting the Young diagram of λ along the diagonal. For example for the partition $\lambda = (3,1)$ we will write $s_{\ydiagram{3,1}}$ instead of $s_{(3,1,0,0)}$. For this partition the conjugate is $(2,1,1)$, whose Young diagram $\ydiagram{2,1,1}$.

The Schur polynomial of $\lambda = (2,1)$, a partition of 3, then

$$\begin{aligned} s_{\ydiagram{2,1}} &= \det \begin{pmatrix} h_2(x) & h_3(x) \\ h_0(x) & h_1(x) \end{pmatrix} = h_2(x)h_1(x) - h_3(x) \\ &= x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3. \end{aligned}$$

It is a well-known result that the collection of all Schur functions generates $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$, the ring of complex polynomials in n variables invariant under permuting the x_j variables. We will make use of this fact several times, especially when dealing with the Segre-Schwartz-MacPherson classes.

In our case, instead of considering the homogeneous symmetric polynomials h_j, we will be using the j^{th} Chern class. Doing this, we can notice that in our first example of $\Sigma^r \subset \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$, that $[\bar{\Sigma}^r] = s_\lambda$, for $\lambda = (r+\ell, r+\ell, \ldots, r+\ell)$.

A good reference for Schur functions is [FP].

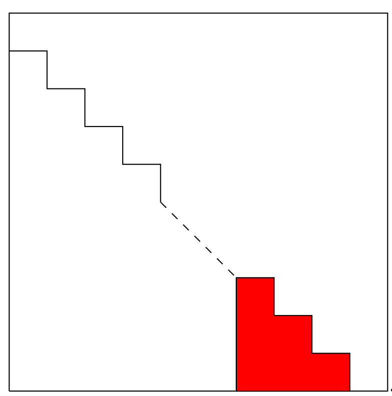
III.2 Example

Let GL_{2n} act on $\mathbb{C}^{2n}$ in the natural way. Take Λ^2 of this representation to have $G = GL_{2n}$ act on $V = \Lambda^2\mathbb{C}^{2n}$, i.e. $g.(e_i \wedge e_j) = ge_i \wedge ge_j$. Because of the alternating property, we can naturally associate $\Lambda^2\mathbb{C}^{2n}$ with the space of $n \times n$ skew-symmetric matrices. For visual representations we will examine V as the symmetric matrices, and to describe the action we will use the second exterior power representation.

Again, there are finitely many orbits of this action, denoted by Σ^r for $0 \leq r \leq 2n$ and r even. A representative of an orbit is

$$\Sigma^r \ni \left(\begin{array}{cccc|c} H & 0 & & 0 & \\ 0 & H & & 0 & 0 \\ & & \ddots & \vdots & \\ 0 & 0 & \dots & H & \\ \hline & & 0 & & 0_{r\times r} \end{array}\right), \tag{22}$$

where $H = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The normal space is the part of the $r \times r$ block below the main diagonal and the tangent space is the rest of the matrix below the main diagonal. In a more pictorial fashion, this is viewed as

,

where the white area below the diagonal line can be identified with tangent space and the red area with the normal space.

Again we restrict to the torus subgroup $T < G$. A generic element of T would be

$$t = \begin{pmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_{2n} \end{pmatrix} \in T, \qquad \gamma_j \in \mathbb{C}^\times. \tag{23}$$

If j is odd and $j \leq 2(n-r)$ then $t.e_j \wedge e_{j+1} = \gamma_j e_j \wedge \gamma_{j+1} e_{j+1} = \gamma_j \gamma_{j+1} e_j \wedge e_{j+1}$. Therefore, to stabilize the representative of Σ^r it is necessary that $\gamma_j = \gamma_{j+1}^{-1}$ for such j.

Moreover, the restriction homomorphisms we concern ourselves with when examining Σ^r are the φ which map γ_j and γ_{j+1} to additive inverses for odd $j \leq 2(n-r)$ and the rest of the γ_j's to a separate set of indeterminates. Explicitly one of the $\varphi : \mathbb{C}[\gamma_1, \ldots, \gamma_{2n}] \to \mathbb{C}[\rho_1, \ldots, \rho_{n-r}, \alpha_1, \ldots, \alpha_{2r}]$ will be of the form

$$\varphi : \begin{cases} \gamma_j \mapsto (-1)^{j+1} \rho_{\lceil \frac{j}{2} \rceil}, & j \leq 2(n-r) \\ \gamma_j \mapsto \alpha_{j-2(n-r)}, & j > 2(n-r) \end{cases} .$$

With these explicit homomorphisms and the interpolation theorems, we can calculate the CSM class of Σ^r, $c^{SM}(\Sigma^r)$, if we can describe how to find the weights for the Euler and Chern classes in the normal and tangent spaces. Since we will exclusively make use of these objects after we apply the restriction homomorphisms, we will describe them in terms of the ρ's and α's.

In a similar fashion to how the weights for the Euler class were calculated in the fundamental class case, we write how t acts on $e_j \wedge e_k$ in additive notation. After we apply the homomorphism φ, the weights will be $\alpha_j + \alpha_k$ for $1 \leq j < k \leq 2r$. The weight for the Chern class is calculated in a similar fashion to the Euler class. If $e_j \wedge e_k$ is located in the tangent space, and the weight of the Euler class is w, then the weight for the Chern class would be $1+w$. So these weights are of the form $1 + \alpha_j \pm \rho_i$ or $1 \pm \rho_j \pm \rho_i$ for an appropriate collection of j, i.

Therefore we find that the Chern classes and the Euler class are the following polynomials

$$\begin{aligned} c(T_{\Sigma^k}) &= \left(\prod_{i=1}^{2r} \prod_{j=1}^{2(n-k)} (1 + \alpha_i \pm \rho_j) \right) \left(\prod_{1 \leq i < j \leq 2(n-k)} (1 \pm \rho_i \pm \rho_j) \right) \\ e(\nu_{\Sigma^r}) &= \prod_{1 \leq i < j \leq 2r} (\alpha_i + \alpha_j). \end{aligned}$$

Recall that $c^{SM}(\Sigma^2 \subset \Lambda^2 \mathbb{C}^{2n}) \in H^*_{GL_{2n}(\mathbb{C})}(\Lambda^2 \mathbb{C}^{2n}) \cong H^*_{GL_{2n}(\mathbb{C})}(pt) = \mathbb{Z}[c_1, \ldots, c_{2n}]$ where the c_i are the universal Chern classes of $GL_n(\mathbb{C})$ as before. Since these uniquely determine the CSM class, making use of a computer algebra program we find

Proposition III.4. *For $\Sigma^2 \subset \Lambda^2 \mathbb{C}^4$ we have*

$$c^{SM}(\Sigma^2) = c_1 + 2c_1^2 + c_1^3 - 4c_1c_4 + 2c_1^2c_2 + c_1^2c_3 + c_1c_2^2 + 2c_1c_2, \tag{24}$$

where c_j is the j^{th} Chern class.

The above formula does not seem to have a distinct pattern; however, if we instead consider the Segre-Schwartz-MacPherson (SSM) class of Σ^2 we see that

$$s^{SM}(\Sigma^2) = c_1 - c_1^2 + c_1^3 - c_1^4 + c_1^5 - c_1^6 + (c_1^7 + c_1^3 c_4 - c_1^2 c_2 c_3 + c_1 c_3^2) + \ h.o.t. \tag{25}$$

This appears to be in an even nicer form when we write the SSM class in terms of the Schur polynomials. When we do this we have

$$\begin{aligned} s^{SM}(\Sigma^2) = s_{\ydiagram{1}} - \left(s_{\ydiagram{2}} + s_{\ydiagram{1,1}}\right) + \left(s_{\ydiagram{3}} + 2s_{\ydiagram{2,1}} + s_{\ydiagram{1,1,1}}\right) - \Big(&s_{\ydiagram{4}} + 3s_{\ydiagram{3,1}} + 3s_{\ydiagram{2,1,1}} + s_{\ydiagram{1,1,1,1}}\Big) \\ &- 2s_{\ydiagram{2,2}} \\ + \mathit{higher\,order\,terms}. \end{aligned}$$

We notice that the partitions $\lambda \ngeq \ydiagram{2,2}$ appear to have coefficient $\binom{|\lambda|-1}{\lambda_1-1}$. This leads us to the make the first conjecture in the next section.

We can do the same process to calculate these classes for larger values of n, but, since the CSM class of $\Sigma^2 \subset \Lambda^2\mathbb{C}^6$ is degree $\binom{6}{2} - 1 = 14$, we will not write it out explicitly. We will, however, list the first few terms of the SSM class.

Proposition III.5. *For $\Sigma^2 \subset \Lambda^2\mathbb{C}^6$ we have*

$$s^{SM}(\Sigma^2) = c_1 - c_1^2 + c_1^3 - c_1^4 + c_1^5 - c_1^6 + (c_1^7 + c_1^3 c_4 - c_1^2 c_2 c_3 + c_1 c_3^2 - c_1^2 c_5) + \dots \tag{26}$$

If we substitute c_5 to 0 then this is clearly agrees, up to degree 7, with the SSM class of $\Sigma^2 \subset \Lambda^2\mathbb{C}^4$.

III.3 Conjectures

Consider $\Sigma^2 \subset \Lambda^2\mathbb{C}^{2n}$ for $n \geq 2$.

Conjecture III.6. *Suppose we write $s^{SM}(\Sigma^2) = \Phi_1 + \Phi_2 + \Phi_3 + \dots$, where Φ_1 is the sum of all the Schur polynomials of the partitions λ such that $(2,2) \nleq \lambda$, Φ_2 is the sum of Schur polynomials of the partitions μ such that $(3,3,3) \nleq \mu$ but $(2,2) \leq \mu$, and so on. Then the coefficients of a Schur function of λ in Φ_1 are, up to sign, $\binom{|\lambda|-1}{\lambda_1-1}$.*

Specifically,

$$
\begin{aligned}
\Phi_1 = \quad & 1s_{\ydiagram{1}} \\
& - (1s_{\ydiagram{1,1}} + 1s_{\ydiagram{2}}) \\
& + (1s_{\ydiagram{1,1,1}} + 2s_{\ydiagram{2,1}} + 1s_{\ydiagram{3}}) \\
& - (1s_{\ydiagram{1,1,1,1}} + 3s_{\ydiagram{2,1,1}} + 3s_{\ydiagram{3,1}} + 1s_{\ydiagram{4}}) + \dots
\end{aligned}
$$

Conjecture III.7. *If we write $s^{SM}(\Sigma^2) = \sum_\lambda \alpha_\lambda s_\lambda$, where λ sums over all partitions then $\alpha_\lambda = \alpha_{\lambda^t}$, where λ^t is the conjugate partition of λ, i.e., the SSM class is invariant if we transpose the partitions.*

Conjecture III.8. *If we write $s^{SM}(\Sigma^2) = \sum_\lambda \alpha_\lambda s_\lambda$, again then if $|\lambda|$ is even implies that $\alpha_\lambda \leq 0$ and if $|\lambda|$ is odd then $\alpha_\lambda \geq 0$.*

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