PERIODIC SOLUTIONS OF NONLINEAR SECOND-ORDER DIFFERENCE EQUATIONS

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We establish conditions for the existence of periodic solutions of nonlinear, second-order difference equations of the form $y(t+2)+by(t+1)+cy(t) = f(y(t))$, where $c \neq 0$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous. In our main result we assume that f exhibits sublinear growth and that there is a constant $\beta > 0$ such that $uf(u) > 0$ whenever $|u| \geq \beta$. For such an equation we prove that if N is an odd integer larger than one, then there exists at least one N-periodic solution unless all of the following conditions are simultaneously satisfied: $c = 1$, $|b| < 2$, and $N \arccos^{-1}(-b/2)$ is an even multiple of π.

1. Introduction

In this paper, we study the existence of periodic solutions of nonlinear, second-order, discrete time equations of the form

$$y(t+2)+by(t+1)+cy(t) = f(y(t)), \quad t = 0,1,2,3,\dots, \tag{1.1}$$

where we assume that b and c are real constants, c is different from zero, and f is a real-valued, continuous function defined on $\mathbb{R}$.

In our main result we consider equations where the following hold.

(i) There are constants a_1, a_2, and s, with $0 \leq s < 1$ such that

$$|f(u)| \leq a_1|u|^s + a_2 \quad \forall u \text{ in } \mathbb{R}. \tag{1.2}$$

(ii) There is a constant $\beta > 0$ such that

$$uf(u) > 0 \quad \text{whenever } |u| \geq \beta. \tag{1.3}$$

We prove that if N is odd and larger than one, then the difference equation will have a N-periodic solution unless all of the following conditions are satisfied: $c = 1$, $|b| < 2$, and $N \arccos^{-1}(-b/2)$ is an even multiple of π.

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As a consequence of this result we prove that there is a countable subset S of $[-2,2]$ such that if $b \notin S$, then

$$y(t+2)+by(t+1)+cy(t) = f(y(t)) \tag{1.4}$$

will have periodic solutions of every odd period larger than one.

The results presented in this paper extend previous ones of Etheridge and Rodriguez [4] who studied the existence of periodic solutions of difference equations under significantly more restrictive conditions on the nonlinearities.

2. Preliminaries and linear theory

We rewrite our problem in system form, letting

$$\begin{aligned} x_1(t) &= y(t), \\ x_2(t) &= y(t+1), \end{aligned} \tag{2.1}$$

where t is in $\mathbb{Z}^+ \equiv \{0,1,2,3,\dots\}$. Then (1.1) becomes

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ f(x_1(t)) \end{bmatrix} \tag{2.2}$$

for t in $\mathbb{Z}^+$. For periodicity of period $N > 1$, we must require that

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_1(N) \\ x_2(N) \end{bmatrix}. \tag{2.3}$$

We cast our problem (2.2) and (2.3) as an equation in a sequence space as follows.

Let X_N be the vector space consisting of all N-periodic sequences $x : \mathbb{Z}^+ \to \mathbb{R}^2$, where we use the Euclidean norm $|\cdot|$ on $\mathbb{R}^2$. For such x, if $\|x\| = \sup_{t\in\mathbb{Z}^+} |x(t)|$, then $(X_N, \|\cdot\|)$ is a finite-dimensional Banach space.

The "linear part" of (2.2) and (2.3) may be written as a linear operator $L : X_N \to X_N$, where for each $t \in \mathbb{Z}^+$,

$$Lx(t) = \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} - A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{2.4}$$

the matrix A being

$$\begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}. \tag{2.5}$$

The "nonlinear part" of (2.2) and (2.3) may be written as a continuous function $F : X_N \to X_N$, where for $t \in \mathbb{Z}^+$,

$$F(x)(t) = \begin{bmatrix} 0 \\ f(x_1(t)) \end{bmatrix}. \tag{2.6}$$

We have now expressed (2.2) and (2.3) in an equivalent operator equation form as

$$Lx = F(x). \tag{2.7}$$

Following [4, 5], we briefly discuss the purely linear problems $Lx = 0$ and $Lx = h$.

Notice that $Lx = 0$ if and only if

$$\begin{aligned} x(t+1) &= Ax(t) \quad \forall t \text{ in } \mathbb{Z}^+, \\ x(0) &= x(N), \end{aligned} \tag{2.8}$$

where $x(t)$ is in $\mathbb{R}^2$. But solutions of this system must be in the form $x(t) = A^t x(0)$, for $t = 1,2,3,\ldots$, where $(I - A^N)x(0) = 0$. Accordingly, the kernel of L (henceforth called $\ker(L)$) consists of those sequences in X_N for which $x(0) \in \ker(I - A^N)$ and otherwise $x(t) = A^t x(0)$.

To characterize the image of L (henceforth called $\operatorname{Im}(L)$), we observe that if h is an element of X_N, and $x(t)$ is in $\mathbb{R}^2$ for all t in $\mathbb{Z}^+$, then h is an element of $\operatorname{Im}(L)$ if and only if

$$x(t+1) = Ax(t) + h(t) \quad \forall t \text{ in } \mathbb{Z}^+, \tag{2.9}$$

$$x(0) = x(N). \tag{2.10}$$

It is well known [1, 6, 7] that solutions of (2.9) are of the form

$$x(t) = A^t x(0) + A^t \sum_{l=0}^{t-1} \left(A^{l+1}\right)^{-1} h(l) \tag{2.11}$$

for $t = 1,2,3,\ldots$. For such a solution also to satisfy the N-periodicity condition (2.10), it follows that $x(0)$ must satisfy

$$(I - A^N)x(0) = A^N \sum_{l=0}^{N-1} \left(A^{l+1}\right)^{-1} h(l), \tag{2.12}$$

which is to say that $A^N \sum_{l=0}^{N-1} (A^{l+1})^{-1} h(l)$ must lie in $\operatorname{Im}(I - A^N)$. Because $\operatorname{Im}(I - A^N) = [\ker(I - A^N)^T]^{\perp}$, it follows that if we construct matrix W by letting its columns be a basis for $\ker(I - A^N)^T$, then for h in X_N, h is an element of $\operatorname{Im}(L)$ if and only if $W^T A^N \sum_{l=0}^{N-1} (A^{l+1})^{-1} h(l) = 0$. See [4].

Following [4], we let

$$\begin{aligned} \Psi(0) &= \left(A^N\right)^T W, \\ \Psi(l+1) &= \left(A^{l+1}\right)^{-T} \left(A^N\right)^T W \quad \text{for } l \text{ in } \mathbb{Z}^+. \end{aligned} \tag{2.13}$$

Then h is in $\operatorname{Im}(L)$ if and only if $\sum_{l=0}^{N-1} \Psi^T(l+1) h(l) = 0$.

As will become apparent in Section 3, in which we construct the projections U and $I - E$ for specific cases, it is useful to know that the columns of $\Psi(\cdot)$ span the solution space of the homogeneous "adjoint" problem

$$\hat{L}\hat{x} = 0, \tag{2.14}$$

where $\widehat{L} = X_N \to X_N$ is given by

$$\widehat{L}\widehat{x}(t) = \widehat{x}(t+1) - A^{-T}\widehat{x}(t) \quad \text{for } t \text{ in } \mathbb{Z}^+. \tag{2.15}$$

Further, this solution space and $\ker(L)$ are of the same dimension. See [4, 5, 9].

The proof appears in Etheridge and Rodriguez [4]. One observes that $x(t+1) = (A^{-T})x(t)$ if and only if $x(t) = (A^{-T})^t x(0)$ and next, by direct calculation, that

$$\Psi(t+1) = (A^{-T})\Psi(t). \tag{2.16}$$

Furthermore,

$$\left[I - (A^{-T})^N\right]\Psi(0) = 0 \tag{2.17}$$

so that $\Psi(0) = \Psi(N)$, whence the columns of $\Psi(\cdot)$ lie in X_N. One then observes that, just as the dimension of $\ker(L)$ is equal to that of $\ker(I - A^N)$, the dimension of $\ker(\widehat{L})$ is equal to that of $\ker(I - (A^{-T})^N)$. The two matrices have kernels of the same dimension.

Our eventual aim is to analyze (2.7) using the alternative method [2, 3, 8, 9, 10, 11, 12] and degree-theoretic arguments [3, 12, 13]. To begin, we will "split" X_N using projections $U : X_N \to \ker(L)$ and $E : X_N \to \operatorname{Im}(L)$. The projections are those of Rodriguez [9]. See also [4, 5]. A sketch of their construction is given here.

Just as we let the columns of W be a basis for $\ker((I - A^N)^T)$, we let the columns of the matrix V be a basis for $\ker(I - A^N)$. Note that the dimensions of these two spaces are the same. Let C_U be the invertible matrix $\sum_{l=0}^{N-1}(A^lV)^T(A^lV)$ and C_{I-E} the invertible matrix $\sum_{l=0}^{N-1}\Psi^T(l+1)\Psi(l+1)$. For x in X_N, define

$$Ux(t) = (A^tV)C_U^{-1}\sum_{l=0}^{N-1}(A^lV)^T x(l), \tag{2.18}$$

$$(I-E)x(t) = \Psi(t+1)C_{I-E}^{-1}\sum_{l=0}^{N-1}\Psi^T(l+1)x(l) \tag{2.19}$$

for each t in $\mathbb{Z}^+$. Rodriguez [9] shows that these are projections which split X_N, so that

$$\begin{aligned} X_N &= \ker(L) \oplus \operatorname{Im}(I-U), \\ X_N &= \operatorname{Im}(L) \oplus \operatorname{Im}(I-E), \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} \operatorname{Im}(E) &= \operatorname{Im}(L), \\ \operatorname{Im}(U) &= \ker(L), \end{aligned} \tag{2.21}$$

the spaces $\operatorname{Im}(I-E)$ and $\operatorname{Im}(U)$ having the same dimension.

Note that if we let $\tilde{L}$ be the restriction to $\operatorname{Im}(I-U)$ of L, then $\tilde{L}$ is an invertible, bounded linear map from $\operatorname{Im}(I-U)$ onto $\operatorname{Im}(E)$. If we denote by M the inverse of $\tilde{L}$, then it follows that

(i) $LMh = h$ for all h in $\operatorname{Im}(L)$,
(ii) $MLx = (I-U)x$ for all x in X_N,
(iii) $UM = 0$, $EL = L$, and $(I-E)L = 0$.

3. Main results

We have $X_N = \ker(L) \oplus \mathrm{Im}(I - U)$. Letting the norms on $\ker(L)$ and $\mathrm{Im}(I - U)$ be the norms inherited from X_N, we let the product space $\ker(L) \times \mathrm{Im}(I - U)$ have the max norm, that is, $\|(u, v)\| = \max(\|u\|, \|v\|)$.

Proposition 3.1. *The operator equation $Lx = F(x)$ is equivalent to*

$$\begin{aligned} v - MEF(u+v) &= 0, \\ Q(I-E)F\big(u + MEF(u+v)\big) &= 0, \end{aligned} \tag{3.1}$$

where u is in $\ker(L) = \mathrm{Im}(U)$, $v \in \mathrm{Im}(I - U)$, and Q maps $\mathrm{Im}(I - E)$ linearly and invertibly onto $\ker(L)$.

Proof.

$$Lx = F(x) \tag{3.2}$$

$$\iff \begin{cases} E\big(L(x) - F(x)\big) = 0, \\ (I-E)\big(Lx - F(x)\big) = 0 \end{cases} \tag{3.3}$$

$$\iff \begin{cases} L(x) - EF(x) = 0, \\ (I-E)F(x) = 0 \end{cases} \tag{3.4}$$

$$\iff \begin{cases} \big(MLx - MEF(x)\big) = 0, \\ Q(I-E)F(x) = 0 \end{cases} \tag{3.5}$$

$$\iff \begin{cases} x = Ux + MEF(x), \\ Q(I-E)F(x) = 0 \end{cases} \tag{3.6}$$

$$\iff \begin{cases} (I-U)x - MEF(x) = 0, \\ Q(I-E)F\big(Ux + MEF(x)\big) = 0. \end{cases} \tag{3.7}$$

Now, each x in X_N may be uniquely decomposed as $x = u + v$, where $u = Ux \in \ker(L)$ and $v = (I - U)x$. So (3.7) is equivalent to

$$v - MEF(u+v) = 0, \tag{3.8}$$

$$Q(I-E)F\big(u + MEF(u+v)\big) = 0. \tag{3.9}$$

By means of (2.18) and (2.19), we have split our operator equation (2.7); $v - MEF(u + v)$ is in $\mathrm{Im}(I - U)$, while $Q(I - E)F(u + MEF(u + v))$ is in $\mathrm{Im}(U)$. □

Proposition 3.2. *If N is odd, $c \neq 0$, and $N \arccos(-b/2)$ is not an even multiple of π when $c = 1$ and $|b| < 2$, then either $\ker(L)$ is trivial or both $\ker(L)$ and $\mathrm{Im}(I - E)$ are one-dimensional. In the latter case, the projections U and $I - E$ and the bounded linear mapping*

$Q(I-E)$ may be realized as follows. For x in X_N, and for all $t \in \mathbb{Z}^+$

$$Ux(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left\{ \left(\frac{1}{2N}\right) \left(\sum_{l=0}^{N-1} x_1(l) \right) + \left(\sum_{l=0}^{N-1} x_2(l) \right) \right\}, \tag{3.10}$$

$$(I-E)x(t) = \begin{bmatrix} -c \\ 1 \end{bmatrix} \left\{ \left(\frac{1}{N(c^2+1)}\right) \left((-c) \left(\sum_{l=0}^{N-1} x_1(l) \right) + \left(\sum_{l=0}^{N-1} x_2(l) \right) \right) \right\}. \tag{3.11}$$

Proof. The "homogeneous linear part" of our scalar problem (corresponding to $Lx = 0$) is

$$y(t+2) + by(t+1) + cy(t) = 0, \tag{3.12}$$

where

$$y(0) = y(N), \qquad y(1) = y(N+1). \tag{3.13}$$

Calculations, detailed in the appendix of this paper, show that under the hypotheses of Proposition 3.2, the homogeneous linear part of our scalar problem has either only the trivial solution $y(t) = 0$ for all t in $\mathbb{Z}^+$ or the constant solution $y(t) = 1$ for all t in $\mathbb{Z}^+$.

In the latter case, the constant function

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{3.14}$$

spans $\ker(L)$, so that for every $t \in \mathbb{Z}^+$, $A^t V$ of (2.18) may be taken to be

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{3.15}$$

Then

$$\begin{aligned} C_U^{-1} &= \left[\sum_{l=0}^{N} (A^l V)^T (A^l V) \right]^{-1} = \left(\sum_{l=0}^{N-1} [1,1] \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{-1} = (2N)^{-1}, \\ \sum_{l=0}^{N-1} (A^l V)^T x(l) &= \sum_{l=0}^{N-1} [1,1] \begin{bmatrix} x_1(l) \\ x_2(l) \end{bmatrix} \\ &= \left(\sum_{l=0}^{N-1} x_1(l) \right) + \left(\sum_{l=0}^{N-1} x_2(l) \right) \end{aligned} \tag{3.16}$$

whenever x is in X_N. Therefore

$$Ux(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left\{ \left(\frac{1}{2N}\right) \left(\sum_{l=0}^{N-1} x_1(l) \right) + \left(\sum_{l=0}^{N-1} x_2(l) \right) \right\} \tag{3.17}$$

for $t \in \mathbb{Z}^+$, a constant multiple of

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{3.18}$$

In the appendix, we also show that under the hypotheses of Proposition 3.2, the homogeneous adjoint problem $\widehat{L}x = 0$ has either only the trivial solution or a one-dimensional solution space spanned by the constant function

$$\begin{bmatrix} -c \\ 1 \end{bmatrix}. \tag{3.19}$$

Therefore in (2.19), we may take

$$\Psi(t) = \begin{bmatrix} -c \\ 1 \end{bmatrix} \tag{3.20}$$

for all t in $\mathbb{Z}^+$, so that

$$\begin{aligned} (C_{I-E})^{-1} &= \left(\sum_{l=0}^{N-1} [-c \quad 1] \begin{bmatrix} -c \\ 1 \end{bmatrix} \right)^{-1} = [N(c^2+1)]^{-1}, \\ \sum_{l=0}^{N-1} \Psi^T(l+1)x(l) &= \sum_{l=0}^{N-1} [-c \quad 1] \begin{bmatrix} x_1(l) \\ x_2(l) \end{bmatrix} \\ &= (-c)\left(\sum_{l=0}^{N-1} x_1(l) \right) + \left(\sum_{l=0}^{N-1} x_2(l) \right). \end{aligned} \tag{3.21}$$

Therefore for x in X_N, for all $t \in \mathbb{Z}^+$

$$(I-E)x(t) = \begin{bmatrix} -c \\ 1 \end{bmatrix} \left(\frac{1}{N(c^2+1)} \right) \left\{ (-c)\left(\sum_{l=0}^{N-1} x_1(l) \right) + \left(\sum_{l=0}^{N-1} x_2(l) \right) \right\}, \tag{3.22}$$

a constant multiple of

$$\begin{bmatrix} -c \\ 1 \end{bmatrix}. \tag{3.23}$$

Furthermore, since Q must map $\operatorname{Im}(I-E)$ linearly and invertibly onto $\ker(L) = \operatorname{Im}(U)$, our simplest choice for Q is as follows.

Each element of $\operatorname{Im}(I-E)$ is of the form (3.11) for some x in X_N. Now let

$$Q(I-E)x(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left\{ \frac{1}{N(c^2+1)} \left((-c)\left(\sum_{l=0}^{N-1} x_1(l) \right) + \left(\sum_{l=0}^{N-1} x_2(l) \right) \right) \right\}, \tag{3.24}$$

for t in $\mathbb{Z}^+$. We notice that Q is clearly linear, bounded, and maps onto $\operatorname{Im}(U)$, and that $Q((I-E)x) = 0$ if and only if $(I-E)x = 0$. □

Remark 3.3. In the case for which $\ker(L) = \{0\}$, each of U and $I - E$ is the zero projection on X_N, E is the identity on X_N, and M is L^{-1}. Equation (3.9) then becomes trivial and (3.8) becomes $L^{-1}F(v) = v$, obviously equivalent to (2.7).

THEOREM 3.4. *Suppose that $N \geq 3$ is odd, $c \neq 0$, and $f : \mathbb{R} \to \mathbb{R}$ is continuous. Assume also that*

(i) *there are nonnegative constants, $\tilde{a}$, $\tilde{b}$, and s with $s < 1$ such that $|f(z)| \leq \tilde{a}|z|^s + \tilde{b}$ for all $z \in \mathbb{R}$,*

(ii) *there is a positive number β such that for all $z > \beta$, $f(z) > 0$ and $f(-z) < 0$,*

(iii) *when $c = 1$ and $|b| < 2$, then $N \arccos(-b/2)$ is not an even multiple of π.*

Then there is at least one N-periodic solution of $y(t+2) + by(t+1) + cy(t) = f(y(t))$.

Proof. We have already seen that this scalar problem may be written equivalently as equations of the form

$$\begin{aligned} 0 &= Q(I-E)F\big(u + MEF(u+v)\big), \\ 0 &= v - MEF(u+v). \end{aligned} \tag{3.25}$$

Recall that our norm on $\ker(L) \times \operatorname{Im}(I-U)$ is $\|(u,v)\| = \max\{\|u\|, \|v\|\}$, where $\|u\|$ and $\|v\|$ are, respectively, the norms on u and v as elements of X_N.

We define $H : \ker(L) \times \operatorname{Im}(I-U) \to \ker(L) \times \operatorname{Im}(I-U)$ by

$$H(u,v) = \begin{bmatrix} Q(I-E)F\big(u + MEF(u+v)\big) \\ v - MEF(u+v) \end{bmatrix}. \tag{3.26}$$

We know that solving our scalar problem is equivalent to finding a zero of the continuous map H.

We have shown that under the hypotheses of this theorem, $\ker(L)$ is either trivial or one-dimensional, and that when $\ker(L)$ is one-dimensional, it consists of the span of the constant function

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{3.27}$$

We will establish the existence of a zero of H by constructing a bounded open subset, Ω, of $\ker(L) \times \operatorname{Im}(I-U)$ and showing that the topological degree of H with respect to Ω and zero is different from zero. We will do this using a homotopy argument.

The reader may consult Rouche and Mawhin [13] and the references therein as a source of ideas and techniques in the application of degree-theoretic methods in the study of nonlinear differential equations.

We write $H(u,v) = (I-G)(u,v)$, where I is the identity and

$$G(u,v) = \begin{bmatrix} u - Q(I-E)F\big(u + MEF(u+v)\big) \\ MEF(u+v) \end{bmatrix}. \tag{3.28}$$

It is obvious that if Ω contains $(0,0)$, then the topological degree of I with respect to Ω and zero is one.

For $0 \le \tau \le 1$,

$$\tau H + (1-\tau)I = \tau(I-G) + (1-\tau)I = I - \tau G. \tag{3.29}$$

Therefore, if we can show that $\|(I-\tau G)(u,v)\| > 0$ for all (u,v) in the boundary of Ω, then by the homotopy invariance of the Brouwer degree, it follows that the degree of H with respect to Ω and zero will be one, and consequently $H(u,v) = (0,0)$ for some (u,v) in Ω. Since, for $0 \le \tau \le 1$,

$$\|(I-\tau G)(u,v)\| > \big|\|(u,v)\| - \tau\|G(u,v)\|\big|, \tag{3.30}$$

it suffices to show that $\|G(u,v)\| < \|(u,v)\|$ for all (u,v) in the boundary of Ω.

We will let Ω be the open ball in $\ker(L) \times \operatorname{Im}(I-U)$ with center at the origin and radius r, where r is chosen such that $r/\sqrt{2} > \beta + (2\tilde{a}r^s + \tilde{b})(1 + \|ME\|)$. Observe that since $0 < s < 1$, such a choice is always possible.

We will show that the second component function of G maps each boundary point of Ω into Ω itself and then, by breaking up the boundary of Ω into separate pieces, consider the effect of the first component function of G on those pieces.

The pieces will be, respectively, those boundary elements (u,v) for which $\|u\| \in [\hat{r}, r]$ and those for which $\|u\| \in [0, \hat{r})$, where $\hat{r} = \sqrt{2}(\beta + \|ME\|(2\tilde{a}r^s + \tilde{b})) < r$.

Observation 3.5. For $(u,v) \in \Omega$,
(i) $\|F(u+v)\| \le 2\tilde{a}r^s + \tilde{b}$,
(ii) $\|MEF(u+v)\| \le \|ME\|(2\tilde{a}r^s + \tilde{b})$.

Proof. For $(u,v) \in \Omega$,

$$\begin{aligned} \|F(u+v)\| &= \sup_{t \in \mathbb{Z}^+} |f(u_1(t) + v_1(t))| \\ &\le \sup_{t \in \mathbb{Z}^+} \left(\tilde{a}|u_1(t) + v_1(t)|^s + \tilde{b}\right) \\ &\le 2^s \tilde{a}\|(u,v)\|^s + \tilde{b} \le 2\tilde{a}r^s + \tilde{b}. \end{aligned} \tag{3.31}$$

This establishes (i), from which (ii) follows immediately. □

Observation 3.6. If (u,v) is in the boundary of Ω, then $\|MEF(u+v)\| < r$.

Proof.

$$\|MEF(u+v)\| \le \|ME\|(2\tilde{a}r^s + \tilde{b}) < (1 + \|ME\|)(2\tilde{a}r^s + \tilde{b}) + \beta < \frac{r}{\sqrt{2}} < r. \tag{3.32}$$

For convenience's sake, we will let $g(u,v)(t) = [MEF(u+v)]_1(t)$ for each $t \in \mathbb{Z}^+$. The function g maps $\ker(L) \times \operatorname{Im}(I-U)$ continuously into $\mathbb{R}$. Keep in mind that for each u in $\ker(L)$, there is a uniquely determined α for which u is the constant function

$$\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{3.33}$$

□

Observation 3.7. For (u,v) in the boundary of Ω, and for every $l \in \mathbb{Z}^+$, if $\alpha > \beta + \|ME\|(2\tilde{a}r^s + \tilde{b})$, then $f(\alpha + g(u,v)(l)) > 0$, while if $\alpha < -(\beta + \|ME\|(2\tilde{a}r^s + \tilde{b}))$, then $f(\alpha+g(u,v)(l)) < 0$.

Proof. When (u,v) lies in the boundary of Ω and $\alpha \geq \beta + \|ME\|(2\tilde{a}r^s + \tilde{b})$, we have for each $l \in \mathbb{Z}^+$,

$$\begin{aligned}
0 < \beta &= [\beta + \|ME\|(2\tilde{a}r^s + \tilde{b})] - \|ME\|(2\tilde{a}r^s + \tilde{b}) \\
&\leq \alpha - \|ME\|(2\tilde{a}r^s + \tilde{b}) \leq \alpha - \|ME\|\|F(u+v)\| \\
&\leq \alpha - |MEF(u+v)(l)| \leq \alpha - |[MEF(u+v)]_1(l)| \\
&= \alpha - |g(u,v)(l)| \leq \alpha + g(u,v)(l)
\end{aligned} \tag{3.34}$$

so that for each l, $f(\alpha+g(u,v)(l)) > 0$.

Similarly, if (u,v) lies in the boundary of Ω and $\alpha \leq -\beta - \|ME\|(2\tilde{a}r^s + \tilde{b})$, then for each l in $\mathbb{Z}^+$,

$$\begin{aligned}
0 > -\beta &= -[\beta + \|ME\|(2\tilde{a}r^s + \tilde{b})] + \|ME\|(2\tilde{a}r^s + \tilde{b}) \\
&\geq \alpha + \|ME\|(2\tilde{a}r^s + \tilde{b}) \geq \alpha + \|ME\|\|F(u+v)\| \\
&\geq \alpha + |MEF(u+v)(l)| \geq \alpha + |[MEF(u+v)]_1(l)| \\
&= \alpha + |g(u,v)(l)| \geq \alpha + g(u,v)(l)
\end{aligned} \tag{3.35}$$

so that for each l, $f(\alpha+g(u,v)(l)) < 0$. □

Observation 3.8. If (u,v) is in the boundary of Ω,

$$\|u - Q(I-E)F(u+MEF(u+v))\| = \sqrt{2}\left|\alpha - \frac{1}{N(c^2+1)}\sum_{l=0}^{N-1} f(\alpha+g(u,v)(l))\right|, \tag{3.36}$$

where

$$u = \alpha\begin{bmatrix}1\\1\end{bmatrix}. \tag{3.37}$$

Proof. Since for all t in $\mathbb{Z}^+$,

$$\begin{gathered}
F(x)(t) = \begin{bmatrix}0\\ f(x_1(t))\end{bmatrix}, \\
F(u+MEF(u+v))(t) = \begin{bmatrix}0\\ f(u_1(t)+[MEF(u+v)]_1(t))\end{bmatrix} = \begin{bmatrix}0\\ f(\alpha+g(u,v)(t))\end{bmatrix}
\end{gathered} \tag{3.38}$$

so that

$$\begin{aligned}
&u(t) - Q(I-E)F(u+v)(t) \\
&\quad = \left(\alpha - \left(\frac{1}{N(c^2+1)}\right)\left[(-c)(0) + \left(\sum_{l=0}^{N-1} f(\alpha+g(u,v)(l))\right)\right]\right)\begin{bmatrix}1\\1\end{bmatrix},
\end{aligned} \tag{3.39}$$

a constant function of t, hence

$$\|u - Q(I-E)F(u+MEF(u+v))\| = \sqrt{2}\left|\alpha - \frac{1}{N(c^2+1)}\sum_{l=0}^{N-1} f(\alpha+g(u,v))(l)\right|. \quad (3.40)$$

□

Observation 3.9. For (u,v) in the boundary of Ω, $\|u - Q(I-E)F(u+MEF(u+v))\| < r$.

Proof. For (u,v) in the boundary of Ω, $\|(u,v)\| = \max\{\|u\|,\|v\|\} = r$. We consider first those elements of the boundary of Ω for which $\|u\| \in [\hat{r}, r]$, and then those for which $\|u\| \in [0,\hat{r})$.

For (u,v) in the boundary of Ω,

$$\|u\| \quad \text{is in } [\hat{r}, r] = [\sqrt{2}(\beta + \|ME\|(2\tilde{a}r^s + \tilde{b})), r] \quad (3.41)$$

if and only if

$$|\alpha| \quad \text{is in } \left[\beta + \|ME\|(2\tilde{a}r^s + \tilde{b}), \frac{r}{\sqrt{2}}\right] = \left[\frac{\hat{r}}{\sqrt{2}}, \frac{r}{\sqrt{2}}\right]. \quad (3.42)$$

We consider the subcases (i) $\alpha > 0$ and (ii) $\alpha < 0$.

(i)

$$\alpha \quad \text{is in } \left[\beta + \|ME\|(2\hat{a}r^s + \tilde{b}), \frac{r}{\sqrt{2}}\right] = \left[\frac{\hat{r}}{\sqrt{2}}, \frac{r}{\sqrt{2}}\right]. \quad (3.43)$$

Then by Observation 3.7, we have $f(\alpha + g(u,v)(l)) > 0$ for each l in $\mathbb{Z}^+$, so that

$$\alpha - \left(\frac{1}{N(c^2+1)}\right)\sum_{l=0}^{N-1} f(\alpha + g(u,v)(l)) < \alpha \le \frac{r}{\sqrt{2}}. \quad (3.44)$$

To show that

$$\left|\alpha - \left(\frac{1}{N(c^2+1)}\right)\sum_{l=0}^{N-1} f(\alpha + g(u,v)(l))\right| < \frac{r}{\sqrt{2}}, \quad (3.45)$$

it remains to show that

$$\frac{1}{N(c^2+1)}\sum_{l=0}^{N-1} f(\alpha + g(u,v)(l)) < \alpha + \frac{r}{\sqrt{2}}. \quad (3.46)$$

Now since $\beta + \|ME\|(2\tilde{a}r^s + \tilde{b}) \le \alpha$, and each $f(\alpha + g(u,v)(l))$ in the sum just above is positive, it suffices to show that

$$\frac{1}{(c^2+1)}[2\tilde{a}r^s + \tilde{b}] < \frac{r}{\sqrt{2}} + \beta + \|ME\|(2\tilde{a}r^s + \tilde{b}) \quad (3.47)$$

or equivalently, that

$$(2\tilde{a}r^s + \tilde{b})\left[\frac{1}{(c^2+1)} - \|ME\|\right] - \beta < \frac{r}{\sqrt{2}}. \tag{3.48}$$

This follows, of course, from our having chosen r so that

$$\frac{r}{\sqrt{2}} > (2\tilde{a}r^s + \tilde{b})[1 + \|ME\|] + \beta. \tag{3.49}$$

(ii)

$$\alpha \quad \text{is in} \left[-\frac{r}{\sqrt{2}}, -\beta - \|ME\|(2\tilde{a}r^s + \tilde{b})\right] = \left[-\frac{r}{\sqrt{2}}, -\frac{\hat{r}}{\sqrt{2}}\right]. \tag{3.50}$$

Then by Observation 3.7, we have $f(\alpha + g(u,v)(l)) < 0$ for each $l \in \mathbb{Z}^+$, so that

$$\alpha - \left(\frac{1}{N(c^2+1)}\right)\sum_{l=0}^{N-1} f(\alpha + g(u,v)(l)) > \alpha \geq -\frac{r}{\sqrt{2}}. \tag{3.51}$$

To show that

$$\left|\alpha - \left(\frac{1}{N(c^2+1)}\right)\sum_{l=0}^{N-1} f(\alpha + g(u,v)(l))\right| < \frac{r}{\sqrt{2}}, \tag{3.52}$$

it remains to show that

$$\alpha - \left(\frac{1}{N(c^2+1)}\right)\sum_{l=0}^{N-1} f(\alpha + g(u,v)(l)) < \frac{r}{\sqrt{2}}, \tag{3.53}$$

or equivalently, to show that

$$-\left(\frac{1}{N(c^2+1)}\right)\sum_{l=0}^{N-1} f(\alpha + g(u,v)(l)) < \frac{r}{\sqrt{2}} - \alpha. \tag{3.54}$$

Now since $-\beta - \|ME\|(2\tilde{a}r^s + \tilde{b}) \geq \alpha$, so that $\beta + \|ME\|(2\tilde{a}r^s + \tilde{b}) \leq -\alpha$, it suffices to show that

$$-\left(\frac{1}{N(c^2+1)}\right)\sum_{l=0}^{N-1} f(\alpha + g(u,v)(l)) < \beta + \|ME\|(2\tilde{a}r^s + \tilde{b}) + \frac{r}{\sqrt{2}}. \tag{3.55}$$

Further, for each l in the sum just above, $f(\alpha+g(u,v)(l))$ is negative and

$$|f(\alpha+g(u,v)(l))| \le \tilde{a}|\alpha+g(u,v)(l)|^s+\tilde{b} \le \tilde{a}r^s+\tilde{b} \tag{3.56}$$

so that

$$\sum_{l=0}^{N-1} f(\alpha+g(u,v)(l)) = \sum_{l=0}^{N-1}[-|f(\alpha+g(u,v)(l))|] \ge \sum_{l=0}^{N-1}[-(2\tilde{a}r^s+\tilde{b})], \tag{3.57}$$

hence,

$$\begin{aligned} -\left(\frac{1}{N(c^2+1)}\right)\sum_{l=0}^{N-1} f(\alpha+g(u,v)(l)) &\le -\left(\frac{1}{N(c^2+1)}\right)\sum_{l=0}^{N-1}[-(2\tilde{a}r^s+\tilde{b})] \\ &= \left(\frac{1}{(c^2+1)}\right)(2\tilde{a}r^s+\tilde{b}), \end{aligned} \tag{3.58}$$

so that it suffices to show that

$$\frac{1}{(c^2+1)}(2\tilde{a}r^s+\tilde{b}) < \beta+\|ME\|(2\tilde{a}r^s+\tilde{b})+\frac{r}{\sqrt{2}}. \tag{3.59}$$

This, as we have seen in the proof of (i), follows from our choice of r.

Finally, we consider those elements (u,v) of the boundary of Ω for which

$$|\alpha| < \beta+\|ME\|(2\tilde{a}r^s+\tilde{b}) = \hat{r}. \tag{3.60}$$

Clearly

$$\begin{aligned} \left|\alpha-\frac{1}{N(c^2+1)}\sum_{l=0}^{N-1} f(\alpha+g(u,v)(l))\right| &\le |\alpha|+\frac{N}{N(c^2+1)}(2\tilde{a}r^s+\tilde{b}) \\ &\le \beta+(1+\|ME\|)(2\tilde{a}r^s+\tilde{b}) \\ &< \frac{r}{\sqrt{2}} \end{aligned} \tag{3.61}$$

so that $\|u-Q(I-E)F(u+MEF(u+v))\| < r$.

For (u,v) in the boundary of Ω, $\|(u,v)\| = r$. For each such (u,v), we have shown by means of Observation 3.6, that $\|MEF(u+v)\| < r$, and, by means of Observation 3.9, that $\|u-Q(I-E)F(u+MEF(u+v))\| < r$. Therefore for such (u,v), $\|G(u,v)\| < \|(u,v)\|$, so that no element of the boundary of Ω is a zero of H; hence the degree of H with respect to Ω and zero is 1, so that at least one solution of (2.7) exists inside Ω. □

Remark 3.10. If in Theorem 3.4, we change (ii) so that we require $f(z) < 0$ and $f(-z) > 0$ for all $z > \beta$, the conclusions of the theorem still hold.

We let

$$\Delta = \left\{\frac{2k\pi}{j} : k \text{ and } j \text{ are integers, } 0 \le 2k < j \text{ and } j \text{ is odd}\right\}. \tag{3.62}$$

It is easy to see that if $\arccos(-b/2)\notin\Delta$, then for any odd integer N, $N\arccos(-b/2)$ cannot be an even multiple of π. It is also obvious that $S \equiv \{b : \arccos(-b/2) \in \Delta\}$ is a countable subset of $[-2,2]$. The following result is now evident.

Corollary 3.11. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, $c \neq 0$, and the following conditions hold:*

(i) *there are constants $\tilde{a}$, $\tilde{b}$, and s, with $0 \le s < 1$ such that*

$$|f(u)| \le \tilde{a}|u|^s + \tilde{b} \quad \forall u \text{ in } \mathbb{R}, \tag{3.63}$$

(ii) *there is a constant $\beta > 0$, such that $uf(u) > 0$ whenever $|u| \ge \beta$.*
Then, if either $b \in \mathbb{R}\backslash S$, or $c \neq 1$, then

$$y(t+2) + by(t+1) + cy(t) = f(y(t)) \tag{3.64}$$

will have N-periodic solutions for every odd period $N > 1$. □

Appendix

We demonstrate here that, provided N is odd, $c \neq 0$, and $N\arccos(-b/2)$ is not an even multiple of π when $c = 1$ and $|b| < 2$, the kernel of L is either trivial or is a one-dimensional space spanned by the constant function

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{A.1}$$

We show also that the solution space of the homogeneous adjoint problem (2.14) is, in the latter case, the span of the constant function

$$\begin{bmatrix} -c \\ 1 \end{bmatrix}. \tag{A.2}$$

Of course, if $\ker(L)$ is trivial, so is $\ker(\hat{L})$.

As we have seen, L is invertible if and only if the matrix $I - A^N$ is invertible. That matrix is invertible if and only if no eigenvalue of A is an Nth root of unity. Those eigenvalues may be complex conjugates, real and repeated, or real and distinct. We will consider each of those three cases after we examine the kernels of L and of $\hat{L}$ in more detail than before.

(i) The kernel of L consists of all functions x in X_N for which $x(t) = A^t x(0)$ for t in $\mathbb{Z}^+$, where

$$A = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \tag{A.3}$$

and $x(0)$ is an element of $\ker(I - A^N)$. Similarly, the kernel of $\widehat{L}$ consists of all functions $\widehat{x}$ in X_N for which $\widehat{x}(t) = (A^{-T})^t \widehat{x}(0)$ for t in $\mathbb{Z}^+$, where

$$A^{-T} = \begin{bmatrix} -\dfrac{b}{c} & 1 \\ -\dfrac{1}{c} & 0 \end{bmatrix} \tag{A.4}$$

and $\widehat{x}(0)$ is an element of $\ker(I - (A^{-T})^N)$.

We will see that it is sometimes convenient to consider instead the scalar boundary value problems corresponding to $Lx = 0$ and $\widehat{L}\widehat{x} = 0$. We have already seen that $Lx = 0$ if and only if $y(t+2) + by(t+1) + y(t) = 0$ for t in $\mathbb{Z}^+$, subject to $y(0) = y(N)$ and $y(1) = y(N+1)$, where $y(t) = x_1(t)$ and $y(t+1) = x_2(t)$.

Similarly, if we let $\widehat{y}(t) = \widehat{x}_2(t)$ and $-c\widehat{y}(t+1) = \widehat{x}_1(t)$, we find that the scalar boundary value problem $(-c)\widehat{y}(t+2) + (-b)y(t+1) + (-1)y(t) = 0$, subject to $\widehat{y}(0) = \widehat{y}(N)$ and $\widehat{y}(1) = \widehat{y}(N+1)$ is equivalent to $\widehat{L}\widehat{x} = 0$.

In cases (ii), (iii), and (iv) we consider the various cases for which $\ker(L)$ is nontrivial. We make the same hypotheses herein as we did in Proposition 3.2.

(ii) The eigenvalues of A are the solutions of $\lambda^2 + b\lambda + c = 0$. Here in case (ii), we consider the case in which they are complex conjugates $\lambda_1 = (-b/2) - (\sqrt{4c - b^2}/2)i$ and $\lambda_2 = (-b/2) + (\sqrt{4c - b^2}/2)i$. If either were an Nth root of unity, both would be, and each would necessarily have modulus 1, where we have $|\lambda_1| = |\lambda_2| = \sqrt{b^2/4 + (4c - b^2)/4} = \sqrt{c}$. Therefore $|\lambda_j| = 1$ with $\lambda_j \in \mathbb{C}\backslash\mathbb{R}$ if and only if $c = 1$. Now $c = 1$ and $4c - b^2 = 4 - b^2 > 0$ if and only if $c = 1$ and $-2 < b < 2$. Therefore if $\lambda^2 + b\lambda + c = 0$ has nonreal roots, and $\lambda_j^N = 1$, we must have $c = 1$ and $-2 < b < 2$.

Only along that line segment in the "parameter space" is there a possibility that L may be singular.

For $c = 1$ and $-2 < b < 2$, we look more closely at solutions of $Lx = 0$, or equivalently, of $y(t+2) + by(t+1) + 1y(t) = 0$ where $y(0) = y(N)$ and $y(1) = y(N+1)$.

It is well known [6] that solutions of (1.1), the unconstrained scalar homogeneous problem (with complex conjugate eigenvalues $\lambda_j = (-b/2) + (-1)^j\sqrt{4c - b^2}/2$) are of the form $y(t) = c_1 r^t \cos(\theta t) + c_2 r^t \sin(\theta t)$, where, of course, $r = |\lambda_1| = |\lambda_2|$ and $\cos(\theta) = -b/2r$, $\sin(\theta) = \sqrt{4c - b^2}/2r$. Here, $r = 1$ whenever L may be singular, so we have $y(t) = k_1\cos(\theta t) + k_2\sin(\theta t)$.

For such y, the periodicity conditions $y(0) = y(N)$, $y(1) = y(N+1)$ are satisfied if and only if

$$\begin{bmatrix} \cos(N\theta) - 1 & \sin(N\theta) \\ \cos\left((N+1)\theta\right) - \cos(\theta) & \sin\left((N+1)\theta\right) - \sin(\theta) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{A.5}$$

If $\cos(N\theta)$ were 1, then the matrix above would clearly be singular; in fact, it would be the zero matrix, resulting in a two-dimensional solution space for (2.2) and (2.3), hence a two-dimensional kernel for L. The conditions of Theorem 3.4, however do not allow that $\cos(N\theta)$ be 1; in fact, they ensure, as we will show, that the matrix above has a nonzero determinant, whence the system (2.2) and (2.3) has only the trivial solution $y(t) = 0$ for all $t \in \mathbb{Z}^+$, so that L is invertible. The determinant of the matrix above is

$$\begin{aligned}
&[\cos(N\theta)-1][\sin(N\theta)\cos(\theta)+\cos(N\theta)\sin(\theta)-\sin(\theta)]\\
&\qquad -[\cos(N\theta)\cos(\theta)-\sin(N\theta)\sin(\theta)-\cos(\theta)][\sin(N\theta)]\\
&\quad =[\cos(N\theta)-1][\sin(\theta)(\cos(N\theta)-1)+\sin(N\theta)\cos(\theta)]\\
&\qquad -\sin(N\theta)[\cos(\theta)(\cos(N\theta)-1)-\sin(N\theta)\cos(\theta)]\\
&\quad =\cos(N\theta)\sin(\theta)(\cos(N\theta)-1)-\sin(\theta)(\cos(N\theta)-1)\\
&\qquad +[\cos(N\theta)-1]\sin(N\theta)\cos(\theta)\\
&\qquad -[\cos(N\theta)-1]\sin(N\theta)\cos(\theta)\\
&\qquad +\sin^2(N\theta)\sin(\theta)\\
&\quad =\left([\cos(N\theta)-1]^2+\sin^2(N\theta)\right)\sin(\theta)\\
&\quad =[1-2\cos(N\theta)+\cos^2(N\theta)+\sin^2(N\theta)]\sin(\theta)\\
&\quad =2(1-\cos(N\theta))\sin(\theta)\\
&\quad =2(1-\cos(N\theta))\sqrt{4c-b^2}\neq 0.
\end{aligned} \tag{A.6}$$

(iii) In this case, we suppose that the roots of $\lambda^2+b\lambda+c=0$ are real and repeated with $\lambda_1=\lambda_2=-b/2$ and $b^2=4c$. L will be singular if and only if an eigenvalue is an Nth root of unity. For odd values of N, as in Theorem 3.4, the sole real Nth root of unity is 1, so that under the conditions of that theorem, L will be singular if and only if $b=-2$ so that $\lambda_1=\lambda_2=1$.

When the characteristic equation has a repeated real root, there is exactly one instance in which L is singular, the case in which

$$y(t+2)+by(t+1)+cy(t)=y(t+2)-2y(t+1)+1y(t). \tag{A.7}$$

It is well known [6] that the general solution of $y(t+2)+by(t+1)+cy(t)=0$ when $\lambda_1=\lambda_2$ is $y(t)=k_1(\lambda_1)^t+k_2t(\lambda_1)^t$, for some real constants k_1 and k_2 and for $t\in\mathbb{Z}^+$.

L will be singular if and only if

$$y(t)=k_1(1^t)+k_2[t(1^t)]=k_1+k_2t, \tag{A.8}$$

where $y(0)=y(N)$ and $y(1)=y(N+1)$. The latter periodicity condition forces k_1 and k_2 to satisfy

$$\begin{bmatrix}0 & N\\ 0 & N\end{bmatrix}\begin{bmatrix}k_1\\ k_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \tag{A.9}$$

so that k_2 must be 0 and $y(t)=k_1$ for all $t\in\mathbb{Z}^+$.

Corresponding to this constant solution of (2.2) and (2.3) is the constant solution to $Lx=0$ given by

$$x(t)=\begin{bmatrix}y(t)\\ y(t+1)\end{bmatrix}=\begin{bmatrix}k_1\\ k_1\end{bmatrix} \tag{A.10}$$

for every $t \in \mathbb{Z}^+$. We have now shown, in the sole case for which the eigenvalue is real and repeated and in which, also, L is singular, that $\ker(L)$ is the span of the constant function

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{A.11}$$

We must now demonstrate that the solutions of the homogeneous adjoint problem (2.14) are multiples of the constant function

$$\begin{bmatrix} -c \\ 1 \end{bmatrix}. \tag{A.12}$$

Problem (2.14) becomes, for $c = 1$ and $b = -2$,

$$\begin{bmatrix} \hat{x}_1(t+1) \\ \hat{x}_2(t+1) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix}, \tag{A.13}$$

where $x \in X_N$.

Corresponding to this system is the scalar problem

$$-\hat{y}(t+2) + 2\hat{y}(t+1) - \hat{y}(t) = 0 \tag{A.14}$$

subject to $\hat{y}(0) = \hat{y}(N)$ and $\hat{y}(1) = \hat{y}(N+1)$, where $\hat{y}(t) = x_2(t)$ and $(-1)\hat{y}(t+1) = x_1(t)$. This scalar problem is the same as that discussed above for L, so we know that its solutions are all of the form $y(t) = k_1$. The corresponding solutions of (2.14) are of the form

$$\begin{bmatrix} \hat{x}_1(t+1) \\ \hat{x}_2(t+1) \end{bmatrix} = \begin{bmatrix} (-1)y(t+1) \\ y(t) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -c \\ 1 \end{bmatrix}. \tag{A.15}$$

(iv) In this case, we suppose that the roots of $\lambda^2 + b\lambda + c = 0$ are real and distinct with $\lambda_1 = -(b/2) - (\sqrt{b^2 - 4c}/2)$ and $\lambda_2 = -(b/2) + (\sqrt{b^2 - 4c}/2)$, where $b^2/4 > c$. As before, L will be singular if and only if at least one of the eigenvalues is an Nth root of unity, for which it is necessary that $|\lambda_1| = 1$ or $|\lambda_2| = 1$, that is, that $2 = |-b \pm \sqrt{b^2 - 4c}|$.

It follows then that $2 + b = \pm\sqrt{b^2 - 4c}$, in which case $c + 1 = -b$, or that $2 - b = \pm\sqrt{b^2 - 4c}$, in which case $c + 1 = b$.

Direct computation shows that when $c + 1 = b$, we have $\lambda_1 = -1$ and $\lambda_2 = -c = -b + 1$, while when $c + 1 = -b$, we have $\lambda_1 = 1$ and $\lambda_2 = c = -b - 1$.

It is well known [6] that when the roots of λ_1 and λ_2 of the characteristic equation $\lambda^2 + b\lambda + c = 0$ are real and distinct, then the solutions of $y(t+2) + by(t+1) + cy(t) = 0$ are of the form $y(t) = k_1(\lambda_1)^t + k_2(\lambda_2)^t$, where k_1 and k_2 are real constants and $t \in \mathbb{Z}^+$.

(a) Suppose herein that $b = c + 1$, so that we have eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -c = -b + 1$, with $\lambda_2 \neq \lambda_1$. L will be singular if and only if one of these eigenvalues is an Nth root of unity; however, for odd values of N, as in Theorem 3.4, this can occur if and only if $\lambda_2 = 1$.

Therefore we need only consider here the case for which $\lambda_1 = -1$ and $\lambda_2 = 1 = -c = -b + 1$. The corresponding scalar equation is $y(t+2) + 0y(t+1) - y(t) = 0$, subject to $y(0) = y(N)$ and $y(1) = y(N+1)$.

Solutions of this problem take the form $y(t) = k_1(-1)^t + k_2(1)t$, where $y(0) = y(N)$ and $y(1) = y(N+1)$. Therefore k_1 and k_2 must satisfy $k_1 + k_2 = k_1(-1)^N + k_2$ and $-k_1 + k_2 = k_1(-1)^N(-1) + k_2$. For odd values of N, the first of these forces k_1 to be 0, whence the second of these is an identity.

Therefore solutions of the homogeneous scalar problem with N-periodicity take the form $y(t) = k_2$ for each $t \in \mathbb{Z}^+$. It follows that solutions of $Lx = 0$ are of the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{A.16}$$

for $t \in \mathbb{Z}^+$.

Now we turn to the homogeneous adjoint problem (2.14), which here takes the form $\hat{x}(t+1) = A^{-T}\hat{x}(t)$ for $t \in \mathbb{Z}^+$, where $x \in X_N$. Here,

$$A = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A^{-T}. \tag{A.17}$$

Solutions of (2.14) are of the form $\hat{x}(t) = (A^{-T})^t\hat{x}(0)$, where $\hat{x}(0)$ must lie in $\ker(I - (A^{-T})^N)$. It is easy to check that any even power of A^{-T} is the identity matrix, while any odd power of A^{-T} is A^{-T} itself, so that

$$I - (A^{-T})^N = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{A.18}$$

For a solution of (2.14), then, $\hat{x}(0)$ must be a real constant multiple of

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{A.19}$$

It follows that solutions of (2.14) are of the form

$$\hat{x}(t) = (A^{-T})^t \begin{bmatrix} k_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = k_2 \begin{bmatrix} -c \\ 1 \end{bmatrix} \tag{A.20}$$

because in this case $c = -1$.

(b) Suppose herein that $b = -c - 1$, so that we have eigenvalues $\lambda_1 = 1$ and $\lambda_2 = c = -b - 1$, with $\lambda_2 \neq \lambda_1$. L will be singular and the scalar equation corresponding to $Lx = 0$ is

$$y(t+2) + (-c-1)y(t+1) + cy(t) = 0 \tag{A.21}$$

subject to $y(0) = y(N)$ and $y(1) = y(N+1)$.

Solutions are of the form $y(t) = k_1(1)^t + k_2(c)^t$, for $t \in \mathbb{Z}^+$, where k_1 and k_2 must satisfy

$$\begin{bmatrix} 0 & 1 - c^N \\ 0 & c(1 - c^N) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{A.22}$$

By the assumptions that $\lambda_2 \neq \lambda_1$ (so that $c \neq 1$) and that N is odd, we know that $1 - c^N \neq 0$, so that k_2 must be 0, and solutions are of the form $y(t) = k_1$ for $t \in \mathbb{Z}^+$. Corresponding solutions of $Lx = 0$ are of the form

$$x(t) = k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{A.23}$$

for every $t \in \mathbb{Z}^+$.

Finally, we must discuss the homogeneous adjoint problem, which in scalar form is $(-c)\hat{y}(t+2) + (c+1)\hat{y}(t+1) - \hat{y}(t) = 0$, subject to $\hat{y}(0) = \hat{y}(N)$ and $\hat{y}(1) = \hat{y}(N+1)$. The corresponding problem $\hat{L}\hat{x} = 0$ is of the form $\hat{x}(t+1) = A^{-T}\hat{x}(t)$ with $\hat{x} \in X_N$, where

$$A^{-T} = \begin{bmatrix} \dfrac{c+1}{c} & 1 \\ -\dfrac{1}{c} & 0 \end{bmatrix}. \tag{A.24}$$

The eigenvalues of A^{-T} are 1 and $1/c$, so that the solutions of the unconstrained homogeneous scalar problem are of the form $\hat{y}(t) = k_1(1)^t + k_2(1/c)^t$ for $t \in \mathbb{Z}^+$. The periodicity conditions force k_1 and k_2 to satisfy

$$\begin{bmatrix} 0 & \left(1 - \left(\dfrac{1}{c}\right)^N\right) \\ 0 & \left(\dfrac{1}{c}\right)\left(1 - \left(\dfrac{1}{c}\right)^N\right) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{A.25}$$

Because, by assumption in this case, $c \neq 1$, it follows that $k_2 = 0$. Our N-periodic homogeneous scalar problem has solutions $\hat{y}(t) = k_1$, where $k_1 \in \mathbb{R}$. The corresponding solutions of $\hat{L}x = 0$ take the form

$$\begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} = \begin{bmatrix} -cy(t+1) \\ y(t) \end{bmatrix} = \begin{bmatrix} -ck_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -c \\ 1 \end{bmatrix}. \tag{A.26}$$

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