ON THE BIAS IN FLEXIBLE FUNCTIONAL FORMS AND AN ESSENTIALLY UNBIASED FORM

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Received August 1980, final version received November 1980

The Fourier flexible form and its derived expenditure system are introduced. Subject to smoothness conditions on the consumer's true indirect utility function, the consumer's true expenditure system must be of the Fourier form over the region of interest in an empirical investigation. Arbitrarily accurate finite parameter approximations of the consumer's true expenditure system are obtained by dropping all high-order terms of the Fourier expenditure system past an appropriate truncation point. The resulting finite parameter system is tractable in empirical studies. The reader who is primarily interested in applications need only read the second and fifth sections. The remainder of the article is concerned with the verification of these claims and an investigation of some aspects of the bias in Translog specifications.

1. Introduction

Much recent work on the specification of empirical expenditure systems has focused on an attempt to find an (indirect) utility function whose derived expenditure system will adequately approximate systems resulting from a broad class of utility functions. Examples of this approach are in Diewert (1974) and Christensen, Jorgenson and Lau (1975). The (indirect) utility function chosen for this task is termed a flexible functional form.

There are two methods for approximating a function that are used frequently in applications. These are Taylor's series approximations and the general class of Fourier series approximations. As examples of the latter, there is the familiar sine/cosine expansion and the possibly less familiar Jacobi, Laguerre, and Hermite expansions. The work in flexible functional

*Much of this work was done while visiting the Institut National de la Statistique et des Études Économiques and while on leave at the Department of Economics, Duke University. I wish to thank Robert Wolpert, Guy Laroque, Alain Monfort, Grace Wahba, and John Monahan for helpful discussions on approximation methods when derivatives are incorporated in the notion of distance. Special thanks are due Maryvonne Deffieux. The many helpful comments received when this paper was read at the Research Triangle Econometrics Seminar Series and the Fall 1980 Econometric Society North American Meeting are gratefully acknowledged. This research was supported by North Carolina Agricultural Experiment Station Project NC03641 and by National Science Foundation Grant SES80-14239.

forms appearing to date has used a Taylor's expansion as the approximating mechanism.

Taylor's theorem only applies locally. It applies on a neighborhood of unspecified size containing a specified value of the argument of the function being approximated — the commodity vector of a direct utility function or income normalized prices of an indirect utility function. The local applicability of the approximation suffices to translate propositions from the theory of demand into restrictions on the parameters of the approximating expenditure system; see especially Christensen, Jorgenson and Lau (1975) in this connection. However, Taylor's theorem fails rather miserably as a means of understanding the statistical behavior of parameter estimates and test statistics; see especially Section 2 of White (1980). If one insists on using Taylor's theorem as a means of understanding statistical behavior one is lead into an algebraic morass; see Section IV of Simmons and Wierserbs (1979) for an example.

The reason for this failure is that statistical regression methods essentially expand the true function in a (general) Fourier series — not in a Taylor's series. As the sample size tends to infinity, a regression estimator $\hat{\theta}$ of the typical sort converges to that parameter value θ^* which minimizes a measure of average distance $\mathscr{B}(\theta)$ of the form

$$\mathscr{B}(\theta)=\int_{\mathscr{X}} \rho[f^*(x), f(x,\theta)]w(x)\mathrm{d}x,$$

where $\rho(y,\hat{y})$ is a measure of the distance between the true and predicted values of the dependent variable determined by the estimation procedure, $\mathscr{X}$ is a set containing all possible values of the independent variable, and $w(x)$ is a density function defined on $\mathscr{X}$ giving the relative frequency with which values of the independent variable occur as sample size tends to infinity [Souza and Gallant (1979)]. This is precisely the defining property of a (general) Fourier approximation of $f^*(x)$ by $f(x,\theta)$. A Fourier approximation attempts to minimize the average prediction bias $\mathscr{B}(\theta)$.

Due to this fact, Fourier series methods permit a natural transition from demand theory to statistical theory. The classical multivariate sine/cosine expansion of the indirect utility function leads directly to an expenditure system with the property that the average prediction bias may be made arbitrarily small by increasing the number of terms in the expansion. The key fact which permits this transition is that the classical Fourier sine/cosine series expansion approximates not only the indirect utility function to within arbitrary accuracy in terms of the $\mathscr{L}_2$ norm but also its first derivatives.

The Fourier expenditure system is used as a vehicle to study potential biases resulting from the use of the Translog expenditure system. The Translog test of the theory of demand based on the equality and symmetry of coefficients as reported in Christensen, Jorgenson and Lau (1975) is repeated using the Fourier expenditure system. Their result is confirmed. The asymptotic power curve of the Translog test of additivity is derived in terms of Fourier parameters. Parameter settings compatible with the data of Christensen, Jorgenson and Lau are used to obtain tabular values for the power curve of the Translog additivity test. Substantial bias is found. The power curve exceeds the nominal significance level of the test when the null hypothesis is true and is relatively flat with respect to departures from the null case.

2. Multi-indexes and multivariate Fourier series

The notion of multi-indexes materially reduces the complexity of the notation required to denote high-order partial differentiation and multivariate Fourier series expansions. A multi-index is an N-vector with integer components. For example,

$$k'=(5,2,7) \quad \text{or} \quad k'=(-4,0,6).$$

The length of a multi-index is defined as

$$|k|^*=\sum_{i=1}^{N}|k_i|.$$

Let λ be a multi-index with non-negative components. Partial differentiation of a function $f(x)$ is denoted as

$$D^\lambda f=\frac{\partial^{|\lambda|^*}}{\partial x_1^{\lambda_1}\partial x_2^{\lambda_2}\ldots\partial x_N^{\lambda_N}}f(x).$$

For example, the multi-index

$$\lambda'=(5,2,7)$$

generates the fourteenth-order partial derivative

$$D^\lambda f=\frac{\partial^{14}}{\partial x_1^5\partial x_2^2\partial x_3^7}f(x_1,x_2,x_3).$$

Differentiation is taken in a generalized sense [Rudin (1973, ch. 6)] in the literature cited in the later sections. Our preference is to sacrifice some generality in exchange for simplicity. Now if f possesses continuous partial derivatives of all orders up to and including $|\lambda|^*$ in the classical sense then the classical notion of differentiation and the generalized notion are coincident for our purposes [Rudin (1973, sect. 6.13)]. The classical notion and the requisite continuity are imposed on the symbol $D^\lambda f$ throughout.

A typical term of a multivariate Fourier series expansion is

$$e^{ik'x} = \cos(k'x) + i\sin(k'x),$$

where i denotes the imaginary unit. For example, if $k' = (-4, 0, 6)$ then

$$e^{ik'x} = \cos(-4x_1 + 6x_3) + i\sin(-4x_1 + 6x_3).$$

A multivariate Fourier series expansion of order K is denoted as

$$\sum_{|k|^* \leqq K} a_k e^{ik'x}.$$

The sum is over those multi-indexes k whose length $|k|^*$ is less than or equal to K. The a_k are complex valued coefficients of the form

$$a_k = u_k + iv_k,$$

where u_k and v_k are real valued.

Excepting the multi-index $0 = (0, 0, \ldots, 0)'$, the multi-indexes k with $|k|^* \leqq K$ will occur in pairs of opposite sign — k, $-k$. Thus the restrictions

$$a_0 \text{ real valued}, \qquad a_k = \bar{a}_{-k},$$

or equivalently,

$$v_0 = 0, \qquad u_k = u_{-k}, \quad v_k = -v_{-k},$$

will cause

$$\sum_{|k|^* \leqq K} a_k e^{ik'x}$$

to be real valued. That is, with these restrictions,

$$a_0 e^{i0'x} = u_0,$$

which is real valued, and

$$a_k e^{ik'x} + a_{-k} e^{-ik'x} = 2u_k \cos(k'x) - 2v_k \sin(k'x),$$

which is real valued.

The notation $\sum_{|k|^* \leqq K} a_k e^{ik'x}$ is conventional but it conceals some structure which is useful later. A more useful form results when the sum $\sum_{|k|^* \leqq K} a_k e^{ik'x}$ is re-expressed as a double sum

$$\sum_{\alpha=1}^{A} \sum_{j=-J}^{J} a_{j\alpha} e^{ijk'_\alpha x}.$$

The idea is to construct a sequence of multi-indexes $\{k_\alpha\}$ and choose values of A and J such that

$$\{k\colon |k|^* \leqq K\} \subset \{jk_\alpha\colon \alpha=1,\ldots,A;\ j=0, \pm 1,\ldots, \pm J\}.$$

The doubly indexed sum may contain more terms than the singly indexed sum but this causes no problems as the coefficients in the extra terms may be set to zero to obtain equality.

The requisite sequence of multi-indexes,

$$\{k_\alpha\colon \alpha=1,2,\ldots,A\},$$

may be constructed from the set

$$\mathscr{K} = \{k\colon |k|^* \leqq K\}$$

as follows. First, delete from $\mathscr{K}$ the zero vector and any k whose first non-zero element is negative; i.e. $(0, -1, 1)$ would be deleted but $(0, 1, -1)$ would remain. Second, delete any k whose components have a common integral divisor; i.e., $(0, 2, 4)$ would be deleted but $(0, 2, 3)$ would remain. Third, arrange the k which remain into a sequence

$$\{k_\alpha\colon \alpha=1,2,\ldots,A\},$$

such that $|k_\alpha|^*$ is non-decreasing in α and such that $k_1, k_2, \ldots, k_N$ are the elementary vectors. The sequence $\{k_\alpha\}$ is displayed in table 1 for $N=3$ and $K=3$.

The sum

$$\sum_{\alpha=1}^{A} \sum_{j=-J}^{J} a_{j\alpha} e^{ijk'_\alpha x}$$

is real valued if the restrictions

$$a_{0\alpha} \text{ real valued}, \qquad a_{j\alpha} = \bar{a}_{-j\alpha},$$

Table 1

The sequence $\{k_\alpha'\}$ for $N=3$ and $K=3$.

$\lvert k_\alpha \rvert^* =$	1	2	3	
	(1,0,0)	(1,1,0)	(1,1,1)	(0,1,2)
	(0,1,0)	(1,0,1)	(1,−1,1)	(0,2,1)
	(0,0,1)	(0,1,1)	(1,1,−1)	(1,2,0)
		(1,−1,0)	(1,−1,−1)	(1,0,2)
		(1,0,−1)		(2,1,0)
		(0,1,−1)		(2,0,1)
				(0,1,−2)
				(0,2,−1)
				(1,−2,0)
				(1,0,−2)
				(2,−1,0)
				(2,0,−1)

are imposed; equivalently, if

$$v_{0\alpha}=0, \qquad u_{j\alpha}=u_{-j\alpha}, \quad v_{j\alpha}=-v_{-j\alpha},$$

where

$$a_{j\alpha}=u_{j\alpha}+iv_{j\alpha}, \qquad \alpha=1,2,\ldots,A, \quad j=0,\pm 1,\pm 2,\ldots,\pm J.$$

With these restrictions

$$\sum_{\alpha=1}^{A}\sum_{j=-J}^{J} a_{j\alpha}\mathrm{e}^{ijk_\alpha' x}=\sum_{\alpha=1}^{A}\left\{u_{0\alpha}+2\sum_{j=1}^{J}\left[u_{j\alpha}\cos(jk_\alpha' x)-v_{j\alpha}\sin(jk_\alpha' x)\right]\right\}.$$

3. Fourier series approximation of an indirect utility function

Let q denote an N-dimensional vector of commodities, let p denote the vector of corresponding (rental) prices, let Y denote the consumer's 'income' or expenditure on the N commodities during the period under consideration, and let $x=p/Y$ be the vector of normalized prices. Finally, let $g^*(x)$ denote the consumer's true indirect utility function.

The consumer's utility is maximized when expenditures are allocated according to the expenditure system (Roy's identity)

$$p_i q_i/Y=\left[\sum_{i=1}^{N} x_i(\partial/\partial x_i)g^*(x)\right]^{-1} x_i(\partial/\partial x_i)g^*(x), \qquad i=1,2,\ldots,N,$$

provided certain regularity conditions are satisfied [Diewert (1974)]. No formal use is made here of these regularity conditions but it is required that the formula for the expenditure system make sense. Therefore, it is assumed that $g^*(x)$ has continuous partial derivatives and that

$$(\partial/\partial x_i)g^*(x_i)<0,$$

for all $x \in \bar{\mathscr{X}}$ where $\mathscr{X}$ is the region of approximation; the overbar denotes closure of a set. Throughout, functions $g(x)$ are used to approximate $g^*(x)$, a Translog for example. There may be no guarantee that $(\partial/\partial x_i)g(x)<0$ over the region of approximation. To prevent technical difficulties later on, the following conventions are adopted. With respect to the expenditure system formula only, if $(\partial/\partial x_i)g(x)>0$ set $(\partial/\partial x_i)g(x)=0$, define $1/0=\infty$, and define $0 \cdot \infty = 0$. With these conventions, expenditure shares will always be between zero and one.

The scaling of the data is important. A Fourier series is a periodic function in each of its arguments and an indirect utility function is not. A Fourier series approximation of $g^*(x)$ can be made as accurate as desired on a region $\mathscr{X}$ which is completely within the cube $\times_{i=1}^{N}[0,2\pi]$ but the approximation will diverge from $g^*(x)$ on $\times_{i=1}^{N}[0,2\pi] \sim \mathscr{X}$ due to its periodic nature, the so-called Gibb's phenomenon. One compensates for this feature of Fourier series expansions by rescaling the units of the commodities so that the income normalized prices are between 0 and 2π. Specifically, let (Y_l, Y_u) with $Y_l>0$ be the interval of incomes over which an approximation is desired and let (p_{li}, p_{ui}) with $p_{li}>0$ be the price intervals. Having made these choices, rescale the units of the commodities and the prices per unit such that the rescaled prices satisfy

$$0<p_{li}/Y_u<p_{ui}/Y_l<2\pi.$$

The region of approximation is, then,

$$\mathscr{X} = \mathop{\times}_{i=1}^{N} (p_{li}/Y_u, p_{ui}/Y_l) = \mathop{\times}_{i=1}^{N} (x_{il}, x_{iu}).$$

It is emphasized that the rescaling step cannot be omitted in applications. Should one prefer to use the original units when reporting results one can return to them by reverse scaling after the coefficients of the expansion have been estimated.

A familiar method for obtaining an expenditure system for empirical work is to set forth an indirect utility function $g(x)$ which is thought to adequately approximate $g^*(x)$ and then apply Roy's identity,

$$p_i q_i/Y = \left[\sum_{i=1}^{N} x_i(\partial/\partial x_i)g(x)\right]^{-1} x_i(\partial/\partial x_i)g(x), \qquad i=1,2,\ldots,N,$$

to obtain the approximating expenditure system. One can see from Roy's identity that if this approach is to succeed it is actually the partial derivatives of the indirect utility function which need to be accurately approximated by the partial derivatives $(\partial/\partial x_i)g(x)$ and not just the function $g^*(x)$. The starting point for an orderly attack on the problem along these lines is to find some measure of distance between $g^*(x)$ and $g(x)$ which takes derivatives into account. A global approximation over $\mathscr{X}$ is sought so the measure should be global. A measure which satisfies these requirements is the Sobolov norm, defined next.

Sobolov norm. If $1 \leqq p < \infty$ let $W^{m,p}(\mathscr{X})$ denote the collection of all complex valued functions $f(x)$ with $\int_{\mathscr{X}} |D^\lambda f|^p \mathrm{d}x < \infty$ for all λ with $|\lambda|^* \leqq m$; for $f \in W^{m,p}(\mathscr{X})$ define

$$\|f\|_{m,p,\mathscr{X}} = \left(\sum_{|\lambda|^* \leqq m} \int_{\mathscr{X}} |D^\lambda f|^p \mathrm{d}x \right)^{1/p}.$$

If $p = \infty$ let $W^{m,\infty}(\mathscr{X})$ denote the collection of all complex valued functions f with $\sup_{x \in \mathscr{X}} |D^\lambda f| < \infty$ for all λ with $|\lambda|^* \leqq m$; for $f \in W^{m,\infty}(\mathscr{X})$ let

$$\|f\|_{m,\infty,\mathscr{X}} = \sum_{|\lambda|^* \leqq m} \sup_{x \in \mathscr{X}} |D^\lambda f|.$$

A Fourier series expansion has the ability to approximate the consumers' true indirect utility function $g^*(x)$ as closely as derived in Sobolov norm, $1 \leqq p \leqq \infty$. Our work is motivated by this fact. It follows directly from Corollary 1 of Edmonds and Moscatelli (1977) and is formally stated as follows:

Theorem 1. Let $m \geqq 2$ and for each multi-index k set

$$\varphi_k(x) = \mathrm{e}^{ik'x},$$

where i denotes the imaginary unit and suppose that $f \in W^{m,p}(\mathscr{X})$ for some $p \geqq 1$. Then there is a sequence of coefficients $\{a_k\}$ such that

$$\lim_{K \to \infty} \left\| f - \sum_{|k|^* \leqq K} a_k \varphi_k \right\|_{m-1,p,\mathscr{X}} = 0,$$

for all p with $f \in W^{m,p}(\mathscr{X})$, $1 \leqq p \leqq \infty$. The sequence of coefficients does not depend on p. When f is real valued the restriction $a_k = \bar{a}_{-k}$ does not affect the validity of the result; the overbar denotes the complex conjugate of a_k.

This result motivates the consideration of a series expansion of the form $\sum_{|k|^* \leq K} a_k \, e^{ik'x}$ as an approximation to the consumers' indirect utility function $g^*(x)$. An equivalent form is $\sum_{\alpha=1}^{A} \sum_{j=-J}^{J} a_{j\alpha} e^{ijk'_\alpha x}$ as was seen in the previous section. Experience acquired in other contexts suggests that the number of sine/cosine terms in a Fourier approximation of a non-periodic function can be reduced considerably if a linear term $b'x$ is included. As seen later, if a quadratic term $x'Cx$ is included as well then curvature restructions may be imposed.

Fourier flexible form. In light of these remarks, consider as an approximation of $g^*(x)$,

$$g_K(x) = a_0 + b'x + \tfrac{1}{2}x'Cx + \sum_{\alpha=1}^{A} \sum_{j=-J}^{J} a_{j\alpha} e^{ijk'_\alpha x},$$

where

$$a_{j\alpha} = \bar{a}_{-j\alpha}, \qquad C = -\sum_{\alpha=1}^{A} a_{0\alpha} k_\alpha k'_\alpha,$$

and a_0, $a_{0\alpha}$, and b are real valued. The derivatives of $g_K(x)$ are

$$(\partial/\partial x) g_K(x) = b + Cx + i \sum_{\alpha=1}^{A} \left(\sum_{j=-J}^{J} j a_{j\alpha} e^{ijk'_\alpha x} \right) k_\alpha,$$

$$(\partial^2/\partial x \partial x') g_K(x) = -\sum_{\alpha=1}^{A} \left(a_{0\alpha} + \sum_{j=-J}^{J} j^2 a_{j\alpha} e^{ijk'_\alpha x} \right) k_\alpha k'_\alpha.$$

Recall that A and J are functions of K as described in the previous section.

It will be necessary to extend Theorem 1 slightly so that it meshes with asymptotic theory in subsequent sections. The problem centers in the sequence of independent variables $\{x_t\}$. Following Malinvaud (1970) and Gallant and Holly (1980) a reasonable assumption in regression situations is that the limit of an average of the form $(1/n)\sum_{t=1}^{n} f(x_t)$ can be computed as

$$\lim_{n \to \infty} (1/n) \sum_{t=1}^{n} f(x_t) = \int_{\mathscr{X}} f(x) \mathrm{d}\mu(x),$$

where $\mu(x)$ is a probability distribution giving the relative frequency with which values of the independent variable occur as sample size tends to infinity. (The computation is assumed to be valid for all continuous $f(x)$ that are dominated by some fixed μ integrable function $b(x)$ [Gallant and

Holly (1980)].) Consequently, a modification of Theorem 1 to cover a measure of distance based on μ is required.

Definition. If $1 \leqq p < \infty$, $f \in W^{m,p}(\mathscr{X})$ and μ is a probability distribution defined on $\mathscr{X}$ with a bounded probability density function, let

$$\|f\|_{m,p,\mu} = \left(\sum_{|\lambda|^* \leqq m} \int_{\mathscr{X}} |D^{\lambda} f|^p \mathrm{d}\mu \right)^{1/p}.$$

If $p = \infty$ and $f \in W^{m,\infty}(\mathscr{X})$, let

$$\|f\|_{m,\infty,\mu} = \sum_{|\lambda|^* \leqq m} \sup_{x \in \mathscr{X}} |D^{\lambda} f|.$$

Corollary. Let the probability distribution μ defined on $\mathscr{X}$ have a bounded probability density function. Let $m \geqq 2$ and let the true indirect utility function $g^* \in W^{m,p}(\mathscr{X})$ for some $p \geqq 1$. Then there is a sequence of coefficients such that

$$\lim_{K \to \infty} \|g^* - g_K\|_{m-1,p,\mu} = 0,$$

for all p with $f \in W^{p,m}(\mathscr{X})$, $1 \leqq p \leqq \infty$.

The standard definition of a flexible function form is that it can provide a second-order approximation to an arbitrary twice differentiable $g^*(x)$ at any given point x^* [Diewert (1974)]. One can see by inspecting the form of C that this definition is satisfied by the Fourier flexible form for A large enough. However, no useful statistical properties flow from this definition. As White (1980, sect. 2) points out, when parameters are estimated by regression methods the estimated flexible form need not provide a second-order approximation to $g^*(x)$ at any point in $\mathscr{X}$ and even if it did, the point of approximation is not known. As his examples show, we are not speaking of small discrepancies, the errors can be quite large. We argue that a flexible functional form ought to meet a higher standard: Given a region $\mathscr{X}$ of the user's choosing, given $g^*(x)$ thrice continuously differentiable on an open set containing $\bar{\mathscr{X}}$, and given $\varepsilon > 0$ of the user's choosing, a flexible form g_X ought to be able to satisfy

$$\int_{\mathscr{X}} |D^{\lambda} g^* - D^{\lambda} g_K|^p \mathrm{d}\mu < \varepsilon, \qquad \text{all} \quad 1 \leqq p < \infty,$$

and

$$|D^{\lambda} g^*(x) - D^{\lambda} g_K(x)| < \varepsilon, \qquad \text{all} \quad x \in \bar{\mathscr{X}},$$

for all partial derivatives D^λ up to and including the second order. In a later section we show that interesting statistical properties flow from this definition. But for now, consider the implications for the approximation of the elasticities of substitution.

The partial elasticity of substitution may be computed from an indirect utility function as

$$\sigma_{ij}=\frac{\left[\sum_k x_k g_k\right] g_{ij}}{g_i g_j}-\frac{\sum_k x_k g_{jk}}{g_j}-\frac{\sum_k x_k g_{ik}}{g_i}+\frac{\sum_m\sum_k x_k g_{km} x_m}{\sum_n x_n g_n},$$

where g_i and g_{ij} denote elements of $(\partial/\partial x)g(x)$ and $(\partial^2/\partial x\partial x')g(x)$, respectively [Diewert (1974)]. Let $\sigma^*_{ij}(x)$ correspond to the true indirect utility function and let $\sigma_{ijK}(x)$ correspond to a flexible form which satisfies our proposed definition. Now σ_{ij}, considered as a function g_i, g_{ij}, x_k, is uniformly continuous over any cube of the form $-a\leqq g_i\leqq -b<0$, $c\leqq g_{ij}\leqq d$, $0<e\leqq x_k\leqq f$. The true indirect utility function satisfies bounds of this sort for all $x\in\mathscr{X}$ whence so must $g_K(x)$ for some K (by choosing ε sufficiently small). Then it follows that given $\varepsilon>0$ of the users choosing there is a K with

$$|\sigma^*_{ij}(x)-\sigma_{ijK}(x)|<\varepsilon, \quad \text{all} \quad x\in\mathscr{X}.$$

Global approximation to within arbitrary accuracy of the elasticities of substitution appears to us to be a far more appealing property than equality at a single point.

The main propositions of the neoclassical theory of consumer demand may be summarized by the statement that the matrix, with typical entry

$$(\partial/\partial p_i)q_j(p,Y)+x_i(\partial/\partial y)q_j(p,Y),$$

is a symmetric, negative semi-definite matrix where $q(p,Y)$ is the consumer's (Marshallian) demand system [Varian (1978, sect. 3.7)]. If $(\partial/\partial x_i)g_K(x)<0$ for all $x\in\mathscr{X}$ (monotonicity) and $(\partial^2/\partial x\partial x')g_K(x)$ is positive semi-definite for all $x\in\mathscr{X}$ (convexity), and Roy's identity is used to obtain $q(p,Y)$ then this matrix will be symmetric and negative semi-definite for all $x\in\mathscr{X}$ [Diewert (1977)]. It is of interest, then, to be able to impose these conditions on g_K.

To impose convexity on g_K let the limit of summation J on j be an even number, say $2I$, and rewrite the Fourier indirect utility function as

$$g_K(x)=a_0+b'x+\sum_{\alpha=1}^{A}\mu_\alpha(k'_\alpha x),$$

where

$$\mu_\alpha(z) = -\tfrac{1}{2}a_{0\alpha}z^2 + \sum_{j=-2I}^{2I} a_{j\alpha}e^{ijz}.$$

Convexity is imposed by setting

$$a_{0\alpha} = -\sum_{s=-I}^{I} c_{s\alpha}\bar{c}_{s\alpha}, \qquad \alpha = 1, 2, \ldots, A,$$

$$a_{j\alpha} = (-1/j^2)\sum_{s=-I}^{I} c_{s\alpha}\bar{c}_{s-j,\alpha}, \qquad \alpha = 1, 2, \ldots, A, \quad j = 1, 2, \ldots, 2I,$$

where the free parameters satisfy

$$c_{j\alpha} = \bar{c}_{-j\alpha}, \qquad \alpha = 1, 2, \ldots, A, \quad j = 1, 2, \ldots, I,$$

$$c_{0\alpha} \geqq 0, \qquad \alpha = 1, 2, \ldots, A,$$

$$c_{j\alpha} = 0, \qquad \alpha = 1, 2, \ldots, A, \quad |j| > I.$$

To see this, observe that the Hessian of $g_K(x)$ is

$$(\partial^2/\partial x \partial x')g(x) = \sum_{\alpha=1}^{A} (d^2/dz^2)\mu_\alpha(k'_\alpha x)k_\alpha k'_\alpha,$$

and a sufficient condition for a positive semi-definite Hessian is that, for each α,

$$(d^2/dz^2)\mu_\alpha(z) \geqq 0, \qquad 0 \leqq z \leqq 2\pi.$$

But, under the restrictions,

$$\begin{aligned}(d^2/dz^2)\mu_\alpha(z) &= -a_{0\alpha} - \sum_{j=-2I}^{2I} a_{j\alpha}j^2e^{ijz} \\ &= \sum_{j=-2I}^{2I}\left(\sum_{s=-I}^{I} c_{s\alpha}\bar{c}_{s-j,\alpha}\right)e^{ijz} \\ &= \sum_{s=-I}^{I}\sum_{p=-2I-s}^{2I-s} c_{s\alpha}\bar{c}_{-p\alpha}e^{ipz+isz}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{s=-I}^{I} c_{s\alpha} e^{isz} \sum_{p=-I}^{I} \bar{c}_{-p\alpha} e^{ipz} \\
&= \left(\sum_{s=-I}^{I} c_{s\alpha} e^{isz} \right) \overline{\left(\sum_{p=-I}^{I} c_{-p\alpha} e^{-ipz} \right)} \\
&= \left(\sum_{s=-I}^{I} c_{s\alpha} e^{isz} \right) \overline{\left(\sum_{s=-I}^{I} c_{s\alpha} e^{isz} \right)} \\
&\geqq 0.
\end{aligned}
$$

The restriction is sufficient for $g_K(x)$ to be a convex function; it is not necessary save in the case when $A \leqq N$. In view of the fact that the restriction is only sufficient for convexity and that, in turn, convexity is only sufficient for consistency with the theory of demand in conjunction with monotonicity, one would not likely invoke convexity without testing it as a hypothesis. It is, however, a plausible restriction which reduces the number of parameters by about one-half when it can be invoked. As yet a convenient means of imposing monotonicity has not been found.

One might note in passing that when the Fourier flexible form is written as

$$
g_K(x) = a_0 + b'x + \sum_{\alpha=1}^{A} \mu_\alpha (k'_\alpha x),
$$

one sees that it is an additive indirect utility function not in prices per se but in price indexes $k'_\alpha x$; note that b can be decomposed as $\sum_{\alpha=1}^{A} \beta_\alpha k_\alpha$ for large enough A. Thus, the elements of a multi-index k_α can be thought of as the weights of a price index.

4. The Fourier expenditure system: An expenditure system with arbitrarily small average prediction bias

As noted earlier, a common method for obtaining an expenditure system for empirical work is to apply Roy's identity to a flexible functional form. If the Fourier flexible form is chosen the resulting expenditure system has a feature which distinguishes it from other flexible form expenditure systems. When estimated, it will approximate the true expenditure system to within an average prediction bias which may be made arbitrarily small by increasing the number of terms in the Fourier expansion. This claim is verified in this section.

Assume that the observed expenditures and normalized prices are generated according to the stochastic specification

$$y_t = f^*(x_t) + e_t, \qquad t = 1, 2, \ldots, n,$$

where

$$f^*(x) = \left[\sum_{i=1}^{N} x_i (\partial/\partial x_i) g^*(x) \right]^{-1} \begin{pmatrix} x_1 (\partial/\partial x_1) g^*(x) \\ x_2 (\partial/\partial x_2) g^*(x) \\ \vdots \\ x_{N-1} (\partial/\partial x_{N-1}) g^*(x) \end{pmatrix},$$

and

$$y = \begin{pmatrix} p_1 q_1 / Y \\ p_2 q_2 / Y \\ \vdots \\ p_{N-1} q_{N-1} / Y \end{pmatrix}$$

Note that y and f^* are $(N-1)$-dimensional; the expenditures on the Nth commodity are obtained from $1 - \sum_{i=1}^{N-1} y_i$ for the observed expenditure and from $1 - \sum_{i=1}^{N-1} f_i^*(x)$ for the predicted expenditure. Let the errors e_t be independent and identically distributed each with zero mean vector and variance–covariance matrix Σ.

The Fourier expenditure system, obtained by applying Roy's identity to $g_K(x)$, is

$$f_i(x, \theta) = \frac{x_i b_i + \sum_{\alpha=1}^{A} \left[-a_{0\alpha} x' k_\alpha + i \sum_{j=-J}^{J} j a_{j\alpha} e^{ijk'_\alpha x} \right] k_{i\alpha} x_i}{b'x + \sum_{\alpha=1}^{A} \left[-a_{0\alpha} x' k_\alpha + i \sum_{j=-J}^{J} j a_{j\alpha} e^{ijk'_\alpha x} \right] k'_\alpha x},$$

$$i = 1, 2, \ldots, N-1.$$

The system is homogeneous of degree zero in its parameters and is therefore not identified without normalization; setting $b_N = -1$ is a convenient normalization rule. Let

$$a_\alpha = (a_{0\alpha}, a_{1\alpha}, \ldots, a_{J\alpha})', \qquad \alpha = 1, 2, \ldots, A.$$

The parameters of the system are

$$\theta = (b_1, b_2, \ldots, b_{N-1}, \alpha'_1, \alpha'_2, \ldots, \alpha'_A)',$$

a vector of length $N-1+A(1+J)$. There are $N-1+A(1+2J)$ free parameters in the vector since the complex parameters have both a real and an imaginary part.

Let $\hat{S}$ be a random matrix of order $(N-1)\times(N-1)$ with $\sqrt{n}(\hat{S}-S^*)$ bounded in probability for some positive definite matrix S^*. The Seemingly Unrelated Nonlinear Regressions estimator of θ [Gallant (1975)] is $\hat{\theta}$ which maximizes

$$s_n(\theta)=(1/n)\sum_{t=1}^{n} s(y_t, x_t, \hat{S}, \theta),$$

where

$$s(y, x, S, \theta) = -\tfrac{1}{2}[y-f(x,\theta)]'S^{-1}[y-f(x,\theta)].$$

Subject to regularity conditions stated in Souza and Gallant (1979), $\hat{\theta}$ converges almost surely to that value θ^* which minimizes the average prediction bias

$$\mathscr{B}(\theta)=\int_{\mathscr{X}}[f^*(x)-f(x,\theta)]'(S^*)^{-1}[f^*(x)-f(x,\theta)]\,\mathrm{d}\mu(x).$$

The Corollary of Theorem 1 and Theorem 2, below, taken together imply that the average prediction bias $\mathscr{B}(\theta^*)$ of the Seemingly Unrelated Nonlinear Regressions estimator of the Fourier expenditure system may be made as small as desired by taking A and J sufficiently large.

Theorem 2. Let $g^* \in W^{2,2}(\mathscr{X})$ with continuous $(\partial/\partial x_i)g^*(x)<0$ for all $x\in\bar{\mathscr{X}}$ and let $f^*(x)$ be the corresponding expenditure system. Let μ be a probability distribution on $\mathscr{X}$ with a bounded probability density function. Let $g_K(x,\theta_1,\theta_2,\dots,\theta_K)$ be a sequence of functions with continuous partial derivatives in x and let $f_K(x,\theta_1,\theta_2,\dots,\theta_K)$ be the corresponding expenditure system. Let the triangular array

$$\begin{array}{l}
\theta^*_{1,1},\\
\theta^*_{1,2},\theta^*_{2,2},\\
\theta^*_{1,3},\theta^*_{2,3},\theta^*_{3,3},\\
\vdots
\end{array}$$

minimize

$$\mathscr{B}_K(\theta_1,\theta_2,\dots,\theta_K)=\int_{\mathscr{X}}(f^*-f_K)'(S^*)^{-1}(f^*-f_K)\,\mathrm{d}\mu,$$

for $K=1,2,\ldots$; *note* $\theta^*_{i,K}$ *need not equal* $\theta^*_{i,K+1}$. *If there exists a triangular array* $\{\bar\theta_{i,K}: i=1,2,\ldots,K;\, K=1,2,\ldots\}$ *such that* $\bar g_K(x)=g_K(x,\bar\theta_{1,K},\bar\theta_{2,K},\ldots,\bar\theta_{K,K})$ *satisfies* $\bar g_K\in W^{2,2}(\mathscr{X})$ *and*

$$\lim_{K\to\infty}\|g^*-\bar g_K\|_{1,2,\mu}=0,$$

then

$$\lim_{K\to\infty}\mathscr{B}(\theta^*_{1,K},\theta^*_{2,K},\ldots,\theta^*_{K,K})=0.$$

Proof. First it is shown that

$$\lim_{K\to\infty}\mathscr{B}(\bar\theta_{1,K},\bar\theta_{2,K},\ldots,\bar\theta_{K,K})=0.$$

By hypothesis, $(\partial/\partial x_i)g^*(x)\leqq\delta<0$ for all $x\in\bar{\mathscr{X}}$. Now the convention that a positive $(\partial/\partial x_i)\bar g_K(x)$ is to be set to zero when forming $\bar f_K(x)$ merely improves the approximation and it remains true that $\lim\|g^*-\bar g_K\|_{1,2,\mu}=0$ after the modification. Expenditure shares are bounded by one whence $|f_i^*(x)-\bar f_{i,K}(x)|\leqq 2$ for all $x\in\bar{\mathscr{X}}$. Further,

$$\lim\int_{\mathscr{X}}|x_i(\partial/\partial x_i)g^*(x)-x_i(\partial/\partial x_i)\bar g_K(x)|\,d\mu$$

$$\leqq\lim\left(\int_{\mathscr{X}}|x_i|^2 d\mu\right)^{\frac12}\left(\int_{\mathscr{X}}|(\partial/\partial x_i)g^*-(\partial/\partial x_i)\bar g_K|^2 d\mu\right)^{\frac12}$$

$$\leqq\lim(2\pi)\|g^*-\bar g_K\|_{1,2,\mu}=0.$$

Now $(\mathscr{X},\mu)$ is a finite measure space so that convergence in $\mathscr{L}_1(\mathscr{X},\mu)$ implies convergence in measure. Thus, $x_i(\partial/\partial x_i)\bar g_K(x)$ converges in measure to $x_i(\partial/\partial x_i)g^*(x)$ as $K\to\infty$ for $i=1,2,\ldots,N$. It follows immediately that the expenditure shares $\bar f_{iK}(x)$ converge in measure to $f_i^*(x)$. Since $|f_i^*-\bar f_{iK}|^2$ is dominated by 4, the dominated convergence theorem for convergence in measure implies

$$\lim\sum_{i=1}^{N-1}\int_{\mathscr{X}}|f_i^*-\bar f_{iK}|^2 d\mu=0.$$

Let δ be the largest eigenvalue of $(S^*)^{-1}$. Then

$$0\leqq\lim\mathscr{B}_K(\bar\theta_{1,K},\ldots,\bar\theta_{K,K})\leqq\lim\delta\sum_{i=1}^{N-1}\int_{\mathscr{X}}|f_i^*-\bar f_{iK}|^2 d\mu=0.$$

The theorem follows from the fact that $(\theta^*_{1,K},\ldots,\theta^*_{K,K})$ minimizes $\mathscr{B}_K(\theta_1,\ldots,\theta_K)$ whence

$$0 \leqq \lim \mathscr{B}_K(\theta^*_{1,K},\ldots,\theta^*_{K,K}) \leqq \lim \mathscr{B}_K(\bar{\theta}_{1,K},\ldots,\bar{\theta}_{K,K}) = 0. \quad \square$$

This is an extremely powerful result. Let us appeal to a time series analogy in order to make our point forcefully. If a stationary time series has a bounded spectral density which is bounded away from zero as well then this time series has an infinite autoregressive representation in mean square

$$\lim_{K\to\infty} \mathscr{E}\left|\sum_{k=0}^{K} a_k x_{t-k} - e_t\right|^2 = 0, \qquad t=0, \pm 1, \pm 2, \ldots .$$

That is, the autoregressive model exhausts the possibilities. It usually suffices to truncate at some K and fit

$$\sum_{k=0}^{K} a_k x_{t-k} = e_t$$

in applications. Granted that in practice one may be able to fit an autoregressive/moving average model which fits as well with fewer parameters but, in principle, there is no need to.

The analogy is perfect. Say one is willing to presume that the true indirect utility function is twice continuously differentiable over $\bar{\mathscr{X}}$. Then, quite simply, the Fourier expenditure system exhausts the possibilities. There is no need to consider anything else if one is willing to accept

$$\lim_{K\to\infty} \int_{\mathscr{X}} (f^* - f_K)'(S^*)^{-1}(f^* - f_K)\,\mathrm{d}\mu = 0$$

as the appropriate measure of distance. One should expect that it would suffice to truncate at some K and fit $f_K(x,\theta)$ in applications. Granted that in practice one may be able to use a CES, Translog, etc., and fit as well with fewer parameters but, in principle, there is no need to.

5. Computational considerations

In the computations it is more convenient to work with sine/cosine representations than with the exponential representations of the previous sections. Recall that q denotes an N-vector of commodities, p the N-vector of corresponding prices, Y income, x the income normalized prices $x=p/Y$, and y an $N-1$ vector of budget shares ($y_i = p_i q_i / Y$, $i=1,2,\ldots,N-1$). Recall also that the data have been scaled so that $0 < x_{il} \leqq x_i \leqq x_{iu} < 2\pi$ for $i = 1, 2, \ldots, N$. It is re-emphasized that the scaling step cannot be omitted.

The Fourier flexible form and its derivatives may be written as

$$g_K(x)$$

$$= u_0 + b'x + \tfrac{1}{2}x'Cx + \sum_{\alpha=1}^{A} \left\{ u_{0\alpha} + \sum_{j=1}^{J} [u_{j\alpha} \cos(jk'_\alpha x) - v_{j\alpha} \sin(jk'_\alpha x)] \right\},$$

$$(\partial/\partial x) g_K(x)$$

$$= b + Cx - 2 \sum_{\alpha=1}^{A} \sum_{j=1}^{J} j [u_{j\alpha} \sin(jk'_\alpha x) + v_{j\alpha} \cos(jk'_\alpha x)] k_\alpha,$$

$$(\partial^2/\partial x \partial x') g_K(x)$$

$$= - \sum_{\alpha=1}^{A} \left\{ u_{0\alpha} + 2 \sum_{j=1}^{J} j^2 [u_{j\alpha} \cos(jk'_\alpha x) - v_{j\alpha} \sin(jk'_\alpha x)] \right\} k_\alpha k'_\alpha,$$

where

$$C = - \sum_{\alpha=1}^{A} u_{0\alpha} k_\alpha k'_\alpha,$$

and u_0, $u_{0\alpha}$, $u_{j\alpha}$, $v_{j\alpha}$ are real valued. The correspondence with the notation of the previous sections is

$$a_{0\alpha} = u_{0\alpha}, \qquad \alpha = 1, 2, \ldots, A,$$

$$a_{j\alpha} = u_{j\alpha} + i v_{j\alpha}, \qquad \alpha = 1, 2, \ldots, A, \quad j = 1, 2, \ldots, J,$$

$$a_{-j\alpha} = u_{j\alpha} - i v_{j\alpha}, \qquad \alpha = 1, 2, \ldots, A, \quad j = 1, 2, \ldots, J.$$

When parameter estimates are in hand the quantities typically of interest in a demand study are obtained from these derivatives as follows. The ordinary (Marshallian) demand system is

$$q(p, Y) = [x'(\partial/\partial x) g_K(x)]^{-1} (\partial/\partial x) g_K(x).$$

Writing q for $q(p, Y)$, ∇g for $(\partial/\partial x) g_K(x)$, and $\nabla^2 g$ for $(\partial^2/\partial x \partial x') g_K(x)$ all evaluated at the point $x = p/Y$ we have

$$(\partial/\partial p')q(p, Y) = (p'\nabla g)^{-1}[\nabla^2 g - qx'\nabla^2 g - (x'\nabla g)qq'],$$

$$(\partial/\partial Y)q(p, Y) = -[(\partial/\partial p')q(p, Y)]x.$$

From these expressions and Slutsky's equation the derivatives of the compensated (Hicksian) demand curve are

$$(\partial/\partial p')h(p, u)\big|_{u=g(x)} = [(\partial/\partial p')q(p, Y)](I - xq').$$

Let

$$P = \mathrm{diag}(p_1, p_2, \ldots, p_N), \qquad Q = \mathrm{diag}(q_1, q_2, \ldots, q_N).$$

The uncompensated price elasticities are

$$\eta_M = Q^{-1}[(\partial/\partial p')q(p, Y)]P,$$

and the compensated price elasticities are

$$\eta_H = Q^{-1}[(\partial/\partial p')q(p, Y)](I - xq')P,$$

where in each case the row index indexes quantities and the column index indexes prices. The income elasticities are

$$\eta_Y = -Q^{-1}[(\partial/\partial p')q(p, Y)]p.$$

It remains to estimate the parameters of $g_K(x)$ in order to use these formulas.

The parameters of the Fourier flexible form are estimated by fitting the Fourier expenditure system to budget share data. In computations, it is advantageous to recognize that each expenditure share is actually the ratio of two linear functions of the parameters. To see this, rewrite the Fourier expenditure system in terms of sines and cosines

$$f_i(x, \theta) =$$

$$\left(x_i b_i - \sum_{\alpha=1}^{A} \left\{u_{0\alpha} x' k_\alpha + 2 \sum_{j=1}^{J} j[u_{j\alpha} \sin(jk'_\alpha x) + v_{j\alpha} \cos(jk'_\alpha x)]\right\} k_{i\alpha} x_i\right) \Big/$$

$$\left(b'x - \sum_{\alpha=1}^{A} \left\{u_{0\alpha} x' k_\alpha + 2 \sum_{j=1}^{J} j[u_{j\alpha} \sin(jk'_\alpha x) + v_{j\alpha} \cos(jk'_\alpha x)]\right\} k'_\alpha x\right),$$

where $i = 1, 2, \ldots, N-1$ and $b_N = -1$. A convenient arrangement of the parameters is obtained by setting

$$\theta_{(0)} = (b_1, b_2, \ldots, b_{N-1})',$$

$$\theta_{(\alpha)} = (u_{0\alpha}, u_{1\alpha}, v_{1\alpha}, u_{2\alpha}, v_{2\alpha}, \ldots, u_{J\alpha}, v_{J\alpha})',$$

and

$$\theta = (\theta'_{(0)}, \theta'_{(1)}, \theta'_{(2)}, \ldots, \theta'_{(A)})',$$

which is a vector of length $N-1+A(1+2J)$. For $i = 1, 2, \ldots, N-1$, let

$$N_{i0} = \text{an } N-1 \text{ vector with } x_i \text{ in the } i\text{th position and zeroes elsewhere},$$

and for $i = N$, let

$$N_{N0} = \text{the zero vector of length } N-1.$$

For $i = 1, 2, \ldots, N$, let

$$N'_{i\alpha} = -(k_{i\alpha} x_i)[x' k_\alpha, 2\sin(k'_\alpha x), 2\cos(k'_\alpha x), \ldots,$$
$$2J\sin(Jk'_\alpha x), 2J\cos(Jk'_\alpha x)],$$

and

$$N'_i = (N'_{i0}, N'_{i1}, N'_{i2}, \ldots, N'_{iA}).$$

Note that N'_i is a vector of length $N-1+A(1+2J)$.

Let

$$D' = \sum_{i=1}^{N} N'_i.$$

Then each expenditure share is given by

$$f_i(x, \theta) = N'_i \theta / (-x_N + D'\theta), \qquad i = 1, 2, \ldots, N-1,$$

and the expenditure system is given by

$$f(x, \theta) = (-x_N + D'\theta)^{-1} N'\theta,$$

where

$$N' = \begin{pmatrix} N_1' \\ N_2' \\ \vdots \\ N_{N-1}' \end{pmatrix} \qquad (N-1) \times [N-1+A(1+2J)].$$

Fitting the system by Seemingly Unrelated Nonlinear Regressions [Gallant (1975)] is straightforward but three features deserve comment. The first is that analytic derivatives are easily computed as

$$(\partial/\partial\theta')f(x,\theta) = (-x_N + D'\theta)^{-1} N' - (-x_N + D'\theta)^{-2} N'\theta D',$$

so that one should prefer to use a nonlinear optimization algorithm which uses analytic derivatives to one which estimates them numerically. The second is that one can exploit the fact that the Fourier expenditure system is the ratio of two linear functions to obtain starting values for the optimizations. The third is that there can be linear dependencies of the form $N_1'\delta = N_2'\delta = \ldots = N_N'\delta = D'\delta = 0$ which hold for some fixed δ at every data point in the sample. These last two points need further discussion.

Neglecting errors, we have that

$$y_i = N_i'\theta/(-x_N + D'\theta), \qquad i = 1, 2, \ldots, N-1,$$

which may be rewritten as

$$x_N y_i = (y_i D' - N_i')\theta, \qquad i = 1, 2, \ldots, N-1.$$

This latter may be fit by Seemingly Unrelated (linear) Regressions. (It would be interesting to study the properties of Three Stage Least Squares estimators for this model.) Good starting values may be obtained by fitting this linear system by Seemingly Unrelated Regressions with across equation constraints imposed and with the variance–covariance matrix set to the identity matrix. In other words, fit the system by Ordinary Least Squares with across equation constraints imposed and use the resulting parameter estimates as starting values for the nonlinear optimizations.

The problem of a linear dependency which holds for each data point,

$$N_1'\delta = N_2'\delta = \ldots = N_N'\delta = D'\delta = 0, \qquad x = x_t, \quad t = 1, 2, \ldots, n,$$

is handled the same as in linear regression. Choose a j for which δ_j is a non-zero element of δ and delete the jth element of $N_1', N_2', \ldots, N_N', D'$, and θ to remedy the problem. This is equivalent to imposing the constraint that $\theta_j = 0$. If the problem persists, repeat and delete another column.

Bear in mind that the parameters of the Fourier flexible form are of no intrinsic interest in and of themselves. One is attempting to estimate the surfaces $(\partial/\partial x')g^*(x)$ and $(\partial^2/\partial x\partial x')g^*(x)$ accurately, not the parameters of $g_K(x)$. If some parameters are indeterminate in a given data set it is of no consequence. Carrying this observation further, one may try various values of J and A and deletions of some k_α using hypothesis testing as an aid in an attempt to obtain an adequate fit with as few parameters as possible.

Convexity may be imposed on the Fourier flexible form as was shown previously. For computations the convexity restriction should be expressed as a functional dependency,

$$\theta = g(\rho),$$

and the Jacobian of the transformation,

$$G(\rho) = (\partial/\partial\rho')g(\rho),$$

is required [Gallant (1975, sect. 2)]. These two functions defy description despite the fact that it is easy to code them at sight using the formulas of section 3. The solution to this dilemma is simply to publish the code itself and skip the formulas for $g(\rho)$ and $G(\rho)$. Accordingly, the appendix gives the (IBM) FORTRAN code. Referring to the appendix, the index J of the Fourier flexible form must be even. On entry to $CONVEX$, $RHO=\rho$, $NN=N$, $IAA=A$, and $II=J/2$; LR and LG give the dimensions of RHO, G, and $DELG$ as declared in the calling program — $RHO(LR)$, $G(LG)$, $DELG(LG, LR)$. On return, $G=g(\rho)$ and $DELG=G(\rho)$. The arrangement of the elements of ρ is

$$\rho(0) = (b_1, b_2, \ldots, b_{N-1})',$$

$$\rho_{(\alpha)} = (p_{0\alpha}, p_{1\alpha}, q_{1\alpha}, p_{2\alpha}, q_{2\alpha}, \ldots, p_{I\alpha}, q_{I\alpha})',$$

and

$$\rho = (\rho'_{(0)}, \rho'_{(1)}, \rho'_{(2)}, \ldots, \rho'_{(A)})',$$

with $I=J/2$, a vector of length $N-I+A(1+2I)$. The correspondence with the notation of section 3 is

$$c_{0\alpha} = p_{0\alpha}, \qquad \alpha = 1, 2, \ldots, A,$$

$$c_{s\alpha} = p_{s\alpha} + iq_{s\alpha}, \qquad \alpha = 1, 2, \ldots, A, \quad s = 1, 2, \ldots, I,$$

$$c_{-s\alpha} = p_{s\alpha} - iq_{s\alpha}, \qquad \alpha = 1, 2, \ldots, A, \quad s = 1, 2, \ldots, I.$$

Starting values for the constrained optimizations are obtained by finding ρ_0 to minimize $[\hat{\theta}-g(\rho)]'[\hat{\theta}-g(\rho)]$ using Ordinary Nonlinear Least Squares where $\hat{\theta}$ is the unconstrained estimate of θ.

By way of illustration, for

$$(\hat{S})^{-1}=\begin{pmatrix}47086.7 & 22362.8\\ 22362.8 & 20422.3\end{pmatrix},$$

the data of the appendix scaled so that

$$\max\{x_{it}: t=1,2,\ldots,n\}=6, \quad i=1,2,3,$$

the multi-indexes,

$$k_\alpha=\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix},$$

$A=4$, $J=2$, $i=1$ for durables, $i=2$ for non-durables, and $i=3$ for services, one obtains the results reported in table 2. There is no rounding in $(\hat{S})^{-1}$ or the data of the appendix so that one should be able to reproduce these results exactly. With the unconstrained fit monotonicity holds at every data point and convexity fails at 29 of the 44 data points. Monotonicity holds at every data point when convexity is imposed. A test for convexity is to reject if

$$\begin{aligned}L&=-2n\{s_n[g(\tilde{\rho})]-s_n(\hat{\theta})\}\\ &=-(2)(44)[-1.4179+0.84330]=50.564\end{aligned}$$

exceeds a chi-square critical point with 8 degrees freedom [Souza and Gallant (1979)]. Convexity is rejected at the 1% level. By way of comparison, a Translog fit has $s_n(\hat{\lambda})=-2.4146$.

6. A test of the theory of demand

There have been many studies that have tested the theory of demand statistically. A concise account of these studies is found in the introduction of Christensen, Jorgenson and Lau (1975). Setting aside the well-known problems with the use of aggregate data for such tests, there remains the problem of bias induced by the choice of a functional form for the expenditure system. Rejection of the null hypothesis implies rejection of

Table 2

Fourier fit to the data of Christensen, Jorgenson and Lau (1975).

Unconstrained $\hat{\theta}$		Convexity constrained $\tilde{\rho}$	
b_1	-0.56604	b_1	-0.40262
b_2	-0.60211	b_2	-0.95697
u_{01}	-0.0088414	p_{01}	0.13908
u_{11}	0.016430	p_{11}	-0.018227
v_{11}	0.00093173	q_{11}	-0.083372
u_{21}	0.00065799		
v_{21}	-0.0041358		
u_{02}	0.18283	p_{02}	0.069018
u_{12}	0.12537	p_{12}	-0.029077
v_{12}	-0.27185	q_{12}	0.18589
u_{22}	0.0090124		
v_{22}	-0.0090039		
u_{03}	0.054104	p_{03}	0.28863
u_{13}	0.10203	p_{13}	-0.032041
v_{13}	-0.094070	q_{13}	-0.076370
u_{23}	-0.014374		
v_{23}	-0.014726		
u_{04}	-0.019725	p_{04}	0.055923
u_{14}	-0.028701	p_{14}	0.054243
v_{14}	-0.0054340	q_{14}	0.022656
u_{24}	-0.0037969		
v_{24}	-0.011059		
$s_n(\hat{\theta})$	-0.84330	$s_n[g(\tilde{\rho})]$	-1.4179

either the choice of a functional form or rejection of the theory of demand or both. The implication of a significant test statistic is unclear; rejection of the theory of demand is not necessarily implied. Of these studies, the most interesting is that of Christensen, Jorgenson and Lau (1975) because it attempts to evade this problem by using a flexible functional form in an attempt to reduce model specification bias. They reject the theory of demand. As noted, the data is aggregate so that the theory is not seriously threatened but, nonetheless, it is of interest to repeat these tests with the Fourier flexible form in an attempt to demonstrate that their significant test statistics can be attributed to specification bias.

Following along the same lines as Christensen, Jorgenson and Lau, a test of the theory of demand may be constructed as follows. Let

$$f(x,\theta_1,\theta_2,\ldots,\theta_{N-1}) = \begin{pmatrix} f_1(x,\theta_1) \\ f_2(x,\theta_2) \\ \vdots \\ f_{N-1}(x,\theta_{N-1}) \end{pmatrix},$$

where $f_i(x,\theta)$ is the Fourier expenditure share of the ith commodity. Note that if

$$\theta = \theta_1 = \theta_2 = \ldots = \theta_{N-1},$$

then

$$f(x, \theta_1, \theta_2, \ldots, \theta_{N-1}) = f(x, \theta).$$

Following previous usage, the restriction

$$\theta_1 = \theta_2 = \ldots = \theta_{N-1}$$

is termed the hypothesis of equality and symmetry here.

A test statistic for the hypothesis,

$$\theta_1 = \theta_2 = \ldots = \theta_{N-1},$$

may be constructed from the Seemingly Unrelated Nonlinear Regression estimator. Let $\hat{S}$ be the random matrix of section 4. The unconstrained estimator is $(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_{N-1})$ which maximizes

$$s_n(\theta_1, \theta_2, \ldots, \theta_{N-1}) = (1/n) \sum_{t=1}^{n} s(y_t, x_t, \hat{S}, \theta_1, \theta_2, \ldots, \theta_{N-1}),$$

where

$$s(y, x, S, \theta_1, \theta_2, \ldots, \theta_{N-1}) = -\tfrac{1}{2}[y - f(x, \theta_1, \theta_2, \ldots, \theta_{N-1})]' S^{-1} \times [y - f(x, \theta_1, \theta_2, \ldots, \theta_{N-1})].$$

The constrained estimator is $\hat{\theta}$ which maximizes $s_n(\theta)$ as defined in section 4. The test statistic for equality and symmetry is

$$L = -2n[s_n(\hat{\theta}) - s_n(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_{N-1})].$$

One rejects the null hypothesis when L exceeds the upper $\alpha \times 100$ percentage point of a chi-square random variable with $(N-2)(N-1+A(1+2J))$ degrees of freedom.

The Fourier expenditure system was fitted to the data of Christensen, Jorgenson and Lau (1975). These data were obtained from Tibibian (1980) and are given in the appendix. The multi-indices employed were

$$k_\alpha = \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix},$$

$A=7$ and $J=1$. These choices result in an estimate of Σ which is one-half the magnitude of Σ estimated from Translog nonlinear least squares residuals; equality and symmetry constraints having been imposed in both cases. Fourier scaling as discussed in section 3 is used to estimate Σ with the Fourier expenditure system; with the Translog, prices are scaled so that each series $x_i = p_{i/Y}$ has a mean of one. There is a singularity with these data which is accommodated by fixing a_{07} at zero as discussed in the previous section; the degrees of freedom of the test statistic are 22.

The computed value of the test statistic for equality and symmetry is

$$L = -2(44)(-0.89053 + 0.12783) = 67.117,$$

which is significant at a level of 1%. After a correction for serial correlation the statistic is

$$L = -2(44)(-0.91160 + 0.18207) = 64.198,$$

which is significant at a level of 1%.

One concludes that the rejection of the theory of demand reported in Christensen, Jorgenson and Lau cannot be shown to result from a bias in favor of rejection induced by a choice of the Translog functional form. One is not permitted to conclude that the Translog expenditure system is free of bias from these tests, only that a bias has not been demonstrated in this instance with these data. In fact, a test of the theory of demand against an unspecified alternative is not a convenient setting in which to deal with the question of bias. The number of parameters is large, computations are therefore extremely costly, and there is no convincing means to parameterize the alternative. The Translog test for an additive indirect utility function is a much more tractable setting for an examination of bias. In the next section, a substantial bias is discovered.

7. The power curve of the translog additivity test

If the true indirect utility function is additive then additivity may be imposed on the Fourier flexible form without affecting the ability of the Fourier expenditure system to approximate the true expenditure system. This fact allows the determination of an analytic expression for the power curve of the Translog test of additivity in terms of the parameter θ^* of the Fourier expenditure system. This power curve turns out to be shallow and biased in favor of rejection. The details follow.

Suppose that the indirect utility function is additive,

$$g^*(x)=F\left[\sum_{\alpha=1}^{N} g_\alpha^*(x_\alpha)\right].$$

The same expenditure system will result regardless of the choice of strictly increasing function F so it is impossible to distinguish between additivity and explicit additivity,

$$g^*(x)=\sum_{\alpha=1}^{N} g_\alpha^*(x_\alpha),$$

from expenditure data. Therefore, only the stronger hypothesis of explicit additivity is considered here. (The same is, of course, true of homotheticity and homogeneity; the same expenditure system results in either case.)

An explicitly additive form of the Fourier indirect utility function results when A is set to $A=N$; recall that the first N multi-indices k_α are the elementary vectors. With $A=N$ the Fourier indirect utility function may be rewritten as

$$g(x)=\sum_{\alpha=1}^{N}\left\{a_\alpha+b_\alpha x_\alpha-\tfrac{1}{2}a_{0\alpha}x_\alpha^2+\sum_{j=-J}^{J} a_{j\alpha}e^{ijx_\alpha}\right\}=\sum_{\alpha=1}^{N} g_\alpha(x).$$

The Corollary of Theorem 1 may be applied successively to conclude that there are coefficients such that

$$\lim_{J\to\infty} ||g_\alpha^*-g_\alpha||_{1,2,\mu}=0, \qquad \alpha=1,2,\ldots,N.$$

By the triangle inequality,

$$\lim_{J\to\infty} ||g^*-g||_{1,2,\mu}\leq \lim_{J\to\infty}\sum_{\alpha=1}^{N} ||g_\alpha^*-g_\alpha||_{1,2,\mu}=0.$$

Thus, the hypotheses of Theorem 2 are satisfied and the Fourier system is seen to retain the ability to approximate the true utility system with arbitrarily small average prediction bias.

The Translog indirect utility function yields expenditure shares

$$f_i(x,\lambda)=\left(\alpha_i+\sum_{j=1}^{N}\beta_{ij}\ln x_j\right)\Big/\left(-1+\sum_{j=1}^{N}\beta_{Mj}\ln x_j\right),$$

$$i=1,2,\ldots,N-1.$$

There are $N-1+N(N+1)/2$ free parameters,

$$\lambda=(a_1,a_2,\ldots,a_{N-1},\beta_{11},\beta_{12},\beta_{22},\beta_{13},\beta_{23},\beta_{33},\ldots,\beta_{1N},\beta_{2N},\ldots,\beta_{NN})'.$$

The dependent parameters are

$$\alpha_N=-1-\sum_{j=1}^{N-1}a_j,$$

$$\beta_{ji}=\beta_{ij}\quad\text{for}\quad i<j,$$

$$\beta_{Mj}=\sum_{i=1}^{N}\beta_{ij}.$$

The hypothesis of explicit additivity for the Translog expenditure system takes the form $\beta_{ij}=0$ for $i\neq j$. This hypothesis may be represented as

$$h(\lambda)=H\lambda=0,$$

where H is of order $[N(N-1)/2]\times[N-1+N(N+1)/2]$ and is obtained from the identity of order $N-1+N(N+1)/2$ by deleting the $N-1+N$ rows corresponding to $a_1,\ldots,a_{N-1}$ and $\beta_{11},\ldots,\beta_{NN}$ of λ.

As before, let $\sqrt{n}(\hat{S}-S^*)$ be bounded in probability. The Seemingly Unrelated Nonlinear Regressions estimator of λ is $\hat{\lambda}$ which maximizes

$$s_n(\lambda)=(1/n)\sum_{t=1}^{n}s(y_t,x_t,\hat{S},\lambda),$$

where

$$s(y,x,S,\lambda)=-\tfrac{1}{2}[y-f(x,\lambda)]'S^{-1}[y-f(x,\lambda)].$$

Then, as for $\hat{\theta}$, $\hat{\lambda}$ converges almost surely to that value λ° which minimizes

$$\mathscr{B}(\lambda)=\int_{\mathscr{X}}[f^*(x)-f(x,\lambda)]'(S^*)^{-1}[f^*(x)-f(x,\lambda)]\,\mathrm{d}\mu.$$

To approximate $\mathscr{B}(\lambda)$, one may use $\mathscr{B}(\lambda, \theta^*)$ where

$$\mathscr{B}(\lambda, \theta) = \int_{\mathscr{X}} [f(x, \theta) - f(x, \lambda)]'(S^*)^{-1}[f(x, \theta) - f(x, \lambda)]\, \mathrm{d}\mu.$$

The argument runs as follows. Note that

$$|\mathscr{B}(\lambda) - \mathscr{B}(\lambda, \theta^*)| \leqq \mathscr{B}(\theta^*) + 2\mathscr{B}^{\frac{1}{2}}(\theta^*)\mathscr{B}^{\frac{1}{2}}(\lambda),$$

and $\mathscr{B}(\lambda)$ is bounded over Λ by $\delta(N-1)(2)^2$ where δ is the largest eigenvalue of $(S^*)^{-1}$. As seen in the previous section, $\mathscr{B}(\theta^*)$ may be made arbitrarily small by taking A and J sufficiently large independently of the value of $\lambda \in \Lambda$. Thus, λ° can be computed as that value of λ which minimizes $\mathscr{B}(\lambda, \theta^*)$ and the error of approximation may be made arbitrarily small by taking A and J sufficiently large.

If one assumes that the random vectors $v_t = (e_t', x_t')'$, $t = 1, 2, \ldots$, are independently and identically distributed and does not condition on the sequence x_t, $t = 1, 2, \ldots$, then the Wald test and the Lagrange multiplier test for the hypothesis

$$h(\lambda^\circ) = 0,$$

are distributed asymptotically as non-central chi-squared random variables each with $N(N-1)/2$ degrees of freedom [Souza and Gallant (1979)], the non-centrality parameter is

$$\alpha^\circ = n\lambda^{\circ\prime} H'(HV^\circ H')^{-1} H\lambda^\circ/2,$$

where

$$V^\circ = (\mathscr{J}^\circ)^{-1}\mathscr{I}^\circ(\mathscr{J}^\circ)^{-1},$$

$$\mathscr{I}^\circ = (1/n)\sum_{t=1}^{n} [(\partial/\partial\lambda')f(x_t, \lambda^\circ)]'(S^*)^{-1}$$

$$\times [\Sigma + \delta(x_t, \lambda^\circ, \theta^*)\delta'(x_t, \lambda^\circ, \theta^*)](S^*)^{-1}[(\partial/\partial\lambda')f(x_t, \lambda^\circ)],$$

$$\mathscr{J}^\circ = (1/n)\sum_{t=1}^{n} [(\partial/\partial\lambda')f(x_t, \lambda^\circ)]'(S^*)^{-1}[(\partial/\partial\lambda')f(x_t, \lambda^\circ)]$$

$$-(1/n)\sum_{t=1}^{n}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} \delta_i(x_t, \lambda^\circ, \theta^*)S^{*ij}(\partial^2/\partial\lambda\partial\lambda')f_j(x_t, \lambda^\circ),$$

$$\delta(x_t, \lambda^\circ, \theta^*) = f(x_t, \theta^*) - f(x_t, \lambda^\circ),$$

and S^{*ij} denotes the elements of S^{*-1}. The asymptotic non-null distribution of the analog of the likelihood ratio test is also given in Souza and Gallant (1979), but it does not have a tabled null distribution in this case. Thus, it is of no practical importance. The arguments supporting the substitution of $f(x,\theta^*)$ for $f^*(x)$ in these formulas are similar to those supporting the use of $\mathscr{B}(\lambda,\theta^*)$ for $\mathscr{B}(\lambda)$.

The choice of S^* for use in these formulas presents somewhat of a problem. The simplest choice is to take $S^*=\Sigma$ which is equivalent to assuming that either Σ is known or that it may be estimated with negligible bias. It is, of course, always possible to obtain Σ with negligible bias, one need only fit a polynomial in x of suitably high degree to each expenditure share y_i and compute $\hat{S}$ from the residuals [Gallant (1979)]. The alternative approach is to assume that $\hat{S}$ was computed from translog residuals and account for the resulting bias. For example, one might compute $\hat{S}$ from nonlinear least squares residuals subject to the equality and symmetry across equation constraint. This is equivalent to taking $\hat{S}=I$ in the Seemingly Unrelated Nonlinear Regressions method whence

$$S^*=\Sigma+\int_{\mathscr{X}}\delta(x,\lambda^{\circ\circ},\theta^*)\delta'(x,\lambda^{\circ\circ},\theta^*)\mathrm{d}\mu,$$

where $\lambda^{\circ\circ}$ minimizes

$$\int_{\mathscr{X}}\delta'(x,\lambda,\theta^*)\delta(x,\lambda,\theta^*)\mathrm{d}\mu.$$

Another possibility is to compute $\hat{S}$ from unconstrained Translog residuals. In view of the variety of choices available for S^* and the additional complexity entailed, it seems that the simplest choice $S^*=\Sigma$ contributes more to understanding. From the data of the appendix, a variance–covariance matrix Σ was computed from Fourier expenditure system residuals with equality and symmetry imposed on the fit; $A=7$ and $J=1$. This variance–covariance matrix was rescaled upward by a factor of two.

A smooth transition between the extremes of additivity and its absence was obtained as follows. The parameter θ^* was computed by fitting the Fourier expenditure system to the data of the appendix by Seemingly Unrelated Nonlinear Regressions with this choice of Σ, with equality and symmetry imposed, and with the constraint

$$\left(\sum_{\alpha=4}^{7}\sum_{j=-1}^{1}|a_{j\alpha}|^2\right)^{\frac{1}{2}}=K$$

imposed. The choice $K=0$ yields the null case. The remaining lines of table 3 correspond to increasingly larger values of K and the last line corresponds to an unconstrained fit. These parameter choices are realistic in that they yield expenditure shares in accord with the expenditure shares in the data of the appendix as revealed by visual inspection of plots of observed and predicted shares against time.

The Translog test of explicit additivity (with equality and symmetry as a maintained hypothesis) is seriously flawed as seen in table 3. The actual size of the test is much larger than the nominal significance level of 0.010 and the power curve is relatively flat compared to the power of a test based on the Fourier expenditure system. The Translog power curve does increase locally, as one might expect, but it falls off again as departures from the null case become more extreme.

Table 3

Tests for an additive indirect utility function.

	Fourier		Translog	
K	Non-centrality	Power	Non-centrality	Power
0.0	0.0	0.010	8.9439	0.872
0.00046	0.0011935	0.010	8.9919	0.874
0.0021	0.029616	0.011	9.2014	0.884
0.0091	0.63795	0.023	10.287	0.924
0.033	4.6689	0.260	14.268	0.987
0.059	7.8947	0.552	15.710	0.993
0.084	82.875	1.000	13.875	0.984
Unconstrained	328.61	1.000	10.230	0.922

An interesting observation follows from the results of this section. Suppose that the consumers' indirect utility function were truly additive in some set of price indexes $w'_\beta x$, $\beta = 1, 2, \ldots, B$. Then the indirect utility function

$$g_J(x) = a_0 + \sum_{\beta=1}^{B} v_\beta(w'_\beta x),$$

where

$$v_\beta(z) = b_\beta z - \tfrac{1}{2} a_{0\beta} z^2 + \sum_{j=J}^{J} a_{j\beta} e^{ijz},$$

would have all the properties of the Fourier flexible form. The role of the multi-indexes k_α in the Fourier flexible form is to add more and more price indexes $k'_\alpha x$ so that eventually all based are covered, so to speak. Each index is then expanded in a univariate Fourier series augmented by a quadratic. The annoyance associated with this intuitive view of the Fourier flexible form is that some of the k_α have negative entries whereas one usually thinks of price indexes as having positive entries. Oddly enough however, in our specification searches those k_α with negative entries were discarded in favor of others with positive entries on the basis of reduction in residual sum of squares.

Appendix

Table A.1

Convexity constraint.

```
      SUBROUTINE CONVEX(RHO,NN,IAA,II,LR,LG,G,DELG)
      IMPLICIT REAL*8 (A-H,O-Z,$)
      REAL*8 RHO(LR),G(LG),DELG(LG,LR),RECS,IMCS,RECSJ,IMCSJ,RETJ,IMTJ
C     CHECK FOR ADEQUATE STORAGE SPACE.
      IF(LR.LT.NN-1+IAA*(1+2*II)) RETURN
      IF(LG.LT.NN-1+IAA*(1+4*II)) RETURN
C     CLEAR ACCUMULATORS.
      DO 10 I=1,LG
      G(I)=0.D0
      DO 10 J=1,LR
10    DELG(I,J)=0.D0
C     COPY B FROM RHO TO G.  THE NORTHWEST CORNER OF DELG IS THE
C     IDENTITY.
      LIM=NN-1
      DO 20 I=1,LIM
      G(I)=RHO(I)
20    DELG(I,I)=1.D0
C     LOOP 40 STEPS THROUGH THE ROWS OF G AND DELG.
C     IA INDEXES A_J_ALPHA, C_S_ALPHA, AND C_S-J_ALPHA WHERE ALPHA=IA.
      DO 40 IA=1,IAA
C     IRA0 IS THE STORAGE LOCATION OF C_0_ALPHA IN THE VECTOR RHO.
C     ITA0 IS THE STORAGE LOCATION OF A_0_ALPHA IN THE VECTOR G.
      IRA0=NN+(1+2*II)*(IA-1)
      ITA0=NN+(1+4*II)*(IA-1)
      LIM=2*II+1
      DO 40 L=1,LIM
C     J INDEXES A_J_ALPHA.
      J=L-1
C     A_J_ALPHA=COEF*SUM(C_S_ALPHA*CONJ(C_S-J_ALPHA):S=0,+-1,...,+-II).
      COEF=-1.D0
      IF(J.GT.0) COEF=-1.D0/DFLOAT(J**2)
C     LOOP 30 IS THE ACCUMULATION LOOP OVER THE INDEX S.
      DO 30 K=1,LIM
C     IS INDEXES C_S_ALPHA.  ISJ INDEXES C_S-J_ALPHA.
      IS=K-II-1
      ISJ=IS-J
      IF(IABS(ISJ).GT.II) GO TO 30
C     IRECS IS THE LOCATION OF THE REAL PART OF C_S_ALPHA IN RHO.
C     IIMCS IS THE LOCATION OF THE IMAGINARY PART OF C_S_ALPHA.
C     IRECSJ IS THE LOCATION OF THE REAL PART OF C_S-J_ALPHA.
C     IIMCSJ IS THE LOCATION OF THE IMAGINARY PART OF C_S-J_ALPHA.
      IRECS=IRA0
      IF(IS.NE.0) IRECS=IRA0+2*IABS(IS)-1
      IF(IS.NE.0) IIMCS=IRECS+1
      IRECSJ=IRA0
      IF(ISJ.NE.0) IRECSJ=IRA0+2*IABS(ISJ)-1
      IF(ISJ.NE.0) IIMCSJ=IRECSJ+1
```

Table A.1—*continued*

```
C     COMPUTE REAL AND IMAGINARY PARTS OF C_S_ALPHA AND C_S-J_ALPHA.
      RECS=RHO(IRECS)
      IMCS=0.D0
      IF(IS.GT.0) IMCS=RHO(IIMCS)
      IF(IS.LT.0) IMCS=-RHO(IIMCS)
      RECSJ=RHO(IRECSJ)
      IMCSJ=0.D0
      IF(ISJ.GT.0) IMCSJ=RHO(IIMCSJ)
      IF(ISJ.LT.0) IMCSJ=-RHO(IIMCSJ)
C     COMPUTE THE REAL PART OF COEF*C_S_ALPHA*CONJ(C_S-J_ALPHA) AND
C     ITS DERIVATIVES WITH RESPECT TO THE REAL AND IMAGINARY PARTS
C     OF C_S_ALPHA AND C_S-J_ALPHA.  THESE ARE ACCUMULATED IN G AND
C     DELG RESPECTIVELY.  IT IS THE STORAGE LOCATION OF THE REAL PART
C     OF A_J_ALPHA IN THE VECTOR G.
      IT=ITA0
      IF(J.NE.0) IT=ITA0+2*J-1
      G(IT)=G(IT)+COEF*(RECS*RECSJ+IMCS*IMCSJ)
      DELG(IT,IRECS)=DELG(IT,IRECS)+COEF*RECSJ
      IF(IS.GT.0) DELG(IT,IIMCS)=DELG(IT,IIMCS)+COEF*IMCSJ
      IF(IS.LT.0) DELG(IT,IIMCS)=DELG(IT,IIMCS)-COEF*IMCSJ
      DELG(IT,IRECSJ)=DELG(IT,IRECSJ)+COEF*RECS
      IF(ISJ.GT.0) DELG(IT,IIMCSJ)=DELG(IT,IIMCSJ)+COEF*IMCS
      IF(ISJ.LT.0) DELG(IT,IIMCSJ)=DELG(IT,IIMCSJ)-COEF*IMCS
C     IF J=0 THEN A_J_ALPHA HAS NO IMAGINARY PART.
      IF(J.EQ.0) GO TO 30
C     COMPUTE THE IMAGINARY PART OF COEF*C_S_ALPHA*CONJ(C_S-J_ALPHA) AND
C     ITS DERIVATIVES WITH RESPECT TO THE REAL AND IMAGINARY PARTS
C     OF C_S_ALPHA AND C_S-J_ALPHA.  THESE ARE ACCUMULATED IN G AND
C     DELG RESPECTIVELY.  IT IS THE STORAGE LOCATION OF THE IMAGINARY
C     PART OF A_J_ALPHA IN THE VECTOR G.
      IT=IT+1
      G(IT)=G(IT)+COEF*(-RECS*IMCSJ+IMCS*RECSJ)
      DELG(IT,IRECS)=DELG(IT,IRECS)-COEF*IMCSJ
      IF(IS.GT.0) DELG(IT,IIMCS)=DELG(IT,IIMCS)+COEF*RECSJ
      IF(IS.LT.0) DELG(IT,IIMCS)=DELG(IT,IIMCS)-COEF*RECSJ
      DELG(IT,IRECSJ)=DELG(IT,IRECSJ)+COEF*IMCS
      IF(ISJ.GT.0) DELG(IT,IIMCSJ)=DELG(IT,IIMCSJ)-COEF*RECS
      IF(ISJ.LT.0) DELG(IT,IIMCSJ)=DELG(IT,IIMCSJ)+COEF*RECS
30    CONTINUE
40    CONTINUE
      RETURN
      END
```

Table A.2

Data of Christensen, Jorgenson and Lau (1975).[a]

	Durables		Non-durables		Services	
Year	Quantity	Price	Quantity	Price	Quantity	Price
1929	28.9645	33.9	98.1	38.4	96.1	31.6
1930	29.8164	32.2	93.5	36.4	89.5	32.1
1931	28.9645	31.4	93.1	31.1	84.3	30.9
1932	26.8821	23.9	85.9	26.5	77.1	28.8
1933	25.3676	31.3	82.9	26.8	76.8	26.1
1934	24.6104	27.7	88.5	30.2	76.3	26.8
1935	22.3387	28.8	93.2	31.5	79.5	26.8
1936	24.1371	32.9	103.8	31.6	83.8	27.2
1937	24.1371	29.0	107.7	32.7	86.5	28.3
1938	26.6928	28.4	109.3	31.1	83.7	29.1
1939	26.4088	30.5	115.1	30.5	86.1	29.2
1940	27.0714	29.4	119.9	30.9	88.7	29.5
1941	28.4912	28.9	127.6	33.6	91.8	30.8
1942	29.5325	31.7	129.9	39.1	95.5	32.4
1943	28.6806	38.0	134.0	43.7	100.1	34.2
1944	28.8699	37.7	139.4	46.2	102.7	36.1
1945	28.3966	39.0	150.3	47.8	106.3	37.3
1946	26.6928	44.0	158.9	52.1	116.7	38.9
1947	28.3966	65.3	154.8	58.7	120.8	41.7
1948	31.6149	60.4	155.0	62.3	124.6	44.4
1949	35.8744	50.4	157.4	60.3	126.4	46.1
1950	38.9980	59.2	161.8	60.7	132.8	47.4
1951	43.5414	60.0	165.3	65.8	137.1	49.9
1952	48.0849	64.2	171.2	66.6	140.8	52.6
1953	49.8833	57.5	175.7	66.3	145.5	55.4
1954	53.1016	68.3	177.0	66.6	150.4	57.2
1955	55.4680	63.5	185.4	66.3	157.5	58.5
1956	58.8756	62.2	191.5	67.3	164.8	60.2
1957	61.6206	56.5	194.8	69.4	170.3	62.2
1958	65.3122	66.7	196.8	71.0	175.8	64.2
1959	65.7854	63.3	205.0	71.4	184.7	66.0
1960	68.6251	73.1	208.2	72.6	192.3	68.0
1961	70.6129	72.1	211.9	73.3	200.0	69.1
1962	71.5594	72.4	218.5	73.9	208.7	70.4
1963	73.5472	72.5	223.0	74.9	217.6	71.7
1964	77.2387	76.3	233.3	75.8	229.7	72.8
1965	81.9715	82.3	244.0	77.3	240.7	74.3
1966	87.4615	84.3	255.5	80.1	251.6	76.5
1967	93.8981	81.0	259.5	81.9	264.0	78.8
1968	99.5774	81.0	270.2	85.3	275.0	82.0
1969	106.7710	94.4	276.4	89.4	287.2	86.1
1970	109.1380	85.0	282.7	93.6	297.3	90.5
1971	115.2900	88.5	287.5	96.6	306.3	95.8
1972	122.2000	100.0	299.3	100.0	322.4	100.0

[a]*Source:* Tibibian (1980).

References

Christensen, Laurits R., Dale W. Jorgenson and Lawrence J. Lau, 1975, Transcendental logarithmic utility functions, The American Economic Review 65, 367–383.

Diewert, W.E., 1974, Applications of duality theory, in: M.D. Intriligator and D.A. Kendrick, eds., Frontiers of quantitative economics, Vol. II (North-Holland, Amsterdam).

Diewert, W.E., 1977, Generalized Slutsky conditions for aggregate demand functions, Journal of Economic Theory 15, 353–362.

Edmonds, D.E. and V.B. Moscatelli, 1977, Fourier approximation and embeddings of Sobolov spaces, Dissertationes Mathematyczne CXLV.

Gallant, A. Ronald, 1975, Seemingly unrelated nonlinear regressions, Journal of Econometrics 3, 35–50.

Gallant, A. Ronald, 1979, A note on the interpretation of polynomial regressions, Institute of Statistics Mimeograph Series no. 1245 (North Carolina State University, Raleigh, NC).

Gallant, A. Ronald and Alberto Holly, 1980, Statistical inference in an implicit, nonlinear, simultaneous equation model in the context of maximum likelihood estimation, Econometrica 48, 697–720.

Malinvaud, E., 1970, The consistency of nonlinear regressions, The Annals of Mathematical Statistics 41, 956–969.

Rudin, Walter, 1973, Functional analysis (McGraw-Hill, New York).

Simmons, Peter and Daniel Weiserbs, 1979, Translog flexible functional forms and associated demand systems, The American Economic Review 69, 892–901.

Souza, Geraldo and A. Ronald Gallant, 1979, Statistical inference based on M-estimators for the multivariate nonlinear regression model in implicit form, Institute of Statistics Mimeograph Series no. 1229 (North Carolina State University, Raleigh, NC).

Tibibian, Mohammed, 1980, An empirical investigation in the theory of consumer behavior, Ph.D. Dissertation (Duke University, Durham, NC).

Varian, Hal R., 1978, Microeconomic analysis (W.W. Norton, New York).

White, Halbert, 1980, Using least squares to approximate unknown regression functions, International Economic Review 21, 149–170.