Modified Gravity with Torsion

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Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text. Throughout this dissertation the plural pronoun ‘we’ is used for stylistic reasons and should be taken to refer to either the singular author, or to the author and her thesis advisors.

It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text.

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Summary

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We are at a point in time when alternative gravitational theories are beginning to be constrained by high precision cosmological and astrophysical data. The work in this thesis focuses on applications of a particular modified gravity theory, the extended Weyl Gauge Theory (eWGT), that was recently developed by Prof. Lasenby and Prof. Hobson. The applications corresponds to subsets of the full theory that are applicable to different cosmological and astrophysical sectors.

We start by investigating a simplified scenario that simulates a famous alternative theory of gravity, Weyl^2 Gravity. Recently a couple of issues have been raised regarding the validity of the theory. Starting from a gauge theory perspective we bring a fresh contribution to the debate. We argue against the classical formulation by showing that the theory cannot support astrophysical matter (introduced by a perfect fluid). Furthermore we extend the theory and show that even if we allow torsion to be present we cannot reach a physical setup. In this process we have discovered interesting properties of the torsion field that could play an important role in generalised cosmological setups.

In the next application we consider a cosmology dictated by a Riemann^2 Lagrangian that can accommodate only radiation. We find new physical behaviour in the perturbed regime that discretises the power spectrum. We prove that the setup admits gravitational waves.

Finally, we construct a new Lagrangian theory for spinning fluids. We show that it is compatible with current literature for a flat space time. Considering its extensibility we believe that it can be widely used in future research.

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I am proud to say that I have been part of the Cavendish Astrophysics Group. I got the chance to meet great scientists who take pride in their work and are diligently trying to improve the world. I would like to thank everyone for the great conversations that have undeniably changed my worldview.

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Another thesis could have been written from all the encouraging messages my friends have sent me throughout the years. Do Young Kim and Jordan Boddy have helped me through the hardest times and I hope that one day I can return the favour.

Last but by no means least, thanks go to my mom, my dad and my husband for almost unbelievable support. This thesis is dedicated to them.

Contents

CHAPTER 1

INTRODUCTION

The General Theory of Relativity (GR) has undoubtedly revolutionised our understanding of the Universe and from 1915 until this day Einstein's field equations are still the norm when it comes to describing the behaviour of space-time on macroscopic scales. The considerable success of the theory has not stopped alternatives being proposed in order to overcome the worrying shortcomings (e.g. the need for Dark Matter and Dark Energy in our cosmological model) and in time a class of theories labelled "modified gravity theories" emerged.

A more formal definition can be found in [1]: " the effect of gravity on matter is tightly constrained to be mediated by interactions of the matter fields with a single rank-2 tensor. The term "gravitational theory" can then by functionally defined by the set of field equations obeyed by the rank-2 tensor, and any other non-matter fields it interacts with. If these equations are anything other than Einstein's equations, then it is considered to be a modified theory of gravity". There is a vast number of theories that satisfy this definition and each have merits and shortcomings, without an obvious candidate to replace (or generalize) Einstein's theory. In our work we actually employ a theory that lies outside this sphere, where we describe gravity as a gauge field and our interactions are not defined by a single rank-2 tensor. To ensure some uniformity we start this thesis by making a general introduction to the main considerations behind theories of gravity. This description will be heavily based on a recent review of theories of gravitation, [1] and the staple book in Gauge Theories of Gravity [2].

The premise behind theories of gravity

Any viable relativistic theory of gravity has to satisfy the foundational requirements (e.g. universality of free fall and isotropy of space) as well as be compatible with observational results. In this section we will briefly outline the current results and their importance in formulating a theory of gravity. We refer the reader to [1] for a more detailed description.

When looking at the foundational requirements the equivalence principle plays an important role. In the following, we will employ the definitions of *Weak Equivalence Principle* (WEP) - once an initial position and velocity have been prescribed all uncharged, freely falling test particles follow the same trajectories; *Einstein Equivalence Principle* (EEP) - the WEP is valid and "the outcome of any local non-gravitational experiment in a freely falling laboratory is independent of the velocity of the laboratory and its location in spacetime." (*Einstein*); *Strong Equivalence Principle* (SEP) - the EEP is valid and "the gravitational motion of a small test body depends only on its initial position in spacetime and velocity, and not on its constitution."(*Einstein*)

From *Eötvös*-type experiments we have a tight constraint on the relative difference in accelerations of two bodies to be of order 10^{-13} [3], which validates the Weak Equivalence Principle. Moreover, it is worth noting that by having a theory that satisfies the WEP the test proposed by Einstein in 1916 involving the gravitational redshifting of light is automatically passed: since energy-momentum is conserved in a closed system then it is only a test of WEP, and the *Eötvös*-type experiments have higher accuracy.

Einstein's Equivalence Principle is considerably more challenging to test than the WEP. Hughes-Drever experiments (see [4]) provide the most accurate evidence by testing for local spatial anisotropies by observing the shape and spacing of atomic spectral lines. Beyond direct experiments there are theoretical reasons that support the EEP as well, namely Schiff's conjecture that states "any complete and self-consistent gravitational theory that obeys the WEP must also satisfy EEP".

The theories that obey the EEP are often described as being 'metric' theories of gravity. This is misleading as theories with torsion, such as the *Poincaré Gauge Theory (PGT)*, should also be included. Observational considerations can be used to differentiate between the various alternatives. The Solar system tests are powerful tools for investigating long-range modifications to Newtonian gravity. For example, good constraints for the 'unit curvature' of space can be found from the tests measuring the spatial deflection of star light by the Sun and the Shapiro time-delay effect (see [5]). Also from the value of the perihelion procession of Mercury the non-linear terms in the space-time geometry as well as preferred frame effects can be determined.

Other types of experiments that are worth mentioning focus on testing whether the Strong Equivalence Principle (SEP) holds, since although the ‘metric’ theories must satisfy the WEP and EEP, they can violate the SEP (the effect is called the ‘Nordtvedt effect’). The most successful experiment for this effect (a giant *Eötvös* experiment that uses the Earth-Moon system in the gravitational system of the Sun) produces no support for the violation of the SEP and can be used to constrain possible deviations from General Relativity.

Until now we have discussed the constraints involving either the Equivalence Principles or generalisations of Newtonian gravity, but have yet to mention the ultimate generic prediction of all known relativistic theories of gravity: gravitational waves. The current theories predict different types of gravitational radiation and thus the most powerful constraint will be the nature of the gravitational wave. By measuring the propagation speed (in General Relativity it is predicted to be the speed of light in vacuum, but not all theories predict null gravitational radiation), or the polarization of the gravitational wave one could dismiss a significant fraction of the available theories.

Recently a significant discovery was made by the Laser Interferometer Gravitational-Wave Observatory(LIGO). At the time this thesis was written the collaboration detected gravitational waves from binary black hole coalescence (GW150914, GW151226, GW170104, GW170608) and from binary neutron star inspiral (GW170817). Their analysis strongly agrees with the gravitational wave representation of General Relativity as explained in [6]. Alternative theories should therefore have the same prediction.

For instance in the case of General Relativity where only quadrupole radiation with a positive energy should be emitted, in most other theories dipole gravitational radiation is also expected and sometimes can carry away negative energy. No evidence for dipolar radiation exists at this moment and therefore experimental limits can be set on theories that predict such radiation.

Finally, when evaluating a theory one should take into account theoretical considerations. In the case of gravitational theories this involves the study of classical and quantum fluctuations about classical solutions.

A significant number of theories have attempted to explain dark energy by employing *ghosts*. To have a cosmic acceleration, an additional repulsive force needs to act between massive objects at large distance. If the force is mediated by a particle of even spin (a scalar or a tensor) then the corresponding kinetic term must have the opposite sign, making it a ghost. To not be in contradiction with the quantum formalism one needs to accept that the eigenvalues of the ghost are negative. The ghost will generate instabilities if it couples to other fields and when these fields are already excited the ghost will keep giving away its energy. This is a problem especially in theories based on Lorentz invariance because the process leads to a production of ghost-non-ghost pairs at a divergent rate [7].

Strong coupling is another problem/virtue for some theories. Although initially considered a problem since the quantum fluctuations of a classical solution become strongly coupled at an unacceptably low scale, recently merit has been given for strong coupling on the vacuum. For example, consider a model that deviates from General Relativity at large distances. This deviation must be at least $O(1)$ on cosmological scales (dark energy) and suppressed to $\leq O(10^{-5})$ on Solar System scales. Therefore the fields that are responsible for the modification must be screened within the Solar System and this can happen if the fields are so strongly interacting that they are frozen together, so they cannot freely propagate (Vainshtein mechanism).

1.1 A brief encounter with Gauge Theories of Gravity

"...the essential achievement of general relativity, namely to overcome 'rigid' space is only indirectly connected with the introduction of a Riemannian metric. The directly relevant conceptual element is the 'displacement field', which expresses the infinitesimal displacement of vectors. It is this which replaces the parallelism of spatially arbitrarily separated vectors fixed by the inertial frame by an infinitesimal operation. This makes it possible to construct tensors by differentiation and hence to dispense with the introduction of rigid space. In the face of this it seems to be of secondary importance in some sense that some particular Γ field can be deduced from a Riemannian metric..."(*Einstein, 1955*)

As early as 1920 É. Cartan had managed to make the connection a concept that is geometrically independent a priori. Inspired by the earlier work of the Cosserat brothers who generalised the 3-D classical continuum of elasticity and fluid dynamics by additionally attaching each material point a new director field (spin), Cartan found a new geometry - the Einstein-Cartan space time. The new setup mimics Einstein's 'laboratory' with two major differences: the point masses are phased by a Cosseratum continuum and the coordinate systems are substituted by two coframes. The effect is that Einstein's procedure is now applied to more general objects (for instance the spin of matter is included).

In standard particle physics all gauge groups are internal, but when working with the gauge theories we are not necessarily restricted to internal groups when localizing a symmetry. If we have a conserved current and a corresponding group, we can describe spacetime using an external group.

The link between the gauge theory and gravity relies on energy-momentum conservation of Special Relativity (Alamorgodo,1945) that superseded the mass conservation of classical physics (Lavoisier, 1790). Therefore the Poisson equation describing Newton's gravitational potential has to be replaced by an equation that uses the energy density of matter (and/or radiation). According to SR this is the time-time component of the symmetric energy-momentum current $t_{ij} = t_{ji}$ of matter (and/or radiation).

The action of an isolated physical system is invariant under translations in time and space which is equivalent to saying that the energy momentum current is conserved. Thus the conserved energy-momentum current and the translational group $T(4)$ acting on Minkowski space should fully define gravity.

Utiyama realised this early on and started to formulate gravity as a gauge theory in 1956 (see [8]) based on Cartan's work. He was the first to attempt to recover general relativity by gauging the Lorentz group *SO(1,3)*. As Hehl points out in [2], Utiyama's approach had some problematic aspects - the tetrads e_i^α were introduced ad hoc, the connection $\Gamma_{i\beta}^\alpha$ of spacetime was assumed to be Riemannian, and the angular momentum current $\partial_k \mathcal{J}_{ij}^k$ was assumed to be conserved, where $\mathcal{J}_{ij}^k = -\mathcal{J}_{ji}^k$ is the current linked to the Lorentz group. Gravity is coupled to the conserved and symmetric energy-momentum current t_{ik}, taken by Einstein as the source of gravity in his field equation, and not the angular momentum current, contradicting Utiyama's approach.

A successful attempt was made by Kibble (see [9]) who was the first to gauge the Poincaré group (the semidirect product of the translation group and the Lorentz group) leading to the Riemann-Cartan geometry: "Starting from special relativity and applying the gauge principle to its Poincaré-group symmetries leads most directly not precisely to Einstein's general relativity, but to a variant, originally proposed by Élie Cartan, which instead of a pure Riemannian space-time uses a space-time with torsion. In general relativity, curvature is sourced by energy and momentum. In the Poincaré gauge theory, in its basic version, additionally torsion is sourced by spin."(*Kibble, 1962*).

Kibble's laboratory encompasses the following setup:

- *An unquantized Dirac spinor*. An impressive feat as it is more than a decade prior to the "Colella, Overhauser and Werner" experiment and a significant leap from Einstein's point mass (even without quantisation).
- *An inertial frame*. A generalization of the frame in Einstein's lab, the inertial frame is defined by four linearly independent (co)vectors (tetrads), which are chosen to be orthonormal over the Minkowski space (before the gauge procedure). While in Einstein's lab there are four coordinates, in Kibble's lab there is a coframe consisting of four covectors (or one-forms).
- *A translational and rotational accelerated frame*. Compared to Einstein's lab where we have only four translational degrees of freedom, we gain six additional rotational Lorentz degrees of freedom.

- *Homogeneous gravitational fields*. Now the local frames inherit non-integrable relative translations and relative rotations from the torsion and curvature.
- *Light rays*. Since the connection does not couple to the electromagnetic field, in this respect there is no difference when compared to Einstein's lab.

In Kibble's model by demanding invariance under Poincaré local transformations of the matter action one needs to introduce two new gravitational gauge fields (h_a^μ and A_μ^{ab}) which correspond to the translational and rotational components of the Poincaré group. Using these two fields one constructs a covariant derivative and can proceed to find the field strength tensors from the commutator of the covariant derivatives. In the case of Poincaré Gauge Theory there are two strength tensors: $\mathcal{R}^{ab}_{cd}$ for the rotational component and $\mathcal{T}^{\,a}_{bc}$ for the translational component, which can be interpreted as the'curvature' and the 'torsion' terms.

In addition to the matter action, the total action must contain also a term that describes the dynamics of the gravitational fields. The gravitational Lagrangian is expressed in terms of the field strength tensors, and for simplicity Kibble chose to use the invariant $L_G \propto \mathcal{R}$ which leads to Einstein-Cartan space time. Although using this form is appealing, other higher-order invariants can be found from the strength field tensors and cannot be discarded. A constraint on the order can be made by demanding that the theory produces wave-type field equations. This implies that the second order partial differential equations obtained from the Lagrangian must be quasi-linear (here linear in the second derivatives) and thus the Lagrangian must be quadratic. The most general quadratic Lagrangian has been found in [10] and consists of a term that describes weak gravity and a (speculative) term that describes strong gravity. If we ignore the second term we are left with:

$$L_G = \kappa^{-1}(a\mathcal{R} + \Lambda) + L_{\mathcal{R}^2} + \kappa^{-1}L_{\mathcal{T}^2}, \tag{1.1}$$

where κ is Einstein's constant, Λ is the cosmological constant, a a dimensionless free parameter, and the Lagrangians are constructed from invariants that do not violate parity. The form of the Lagrangian defines the Poincaré theory, and popular choices are found as particular cases of the $\mathcal{R} + \mathcal{R}^2 + \mathcal{T}^2$ class of Lagrangians (see [11]).

The next important step in the evolution of gauge theories was based on Weyl's earlier work. In the early 1920's Weyl reconsidered the foundation of the geometry. He realised that the transition to ds^2 is only part of the process to obtain an 'infinitesimal' geometry and one also needs to allow a non-Euclidean 'recalibration' of spacetime in which the measure of length will not remain fixed under parallel transport. In his work, Weyl constructed a new spacetime that is described by non-zero metricity. The new structure can be thought as an extension of Riemann-Cartan spacetime by local scale invariance.

From a gauge point of view, the Weyl group is constructed by adding the group of dilations to $P(1,3)$, assuming scale covariance in addition to the $P(1,3)$ covariance. The associated current is the dilation (or scale) current (which Weyl had mistaken for the electric current) that leads to a Weyl-Cartan spacetime with a gauge field that has the dilation current as source.

1.1.1 Making the case for scale invariance

Several authors (including [12], [13], [14]) have taken the route of scale invariance. As this is the core of our model for gravity we would like to start by bringing arguments in its favour. We will make use of 't Hooft's reasoning as presented in [15].

In a perfectly symmetric world if we had a clear understanding of how one particular domain worked we would know how the whole universe unfolded by applying a symmetry transformation. We could use translations to cover the entire space and time, rotations to span all directions, Lorentz transformations to know how a particle could move if we know how it is at rest. Unfortunately the microscopic and macroscopic worlds do not seem to agree and it would be tempting to just discard the concept of symmetry. Scale invariant enthusiasts argue that instead we could blame the lack of understanding of the symmetry of scale transformations.

In a symmetry-imperfect world, the symmetry (which locally is conformal) must be spontaneously broken. This property is induced by further field transformations which can be shown to leave the vacuum not invariant [16]. Lorentz defined the invariance group as a consequence of electro-magnetism, so the setup (fully described by Maxwell's equations) has conformal symmetry. In practice this leads to having only light rays as measures (only relative size and time can be observed).

It turns out that spontaneously breaking symmetry is naturally occurring in the Einstein-Hilbert action by multiplying the metric tensor with the square of a scalar dilaton field ϕ. It is argued in [16] that the functional integral over the dilaton field has to be shifted to a complex contour such that the vacuum value is $\langle\phi\rangle = \pm\sqrt{3/4\pi G}$. So by fixing the field to have this exact value we recover Einstein-Hilbert gravity, but one might wonder what would happen otherwise. 't Hooft argues that we should consider this 'standard' gauge as a "unitarity gauge" and we can get a "renormalizable gauge" if the conformal factor is chosen such that "the amount of activity in a given space-time volume element is fixed or at least bounded. How to implement such a gauge choice is not known today"(*'t Hooft, 2013*).

1.1.2 The world of eWGT

Inspired by these arguments, a new theory of gravity was constructed in the last few years - the *extended Weyl Gauge Theory (eWGT)*. This description is a summary of eWGT as first introduced in [13].

eWGT was constructed to encompass an extended transformation law for the rotational gauge field under local dilations for which the transformation law in WGT is a particular case. In WGT the gauge fields h_a^μ and A_μ^{ab} transform covariantly with weights of w ,-1 and 0. In our case A_μ^{ab} transforms inhomogeneously. This transformation was designed such that the PGT matter action (for Dirac or for electromagnetic fields) is invariant under local dilations under the 'extended' law as it is for the 'standard' law assumed in WGT. Under a global scale transformation the laws coincide and thus both can be used to gauge the Weyl group.

Thus in both WGT and eWGT, h transforms as

$$h_a^{'\mu} = e^{-\rho} h_a^\mu, \tag{1.2}$$

whereas A's transformation is changed from

$$A_\mu^{'ab} = A_\mu^{ab} \quad \text{to} \quad A_\mu^{'ab} = A_\mu^{ab} + \theta(b_\mu^a \mathcal{P}^b - b_\mu^b \mathcal{P}^a), \tag{1.3}$$

for ρ the local physical dilation, θ is an arbitrary parameter, $P_\nu \equiv \partial_\nu \rho$ and $\mathcal{P}_a \equiv h_a^\mu P_\nu$.

A consequence of using this setup is that we end up with transformations for the curvature and torsion that are more appealing. Under the 'standard' transformations the PGT curvature transforms covariantly with weight -2, whereas the PGT torsion transforms inhomogeneously.

It can be shown that for $\theta = 1$ we find a covariant transformation law with weight -1 for the PGT torsion. For $\theta = 0$ we recover the 'standard' setup and (as previously mentioned) have a covariant transformation for the curvature. Thus we can accommodate a more balanced setup. In this work we will ocassionally focus on particular values of θ if we are trying to link our findings with other approaches (in Chapter 2 we will use $\theta = 0$ and in Chapter 3 $\theta = 1$).

Similarly to other gauge theories, we start with a matter action that is invariant under global transformations. We want to make it invariant under local transformations where h and A transform as previously stated. The standard approach is to construct a new covariant derivative which transforms in the same way as the standard partial derivative under global Weyl transformations.

We start by introducing a dilation vector gauge field V_μ alongside a new field (that is not fundamental, but simplifies computation),

$$A_\mu^{\dagger ab} \equiv A_\mu^{ab} + (\mathcal{V}^a b_\mu^b - \mathcal{V}^b b_\mu^a), \text{ where } \mathcal{V}_a = h_a^\mu V_\mu. \tag{1.4}$$

Using this form it can be shown as in [13] that the new covariant derivative takes the form

$$D_\mu^\dagger \phi \equiv (\partial_\mu + \frac{1}{2} A_\mu^{\dagger ab} \Sigma_{ab} - w V_\mu - \frac{1}{3} w T_\mu)\phi, \tag{1.5}$$

where w is the Weyl weight of the field ϕ and $\mathcal{T}_a$ is the trace of the PGT torsion for which we define $T_\mu = b^a_\mu \mathcal{T}_a$.

Furthermore we have a easy transformation law,

$$T'_\mu = T_\mu + 3(1-\theta)P_\mu, \tag{1.6}$$

which verifies that the new covariant derivative is as intended given the dilation gauge field transforms as

$$V'_\mu = V_\mu + \theta P_\mu. \tag{1.7}$$

These are the fundaments on which eWGT is built. Throughout this thesis we will look at various gravitational setups and we will specify at each stage the relevant construction that we will employ. Although we will hopefully offer convincing results to support the use of the new theory it is worth mentioning a few key features as advertised in [13]. (i) The equations of motion from a limited action (that is at most quadratic in the field strength tensors) are linear in the second derivatives of the gauge fields. This leads to a Hamiltonian that does not have Ostrogradsky's instability, unlike other approaches as discussed in [17]. (ii) The trace of the eWGT translational gauge field strength vanishes. This feature produces highly desirable simplifications at the level of the torsion-squared part of the free gravitational action and the Bianchi identities. (iii) Although the energy-momentum tensors derived from the free gravitational and matter actions are not initially covariant, they become so when accompanying terms from the gravitational sector are also considered ($A^\dagger$). (iv) The dilation field V does not explicitly appear in the Lagrangian density and an alternative dilation current can be identified (which has the added benefit of vanishing identically). (v) eWGT has only two independent field equations - the h-equation and the A-equation as the ϕ-equation is just the contraction of the h equation. (vi) Geometrically eWGT introduces a new space time which is an extension of Weyl-Cartan Y_4. However, like Gauge Theory Gravity [18] on which it is based, the theory can also be considered as a theory of forces in a flat spacetime, thus simplifying interpretation.

In this work we directly apply eWGT to various scenarios. Our aim is to understand how specific types of torsion change the cosmology and whether there are general attributes. To our knowledge we have produced the first rigorous attempt at fully exploring the torsion profile for various reduced Lagrangians. Nonetheless we will start by conducting a survey of the current development in torsion theories and we will primarily build on two well known classifications.

1.2 Torsion Review

When we talk about General Relativity torsion is not usually part of the conversation. It might come as a surprise that Einstein worked on introducing torsion (teleparallel theories) in collaboration with Cartan (as their collection of letters attest [19]) in his later research. Furthermore it influenced Schrodinger to start his work on unifying gravity and electromagnetism [20]. In his formalism torsion was related to the electromagnetic potential which led to photons gaining mass and thus finding the best experimental upper bound for photon mass.

The way most contemporary researchers think about torsion was introduced by McCrea in the early 90s [21]. As we are also using this formalism we will start by offering a brief presentation.

By definition, the torsion tensor $T^{\ c}_{ab}$ is the antisymmetric part of the connection coefficients Γ^c_{ab},

$$T^{\ c}_{ab} \equiv \frac{1}{2}\left(\Gamma^c_{ab} - \Gamma^c_{ba}\right) \equiv \Gamma^c_{[ab]}, \tag{1.8}$$

where metric compatibility is assumed. Another useful quantity is the contortion tensor, defined as

$$K_{abc} = T_{abc} - T_{bca} + T_{cab}, \tag{1.9}$$

since we can just write $\Gamma^c_{ab} = \tilde{\Gamma}^c_{ab} - K^{\ c}_{ab}$, where $\tilde{\Gamma}$ is the GR version.

Now an important property of torsion is that it can be decomposed (with respect to the Lorentz group) into three irreducible tensors,

$$T^{\alpha} = T^{(1)} + T^{(2)} + T^{(3)} \equiv \textbf{tentor} + \textbf{trator} + \textbf{axitor}. \tag{1.10}$$

In 4D, the torsion has 24 components out of which 16 are inherited by $T^{(3)}$, 4 by $T^{(2)}$ and 4 by $T^{(1)}$. By construction, we define the axitor as the totally anti-symmetric torsion such that

$$T^{\ c(3)}_{ab} \equiv g^{cd} S_{[abd]}, \tag{1.11}$$

the trator as

$$T^{\ c(2)}_{ab} \equiv \frac{1}{3}\left(T_a \delta^c_b - T_b \delta^c_a\right), \tag{1.12}$$

and the remaining components get allocated to the tentor

$$T^{\ c(1)}_{ab} = T^{\ c}_{ab} - T^{\ c(2)}_{ab} - T^{\ c(3)}_{ab}. \tag{1.13}$$

Such a representation is particularly attractive as it gives us a simple way to restrict the torsion tensor in applications where working with the general torsion would be too involved. In recent studies torsion classifications have been found and in this thesis we will introduce two that are relevant to our work. We are primarily interested in the classification in terms of irreducible tensors in 4D as presented in [22] and the classification based on physical manifestations and experiment as introduced in [23].

Although the analysis in [22] is broad, we will introduce a few results in order to offer a brief overview of what we currently know about torsion.

Scalar-tensor theories (STT) are a branch we particularly focus on in our research. A (scalar) field ϕ generates torsion only in non-minimally coupled theories with a $\zeta\phi^2 R$ term in the Lagrangian density, which makes the torsion related to the gradient of the field. The $\zeta\phi^2 R$ term is just an example and more general options are available. A great example of STT is presented in [24] where a generalized teleparallel theory of gravitation is constructed by using an arbitrary function of the torsion scalar and a scalar field.

Another approach is Soleng's work [25] which is focused on a theory with spin-torsion where he employed a scale invariant gravitational Lagrangian. Among the interesting aspects of his theory one can show that in homogeneous cosmologies a sufficient form for torsion is given by a time-like $T^{(2)}$, whereas for a Schwarzschild solution one would have a space-like $T^{(2)}$.

If we look at a Friedmann-Lemaitre-Robertson-Walker universe, we should explore the formalism presented in [26]. The authors construct a thorough study of spatially homogeneous and SO(3)-isotropic exact solutions of the 10-parameter Lagrangian of the 'Poincaré gauge theory'. One important result is that the torsion must be comprised at most of a time-like $T^{(2)}$ and $T^{(3)}$.

When we discuss torsion we must mention electromagnetism. Although various 'naive' attempts to imitate electromagnetism in gravity were less successful in the 20th century, in recent times we had more success. A significant step was made by Hammond [27], who inspired by electromagnetism introduces a torsion potential. Poplawski [28] followed his approach and formulated the classical Einstein-Maxwell-Dirac theory of spinors interacting with the gravitational and electromagnetic fields as the *Einstein-Cartan-Kibble-Sciama Theory* (ECKS). A more detailed analysis conducted by Baker et al [29] showed that for Riemann-Cartan spaces there is a restricted class of torsion for which plane null electromagnetic solutions exist, namely for light-like $T^{(2)}$ and $T^{(3)}$.

We can generate $T^{(3)}$ type torsion using Dirac particles as sources. In [30] Hehl et al. generalise GR to include dynamical mass with spin in a U_4 theory that arises as a local gauge theory for the Poincaré group. The authors show that we can source space-like $T^{(3)}$ from massive classical Dirac particles, light-like $T^{(3)}$ from massless neutrino and time-like $T^{(3)}$ from tachyon particles.

Significant work was presented in [31] where Hammond proves that the conservation law of total angular momentum is correct only if torsion (from intrinsic spin) is present. Furthermore torsion is the gauge field for the local invariance of the chiral transformation that sends "mass to negative mass" and leaves the Dirac equation invariant. An immediate consequence is a possible influence on neutron phase shifts.

In the context of Gauge Theories of Gravity, Challinor et al [32] find massive, non-ghost solutions for the Dirac field coupled self-consistently to gravity. They show that torsion makes an impact only if the Compton wavelength of the Dirac field is larger than the Hubble radius.

Following this work the authors in [33] conjecture that the accelerated expansion of the universe possibly arises due to the spin correction to the energy momentum tensor for the Dirac field. Furthermore in [34] it is shown that Dirac fields with torsion have interesting properties for the problem of Dark Matter: (i) The (space-like) torsion tensors related to spin are generated by the Weyssenhoff spinning particle and the classical Dirac particle. The first attempt was presented by Trautman in [35] and interesting cosmological solutions have been found (e.g big bounce instead of big bang in [36]). (ii) A significant use for $T^{(1)}$ torsion is in supergravity. Introductory work is presented in [37], where supergravity is sourced only by vierbein and Rarita-Schwinger fields with supersymmetry transformations for these fields. For this formalism $T^{(1)}$ is the only surviving form. (iii) The authors in [22] show that since space-like $T^{(2)}$ and $T^{(3)}$ act in a prefered direction they introduce anisotropies. (iv) Time-like $T^{(3)}$ type torsion is used in a thorough analysis on cosmological perturbations. Cappoziello et al. [38] show that the characteristic scales are enhanced by torsion. (v) The presence of light-like $T^{(3)}$ type torsion has been shown to flip the helicity of fermions. Hammond [39] shows that the torsion field gives the Dirac equation an extra coupling that acts as a mechanism for spin flipping. (vi) Gasperini [40] and Trautman [41] show that in ECKS gravity, the presence of torsion can eliminate the singularity at the big bang and replace it with a bounce. On the other hand Kerlick [42] has shown that by using a Dirac coupling leads to an enhancement of the singularity. Although a supporter of inflation, the author shows that torsion has an important role in particle creation in the early Universe. (vii) Several authors [43], [44] have shown how problems such as the flatness and horizon problems, the nature of dark matter and dark energy can be solved by introducing torsion.

In this section we have presented the current progress made to understand the behaviour and effects of different types of torsion. Although we are far from reaching a consensus on what is the best approach, there are undeniable torsion effects that cannot be neglected. These can be split into three categories (Quantum effects, laboratory tests and large-scale tests) as introduced in [23].

Quantum effects

As we have previously mentioned, torsion couples to the Dirac equation and thus it should nurture quantum effects. Unfortunately it has been shown in [45] that the effect of torsion is 17 orders or magnitude smaller than that of the classical magnetic field, which makes experiments complicated. Instead, we could look for certain secondary effects as the ones we have pointed to in the previous classification, such as the neutron phase shifts and leptonic current anomalies. Currently we do not have the experimental power to conduct such tests, but some attempts have been made. By using atom interferometry in Hughes-Drever-type experiments, Lammerzahl et al have made a convincing start that will hopefully lead to a new breakthrough in the near future.

On the other hand, neutrino spin flip has been disproved. When observing neutrinos travelling outward through the Sun it was found that the spin flip probability is 50% if the coupling constant is $\propto 10^{21}$, seven orders of magnitude too big. Thus there should be a preferential coupling constant between neutrinos and nucleons, significantly greater than the one between electrons and nucleons. This work has been explored in [46].

Laboratory tests

As gauge invariance requires the torsion quantum be massless we have a long-range force. Among the many attempts to measure such a force it is worth mentioning the "Anselm and Uratsev experiment". By working with a Lagrangian of arions one can find a testable interaction energy. By using an electromagnet with a ferromagnetic core and alternating the polarity one could have a setup where electrons spin flip (from the presence of the arion field) resulting in a measurable emf. The most recent version of this experiment produced a constraint for the g-factor (also known as the dimensionless magnetic moment) $g < 10^{-3} G_F$, where G_F is the Fermi constant [47].

Other experiments have attempted to measure the torsion effect directly. The "Spin pendulum experiment" and the "Moody experiment" (for macroscopic scalar and pseudoscalar interactions that were conducted using ion spectroscopy) have found ‘new’ physics that could be explained with torsion, but several other explanations are as viable.

Large-scale test

Several experiments are awaiting approval in the next decade that would have a better chance to prove the existence of torsion. Firstly, for an astrophysical source there could exist a ferromagnetic state in which neutrons are aligned and a torsion field is created. It is hoped that wraith-like neutrinos (which have been shown to be immune to other forces) could be affected by torsion and thus experimentally relevant. Secondly, the radiation from spinning neutron polarized stars (that some theories predict to be strongly linked to torsion) could be observed. Similarly the case can be made for pulsars.

Considering the wealth of experiments that have a fighting chance to prove (or disprove) the existence of torsion, it is a great time to focus on having a better theoretical understanding of torsion.

In this thesis we have looked at various applications for the eWGT, aiming to answer a few questions

- how does eWGT relate to other work,
- what can we learn about torsion under this setup,
- does this theory predict new, viable physics .

Considering the nature of eWGT it is unsurprising that one of the first applications used a Lagrangian dictated by the $Weyl^2$. Prof. Lasenby and Prof. Hobson had focused on creating a general theory (with unrestricted coupling constants for the matter fields) and believe from numerical tests that such a formalism should be viable. The current standard in $Weyl^2$ gravity (or Conformal Gravity) is the formalism presented by Mannheim and Kazanas, [48]. In recent years the theory attracted significant criticism and thus we decided to join the conversation. In Chapter 2 we show that if we simplify our theory to emulate Conformal Gravity we cannot obtain a viable perfect fluid representation (even by introducing the most general type of torsion). As this is a key point in any gravitational theory we have shown that we should rethink Conformal Gravity and work towards constructing a viable theory. Fortunately, since we worked with a simple setup we had a great opportunity to explore the behaviour of torsion and construct a detailed analysis that can help us choose forms of torsion for future models.

In Chapter 3 we start exploring a different Lagrangian, namely *Riemann*^{2}, as a natural continuation to Chapter 2. As we have presented the first model in standard tensor algebra, and our work is primarily in geometric algebra, we will present this work in geometric algebra as a good parallel. In this application we use perturbation theory in a radiation-only setup to uncover interesting new physics. In this scenario we use two forms of torsion, find pressure and density profiles that force a discrete power spectrum and recover gravitational waves.

In Chapter 4 we focus on constructing a viable Lagrangian that describes a spinning fluid. We decided to pursue this project as current representations in the literature are not easy to use, especially if we are looking for extensions. In geometric algebra, the formalism simplifies considerably and we end up with a manageable form that incorporates additional fields. We focus on showing it is consistent with literature, namely that it represents a Weyssenhoff fluid, and use it in simple but important applications.

Finally, we summarise the key results and discuss future applications. In this thesis we have started the groundwork for employing eWGT in more general scenarios. We have built various forms of torsion and we believe that we have improved our understanding of how to construct viable representations. With observational data improving at a fast pace, in the close future we will be able to test eWGT and hopefully start a new path to a unified theory of gravity.

CHAPTER 2

WEYL-SQUARED GRAVITY

2.1 Introduction

Considering that the infancy of gauge theories is contemporary with General Relativity, one would not be surprised to hear the Weyl^2-gravity has a long history. Beginning in 1918, Weyl proposed the concept of parallel transfer [49], [50] that let vectors associated with different points on a manifold be compared. He attempted to unify electromagnetism and gravitation by starting with a quadratic curvature action. Unfortunately when trying to match gauge field dilatations with the electromagnetic potential he reached a non-integrable form (as pointed out by Einstein).

Bach found a new quantity, the Weyl curvature [51], that is invariant under the scale transformations proposed by Weyl. He perservered by constructing a quadratic action in terms of the new curvature that led to the well known Bach equation. Fifty years later this theory is the foundation to several modern approaches (such as supergravity [52], biconformal gravity [53], auxiliary conformal gravity [54]).

Somewhat unexpectedly, a significant revival for "classical" Conformal Gravity started in the early 90s when Prof. Kazanas and Prof. Mannheim published the equations of motion in an allegedly *viable* fourth-order theory of gravity that is based on local conformal Weyl invariance of the gravitational action. This discovery impressed further with a set of unexpected predictions and 20 years onwards we are still focusing on this research.

Unlike his predecessors, Mannheim's primary focus was to develop a testable theory that was rooted in modern cosmology. In this introduction we will build up on the motivation behind

Mannheim's work [55] and construct a faithful representations of current developments.

2.1.1 A short motivation

If we go back to first principles with Einstein's theory the starting point must be the Christoffel symbols. With their aid one constructs the geodesic equation that enables us to judge distances intrinsic of the geometry. As we would like to work with a true coordinate tensor, we can use Christoffel symbols to build a new quantity, namely the Riemann tensor. We now have a tensor that vanishes iff the spacetime is flat.

Now when the Riemann vanishes the geodesic equation becomes Newton's second law of motion for a free particle, in the absence of gravity as viewed in an accelerating coordinate system. When it is non-zero, there is a choice of values for the Christoffel symbols that enables this equation to describe Newton's law of gravity. As the Schwarzschild metric also provides relativistic corrections to Newtonian gravity, the observation of the predicted gravitational bending of light by the Sun validated the above formalism.

This leads to the belief that gravity is a covariant metric theory in which the metric describes the gravitational field. In the vicinity of the Sun it is given by the Schwarzschild metric.

Up to this point nothing has been said about gravitational field equations. Einstein postulated that gravity obeys,

$$-\frac{1}{8\pi G}\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R^{\alpha}_{\alpha}\right)=T^{\mu\nu}, \tag{2.1}$$

that originates from the action

$$I=-\frac{1}{16\pi G}\int d^4x(-g)^{1/2}R^{\alpha}_{\alpha}+I_M,\ \text{where } I_M \text{ is the matter contribution.} \tag{2.2}$$

We can easily check that if the energy momentum tensor vanishes the Ricci must vanish and we recover the Schwarzschild solution we previously mentioned. Einstein found a solution that satisfies our constraint, but he never argued for uniqueness - in his late work he actually looked for alternatives. Thus it is not surprising that when Conformal Gravity was presented as an alternative solution it attracted a lot of attention.

In Mannheim's formalism we start from the action given by

$$I=-\alpha_g\int d^4x(-g)^{1/2}C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa}+I_M,\ \text{where } C_{\lambda\mu\nu\kappa} \text{ is the Weyl tensor,} \tag{2.3}$$

$$C_{\lambda\mu\nu\kappa}\equiv R_{\lambda\mu\nu\kappa}+\frac{1}{6}R^{\alpha}_{\alpha}(g_{\lambda\mu}g_{\mu\kappa}-g_{\lambda\kappa}g_{\mu\nu})-\frac{1}{2}(g_{\lambda\nu}R_{\mu\kappa}-g_{\lambda\kappa}R_{\mu\nu}g_{\mu\nu}R_{\lambda\kappa}+g_{\mu\kappa}R_{\lambda\nu}). \tag{2.4}$$

Varying this action with respect to the metric leads to an equation in terms of the Bach tensor, $-4\alpha_g W^{\mu\nu}+T^{\mu\nu}=0$. As the Bach tensor is defined as

$$W_{\mu\nu} = 2\nabla^\alpha\nabla^\beta C_{\alpha\mu\nu\beta} + C_{\alpha\mu\nu\beta}R^{\alpha\beta}, \tag{2.5}$$

we can easily see, from the Bianchi equations, that this theory also has a vanishing Ricci as a vacuum solution.

2.1.2 *Pure* Weyl^2 - the first frontier

Unsurprisingly the idea behind using a Weyl^2 Lagrangian is not aleatory. When we build a theory of gravity, we must prioritise that our physics is invariant under any arbitrary change of local coordinates, or general coordinate transformation (GCT). A subclass of GCTs are the conformal transformations, which are defined as preserving the oriented angles between curves at each point. Bearing this in mind, we can build a conformal geometry by defining a conformal manifold as a differentiable manifold that has an equivalence class of Riemannian metrics, where

$$h \propto g \text{ iff } h_{\mu\nu}(x) = \lambda^2(x)g_{\mu\nu}(x), \tag{2.6}$$

i.e. two conformal metrics are identical up to a *Weyl transformation*, a local change of scale. Although one could construct transformations which preserve angles but not scales, it turns out that such transformations are not well posed. Thus "Weyl" and "conformal" are used interchangeably in literature, although we would argue that this is a misnomer as the underlying groups are not equivalent.

An important aspect is that a conformal metric is conformally flat if one of its representative metrics is flat. In 2D every conformal metric is locally conformally flat, but in higher dimensions we need to look only at the trace-free part of the curvature tensor (as this part is volume-preserving, but accounts for shape distortions under a GCT).

Thus in 3D we define the Cotton tensor and in 4D the Weyl tensor that have the property that conformal flatness is achieved if they vanish. The Weyl tensor has the extra property that it is completely invariant under a conformal transformation. The only other conformally invariant tensor which is algebraically independent of the Weyl tensor is the Bach tensor. Thus the gravitational equations become even more appealing.

Returning to the Lagrangian, we are now in a better position to argue for invariance. Under a rescaling of the metric $g_{\mu\nu} \to g_{\mu\nu}e^{\phi(x)}$ the Weyl tensor $C^\mu_{\nu\rho\sigma}$ will be invariant. Thus the square of the Weyl will transform as $C^2_{\mu\nu\rho\sigma} \to e^{-2\phi}C^2_{\mu\nu\rho\sigma}$ and together with the transformation $\sqrt{-g} \to e^{2\phi}\sqrt{-g}$ we show that the action in 2.3 is invariant.

Considering the appealing features one could believe that *Pure* Weyl^2 Gravity should be the new answer to General Relativity. Unfortunately as in the case for many alternative theories of gravity, the simple route turns out to not be the most fruitful. Niederle [56] shows that the theory has ghost states that come from the opposite-sign propagators to the physical graviton. If we introduce an Einstein-Hilbert term ($m^2 R$) in the action we can show that the ghost states become massive and the $1/p^4$ propagators can be resolved. The relevant part of the propagator will read

$$\frac{1}{p^4} \to -\frac{1}{m^2}\Big(\frac{1}{p^2} - \frac{1}{p^2 - m^2}\Big), \tag{2.7}$$

which exemplifies the problematic relative sign between the massless and massive states. If we look at the limiting cases for m, as $m \to \infty$ we enter General Relativity territory and the massive ghost decouples. Unfortunately for Conformal Gravity ($m \to 0$) we have a pole at $1/m^2$ and the limit is not quite smooth as the theory has seven states (massless+massive graviton) that become six in the $m \to 0$ limit. The reason for the removal of one state is the Weyl symmetry, or local conformal symmetry, that emerges in this limit.

In order to tackle this problem several generalisations have been proposed. The most common formalisms focus on a Weyl Theory with either the addition of matter or supersymmetry, or the addition of non-minimal couplings, or by considering higher-dimensional versions, or by introducing a Yang-Mills sector.

2.1.3 Recent developments

Bearing in mind all the possibilities that start from a "pure" Conformal Gravity, it is not surprising that many promising results have been found. We will proceed to briefly introduce some of the most famous ones, particularly from the vast research carried by Mannheim et al. We will use a recent review [57] to present their recent efforts.

A few years ago, Mannheim and O'Brien have claimed that Conformal Gravity can be used to accurately describe the rotation curves of galaxies without the need to invoke dark matter [58]. Recently they have extended their repertoire by fitting the LITTLE THINGS survey (25 dwarf galaxies) successfully.

The formalism is based on solving the fourth order Bach equation with a source term, in this case a galaxy. Both local matter inside the galaxy, modelled with a modified Newtonian potential (extra linear term), and global matter exterior to it, given by the background cosmology, contribute. For comparison, the General Relativity prediction is given by [59],

$$v_{GR}(R) = \sqrt{\frac{N^* M_\odot G R^2}{2R_0^3}\Big(\Big(I_0\Big(\frac{R}{2R_0}\Big)K_0\Big(\frac{R}{2R_0}\Big) - I_1\Big(\frac{R}{2R_0}\Big)K_1\Big(\frac{R}{2R_0}\Big)\Big)}, \tag{2.8}$$

and the conformal gravity prediction by [58],

$$v_{cg}(R) = \sqrt{v_{gr}^2 + \frac{N^*\gamma^* c^2 R^2}{2R_0} I_1\Big(\frac{R}{2R_0}\Big) K_1\Big(\frac{R}{2R_0}\Big) + \frac{\gamma_0 c^2 R}{2} - \kappa c^2 R^2}. \tag{2.9}$$

In the above formulae N^* is the estimated number of stars in the galaxy, I and K Bessel functions, R_0 the galactic scale length and $\gamma^* = 5.43 \times 10^{-41}\mathrm{cm}^{-1}$, $\gamma_0 = 3.06 \times 10^{-30}\mathrm{cm}^{-1}$, and $\kappa = 9.54 \times 10^{-54}\mathrm{cm}^{-2}$ are Mannheim's fitted constants.

Using this form for the velocity they show that for a sample of 200 galaxies the scaled centripetal acceleration of the last observed data point seems to be order one. This universality is supported by the work of McGaugh et al [60] where it is postulated that such a universality could be explained by new physics outside GR.

Another significant result regards the missing mass problem in cluster motion. In 1933, Zwicky applied the virial theorem to his observations on the Coma Cluster and found that the theoretical mass estimate resulted in a mass to light ratio that was an order of magnitude larger than the theoretical prediction. Mannheim and O'Brien show that Conformal Gravity solves the problem without the need for dark matter [61]. They show that the driving term in the dispersion velocity, in the long range, becomes

$$3\sigma^2 \approx \kappa c^2 R^2, \text{ where } k = 9.54 \times 10^{-50} m^{-2} \text{ is found from galactic rotation fitting.} \tag{2.10}$$

This relation makes the same velocity prediction without assuming any dark matter for observed measured velocities.

Finally, we should mention the applications for the local solar system. Investigating gravitational light bending, O'Brien et al show [57] that the total angular deflection consists of the general relativity prediction and a conformal correction,

$$\theta \approx \frac{4GM}{c^2 r_0} + \frac{M\gamma^* r_0}{2M_\odot} + \gamma_0 r_0 - \pi k r_0^2. \tag{2.11}$$

For large impact parameters the equation can be rewritten as

$$M \approx \Big(\frac{\pi k r_0}{\gamma^*}\Big) M_\odot, \tag{2.12}$$

which is consistent with the mass one would obtain in clusters by using the virial theorem. This presents a consistent framework for the constants which were originally described in terms of rotation curves, with clusters and gravitational bending of light.

Furthermore, Sultana et al [62] have shown that the procession of the perihelion of Mercury can be accurately predicted. The global parameter of Conformal Gravity (γ) can be constrained by the Mercury observations and is consistent with the value originally obtained from rotation curves.

Although these results are impressive they have attracted as much criticism as praise. Recently a couple of issues have been raised that have not been resolved to date.

A first concern is outlined in Horne's work [63]. As we have previously mentioned, Conformal Gravity should not be used in its pure form and particular care should be carried when exploring the matter sector. When Mannheim and O'Brien fitted the rotation curves they neglected the non-constant Higgs field, whose conformal coupling is required. Thus their galaxy rotation curves have a Higgs field that varies with radius, implying that particle masses change with radius. When using a conformal transformation to make the Higgs field constant it is found that there is an additional term from the Higgs field. This term cancels out the contribution from the linear potential that enables Conformal Gravity to reproduce the flat rotation curves of galaxies.

In more recent work Horne et al [64] have compared Conformal Gravity cosmology to the standard ΛCDM cosmology. In the work we have previously mentioned, Mannheim claimed close agreement with ΛCDM for the distance moduli at redshift $z < 1$. Using Gamma Ray Bursts and Quasar data, the Hubble diagram was extended up to $z = 8$ and it was shown that at redshift $z \geq 2$ ΛCDM significantly outperformed Conformal Gravity.

Furthermore, Phillips [65] has shown that under closer examination the predictions for the Solar System do not hold as well as we previously thought. The author argues that the form of the linear potential as chosen by Mannheim is incorrect. His claim is supported by other authors such as Yoon (2013) and Walker (1994). By working in the weak field limit, where a linear approximation is appropriate, Phillips shows that near the Sun the solution is incompatible with observations.

Although there seem to be several questions left unanswered, one cannot deny the potential of Conformal Gravity - a quantum gravity theory in the making. In this brief introduction we have presented the way this theory is viewed from a "traditional" perspective, in the hope that it will offer some insight. From now on we will shift to a different approach as we will discuss Conformal Gravity in the context of Gauge Theories of Gravity.

As we have previously mentioned, in our work we use a specific GTG, namely the 'extended'-Weyl Gauge Theory (eWGT). Considering the nature of Conformal Gravity we have decided that it would be ideal as one of the first applications of eWGT. We will work with a restricted theory that simulates Mannheim's Conformal Gravity for static, spherically symmetric systems that are torsion-free. We aim to bring a new perspective to the "traditional" formalism and then extend to a new regime that includes torsion.

2.2 A different perspective

"The effect of gravity on matter is tightly constrained to be mediated by interactions of the matter fields with a single rank-2 tensor. The term "gravitational theory" can then be functionally defined by the set of field equations obeyed by the rank-2 tensor, and any other non-matter fields it interacts with. If these equations are anything other than Einstein's equations, then it is considered to be a modified theory of gravity." [1]

Under the large umbrella of alternative theories of gravity several approaches have tried to address various shortcomings of GR. One such class of theories are the Gauge Theories of Gravity (GTG) that arose from the need to construct a unified theory with quantum mechanics. The quintesscence of a gauge theory is the requirement that the global symmetry group linked to the conserved current under consideration has to be local. As the Lagrangian needs to remain invariant, we have to introduce a gauge potential that transforms accordingly. This leads to a mechanism that creates a new interaction from a conserved current via the Noether theorem and its symmetry group.

Fortunately we can also use this structure to describe gravity. The source of gravity is stored in the energy-momentum current of matter. For an isolated system this quantity is conserved with the action being invariant under translations in space and time. So we can describe gravity with the aid of the conserved energy-momentum current and the translational group acting in Minkowski spacetime.

As early as 1920 É. Cartan had managed to make the connection a concept that is geometrically independent a priori which led to a new geometry, namely the Einstein-Cartan spacetime. As discussed in Chapter 1, based on this new construction Utiyama started to formulate gravity as a gauge theory in 1956 by requiring that the global symmetry group (that is related to the conserved current under consideration) to be made local.

A first successful attempt, as discussed in Chapter 1, was made by Kibble who gauged the Poincare group. He started from a Minkowski spacetime in which a matter field with energy momentum and spin-angular-momentum is distributed continuously. The corresponding action is then demanded to be invariant with respect to Poincare transformations and thus one arrives to new gravitational gauge field variables (h_a^μ for translational part of the Poincare group and A_μ^{ab} for the rotational part). In turns these fields lead to two field strength tensors respectively $\mathcal{T}_{bc}^{a}(h, \partial h, A)$ and $\mathcal{R}_{cd}^{ab}(h, A, \partial A)$.

Although significant effort has been made in quantising PGT, no approach has yet yielded a theory that has all the desired characteristics of a theory compatible with quantum mechanics and with the correct classical limits. In recent years alternative theories have become quite popular. Extensions of the theory can be reached by imposing new gauge symmetries such as local scale invariance. This can be obtained by working with the Weyl group and a new gravitational gauge field B_μ corresponding to the dilation part of the group. Furthermore by requiring a more general transformation for the rotational gauge field under local dilations one can further extend to the eWGT. In WGT the transformations are given by,

$$\begin{aligned} A'^{ab}_{\mu}(x') &= \frac{\partial x^\mu}{\partial x'^\mu}\Big(\Lambda^a_c \Lambda^b_d A^{cd}_\nu - \Lambda^{bc}\partial_\nu \Lambda^a_c\Big), \\ B'_\mu(x') &= \frac{\partial x^\nu}{\partial x'^\mu}\Big(B_\nu(x) - \partial_\nu \rho(x)\Big), \\ h'^{\mu}_a(x') &= \frac{\partial x'^\mu}{x^\nu} e^{-\rho(x)} \Lambda^b_a h^\nu_b, \end{aligned} \tag{2.13}$$

where Λ is the local Lorentz rotationand ρ the local physical dilation.

It can be shown that we do not lose the invariance property if we generalise the dilation part of the transformation of A to

$$A'^{ab}_{\mu} = A^{ab}_\mu + \theta(b^a_\mu \mathcal{P}^b - b^b_\mu \mathcal{P}^a), \tag{2.14}$$

where $P_\nu \equiv \partial_\nu \rho$, $\mathcal{P}_a \equiv h^\mu_a P_\mu$ and θ is an arbitrary parameter that can take any value. In the Space Time Algebra description of Gauge Theories of Gravity, [18], we write A as Ω. This extension is the cornerstone for the new theory.

From now on we will call the dilation vector gauge field V_μ to differentiate from the one in WGT. We will require the modified covariant derivative to transform covariantly for which we need V_μ to transform as

$$V'_\mu = V_\mu + \theta P_\mu, \text{ the 'extended' part of (2.13).} \tag{2.15}$$

2.3 Our framework

In any new theory we start by looking at a simple scenario to get a taste of the things to come. As we are working in Gauge Theories of Gravity we are employing a“Palatini-type” approach so we will be treating the spin connection and tetrad as independent concepts. We will find the field equations by varying with respect to both of these which leads to two sets of equations of motion describing the spacetime. As a consequence in our formalism having a rigorous understanding of the choice of Lagrangian and metric is fundamental. In the remainder of this section we will discuss our choices and set the groundwork for the cosmological model.

2.3.1 The Lagrangian

The most general form for the Lagrangian, as presented in [13], is daunting at first sight. As usual we construct the total action as the sum of the matter and free gravitational actions. Considering the structure of eWGT, we will write the free gravitational action in terms of the three gauge fields and their derivatives. In the matter part, most generally we will have a matter field ψ and a 'compensator' scalar field ϕ. As explained in [13] the total Lagrangian density will thus take the form

$$L_T = L_G(h, \partial h, \partial^2 h, A, \partial A, V, \partial V) + L_M(\psi, \partial\psi, \phi, \partial\phi, h, \partial h, A, \partial A, V, \partial V). \tag{2.16}$$

A suitable form can be found by employing Dirac's formalism, as explained in [13]. Note that $\partial^2 h$ is not used in the gravitational part whereas ∂h is not used in matter part. In our simplified scenario we have no torsion so we will be looking at

$$L_M = L_\psi + L_\phi + \phi^2 L_{\mathcal{R}^\dagger}, \tag{2.17}$$

where

$$\begin{aligned} L_\psi &= L_\psi(\psi, \partial\psi, h, \partial h, A, V, \phi), \\ L_\phi &= \frac{1}{2}\nu \mathcal{D}^\dagger_a \mathcal{D}^{\dagger a}\phi - \lambda\phi^4, \\ L_{\mathcal{R}^\dagger} &= -\frac{1}{2}a\mathcal{R}^\dagger. \end{aligned} \tag{2.18}$$

In this notation we introduce $\mathcal{D}^\dagger$ as the covariant derivative under Weyl transformations. For the moment we keep the Lagrangian for the matter field in the most general form in order to make the connection to [13] obvious.

For the gravitational sector the Lagrangian density will be written in terms of the field strength tensors and in the most general case it will be at most quadratic in $\mathcal{R}^\dagger$ and $\mathcal{H}$ (which represent the rotational part of the Poincare group and respectively the dilation part of the Weyl group), namely

$$L_G = L_{\mathcal{R}^{\dagger 2}} + L_{\mathcal{H}^{\dagger 2}}. \tag{2.19}$$

As a first application we would like to look at the toy model in which we discard the former term. In this setup the dilations will be present intrinsically but not at the level of the gravitational action. This simplification leads to the following Lagrangian

$$L_G = \alpha_1 \mathcal{R}^{\dagger 2} + \alpha_2 \mathcal{R}^{\dagger}_{ab}\mathcal{R}^{\dagger ab} + \alpha_3 \mathcal{R}^{\dagger}_{ab}\mathcal{R}^{\dagger ba} + \alpha_4 \mathcal{R}^{\dagger}_{abcd}\mathcal{R}^{\dagger abcd}$$
$$+\alpha_5 \mathcal{R}^{\dagger}_{abcd}\mathcal{R}^{\dagger acbd} + \alpha_6 \mathcal{R}^{\dagger}_{abcd}\mathcal{R}^{\dagger cdab} \tag{2.20}$$

Furthermore the field strength $\mathcal{R}^{\dagger}_{abcd}$ satisfies the Gauss-Bonnet identity, so we are free to set one of α_1, α_3 or α_6 to zero without loss of generality.

In order to recover Weyl^2, Mannheim [48] employed Lanczos's formalism [66]. The author showed that the quantity,

$$(-g)^{1/2}\Big(R_{\lambda\mu\nu\kappa}R^{\lambda\mu\nu\kappa} - 4R_{\mu\kappa}R^{\mu\kappa} + (R^{\alpha}_{\alpha})^2\Big) \tag{2.21}$$

is a total divergence and thus the Weyl action can be represented in terms of more familiar forms.

In this chapter we will employ the following notation for our work,

$$\mathcal{L}_{full} = \beta\mathcal{R}_{abcd}\mathcal{R}^{abcd} + \alpha\mathcal{R}^2 + \frac{1}{2}\kappa\phi^2\mathcal{R} + \lambda\phi^4 - 8\pi\mathcal{L}_{mat} - \frac{1}{2}\eta\mathcal{D}^{\dagger}_{a}\mathcal{D}^{\dagger a}\phi. \tag{2.22}$$

By setting $\beta + 6\alpha = 0$ we produce the 'Weyl^2' Lagrangian.

2.3.2 Framework for Spherical Systems

In our work we employ a tetrad-based method for solving the field equations. This approach was originally constructed in geometric algebra in [18] and has recently been used in spherically symmetric applications in cosmology [67], [68].

If we look at each point in spacetime we have coordinate basis vectors $\mathbf{e}_\mu$ that are related to the metric via $\mathbf{e}_\mu \cdot \mathbf{e}_\nu$. We can also attach a local Lorentz frame with another set of orthogonal basis vectors $\hat{\mathbf{e}}_i$ that are instead related to the Minkowski metric via $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \eta_{\mu\nu}$. These sets of basis vectors can be linked by tetrads (or vierbeins):

$$\hat{\boldsymbol{e}}_k = e^{\mu}_{k}\boldsymbol{e}_\mu \qquad \boldsymbol{e}_\mu = e^{k}_{\mu}\hat{\boldsymbol{e}}_k. \tag{2.23}$$

It follows that we can write the metric elements in terms of tetrads as $g_{\mu\nu} = \eta_{ij}e^i_\mu e^j_\nu$ or $g_{\mu\nu} = e_\mu \cdot e_\nu$. For a spherically-symmetric system, the tetrads can be defined in terms of unknown functions f_1, f_2, g_1 and g_2 such that the non-zero tetrad components are

$$\begin{aligned} e^0_0 &= f_1, \qquad e^0_1 = f_2, \qquad e^1_0 = g_2, \\ e^1_1 &= g_1, \qquad e^2_2 = 1/r, \qquad e^3_3 = 1/(r\sin\theta). \end{aligned} \tag{2.24}$$

It has been shown that a natural gauge choice is one in which $f_2 = 0$, commonly known as the 'Newtonian gauge'. By using the tetrads we can calculate the coefficients and find the line element

$$ds^2 = \frac{g_1^2 - g_2^2}{f_1^2 g_1^2} dt^2 + \frac{2g_2}{f_1 g_1^2} dr dt - \frac{1}{g_1^2} dr^2 - r^2 d\Omega^2. \tag{2.25}$$

In our simplified scenario we can further simplify $g_2 = 0$ as the system is static, which leads to the final form for the metric:

$$ds^2 = \frac{1}{f_1^2} dt^2 - \frac{1}{g_1^2} dr^2 - r^2 d\Omega^2. \tag{2.26}$$

Furthermore we need to set up the apparatus relevant for the astrophysical model. It will be useful to define two linear differential operators

$$\begin{aligned} L_t &\equiv f_1 \partial_t + g_2 \partial_r, \\ L_r &\equiv g_1 \partial_r. \end{aligned} \tag{2.27}$$

Additionally it is also useful to construct explicitly the spin-connection coefficients F and G as described in [18]. The coefficients are position-gauge convariant and can let us write

$$[L_t, L_r] = G L_t - F L_r. \tag{2.28}$$

Since we are assuming there is no torsion, the spin connection can be written entirely in terms of tetrad components and their derivatives. The full equations for matter in the form of a perfect fluid can be found in [18]. By looking at the Bianchi identities one can notice that for a radially symmetric field, G can be identified as the radial acceleration and F as the Hubble function. In this static case G will be effectively playing the same role as F does in the cosmological case. We are particularly interested in the relation that fully defines G:

$$L_r(f_1) = f_1 G \Rightarrow G = \frac{g_1 \partial_r f_1}{f_1}. \tag{2.29}$$

Thus we would like to express our system in terms of a scale-gauge covariant quantity linked to G. We can show that

$$g \equiv rG - g_1, \tag{2.30}$$

is a suitable candidate as it is scale-gauge covariant and in fact invariant.

We can make the transition between our work and the "standard" approach by writing in terms of metric coefficients A and B as presented in [48]:

$$ds^2 = A(r)dt^2 - \frac{1}{B(r)} dr^2 - r^2(d\theta^2 + \sin\theta^2 d\phi^2), \text{where } A(r) = B(r) \ \ [48] \tag{2.31}$$

Our covariant quantity thus takes the form

$$g \equiv \frac{\sqrt{B}}{2A}(rA' - 2A), \tag{2.32}$$

and we can show that if we insist that $A = B$ then $g = rg_1' - g_1$.

We will proceed to write the equations of motion governing the gravitational sector in terms of g, g_1 and V.

2.4 Looking at vacuum

In this section we want to show that we can recover Mannheim's solution in vacuum as presented in [48]. Considering that we are looking at vacuum solutions we will be working with the gravitational Lagrangian only. As we have mentioned in the Introduction in eWGT we find equations by just varying with respect to h and A. We introduce V as the dilation gauge field. The full derivation in Geometric Algebra for a Riemann^2 Lagrangian can be found in Chapter 3 and thus we will just present the final forms for our equations. Variation of the action with respect to the h field reduces to three coupled gravitational field equations

$$(rg_1g' - 1 + g^2)(-2g_1^2 + rg_1g' + 2r^2g_1V' - 2rVg_1 + 2rg_1g_1' - 2r^2V^2 + g^2 + 1) = 0, \tag{2.33}$$

$$(rg_1g' - 1 + g^2)(-g_1^2 + rg_1g' + r^2g_1V' - rVg_1 + rg_1g_1' - r^2V^2 + rVg + gg_1 + g^2 + 1) = 0, \tag{2.34}$$

$$(rg_1g' - 1 + g^2)(rg_1g' + 2gg_1 + 2rVg + g^2 + 1) = 0, \tag{2.35}$$

and similarly we can show that the A-variation becomes,

$$g'' = \frac{1}{r^2g_1^2}\Big(- rg_1(2g_1 + Vr + 2g + g_1'r)g' - (g^2 - 1)(Vr + g_1)\Big). \tag{2.36}$$

We can easily see that a solution to the equations is obtained by setting $rg_1g' - 1 + g^2$ to zero.

2.4.1 Recovering the $A = B$ solution

If we introduce the extra constraint that $g_1 = f_1^{-1} = \sqrt{\frac{1}{B}}$ we can find an equation for B

$$\frac{1}{2}r^2B'' - rB' + B - 1 = 0, \tag{2.37}$$

which has the general solution

$$B = 1 + c_1r + c_2r^2, \text{ for some constants } c_1 \text{ and } c_2. \tag{2.38}$$

Although this is a viable solution, we will not consider it in our work since it does not admit a central singularity.

Another solution can be found by looking at the remainder of the coupled equations,

$$-2g_1^2 + rg_1g' + 2r^2g_1V' - 2rVg_1 + 2rg_1g_1' - 2r^2V^2 + g^2 + 1 = 0, \tag{2.39}$$

$$-g_1^2 + rg_1g' + r^2g_1V' - rVg_1 + rg_1g_1' - r^2V^2 + rVg + gg_1 + g^2 + 1 = 0, \tag{2.40}$$

$$rg_1g' + 2gg_1 + 2rVg + g^2 + 1 = 0. \tag{2.41}$$

It can be shown that this system is in fact a linear combination of the following two equations,

$$rg_1g' + 1 + g^2 + 2(g_1 + rV)g = 0, \tag{2.42}$$

$$V'r^2g_1 + rg_1g_1' - g_1^2 - (g + rV)(g_1 + rV) = 0. \tag{2.43}$$

We now have two equations for three quantities and thus inherit a residual gauge freedom. In order to find a solution we would need further constraints, like imposing $A = B$. The above equations would lead to an equation for V

$$V' = \frac{V(g_1' + V)}{g_1} \Rightarrow V = \frac{g_1}{c_1 - r}, \tag{2.44}$$

and then an equation for B

$$\frac{1}{2}r^2(r - c_1)B'' - r^2B' + (r + c_1)B + r - c_1 = 0, \text{ with the solution} \tag{2.45}$$

$$B = c_1\frac{2 + 3c_2c_1}{3r} - 1 - 3c_1c_2 + 3c_2r + c_3r^2. \tag{2.46}$$

We can rewrite this solution to correspond to the one found by Mannheim and Kazanas [48] by employing the following substitution

$$c_1 = \frac{3\beta'\gamma - 2}{\gamma}, \qquad c_2 = \frac{\gamma}{3}, \qquad c_3 = -k. \tag{2.47}$$

$$\Rightarrow B = 1 - \frac{\beta'(2 - 3\beta'\gamma)}{r} - 3\beta'\gamma + \gamma r - kr^2. \tag{2.48}$$

We notice straight away that for $\gamma = 0$ and $k \neq 0$ we regain the regime in General Relativity (with a cosmological constant) so the value of $1 - 3\beta'\gamma$ will account for the departure of the Weyl theory from the norm. We could worry that this structure does not support the standard representation of flat space at infinity. Considering models with a cosmological constant-type term do not have this property either, a linear term can't be criticised on these grounds.

Some readers might recognise this result as the trademark for Conformal Gravity as proposed by Mannheim and Kazanas [48]. This solution has a remarkable property, namely that it satisfies the 4th order version of the vacuum Poisson equation

$$\nabla^4 B = 0. \tag{2.49}$$

Taking advantage of this form, Mannheim and O'Brien replace the righthand side with a source term and derive rotational curves for a large sample of galaxies as described in the Introduction of this chapter.

2.4.2 A dangerous constraint

When we recovered Mannheim's vacuum solution we started from the strong assumption that we should be working with a 'conformally extended' Schwarzschild metric from General Relativity:

$$ds^2 = A(r)dt^2 - \frac{1}{B(r)}dr^2 - r^2(d\theta^2 + \sin\theta^2 d\phi^2), \text{where } A(r) = B(r) \;\; [48] \tag{2.50}$$

In this section we will start from first principles and show that by using this metric ansatz we do not accomplish an extension, but actually the Schwarzschild de Sitter solution. As a result we believe that keeping $A(r) \neq B(r)$ is crucial and we will base our further investigation on the corrected metric.

In [48] Mannheim et al find that the 4^{th} order equation 2.49 reduces to

$$B^{-1}W^{rr} = \frac{1}{3r^4}\Big(1 + y^3\frac{dy}{dr}\Big), \text{ for } y^2(r) \equiv r^4\frac{d(B(r)/r^2)}{dr}, \tag{2.51}$$

and can be further simplified in vacuum since $W^{rr} = 0$ to

$$\begin{aligned} \gamma &= y'^2 - \frac{1}{y^2}, \\ \gamma r &= (1+\gamma y^2)^{1/2} + 3\beta\gamma - 1, \end{aligned}$$

for *conveniently* defined integration constants γ, β. This leads to the general solution

$$A(r) = B(r) = 1 - 3\beta\gamma - \frac{\beta(2-3\beta\gamma)}{r} + \gamma r - kr^2. \tag{2.52}$$

Clearly we can also have a degenerate solution by setting $\gamma = 0$,

$$A(r) = B(r) = 1 - \frac{2\beta_2}{r} - k_2 r^2, \tag{2.53}$$

which is just the Schwarzschild de Sitter solution if we identify $\beta_2 = M$ and $k_2 = \frac{1}{3}\lambda$.

As previously stated, the Conformal Gravity action is invariant under both local transformations $g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(x) \equiv \Omega^2(x) g_{\mu\nu}$ and under general coordinate transformations $x^\mu \to x'^\mu$ for which

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho(x')}{\partial x'^\mu}\frac{\partial x^\sigma(x')}{\partial x'^\nu} g_{\rho\sigma}(x(x')). \tag{2.54}$$

So the combined transformation leads to

$$\tilde{g}'_{\mu\nu} = \Omega^2(r') g'_{\mu\nu}(x') = \Omega^2(r')\frac{\partial x^\rho(x')}{\partial x'^\mu}\frac{\partial x^\sigma(x')}{\partial x'^\nu} g_{\rho\sigma}(x(x')), \text{ for } r = \frac{r'}{\Omega(r')}. \tag{2.55}$$

Going back to the general line element 2.50 we can write after the combined transformation

$$ds^2 = \tilde{A}'(r')dt^2 - \frac{1}{\tilde{B}'(r')}dr'^2 - r'^2(d\theta^2 + \sin\theta^2 d\phi^2), \text{ where} \tag{2.56}$$

$$\tilde{A}'(r') = \Omega^2(r')A(r(r')), \text{ and} \tag{2.57}$$

$$\tilde{B}'(r') = \left(1 - r'\frac{d\ln\Omega}{dr'}\right)^{-2} B(r(r')). \tag{2.58}$$

We proceed to show that we can find a combined transformation (as the one presented in 2.56) that relates solutions 2.57 and 2.58. This proves that when we set $A = B$ in the metric we lose the extra degree of freedom required to reach a novel solution.

Demanding the two solutions to be equivalent,

$$1 - 3\beta\gamma - \frac{\beta(2-3\beta\gamma)}{r'} + \gamma r' - kr'^2 = \left(1 - r'\frac{d\ln\Omega}{dr'}\right)^{-2}\left(1 - \frac{2\beta_2\Omega}{r'} - \frac{k_2 r'^2}{\Omega^2}\right) \tag{2.59}$$

leads to the equation

$$\begin{aligned}&k\Omega'^2 r^5 - \Omega'(2k\Omega + \gamma\Omega')r^4 + (k\Omega^2 + 2\gamma\Omega'\Omega + 3\beta\gamma\Omega'^2 - \Omega'^2 - k_2)r^3 + \\ &(2\Omega\Omega' - 3\beta^2\gamma\Omega'^2 - \gamma\Omega^2 - 6\beta\gamma\Omega'\Omega + 2\beta\Omega'^2)r^2 + \beta\Omega(6\beta\gamma\Omega' + 3\gamma\Omega - 4\Omega')r - \\ &\Omega^2(2\beta_2\Omega - 2\beta + 3\beta^2\gamma) = 0, \text{ where we have dropped the prime on } r \text{ for convenience.}\end{aligned}$$

By imposing the trial solution $\Omega = ar + b$, for constants a and b, the above simplifies to

$$r^3(kb^2 - k_2 + a^2 - 2\beta_2 a^3) + r^2(2a - \gamma b - 6\beta_2 a^2)b + r(3\beta\gamma - 6\beta_2 a)b^2 + b^2(2\beta - 3\beta^2\gamma - 2\beta_2 b) = 0. \tag{2.60}$$

Now we demand the coefficients of r^i to vanish and find

$$
\begin{aligned}
b &= \frac{\beta(2-3\beta\gamma)}{2\beta_2} \text{ from the coefficient of } r^0, \\
a &= \frac{\beta\gamma}{2\beta_2} \text{ from the coefficient of } r^1, \\
k &= \frac{\beta^2\gamma^2(\beta\gamma-1)+4\beta_2^2 k_2}{\beta^2(2-3\beta\gamma)^2} \text{ from the coefficient of } r^3.
\end{aligned}
$$

Finally, we need to ensure that $A(r)$ transforms correctly. Using 2.56, if the ansatz for Ω is already satisfied, when we transform from 2.58 to 2.57 we require that

$$
\left(1-r'\frac{d\ln\Omega}{dr'}\right)^{-2}\left(1-\frac{2\beta_2\Omega}{r'}-\frac{k_2 r'^2}{\Omega^2}\right)=\Omega^2\left(1-\frac{2\beta_2\Omega}{r'}-\frac{k_2 r'^2}{\Omega^2}\right) \Rightarrow \Omega = cr\pm 1, \tag{2.61}
$$

so we can identify $c = a$ and $b = 1$ considering that b is positive for $\beta\gamma \ll 1, \beta_2 > 0$.

Now since $b = 1$,

$$
\beta_2 = \beta\left(1+\frac{3}{2}\beta\gamma\right) \Rightarrow a = \frac{\gamma}{2-3\beta\gamma} \text{ and } k_2 = k + \frac{\gamma^2(1-\beta\gamma)}{(2-3\beta\gamma)^2}.
$$

So we have found a combined solution $\Omega = 1 + \dfrac{\gamma r}{2-3\beta\gamma}$

for which the two solutions are equivalent.

We have shown that using the Mannheim-Kazanas metric does not give us more freedom than just by using Schwarzschild de Sitter. As we have been able to go from one to another just by gauge transformations within the theory the two must be physically equivalent. This result was anticipated, but to our knowledge not proven before. Therefore we will keep the metric parameters independent and conduct a general analysis.

2.5 Adding Matter

2.5.1 The original formalism

When we were discussing the vacuum solution we touched on the fact that in order to find galactic rotation curves we do not have to directly introduce matter. The approach employed can be found in [58]. In short, the authors use the vacuum static, spherically symmetric solution and introduce a source term, i.e.

$$
\nabla^4 B = \frac{3}{4\alpha_g B}(T^0_0 - T^r_r) = f(r). \tag{2.62}
$$

For a localized system such as a star the source function $f(r)$ is restricted to its interior region. Unlike the standard Poisson equation, where only local material within a galaxy contributes to the gravitational force, material from outside the galaxy also contributes. This is because the fourth-order solution is sensitive to the homogeneous cosmological background and the inhomogeneities in it. In the weak limit, keeping linear and quadratic potential terms, one finds a generalised centripetal velocity,

$$v_{TOT}^2(R) = v_{LOC}^2(R) + \frac{\gamma_0 c^2 R}{2} - \kappa c^2 R^2, \tag{2.63}$$

where v_{LOC} is the contribution of the material in a galaxy.

The authors could use this simple formalism, as on galactic scales one has the luxury to work in the weak limit and the sources are fairly uninvolved. If we want to explore other sectors we need to make appropriate modifications at the level of the Lagrangian to accommodate matter intrinsically. This is a priori non-trivial since the SET of ordinary matter has non-vanishing trace, which could be expected to be incompatible with a scale-invariant theory.

The initial approach was developed in [69] and has been used widely as the groundwork for many studies, including [63], [70].

The main idea is to introduce mass that is developed by fields that spontaneously break symmetry. It is convenient to start working with massive scalar fields (S) that are introduced via a non-minimally coupled curved space action,

$$I_M = -\int d^4x(-g)^{1/2}\left(\frac{1}{2}S^{;\mu}S^*_{;\mu} + \frac{1}{2}m^2 SS^* - \frac{\zeta}{12}SS^* R^\mu_\mu\right). \tag{2.64}$$

$\zeta = 1$ is imposed such that the coupling of the scalar field to the geometry is conformal, with the massless action ($m = 0$) being invariant under local conformal transformations.

In a plane wave solution to the $\partial_\mu \partial^\mu S = m^2 S$ wave equation of the form $S(x) = e^{ik\cdot x/V^{1/2}E_k^{1/2}}$, where $k^\mu k_\mu = -m^2$,$E_k = (k^2 + m^2)^{1/2}$ and V is the 3-volume, $T_{\mu\nu}$ becomes

$$T_{\mu\nu} = \frac{k_\mu k_\nu}{VE_k}. \tag{2.65}$$

In order to recover a perfect fluid one needs to incoherently add a set of six plane waves moving in the $\pm x$, $\pm y$, $\pm z$ directions, with the same k and E_k, such that the energy-momentum reads

$$T_{00} = \frac{6E_k}{V}, \qquad T_{xx} = T_{yy} = T_{zz} = \frac{2k^2}{E_k V}, \qquad T^\mu_\mu = -\frac{6m^2}{E_k V}. \tag{2.66}$$

This is indeed a perfect fluid with $\rho = 6E_k/V$ and $p = 2k^2/E_k V$.

In the above analysis it is shown how one can recover a perfect fluid from a field theory. Unfortunately it does not recover the process that develops mass. To investigate what is to happen in the dynamical mass case, it is convenient to consider a spin one-half matter field fermion (ψ). We get its mass through a real spin-zero Higgs scalar field (S). The action used reads as

$$I_M = -\int d^4x(-g)^{1/2}\Big(\frac{1}{2}S^{;\mu}S_{;\mu} - \frac{1}{12}S^2R^\mu_\mu + \lambda S^4 + i\overline{\psi}\gamma^\mu(x)[\partial_\mu + \Gamma_\mu(x)]\psi - hS\overline{\psi}\psi\Big), \quad (2.67)$$

where γ^μ are the Dirac matrices, Γ_μ the fermion spin connection,and h and λ dimensionless constants. The action chosen is the matter action for ψ and S that is invariant under the local conformal transformations.

Varying the action with respect to the matter fields yields the equations of motion

$$i\gamma^\mu(x)\Big(\partial_\mu + \Gamma_\mu(x)\Big)\psi - hS\psi = 0, \quad (2.68)$$

and

$$S^{;\mu}_{;\mu} + \frac{1}{6}SR^\mu_\mu - 4\lambda S^3 + h\overline{\psi}\psi = 0. \quad (2.69)$$

Varying the action with respect to the metric gives the conformal stress-energy tensor

$$\begin{aligned} T^{\mu\nu} \;=\;& i\psi\gamma^\mu(x)(\partial^\nu + \Gamma^\nu)\psi + \frac{2}{3}S^{;\mu}S^{;\nu} - \frac{1}{6}g^{\mu\nu}S^{;\alpha}S_{;\alpha} \\ & -\frac{1}{3}SS^{;\mu;\nu} + \frac{1}{3}g^{\mu\nu}SS^{;\alpha}_{;\alpha} - \frac{1}{6}S^2\Big(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R^\alpha_\alpha\Big) - g^{\mu\nu}\lambda S^4. \end{aligned} \quad (2.70)$$

In the presence of a spontaneously broken non-zero constant expectation value S_0 for the scalar field, for $\lambda = 0$, the energy momentum simplifies to

$$T_{\mu\nu} = i\overline{\psi}\gamma_\mu(x)[\partial_\nu + \Gamma_\mu(x)]\psi - \frac{1}{4}g_{\mu\nu}hS_0\overline{\psi}\psi - \frac{1}{6}S_0^2\Big(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R^\alpha_\alpha\Big), \quad (2.71)$$

In the flat space limit, the energy-momentum becomes

$$T_{\mu\nu} = i\overline{\psi}\gamma_\mu\partial_\nu\psi - \frac{1}{4}\eta_{\mu\nu}hS_0\overline{\psi}\psi, \text{ and the fermion obeys} \quad (2.72)$$

$$u\gamma^\mu\partial_\mu\psi - hS_0\psi = 0. \quad (2.73)$$

This translates in a straightforward quantization - a free fermion with mass $m = hS_0$ and yields one particle plane wave eigenstates of four-momentum $k^\mu = (E_k, k)$ with $E_k = (k^2 + m^2)^{1/2}$. Thus the energy-momentum reads

$$T_{\mu\nu} = \frac{1}{V}\begin{pmatrix} E_k & 0 & 0 & -k \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & 0 & k^2/E_k \end{pmatrix} + \frac{1}{V}\begin{pmatrix} -m^2/4E_k & 0 & 0 & 0 \\ 0 & m^2/4E_k & 0 & 0 \\ 0 & 0 & m^2/4E_k & 0 \\ 0 & 0 & 0 & m^2/4E_k \end{pmatrix} \quad (2.74)$$

We recognise a two-component structure to the fermion-energy momentum tensor, a standard kinematic piece and a dynamic part coming from symmetry breaking. On incoherently averaging over the directions of k, the energy momentum tensor is then found to take the form

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu + p\eta_{\mu\nu} + \Lambda\eta_{\mu\nu}, \text{ where} \tag{2.75}$$

$$\rho = \frac{6E_k}{V}, \qquad p = \frac{2k^2}{VE_k}, \qquad \Lambda = \frac{3m^2}{2VE_k}. \tag{2.76}$$

This shows how we can recover a perfect fluid with a cosmological constant Λ that arises naturally.

This proof shows how in Conformal Gravity one can use field theory to imitate a perfect fluid. The method proposed started with the most general (suitable) Lagrangian for the scalar and fermion fields. After standard analysis one ends up with a energy-momentum tensor that is initially not compatible with a perfect fluid. Mannheim proceeds to use a plane-wave formalism and then by incoherently adding a set of such plane waves he recovers the desired perfect fluid form. Thus we are at liberty to use this mechanism to produce matter and treat as a perfect fluid source.

2.5.2 Perfect Fluids - a classical treatment

As we would like to explore astrophysical and cosmological setups it is imperative to have a working pressure and density. A straightforward approach would be to introduce matter via a perfect fluid that acts as a source. Several authors have focused on such a spherically-symmetric perfect fluid solutions and we will introduce the work presented in [63], [71].

Going back to the gravitational Lagrangian, $L_G = -\frac{1}{2}C_{\kappa\lambda\mu\nu}C^{\kappa\lambda\mu\nu}$, we recall that the key relation we had was that the Bach tensor is sourced by the energy-momentum tensor via $W_{\mu\nu} = \frac{\alpha}{2}T_{\mu\nu}$. One can easily show, from the definition of the Bach tensor, that

$$W^0_0 - W^r_r = -\frac{B(rB)''''}{3r}. \tag{2.77}$$

For a fluid source described by T^μ_ν =diag$(\rho, -P_r, -P_\perp, -P_\perp)$ that has $T^\mu_\mu = 0$ to be conformal, this condition translates to a single simple field equation,

$$\frac{(rB)''''}{r} = -\frac{3\alpha}{2B}(\rho + P_r). \tag{2.78}$$

Another equation can be found by requiring the standard conservation equation, which in our setup reads as

$$P_r' + \frac{1}{r}(3P_r - \rho) + \frac{B'}{2B}(\rho + P_r) = 0. \tag{2.79}$$

This system does not produce a fully constrained setup, so the authors in [63], [71]introduce additional equations of state which relate ρ, P_r and $P_\perp$ such as polytropes to model stars.

Now, returning to the extended Lagrangian that contains a scalar field, the authors in [71] find that the modification is just an additional energy-momentum tensor $S_{\mu\nu}$, i.e.

$$W_{\mu\nu} = \frac{\alpha}{2}T_{\mu\nu} + T_{\mu\nu}(S), \text{ where} \tag{2.80}$$

$$T^\mu_\nu(S) = \frac{2}{3}S^{;\mu}_{;\nu} - \frac{1}{3}SS^{;\mu}_{;\nu} - \frac{1}{6}S^2R^\mu_\nu - \delta^\mu_\nu\Big(\frac{1}{6}S^{;\alpha}_{;\alpha} - \frac{1}{3}SS^{;\alpha}_{;\alpha} - \frac{1}{12}RS^2 + \lambda S^4\Big) \tag{2.81}$$

The authors now argue that as the scalar field should be considered as a gravitational degree of freedom (stressed by the absence of α in equation 2.81), when constructing the evolution equations one should consider it as an part of the gravitational sector.

Thus the equivalent of equation 2.78 becomes,

$$\frac{(rB)''''}{r} + \frac{1}{B}\left(BS'^2 - 2\lambda S^4 + \frac{1}{4}\Big(B' + \frac{2B}{r}\Big)(S^2)' - \frac{R}{12}S^2\right) = -\frac{3\alpha}{2B}(\rho + P_r). \tag{2.82}$$

The full system of equations is given by the above equation together with the equation

$$\frac{(r^2BS')'}{r^2} - \frac{R}{6}S - 4\lambda S^3 = 0, \tag{2.83}$$

the conservation equation 2.79 and imposed equations of state.

Focusing on just equations 2.82 and 2.83 we can make the following substitutions,

$$V(r) \equiv \frac{B(r)}{r^2}, \qquad u \equiv \frac{1}{r}, \qquad \Sigma(u) \equiv \frac{S(1/u))}{u}, \tag{2.84}$$

to find a more manageable version,

$$\begin{aligned} V'''' + (\Sigma')^2 - \frac{1}{2}\Sigma\Sigma'' &= -\frac{3\alpha u^2}{2V}(\rho + P_r), \\ (V\Sigma)' + \frac{V'' - 2}{6}\Sigma - \nu\Sigma^3 &= 0. \end{aligned} \tag{2.85}$$

The authors proceed to show that for vacuum the system possessses a three-parameter family of explicit solutions given by

$$\begin{aligned} B(r) &= (1 + r/a)^2 - \frac{r_h}{r}\frac{(1 + r/a)^3}{(1 + r_h/a)} + \frac{\nu S_0^2 r_h^2}{2}\left(\frac{r^2}{r_h^2} - \frac{r_h}{r}\frac{(1 + r/a)^3}{(1 + r_h/a)^3}\right), \\ S(r) &= \frac{S_0}{1 + r/a}, \text{ where } r_h, a, S_0 \text{ are free parameters.} \end{aligned} \tag{2.86}$$

The system can be further extended to the full Lagrangian as presented in [63]. When we introduce the fermion field, the energy-momentum tensor separates further, and we can write

$$W^\mu_\nu = \frac{\alpha}{2}T_{\mu\nu}(\psi) + T_{\mu\nu}(S), \tag{2.87}$$

where the fermion contribution can be represented as a perfect fluid with

$T_{\mu\nu} = diag(-\rho_f, p_{rf}, p_{\perp f}, p_{\perp f})$, whose components are constrained by the Dirac equation 2.73 such that,

$$p_{rf} + 2p_{\perp f} - \rho_f = \overline{\psi} u \gamma^\mu \partial_\mu \psi = hS\overline{\psi}\psi. \tag{2.88}$$

The authors in [63] follow the same approach as [71] and transfer the contributions from the scalar field to the gravitational sector. Unlike [71] who introduce the perfect fluid externally, in [63] the perfect fluid is represented with the aid of the fermion field at the level of the Lagrangian. The setups presented become analogous at the level of the pressure and density.

2.5.3 A different approach

Although the setup presented is interesting, solutions to the system are very complicated and the authors in [71] discuss at length numerical solutions for polytropes. As we have shown in the vacuum case, we can recover the standard result presented in [48]. To explore Mannheim's argument further, we have continued our work by adding matter to the system via a perfect fluid. Unlike [71] and [63] we will introduce a 'true' perfect fluid that has $P_r = P_\perp$ and our system will be significantly more constrained.

Working in the context of a gauge theory has proven fruitful as we have discovered a possible way to reach full differential equations for the pressure and density that describe the perfect fluid. We have also found that by rewriting the setup in terms of invariant quantities one can significantly simplify the system of equations.

By returning to the gauge transformations presented in the previous sections and recalling the definition of the dilation vector field, one can find in the new notation that the gauge transformation takes the form

$$V(r') = \phi\Big(V'(r') + g_1'(r')\frac{d\ln\phi}{dr'}\Big). \tag{2.89}$$

In the original coordinates we can write

$$V(r) = \frac{1}{1 + r\frac{d\ln\phi}{r}}\Big(\phi V'(r) + \frac{d\ln\phi}{dr}(g_1'(r) + r\phi V'(r))\Big). \tag{2.90}$$

Finally, by combining the above equations it becomes transparent that we have found an invariant quantity, namely $g_1 + rV$ as

$$g_1(r) + rV(r) = g_1'(r') + r'V'(r'). \tag{2.91}$$

We have reached a point where we have enough invariant variables to redefine the setup in a convenient form. We will call the new invariant X and together with g (as previously defined in equation 2.30) we will proceed to find solutions for the *physical* pressure and density (namely $\rho^P \equiv \rho/\phi^3$ and $P^P \equiv P/\phi^3$). We consider them physical because they possess the appropriate scaling.

We need to redefine our system in terms of a new variable, such that the intrinsic derivative in the r direction is scale-gauge and position-gauge change covariant. The obvious choice is to define $s = \ln r$, such that

$$L_s = rL_r = rg_1\frac{d}{dr}. \tag{2.92}$$

The A-equations become two constraints, namely

$$\begin{aligned} L_s^2 g + L_s g(X+2g) + X(g^2-1) &= 0 \\ L_s\phi &= \phi r V1. \end{aligned} \tag{2.93}$$

By examining the second order equation for g we can find that the following ansatz provides a solution for $L_s g$:

$$L_s g = 1 - g^2 - f, \text{ given that } f(r) \text{ satisfies } L_s f = -Xf. \tag{2.94}$$

Looking at the first equation we can easily see that we can re-write it as $L_s(r\phi) = r\phi X$, and thus we have found a perfect candidate for f.

Hence our system of equations becomes

$$L_s g = 1 - g^2 - \frac{C}{r\phi}, \text{ with } C \text{ a constant.} \tag{2.95}$$

We now proceed to look at the h-equation. After some manipulations we can reach a constraint equation,

$$4\beta C(C - 2r\phi Xg - 2r\phi) + 24\pi(r\phi)^6 P^p - 3(r\phi)^4\Big((r\phi)^2\lambda + 3\kappa X^2 - \kappa + 2\kappa Xg\Big) = 0. \tag{2.96}$$

Considering we are using gauge invariant variables, we can go in the frame where $X \to g_1$ and $\phi \to 1$ (we are free to transform to the Einstein gauge). The A-equations give us all the information we need to constrain the system. We find a relation for g_1,

$$\frac{dg_1}{dr} = \frac{-24\pi r^6\rho - 3g_1^2\kappa r^4 - 3\lambda r^6 + 8Cg_1^2\beta r + 3\kappa r^4 + 4C^2\beta - 8C\beta r}{2g_1 r^2(3r^3\kappa + 4C\beta)}, \tag{2.97}$$

and a further constraint

$$-72\rho\pi\kappa r^9 - 72\pi P\kappa r^9 + 18g_1 g\kappa^2 r^7 + 48C\rho\pi\beta r^6 - 96C\pi P\beta r^6 + 54Cg_1^2\beta\kappa r^4$$
$$+12Cg_1 g\beta\kappa r^4 + 18C\beta\lambda r^6 - 9C\kappa^2 r^6 + 18\kappa^2 r^7 - 16C^2 g_1 g\beta^2 r$$
$$+12C^2\beta\kappa r^3 - 42C\beta\kappa r^4 + 8C^3\beta^2 - 16C^2\beta^2 r = 0. \qquad (2.98)$$

Introducing these constraints in the h-equations, we find explicit solutions for P and ρ:

$$P = -\frac{-9g_1^2\kappa r^4 - 6r^4 g\kappa g_1 - 3\lambda r^6 - 8Cg_1 g\beta r + 3\kappa r^4 + 4C^2\beta - 8C\beta r}{24\pi r^6}, \qquad (2.99)$$

$$\rho = -\frac{1}{8\pi r^6(2C\beta - 3r^3\kappa)}\Big(-9g_1^2\kappa^2 r^7 - 3\kappa\lambda r^9 + 6Cg_1^2\beta\kappa r^4 - 12Cg_1 g\beta\kappa r^4 +$$
$$2C\beta\lambda r^6 - 3C\kappa^2 r^6 + 9\kappa^2 r^7 - 16C^2 g_1 g\beta^2 r + \qquad (2.100)$$
$$8C^2\beta\kappa r^3 - 18C\beta\kappa r^4 + 8C^3\beta^2 - 16C^2\beta^2 r \Big).$$

Thus in our formalism we are not at liberty to write down a density profile - we are required to integrate g and g_1. In GR either we can specify a ρ and P then has to follow, or we can specify an equation of state and then both ρ and P follow from this. Here we don't seem to be able to pursue either route. The system we are looking at is given by

$$\frac{dg}{dr} = -\frac{g^2 r - r + C}{g_1 r^2}$$
$$\frac{dg_1}{dr} = \frac{-6g_1^2\kappa r^4 + 4Cg_1^2\beta r - 12Cg_1 g\beta r - 3C\kappa r^3 + 6\kappa r^4 + 8C^2\beta - 16C\beta r}{2r^2 g_1(2C\beta - 3r^3\kappa)} \qquad (2.101)$$

We would like to switch to the conventional metric variables A and B . One can recall from the definition of g that we can express it directly as,

$$g \equiv \frac{\sqrt{B}}{2A}(rA' - 2A), \qquad (2.102)$$

and thus our system becomes

$$3A\frac{dB}{dr}\kappa r^5 - 6B\kappa r^4 A - 6CB\frac{dA}{dr}\beta r^2 - 2CA\frac{dB}{dr}\beta r^2 - 3C\kappa r^3 A$$
$$+6\kappa r^4 A + 16CB\beta rA + 8C^2\beta A - 16C\beta rA = 0, \qquad (2.103)$$

$$-\Big(\frac{dA}{dr}\Big)^2 Br^3 + 2AB\frac{d^2A}{dr^2}r^3 - 2\frac{dA}{dr}ABr^2 + \frac{dA}{dr}A\frac{dB}{dr}r^3$$
$$-2A^2\frac{dB}{dr}r^2 + 4BA^2 r + 4CA^2 - 4rA^2 = 0. \qquad (2.104)$$

Now can find A' from the first equation and then use the second equation to find an relation for B'' only in terms of B.

We are finally equiped to find P and ρ as functions of B only and they read as:

$$\begin{aligned} P = \frac{1}{144C\beta\pi r^6} \quad & \Big(9\frac{dB}{dr}\kappa^2 r^8 - 18B\kappa^2 r^7 + 6C\frac{dB}{dr}\beta\kappa r^5 + 18C\beta\lambda r^6 \\ & -9C\kappa^2 r^6 + 18\kappa^2 r^7 + 42CB\beta\kappa r^4 - 8C^2\frac{dB}{dr}\beta^2 r^2 \\ & +C^2\beta\kappa r^3 - 42C\beta\kappa r^4 + 16C^2B\beta^2 r + 8C^3\beta^2 - 16C^2\beta^2 r \Big) \end{aligned} \quad (2.105)$$

$$\rho = \frac{-3\frac{dB}{dr}\kappa r^5 - 3\lambda r^6 - 3B\kappa r^4 - 4C\frac{dB}{dr}\beta r^2 + 3\kappa r^4 + 8CB\beta r + 4C^2\beta - 8C\beta r}{24r^6\pi} \quad (2.106)$$

It would be nice to gain some insight into this solution. If we expand B as a series,

$$B(r) = \Sigma_{i=0}^{n} a_i r^i + \frac{a_-}{r}, \quad (2.107)$$

and substitute in the second order equation for B, we find that a minimum representation is given by

$$B(r) = -\frac{C}{3r} + a_2 r^2 + a_7 r^7 + \frac{5a_7}{C} r^8 + \frac{135a_7}{7C^2} r^9 + \frac{15a_7(8C^2 a_2\beta + 5C^2\kappa + 144\beta)}{32C^3\beta} r^{10}. \quad (2.108)$$

We call it a minimum representation since when we introduce more terms we do not find a change in the limit we are interested in. By looking at the expansion for pressure and density, we find that there is a preferred a_2 that cancels undesirable contributions, namely $a_2 = -\frac{\lambda}{3\kappa}$. We are mostly interested in the equation of state and thus we look at $w \equiv \frac{P}{\rho}$. We find that

$$w = \frac{1}{3} - \frac{11}{10}\frac{\kappa}{C\beta} r^3 + O(r^4). \quad (2.109)$$

We have shown that the general setup produces the equation of state for radiation at the origin. We will now proceed to explore the solution in the "standard" case where $A = B$.

From A' equation we get a simplified constraint,

$$\frac{(8C\beta + \kappa 3r^3)(-\frac{dB}{dr} r^2 + 2rB + C - 2r)}{6C\beta r^2} = 0, \quad (2.110)$$

which produces the solutions

$$\begin{aligned} B &= -\frac{C}{3r} + 1 + r^2 C_1, \\ P &= \frac{3C_1\kappa + \lambda}{8\pi} = -\rho. \end{aligned} \quad (2.111)$$

Unfortunately we have found that in this regime the solution is not physical. As we have previously shown, the case where $A = B$ describes a Schwarzschild de Sitter cosmology and thus we would like to have a viable solution in this setup. Mannheim's assumption that we can always use conformal rescaling to make A=B clearly cannot apply in this case since it leads to a non-physical form of fluid. Although discouraging, our work is not over as we have not explored all of our options. As we have previously emphasized, in our approach torsion plays a vital part and thus we are hopeful that we will find a viable solution once we introduce torsion.

We will focus on the full analysis in the next sections, but in the meantime we would like to comment on the link with the work that we have previously mentioned.

Let us start by looking at the form for g_1 in terms of X and ϕ, i.e.

$$g_1 = \frac{X}{1 + r\frac{d\ln\phi}{dr}}. \tag{2.112}$$

If we have a solution to our system in terms of invariant functions $X(r\phi)$ and $g(r\phi)$, the non-invariant g_1 is generated from the physical X and the gauge choice ϕ. Using the equation for g,

$$g = rg_1' - g1, \text{ we can write that} \tag{2.113}$$

$$g(r\phi) = r\frac{d}{dr}\left(\frac{X(r\phi)}{1 + r\frac{d\ln\phi}{dr}}\right) - \frac{X(r\phi)}{1 + r\frac{d\ln\phi}{dr}}. \tag{2.114}$$

As we know g and X we now have a second order differential equation we can solve for ϕ. If we define $z = 1/\phi$, we can write

$$g(r\phi) = r\phi X'(r\phi) - X(r\phi)\left(1 - r^2\frac{z\frac{d^2z}{dr^2}}{(z - r\frac{dz}{dr})^2}\right), \text{ where differentiation is with respect to } r. \tag{2.115}$$

This equation shows that we have the same relationship between g and X as that between g and g_1, if and only if z is of the form $z = ar + b$, or

$$\phi = \frac{1}{ar + b}. \tag{2.116}$$

One can notice that this is the result presented in [63] and [71]. By imposing a perfect fluid we have a fully constrained system and we are not at liberty to write down a spare relation between P and ρ, unlike [71] and [63]. We show that when we keep the metric general we find a solution that behaves as radiation in the $B \to 0$ limit. Thus the theory cannot accommodate the sort of matter we deal with in astrophysical problems.

2.6 Introducing Torsion

Until now we have focused on using our gravitational representation to place ourselves in line with current research. Although the general approach is to introduce a 'partial' perfect fluid that allows extra freedom to reach viable solutions, we hope that we could be successful by instead generalising the setup.

At the core of any GTG is the presence of torsion. In particular, as we have shown in the Introduction, from the construction of eWGT we are in a particularly favourable position to explore various torsion profiles and their implications for the cosmology. Thus it is no surprise that we will dedicate the reminder of this chapter to extending our theory beyond the standard approach and focus on a more complete model that incorporates torsion.

We will start by presenting the basic definitions that allow us to construct our model for torsion. We will build upon the foundation that we have set out in the Introduction and employ the notation presented in [21]and [22].

Fundamentally a connection specifies how a vector field is transported along a curve. In a local coordinate chart with basis vectors $e_\mu = \partial_\mu$, the connection coefficients $\Gamma^\lambda_{\nu\mu}$ are defined by $\nabla_{e_\nu} e_\mu = e_\lambda \Gamma^\lambda_{\nu\mu}$. A connection is said to be metric compatible if

$$\nabla_\lambda g_{\mu\nu} \equiv \partial_\lambda g_{\mu\nu} - \Gamma^\rho_{\lambda\mu} g_{\rho\nu} - \Gamma^\rho_{\lambda\nu} g_{\rho\mu} = 0. \tag{2.117}$$

On the other hand we define the spin connection A_μ for the Lorentz group as $A_\mu = \frac{1}{2} A^{ab}_\mu S_{ab}$ with S_{ab} a representation of the Lorentz generators. The general connection can be related to the spin connection with the aid of tetrads as

$$\Gamma^\rho_{\mu\nu} = h^\rho_a \partial_\mu h^a_\nu + h^\rho_a A^a_{b\mu} h^b_\nu. \tag{2.118}$$

A considerate approach is to separate the space and connections as curvature and torsion are in fact properties of the connection. In General Relativity the Levi-Civita connection can be interpreted as part of the spacetime definition as all particles and fields feel this connection the same. However, we can accommodate several connections with different curvature and torsion and thus it is more convenient to take spacetime simply as a manifold and connections as additional structures. This formalism is embedded in the $Palatini$-style framework we work with. When torsion was not present the structure was more subtle (comparatively to a standard metric-based formalism), but in the full case it becomes the foundation block.

Bearing this in mind we can define the torsion tensors of the connection $A^a_{b\mu}$ as

$$T^a_{\nu\mu} = \partial_\nu h^a_\mu - \partial_\mu h^a_\nu + A^a_{e\nu} h^e_\mu - A^a_{e\mu} h^e_\nu, \tag{2.119}$$

$$\text{and } T^\rho_{\mu\nu} \equiv \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}. \tag{2.120}$$

We have made a slight abuse of notation by incorporating the $\frac{1}{2}$ in the definition of T (compared to equation 1.8). A vital property of torsion is that it can be decomposed with respect to the Lorentz group into three irreducible tensors, that we will represent as

$$T \equiv T^{(1)} + T^{(2)} + T^{(3)} = \text{tentor} + \text{trator} + \text{axitor}. \tag{2.121}$$

The torsion tensor has 24 components, of which $T^{(1)}$ has 16, $T^{(2)}$ has 4 and $T^{(3)}$ the remaining 4.

These quantities are defined as

$$\begin{aligned} T^{\alpha(2)} &= \frac{1}{n-1}\vartheta^{\alpha} \wedge (e_{\beta}\rfloor T^{\beta}) \\ T^{\alpha(3)} &= (-1)^{s}\frac{1}{3}\{\vartheta^{\alpha} \wedge (T^{\beta} \wedge \vartheta^{\beta})\} \\ T^{\alpha(1)} &= T^{\alpha} - T^{\alpha(2)} - T^{\alpha(3)}, \end{aligned} \tag{2.122}$$

where $\vartheta^{\alpha} \equiv e^{\beta}_{j} dx^{j}$ is the coframe, $e_{\beta}\rfloor T^{\beta}$ is the trace torsion one-form and n the number of dimensions (here 4). This makes $T^{\alpha(2)}$ the totally anti-symmetric torsion and $T^{\alpha(1)}$ the traceless non totally anti-symmetric part of torsion.

The most general form for torsion for spherically symmetric systems can be written elegantly in Geometric Algebra as introduced in [18]. For a representation given in terms of $e = [e_t, e_r, e_{\theta}, e_{\phi}]$, we write the torsion as $T = [T_0, T_1, T_2, T_3]$. The components can be defined as,

$$\begin{aligned} T_0(t,r) &= G(t,r)e_r e_t + I\tilde{G}(t,r)e_r e_t, \\ T_1(t,r) &= F(t,r)e_t e_r + I\tilde{F}(t,r)e_r e_t, \\ T_2(t,r) &= -\frac{1}{2}F(t,r)e_{\theta}e_t - \frac{1}{2}G(t,r)e_r e_{\theta} + I(\tilde{S}(t,r)e_{\theta}e_t + \tilde{T}(t,r)e_r e_{\theta}), \\ T_3(t,r) &= -\frac{1}{2}F(t,r)e_{\phi}e_t - \frac{1}{2}G(t,r)e_r e_{\phi} + I(\tilde{S}(t,r)e_{\phi}e_t + \tilde{T}(t,r)e_r e_{\phi}) \end{aligned} \tag{2.123}$$

We will work with two simplifications of this form - the 'cosmological' torsion and the'real' torsion. We have written some components in this form to clearly separate the two types of torsion. For cosmological torsion we have only $\tilde{G}, \tilde{F}, \tilde{S}, \tilde{T}$ non-zero and in particular if $\tilde{G} = \tilde{T}$ and $\tilde{F} = \tilde{S}$ we recover $T^{\alpha(3)}$. For 'real' torsion we have only G and F and this case corresponds to $T^{\alpha(2)}$. In our work we do not use $T^{\alpha(1)}$. Thus our most general torsion is given by

$$T^{\alpha} \equiv T^{\alpha(2)} + T^{\alpha(3)} \text{ and we will treat the components independently.} \tag{2.124}$$

New variables need to be introduced in our formalism. χ and ζ are the coefficients for the torsion Lagrangian entries. Specifically the Lagrangian contains new terms,

$$L_T = \chi Q'(\partial_a)Q'(a) + \frac{1}{2}\zeta T^2, \tag{2.125}$$

where $Q'(a)$ is the full eWGT torsion (with the contraction part automatically subtracted) and T is the trivector bit. Both of these are multiplied by ϕ^2 to give the right weight in the Lagrangian.

2.7 Cosmological Torsion

Our first attempt at understanding how the cosmology changes with the presence of torsion is to introduce the simplest form - cosmological-style torsion.

$$\tilde{G} = \tilde{T} = Q_1, \qquad \tilde{F} = \tilde{S} = -Q_0 \tag{2.126}$$

The system of governing equations simplifies significantly and we start by looking at two equations in particular, the generators for the torsion components,

$$\frac{2\beta}{r}\Big(- rQ_1 g_1 \frac{d}{dr}Q_0 + rg_1 \frac{d}{dr}Q_1 Q_0 + 2Q_0 Q_1 g_1 + 2rQ_0 V Q_1\Big) = 0, \tag{2.127}$$

$$\frac{\beta}{r}\Big(3Q_0 Q_1 g + 7Q_0 Q_1 g_1 + rQ_1 g_1 \frac{d}{dr}Q_0 + 2rg_1 \frac{d}{dr}Q_1 Q_0 + 4rQ_0 V Q_1\Big) = 0. \tag{2.128}$$

By direct manipulation we find the equations reduce to

$$\begin{aligned} \frac{dQ_0}{dr} &= -\frac{Q_0(g+g_1)}{rg_1}, \\ \frac{dQ_1}{dr} &= -\frac{Q_1(g+3g_1+2Vr)}{rg_1}. \end{aligned} \tag{2.129}$$

These equations have two desirable properties. Firstly we notice that the evolution equations are independent (i.e. the equation for Q_0 does not depend on Q_1 and viceversa) which lets us shut down either component without damaging the setup. Secondly neither equation depends explicitly on the scalar field. In our research we have noticed that cosmological torsion setups are either free-propagating or non-interacting with the other fields that introduce matter. In our analysis that field would be the scalar field and thus our results are consistent with our previous understanding.

Introducing the torsion constraints in the remaining set of equations we can find a relation for $\frac{dg}{dr}$:

$$\frac{dg}{dr} = \frac{1}{4}\frac{4\beta + 3r^2\phi^2\kappa + 18\zeta\phi^2 r^2 + 12\chi\phi^2 r^2 - 4\beta g^2}{\beta g_1 r} \tag{2.130}$$

At this point we introduce new notation, $x \equiv \eta + 6\kappa$. With these simplifications the A-equations become

$$\frac{\phi}{3r}\Big(12r\kappa g_1\frac{d}{dr}\phi + rg_1\frac{d}{dr}\phi x + 6\phi r\kappa V - r\phi V x + 72rg_1\zeta\frac{d}{dr}\phi + 48rg_1\chi\frac{d}{dr}\phi$$
$$+18g_1\phi\kappa + 108g_1\zeta\phi + 72g_1\chi\phi + 36Vr\zeta\phi + 24Vr\chi\phi\Big) = 0 \qquad (2.131)$$

$$\frac{\phi}{3r}\Big(6r\kappa g_1 d\frac{d}{dr}\phi - rg_1 d\frac{d}{dr}\phi x + 3\phi r\kappa V + r\phi V x + 36rg_1\zeta\frac{d}{dr}\phi + 24rg_1\chi d\frac{d}{dr}\phi$$
$$+9g_1\phi\kappa + 54g_1\zeta\phi + 36g_1\chi\phi + 18Vr\zeta\phi + 12Vr\chi\phi\Big) = 0 \qquad (2.132)$$

$$Q_1\phi^2(6\zeta + 4\chi + \kappa) = 0 \qquad (2.133)$$

From the first two equations we find

$$rx(g_1\frac{d}{dr}\phi - \phi V) = 0. \qquad (2.134)$$

From the third we find a constraint for the constants, namely $\zeta = -\frac{1}{6}(4\chi + \kappa)$. This leads to an interesting form for g (using equation 2.30),

$$\frac{dg}{dr} = \frac{1 - g^2}{rg_1}. \qquad (2.135)$$

This is the exact form we find in the case where we do not have torsion if we require the constant C per equation 2.93, from the previous section, to vanish. Thus although one would expect that adding torsion would introduce more freedom, it actually further constrains the setup.

Equipped with the above evolution equations we can proceed to look at the h-equations. Depending on the choice we make in equation 2.134, i.e whether x is zero, we find very different behaviour. We proceed to split our analysis in two cases and we start by looking at the most general case.

2.7.1 x non-zero

From equation 2.134 we find directly an equation for $\frac{d}{dr}\phi$, namely

$$\frac{d\phi}{dr} = \frac{\phi V}{g_1}. \qquad (2.136)$$

This constraint is the familiar setup we also had in the torsion-less case. By employing this constraint we can return to the h and A equations and solve to find evolution equations for P,

ρ and g_1. Instead in order to compare directly with the non-torsion case we will move to the conformal variable X, which we defined as

$$X \equiv rV + g_1. \tag{2.137}$$

Our solution reads

$$P = \frac{\phi\left(\phi^2\lambda r^2 + 3X^2\kappa + 2\kappa g X - \kappa\right)}{8\pi r^2} \tag{2.138}$$

$$\rho = -\frac{\phi\left(\phi^2\lambda r^2 + 3X^2\kappa - 3\kappa\right)}{8\pi r^2} \tag{2.139}$$

$$\frac{dX}{dr} = \frac{X^2 - 1}{rg_1} \tag{2.140}$$

These equations are the same as the ones that we have found in the non-torsion case. Although the solution might not seem appealing, this is an important result as far as torsion construction is concerned. We have shown that we are at liberty to introduce a form of cosmological torsion, that does not have an explicit source, and does not interact directly with the matter. The only role of this type of torsion is to restrict the evolution equation for g, which creates a stronger tie between the metric generators.

2.7.2 Vanishing x

The A-equations we work with are given by

$$\begin{aligned}\frac{-1}{r^2}\Big(&2r^2\phi g_1^2\partial_{rr}\phi\kappa + 4r\phi g_1^2\frac{d}{dr}\phi\kappa + 8\pi\phi r^2\rho - r^2 g_1^2\frac{d}{dr}\phi^2\kappa + \phi^2\kappa g_1^2 + \lambda\phi^4 r^2\\ &-\kappa\phi^2 + 2r^2\phi g_1\frac{d}{dr}g_1\frac{d}{dr}\phi\kappa + 2r\phi^2\kappa g_1\frac{d}{dr}g_1\Big) = 0,\end{aligned} \tag{2.141}$$

$$\begin{aligned}-\frac{1}{r^2}\Big(&6r\phi g_1^2\frac{d}{dr}\phi\kappa - 8\pi\phi P r^2 + 3r^2 g_1^2\frac{d}{dr}\phi^2\kappa + 3\phi^2\kappa g_1^2 2r\phi g_1\frac{d}{dr}\phi\kappa g\\ &+\lambda\phi^4 r^2 - \kappa\phi^2 + 2g_1\phi^2\kappa g\Big) = 0,\end{aligned} \tag{2.142}$$

$$\begin{aligned}\frac{-1}{r^2}\Big(&2r^2\phi g_1^2\partial_{rr}\phi\kappa + 4r\phi g_1^2\frac{d}{dr}\phi\kappa - 8\pi\phi P r^2 - r^2 g_1^2\frac{d}{dr}\phi^2\kappa + \phi^2\kappa g_1^2 + 2r\phi g_1\frac{d}{dr}\phi\kappa g\\ &+\lambda\phi^4 r^2 + \kappa\phi^2 + 2r^2\phi g_1\frac{d}{dr}g_1\frac{d}{dr}\phi\kappa + 2r\phi^2\kappa g_1\frac{d}{dr}g_1 + 2g_1\phi^2\kappa g\Big).\end{aligned} \tag{2.143}$$

We can find an expression for $\frac{d}{dr}\phi$, by direct substitution,

$$\frac{d\phi}{dr} = \frac{-4\pi r^2 \rho + \phi\kappa - 4\pi P r^2 + g_1 \phi \kappa g}{r g_1 \kappa g}. \tag{2.144}$$

The h-equations reduce to two main equations,

$$\begin{aligned} r^2\phi^4\lambda g^2\kappa + 8\pi\phi r^2\rho g^2\kappa + 48P^2\pi^2 r^4 + 96P\rho\pi^2 r^4 \\ +48\rho^2\pi^2 r^4 - 3\phi^2\kappa^2 g^2 - 24\phi P\pi\kappa r^2 - 24\phi\rho\pi\kappa r^2 + 3\phi^2\kappa^2 = 0, \end{aligned} \tag{2.145}$$

$$\begin{aligned} r^2\phi^4\lambda g^2\kappa + 8\phi\frac{dP}{dr}g_1\pi g\kappa r^3 + 8\phi\frac{d\rho}{dr}g_1\pi g\kappa r^3 + 24\phi g_1 P\pi g\kappa r^2 \\ +24\phi g_1\rho\pi g\kappa r^2 + 8\phi\pi g^2\kappa r^2 + 16\pi\phi r^2\rho + g^2\kappa - 16P^2\pi^2 r^4 - 32P\rho\pi^2 r^4 \\ -16\rho^2\pi^2 r^4 - 3\phi^2\kappa^2 g^2 - 8\phi P\pi\kappa r^2 - 8\phi\rho\pi\kappa r^2 + 3\phi^2\kappa^2 = 0. \end{aligned} \tag{2.146}$$

Using this system of equations we can write the evolution equations for the pressure and density as

$$\begin{aligned} \frac{dP}{dr} &= -\frac{4\pi\rho^2 r^2 - 4\pi\rho P r^2 - 8\pi P^2 r^2 + \rho g^2\phi\kappa + g^2 P\phi\kappa + 3gP\phi g_1\kappa - \rho\phi\kappa + 2P\phi\kappa}{9\phi\kappa r g_1} \\ \frac{d\rho}{dr} &= \frac{3\rho(4r^2\pi\rho + 4r^2\pi P - \phi g\kappa g_1 - \phi\kappa)}{g\phi\kappa r g_1} \end{aligned} \tag{2.147}$$

We can now define new variables, which we can think of as “physical” variables, $\tilde{\rho} = \frac{\rho}{\phi^3}$ and $\tilde{P} = \frac{P}{\phi^3}$. Using these quantities we can easily notice that $\tilde{\rho}$ is a constant and thus $\rho \propto \phi^3$ which makes the density to be fully driven by the scalar field. If we write $\tilde{\rho} \equiv \rho_0$, a constant we can re-write the other evolution equations as

$$\frac{d\tilde{P}}{dr} = -\frac{(\tilde{P} + \rho_0)(4\pi\tilde{P}\phi^2 r^2 + 4\pi\rho_0\phi^2 r^2 + \kappa g^2 - \kappa)}{g\kappa r g_1}, \tag{2.148}$$

$$\frac{d\phi}{dr} = \frac{\phi(4\pi\tilde{P}\phi^2 r^2 + 4\pi\rho_0\phi^2 r^2 - g g_1\kappa - \kappa)}{g g_1 \kappa r}. \tag{2.149}$$

When we constrained x we freed g and without any further information we have an incomplete system of equations. One option would be to impose the usual relation between V and ϕ, $\frac{d\phi}{dr} = \frac{\phi V}{g_1}$. As in the non-torsion case, we can show that the following relation satisfies the field equations,

$$\frac{dg}{dr} = \frac{1 - g^2}{g_1 r} - \frac{C}{r^2\phi g_1}. \tag{2.150}$$

Thus we have forced g to behave in the “standard” way at the expense of enslaving the dilaton to the scalar field.

A scenario that is worth exploring is the one where we just have a constant density profile, and thus a constant $\phi \equiv \phi_0$. Although an oversimplification, it is important to understand the effect of introducing a small constant scalar field to the system as it could act as a fine-tuning parameter. In this case, the physical pressure reduces to

$$\tilde{P} = -\frac{4\pi\rho_0\phi_0^2 r^2 - g g_1 \kappa - \kappa}{4\pi\phi_0^2 r^2}, \tag{2.151}$$

and acts as a constraint for g_1,

$$\kappa g\Big(g_1 \frac{dg_1}{dr} r - g_1^2 + 1\Big) = 0. \tag{2.152}$$

This restricts the metric to $g_1 = \sqrt{Cr^2 + 1}$. Since we know g_1 we can easily find a form for g by solving equation 2.150,

$$g = \tanh\left(C_1 - \arctan\left(\frac{1}{\sqrt{1 + Cr^2}}\right)\right) = \frac{\sinh(C_1)\sqrt{1 + Cr^2} - \cosh(C_1)}{\cosh(C_1)\sqrt{1 + Cr^2} - \sinh(C_1)}. \tag{2.153}$$

We can show the solution is well behaved by verifying the expansion around the origin,

$$g = -1 + \frac{C}{2}\Big(2\cosh(C_1)^2 + 2\sinh(C_1)\cosh(C_1) - 1\Big)r^2 + O(r^4). \tag{2.154}$$

Finally from to the constraint equation we can show that C must satisfy

$$C = -\frac{\phi_0^2(8\pi\rho_0 + \lambda)}{3\kappa}. \tag{2.155}$$

For C negative, we are working in the domain where

$$1 + Cr^2 > 0 \Rightarrow r^2 < \frac{3\kappa}{\phi_0^2(8\pi\rho_0 + \lambda)}, \tag{2.156}$$

otherwise if λ is chosen such that C is positive we do not have a restriction on r. This would require $\lambda < -8\pi\rho_0$ which is not a desirable feature.

Returning to the pressure equation, we find that

$$\tilde{P} = \frac{-4\pi\cosh(C_1)\phi_0^2\rho_0\sqrt{1 + Cr^2} + 4\pi\sinh(C_1)\phi_0^2\rho_0 + C\kappa\sinh(C_1)}{4\pi\phi_0^2(\cosh(C_1)\sqrt{1 + Cr^2} - \sinh(C_1)} \tag{2.157}$$

The limit as r gets large is given by

$$\lim_{r\to\infty}\tilde{P} = -\rho_0, \tag{2.158}$$

and the pressure remains bounded as exemplified in Figure 2.1.

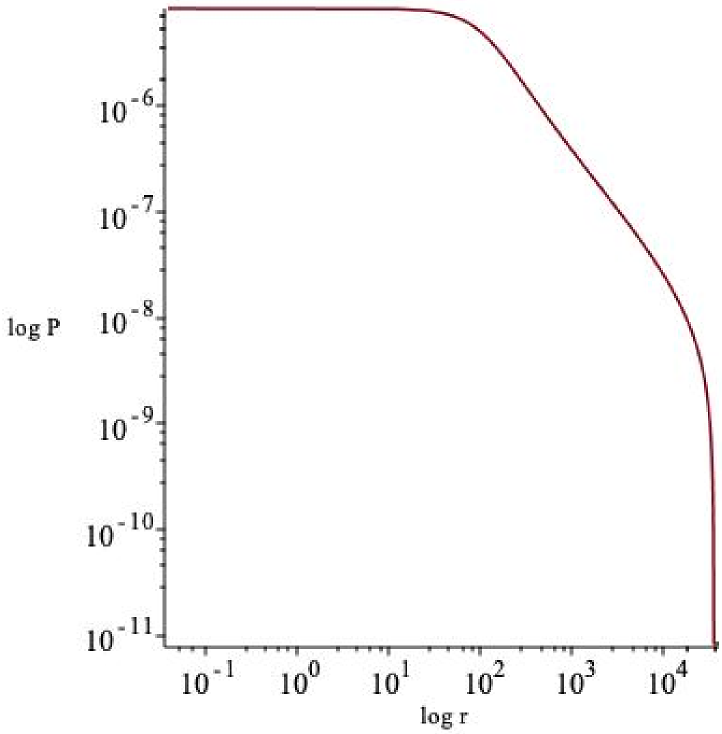

Figure 2.1: Physical pressure profile for κ=ϕ_0=1, $\rho_0 = 10^{-8}$, $\lambda = -10^{-4}$.

Thus by introducing a small constant density leads to a remnant pressure profile of similar order of magnitude. This result is encouraging as it suggests that such a setup could be incorporated in more advanced cosmological setups that require fine-tuning, such as [72].

Although promising, this simplistic form is not physically viable. We can check the orbital velocities generated by this setup, as $v = \frac{g_1+g}{g_1}$ lead to

$$v = \frac{C\cosh(C_1)r^2}{\sqrt{1+Cr^2}\Big(\cosh(C_1)\sqrt{1+Cr^2} - \sinh(C_1)\Big)}. \tag{2.159}$$

For small r we have

$$v = C(\sinh(C_1) + \cosh(C_1))\cosh(C_1)r^2 + O(r^4), \tag{2.160}$$

and thus we require C to be positive and λ negative. Although we might accommodate a negative λ, we find that the orbital velocity behaves as an escape velocity as shown in Figure 2.2.

Despite our apparent setback, not finding viable orbital velocities is not unexpected. As we are working with an overly simplified scenario we are missing the relevant physics that would dictate the velocity profile. Our study just shows that the effect of introducing a small constant

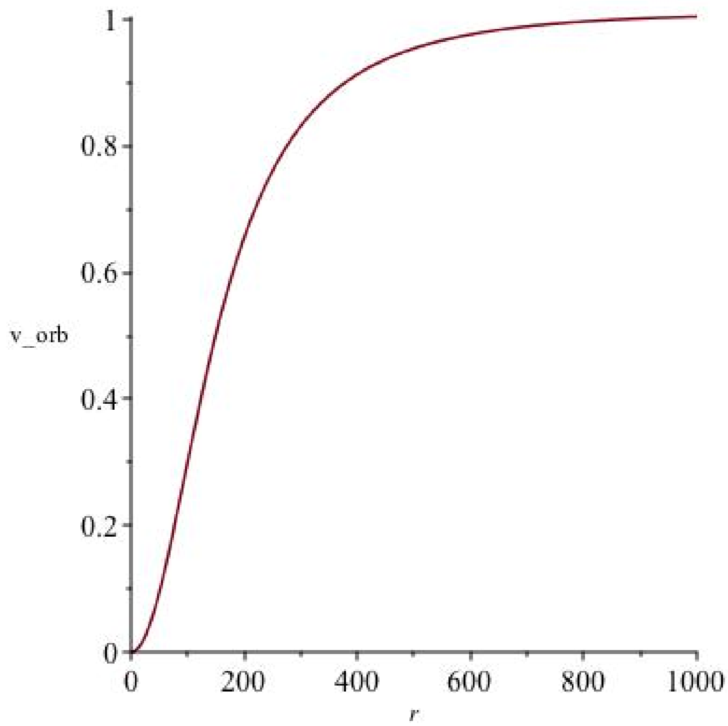

Figure 2.2: Orbital velocity for κ=ϕ_0=1, $\rho_0 = 10^{-8}$, $\lambda = -10^{-4}$.

density is to add a constant velocity contribution at high r. This result further reinforces the premise that this setup is not self-supporting.

As a final point we should return to the $Weyl$ tensor. Structurally we find interesting relations between the dilaton and torsion contributions. The main components are given in terms of

$$\begin{aligned} &\frac{dV}{dr} g_1 r + g_1 V + gV, \\ &\frac{dQ_1}{dr} g_1 r + Q_1 g_1 + gQ_1, \\ -&\frac{dQ_0}{dr} g_1 r + Q_0 g_1 + Q_0 g. \end{aligned} \tag{2.161}$$

This depicts the duality between V and Q-torsion. We can show that if we cancel these contributions the $Weyl$ vanishes, as expected. More surprisingly we find that by tying the dilaton to the scalar field, the Weyl tensor reduces to the form that we have for vacuum around a mass of $\frac{C}{6\phi}$.

In conclusion, when we introduce cosmological type torsion we find that if we keep a general setup (with x non-zero) we recover the standard setup with a more constrained metric. Instead if we work with a particular model (where x vanishes), we discover a significantly different cosmology. Either case is important in its own right. On one hand we have proved that we can introduce a torsion that is decoupled from the matter sector and thus although present would be impossible to measure. On the other hand we have shown that we can have a very simple model (that can be included in advanced theories) that introduces highly desirable features for theories that require a fine-tuning mechanism. Such a theory is the one presented in [72] where it is postulated that by introducing torsion we could have a more manageable fine-tuning problem when we deal with inflation. In our work we have found a potential mechanism that could create a remnant pressure from cosmological torsion.

2.8 Real Torsion

Motivated by the interesting features we have found when we worked with cosmological torsion we will proceed to investigate the effects of introducing "real" torsion. Considering its more convoluted structure, the elegance we became accustomed to in the previous section will not be attained and thus throughout this section we will investigate the $x = 0$ regime.

2.8.1 Conventional setup

We will start our analysis by exploring a different sector - a conventional setup where we turn off our torsion contribution.

The A-equations become our usual constraint equation for V

$$2\phi\Big(V\phi - \frac{d\phi}{dr}g_1\Big) = 0 \qquad \Rightarrow \qquad V = \frac{\frac{d\phi}{dr}g_1}{\phi}. \tag{2.162}$$

Using this constraint in the h-equation, we find forms for the pressure P,

$$P = \frac{1}{8\pi\phi r^2}\Big(\phi^4\lambda r^2 + 3\frac{d\phi}{dr}^2 g_1^2\kappa r^2 + 2g\phi\frac{d\phi}{dr}g_1\kappa r + 6\phi\frac{d\phi}{dr}g_1^2\kappa r \\ +2g\phi^2 g_1\kappa + 3\phi^2 g_1^2\kappa - \kappa\phi^2\Big), \tag{2.163}$$

and for the density ρ,

$$\rho = \frac{1}{8\pi\phi r^2}\Big(2\frac{dg}{dr}\phi\frac{d\phi}{dr}g_1\kappa r^2 + \phi^4\lambda r^2 + 2\phi\frac{d\phi}{dr}g_1^2\kappa r^2 - \frac{d\phi}{dr}^2 g_1^2\kappa r^2 \\ +2\frac{dg_1}{dr}\phi^2 g_1\kappa r + 4\phi\frac{d\phi}{dr}g_1^2\kappa r + \phi^2 g_1^2\kappa - \kappa\phi^2\Big) \tag{2.164}$$

The reason we started with this setup will become obvious shortly as we will show that disregarding of the type of torsion that we use we will always recover these pressure and density profiles. The significant difference is that the presence of torsion frees the scalar field ϕ from the vector field V. Thus we have found a first setup where we can have a better understanding of the role of torsion in $Weyl$-type cosmologies.

2.8.2 Governing equations

2.8.2.1 Full torsion

We start by analysing the most general form. We make an early simplification by imposing $\chi = 2\kappa$, i.e. we remove the torsion component at the level of the Lagrangian. We introduce this simplification as in the cases where the form of torsion is constrained it turns out to be a requirement (as we will show in the following sections). In the full torsion case, the equations are significantly more complicated if we keep them general and it turns out that we cannot find a physical set of evolution equations.

We start by looking at one of the A-equations where we can read directly the derivative of G:

$$\begin{aligned}\frac{dG}{dr} = -\frac{1}{6Fg_1r^2}\Big(&3F^3r^2 - 36FV^2r^2 - 24FVGr^2 - 3FG^2r^2 - 12g_1V\frac{dF}{dr}r^2 \\ &-6g_1\frac{dF}{dr}Gr^2 - 84Fg_1Vr - 24Fg_1Gr - 4Fg_1\frac{dg}{dr}r - 12FVgr \\ &-12g_1^2\frac{dF}{dr}r - 48Fg_1^2 - 12Fg_1g - 4Fg^2 + 4F\Big)\end{aligned} \tag{2.165}$$

By substituting this expression in the full A-equations we find a full constraint,

$$(3rVF + rg_1\frac{dF}{dr} + 4Fg_1 + Fg)(F^2r + 2VGr + rG^2 + 2g_1G) = 0. \tag{2.166}$$

This leads to two options. Firstly, if

$$3rVF + rg_1\frac{dF}{dr} + 4Fg_1 + Fg = 0, \tag{2.167}$$

we recover two simple forms for the torsion derivatives:

$$\begin{aligned}\frac{dF}{dr} &= -\frac{F(3Vr + g + 4g_1)}{rg_1} \\ \frac{dG}{dr} &= \frac{3G^2r^2 + 6GVr^2 - 3F^2r^2 - 6Ggr + 4\frac{d}{dr}gg_1r - 4g^2 - 4}{6r^2g_1}\end{aligned} \tag{2.168}$$

By just imposing these evolutions for torsion in the h-equations we recovered the conventional setup. It is worth noting that the evolution equation for F remains unchanged when G

is introduced, whereas the same cannot be said about G. This asymmetry leads to interesting results as we will show in the following sections.

Otherwise we can establish a relation between the torsion components,

$$F^2 r + 2VGr + rG^2 + 2g_1 G = 0, \tag{2.169}$$

that fixes V to

$$V = -\frac{F^2 r + rG^2 + 2g_1 G}{2Gr}. \tag{2.170}$$

Using this constraint we can re-write the derivative of G as

$$\begin{aligned} \frac{dG}{dr} = \frac{1}{6r^2 g_1 G^2}\Big(&9r^2F^4 + 3F^2G^2r^2 - 6Fg_1\frac{dF}{dr}Gr^2 - 6F^2g_1rG - 6F^2Ggr \\ &-6g_1G^3r + 4g_1\frac{dg}{dr}rG^2 - 6G^3gr + 4g^2G^2 - 4G^2\Big). \end{aligned} \tag{2.171}$$

These new relations can be substituted in the A-equations to find a new constraint,

$$\begin{aligned} &(3F^3r + 3FG^2r - 2g_1\frac{dF}{dr}Gr - 2Fg_1G - 2FgG)(9F^5r^2 - 3r^2F^3G^2 - 6F^2g_1\frac{dF}{dr}Gr^2 \\ &+6r^2G^4F - 6r^2g_1\frac{dF}{dr}G^3 - 6F^3g_1Gr - 6F^3Ggr - 6rG^3Fg_1 + 4Fg_1\frac{dg}{dr}rG^2 \\ &-6FG^3gr + 4Fg^2G^2 - 4FG^2) = 0. \end{aligned} \tag{2.172}$$

We are left with a further two cases.

If we use the constraint,

$$3F^3r + 3FG^2r - 2g_1\frac{dF}{dr}Gr - 2Fg_1G - 2FgG = 0, \tag{2.173}$$

we find a new derivative for F,

$$\frac{dF}{dr} = \frac{F(3F^2r + 3rG^2 - 2g_1G - 2Gg)}{2rGg_1}, \tag{2.174}$$

and thus a new derivative for G,

$$\frac{dG}{dr} = \frac{3r^2F^2 + 3rGg_1 - 2rg_1\frac{dg}{dr} + 3rGg - 2g^2 + 2}{3r^2g_1}. \tag{2.175}$$

One could notice that these are the exact forms we have found in case 1 with an additional constraint on V. As we have shown that the setup remains conventional in the general case, the constraint on V should not provide any new insight.

Otherwise using the remainder constraint equation,

$$9F^5r^2 - 3r^2F^3G^2 - 6F^2g_1\frac{dF}{dr}Gr^2 + 6r^2G^4F - 6r^2g_1\frac{dF}{dr}G^3 - 6F^3g_1Gr$$
$$-6F^3Ggr - 6rG^3Fg_1 + 4Fg_1\frac{dg}{dr}rG^2 - 6FG^3gr + 4Fg^2G^2 - 4FG^2 = 0, \tag{2.176}$$

we can find different relations for the derivatives of F and G,

$$\frac{dF}{dr} = \frac{F}{6r^2g_1G(F^2+G^2)}\Big(9r^2F^4 - 3F^2G^2r^2 + 6r^2G^4 -$$
$$6F^2g_1rG - 6F^2Ggr - 6g_1G^3r + 4g_1\frac{dg}{dr}rG^2 - 6G^3gr + 4g^2G^2 - 4G^2\Big) \tag{2.177}$$

$$\frac{dG}{dr} = \frac{1}{(6(F^2+G^2))r^2g_1}\Big(15r^2F^4 - 3F^2G^2r^2 - 6F^2g_1rG -$$
$$6F^2Ggr - 6g_1G^3r + 4g_1\frac{dg}{dr}rG^2 - 6G^3gr + 4g^2G^2 - 4G^2\Big) \tag{2.178}$$

The new evolution equations are notably different from what we have previously encountered. When making the substitutions in the h equations, the setup becomes significantly more convoluted. Considering that we have found that all sensible roads lead to a solution identical to the conventional setup we decided not to pursue the most general form for torsion. In the following sections we will show that the simplified torsion cases lead to the same pressure and density profile. As our aim is to understand how torsion influences our system we will focus on simplified cases where the significance becomes transparent.

2.8.2.2 F torsion only

From the A equations, we can find directly a relation for the derivative of F,

$$\frac{dF}{dr} = \frac{-F(3rV + g + 4g_1)}{rg_1}. \tag{2.179}$$

By substituting the evolution equation in the A equations we reach an equation for the derivative of g,

$$\frac{dg}{dr} = -\frac{-3r^2\beta F^2 + 6r^2\phi^2\chi - 3\phi^2\kappa r^2 + 4\beta g^2 - 4\beta}{4g_1\beta r}. \tag{2.180}$$

Finally the A equations simplify to a simple constraint,

$$2\chi - \kappa = 0, \tag{2.181}$$

that imposes the torsion contribution at the level of the Lagrangian to be removed. One should note that unlike the case where we have only G present, here we have a direct link between the torsion and g. As we will show in the next section this constrains the form of torsion significantly exemplifying the non-trivial changes that occur when one works only with a simplified type of torsion.

With these considerations in mind, the h-equations lead to a simple form for the derivative of ϕ,

$$\frac{d\phi}{dr} = \frac{-3\kappa\phi r^2 F^2 + 32\pi\rho r^2 + 32r^2\pi P - 8\kappa\phi g g_1 - 8\kappa\phi}{8 g g_1 \kappa r}, \tag{2.182}$$

alongside the conventional pressure and density profiles.

2.8.2.3 G torsion only

From the A equations we find a form for the derivative of G,

$$\frac{dG}{dr} = \frac{6VGr^2 + 3G^2r^2 + 4g_1\frac{dg}{dr}r - 6Grg + 4g^2 - 4}{6r^2 g_1}, \tag{2.183}$$

which reduces the equations to a simple condition

$$G\phi^2(2\xi - \kappa) = 0. \tag{2.184}$$

As in the case where we had F-only torsion, this setup forces $2\xi - \kappa = 0$. By substituting this constraint in the h-equations we recover the conventional pressure and density.

2.8.3 Metric dependent interpretations

From previous attempts we believed that the 'Weyl' squared term in the Lagrangian would not permit physical solutions for the pressure and density. Horne [63], Mannheim [73] have also worked on recovering viable matter solutions and to our knowledge have been unsuccessful. In our work we have shown that in the case where torsion is turned off we are in agreement with Horne's results. In this section we have introduced torsion at an intrinsic level (although we have been forced to remove the torsion contribution at the level of the Lagrangian). We have shown that most sensible roads lead to a conventional setup despite our choice of torsion. Considering the unexpected effect of torsion i.e that it frees the scalar field from the dilaton, we have decided to further analyse this solution.

We will consider four regimes - Minkowski space, Schwarzschild space, de Sitter space and Schwarzschild de Sitter space. We will start by exploring the pressure and density profiles in these regimes by finding solutions for the scalar field ϕ. We will then proceed to analyse the different forms torsion takes depending on metric. We will show how various solutions constrain the dilaton field and how some metric choices are favoured.

Before we start, we would like to summarise what setups we will employ:

Minkowski

$$g_1 = 1, \quad g = -1, \quad \rho = 0, \quad \lambda = 0. \tag{2.185}$$

Schwarzschild

$$g_1 = \sqrt{1 - \frac{2M}{r}}, \quad g = -\frac{1 - \frac{3M}{r}}{\sqrt{1 - \frac{2M}{r}}}, \quad \rho = 0, \quad \lambda = 0. \tag{2.186}$$

de Sitter

$$g_1 = \sqrt{1 - \frac{\lambda}{3} r^2}, \quad g = -\frac{1}{\sqrt{1 - \frac{\lambda}{3} r^2}}, \quad \rho = 0. \tag{2.187}$$

Schwarzschild de Sitter with λ

$$g_1 = \sqrt{1 - \frac{2M}{r} - \frac{\lambda}{3} r^2}, \quad g = -\frac{1 - \frac{3M}{r}}{\sqrt{1 - \frac{2M}{r} - \frac{\lambda}{3} r^2}}, \quad \rho = 0. \tag{2.188}$$

2.8.3.1 The scalar field

We start by rearranging the h-equations in order to write evolution equations for g and g_1.

$$\frac{dg}{dr} = \frac{3g_1^2 \frac{d\phi}{dr}^2 \kappa r^2 + \phi^4 \lambda r^2 + 6g_1^2 r \frac{d\phi}{dr} \phi\kappa + 3g_1^2 \kappa\phi^2 + 8\pi\phi\rho r^2 - \phi^2 g^2 \kappa - 2\kappa\phi^2}{g_1 \phi^2 \kappa r}, \tag{2.189}$$

$$\frac{dg_1}{dr} = -\frac{4g_1^2 r \frac{d\phi}{dr} \phi\kappa - \kappa\phi^2 - g_1^2 \frac{d\phi}{dr}^2 \kappa r^2 + 8\pi\phi\rho r^2 + g_1^2 \kappa\phi^2 + 2g_1^2 \frac{d\phi}{dr} \phi\kappa r^2 + \phi^4 \lambda r^2}{2\kappa g_1 \phi r \frac{d\phi}{dr} r + \phi}. \tag{2.190}$$

In a **Minkowski** setup the equations simplify to

$$\frac{3\frac{d\phi}{dr}^2 \kappa r^2 + 6r \frac{d\phi}{dr} \phi\kappa}{\phi^2 \kappa r} = 0, \tag{2.191}$$

$$-\frac{4r \frac{d\phi}{dr} \phi\kappa - \frac{d\phi}{dr}^2 \kappa r^2 + 2\frac{d^2\phi}{dr^2} \phi\kappa r^2}{2\kappa\phi r(\frac{d\phi}{dr} r + \phi)} = 0. \tag{2.192}$$

We start by solving the ODE from the g constraint and we find that ϕ admits the following solutions:

$$\phi \in \left\{\frac{C_1}{r^2}, C_2\right\}, \quad \text{where } C1 \text{ and } C2 \text{ are constants.} \tag{2.193}$$

Both of these solutions solve the remaining equation. Now we turn to the equation for pressure, which in this setup takes the form

$$P = \frac{\kappa\frac{d\phi}{dr}(3\frac{d\phi}{dr}r + 4\phi)}{8\pi r\phi}. \tag{2.194}$$

The above solutions for ϕ force $P \in \{\frac{C_1\kappa}{2\pi r^4}, 0\}$. From this solutions we can note that in the case when ϕ is not trivial, we have a residual pressure from the perfect fluid. This has been previously discussed in literature in the context of inflation [74]. Since we have C_1 present we could in principle fine tune the pressure to match observational discrepancies as presented in [72].

For the **Schwarzschild** case we can easily find an equation for ϕ from the equation for g,

$$\sqrt{\frac{r-2M}{r}}\frac{d\phi}{dr}\left(\frac{d\phi}{dr}r + 2\phi\right) = 0. \tag{2.195}$$

This has the same solution as in the Minkowski case, but unlike the previous case only the trivial solution verifies the constraint from the g_1 equation. The pressure remains zero.

In the **de Sitter** setup we can find a constraint from the equation for g,

$$\frac{3r\kappa(\frac{\lambda}{3}r^2-1)\frac{d\phi}{dr}^2 + 6\phi\kappa(\frac{\lambda}{3}r^2-1)\frac{d\phi}{dr} - \phi^4\lambda r + \lambda\phi^2\kappa r}{\phi^2\kappa\sqrt{-\frac{\lambda}{3}r^2+1}} = 0. \tag{2.196}$$

This has the general solutions $\phi = \pm\frac{\sqrt{3\kappa\lambda}}{\lambda r}$. We can also notice that $\phi = C_2$ (a constant) satisfies the equation given that $\lambda = \frac{C_2^2\lambda}{\kappa}$.

The constraint from g_1 reads as

$$\frac{r\kappa(\frac{\lambda}{3}r^2-1)\frac{d\phi}{dr}^2 - 6(\frac{\lambda}{3}r^2-2/3)\kappa\phi\frac{d\phi}{dr} - 2\phi(-\phi^3\lambda/2 + 3\frac{\lambda}{3}\kappa\phi/2 + (\frac{\lambda}{3}r^2-1)\frac{d\phi^2}{d^2r}\kappa)r}{2\sqrt{-\frac{\lambda}{3}r^2+1}\kappa\phi(\frac{d\phi}{dr}r+\phi)} = 0. \tag{2.197}$$

We can notice straight away that the general ϕ solution sets the denominator to 0. The constant solution reduces the equation to the previous λ-constraint. The pressure remains zero.

Finally, we reach the most general setup, **Schwarzschild de Sitter** , and the constraints become

$$\frac{3(\frac{\lambda}{3}r^3+2M-r)r\kappa\frac{d\phi}{dr}^2 + 6\kappa(\frac{\lambda}{3}r^3+2M-r)\frac{d\phi}{dr}\phi - \lambda\phi^4r^2 + \lambda\kappa\phi^2r^2}{\sqrt{\frac{-\frac{\lambda}{3}r^3-2M+r}{r}}r\kappa\phi^2} = 0, \tag{2.198}$$

$$\frac{-2(\frac{\lambda}{3}r^3+2M-r)r\frac{d\phi^2}{d^2r}\phi\kappa+(\frac{\lambda}{3}r^3+2M-r)r\frac{d\phi}{dr}^2\kappa-6\kappa(\frac{\lambda}{3}r^3+M-2r/3)\phi\frac{d\phi}{dr}}{2\sqrt{\frac{-\frac{\lambda}{3}r^3-2M+r}{r}}r\kappa\phi(\frac{d\phi}{dr}r+\phi)}$$
$$+\frac{+\phi^4\lambda r^2-\lambda\phi^2\kappa r^2}{2\sqrt{\frac{-\frac{\lambda}{3}r^3-2M+r}{r}}r\kappa\phi(\frac{d\phi}{dr}r+\phi)}=0. \quad (2.199)$$

This system does not have a natural differential solution outside the constant solution which provides the usual constraint for λ.

To summarise, we have found that in most cases ϕ is forced to a constant. In the case where the dilaton was linked to the scalar field, we would be forced to remove the dilaton field. We will discuss such solutions when we investigate the forms for torsion. The Minkowski setup recovers another solution which has been discussed in previous work [72].

2.8.3.2 Exploring torsion solutions

We have previously shown that disregarding of the type of torsion we recover a conventional setup with the unexpected benefit that the scalar and the dilaton fields are not directly linked. We have also explored metric-dependent solutions for the scalar field and have found that if we decided instead to enslave the dilaton to the scalar field we would be forced to remove it. In this section we will explore the metric based solutions for torsion in the hope that we could shed some light on two main questions: Are the certain metrics that are ruled out by inappropriate forms for torsion? Can we find constraints for the dilaton from the form of the metric?

In this work we will consider the torsion "well behaved" if it has the correct behaviour at large r, i.e. if it has a finite limit. We would prefer a solution that has an adaptable non-zero limit at large r as there would be the possibility for fine-tunning.

2.8.3.3 G only torsion

We will start by looking at constrained types of torsion, specifically G-only torsion. We can recall that the evolution equation for torsion was given by

$$\frac{dG}{dr}=\frac{6VGr^2+3G^2r^2+4g_1\frac{dg}{dr}r-6Grg+4g^2-4}{6r^2g_1}. \quad (2.200)$$

In **Minkowski space** the equation simplifies to

$$\frac{dG}{dr}=\frac{G(2Vr+Gr+2)}{2r}. \quad (2.201)$$

The ODE solution reads as

$$G = -\frac{2e^{\int \frac{Vr+1}{r}dr}}{\int e^{\int \frac{Vr+1}{r}dr}dr + C}, \quad \text{where C is a constant.} \tag{2.202}$$

We can rewrite the above expression to localize the V dependent part as

$$G = \frac{-2rS}{\int rSdr + C}, \text{ where S is defined as } S = e^{\int Vdr}. \tag{2.203}$$

In order to have a well behaved form for torsion we end up requiring

$$\lim_{r\to\infty} G = \lim_{r\to\infty} \frac{-2rS}{\int rSdr + C} = -2F, \text{ where F is a constant that we can impose.} \tag{2.204}$$

We can rewrite this limit as,

$$\lim_{r\to\infty} \frac{(rS)'}{rS} = \lim_{r\to\infty} \frac{d\ln S}{dr} = F \Rightarrow \lim_{r\to\infty} V = F. \tag{2.205}$$

Thus if we want to have the possibility to fine tune at large r we would need to have a non-zero dilaton field. For $V = constant$ the solution reduces to

$$G = \begin{cases} -\dfrac{4r}{r^2 + 2C} & \text{if } V = 0 \\ -\dfrac{2V^2 r}{Vr - 1 + \frac{C}{e^{Vr}}} & \text{otherwise} \end{cases}$$

Looking at this solution we notice that we could have also finite torsion as $r \to 0$ if we impose $C = 0$ at the cost of having a pole at $r = 1/V$. Otherwise we can see that the solution behaves as shown in Figure 2.3.

In the **de Sitter** setup, we find that the solution takes the form

$$G = \frac{E}{\int \frac{E\sqrt{1-\frac{\lambda}{3}r^2}}{2(\frac{\lambda}{3}r^2-1)}dr + C_3}, \text{ where } E = e^{\int -\frac{Vr\sqrt{1-\frac{\lambda}{3}r^2}+1}{r(\frac{\lambda}{3}r^2-1)}dr}. \tag{2.206}$$

We could rewrite E as

$$E = Se^{\int \frac{1}{r(1-\frac{\lambda}{3}r^2)}dr} = \frac{Sr}{\sqrt{\frac{\lambda}{3}r^2 - 1}}, \text{ where } S = e^{\int \frac{V}{\sqrt{1-\frac{\lambda}{3}r^2}}dr} \tag{2.207}$$

When we take the limit at large r we find

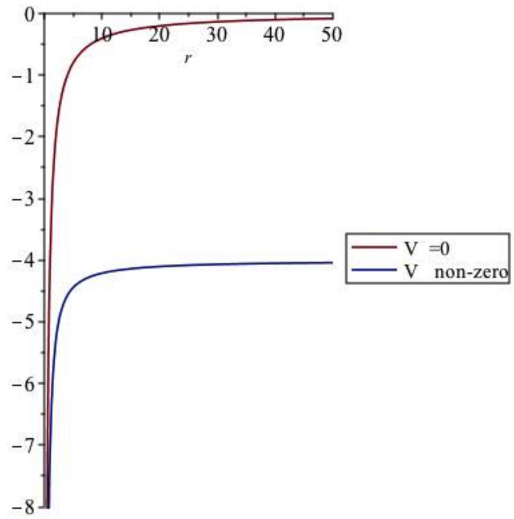

Figure 2.3: V behaviour in Minkowski space

$$\lim_{r\to\infty} G = -2\lim_{r\to\infty}\frac{\frac{rS}{\sqrt{\frac{\lambda}{3}r^2-1}}}{\int\frac{r}{1-\frac{\lambda}{3}r^2}Sdr + C_3} = -2\lim_{r\to\infty}\frac{r(\frac{\lambda}{3}r^2-1)\frac{dS}{dr}-S}{rS\sqrt{\frac{\lambda}{3}r^2-1}}$$

$$= -2\lim_{r\to\infty}\sqrt{\frac{\lambda}{3}r^2-1}\frac{d\ln S}{dr} = -2\lim_{r\to\infty}V, \tag{2.208}$$

which is the same result as in the case where we had no cosmological constant.

The solution is fairly involved even in the case where V is a constant as $\int\frac{1}{\sqrt{1-\frac{\lambda}{3}r^2}}dr$ is an $arctan$ function. For $V = 0$ we find that this simplifies to a more manageable form,

$$G = \frac{4\frac{\lambda}{3}r}{4\frac{\lambda}{3}C_4\sqrt{\frac{\lambda}{3}r^2-1}+\ln(\frac{\lambda}{3}r^2-1)\sqrt{\frac{\lambda}{3}r^2-1}}, \text{ where } C_4 \text{ is a constant.} \tag{2.209}$$

The solution can be seen in Fig 2.4.

In the case where we are looking at **Schwarzschild**, the governing equation is significantly more involved,

$$\frac{dG}{dr} = \frac{(8M - r^3G^2)\sqrt{\frac{r-2M}{r}} + 6rG(M-\frac{r}{3})}{4Mr^2-2r^3} \tag{2.210}$$

The solution is a hypergeometric function. When we explore the solution numerically we can impose that it is well behaved at large r. As expected, we find similar difficulties when we introduce the cosmological constant. In conclusion, employing this type of torsion has limited

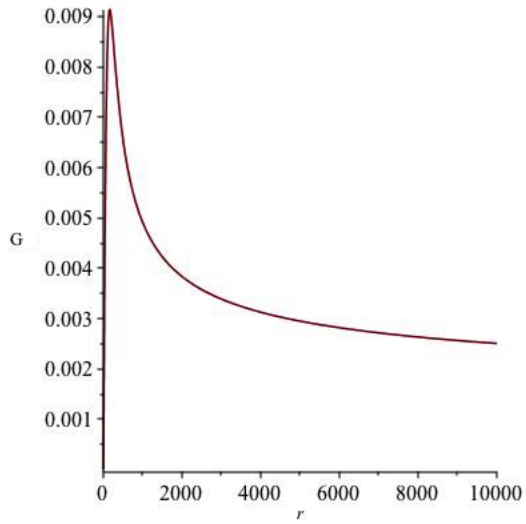

Figure 2.4: G's behaviour in de Sitter space

applicability in Minkowski type cosmologies. As even simple generalisations seem to cause problems one should consider a different type of torsion.

2.8.3.4 F only torsion

As a reminder, the evolution equations were given by,

$$\frac{dF}{dr} = -\frac{F(3rV + g + 4g_1))}{rg_1}, \tag{2.211}$$

$$\frac{dg}{dr} = \frac{3r^2F^2 - 4g^2 + 4}{4rg_1}. \tag{2.212}$$

Since we are inputing g and g_1 we can just read the torsion profile from the second equation. Furthermore by combining the two we can also easily compute the corresponding dilaton field as the equations reduce to

$$V = -\frac{\frac{d\log F^2}{dr} + 2g + 8g_1}{6r}. \tag{2.213}$$

In the **Minkowski** case we find that these equations remove the torsion and do not constrain V. The **Schwarzschild** setup does produce a unique solution, under the caveat that M is negative. Disregarding the presence of the cosmological constant, we find that the torsion takes the form

$$F = 2\sqrt{-\frac{2M}{r^3}}, \text{ and thus we require } M < 0 \text{ as previously mentioned..} \tag{2.214}$$

Using the above relation we can find separate solutions for the dilaton:

$$V = \begin{cases} \dfrac{-3r + 4M}{6r^2\sqrt{1 - \frac{2M}{r}}} & \text{if } \lambda = 0 \\ \dfrac{-11\frac{\lambda}{3}r^3 - 16M + 9r}{6r^2\sqrt{1 - \frac{2M}{r} - \frac{\lambda}{3}r^2}} & \text{otherwise} \end{cases}$$

Although the profiles for the dilaton appear similar, the contribution of the cosmological constant becomes apparent at high radius. It moves the limit $\lim_{r\to\infty} V = 0$ to $\lim_{r\to\infty} V = -\frac{11\frac{\lambda}{3}}{6\sqrt{-\frac{\lambda}{3}}}$ and thus we would also require a negative λ. Interestingly, the torsion is fully decoupled from the cosmological constant and absorbs the scale of M. Although it is pathological as it requires both M and λ to be negative, this could be thought of as the complementary result to what we had in the "cosmological torsion" setup.

Considering the reduced torsion setups seems to be successful in different regimes we can hope that a more general solution will give us better options. We have shown in the previous sections that when we look at the general setup we "free" F from the strong relation with the metric at the expense of tying G and F.

2.8.3.5 F+G torsion

We can proceed to the most general case where we are using the full form of torsion. We have previously shown how by restricting the torsion profile we limit our solution space significantly. Thus it is important to prove there exist manageable forms for torsion that do not restrict the cosmological setup.

We recall the evolution equations for the torsion components are

$$\begin{aligned} \frac{dF}{dr} &= -\frac{F(3Vr + g + 4g_1)}{rg_1} \\ \frac{dG}{dr} &= \frac{3G^2r^2 + 6GVr^2 - 3F^2r^2 - 6Ggr + 4\frac{d}{dr}gg_1r - 4g^2 - 4}{6r^2g_1} \end{aligned} \tag{2.215}$$

We can solve for F in isolation and then substitute in the G equation. Although this might seem the obvious approach, we should consider the equation for G first. This equation inherits the complexity of the isolated G equation and thus we know that we will stuggle to find a solution for non trivial g and g_1. Instead of thinking of the F^2 term as an extra complication we could consider it as a helping hand. We know that the term that produces hypergeometric solutions is G^2 and we could impose that F^2 balances it.

Thus for $F = G$ we find a more manageable system, namely

$$
\begin{aligned}
\frac{dF}{dr} &= -\frac{F(3Vr+g+4g_1)}{rg_1} \\
\frac{dF}{dr} &= \frac{6FVr^2-6Fgr+4\frac{d}{dr}gg_1r-4g^2-4}{6r^2g_1}
\end{aligned}
\tag{2.216}
$$

We find that

$$
\begin{aligned}
V &= -\frac{24g_1rF+4\frac{d}{dr}gg_1r-4g^2-4}{24Fr^2} \\
\frac{dF}{dr} &= \frac{\frac{d}{dr}gg_1r-2r(g+g_1)F-g^2-1}{2r^2g_1}
\end{aligned}
\tag{2.217}
$$

The solutions to the above system are listed below, and can be compared in Fig 2.5 and Fig 2.6.

Minkowski

$$
F = \frac{1}{r} + C_1 \qquad V = -\frac{3C_1r+2}{3r(1+C_1r)}
\tag{2.218}
$$

de Sitter

$$
F = \frac{C_2}{\sqrt{1-\frac{\lambda}{3}r^2}} - \frac{\sqrt{\frac{\lambda}{3}}r\operatorname{arctanh}(\sqrt{\frac{\lambda}{3}}r)-1}{r\sqrt{1-\frac{\lambda}{3}r^2}},
\tag{2.219}
$$

$$
V = \frac{r(1-\frac{\lambda}{3}r^2)(C_2-r\sqrt{\frac{\lambda}{3}}\operatorname{arctanh}(r\sqrt{\frac{\lambda}{3}}))+\frac{2}{3}-\frac{\lambda}{3}r^2}{r\sqrt{1-\frac{\lambda}{3}r^2}(r\sqrt{\frac{\lambda}{3}}\operatorname{arctanh}(r\sqrt{\frac{\lambda}{3}})+C_2-1)}
\tag{2.220}
$$

Schwarzschild

$$
F = \frac{C_3}{\sqrt{1-\frac{2M}{r}}} - \frac{r^2\ln(\frac{2M}{r}-1)+6M^2-6rM}{8Mr^2\sqrt{1-\frac{2M}{r}}},
\tag{2.221}
$$

$$
V = \frac{r^2(2M-r)/8\ln(\frac{r}{2M-r})-M^2/2-5M^2r/4-5Mr^2/12+MC_3r^2(r-2M)}{r^2\sqrt{1-\frac{2M}{r}}(r^2/8\ln(\frac{r}{2M-r})-3M(M-r)/4+MC_3r^2)}
\tag{2.222}
$$

Schwarzschild de Sitter

$$
F = \frac{\int \frac{-\lambda Mr^3+r^2-3rM+3M^2}{r^4(1-\frac{2M}{r}-\frac{\lambda}{3}r^2)}dr + C_4}{\sqrt{1-\frac{2M}{r}-\frac{\lambda}{3}r^2}}
\tag{2.223}
$$

Solving the integral in isolation gives a standard pole-solution:

$$
F = -\frac{3}{4r} - \frac{\ln r}{8M} + \frac{3M}{4r^2} + \Sigma_R\frac{(12\lambda M^2-2\lambda MR+\frac{\lambda}{3}R^2-1)\ln(r-R)}{8M\lambda R^2-1},
\tag{2.224}
$$

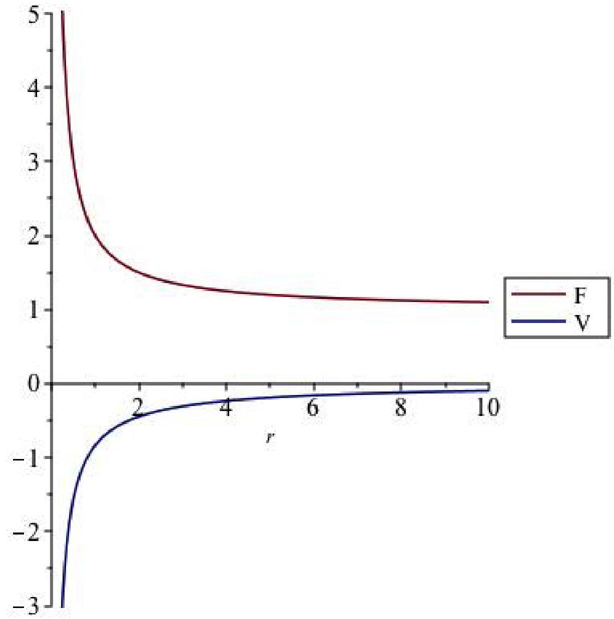

Figure 2.5: F and V profile in Minkowski space

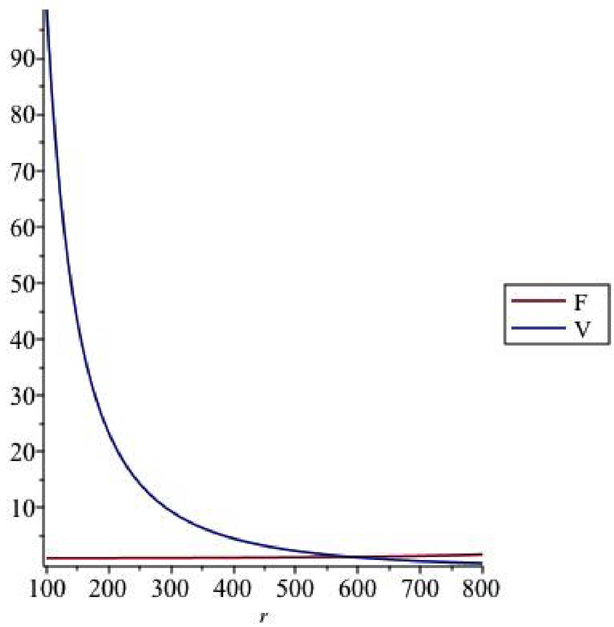

Figure 2.6: F and V profile in de Sitter space

where R solves the equation $\frac{\lambda}{3}R^3 + 2M - R = 0$.

In conclusion, although the solution is pathological, interesting properties of torsion emerge that can hopefully be generalised for more successful models.

The main function of the “real”-torsion is to decouple the scalar and dilaton fields so its role is to act as a support field. Interesting torsion setups arise since although the scalar field can be set to a constant we have the freedom to have a self-evolving dilaton field.

In this study we have shown that the form of torsion is not directly important to the observables and thus we just had to show that physical, convenient forms can be reached. We found it instrumental to look at reduced cases as they helped us understand what key components we needed for the general case.

2.9 Conclusion

In this Chapter we have presented a first application of eWGT. We chose to use our theory to simulate the Weyl^2 formalism that has been thoroughly researched by Mannheim et al in various publications, such as [48], [58], [55], [59], [61], [69], [73].

We started by recovering their predictions in the vacuum case. We proceeded to discuss the validity of using the Mannheim-Kazanas metric and agreed with the arguments presented in [63].

When we tried to introduce matter we found that the setup does not accommodate a matter profile that is compatible with astrophysical problems as it recovers radiation when $B \to 0$. We also restricted our metric to Mannheim-Kazanas and in that case the fluid profile was non-physical. We argue that the previous attempts to solve these problems are somewhat misleading. The authors in [71] use a ‘perfect’ fluid that has different perpendicular and radial pressures to give them extra freedom. In our approach we believe that any viable theory of gravity should be able introduce a perfect fluid as a matter source.

Finally, we wanted to prove that the problem cannot be solved even in our extended setup with torsion. We investigated all the types of torsion our theory allows and could not find physical pressure and density profiles. Instead we reinforced our belief that a ‘cosmological’ torsion should not interact with the matter sector. In this case we found that it redefined the relation between various support fields, an aspect we were not aware of prior to this research. For ‘real’ torsion (the component we expect to effect the matter sector), we found that disregarding of its form, it forces the matter sector into a new regime which in this case was not physical. We explored how different forms can be chosen and found a new way of building torsion components that will hopefully help us when using a general setup.

CHAPTER 3

Riemann-squared Gravity

3.1 Introduction

In 1915 Hilbert presented a novel method for finding Einstein field equations through the principle of least action applied to an action that contained the Ricci scalar. Looking to generalise the Lagrangian for gravity, Weyl introduced the quadratic term of a modified Ricci scalar (that is covariant under scale changes) which led to the formulation of Gauge Theories. Unfortunately Weyl's theory was not successful since it portrayed electromagnetism as the gauge theory that corresponds to changes of scale length, not to local changes of quantum mechanical phase.

We believe that an interesting scenario can be reached by replacing the quadratic Ricci scalar term proposed by Weyl with a modified quadratic Riemann tensor. As Riemann-square alone cannot handle ordinary matter, this work acts as a test ground for radiation-only 'matter' and provides a first insight into the potential of exploring a full theory.

In 1958 Stephenson [75] started looking at quadratic invariants of the Riemann-Christoffel curvature tensor and its contractions in a four-dimensional Riemann space. He found that the derived field equations have a class of solutions that satisfy the source-free Einstein's equations with an arbitrary cosmological term (see [75]). Intrigued by his results Higgs investigated Lagrangians formed by either R^2, $R_{\gamma\beta}R^{\gamma\beta}$ or $R^{\mu}_{\ \gamma\beta\alpha}R_{\mu}^{\ \gamma\beta\alpha}$ and found that for the first two he could find consistent equations of motions, but not for the third(see [76]).

With the belief that we know the missing ingredient in Higgs' analysis, we will be working with a Lagrangian given only by a Riemann-squared term. We will be considering only cosmological solutions where the matter is described by radiation. To take into account coupling to matter, future work should investigate adding terms such as $\phi^2\mathcal{R}$ (proposed by Dirac in 1973 [77]) and ϕ^4, where ϕ is a real scalar field. A starting point for such applications can be found in [13].

We will build on the first attempt, made in the paper [78], where the authors have found the background solutions for the equations of motion. Considering most of our work is constructed in Geometric Algebra, we will introduce this new language as a parallel to the 'standard' approach in Chapter 1.

3.1.1 Notation guide for Gauge Theory of Gravity (GTG) in STA

Space Time Algebra (STA) is the geometric (or Clifford) algebra of Minkowski spacetime. A comprehensive treatment of geometric algebra can be found in [79] and we will offer an outline of the standard notation employed in this Chapter.

Geometric algebra is based on the concept of 'geometric product' from which we define all multivector calculus. For example, for vectors a and b we write the geometric product as ab and we define:

- inner product: $a \cdot b \equiv \frac{1}{2}(ab + ba)$
- outer product (which gives a *bivector*): $a \wedge b \equiv \frac{1}{2}(ab - ba)$

General elements of the algebra are called *multivectors* and these decompose into sums of elements of different grades (scalars have grade 0, vectors have grade 1, etc).

For multivectors A and B we define:

- scalar product: $A * B \equiv \langle AB \rangle_0$, where $\langle M \rangle_r$ denotes projection onto grade r of M.
- commutator product: $A \times B \equiv \frac{1}{2}(AB - BA)$

We can extend the definition of inner/outer product to multivectors that contain only elements of the same grade (we use an index to label the grade of the elements) and employ the notation:

$$
\begin{aligned}
A_r \cdot B_s &\equiv \langle AB \rangle_{|r-s|} \\
A_r \wedge B_s &\equiv \langle AB \rangle_{r+s}
\end{aligned}
$$

An overdot notation is useful in expressions when several multivectors are used. Since in general multivectors do not commute, the overdot shows between which multivectors the operation applies. As an example: $\partial_X(AB) = \dot{\partial}_X \dot{A}B + \dot{\partial}_X A\dot{B}$.

The STA is generated by 4 orthonormal vectors $\{\gamma_\nu\}$ such that $\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = \mathrm{diag}(+ - - -)$. The associated algebra is composed of 1(1 scalar), $\{\gamma_\mu\}$(4 vectors), $\{\gamma_\mu \wedge \gamma_\nu\}$(6 bivectors),

$I \equiv \gamma_0\gamma_1\gamma_2\gamma_3$(1 pseudoscalar), $\{I\gamma_\mu\}$(4 trivectors). Under the STA setup, a novel approach to GTG is introduced in [18]. In this work we will rely on their results and notation and we proceed by just mentioning the relevant definitions and basic concepts without much detail - the reader is encouraged to refer to [18] for a full account.

The central idea of the STA approach is that the physical content of a field equation (in STA) must be invariant under arbitrary local displacements and rotations of the fields. This is referred to as position (respectively rotation)-gauge invariance. In order to preserve the local symmetries one needs to modify only the derivative-operator since all non-derivative relations already satisfy the correct requirements.

The position-gauge field

Let f be an arbitrary map between spacetime vectors and ϕ a scalar field such that:

$$\phi'(x) \equiv \phi(x') = \phi(f(x)) \tag{3.1}$$

Any quantity that has this transformation under arbitrary displacements is referred to as covariant.

If $\underline{f}(a) \equiv a \cdot \nabla f(x)$ and the adjoint is defined as $\overline{f}(a)$(i.e. $a \cdot \underline{f}(b) = \overline{f}(a) \cdot b$) one can find that

$$\nabla_x = \overline{f}(\nabla_{x'}) \Rightarrow \nabla\phi'(x) = \overline{f}[\nabla_{x'}\phi(x')] \tag{3.2}$$

In order to form derivatives of covariant objects which are also covariant, define the position-gauge field $\overline{h}(a, x)$:

$$\overline{h}'(a, x) \equiv \overline{h}(\overline{f}^{-1}(a), f(x)) \tag{3.3}$$

(i.e. require $\overline{h}_x(\nabla_x) \rightarrow \overline{h}_{x'}(\nabla_{x'})$).

The rotation-gauge field

In geometric algebra one can prove that under a rotation (marked by a rotor R defined to satisfy $R\tilde{R} = 1$) a multivector transforms like:

$$M \mapsto RM\tilde{R} \tag{3.4}$$

By looking at the effect of rotations on equations such as Maxwell's equation one finds that in order to have a covariant derivative an extra 'connection' term is needed:

$$\mathcal{D}_a \equiv a \cdot \nabla + \Omega(a)\times \tag{3.5}$$

Here $\Omega(a)$ is a bivector-valued linear function and represents the second gauge field.
It is useful to define the full covariant directional derivative $a \cdot \mathcal{D}$:

$$a \cdot \mathcal{D} \equiv a \cdot \overline{h}(\nabla)M + \omega(a) \times M, \text{ where we define } \omega(a) = \Omega(h(a)). \tag{3.6}$$

Thus one can convert between $\mathcal{D}_a$ and $a \cdot \mathcal{D}$ by using $\underline{h}^{-1}$: $\underline{h}^{-1}(a) \cdot \mathcal{D} = \mathcal{D}_a$.
For the $a \cdot \mathcal{D}$ operator can further define the covariant vector derivative:

$$\mathcal{D}M \equiv \partial_a a \cdot \mathcal{D}M = \overline{h}(\partial_a)\mathcal{D}_a M \tag{3.7}$$

The field strength

The field strength is defined as the commutator of the covariant derivatives:

$$R(a \wedge b) \times M = [\mathcal{D}_a, \mathcal{D}_b]M$$

$$\Rightarrow R(a \wedge b) = a \cdot \nabla\Omega(b) - b \cdot \nabla\Omega(a) + \Omega(a) \times \Omega(b) \tag{3.8}$$

A covariant quantity can be constructed by defining:

$$\mathcal{R}(b \wedge c) \equiv R(\underline{h}(b) \wedge \underline{h}(c)) \tag{3.9}$$

This new quantity is the gauge theory analogue of the Riemann tensor. The equivalent for the Ricci tensor, Ricci scalar and Einstein tensor can be found from contractions:

$$\mathcal{R}(b) = \partial_a \cdot \mathcal{R}(a \wedge b) \tag{3.10}$$

$$\mathcal{R} = \partial_a \cdot \mathcal{R}(a) \tag{3.11}$$

$$\mathcal{G}(a) = \mathcal{R}(a) - \frac{1}{2}a\mathcal{R} \tag{3.12}$$

It is worth noting that all the above quantities are covariant since they inherit the transformation properties of the Riemann tensor.

Results in multivector calculus

Considering our work relies heavily on the action principle, it is important to note some preliminary results for directional derivatives of a field and for derivatives with respect to linear and bivector functions (for $\underline{h}$ and Ω).

For a fixed frame e^j (with $e^j \cdot e_k = \delta^j_k$), the partial derivative of field ψ with respect to the coordinate $x^j = e^j \cdot x$ is given by:

$$\psi_{,j} \equiv e_j \cdot \nabla\psi \tag{3.13}$$

The equivalent frame-free multivector derivative (the derivative of ψ with respect to a) can be defined as:

$$\partial_{\psi,a} \equiv a \cdot e_j \partial_{\psi,j} \tag{3.14}$$

and has the important property:

$$\partial_{\psi,a}\langle b \cdot \nabla\psi M\rangle = a \cdot bP_{\psi}(M), \text{ where } P \text{ is the projection.} \tag{3.15}$$

As a side motivation, in Geometric Algebra most of the properties of the multivector derivative follow from $\partial_X\langle XA\rangle = P_X(A)$, where $P_X(A)$ is the projection of A onto the grades contained in X.

By directly applying to a Lagrangian density $\mathcal{L} = \mathcal{L}(\psi, a \cdot \nabla\psi)$, the Euler-Lagrange equations become:

$$\partial_{\psi}\mathcal{L} = \partial_a \cdot \nabla(\partial_{\psi,a}\mathcal{L}) \tag{3.16}$$

The scalar coefficients for a linear function $\underline{h}$ are given by $h_{ij} \equiv e_i \cdot \underline{h}(e_j)$ which implies that equation 3.14 extends to:

$$\partial_{\underline{h}(a)} \equiv a \cdot e_i e_j \partial_{h_{ij}} \tag{3.17}$$

and has the important property:

$$\partial_{\underline{h}(a)}\underline{h}(b) \cdot c = a \cdot e_j e_i \partial_{h_{ij}}(h_{lk}b^k c^l) = a \cdot e_j e_i c^i b^j = a \cdot bc \tag{3.18}$$

It can be shown that:

$$\partial_{\underline{h}(a)}\langle\underline{h}(A_r)B_r\rangle = \langle\underline{h}(a \cdot A_r)B_r\rangle_1, \tag{3.19}$$

which is particularly important for the field equations of the perturbed case when one needs to express $\partial_{\underline{h}(a)}\det(\underline{h})$.

By recalling the definition of the *inverse function*: $\underline{f}^{-1}(A) = \det(\underline{f})^{-1}\overline{f}(AI)I^{-1}$ and applying the result from above:

$$\partial_{\underline{h}(a)}\det(\underline{h}) = \partial_{\underline{h}(a)}\langle \underline{h}(I)I^{-1}\rangle = \underline{h}(a\cdot I)I^{-1} = \det(\underline{h})\overline{h}^{-1}(a) \tag{3.20}$$

It is worth noting that the derivatives of $\underline{h}$ and $\overline{h}$ give different results in general ($\partial_{\underline{h}(a)}\overline{h}(b) = ba$) which makes the stress-energy tensors for certain fields lose symmetry. Such examples can be seen for Dirac-fields in [18].

Finally, the derivative can be extended for bivector-valued linear functions in a similar fashion:

$$\begin{aligned} \partial_{\Omega(a)}\langle \Omega(b)M\rangle &= a\cdot b\langle M\rangle_2 \\ \partial_{\Omega(b),a}\langle c\cdot\Omega(d)M\rangle &= a\cdot c b\cdot d\langle M\rangle_2 \end{aligned} \tag{3.21}$$

3.1.2 Current work in Riemann-squared gravity

In [78] the authors find analytic background solutions in the conformal gauge for Riemann-squared gravity. The equations have been translated from STA to tensor algebra to illustrate the correspondence between the two languages. Considering we will have a similar approach we give a brief account of their setup and results.

As we have previously mentioned we start from the action integral given by:

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{12}R^{\alpha}_{\beta\mu\nu}R^{\beta\mu\nu}_{\alpha} + \kappa\mathcal{L}_m\right) \tag{3.22}$$

In this representation the gravitational degrees of freedom are the tetrad h^a_μ and the connection ω^{ab}_μ. Although in STA $\underline{h}(a)$ has the purpose to ensure covariance under displacements, it can be used to construct a vierbein (essentially an orthonormal tetrad).

In the coordinate frame e_μ one constructs the vector $g_\mu \equiv \underline{h}^{-1}(e_\mu)$ which gives the metric:

$$g_{\mu\nu} \equiv g_\mu \cdot g_\nu \tag{3.23}$$

This leads to a vierbein e^i_μ given by:

$$e^i_\mu = g_\mu \cdot \gamma^i \tag{3.24}$$

The equivalent of ω^{ab}_μ is $\omega(a) \equiv \Omega\underline{h}(a)$.

In [18] the authors find the most general form for $w(a)$ that is consistent with isotropy and homogeneity. To obtain the equivalent in standard notation one needs to introduce three gravitational degrees of freedom: the scale factor $a(t)$, the potential $\phi(t)$ (which is $\dot{a}$ in the cosmological gauge) and the torsion contribution $\psi(t)$. The only non-zero components are:

$$\omega_{101} = -\omega_{110} = \frac{\phi}{\sqrt{1-kr^2}} \qquad \omega_{132} = -\omega_{123} = \frac{\psi}{\sqrt{1-kr^2}}$$

$$\omega_{202} = -\omega_{220} = r\phi \qquad \omega_{212} = -\omega_{221} = \sqrt{1-kr^2}$$

$$\omega_{213} = -\omega_{231} = r\psi \qquad \omega_{303} = -\omega_{330} = r\phi\sin\theta$$

$$\omega_{321} = -\omega_{312} = r\psi\sin\theta \qquad \omega_{313} = -\omega_{331} = \sin\theta\sqrt{1-kr^2}$$

$$\omega_{323} = -\omega_{332} = \cos\theta$$

Using the same approach as in STA (equation 3.8) one can define the Riemann tensor from the ω's by direct calculation. By assuming the matter content is pure radiation and substituting the form of the Riemann tensor, the reduced action becomes:

$$S = \int dt\Big(a(\dot{\phi}^2 - \dot{\psi}^2) + \frac{1}{a}(\phi^4 - 6\phi^2\psi^2 + \psi^4 + 2k\phi^2 - 2k\psi^2 + k^2) + \frac{\kappa}{3}a^3\rho\Big) \tag{3.25}$$

Under rescaling and reparametrisation a transforms as the inverse of an einbein and thus is an arbitrary function that can be chosen for convenience. To simplify the equations of motion it will be considered unity, i.e. choose to work in the 'conformal' gauge. In this gauge the analytic solutions for the general Euler-Lagrange equations are given in terms of Weierstrass functions. In order to simplify the equations torsion can be turned off (set $\psi = 0$) which leads to:

$$\frac{\kappa}{2c_0}\rho = H^2 + \frac{c_0}{2} + \frac{k}{a^2}, \tag{3.26}$$

where c_0 was introduced as a constant of integration. By redefining ρ the above equation becomes the standard FRW equation with a cosmological constant $\frac{c_0}{2}$. This suggests that the cosmological constant occurs naturally.

When further simplifying by setting $k = \rho = 0$ one obtains two solutions for H: $H = constant$ (de Sitter space) and $H = 1/(2t)$ (a simple big bang model). Prompted by this result the authors investigate how to recover the simple big bang solution from the general solution. An interesting effect is found when the solution includes some initial torsion: the initial big bang singularity is removed and the evolution is made entirely finite.

By looking at the perturbed regime we hope to improve our understanding of the results presented in [78]. The perturbed cosmological parameters will be affected by the free-propagating torsion, so by looking at how they change we could gain a significant insight in the behaviour of the torsion field.

3.2 Perturbation theory in Flat-Λ cosmology

In order to find the unique features of the pure radiation Riemann-squared model, we use the standard pure-radiation model as reference. For clarity we will write this section in standard GR and then link to the work conducted in GA in the following section. We start by giving a full account on how to obtain the perturbed field equations in the general form (by following the standard prescription presented in [80], Chapter 8) and then specialise to the pure radiation flat- Λ universe.

As advertised in [81] we choose to conduct our analysis in the conformal-Newtonian Gauge. In this gauge the linearised equations take a simpler form since there are no residual gauge modes. We impose the notation:

$$ds^2 = a^2((1+2\Psi)d\tau^2 - (1-2\Phi)\delta_{ij}dx^i dx^j), \tag{3.27}$$

where Ψ and Φ are the amplitudes of the metric perturbations in the conformal-Newtonian coordinate system.

Perturbed Energy-Momentum Tensor

For a homogeneous and isotropic universe we know that the Energy-Momentum Tensor (EMT) takes the form:

$$\overline{T}^{\mu}_{\nu} = (\overline{\rho} + \overline{P})\overline{U}^{\mu}\overline{U}_{\nu} - \overline{P}\delta^{\mu}_{\nu}, \tag{3.28}$$

where we use the overline notation to denote the unperturbed quantity. By writing Π^{μ}_{ν} as the anisotropic stress, the perturbed part of the EMT becomes:

$$\delta T^{\mu}_{\nu} = (\delta\rho + \delta P)\overline{U}^{\mu}\overline{U}_{\nu} + (\overline{\rho} + \overline{P})(\delta U^{\mu}\overline{U}_{\nu} + \overline{U}^{\mu}\delta U_{\nu}) - \delta P\delta^{\mu}_{\nu} - \Pi^{\mu}_{\nu} \tag{3.29}$$

By looking at this expression we can note that δU is the generator of energy flux (T^0_j) and momentum density (T^i_0). In order to compute δU we recall the standard result:

$$g_{\mu\nu}U^{\mu}U^{\nu} = \overline{g}_{\mu\nu}\overline{U}^{\mu}\overline{U}^{\nu} = 1 \tag{3.30}$$

$$\Rightarrow \delta g_{\mu\nu}\overline{U}^{\mu}\overline{U}^{\nu} + 2\overline{U}_{\mu}\delta U^{\mu} = 0 \text{ (to first order)} \tag{3.31}$$

For a comoving observer $\overline{U}^{\mu} = \frac{1}{a}\delta^{\mu}_{0}$. Thus we find that U takes the form:

$$U^{\mu} = \frac{1}{a}(1-\Psi, v^i), \tag{3.32}$$

where $v^i \equiv \frac{dx^i}{d\tau}$ is the coordinate velocity.

Using this result we can find from equation 3.29 the components of the EMT:

$$\begin{aligned}
\delta T^0_0 &= \delta\rho \\
\delta T^i_0 &= (\overline{\rho}+\overline{P})v^i \\
\delta T^0_j &= -(\overline{\rho}+\overline{P})v_j \\
\delta T^i_j &= -\delta P\delta^i_j - \Pi^i_j
\end{aligned}$$

EMT conservation

Before we proceed to find the linearised evolution equations we should note the restrictions on ρ and P under the pure-radiation and Λ scenario:

$$\begin{aligned}
\overline{\rho} &= \overline{\rho}_r + \Lambda, \qquad \overline{P} = \overline{P}_r - \Lambda = \frac{1}{3}\overline{\rho}_r - \Lambda \\
\delta\rho &= \delta\rho_r, \qquad \delta P = \delta P_r = \frac{1}{3}\delta\rho_r
\end{aligned}$$

From the conservation of the EMT we can find the continuity and Euler equations. The conserved quantity takes the form:

$$\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_{\mu\alpha}T^\alpha_\nu - \Gamma^\alpha_{\mu\nu}T^\mu_\alpha = 0 \tag{3.33}$$

The Christoffel symbols can be found directly from the definition in terms of the metric:

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\lambda}(\partial_\nu g_{\lambda\rho} + \partial_\rho g_{\lambda\nu} - \partial_\lambda g_{\nu\rho}) \tag{3.34}$$

From $\nabla_\mu T^\mu_0$ we can find the continuity equation. By looking at the zeroth order components we recover the background continuity equation:

$$\overline{\rho}' = -3\mathcal{H}(\overline{\rho}+\overline{P}), \tag{3.35}$$

where the differentiation is made with respect to conformal time and $\mathcal{H} \equiv \frac{a'}{a}$.

The first order component reads as:

$$\delta' + \frac{4}{3}\frac{\overline{\rho}_r}{\overline{\rho}_r+\Lambda}(\nabla\cdot\underline{v} - 3\Phi') + 4\mathcal{H}\frac{\Lambda}{\overline{\rho}_r+\Lambda} = 0 \tag{3.36}$$

As a possible physical interpretation the term containing $\nabla\cdot\underline{v}$ accounts for the local fluid flow due to peculiar velocity, the term containing Φ' provides the change in density caused by the perturbation to the local expansion rate, and the final term adds an extra dilution from the background expansion.

From $\nabla_\mu T^\mu_i$ we can find the Euler equation:

$$\underline{v}' = -\frac{1}{4}\frac{\nabla\delta\rho_r}{\rho_r} - \nabla\Psi \tag{3.37}$$

Thus the rate of change of the velocity is given by the pressure gradient (the first term) and the gravitational infall ($\nabla\Psi$).

Einstein's equations

We derive the final evolution equations from Einstein's equations, which have the same structure as in the unperturbed case:

$$\delta G_{\mu\nu} = 8\pi G\delta T_{\mu\nu} \tag{3.38}$$

We start by looking at the equation corresponding to the ij-component, which provides a constraint for the gauge potentials in our setup. In a pure-radiation universe there is no anisotropic stress and thus the equation can be written as:

$$\partial_{<i}\partial_{j>}(\Phi - \Psi) = 0 \qquad \Rightarrow \Phi = \Psi. \tag{3.39}$$

After rewriting the equations in terms of only one potential, we find that the zeroth order part of the 00-component recovers Friedman's first equation and the first order part gives:

$$\nabla^2\Phi = 4\pi Ga^2\overline{\rho}\delta + 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) \tag{3.40}$$

Furthermore, by assuming the perturbations decrease at infinity the expression from the $0i$-component can be integrated to obtain:

$$\Phi' + \mathcal{H}\Phi = -4\pi Ga^2(\overline{\rho} + \overline{P})v, \tag{3.41}$$

We derive the new form for Poisson's equation by combining the above equation with equation 3.40:

$$\nabla^2\Phi = 4\pi Ga^2(\overline{\rho}\delta - 4\mathcal{H}\overline{\rho}_r v) \tag{3.42}$$

In order to find the final equation we look at the trace-component. As expected, at zeroth order we recover Friedman's second equation and at first order we obtain a new equation, namely:

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = \frac{4\pi G}{3}a^2\delta\rho_r \tag{3.43}$$

We can now find a "master"-equation for Φ from equations 3.40 and 3.43 that can be solved in Fourier Space:

$$\Phi'' + 4\mathcal{H}\Phi' + (2\mathcal{H}' + 2\mathcal{H}^2)\Phi - \frac{1}{3}\nabla^2\Phi = 0 \tag{3.44}$$

The evolution of the scale parameter

Before we proceed to solve the equation for Φ we need to understand the behaviour of a for a pure radiation model. For t defined as coordinate time it can be easily shown that

$$\left(\frac{da}{dt}\right)^2 = H_0^2\left(\frac{\Omega_{r,0}}{a^2} + \Omega_{\Lambda,0}a^2\right), \quad \text{where } \Omega_{r,0} + \Omega_{\Lambda,0} = 1, \tag{3.45}$$

leads to an expression for t as an integral of a:

$$t = \int_0^a \frac{a}{H_0\sqrt{1 - \Omega_{\Lambda,0} + \Omega_{\Lambda,0}a^4}}da. \tag{3.46}$$

This integral can easily be solved by using the substitution $y^2 \equiv \frac{\Omega_{\Lambda,0}}{1-\Omega_{\Lambda,0}}a^4$ and assuming that $\Omega_{\Lambda,0}$ is positive:

$$t = \sinh^{-1}\left(a^2\sqrt{\frac{\Omega_{\Lambda,0}}{1 - \Omega_{\Lambda,0}}}\right) \tag{3.47}$$

Moving back to conformal time we can find:

$$\tau(a) = 2\int_0^a \left(b^4 + \frac{1 - \Omega_{\Lambda,0}}{\Omega_{\Lambda,0}}\right)^{-\frac{1}{2}} db, \tag{3.48}$$

From equation 3.47 it is transparent that the epoch of equality can be reached when:

$$a_{eq} = \left(\frac{1 - \Omega_{\Lambda,0}}{\Omega_{\Lambda,0}}\right)^{\frac{1}{4}} \tag{3.49}$$

It follows that under the transformation $a \mapsto a_{eq}^2/a$:

$$\tau(a_{eq}) = 2\int_{a_{eq}}^{\infty} \left(b^4 + \frac{1 - \Omega_{\Lambda,0}}{\Omega_{\Lambda,0}}\right)^{-\frac{1}{2}} db = \tau_{total} - \tau(a_{eq}), \tag{3.50}$$

which implies that the energy densities are equal at the midpoint in the conformal time evolution. It is convenient to fix the scaling of a by requiring $a_{eq} = 1$. Under this scaling a becomes symmetric about the midpoint and antisymmetric under the transformations $a \mapsto 1/a$ and $\tau \mapsto \tau_{total} - \tau$.

Solution for the Newtonian potential

Under the new scaling it becomes convenient to write equation 3.44 in terms of a. By moving to Fourier space and writing $k = K\sqrt{\Lambda}$ we find the equivalent equation:

$$a(1+a^4)\Phi'' + 2(3a^4+2)\Phi' + a(4a^2+K^2)\Phi = 0, \tag{3.51}$$

where Φ' now denotes the derivative with respect to a.

In order to find the initial conditions we require Φ to be analytic as $a \mapsto 0$ (which must be true considering Φ is the Newtonian Potential) and thus can be written as a series of a for small a:

$$\Phi = a^{\alpha} \sum_{i=0}^{\infty} c_i a^i \tag{3.52}$$

By using this form in equation 3.51 we find that α can be 0 or negative. We cannot accept $\Phi \mapsto \infty$ since it would violate the conditions for linearisation so we have only one possible linear mode:

$$\Phi(a) = c_0\Big(1 - \frac{K^2}{10}a^2 + \ldots\Big) \tag{3.53}$$

Thus we need to solve equation 3.51 under the initial conditions that $\Phi(a \mapsto 0) = 1$ and $\Phi'(a \mapsto 0) = 0$ - we can just set $c_0 = 1$ without loss of generality.

The most general solution can be identified as a multiple of a Heun-function:

$$\Phi(a) = (1+a^4)^{\frac{1}{4}} \exp\left(\frac{1}{2}\tan^{-1}(a^2)\right)\mathrm{HeunG}\Big(-1, \frac{1}{4}(5 - iK^2), 1, \frac{5}{2}, \frac{5}{2}, \frac{1}{2}, a^2 i\Big) \tag{3.54}$$

Currently there is significant research ongoing for a better understanding of Heun-functions, a class of functions that despite being very popular in modern physics are not very well documented. Although several programming languages can identify Heun-functions as solutions to differential equations, there are considerable issues when using them - such as the inability to accurately compute (higher) derivatives or the incapability to describe the solution on the full domain. At an analytic level significant work has been made to find integral forms to Heun-functions and several representations have been discovered for special cases. For our form there is an available representation which can be found by following the work in [82]:

$$\begin{aligned}
\Phi(a) &= \frac{3\sqrt{1+a^2K^2+a^4}}{a^3K\sqrt{K^4-4}} \sin\left(K\sqrt{K^2-4}\psi(a)\right), \text{ where} \\
\psi(a) &= \int_0^a \frac{b^2}{\sqrt{1+b^4}(1+b^2K^2+b^4)} db
\end{aligned}$$

The computational solution for Φ can be seen in *Figure 3.1*, where we have chosen the normalised wavenumber to take the value $K = 10$. This is a representative plot for the behaviour of Φ for any value of K: a solution that after a short period of slight oscillation decays away.

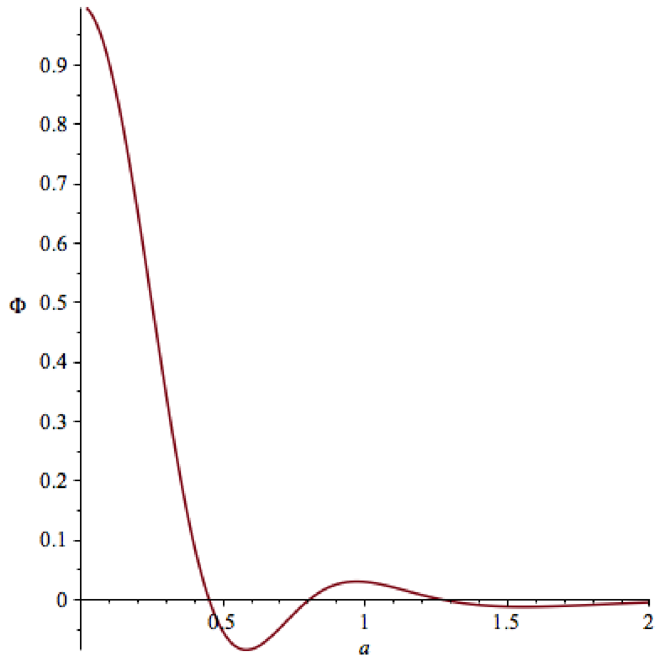

Figure 3.1: Newtonian potential as a function of scale factor for normalised wavenumber $K = 10$

Other cosmological parameters

After finding the value for Φ it is straightforward to find the evolution for the velocity perturbation and the density perturbation. By rewriting equation 3.42 in terms of a we find the equation for v:

$$v = \frac{\sqrt{3}}{2} aK\sqrt{1 + a^4}(\Phi + a\Phi') \tag{3.55}$$

Similarly from equation 3.43 we obtain the equation for δ:

$$\delta = -2(1 + a^2K^2 + a^4)\Phi - 2a(1 + a^4)\Phi' \tag{3.56}$$

When solved with respect to cosmic time, both solutions oscillate initially and then "freeze-out". By looking at high values for t one can notice that the solutions do not depart visibly from the values already reached at about $t = 10$.

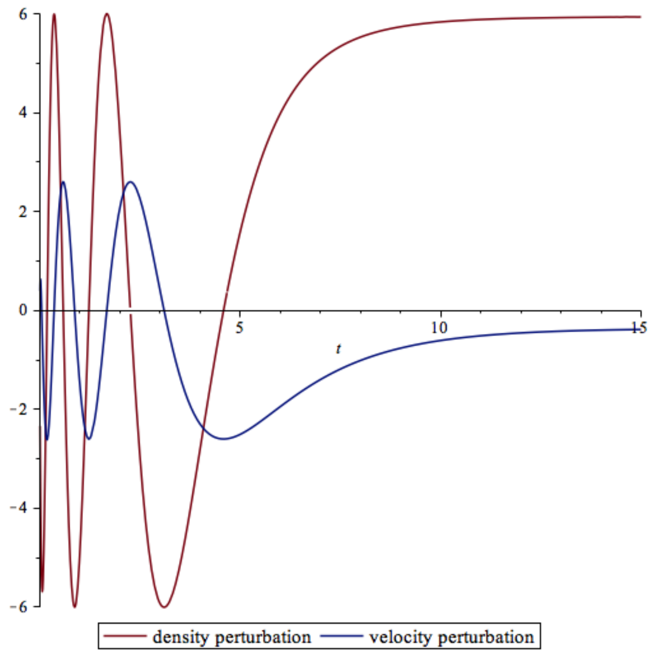

Figure 3.2: Perturbations with respect to cosmic time

A different behaviour can be observed in the evolution with respect to conformal time (τ), where the perturbations are regular-looking sine waves that do not decay. It can easily be seen that the velocity perturbation is phase-orthogonal to the density perturbation as expected from 3.37. We can show from 3.50 that for Λ=1 the total elapsed conformal time is

$$\tau_{total} = \frac{\Gamma(\frac{1}{4})\Gamma(\frac{5}{4})}{\Gamma(\frac{1}{2})} \approx 3.52135, \tag{3.57}$$

and thus our representation in *Figure* 3.3 is confined in this range.

3.3 Perturbation Theory in Riemann-squared Gravity

In order to find the perturbed linear equations in Riemann-squared gravity we employ the theoretical approach presented in [81]. To maintain consistency with our previous work we choose to work in the conformal Newtonian gauge which has the attractive feature of not having residual gauge modes. Unlike in *Section 2* where we have shown that the lack of anisotropic stress implies the potentials are indistinguishable, in this case we will keep both potentials and write:

$$ds^2 = a(\tau)^2((1 + 2\Psi(\tau, \underline{\mathbf{x}}))d\tau^2 - (1 - 2\Phi(\tau, \underline{\mathbf{x}}))\delta_{ij}dx^i dx^j) \tag{3.58}$$

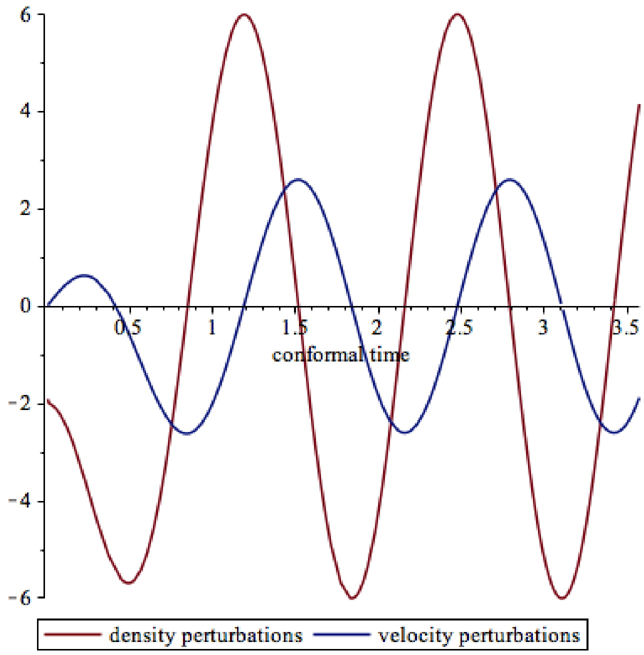

Figure 3.3: Perturbations with respect to conformal time

Setup for the field equations:

We start by stating the *recipe* that leads to field equations. As in the unperturbed case, we encode the information from the metric in the vectors $g_\mu \equiv \underline{h}^{-1}(e_\mu)$ that satisfy $g_{\mu\nu} \equiv g_\mu \cdot g_\nu$:

$$g_0 = a(\tau)\sqrt{1 + 2\Psi(\tau, \underline{\mathbf{x}})}\gamma_0$$

$$g_i = -a(\tau)\sqrt{1 - 2\Phi(\tau, \underline{\mathbf{x}})}\gamma_i$$

In order to compute the Ω's (and respectively the ω's) one uses the 'dual' definition of the connection as prescribed in [18] Appendix C. From a "geometric" point of view, the connection can be written as a function of the covariant derivative as

$$\Gamma^\lambda_{\mu\nu} \equiv g^\lambda \cdot (\mathcal{D}_\mu g_\nu), \text{ where } \mathcal{D}_\mu \equiv g_\mu \cdot \mathcal{D} = \partial_\mu + \omega(g_\mu) \times . \tag{3.59}$$

The connection also 'stores' the structure of the spacetime and thus needs to account for the metric (the Christoffel symbol) and torsion (the contorsion tensor $K^\nu_{\lambda\mu}$) contributions:

$$\Gamma^\nu_{\lambda\mu} = \{\}^\nu_{\lambda\mu} - K^\nu_{\lambda\mu} \tag{3.60}$$

The contorsion tensor is defined from the torsion tensor ($S^{\nu}_{\lambda\mu}$) as :

$$K^{\nu}_{\lambda\mu} = -S^{\nu}_{\lambda\mu} + S_{\lambda}{}^{\nu}{}_{\mu} - S^{\nu}_{\lambda\mu} \tag{3.61}$$

We proceed by defining the torsion bivector $\mathcal{S}$ by using the following ansatz - although this might not be the most general form it produces consistent field equations which suffices for our current purposes:

$$\mathcal{S}_i = T_1(\gamma_i + 3\gamma_0\gamma_i\gamma_0) + T_2(\gamma_i - 3\gamma_0\gamma_i\gamma_0) + FF(\tau, \mathbf{x})\gamma_0\gamma_i, \tag{3.62}$$

where T_i are two general first order vectors :

$$T_1 = \sum_{i=1}^{3} GG_i(\tau, \mathbf{x})\gamma_i \, , T_2 = \sum_{i=1}^{3} U_i(\tau, \mathbf{x})\gamma_i \tag{3.63}$$

and FF is a general function.

It is interesting to note that there is no protraction ($\partial_a \wedge (\mathcal{S} \cdot a) = 0$), but there is a non-zero contraction ($\partial_a \cdot (\mathcal{S} \cdot a)$) that needs to be accounted for - the contraction will appear in the ω's.

Using this form of torsion one can find the ω's by direct substitution in the above equations.

The field equations

In Riemann-squared gravity the scale-invariant gravitational Lagrangian takes the form:

$$\mathcal{L}_G = \frac{1}{4}\det h^{-1}\mathcal{R}(\partial_b \wedge \partial_c) \cdot \mathcal{R}(c \wedge b) \tag{3.64}$$

Before we proceed it is worth showing the invariance of the Lagrangian. Local changes of scale are determined by:

$$h^a \mapsto e^{-\alpha(x)}h^a, \qquad \Omega_a \mapsto \Omega_a, \tag{3.65}$$

where α is a function of position.

By using its definition, it is easy to see that the field strength transforms as:

$$\mathcal{R}_{ab} \mapsto e^{-2\alpha}\mathcal{R}_{ab} \tag{3.66}$$

So all quadratic terms in $\mathcal{R}_{ab}$ pick up a factor of $e^{-4\alpha}$ under scale changes (said to have weight 4). Similarly it can be shown that $\det h^{-1}$ has weight (-4) so the Lagrangian has weight 0.

In any gauge field theory, in order to find the field equations one needs to vary the Lagrangian with respect to the gauge fields, in this case $\overline{h}$ and Ω. This approach enables us to find the form of the stress-energy tensor (SET) and the Ω-equation. We then proceed to determine the density perturbations and the potentials as we have done in *Section 2* .

Finding the stress-energy tensor equations

The covariant gravitational SET is given in terms of the variation of the gravitational Lagrangian with respect to the position gauge:

$$\det(\underline{h})\partial_{\overline{h}(a)}(\mathcal{L}_G \det(\underline{h})^{-1}) = \mathcal{T}\underline{h}^{-1}(a) \tag{3.67}$$

Since the ∂ operator satisfies Leibniz's rule, we need to find expressions for $\partial_{\overline{h}(a)}\det h^{-1}$ and for $\partial_{\overline{h}(a)}\mathcal{R}(\partial_b \wedge \partial_c) \cdot \mathcal{R}(c \wedge b)$.

The first component can be obtained in a straightforward manner by modifying equation 3.20:

$$\partial_{\overline{h}(a)}\det h^{-1} = -\det h^{-1}\underline{h}^{-1}(a) \tag{3.68}$$

By using the definition of the covariant Riemann tensor 3.9 and the definition of the covariant vector derivative for b 3.7 one can write:

$$\mathcal{R}(b \wedge c) = R(\underline{h}(b) \wedge \underline{h}(c)) = b \cdot \overline{h}(\partial_d)R(d \wedge \underline{h}(c)) \tag{3.69}$$

From equation 3.17:

$$\partial_{\overline{h}(a)} b \cdot \overline{h}(\partial_d) = a \cdot \partial_d(b) \tag{3.70}$$

Similar results can be obtained if we do the same calculations for c. By using standard algebra manipulations we can write:

$$\begin{aligned}
\partial_{\overline{h}(a)}\mathcal{R}(b \wedge c) \cdot \mathcal{R}(\partial_c \wedge \partial_b) \quad &= 2[b\mathcal{R}(\underline{h}^{-1}(a) \wedge c) + c\mathcal{R}(b \wedge \underline{h}^{-1}(a))] \cdot \mathcal{R}(\partial_c \wedge \partial_b) \\
&= 4\partial_b \mathcal{R}(\underline{h}^{-1}(a) \wedge \partial_c) \cdot \mathcal{R}(c \wedge b)
\end{aligned}$$

From equation 3.67:

$$\mathcal{T}_G(a) = \partial_b \mathcal{R}(a \wedge \partial_c) \cdot \mathcal{R}(c \wedge b) - \frac{1}{4} a\mathcal{R}(\partial_b \wedge \partial_c) \cdot \mathcal{R}(c \wedge b) \tag{3.71}$$

It is worth noting that the SET is traceless ($\partial_a \cdot \mathcal{T}_G(a) = 0$), as expected from the scale invariance of $\mathcal{L}_G$.

For the background solution one obtains:

$$\begin{aligned}\mathcal{T}_G(\gamma_0) &= \frac{6\ddot{a}(\tau)}{a(\tau)^7}(a(\tau)\ddot{a}(\tau) - 2\dot{a}(\tau)^2) \\ \mathcal{T}_G(\gamma_i) &= -\frac{\mathcal{T}_G(\gamma_0)}{3}\end{aligned}$$

The relationship between $\mathcal{T}_G(\gamma_0)$ and $\mathcal{T}_G(\gamma_i)$ is not surprising considering the pressure aligns with the density in the radiation dominated case ($\rho = P/3$). In the case where $a \propto \tau$, the solution for the above equation corresponds to empty matter. This implies that for a solution to exist it is essential that the cosmological constant is introduced. In the unperturbed case the authors show that the cosmological constant occurs naturally in equation 3.26 and we furthermore believe it is a crucial trait of Riemann-squared gravity.

The Ω-equation

The general Ω-equation is given by:

$$\partial_{\Omega(a)}\mathcal{L} = \partial_b \cdot \nabla[\partial_{\Omega(a),b}\mathcal{L}_G] \tag{3.72}$$

We start by looking at the LHS of the equation. From the form of R in terms of Ω:

$$R(c \wedge d) = c \cdot \nabla\Omega(d) - d \cdot \nabla\Omega(c). \tag{3.73}$$

We notice that $\partial_{\Omega(a)}((\Omega(c) \times \Omega(d)) \cdot B)$ (where B will be replaced by the Riemann tensor in future computations) is the component we need to look at.

By definition,

$$\dot{\partial}_{\Omega(a)}((\dot{\Omega}(c) \times \Omega(d)) \cdot B) = \frac{1}{2}\partial_{\Omega(a)}\langle\dot{\Omega}(c)\Omega(d)B - \Omega(d)\dot{\Omega}(c)B\rangle \tag{3.74}$$

From equation 3.21:

$$\dot{\partial}_{\Omega(a)}((\dot{\Omega}(c) \times \Omega(d)) \cdot B) = a \cdot c\Omega(d) \times B \tag{3.75}$$

$$\Rightarrow \partial_{\Omega(a)}((\Omega(c) \times \Omega(d)) \cdot B) = \Omega(a \cdot (c \wedge d)) \times B \tag{3.76}$$

It is convenient to change $c \mapsto \underline{h}(\partial_c)$ and $d \mapsto \underline{h}(\partial_d)$ so we actually use $\mathcal{R}(\partial_c \wedge \partial_d)$ and set $B = \mathcal{R}(c \wedge d)$:

$$\Rightarrow \det h\, \partial_{\Omega(a)}\mathcal{L}_G = \frac{1}{2}\Omega(a \cdot \underline{h}(\partial_b \wedge \partial_c)) \times \mathcal{R}(c \wedge b) \tag{3.77}$$

By using the identity:

$$a \cdot \underline{h}(\partial_b \wedge \partial_c) = \partial_d((d \wedge a) \cdot \underline{h}(\partial_b \wedge \partial_c)) \tag{3.78}$$

$$\Rightarrow \det h\, \partial_{\Omega(a)}\mathcal{L}_G = \Omega(\partial_d) \times \mathcal{R}(\overline{h}(d \wedge a)) \tag{3.79}$$

For the RHS part start by noting that for a general bivector B:

$$\partial_{\Omega(a),b}(R(c \wedge d) \cdot B) = \partial_{\Omega(a),b}\langle c \cdot \nabla\Omega(d)B - d \cdot \nabla\Omega(c)B\rangle \tag{3.80}$$

From equation 3.21:

$$\Rightarrow \partial_{\Omega(a),b}(R(c \wedge d) \cdot B) = (a \wedge b) \cdot (c \wedge d)B \tag{3.81}$$

As previously, for $c \mapsto \underline{h}(\partial_c)$ and $d \mapsto \underline{h}(\partial_d)$, set $B = \mathcal{R}(c \wedge d)$ to get:

$$\det h\, \partial_{\Omega(a),b}\mathcal{L}_G = \frac{1}{2}(a \wedge b) \cdot \underline{h}(\partial_c \wedge \partial_d)\mathcal{R}(d \wedge c) = \mathcal{R}(\overline{h}(a \wedge b)) \tag{3.82}$$

To introduce these results in the general Ω-equation 3.72 we write it as:

$$\det h\, \partial_{\Omega(a)}\mathcal{L}_G = \partial_b \cdot \nabla(\det h\, \partial_{\Omega(a),b}\mathcal{L}_G) - \partial_{\Omega(a),b}\mathcal{L}_G\partial_b \cdot \nabla\det h \tag{3.83}$$

By defining a vector (which can be shown to be equal to the torsion contraction) $\mathcal{U} \equiv \det h\, \partial_b(\overline{h}(\partial_b)\det h^{-1})$ and employing our previous results, the above equation can be written in the elegant form:

$$\dot{\mathcal{D}}_b\dot{\mathcal{R}}(\overline{h}(\partial_b) \wedge \overline{h}(a)) + \mathcal{R}((\mathcal{D} + \mathcal{U}) \wedge \overline{h}(a)) = 0 \tag{3.84}$$

General solutions from the field equations

The SET-equations and Ω-equation form the full linearised evolution equations. In order to obtain the constraints for the general functions that were used to define torsion one works in Fourier space. The previously defined variables take the form:

$$\begin{aligned}
\Phi(\tau, \mathbf{x}) &= \Phi(\tau)e^{i\mathbf{k}\cdot\mathbf{x}} \\
\Psi(\tau, \mathbf{x}) &= \Psi(\tau)e^{i\mathbf{k}\cdot\mathbf{x}} \\
FF(\tau, \mathbf{x}) &= FF(\tau)e^{i\mathbf{k}\cdot\mathbf{x}} \\
GG_i(\tau, \mathbf{x}) &= -i\frac{k_i}{k}GG(\tau)e^{i\mathbf{k}\cdot\mathbf{x}} \\
U_i(\tau, \mathbf{x}) &= -i\frac{k_i}{k}U(\tau)e^{i\mathbf{k}\cdot\mathbf{x}} \\
v_i(\tau, \mathbf{x}) &= -i\frac{k_i}{k}V(\tau)e^{i\mathbf{k}\cdot\mathbf{x}}
\end{aligned}$$

The relevant physical quantities can be expressed in STA as:

$$\begin{aligned} \text{non-relativistic velocity:} \quad & v_{non} \equiv v_1(\tau)\gamma_1 + v_2(\tau)\gamma_2 + v_3(\tau)\gamma_3 \\ \text{SET:} \quad & T_0 \equiv (\delta\rho(\tau)\gamma_0 + (\rho(\tau) + P(\tau))v_{non})e^{i\mathbf{k}\cdot\mathbf{x}} \\ & T_i \equiv (-\delta P(\tau)\gamma_i + (\rho(\tau) + P(\tau))(\gamma_i v_{non})\gamma_0)e^{i\mathbf{k}\cdot\mathbf{x}} \end{aligned}$$

In a radiation-dominated universe the following relations hold:

$$\begin{aligned} \delta\rho(\tau) &\equiv \delta(\tau)\rho(\tau) \text{ (the definition of the density contrast)} \\ P(\tau) &= \frac{1}{3}\rho(\tau) \text{ (the equation of state)} \\ \Rightarrow \delta P(\tau) &= \frac{1}{3}\rho(\tau)\delta(\tau) \end{aligned}$$

For the background solution one finds from the SET:

$$\rho_0 = \rho(\tau) = \frac{3\beta}{4\pi a^7}\ddot{a}(a\ddot{a} - 2\dot{a}^2) \tag{3.85}$$

and from the Ω-equation:

$$\ddot{H} = -3Ha\dot{H} \tag{3.86}$$

Confident we have the right general equations in order to obtain the cosmological quantities we are left to play a game of finding the right substitutions. We will give an account of all the steps, but we will not write partial forms unless they are essential in later applications.

We start by finding a relation for Ψ from the SET-equations:

$$\Psi(\tau) = \frac{a}{k(aH^2 + \dot{H})}(2\dot{H}U + k\Phi H^2 - 4\dot{H}GG - 2aUH^2 - 2aGGH^2) \tag{3.87}$$

We substitute the new form in the remaining SET and Ω-equations. From the new Ω-equation we can find coupled differential equations for GG (first order) and Φ (second order). Using these relations we modify the Ω-equation and find a form for GG.

A different way to find GG is to use a relation for $\dot{\Psi}$ from the Ω-equation:

$$\dot{\Psi} = -\frac{1}{k}(4a\dot{GG} - 2a^2HU + kaH\Psi - kaFF + k\dot{\Phi} + kaH\Phi + 4a^2HGG - 2a\dot{U}) \tag{3.88}$$

We can use this substitution to change the Ω-equation and find two coupled second order differential equations for GG and Φ. These can be used to further change the Ω-equation. In order to modify the SET equations we need to use the form we have previously found for Ψ (equation 3.87) and the equations for $\ddot{\Phi}$ and $\ddot{GG}$.

From the new SET-equations we can find an equation for $\dot{\Phi}$ and:

$$\delta(\tau) = -\frac{4}{k(2aH^2+\dot{H})}(2VaH^3 - 4aGGH^2 + 2aUH^2 - 2H\dot{G}G + VH\dot{H} \\ +4H\dot{U} - kHFF + 2\dot{H}GG - 4aU\dot{H}) \tag{3.89}$$

When we require consistency between the two methods we find a simpler form for GG from comparing the derivatives of Ψ found from the different approaches:

$$GG = \frac{1}{6aH}(aVH^2 + 12aUH - 3kH\Phi + V\dot{H}) \tag{3.90}$$

We can proceed to find a form for V from the SET-equations by substituting the previously found identities for $\ddot{\Phi}$, $\ddot{G}G$, Ψ, $\dot{\Phi}$, δ. By constructing a first order differential equation for GG (by comparing the form we have found for GG in the first method and the new form) we can find a simpler form for V:

$$\dot{V} = -\frac{1}{H(4aH^2+\dot{H})}(3a^2H^4V + 3akH^3\Phi - 3akH^2FF + 12aH^2\dot{U} - \dot{H}^2V) \tag{3.91}$$

Similarly, we can find that U solves:

$$\dot{U} = \frac{1}{24aH^2}(6H^4a^2V + 72H^3a^2U - 42kaH^3\Phi + 19aH^2V\dot{H} + 6kaH^2FF \\ +18aHU\dot{H} - 9kH\Phi\dot{H} + 6V\dot{H}^2) \tag{3.92}$$

Substituting this form for $\dot{U}$ we can find more appealing forms for δ, $\dot{\Phi}$ and $\dot{V}$:

$$\delta(\tau) = -\frac{2}{3kH}(7aH^2V + 18aHU - 15kH\Phi + 8\dot{H}V) \tag{3.93}$$

$$\dot{\Phi} = \frac{a}{2k}(aH^2V + 18aHU - 9kH\Phi + 2kFF + 2\dot{H}V) \tag{3.94}$$

$$\dot{V} = -\frac{1}{2H}(3aH^2V + 18aHU - 9kH\Phi + 4V\dot{H}) \tag{3.95}$$

Using the above equations and the further derivatives of GG from equation 3.90 we can finally reach a constraint for FF:

$$FF = -\frac{1}{18kaH^2}(14a^2H^4V + 288a^2H^3U - 162kaH^3\Phi + 151aH^2V\dot{H} + 8k^2H^2V \\ +234aHU\dot{H} - 117kH\dot{H}\Phi + 78\dot{H}^2V)$$

In order to obtain first order differential equations for Φ and V we eliminate U by using equation 3.95. By first solving $\dot{\Phi}$ for V and then substituting the result into the $\dot{V}$ equation one finds a second order differential equation for Φ. Since the equation expresses Φ only in terms of $H(\tau)$, $a(\tau)$, k and the constant GG, it becomes the equivalent of the "master"-equation we have found in *Section 2*.

We have finally reached a form that can be used to find Φ by imposing constraints on the background cosmology. We start by looking at the simplest case, which corresponds to a de Sitter universe.

de Sitter cosmology

In a de Sitter cosmology the dynamics of the universe are dominated by the cosmological constant (Λ) that sets the expansion rate:

$$H_0 = H, \qquad a(\tau) = -\frac{1}{H_0 \tau} \tag{3.96}$$

We start by assuming $GG = 0$. The "master"-equation takes the simple form:

$$\ddot{\Phi}(\tau) = -\frac{99k^2 \frac{\Phi(\tau)}{\tau^2} - 16\Phi(\tau)k^4 - 32k^2 \frac{\dot{\Phi}}{\tau} + 44 \frac{\Phi(\tau)}{t^3}}{(11\tau^2 - 16k^2)} \tag{3.97}$$

The independence from H_0 is an interesting feature of the equation - at a first glance it might seem to be in contradiction with our previous claim (we have stated we required Λ in order to have consistent field equations). This is not the case since the evolution of τ is dependent on Λ and thus Φ will inherit this dependence. The equation has the solution:

$$\Phi = \frac{C_1}{k^3\tau^3}(4k^2\tau^2 \cos k\tau - 11 \cos k\tau - 11k\tau \sin k\tau) + \frac{C_2}{k^3\tau^3}(11k\tau \cos k\tau + 4k^2\tau^2 \sin k\tau - 11 \sin k\tau) \tag{3.98}$$

To find the unknown constants we can apply a series expansion at $\tau = 0$:

$$\Phi(\tau) = -C_1\left(\frac{11}{k^3\tau^3} + \frac{3}{2k\tau} + \frac{5k\tau}{8}\right) + C_2\left(\frac{1}{3} - \frac{3k^2\tau^2}{10}\right) + O(\tau^3) \tag{3.99}$$

It is straightforward to see that in order to have a finite potential we need to set $C_1 = 0$. The solution oscillates and then decreases at large τ as exhibited in *Figure 3.4*.

When we keep GG a non-zero constant and solve the "master"-equation we obtain:

$$\Phi_{GG}(\tau) = \Phi(\tau) - \frac{2GG}{kH_0\tau} \tag{3.100}$$

In order to have a finite potential at $\tau = 0$ we need to restrict $GG = 0$. Considering that a non-zero GG does not add physical content to the equations (just acts like a translation) this restriction is not of consequence.

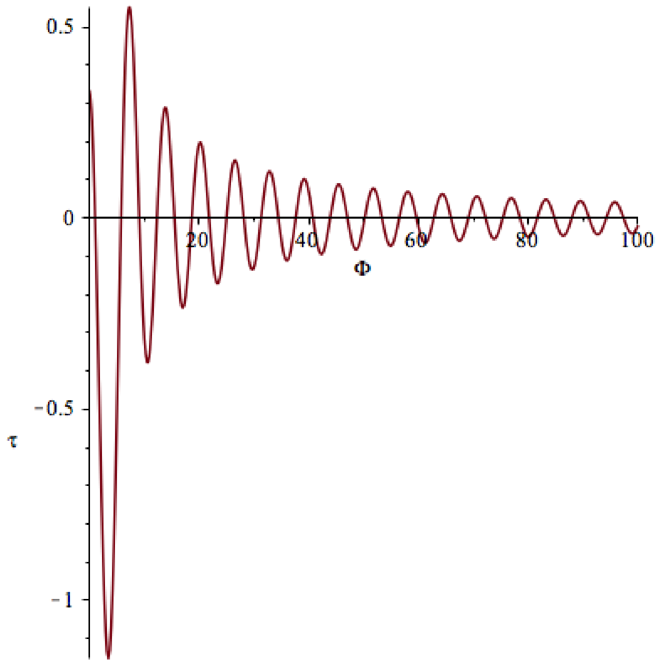

Figure 3.4: Φ Potential for de Sitter for GG=0

It is worth noting that in a de Sitter setup the Future Conformal Boundary occurs at $\tau = 0$ (from equation 3.96 we can easily see that $a \to \infty$ as $\tau \to 0$). In our model we have been able to use a series expansion around the origin which implies that the boundary might not be as essential as we might have previously thought. This feature will be important if we proceed to investigate gravitational waves under the de Sitter setup.

"Flat-Λ cosmology

We now look at the setup we have discussed in *Section 2*: a pure radiation flat - Λ universe that starts with a big bang and ends with an asymptotic de Sitter phase.

Under this setup we can show that the background equation for H in terms of a is given by:

$$aH(a)\partial_a H(a) - \frac{2\Lambda}{3} + 2H(a)^2 = 0, \tag{3.101}$$

and has the solution:

$$H(a) = \frac{1}{3a^2}\sqrt{3a^4\Lambda + 3\Lambda}. \tag{3.102}$$

Now we can obtain the "master"-equation in terms of a. From the chain rule:

$$\partial_\tau \Phi = a^2 H(a)\partial_a \Phi$$

$$a^2 H(a)\partial_a (a^2 H(a)\partial_a \Phi) - \partial_{\tau\tau}\Phi(a) = 0$$

The desired equation can be found by plugging in $\partial_a H(a)$ from equation 3.102 and $H(a)$ from equation 3.103 into the "master"-equation for $\partial_{\tau\tau}\Phi(a)$ and then substituting this result into the above equation. Starting with Φ we can now obtain the cosmological parameters.

Finding Φ

From the "master"-equation we can proceed to find an analytic solution for Φ similar in form to the solution in the standard model (equation 3.55). In order to find the constants of integration we find the series expansion as $a \to 0$ and demand it to have a regular solution at the origin. By substituting $k = K\sqrt{\Lambda}$ we obtain:

$$\Phi(a) = \frac{1}{a^3}\left(\sqrt{P_1(a,K)}\sin\left(\sqrt{P_2(K)}\mathcal{I}(a,K)\right)\right), \tag{3.103}$$

where

$$\begin{aligned} P_1(a,K) &= 144K^4a^8 + 99K^2a^{10} + 121a^{12} - 209a^8 - 105a^4 + 144K^4a^4 \\ &\quad -558K^2a^6 + 315K^2a^2 + 225 \\ P_2(K) &= 3K^2(3K^2-2)(3K^2+2) \\ \mathcal{I}(a,K) &= \int_0^a \frac{b^2(-48K^2b^6 + 11b^8 - 48K^2b^2 + 434b^4 - 45)}{\sqrt{b^4+1}P_1(a,K)}db \end{aligned} \tag{3.104}$$

We have seen in the standard model that the Newtonian potential vanishes as $a \to \infty$. Is this the case for Φ?

Ideally we would like to find a discrete set of values for K for which Φ does not blow up as $a \to \infty$. Solving such an equation is very challenging analytically, but a high-precision computational solution is possible for finite, large a. By looking at large values of a we notice that the behaviour of Φ in terms of K is dictated by the oscillating component and thus we are interested in solving:

$$\sin\left(\sqrt{P_2(K)}\mathcal{I}(a,K)\right) = 0 \text{ as } a \to \infty \tag{3.105}$$

When we plot Φ at this boundary we observe that as we keep increasing a the amplitude grows significantly whereas the roots change position very slightly. This indicates that the roots are stable as $a \to \infty$ under the right error-tolerance.

From *Figure* 3.5 we can see that there are a number of interesting properties. We start by noticing there is a single solution between any two integers and the evolution is regular. This

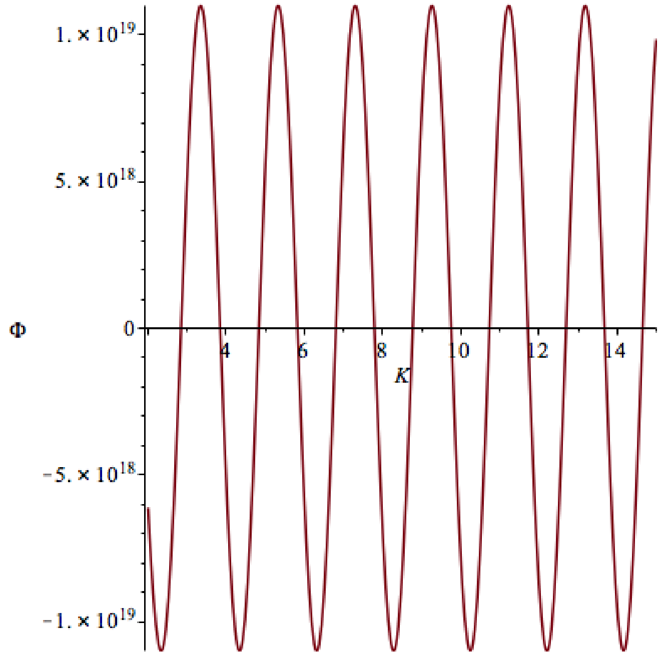

Figure 3.5: Φ for $a = 10^6$. As it can be seen on the boundary given by large a the potential evolves in a regular sinusoidal that intersects the horizontal axis exactly once between any two integers.

is particularly good for the accuracy of root finders that are based on n-point interpolation as is the case for Maple.

Since the sinusoid is very steep we expect that if we choose a value for K slightly different than the root we will have a divergent potential. This behaviour is illustrated in *Figure* 3.6 where we have plotted the solution for $K = 3.8$ and $K = 3.9$.

Considering the roots change in position when a changes value can we actually state that there exists a K for which the potential converges? For a finite a we can keep increasing the precision until satisfactory results are obtained (we are mostly interested in having a stable solution). For instance by using $K = 3.8349724960036489533536892920520690756 43$ while keeping 40 decimal places we can find a solution that does not diverge up to $a = 10^{25}$. By keeping more decimal places the solution improves (for 55 decimal places can improve to $a = 10^{81}$), but the computational cost is too high for our purposes. More importantly, using the ODE interpolator package in Pytorch (via Autograd [83]) we can construct better convergence checks and this solution is deemed stable. Thus we will need to keep in mind when looking at the behaviour of the cosmological quantities that we are missing a very small fraction of the time-span.

Possible solutions of Φ in terms of conformal time are shown in *Figure* 3.7 for values of

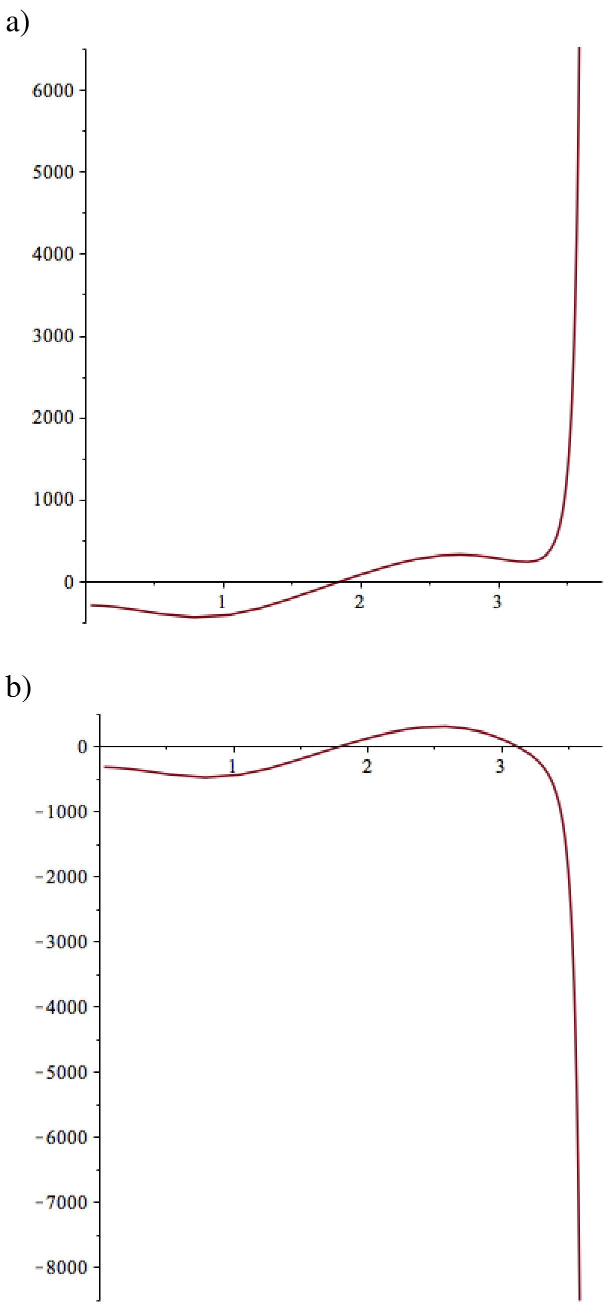

Figure 3.6: Example of the evolution of the Newtonian Potential near a root a)Potential in conformal time for $K = 3.8$b)Potential in conformal time for $K = 3.9$

$K < 20$. As it can be seen all the solutions oscillate and it seems that they cross the Future Conformal Boundary without "noticing". As we increase K an increase in frequency and amplitude can be seen (as expected from *Figure* 3.5). Since Φ is dimensionless we can always normalise to start from Φ unity so we are not worried about Φ's initial value.

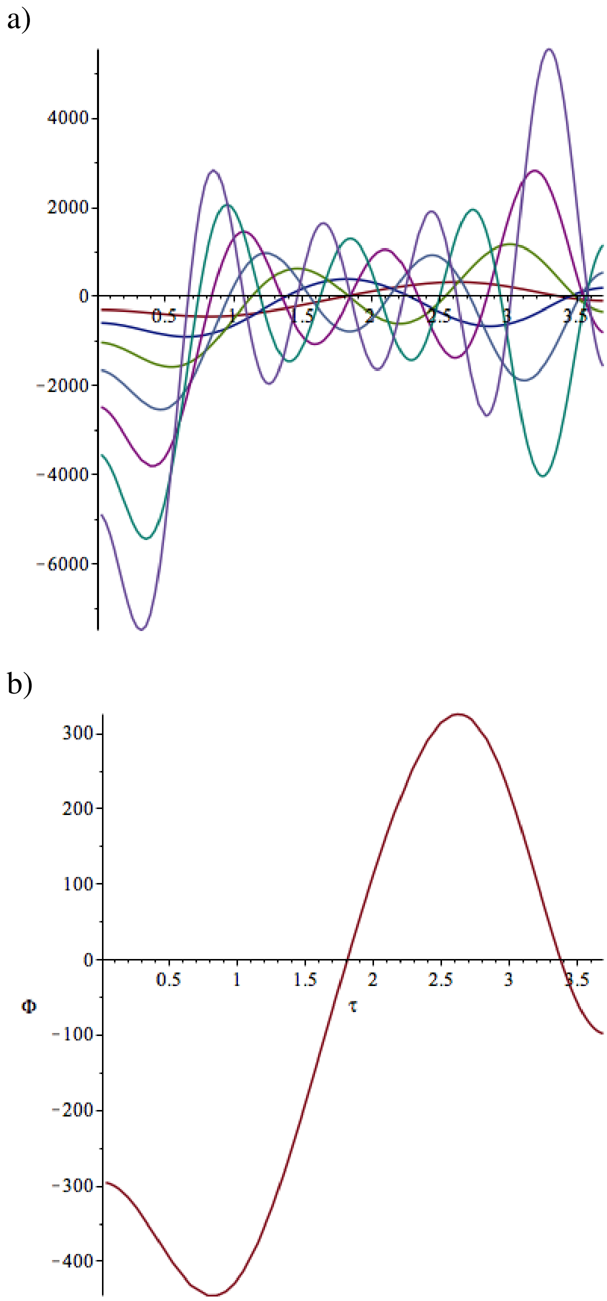

Figure 3.7: Example of the evolution of the Newtonian Potentials in cosmic/conformal time a) Potentials in conformal time for $K \approx \{3.8, 5.2, 8.3, 10.4, 14.1, 16.2, 19.4\}$ b)Potential for $K = 3.83497249600364895335368929205206907564\!3$

We have found an infinitely countable set of discrete values for K that produce non-divergent values for Φ as $a \to \infty$. If this feature persists in a more general setup (one that includes matter), we expect some interesting changes from the predictions of the standard cosmological model. For instance, when looking at the CMB we know that for low k the power spectrum departs from a power-law and several authors have discussed the merits of a cutoff point. In our toy-model the smallest admissible value for k is $k \approx 1.89$ which produces an interesting limit. Although we have not focused on this line of work (we are currently interested in the radiation-only case), we acknowledge the importance of the possible results and leave it for consideration at a later date.

Going back to our model, we proceed to find the other cosmological quantities. Since the other quantities inherit Φ's behaviour at the boundary we expect to find sensible solutions for Ψ, δ and V.

Finding the other cosmological parameters

By setting $GG = 0$, from equation 3.90, U can be expressed in terms of V as:

$$U = \frac{1}{12aH}(3kH\Phi - \dot{H}V - aH^2V) \tag{3.106}$$

With respect to conformal time equations 3.102-3.103 become:

$$\begin{aligned} H^2 &= \frac{\Lambda}{3}\left(1 + \frac{1}{a^4}\right) \\ \dot{H} &= -\frac{2\Lambda}{3a^3} \end{aligned}$$

and can be used to write U independently of H.

Now the constraint equation for FF can be written in terms of V and Φ. By rewriting equation 3.95 in terms of a and substituting FF, V can be expressed in terms of Φ and $\partial_a\Phi$:

$$V(a) = \frac{108K(a^4+1)a^4\left(\frac{1+a^{-4}}{3}\right)^{\frac{1}{2}}}{48a^6K^2 - 11a^8 + 48K^2a^2 - 434a^4 + 45}\left(\Phi(a)\left(\frac{11}{4}\frac{a^3}{a^4+1} - \frac{15}{4}\frac{1}{a(a^4+1)}\right) - \partial_a\Phi\right) \tag{3.107}$$

One can find an equation for Ψ and δ in terms of V and Φ from equations 3.88 and 3.94.

We start by looking at the Newtonian Potentials. Dissimilar from the standard model where in the absence of anisotropic stress the two potentials are indistinguishable, in this theory the difference can be seen in *Figure* 3.8. Ψ dominates on all scales and the fact that it vanishes for large t and behaves as a regular sinusoidal wave in conformal time is encouraging. The fact that Φ does not tend to 0, but freezes out in cosmic time is an interesting feature.

As far as the perturbations are concerned, the behaviour is somehow different from the standard model as it can be seen in *Figure* 3.9. The density perturbation freezes-out in cosmic

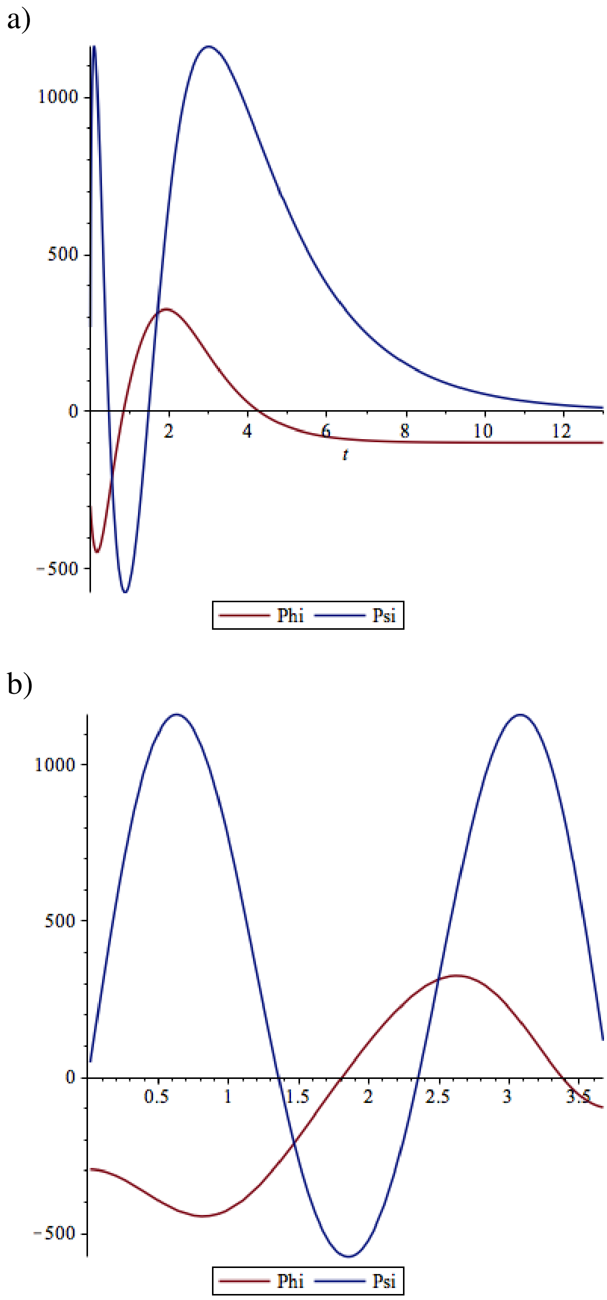

Figure 3.8: Example of the evolution of the Newtonian Potentials in cosmic/conformal time

time, but unlike in the case of the standard model it does not oscillate regularly in conformal time. The velocity perturbation does oscillate regularly in conformal time, but does not freeze out (it just vanishes) in cosmic time.

As it can be seen from the evolution of the cosmological quantities, in the Riemann-squared theory we obtain results that are physically different from the standard model. This outcome

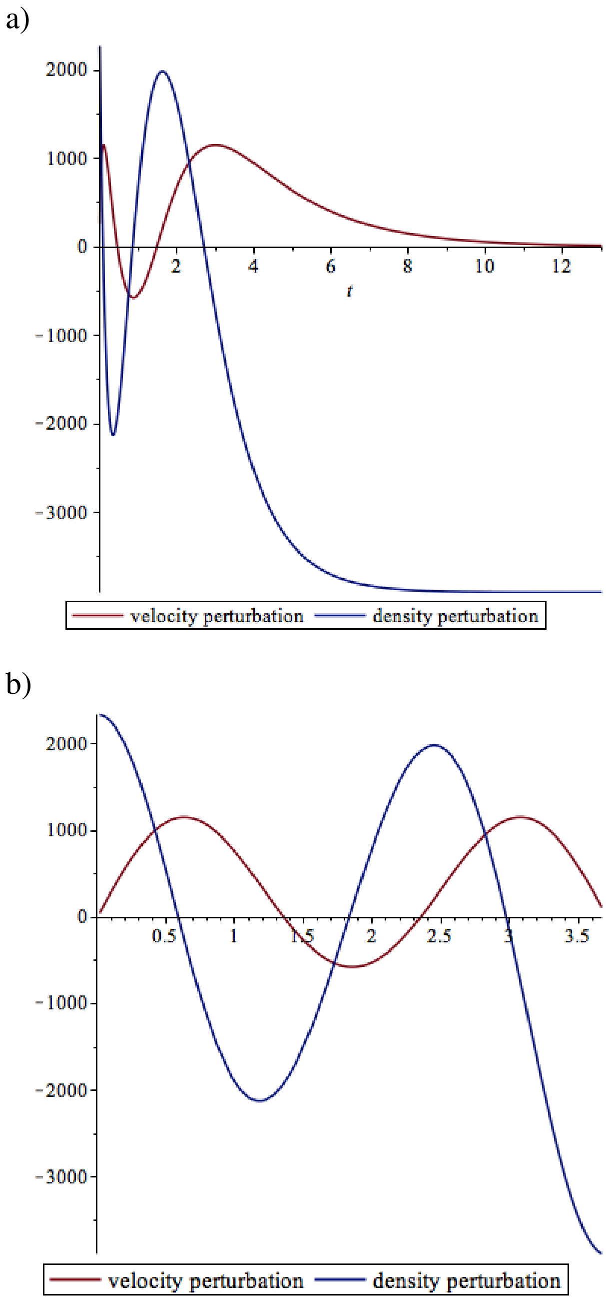

Figure 3.9: Example of the evolution of the perturbations in cosmic/conformal time

is encouraging for future work especially since we have found that even in an overly simplified setup torsion is required and can lead to new physics. In this work we have not used the most general form for torsion and this could pose a problem for the validity of our results. In this case the terms that include GG are significantly smaller, so we are hopeful that a non-zero time-evolving GG would not change the interesting features that we have come across. Furthermore later on in the chapter we will introduce a significantly different torsion profile that supports our current results.

Effective anisotropic stress

From our results an immediate feature is that free-propagating torsion contributes to anisotropic stress. The effective scalar anisotropic stress defined as the difference between the potentials and the anisotropy equation is given by:

$$\Phi - \Psi \equiv \Pi^{(\mathrm{eff})} \tag{3.108}$$

In a source-free setup, under a regime dictated by General Relativity, this quantity vanishes and therefore the inequality of the Newtonian potentials is a "signature" of the departure from Einstein's theory. Several authors have investigated whether this property is inherited by all higher order gravity models and recent work led to the conjecture that suppressing the effective anisotropic stress is impossible to achieve in realistic scenarios. The authors in [84] have shown that in models with a single extra degree of freedom it is not possible to have no effective anisotropic stress except in the GR limit. In more complicated cases they have found that it is possible to cancel the contributions at the price of developing fatal singularities.

Thus by being able to measure accurately the anisotropic stress a clear direction will be found as far as gravitational theories are concerned - a strong anisotropic stress would indicate a modification of GR, whereas the absence of anisotropic stress would require strong fine-tuning.

Until such results are available several theoretical aspects can be considered and an interesting result has been found in [85]. The authors claim that when anisotropic stress is sourced by perfect-fluid matter perturbations at linear level, the propagation of gravitational waves is modified.

3.4 Gravitational Waves

Inspired by their findings, the natural next step in our work is to consider gravitational waves in the case of the simplified Riemann-squared gravity.

Several authors have discussed gravitational waves in the context of Gauge Theories and we believe the following research is relevant to our future study: forms for torsion compatible with PP-waves [86], exact vacuum solutions in three-dimensional gravity with propagating torsion [87] and waves corresponding to AdS spacetime [88].

3.4.1 An unfortunate choice of torsion

When exploring the current literature we have noticed that several authors have come to the conclusion that modified gravity theories carry a certain signature, in particular that the Newtonian Potentials are not equal. We have found that our study on $Riemann^2$ radiation - Λ cosmology respects this conjecture.

When we started looking at gravitational waves we quickly realised that the solutions we were finding were unphysical. We start this study by tracing down the route of the problem in order to better understand the impact of the choice of torsion that we employ. We will build the motivation in standard notation so we can use several classical results and then revert back to our setup, adjusted by the new findings. For simplicity we account only for pp waves. We start from the ansatz that for an arbitrary polarization in Rosen representation the metric takes the following form:

$$ds^2 = -2dudv + S^2(u)\Big(\cosh(B(u))(e^{A(u)}dx^2 + e^{-A(u)}dy^2) + 2\sinh(B(u))dxdy\Big). \quad (3.109)$$

S represents the background factor and for this work we can just set it to 1 to recover Minkowski space-time, otherwise S is zeroth order. B and A are both high order parameters and to first order we find that the above metric reduces to a TT-gauge metric.

The full metric we will be working with takes the form:

$$ds^2 = a^2(\tau)\Big((1+2\epsilon\Psi)d\tau^2 - (1-2\epsilon(\Phi-A))dx^2 - (1-2\epsilon(\Phi+A))dy^2 - (1-2\epsilon\Phi)dz^2 - 2\epsilon Bdxdy\Big)\Big), \quad (3.110)$$

where we have changed from Rosen coordinates $(x, y, u = t - z, v = t + z)$ to conformal coordinates (τ, x, y, z).

Before we begin a few aspects should be noted. By "switching off" A we have only a "cross" polarization, by "switching off" B we have only a "plus" polarization and by "switching off" both we just recover the standard perturbed metric.

Since the gravitational wave does not carry any energy momentum (at first order at least), the energy-momentum tensor ($G_{\mu\nu}$) will be given in its classical form. Considering we can "switch off" the gravitational wave, the EMT should not inherit any new anisotropic stress.

We start by looking at specific Einstein coefficients:

$$G_{12} = -\frac{\epsilon}{4a^2}\Big(-4a^2\partial_x\partial_y\Phi + 4a^2\partial_x\partial_y\Psi + 2a^2\partial_{zz}B - 4\dot{a}a\partial_\tau B + 8\ddot{a}aB - 4\dot{a}^2B - 2a^2\partial_{\tau\tau}B\Big) \quad (3.111)$$

$$G_{13} = \epsilon\partial_x\partial_z(\Phi - \Psi) \quad (3.112)$$

$$G_{23} = \epsilon\partial_y\partial_z(\Phi - \Psi) \quad (3.113)$$

In the absence of anisotropic stress we find that we should keep the potentials equal and obtain a constraint on B. This is consistent with the idea that we can switch off the "cross" polarization.

$$\partial_{zz}B - \partial_{\tau\tau}B - 2\mathcal{H}\partial_\tau B - 2\mathcal{H}^2 B + 4\frac{\ddot{a}}{a}B = 0 \tag{3.114}$$

Under the equal potential setup one can find that the remaining non-zero Einstein coefficients are:

$$G_{00} = 3\mathcal{H}^2 + 2\epsilon(\nabla^2\Phi - 3\mathcal{H}\partial_\tau\Phi) \tag{3.115}$$

$$G_{11} = \mathcal{H}^2 - 2\frac{\ddot{a}}{a} + \epsilon\Big((\partial_{\tau\tau} - \partial_{zz})A + 2\mathcal{H}^2 A + 2\mathcal{H}\partial_\tau A - 4\frac{\ddot{a}}{a}A + 8\frac{\ddot{a}}{a}\Phi + 6\mathcal{H}\partial_\tau\Phi + 2\partial_{\tau\tau}\Phi - 4\mathcal{H}^2\Phi\Big) \tag{3.116}$$

$$G_{22} = \mathcal{H}^2 - 2\frac{\ddot{a}}{a} + \epsilon\Big(-(\partial_{\tau\tau} - \partial_{zz})A - 2\mathcal{H}^2 A - 2\mathcal{H}\partial_\tau A + 4\frac{\ddot{a}}{a}A + 8\frac{\ddot{a}}{a}\Phi + 6\mathcal{H}\partial_\tau\Phi + 2\partial_{\tau\tau}\Phi - 4\mathcal{H}^2\Phi\Big) \tag{3.117}$$

$$G_{33} = \mathcal{H}^2 - 2\frac{\ddot{a}}{a} + \epsilon\Big(8\frac{\ddot{a}}{a}\Phi + 6\mathcal{H}\partial_\tau\Phi + 2\partial_{\tau\tau}\Phi - 4\mathcal{H}^2\Phi\Big) \tag{3.118}$$

$$G_{i0} = 2\epsilon(\partial_{i\tau}\Phi + \mathcal{H}\partial_i\Phi) \tag{3.119}$$

From the Einstein equations we know that the G_{ii}'s store the same information and thus we now have a constraint on A. It is easy to notice that actually both A and B need to satisfy the same equation, namely equation 3.114.

We have found a constraint equation that, if satisfied, lets the radiation evolve without noticing the presence of gravitational waves. Any solution to equation 3.31 will describe a viable representation for the gravitational wave.

We start by looking at Einstein's coefficients. For the form of torsion we currently have we find that Einstein's coefficients are separable as in Section 3. We find that A (and B) must satisfy the constraint:

$$\partial_{\tau\tau}A - \partial_{zz}A + 2\mathcal{H}\partial_\tau A = 0 \tag{3.120}$$

We can notice that the terms depending on A which generate the structure of equation 3.114 vanish. Looking at the ω's we can easily see that the torsion terms cancel these contributions at linear order (makes them second order).

For now assume that equation 3.120 is a viable constraint. When looking at the SET's it is easily noticeable that they are separable and thus a new constraint can be found. For A (and B) one finds:

$$\partial_{\tau\tau}A a\ddot{a} - \partial_{\tau\tau}A\dot{A}^2 + \ddot{a}\dot{a}\partial_\tau A - \dot{a}^2\partial_{zz}A = 0 \tag{3.121}$$

By combining equation 3.114 and 3.120 one can find a solution for A.

By substituting the form of $\partial_{zz}A$ from 3.120 into 3.121, equation 3.121 becomes:

$$(\dot{\mathcal{H}} - \mathcal{H}^2)(a^2\partial_{\tau\tau}A + \dot{a}a\partial_\tau A) = 0, \tag{3.122}$$

which is equivalent to

$$\partial_\tau(a\partial_\tau A) = 0 \tag{3.123}$$

It can easily be seen that the only solution to 3.123 and 3.121 is $A = C_1 z + C_2$, a function independent of τ. Unfortunately this form for A is not complex enough to cater for viable gravitational waves. Thus we need to change our torsion profile to be able to accommodate gravitational waves and we will explore viable profiles in the following section.

3.4.2 A new form for torsion

As we are studying gravitational waves we will work directly with linearised versions for the vierbein as presented in [18]. In this case, for a setup only in terms of t and z we can employ the following notation,

$$\begin{aligned}
\underline{g_t}(t,z) &= \left(\frac{1}{a(t)} + \frac{C(t,z)\epsilon}{a(t)}\right)e_t, \\
\underline{g_x}(t,z) &= \left(-\frac{1}{a} + \epsilon\frac{\cos(2\Theta_0)T(t,z) - S(t,z)}{a(t)}\right)e_x - \frac{T(t,z)\sin(2\Theta_0)\epsilon}{a(t)}e_y, \\
\underline{g_y}(t,z) &= -\frac{T(t,z)\sin(2\Theta_0)\epsilon}{a(t)}e_x - \left(\frac{1}{a(t)} + \epsilon\frac{\cos(2\Theta_0)T(t,z) + S(t,z)}{a(t)}\right)e_y, \\
\underline{g_z}(t,z) &= -\left(\frac{1}{a} + \frac{\epsilon U(t,z)}{a(t)}\right)e_z.
\end{aligned} \tag{3.124}$$

We start by setting up a more general form for torsion, $S = [S_0, S_1, S_2, S_3]$ where

$$\begin{aligned}
S_0 &= \epsilon G_0(\tau,z)\sigma_3 \\
S_1 &= \epsilon\Big(G_1(\tau,z)\sigma_1 + F_1(\tau,z)I\sigma_2\Big) \\
S_2 &= \epsilon\Big(G_2(\tau,z)\sigma_2 + F_2(\tau,z)I\sigma_1\Big) \\
S_3 &= \epsilon G_3(\tau,z)\sigma_3.
\end{aligned} \tag{3.125}$$

By using the same steps that we have previously described we find that despite the setup's cumbersome nature an ansatz solution can be found. We can immediately set $G_0 = G_3 = 0$ as these are 'longitudinal' and form a separate sector that we can just discard. This simplification leads to

$$
\begin{aligned}
\frac{dF_1}{d\tau} &= -\frac{I\Big(3IF_1\beta H'^2 + 3H'HG_1\beta k + a\pi\delta k\Big)}{3H\beta H'}, && (3.126)\\
\frac{dG_1}{d\tau} &= -\frac{1}{3(H'+4aH^2)(2aH^2+H')H'a\beta}\Big(9\beta k^2 SH'^3 - 15H'^3HG_1a^2\beta\\
&\quad -42H'^2G_1H^3a^3\beta + 6H'^2SH^2ak^2\beta + 18IH'^3GF_1ak\beta + 60IH'^2F_1H^2a^2k\beta\\
&\quad -24H'G_1H^5a^4\beta + 48IH'F_1H^4a^3k\beta\\
&\quad +2H'\pi\delta a^2k^2 - 24H'a^2H^4Sk^2\beta + 8H^2a^3k^2\pi\delta\Big), && (3.127)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dS}{d\tau} &= -G_1a, && (3.128)\\
\frac{dU}{d\tau} &= -\frac{aH'G_1 + 3IF_1Hak + 3SHk^2 + a^2H^2G_1}{H'+4aH^2}, && (3.129)\\
\frac{d\delta}{d\tau} &= -\frac{1}{\pi a^2(H'+4aH^2)}\Big(3H'^3G_1a\beta + 15H'^2G_1H^2a^2\beta\\
&\quad +3H'^2HS\beta k^2 + 3I\beta HH'^2aF_1k + 18H'G_1H^4a^3\beta + 4\pi\delta a^3H'H\\
&\quad +6aH'H^3S\beta k^2 + 6Ia^2H^3H'\beta kF_1 + 16\pi\delta a^4H^3\Big), && (3.130)\\
C &= \frac{\pi\delta a^2}{3\beta H'(2aH^2+H')}. && (3.131)
\end{aligned}
$$

By returning to the Ω-equations and we find an additional constraint,

$$
-6Ia^2H^2G_1 + 6Ia^2H^2G_2 + 10aF_2Hk + 10aHF_1k + 3Ik^2G_1 - 3Ik^2G_2 = 0. \qquad (3.132)
$$

By introducing this constraint in the SET-equations we notice that there is a particular solution that simplifies the setup considerably, namely $G_1 = G_2$ and $F_2 = -F_1$.

Before we proceed to investigate the full solution we briefly explore the case without torsion, i.e.

$$
\begin{aligned}
a = -\frac{1}{H_0\tau}, &\qquad H = H_0,\\
F_1 = G_1 = F_2 = G_2 &= G_0 = G_3 = 0. \qquad (3.133)
\end{aligned}
$$

From the SET-equations we find that the system reduces to

$$
\begin{aligned}
C &= -S't, \\
U'' &= \frac{S'k^2t^2 - k^2St + S''t + 2U' - 2S'}{t}, \\
\delta' &= \frac{3H_0^4\beta t(k^2St - U' + S')}{2\pi}.
\end{aligned} \tag{3.134}
$$

Interestingly, as we introduce this form for δ in the Ω-equations we find that it must vanish and thus we do not have a viable setup. This is in fact unsurprising as it is consistent with our belief from Chapter 2: *any non-zero Weyl due to matter needs torsion.*

Returning to our main equations, by working in terms of C and not δ we can simplify our equations significantly. By direct manipulation it can be showed that the setup reduces to,

$$C' = -\frac{3a^2H^2G_1 + IaHF_1k + SHk^2 + aH'G_1}{H' + 4aH^2}, \tag{3.135}$$

$$F_1' = -\frac{2IaH^2Ck - F_1H'a + IHG_1ka + IH'Ck}{aH}, \tag{3.136}$$

$$
\begin{aligned}
G_1' = \; & \frac{1}{a(H' + 4aH^2)}\Big(4a^3H^3G_1 - 8Ia^2H^2F_1k + 4aH^2Sk^2 - 8H^2ak^2C \\
& +5a^2HG_1H' - 6IaH'F_1k - 3SH'k^2 - 2H'Ck^2\Big),
\end{aligned} \tag{3.137}
$$

$$S' = -G_1a, \tag{3.138}$$

$$U' = -\frac{a^2H^2G_1 + 3IF_1Hak + 3SHk^2 + aH'G_1}{H' + 4aH^2}. \tag{3.139}$$

Now this system is fully defined and can even be taken in the de Sitter limit $H = constant$. Thus we have reached a solution that accommodates gravitational waves.

Unsurprisingly when we returned to the initial setup we found that the new form of torsion does not affect our profiles for pressure and density. In this setup we have employed 'cosmological'-type torsion and we recover the same conclusions we drew in Chapter 2. As it is self-propagating, it acts as a support field that does not interact with the matter sector. In this work we chose to present both forms of torsion to account for the way our work progressed and exemplify how we manipulate the torsion field.

3.5 Conclusion

In this Chapter we have looked at a pure Riemann^2 Lagrangian that can only accommodate radiation. As we have shown in Chapter 1, we can represent a Weyl^2 Lagrangian starting from the Riemann in the hope that we can introduce matter. Although the attempt did not succeed we found interesting forms for torsion and we decided to explore a ‘pure’ setup further.

Considering we do not have matter we used only ‘cosmological’-type torsion profiles. We found an interesting density profile that does not oscillate regularly in conformal time and could change our understanding of perturbations. One direct implication is that the power spectrum is discrete in this setup. Proving convergence for numerical ODE solutions is not a simple task even for well behaved functions, and in this case it was an obstacle. We hope that as the steep ODE numerical solvers improve we will have a better handling of our solution and can look for features in the discrete sample.

Finally, we wanted to prove that this theory accepts gravitational waves. In order to do so we had to construct a new torsion profile. An important step was to isolate the effect of torsion and figure out what components caused our solution to diverge. In our future work we will further investigate this solution and add possible extensions.

CHAPTER 4

A new Lagrangian formalism for a perfect fluid with spin

4.1 Introduction

"The ideal Weyssenhoff fluid is a continuous medium, the elements of which are characterised (along with the energy and momentum) by the intrinsic angular momentum (spin) proportional to the volume." (*Obukhov and Korotky [89]*)

In Einstein-Cartan-Kibble-Sciama theory (ECKS), torsion is fully generated by spin sources that are established at the level of the Lagrangian [90]. In the 1970s there was the 'second wave' revival for torsion theories and several authors [41], [91], [92] have thoroughly investigated cosmologies based on a Weyssenhoff model, as first introduced in [93], in the context of ECKS.

After extensive research it seemed that using this theory had mixed levels of success. It offers an attractive alternative to inflation ([43], [94], [40], [36]) as it predicts a big bounce instead of a big bang and a significantly different mechanism for the early universe that seems to solve various cosmological problems (such as the flatness and horizon problems as shown in [43]). Furthermore as it has been shown that a static sphere of a Weyssenhoff fluid sources the Kerr metric (to first order) in [95] it could naturally accommodate black holes. More exotic claims such as "every black hole contains a new Universe" have recently been argued for by Poplawski [96]. Several authors (such as [97] and [98]) have also investigated the effect on rotation of the universe and found that the sense of rotation is flipped. In [99] the authors looked at supernova Ia data to compare the predictions from a dust Weyssenhoff fluid with observations. They found that although the model could provide the correct rate of accelerated expansion, it falls short on providing an alternative to Dark Energy.

In order to further explore the effects on the early Universe it is important to be able to define the Weyssenhoff fluid from first principles, namely to construct a variational (Lagrangian) theory for a spinning fluid in U_4. The first serious attempts did not quite recover a perfect Weyssenhoff fluid as discussed in, for example, [100], [101], [102]. Obukhov and Korotky [89] were the first to construct a viable Lagrangian by refining the work of Smalley et al [103]. In short the authors start from the ECKS action,

$$S = \int d^4xh\left(\frac{1}{2\alpha}\tilde{R} + L_m\right), \tag{4.1}$$

where $\alpha = 8\pi Gc^{-4}$ is the Einstein gravitational coupling constant, $h \equiv {}_{\mu} = \sqrt{-g}$, $\tilde{R} = h^{\mu}_{a}h^{\nu}_{b}\tilde{R}^{ab}_{\mu\nu}$ is the Riemann-Cartan curvature scalar and $L_m = L_m(h^a_\mu, \tilde{\Gamma}^a_{b\mu}, \phi)$ the matter Lagrangian for ϕ. By variation with respect to h^a_μ and $\tilde{\Gamma}^a_{b\mu}$ they find the equations of motion

$$\begin{aligned} \tilde{R}^{\mu}_{a} - \frac{1}{2}h^{\mu}_{a}\tilde{R} &= \alpha T^{\mu}_{a}, \\ Q^{\mu}_{ab} + 2h^{\mu}_{[a}Q_{b]} &= \alpha S^{\mu}_{ab}, \text{ where } Q_a \text{ is the torsion trace,} \\ T^{\mu}_{a} &\equiv \frac{1}{h}\frac{\delta(hL_m)}{\delta h^a_\mu}, \text{ the energy-momentum tensor and} \\ S^{\mu b}_{a} &= \frac{1}{h}\frac{\delta(hL_m)}{\delta\tilde{\Gamma}^a_{b\mu}}, \text{ the tensor of spin.} \end{aligned} \tag{4.2}$$

This equations lead to the generalisation of the special relativistic conservation law of total angular momentum,

$$(\tilde{\nabla}_\alpha - 2Q_\alpha)S^{\alpha}_{\mu\nu} = T_{[\mu\nu]}, \tag{4.3}$$

and the generalisation of the conservation law of the energy-momentum tensor,

$$(\tilde{\nabla} - 2Q_\nu)\tilde{G}^{\nu}_{\mu} + 2Q^{\alpha}_{\mu\beta}T^{\beta}_{\alpha} + S^{\nu}_{\alpha\beta}\tilde{R}^{\alpha\beta}_{\mu\nu} = 0. \tag{4.4}$$

Using the last two relations the authors generalise the initial formalism presented in [93]. The spin density is defined as a second-rank skew-symmetric tensor, $S^{\mu\nu} = -S^{\nu\mu}$, such that the spin pseudovector is spacelike in the fluid rest frame. This constraint is known as the Frenkel condition $S^{\mu\nu}u_\nu = 0$ (for u^μ the 4-velocity).

'Dust' in the context of a Weyssenhoff fluid can be described by

$$S^{\mu}_{\alpha\beta} = u^{\mu}S_{\alpha\beta} \qquad T^{\mu}_{\alpha} = u^{\mu}P_{\alpha}, \tag{4.5}$$

where P_α is the 4-vector density of the energy-momentum. For neutral spinning matter, the fluid if fully characterised in term of ϵ, the internal energy density, ρ, the particle density and μ^{ij}, the spin density per particle. Obukhov and Korotky postulate that a viable Lagrangian can take the form

$$L_m = \epsilon(\rho, s, \mu^{ij}) - \frac{1}{2}\rho\mu^{ij} b_i^\mu (\tilde{\nabla}_\alpha b_j^\nu) u^\alpha g_{\mu\nu} + \lambda_1 \nabla_\mu(\rho u^\mu) + \lambda_2 u^\mu \partial_\mu X + \lambda_3 u^\mu \partial_\mu s \\ + \lambda^{ab}(g_{\mu\nu} b_a^\mu b_b^\nu - \eta_{ab}), \qquad (4.6)$$

where the λ_i are the Lagrange multipliers, X the Lagrangian coordinate, b_i the tetrad, $g_{\mu\nu}$ the metric coefficients, $\eta_{\mu\nu}$ the Minkowski metric coefficients.

The main difference to the previous attempts is that in this formalism (compared to [103]) the spin density vector is not linked to the third axis of the tetrad. This means that the h^a_μ, $\tilde{\Gamma}^a_{b\mu}$, λ's, s, X, ρ, u^μ, b_i's and μ^{ij}'s are independent dynamical variables. Thus they can be constrained only after the Euler-Lagrange equations are obtained. By employing this formalism the description of the Weyssenhoff fluid was recovered.

The authors show that the field equations from this Lagrangian can be portrayed in a more convenient manner. The symmetric part can be expressed as the effective GR field equations with extra spin terms and the antisymmetric part as GR spin field equations. Thus the system becomes

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} = \kappa T^s_{\mu\nu}, \\ \nabla_\lambda(u^\lambda S_{\mu\nu}) = 2u^\rho u_{[\mu}\nabla_{|\lambda}(u^\lambda S_{\rho|\nu]}), \qquad (4.7)$$

where the *effective* stress-energy-momentum tensor of the fluid is defined as

$$T^s_{\mu\nu} \equiv (\rho_s + p_s)u_\mu u_\nu - p_s g_{\mu\nu} - 2(g^{\rho\lambda} + u^\rho u^\lambda)\nabla_\rho[u_{(\mu}S_{\nu)\lambda}], \qquad (4.8)$$

for *effective* energy density and pressure given by

$$\rho_s \equiv \rho - \kappa S^2 + \kappa^{-1}\Lambda, \qquad p_s \equiv p - \kappa S^2 - \kappa^{-1}\Lambda. \qquad (4.9)$$

Brechet et al [36] point out that the spin-density-squared terms contained in the effective stress-energy-momentum tensor explain the behaviour at the early and late times of the universe. As the spin contributions dominate at early times, but are insignificant later on, it should not be surprising to see a significant impact on inflation and no effect on dark matter.

An important result for the understanding of the Weyssenhoff fluid, that was not proved before [89], is regarding the conservation of spin. It can be shown that spin is conserved if and only if the fluid elements have no acceleration (i.e. $a^\mu = u^\nu \tilde{\nabla}_\nu u^\mu = 0$). This is particularly relevant as it provides a filtering for cosmological models with a conserved Weyssenhoff fluid.

The formalism can be directly generalised for a charged spinning fluid. The proposed Lagrangian takes the form,

$$L_m = \tilde{L_m} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e\rho u^\mu A_\mu - \frac{1}{2}\chi\rho b_i^\mu b_j^\nu \mu^{ij} F_{\mu\nu}, \tag{4.10}$$

where $\tilde{L_m}$ is the Lagrangian for a neutral fluid, A_μ is the electromagnetic field with $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, e is the charge of a fluid element of electrical current characterised by spin density $S^{\mu\nu} = -\frac{1}{2}\rho b_i^\mu b_j^\nu \mu^{ij}$ and χ is a constant (not necessarily $e/2m$).

The Euler-Lagrange equations remain unchanged for the λ's, s and X and it turns out that the rest of the equations are naturally extended. The conjugate of μ^{ij}, defined as ω_{ij} becomes the sum of angular velocity and of the magnetic field which leads to a new generalised internal energy density $\bar{\epsilon} = \epsilon - \frac{1}{2}\rho\chi\mu^{ij}F_{ij}$. Another consequence is that the electric current conservation $\nabla_\mu(e\rho u^\mu) = 0$ is directly satisfied due to the particle conservation law.

In this Chapter we will develop a new Lagrangian based on a Weyssenhoff fluid that is extended to include Dirac matter. One of the most attractive features of Geometric Algebra is the way spin is represented (as emphasized in [18]). We will present our formalism in this language as the setup will be considerably less complicated. We will start by presenting the basic model introduced in [79] for a non-spinning relativistic fluid in order to exemplify how Geometric Algebra works on a simpler representation. We will continue by directly extending the model to a spinning fluid. We will discuss our model's constraints and compare with the literature (more precisely with [36] and [89]) at each opportunity. We will also try extend the model to curved spacetime and link to [104]. Finally we present a possible problem that becomes apparent when comparing to [105] .

4.2 Non-spinning relativistic fluids

We start our study by describing non-spinning relativistic fluids as presented in [79]. The authors start from the ansatz that the action is given by:

$$S = \int dx^4(-\epsilon + J \cdot (\nabla\lambda) - \mu J \cdot \nabla\eta), \tag{4.11}$$

where λ and μ are Lagrange multipliers, J is a spacetime current, ϵ the total energy density and η the entropy. Although by no means a unique representation, as we will generalise it directly we will keep close to their formalism.

The study starts by expressing the main characteristics of the fluid, J, ϵ and η. By definition, J can be written is terms of ρ and v, for $v^2 = 1$, as $J = \rho v$. The main assumption of this work is concerning ϵ, namely that the energy density is a function of density and entropy alone. A convenient form for this statement is to write,

$$\epsilon = \rho(1 + e(\rho, \eta)). \tag{4.12}$$

This equation is not fundamentally new as it just represents a form of the equation of state from textbook relativistic thermodynamics, that can be found in classical literature such as [106].

4.2.1 Equations of motion

Armed with a fluid characterisation we can employ the standard formalism to find equations of state. In order to familiarise the reader with the approach in Geometric Algebra we will present the equations step by step.

Constraints from Lagrangian multipliers

We start by looking at $\nabla \cdot (J\lambda)$. Since this is a total divergence and

$$\nabla \cdot (J\lambda) = \lambda \nabla \cdot J + J \cdot \nabla\lambda, \tag{4.13}$$

it follows that the λ multiplier enforces $\nabla \cdot J = 0$. Thus the current is conserved and the total number of particles in the system is constant. The μ multiplier enforces $J \cdot \nabla\eta = 0$, which tells us the entropy is constant along the field lines of J.

Constraint from η variation

From the Euler-Lagrange equation, $\frac{\partial \mathcal{L}}{\partial \eta} = \frac{\partial}{\partial x^\nu} \frac{\partial \mathcal{L}}{\partial(\partial_\nu \eta)}$, we can find directly that

$$\frac{\partial \epsilon}{\partial \eta} = \frac{\partial (J^\nu \mu)}{\partial x^\nu} = J \cdot \nabla\mu \quad \text{since } \nabla \cdot J \text{ vanishes.} \tag{4.14}$$

This leads to the third equation of motion,

$$\frac{\partial e}{\partial \eta} = v \cdot \nabla\mu. \tag{4.15}$$

Constraint from J variation

From the Euler-Lagrange equation, $\frac{\partial \mathcal{L}}{\partial J} = \frac{\partial}{\partial x^\nu} \frac{\partial \mathcal{L}}{\partial(\partial_\nu J)}$, we find directly that

$$\partial_J \epsilon = v\frac{d\epsilon}{d\rho} = v(1 + e + \rho\frac{\partial e}{\partial \rho}) \text{ since } J = \rho v, \tag{4.16}$$

Thus the final equation of motion is given by,

$$v(1 + e + \rho\frac{\partial e}{\partial \rho}) = \nabla\lambda - \mu\nabla\eta. \tag{4.17}$$

It is convenient to define the pressure as

$$p = \rho^2\frac{\partial e}{\partial \rho}, \tag{4.18}$$

in order to have the above equation pressure dependent, i.e. of the form

$$\frac{1}{\rho}v(\epsilon + P) = \nabla\lambda - \mu\nabla\eta. \tag{4.19}$$

Finally we should remove the Lagrange multipliers. If we take $v\cdot$ of the above we find

$$\epsilon + P = \rho v \cdot \nabla\lambda = J \cdot \nabla\lambda. \tag{4.20}$$

Next we can take $J \cdot \nabla$ of equation 4.19 and look at both sides separately. The right-hand side becomes

$$J \cdot \nabla\Big(\frac{1}{\rho}\Big)v(\epsilon + P) + v \cdot \nabla(v(\epsilon + P)) = v \cdot \nabla(v(\epsilon + P)) + v(\epsilon + P)\nabla \cdot v, \tag{4.21}$$

since

$$\nabla \cdot v = \nabla \cdot \Big(\frac{1}{\rho}J\Big) = J \cdot \nabla\Big(\frac{1}{\rho}\Big). \tag{4.22}$$

To explore the left-hand side we start by presenting some intermediate steps. We note that,

$$\nabla(J \cdot \nabla\lambda) = J \cdot \nabla\nabla\lambda + \dot{\nabla}(\dot{J} \cdot \nabla\lambda), \text{and} \tag{4.23}$$

$$\nabla(\mu J \cdot \nabla\eta) = \nabla\mu J \cdot \nabla\eta + \mu\nabla(\dot{J} \cdot \nabla\eta) + \mu J \cdot \nabla\nabla\eta. \tag{4.24}$$

The above equation simplifies to

$$\mu J \cdot \nabla\nabla\eta = -\mu\dot{\nabla}(\dot{J} \cdot \nabla\eta), \text{since } J \cdot \nabla\eta = 0. \tag{4.25}$$

and since

$$J \cdot \nabla(\mu\nabla\eta) = (J \cdot \nabla\mu)\nabla\eta - \mu\dot{\nabla}(\dot{J} \cdot \nabla\eta), \tag{4.26}$$

we can finally write the left-hand side as

$$\begin{aligned} J\cdot\nabla(\nabla\lambda-\mu\nabla\eta) &= \nabla(J\cdot\nabla\lambda)-\dot{\nabla}(\dot{J}\cdot\nabla\lambda)-(J\cdot\nabla\mu)\nabla\eta+\mu\dot{\nabla}(\dot{J}\cdot\nabla\eta) \\ &= \nabla(\epsilon P)-\rho\frac{\partial e}{\partial\eta}\nabla\eta-\partial J\cdot(\nabla\lambda-\mu\nabla\eta) \\ &= \nabla(\epsilon+P)-\rho\frac{\partial e}{\partial\eta}\nabla\eta-\nabla J\cdot v\frac{\epsilon+P}{\rho}. \end{aligned} \tag{4.27}$$

The remainder of the work will focus on showing that the above expression is ∇P, which is equivalent to proving that

$$\nabla\epsilon=\rho\frac{\partial e}{\partial\eta}\nabla\eta+\nabla J\cdot v\frac{\epsilon+P}{\rho}. \tag{4.28}$$

By definition,

$$\begin{aligned} \nabla\epsilon=\nabla\rho(1+\epsilon)+\rho\frac{\partial e}{\partial\rho}\nabla\rho+\rho\frac{\partial e}{\partial\eta}\nabla\eta &= (1+e+\frac{\partial e}{\partial\rho})\nabla\rho+\rho\frac{\partial e}{\partial\eta}\nabla\eta \\ &= (1+e+\frac{P}{\rho})\nabla\rho+\rho\frac{\partial e}{\partial\eta}\nabla\eta, \end{aligned} \tag{4.29}$$

so we still have to show that

$$\frac{\epsilon+P}{\rho}\nabla\rho=\dot{\nabla}\dot{J}\cdot v\frac{\epsilon+P}{\rho}\Leftrightarrow\nabla\rho=\dot{\nabla}\dot{J}\cdot J. \tag{4.30}$$

Since

$$\dot{\nabla}((\dot{\rho v})\cdot v)=\nabla\rho+\rho\dot{\nabla}\dot{v}\cdot v=\nabla\rho,\ \text{as}\ \nabla(v\cdot v)=2\dot{\nabla}(\dot{v}\cdot v)=0, \tag{4.31}$$

we recover the desired form

$$v\cdot\nabla(v(\epsilon+P))+v(\epsilon+P)\nabla\cdot v=\nabla P, \tag{4.32}$$

which is the equation for a perfect fluid. We can understand the equation more clearly if we introduce the relativistic stress-energy tensor.

4.2.2 Stress Energy Tensor

The covariant generalisation of flat space action is given by,

$$S=\int dx^4\det(\mathrm{h})^{-1}(-\epsilon+\mathcal{J}\cdot(\mathcal{D}\lambda)-\mu\mathcal{J}\cdot\mathcal{D}\eta) \tag{4.33}$$

This is indeed the same action - it follows directly from using

$$\mathcal{J}=\det(\mathrm{h})\underline{\mathrm{h}}^{-1}(J)=\rho v\ \text{and}\ \mathcal{D}(\text{scalar})=\overline{\mathrm{h}}(\nabla)(\text{scalar}). \tag{4.34}$$

Now in the covariant setup the dynamical terms will be $\bar{\mathrm{h}}$, ϵ, J, λ, $\mu+\eta$ and therefore energy density depends on the h-field via ρ.

The $\bar{\mathrm{h}}$*-equation*

In order to find the $\bar{\mathrm{h}}$-equation we start from the identity,

$$\rho^2 = \mathcal{J}^2 = \det{}^2(\mathrm{h})\underline{\mathrm{h}}^{-1}(J)\cdot\underline{\mathrm{h}}^{-1}(J), \tag{4.35}$$

which lets us write

$$\partial_{\bar{\mathrm{h}}(a)}\rho^2 = 2\det(\mathrm{h})\underline{\mathrm{h}}^{-1}(J)\cdot\underline{\mathrm{h}}^{-1}(J)\det(\mathrm{h})\underline{\mathrm{h}}^{-1}(a) + 2\det{}^2\mathrm{h}\dot{\partial}_{\bar{\mathrm{h}}(a)}\langle\dot{\underline{\mathrm{h}}}^{-1}(J)\underline{\mathrm{h}}^{-1}(J)\rangle. \tag{4.36}$$

In general, for b and c we have that $\underline{\mathrm{h}}^{-1}(b)\cdot\bar{\mathrm{h}}(c) = b\cdot c$, so

$$\dot{\partial}_{\bar{\mathrm{h}}(a)}\langle\dot{\underline{\mathrm{h}}}^{-1}(b)\bar{\mathrm{h}}(c)\rangle = -\partial_{\bar{\mathrm{h}}(a)}\langle\dot{\bar{\mathrm{h}}}(c)\underline{\mathrm{h}}^{-1}(b)\rangle = -a\cdot c\underline{\mathrm{h}}^{-1}(b). \tag{4.37}$$

We can substitute $b = J$ and $\underline{\mathrm{h}}^{-1}(J) = \bar{\mathrm{h}}(c)$ to write

$$\dot{\partial}_{\bar{\mathrm{h}}(a)}\langle\dot{\underline{\mathrm{h}}}^{-1}(J)\underline{\mathrm{h}}^{-1}(J)\rangle = -a\cdot\bar{\mathrm{h}}^{-1}\underline{\mathrm{h}}^{-1}(J)\underline{\mathrm{h}}^{-1}(J), \tag{4.38}$$

$$\partial_{\bar{\mathrm{h}}(a)}\rho^2 = 2\rho^2(\underline{\mathrm{h}}^{-1}(a) - \underline{\mathrm{h}}^{-1}(a)\cdot vv). \tag{4.39}$$

The full equation becomes,

$$\partial_{\bar{\mathrm{h}}(a)}\mathcal{L} = \det(\mathrm{h}^{-1})\underline{\mathrm{h}}^{-1}(a)\epsilon - \det(\mathrm{h}^{-1})\frac{\partial\epsilon}{\partial\rho}\rho(\underline{\mathrm{h}}^{-1}(a) - \underline{\mathrm{h}}^{-1}(a)\cdot vv), \tag{4.40}$$

and thus we recover the SET

$$\Rightarrow \mathcal{T}(a) = a\epsilon - \rho(a - a\cdot vv)\frac{\partial\epsilon}{\partial\rho} = (\epsilon + P)a\cdot vv - Pa. \tag{4.41}$$

This form makes the physical interpretation transparent and fully compliant with the way we define a relativistic SET for an ideal fluid:

- we have a fluid that has the rest frame defined locally by v,
- as $T(v) = \epsilon v$, ϵ can be understood as the local energy density,
- as $T(n) = -Pn$, the local stress is controlled by isotropic pressure,
- the field equations become a single conservation equation $\dot{T}(\dot{\nabla}) = 0$.

4.3 Spinning relativistic fluids

Having recovered a perfect fluid successfully we proceed to construct a Lagrangian description for a spinning fluid. We would like to generalise the previous action by introducing a Dirac field. Ultimately we hope to build a bridge between the point mass Dirac Lagrangian and the fluid Lagrangian formulated in [89]. In order to proceed we will briefly describe the STA equivalent of Dirac theory as introduced in [79] and [18].

The Dirac matrix operators are defined as

$$\hat{\gamma_0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \hat{\gamma_k} = \begin{pmatrix} 0 & -\hat{\sigma_k} \\ \hat{\sigma_k} & 0 \end{pmatrix}, \qquad \hat{\gamma_5} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

in order to represent Dirac spinors (i.e. they cover 4 complex components and a total of 8 real degrees of freedom). In [79] there is significant emphasis on how one can map normalised spinors to the rotor. We will present their representation by example,

$$|\psi\rangle = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{pmatrix} \leftrightarrow \psi = a^0 + a^k I\sigma_k.$$

This can be extended, in the style of Pauli algebra, such that each spinor is mapped as

$$|\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\eta\rangle \end{pmatrix} \leftrightarrow \psi = \phi + \eta\sigma_3.$$

Using this representation Dirac matrix operators become significantly simpler, as now they take the form

$$\hat{\gamma_\mu}|\psi\rangle \leftrightarrow \gamma_\mu\psi\gamma_0, \qquad i|\psi\rangle \leftrightarrow \psi I\sigma_3, \qquad \hat{\gamma_5}|\psi\rangle \leftrightarrow \psi\sigma_3. \tag{4.42}$$

In Geometric Algebra we can replace the standard Hermitian adjoint $\langle\psi|$ with

$$\langle\psi| \leftrightarrow \psi^\dagger = \gamma_0\tilde{\psi}\gamma_0, \text{ where } \tilde{\psi} \text{ is the reverse of } \psi. \tag{4.43}$$

The representation is particularly interesting when looking at observables. For the current $J_\mu \equiv \langle\overline{\psi}|\hat{\gamma}_\mu|\psi\rangle$ we can show that it ends up mapping to $\gamma_\mu \cdot (\psi\gamma_0\tilde{\psi})$ and thus we obtain a simple form for J,

$$J = \psi\gamma_0\tilde{\psi}. \tag{4.44}$$

In order to better understand J we will use a useful result that holds for any multivector. If we look at $\psi\tilde{\psi}$ we can easily notice, from the definition of a reverse, that

$$\psi\tilde{\psi} = \widetilde{(\psi\tilde{\psi})}, \tag{4.45}$$

and thus $\psi\tilde{\psi}$ is even-grade and equal to its reverse. This implies that it can only be constructed from a scalar and a pseudoscalar,

$$\psi\tilde{\psi} = a + Ib = \rho e^{I\beta}, \text{ where } \rho \neq 0. \tag{4.46}$$

So now we can directly define a spacetime rotor R, that generates a rotation ($a' \leftrightarrow Ra\tilde{R}$), as

$$R = \psi\rho^{-1/2}e^{-I\beta/2} \text{ with } R\tilde{R} = 1. \tag{4.47}$$

This leads to a decomposition of the spinor ψ that separates the density and the rotor,

$$\psi = \rho^{1/2}e^{I\beta/2}R. \tag{4.48}$$

Returning to the current we can now write $J = \rho R\gamma_0\tilde{R}$ and thus we have by construction γ_0 rotated in the direction of the current.

Using similar arguments we can show that the spin bivector S is given by $S = \psi I\sigma_3\tilde{\psi}$.

4.3.1 A toy Lagrangian

A direct generalisation to the previous action to include spin could be given by

$$S = \int dx^4(-\epsilon + J\cdot(\nabla\lambda) - \mu J\cdot\nabla\eta + \sigma\langle v\cdot\nabla\psi i\sigma_3\widetilde{\psi}\rangle + \langle k(J - \psi\gamma_0\widetilde{\psi})\rangle), \tag{4.49}$$

where the new variables are ψ a Dirac spinor, σ a scalar constant, and k a vector Lagrange multiplier. We introduce the σ-term such that there is no extra rotation and the spin is constant along the stream lines. The k-term guarantees that the spinor is aligned with the current. In this setup we have made a significant assumption, that σ is a constant. We aim to treat this setup as a toy model for the final version of the Lagrangian.

We will start by assuming that there is no pseudoscalar part in equation 4.46, i.e. $\psi\widetilde{\psi} = \rho$. We are still going to assume that the energy density is a function of density and entropy only, 4.12. As we are including ψ this constraint requires the spin magnitude to be fixed (by σ) as a function of density. We also keep the same definition for pressure, 4.18.

Considering the new variables are localized we only have to redo the constraint that arises from the variation with respect to the current.

Constraint from J variation

From the action we can easily see there are two extra contributions to the constraint from the J variation,

$$\partial_J\langle v\cdot\nabla\psi i\sigma_3\widetilde{\psi}\rangle = \partial_J\frac{1}{\rho}\langle J\cdot\nabla\psi i\sigma_3\widetilde{\psi}\rangle = -\frac{1}{\rho^2}v\langle J\cdot\nabla\psi i\sigma_3\widetilde{\psi}\rangle + \frac{1}{\rho}\dot{\nabla}\langle\dot{\psi}i\sigma_3\widetilde{\psi}\rangle \tag{4.50}$$

By using our new assumptions, the equation reads:

$$-\frac{\sigma}{\rho}v\langle v\cdot\nabla\psi i\sigma_3\widetilde{\psi}\rangle + \frac{\sigma}{\rho}\dot{\nabla}\langle\dot{\psi}i\sigma_3\widetilde{\psi}\rangle + k - \frac{v}{\rho}(\epsilon+P) + \nabla\lambda - \mu\nabla\eta = 0 \tag{4.51}$$

Constraint from ψ variation

For the new constraint we will start by looking at

$$\partial_\psi\mathcal{L} = (\sigma\widetilde{v\cdot\nabla\psi i\sigma_3}) - 2\gamma_0\widetilde{\psi}k. \tag{4.52}$$

The first term seems to give us the right contribution for equation 4.51 so our aim is to find a usable form. We start by constructing further derivatives,

$$\begin{aligned}\partial_{\psi,a}\mathcal{L} &= \partial_{\psi,a}\sigma\langle v\cdot\nabla\psi i\sigma_3\widetilde{\psi}\rangle = \sigma a\cdot vi\sigma_3\widetilde{\psi}\\ \partial_a\cdot\nabla(\partial_{\psi,a}\mathcal{L}) &= \sigma\nabla\cdot vi\sigma_3\widetilde{\psi} + \sigma i\sigma_3\widetilde{(v\cdot\nabla\psi)}.\end{aligned} \tag{4.53}$$

If we reverse $\psi\to\widetilde{\psi}$ in the above equations we find

$$\sigma v\cdot\nabla\psi i\sigma_3 - 2k\psi\gamma_0 = -\psi i\sigma_3\nabla\cdot v\sigma - \sigma v\cdot\nabla\psi i\sigma_3, \tag{4.54}$$

$$\Rightarrow \sigma(2v\cdot\nabla\psi + \nabla\cdot v\psi) = -2k\psi i\gamma_3, \tag{4.55}$$

$$\Rightarrow \sigma v\cdot\nabla\psi i\sigma_3 = k\psi\gamma_0 - \frac{1}{2}\sigma(\nabla\cdot v)\psi i\sigma_3. \tag{4.56}$$

Now if we right-multiply by $\widetilde{\psi}$ and take the scalar part the equation reduces to

$$\sigma\langle v\cdot\nabla\psi i\sigma_3\widetilde{\psi}\rangle = k\cdot J, \text{ and thus} \tag{4.57}$$

$$\begin{aligned}k &= \frac{\sigma}{\rho}\partial_J\langle J\cdot\nabla\psi i\sigma_3\widetilde{\psi}\rangle,\\ k &= \frac{\sigma}{\rho}\dot{\nabla}\langle\dot{\psi}i\sigma_3\widetilde{\psi}\rangle\end{aligned} \tag{4.58}$$

We can substitute the relevant terms to rewrite equation 4.51 as,

$$-\frac{v}{\rho}k\cdot J + 2k - \frac{v}{\rho}(\epsilon+P) + \nabla\lambda - \mu\nabla\eta = 0 \tag{4.59}$$

and we can further contract with v to find the final form,

$$k \cdot v - \frac{\epsilon + P}{\rho} + v \cdot \nabla \lambda = 0. \tag{4.60}$$

If we would have right-multiplied equation 4.56 instead by $-i\sigma_3\widetilde{\psi}$, we would have found

$$\sigma v \cdot \nabla \psi \widetilde{\psi} = k \psi i \gamma_3 \widetilde{\psi} - \frac{1}{2}\sigma \nabla \cdot v \psi \widetilde{\psi} \tag{4.61}$$

The scalar part is given by $\frac{1}{2}\sigma\langle 2v \cdot \nabla \psi \widetilde{\psi} + \nabla \cdot v \psi \widetilde{\psi}\rangle$. Our initial assumption $\psi\widetilde{\psi} = \rho$ leads to $\nabla \cdot (\rho v) = 0$ which is one of the Euler-Lagrange equations, as desired.

4.3.2 Finding the SET

Similarly to the perfect fluid case, in order to find the SET we need to move to a covariant form. The main change that we need to make is to replace k by $k \to \overline{\mathrm{h}}(k)$. The covariant action then takes the form:

$$S = \int dx^4 \det(\mathrm{h})^{-1}(-\epsilon + \mathcal{J} \cdot \overline{\mathrm{h}}(\nabla)\lambda - \mu \mathcal{J} \cdot \overline{\mathrm{h}}(\nabla)\eta + \sigma\langle v \cdot D\psi i \sigma_3 \widetilde{\psi}\rangle + \overline{\mathrm{h}}(k) \cdot (\mathcal{J} - \psi\gamma_0\widetilde{\psi})). \tag{4.62}$$

The dynamical variables are now J, k, λ, η, μ and ψ, so we need to re-evaluate the constraint equations. It turns out that in this form the k constraint is pretty involved. Instead we can try to find a more manageable form for the Lagrangian. By using equation 4.60 and the derivative definition we can write

$$\begin{aligned} v \cdot D\psi &= v \cdot \overline{\mathrm{h}}(\nabla)\psi + \frac{1}{2}\omega(v)\psi = \frac{1}{\rho}(\mathcal{J} \cdot \overline{\mathrm{h}}(\nabla)\psi + \frac{1}{2}\omega(\mathcal{J})\psi) \\ &= \frac{\det(\mathrm{h})}{\rho}(J \cdot \nabla\psi + \frac{1}{2}\Omega(J)\psi). \end{aligned} \tag{4.63}$$

Now the action can be expressed only in terms of the dynamical variables as

$$S = \int dx^4(-\epsilon \det(\mathrm{h})^{-1} + J \cdot \nabla\lambda - \mu J \cdot \nabla\eta + \frac{\sigma}{\rho}\langle (J \cdot \nabla\psi + \frac{1}{2}\Omega(J)\psi) i\sigma_3\widetilde{\psi}\rangle + k \cdot (J - \det(\mathrm{h})^{-1}\underline{\mathrm{h}}(\psi\gamma_0\widetilde{\psi}))). \tag{4.64}$$

By direct manipulation it can be shown that the new ψ-constraint takes the form

$$\frac{\sigma}{\rho}(J \cdot \nabla\psi + \frac{1}{2}\Omega(J)\psi)i\sigma_3 = \det(\mathrm{h})^{-1}\overline{\mathrm{h}}(k)\psi\gamma_0 - \frac{1}{2}\sigma\nabla \cdot \left(\frac{J}{\rho}\right)\psi i\sigma_3. \tag{4.65}$$

As usual we right-multiply by $\widetilde{\psi}$ and look at the scalar part. By defining $S \equiv \frac{1}{2}\psi i\sigma_3\widetilde{\psi}$, as advertised in the previous section, we find a simple equation

$$\frac{\sigma}{\rho}\langle J \cdot \nabla\psi i\sigma_3\widetilde{\psi}\rangle + \frac{\sigma}{\rho}\Omega(J) \cdot S = k \cdot J, \tag{4.66}$$

and a new form for k,

$$k = \frac{\sigma}{\rho}\partial_J\langle J\cdot\nabla\psi i\sigma_3\widetilde{\psi} + \frac{1}{2}\Omega(J)\psi i\sigma_3\widetilde{\psi}\rangle. \tag{4.67}$$

In order to find the new contributions to the SET we need to look at the $\overline{h}$-equation. We have a modification from the perfect fluid case due to ψ's presence such that we now have to look at

$$\partial_{\overline{h}(a)}\mathcal{L} = -\partial_{\overline{h}(a)}\frac{1}{\rho}(k\cdot J)\rho - \partial_{\overline{\mathrm{h}}(a)}(\det(\mathrm{h})^{-1}k\cdot\underline{\mathrm{h}}(\psi\gamma_0\widetilde{\psi})). \tag{4.68}$$

For the first term the identity remains unchanged from the perfect fluid case,

$$\partial_{\overline{\mathrm{h}}(a)}\frac{1}{\rho} = -\frac{1}{\rho^2}\partial_{\overline{\mathrm{h}}(a)}\rho = \frac{1}{\rho}(\underline{\mathrm{h}}^{-1}(a) - \underline{\mathrm{h}}^{-1}(a)\cdot vv), \tag{4.69}$$

and the second term can be written as

$$\begin{aligned}\partial_{\overline{\mathrm{h}}(a)}(\det(\mathrm{h})^{-1}k\cdot\underline{\mathrm{h}}(\psi\gamma_0\widetilde{\psi})) &= -\det(\mathrm{h})^{-1}\underline{\mathrm{h}}^{-1}(a)k\cdot\underline{\mathrm{h}}(\psi\gamma_0\widetilde{\psi}) + \det(\mathrm{h})^{-1}a\cdot k\psi\gamma_0\widetilde{\psi},\\ &= -\det(\mathrm{h})^{-1}\underline{\mathrm{h}}^{-1}(a)\overline{\mathrm{h}}(k)\cdot\mathcal{J} + \det(\mathrm{h})^{-1}\underline{\mathrm{h}}^{-1}(a)\cdot\overline{\mathrm{h}}(k)\mathcal{J}.\end{aligned} \tag{4.70}$$

We can identify directly the extra component of $\mathcal{T}(a)$ from the above equations to be $\rho v(a\cdot v\overline{\mathrm{h}}(k)\cdot v - a\cdot\overline{\mathrm{k}})$, and we can define

$$\mathcal{T}_{extra}(a) \equiv \rho v(\overline{\mathrm{h}}(k)\wedge v)\cdot(a\wedge v). \tag{4.71}$$

Next we will focus on the symmetries of the new components of $\mathcal{T}$, but firstly we would like to note that k has a σ/ρ prefactor, so in fact $\mathcal{T}_{extra}\propto\sigma$, not ρ.

The extra contribution to the SET is trivially trace-free and we can identify symmetric and antisymmetric components. Since

$$\partial_a\mathcal{T}_{extra}(a) = \rho v\cdot(\overline{\mathrm{h}}(k)\wedge v)v = \rho(v\cdot\overline{\mathrm{h}}(k)v - \overline{\mathrm{h}}(k))v = -\rho\overline{\mathrm{h}}(k)\wedge v, \tag{4.72}$$

we can clearly identify the antisymmetric component. Thus we can write

$$\begin{aligned}\mathcal{T}_{extra}(a) &= \mathcal{T}^{symm}_{extra}(a) + \mathcal{T}^{antisymm}_{extra}(a), \text{ where}\\ \mathcal{T}^{symm}_{extra}(a) &= \rho v(a\cdot v\overline{\mathrm{h}}(k)\cdot v - \frac{1}{2}a\cdot\overline{\mathrm{h}}(k)) - \frac{1}{2}\rho a\cdot v\overline{\mathrm{h}}(k),\\ \mathcal{T}^{antisymm}_{extra}(a) &= -\frac{1}{2}\rho a\cdot(\overline{\mathrm{h}}(k)\wedge v).\end{aligned} \tag{4.73}$$

4.3.3 Making the spin contribution variable independently of the density

As previously discussed, we should consider a Lagrangian where the spin contribution varies independently of the density. We will start by exploring the changes arising at the level of the constraints.

The new variation constraints

The ψ-equation generalises directly to

$$\frac{\sigma}{\rho}(J\cdot\nabla\psi+\frac{1}{2}\Omega(J)\psi)i\sigma_3=\det(\mathrm{h})^{-1}\bar{\mathrm{h}}(k)\psi\gamma_0-\frac{1}{2}\sigma\nabla\cdot\left(\frac{J}{\rho}\right)\psi i\sigma_3-\frac{1}{2}\frac{1}{\rho}(J\cdot\nabla\sigma)\psi i\sigma_3. \quad (4.74)$$

If we pursue the usual routine to right-multiplying by $\tilde{\psi}$ and take the scalar part we will find the same result for k as before. Instead we could right-multiply by $-i\sigma_3\tilde{\psi}$ to find a scalar part of:

$$\sigma(J\cdot\nabla\psi)\tilde{\psi}+\frac{\rho}{2}\sigma\nabla\cdot\left(\frac{J}{\rho}\right)+\frac{1}{2}(J\cdot\nabla\sigma)\psi\tilde{\psi}=0 \quad (4.75)$$

Now if $\psi\tilde{\psi}=\rho$ still holds, the above equation becomes

$$\sigma J\cdot\nabla\rho+\rho\sigma\nabla\cdot J-J\cdot\nabla\rho+\rho J\cdot\nabla\sigma=0 \quad (4.76)$$

As we still want $\nabla\cdot J=0$ (which leads to $\mathcal{D}\cdot J=0$) we require $J\cdot\nabla\sigma=0$. This forces σ to be constant along the flow lines. Another way to view this is to notice that equation 4.76 establishes that

$$\sigma\nabla\cdot J+J\cdot\nabla\sigma=0\Leftrightarrow\nabla\cdot(J\sigma)=0, \quad (4.77)$$

and thus by demanding $\nabla\cdot J=0$ we require $v\cdot\nabla\sigma=0$

By varying the Lagrangian we introduced in 4.64 with respect to σ we find

$$\frac{\partial\epsilon}{\partial\sigma}\det(\mathrm{h})^{-1}=\frac{1}{\rho}\langle(J\cdot\nabla\psi+\frac{1}{2}\Omega(J)\psi)i\sigma_3\tilde{\psi}\rangle. \quad (4.78)$$

Finally, we can look at the Ω constraint,

$$\frac{\partial\mathcal{L}_m}{\partial\Omega(a)}=\frac{1}{2}\frac{\sigma}{\rho}a\cdot J\psi i\sigma_3\tilde{\psi}\det(\mathrm{h}) \quad (4.79)$$

From the definition of the covariant spin tensor, we know that it is sourced directly from the Ω constraint and thus

$$\mathcal{S}(a) = \frac{1}{2}\frac{\sigma}{\rho}\overline{\mathrm{h}}^{-1}(a) \cdot J\psi i\sigma_3\widetilde{\psi}\det(\mathrm{h}). \tag{4.80}$$

We have reached a point where we have a reasonable set of constraints from the Lagrangian setup. Before we can start applying the formalism to various astrophysical setups we need to explore the torsion profile further.

Finding an appropriate ω(a) for $\mathcal{S}(a)$

In applications with spin Doran et al. find viable forms for ω in [107],

$$\omega_{\mathcal{S}}(a) = -\mathcal{S}(a) + \frac{1}{2}a \cdot (\partial_b \wedge \mathcal{S}(b)). \tag{4.81}$$

For the full Dirac case, where we have $\mathcal{S}(a) = \frac{1}{2}a \cdot \psi I\gamma_3\widetilde{\psi} = a \cdot S$ we find

$$\partial_b \wedge (b \cdot S) = 3S \Rightarrow \omega_{\mathcal{S}}(a) = \frac{1}{2}a \cdot S. \tag{4.82}$$

As we have shown in 4.161, in our setup we can write $\mathcal{S}(a) = \sigma a \cdot vS_B$ for $S_B \equiv \frac{1}{2}\psi i\sigma_3\widetilde{\psi}$ the spin bivector. Thus

$$\partial_b \wedge \mathcal{S}(b) = \partial_b \wedge (b \cdot v\sigma S_B) = \langle \partial_b b \cdot v\sigma\rho\rangle_3 = \sigma v S_B = \sigma v \wedge S_B \text{ since we are assuming } \psi\widetilde{\psi} \in \mathbb{R}, \tag{4.83}$$

and finally

$$\omega_{\mathcal{S}}(a) = -\sigma a \cdot vS_B + \frac{1}{2}\sigma a \cdot (v \wedge S_B) = -\frac{1}{2}\sigma(a \cdot vS_B + v \wedge (a \cdot S_B)), \tag{4.84}$$

gives a method to introduce spin at the level of the ω's in applications.

4.3.4 The spin conservation law

We would like to recover the spin conservation law found by Brechet et al in [104]. We start from the characterisation produced in [107] for Symmetries and Conservation Laws. If we introduce $\mathcal{G}(a) = \kappa\mathcal{T}(a)$, the analogue of Einstein's equation, we can show that

$$\partial_a \wedge \mathcal{R}(a) = \partial_a \wedge \mathcal{G}(a). \tag{4.85}$$

The authors in [107] prove that the above relation leads to

$$\partial_a \wedge \mathcal{G}(a) = -\kappa\dot{\mathcal{S}}(\dot{\mathcal{D}}) + \kappa^2(\partial_b \cdot \mathcal{S}(\partial_a)) \wedge (a \cdot \mathcal{S}(b)). \tag{4.86}$$

When $\partial_a \cdot \mathcal{S}(a) = 0$, the above equation becomes

$$\partial_a \wedge \mathcal{G}(a) = -\kappa \dot{\mathcal{S}}(\dot{\mathcal{D}}) - \kappa^2 I \partial_b \wedge \overline{\mathcal{S}}(I\mathcal{S}(b)). \tag{4.87}$$

For our current form of $\mathcal{S}(a)$,

$$\mathcal{S}(a) = \sigma a \cdot v S_B \text{ for } S_B \equiv \frac{1}{2}\psi i \sigma_3 \widetilde{\psi} \Rightarrow \quad \overline{\mathcal{S}}(B) = \sigma v B \cdot S_B \tag{4.88}$$

$$\Rightarrow \mathcal{S}(I\mathcal{S}(b)) = \sigma v \langle I \sigma b \cdot v S_B^2 \rangle. \tag{4.89}$$

Now for $\psi\widetilde{\psi} = 1$, S_B^2 is a scalar and thus the above term vanishes. The conservation equation takes the form,

$$\partial_a \wedge \mathcal{T}(a) + \dot{\mathcal{S}}(\dot{\mathcal{D}}) = 0. \tag{4.90}$$

We show in the Appendix that,

$$\dot{\mathcal{S}}(\dot{\mathcal{D}}) = v \cdot \mathcal{D}(\sigma S_B) + \sigma S_B \mathcal{D} \cdot v, \tag{4.91}$$

which simplifies the conservation equation to

$$\partial_a \wedge \mathcal{T}(a) + v \cdot \mathcal{D}(\sigma S_B) + \sigma S_B \mathcal{D} \cdot v = 0. \tag{4.92}$$

Although the terms might seem simpler, further analysis becomes fairly complicated. Although we could use equation 4.84 and the full Ω-equation the derivation becomes significantly more complicated and thus we will set σ to be a constant. We show in the Appendix that by starting from the identity

$$\overline{\mathrm{h}}(\partial_a)\langle D_a \psi i \sigma_3 \widetilde{\psi}\rangle \wedge v = v \cdot \mathcal{D} S_B + S_B \mathcal{D} \cdot v \tag{4.93}$$

we can find an expression for $v \cdot \mathcal{D}(S_B)$,

$$v \cdot \mathcal{D} S_B = \frac{1}{2} v \cdot \mathcal{D}(\psi i \sigma_3 \widetilde{\psi}). \tag{4.94}$$

For simplicity we will introduce a new linear function $\mathcal{H}$ for which,

$$\begin{aligned} \mathcal{H}(a) &\equiv a - a \cdot v v = a - \frac{1}{2}(av + va)v = \frac{1}{2}(a - vav), \\ \mathcal{H}(a \wedge b) &= (a - a \cdot vv) \wedge (b - b \cdot vv) = (a \wedge b \wedge v) \cdot v, \end{aligned} \tag{4.95}$$

and thus $\mathcal{H}(A) = (A \wedge v) \cdot v$.

For $a = v \cdot \mathcal{D} v$ we find that

$$\omega \equiv \mathcal{D} \wedge v + a \wedge v = \partial_b \wedge (b \cdot \mathcal{D} v) + a \wedge v. \tag{4.96}$$

$$\Rightarrow \omega \cdot v = v \cdot av - \partial_b v \cdot (b \cdot \mathcal{D} v). \tag{4.97}$$

Finally we need to link ω to $\omega_{ab} = D_{[a}u_{b]}$. We start by looking at

$$(v \wedge (\mathcal{D} \wedge v)) \cdot v = v \wedge \dot{\mathcal{D}}\dot{v} \cdot v - v \wedge \dot{v}v \cdot \dot{\mathcal{D}} + \mathcal{D} \wedge v, \tag{4.98}$$

$$\mathcal{H}(\mathcal{D} \wedge v) = ((\mathcal{D} \wedge v) \wedge v) \cdot v = \mathcal{D} \wedge v + a \wedge v, \tag{4.99}$$

$$\mathcal{H}(\mathcal{D} \cdot v) = \mathcal{D} \cdot vv \cdot v = \mathcal{D} \cdot v, \tag{4.100}$$

and define $S_V \equiv IvS_B$. We can find that

$$\mathcal{D} \cdot S_V = -I\mathcal{D} \wedge (v \wedge S_B) = -I(w - a \wedge v) \wedge S_B - I(\mathcal{D} \wedge S_B) \wedge v, \tag{4.101}$$

and

$$\mathcal{D} \wedge S_B = \frac{1}{2}\bar{\mathrm{h}}(e^\mu) \wedge \mathcal{D}_\mu(\psi i\sigma_3\widetilde{\psi}) = \bar{\mathrm{h}}(e^\mu) \wedge \langle (D_\mu\psi) i\sigma_3\widetilde{\psi}\rangle_2. \tag{4.102}$$

Thus the projected streamline derivative of the spin bivector is minus the volume expansion times the spin bivector, i.e.

$$h^\epsilon_\lambda h^\mu_\rho \dot{S}^{\lambda\rho} = -\Theta S^{\epsilon\mu}. \tag{4.103}$$

This expression might seem familiar as it is in fact identical to the one found in [104] for a Weyssenhoff fluid in flat spacetime.

Finally, the new covariant derivative takes the form

$$\mathcal{D}v = \mathcal{D} \cdot v + \mathcal{D} \wedge v = \Theta + w - a \wedge v. \tag{4.104}$$

4.3.5 Revisiting the SET and the Riemann tensor

We will like to rewrite our setup in terms of $\mathcal{H}$. We show in the Appendix that we can find a useful expression from the covariant form of the J-variation constraint,

$$\rho\mathcal{H}(\bar{\mathrm{h}}(\nabla)\lambda - \mu\bar{\mathrm{h}}(\nabla)\eta) = -2\mathcal{H}(\rho\bar{\mathrm{h}}(k)). \tag{4.105}$$

We will start by first looking at the SET. We have previously shown that in this formalism the SET takes the form

$$\mathcal{T}(a) = (\epsilon + P)a \cdot vv - Pa + \mathcal{T}_{extra}(a), \tag{4.106}$$

$$\text{for which} \quad \partial_a \cdot \mathcal{T}(a) = \epsilon + P - 4P = \epsilon - 3P. \tag{4.107}$$

We can now use a compact expression for the new contribution,

$$\mathcal{T}_{extra}(a) = -va \cdot \mathcal{H}(\rho\bar{\mathrm{h}}(k)), \tag{4.108}$$

By introducing a new quantity, $\mathcal{V} = \rho\bar{\mathrm{h}}(k)$, the SET in full can be written in its final form as

$$\mathcal{T}(a) = a \cdot ((\epsilon + P)v - \mathcal{H}(\mathcal{V}))v - Pa. \tag{4.109}$$

We would like to know the form of the matter Riemann tensor, namely $\mathcal{R}_{matter}(a \wedge b)$. We start by looking at $\mathcal{R}(a) \wedge b + a \wedge \mathcal{R}(b)$, which involves

$$(a \cdot A)(v \wedge b) + (a \wedge v)(b \cdot A) = -A \cdot (a \wedge b \wedge v) + A \cdot va \wedge b, \text{ for} \tag{4.110}$$

$$A = (\epsilon + P)v - \mathcal{H}(\mathcal{V}), \qquad B = a \wedge b. \tag{4.111}$$

Therefore

$$\begin{aligned}\mathcal{R}_{matter}(B) &= \frac{1}{2}\kappa(-A \cdot (B \wedge v) + A \cdot vB) + \frac{1}{2}\kappa B(P - \epsilon) + \frac{1}{6}\kappa B(\epsilon - 3P) \\ &= \frac{1}{2}\kappa(-A \cdot (B \wedge v) + A \cdot vB) - \frac{1}{3}\kappa\epsilon B.\end{aligned} \tag{4.112}$$

Now suppose $\mathcal{H}(\mathcal{V})$ vanishes, so we are left with

$$\mathcal{R}_{matter}(B) = \frac{1}{2}\kappa(\epsilon + P)(B - v \cdot (B \wedge v)) - \frac{1}{3}\kappa\epsilon B. \tag{4.113}$$

We can rewrite

$$v \cdot (B \wedge v) = (v \cdot B) \wedge v + B, \tag{4.114}$$

and thus

$$\frac{1}{2}\kappa(\epsilon + P)(-(v \cdot B) \wedge v) - \frac{1}{3}\kappa\epsilon B = \frac{1}{2}\kappa\left((\epsilon + P)B \cdot vv - \frac{2}{3}\epsilon B\right). \tag{4.115}$$

In the general case we have

$$\mathcal{R}_{matter}(B) = \frac{1}{2}\kappa\left(A \cdot vB - A \cdot (B \wedge v) - \frac{2}{3}\epsilon B\right), \tag{4.116}$$

which we can further simplify by noting that

$$A \cdot vB - A \cdot ab \wedge v + A \cdot ba \wedge v - A \cdot vB = -(A \cdot (a \wedge b)) \wedge v = (B \cdot a) \wedge v, \tag{4.117}$$

$$\Rightarrow \mathcal{R}_{matter}(B) = \frac{1}{2}\kappa\left((B \cdot A) \wedge v - \frac{2}{3}\epsilon B\right). \tag{4.118}$$

A direct application from the Riemann tensor is to find a form for $[\mathcal{D}_\alpha, \mathcal{D}_\beta]v$ which is important for the understanding of particle motion.

We can prove, as shown in the Appendix, that for a multivector M,

$$[a \cdot \mathcal{D}, b \cdot \mathcal{D}]M = \mathcal{R}(a \wedge b) \times M, \tag{4.119}$$

and thus

$$[g_\mu \cdot \mathcal{D}, g_\nu \cdot \mathcal{D}]M = R(e_\mu \wedge e_\nu) \times M = R\underline{\mathrm{h}}\underline{\mathrm{h}}^{-1}(e_\mu \wedge e_\nu) \times M = \mathcal{R}(g_\mu \wedge g_\nu) \times M. \tag{4.120}$$

Now using equation 4.118,

$$\mathcal{R}_{matter}(B) \cdot v = \frac{1}{6}\kappa\left((\epsilon + 3P)B \cdot v - 3\mathcal{H}(B \cdot \mathcal{H}(\mathcal{V})\right), \tag{4.121}$$

so if we evaluate for $B = g_\alpha \wedge g_\beta$ we can find $[\mathcal{D}_\alpha, \mathcal{D}_\beta]v$.

4.3.6 Constructing σ

What we have done in the past sections was to generalise

$$\nabla_a \phi = -\dot{\phi} u_a + D_a \phi \quad \text{to} \quad \mathcal{D}\phi = (u \cdot \mathcal{D}\phi)u + \mathcal{H}(\mathcal{D}\phi). \tag{4.122}$$

By direct substitution it can be verified that the identity below holds,

$$\partial_a = \frac{1}{2}\left(\mathcal{H}(e^\mu)e_\mu \cdot \mathcal{D}u + e_\mu \cdot \mathcal{D}u\mathcal{H}(e^\mu)\right) - \frac{1}{3}\partial_a \mathcal{H}(a)\mathcal{H}(e^\mu) \cdot (e_\mu \cdot \mathcal{D}u). \tag{4.123}$$

From the definition of σ we can identify that a suitable form is given by

$$\sigma(a) = \frac{1}{2}\left(a \cdot \mathcal{H}(e^\mu)e_\mu \cdot \mathcal{D}u + \mathcal{H}(e^\mu)a \cdot (e_\mu \cdot \mathcal{D}u)\right) - \frac{1}{3}\mathcal{H}(e^\mu) \cdot (e_\mu \cdot \mathcal{D}u)\mathcal{H}(a), \tag{4.124}$$

and thus we have a closed form for this representation.

4.3.7 An interesting result

In our setup

$$\begin{aligned} a \cdot \mathcal{H}(e^\mu)e_\mu \cdot \mathcal{D}u &= a \cdot \mathcal{H}(\mathcal{D})u \text{ and by symmetry} \\ \mathcal{H}(e^\mu) \cdot (e_\mu \cdot \mathcal{D}u) &= \mathcal{H}(\mathcal{D}) \cdot u \text{ so} \\ \mathcal{H}(\mathcal{D}) \cdot u &= (\mathcal{D} - u \cdot u\mathcal{D}) \cdot u = \mathcal{D} \cdot u. \end{aligned}$$

Now if we consider

$$a \cdot \mathcal{D}u = a \cdot \mathcal{H}(\mathcal{D})u + a \cdot \mathcal{H}(\mathcal{D})u, \tag{4.125}$$

$$a \cdot \mathcal{H}(\mathcal{D})u = a \cdot (\mathcal{D} + ud \cdot \mathcal{D})u = a \cdot \mathcal{D}u + a \cdot uu \cdot \mathcal{D}u, \tag{4.126}$$

and define $A \equiv u \cdot \mathcal{D}u$ as the acceleration, we find that

$$\partial_a(a \cdot \mathcal{H}(\mathcal{D})u - \mathcal{H}(e^\mu)a \cdot (e_\mu \cdot \mathcal{D}u)) = \mathcal{H}(\mathcal{D})u - \dot{u}\mathcal{H}(\dot{\mathcal{D}}) = 2\mathcal{H}(\mathcal{D}) \wedge u. \tag{4.127}$$

If we look at the relation of $\mathcal{H}(\mathcal{D}) \wedge u$ with respect to ω,

$$(\mathcal{D} - uu \cdot \mathcal{D}) \wedge u = \mathcal{D} \wedge u - u \wedge A, \text{ which is the same as } \mathcal{H}(\mathcal{D} \wedge u). \tag{4.128}$$

We find that

$$a \cdot \mathcal{D}u = a \cdot \mathcal{H}(\mathcal{D})u + a \cdot uA, \tag{4.129}$$

where the symmetric part of $a \cdot \mathcal{H}(\mathcal{D})u$ gives $\sigma(a)$ and the trace, and the anti-symmetric part gives the vorticity and another term $a \cdot (u \wedge A)$. This allows us to write

$$\mathcal{D}_\mu u = g_\mu \cdot \mathcal{H}(\mathcal{D})u + g_\mu \cdot uA, \tag{4.130}$$

So $\mathcal{D}_\mu g_\nu - \mathcal{D}_\nu g_\mu = 0$ for a coordinate frame, but not when we have torsion. As an example of a non-zero term we can take

$$\begin{aligned}
&[g_\mu \cdot (\mathcal{D}_\nu u) - g_\nu \cdot (\mathcal{D}_\mu u)]A \\
&= g_\mu \cdot (g_\nu \cdot \mathcal{H}(\mathcal{D})u + g_\nu \cdot uA) - g_\nu \cdot (g_\mu \cdot \mathcal{H}(\mathcal{D})u + g_\mu \cdot uA) \\
&= (g_\mu \wedge g_\nu) \cdot (u \cdot A) + g_\mu \cdot (g_\nu \cdot \mathcal{H}(\mathcal{D})u) - g_\nu \cdot (g_\mu \cdot \mathcal{H}(\mathcal{D})u).
\end{aligned} \tag{4.131}$$

4.3.8 The Ω-equation

A final sector we have not explored yet is given by the Ω-equation. We start by looking at

$$\dot{\mathcal{D}} \wedge \dot{\overline{\mathcal{R}}}(B) = -\kappa\partial_c \wedge S[c \cdot \overline{\mathcal{R}}(B)], \tag{4.132}$$

and we want to show that the above is equal to $\overline{S}(a \wedge b) \cdot \overline{\mathcal{R}}(B) \wedge \partial_b\partial_a$ in order to recover the standard construction presented in [107]. In GA notation we use the overbar to denote the adjoint.

We can prove, as shown in the Appendix, that

$$S[\partial_c \cdot \overline{\mathcal{R}}(B)] \wedge c = -S(\partial_c) \wedge (c \cdot \overline{\mathcal{R}}(B)). \tag{4.133}$$

Now consider a setup for which $\overline{\mathcal{R}}(B) = a \wedge b$. We can treat each side of the equation separately,

$$\begin{aligned}
S[\partial_c \cdot (a \wedge b)] \wedge c &= S[b] \wedge a - S[a] \wedge b, \\
S(\partial_c) \wedge (c \cdot (a \wedge b)) &= S(\partial_c) \wedge (c \cdot ab - c \cdot ba) = S(a) \wedge b - S(b) \wedge a.
\end{aligned} \tag{4.134}$$

We would ideally want $S(c) \wedge \partial_c \cdot \overline{\mathcal{R}}(B) = \partial_c S(c) \wedge \overline{\mathcal{R}}(B)$ and thus

$$
\begin{aligned}
S(c) \wedge \partial_c \cdot (a \wedge b) &= S(a) \wedge b - S(b) \wedge a, \text{ and since} \\
a \cdot (B \wedge C) &= (a \cdot B) \wedge C + B \wedge (a \cdot C), \\
\Rightarrow \partial_c (S(c) \wedge a \wedge b) &= \partial_c \cdot S(c) \wedge a \wedge b + S(c) \wedge \partial_c \cdot (a \wedge b). \qquad (4.135)
\end{aligned}
$$

This identity is satisfied if and only if $\partial_c \cdot S(c)$ vanishes, as we have previously assumed. The fact that torsion has no contraction limits the space of solutions considerably and we have clear candidates in mind, as we will show in applications.

4.3.9 Adding an extra term in the Lagrangian

For future applications we should explore the possibility of adding extra terms. In order to do so we will carry a dimensional analysis argument.

If we start by writing the energy, mc^2, proportional to $\sim L^{-1}$ we recover the dimensions of energy density $\sim L^{-4}$. As $\kappa = 8\pi G \sim L^2$ we can thus write that $\kappa\epsilon \sim L^{-2}$.

By definition, $|\psi|^2 \sim \rho \sim L^{-4}$ so we know that $\sigma \sim L$ and thus $\rho\sigma\kappa \sim L^{-1}$. Finally, the spin energy density, $\frac{I\omega^2}{2L^3} = \frac{mr^2\omega^2}{2L^3} \sim L^{-1}L^2L^{-1}L^{-3} \sim L^{-4}$.

We would like to introduce a terms of the type $\kappa^a\sigma^b\rho^c\psi^d$. One viable option would be to add $2\mathrm{deth}^{-1}\kappa\sigma^2\rho\psi$ to the Lagrangian.

We can show directly that the SET gains an extra term $\mathcal{T}_{new}(a) = -\kappa\sigma^2\rho^2 a$ since

$$\partial_{\overline{\mathrm{h}}(a)} \det(\mathrm{h}^{-1}) = -\det(\mathrm{h}^{-1})\underline{\mathrm{h}}^{-1}(a). \qquad (4.136)$$

Furthermore, if we include $\mathcal{L}_{extra} = -\kappa\sigma^2(\psi\tilde{\psi})^2$ it will just look like a cosmological constant. The equations become

$$
\begin{aligned}
\sigma - \text{equation} \quad &: \quad \frac{\partial\epsilon}{\partial\sigma}\det(\mathrm{h}^{-1}) + 2\kappa\sigma\rho^2\det(\mathrm{h}^{-1}) = \frac{1}{\rho}\langle (J \cdot \nabla\psi + \frac{1}{2}\Omega(J)\psi) i\sigma_3\tilde{\psi}\rangle \\
\psi - \text{equation} \quad &: \quad \text{just add an extra } -2\kappa\sigma^2\rho\psi \text{ to the standard equation.} \qquad (4.137)
\end{aligned}
$$

4.4 Applying the Theory

We have exhausted the analysis that we can perform at the foundational level of the theory. Equipped with a better understanding of what choices we can make we proceed to exemplify how we use the theory with a simple setup.

As presented in Chapter 1, we employ a spherically symmetric setup that can be described by

$$\begin{aligned}\bar{\mathrm{h}}(e^t) &= f_1 e^t + f_2 e^r,\\ \bar{\mathrm{h}}(e^r) &= g_1 e^r + g_2 e^t,\\ \bar{\mathrm{h}}(e^\theta) &= e^\theta,\\ \bar{\mathrm{h}}(e^\phi) &= e^\phi,\end{aligned} \tag{4.138}$$

where f_i and g_i are functions of t and r only.

In order to compute the ω's we start by recalling the way we construct the displacement-gauge covariant object $H(a)$,

$$H(a) = -\bar{\mathrm{h}}(\nabla \wedge \bar{\mathrm{h}}^{-1}(a)). \tag{4.139}$$

By introducing the displacement-gauge-covariant connection $\omega(a) = \Omega(\mathrm{h}(a))$ we can write

$$\partial_b \wedge (\omega(b) \cdot a) = -H(a) \tag{4.140}$$

$$\Rightarrow \omega(a) = H(a) - \frac{1}{2} a \cdot (\partial_b \wedge H(b)), \tag{4.141}$$

enables us to compute $\omega(a)$ directly. Using the notation in [18] we can write

$$\begin{aligned}\omega(e_t) &= G e_r e_t,\\ \omega(e_r) &= F e_r e_t,\\ \omega(e_\theta) &= \frac{g_2(t,r)}{r} e_\theta e_t + \left(-\frac{g_1(t,r)}{r} + \frac{1}{r}\right) e_\theta e_r,\\ \omega(e_\phi) &= \frac{g_2(t,r)}{r} e_\phi e_t + \left(-\frac{g_1(t,r)}{r} + \frac{1}{r}\right) e_\phi e_r.\end{aligned} \tag{4.142}$$

In the presence of spin an additional term built from the spin tensor is added to the right-hand side as we will discuss further. Next we want to find an equation that links the ω's to $\mathcal{D}$. We start by defining the operator

$$L_a = a \cdot \bar{\mathrm{h}}(\nabla). \tag{4.143}$$

The torsion equation,

$$\bar{\mathrm{h}}(\dot{\nabla}) \wedge \dot{\bar{\mathrm{h}}}(\dot{c}) = -\partial_d \wedge (\omega(d) \cdot \bar{\mathrm{h}}(c)), \tag{4.144}$$

becomes

$$(\dot{L}_a \dot{\mathrm{h}}(b) - \dot{L}_b \dot{\mathrm{h}}(a)) \cdot c = (a \cdot \omega(b) - b \cdot \omega(a)) \cdot \bar{\mathrm{h}}(c). \tag{4.145}$$

This leads to a relation for the commutator,

$$[L_a, L_b] = (a \cdot \omega(b) - b \cdot \omega(a) + L_a b - L_b a) \cdot \overline{\mathrm{h}}(\nabla) = L_c, \tag{4.146}$$

$$\text{for } c \equiv a \cdot \omega(b) - b \cdot \omega(a) + L_a b - L_b a = a \cdot \mathcal{D} b - b \cdot \mathcal{D} a. \tag{4.147}$$

The bracket structure summarises the intrinsic content of the torsion equation. If spin is present, the right-hand side of the equation is modified in a straightforward way to include spin-dependent terms.

Note that in the spherical setup we can find the directional derivatives

$$\begin{aligned} l_t &= f_1 \partial_t + g_2 \partial_r, \\ l_r &= g_1 \partial_r. \end{aligned} \tag{4.148}$$

From the definition of the observables we can express the partial derivatives as

$$\begin{aligned} \partial_t &= \frac{1}{det(\mathrm{h})}(g_1(t,r)L_t - g_2(t,r)L_r), \\ \partial_r &= \frac{1}{det(\mathrm{h})}(f_1(t,r)L_r - f_2(t,r)L_t), \end{aligned} \tag{4.149}$$

and then rewrite

$$\begin{aligned} L_r(f_1(t,r)) &= F(t,r)f_2(t,r) - G(t,r)f_1(t,r) + L_t(f_2(t,r)), \\ L_t(g_1(t,r)) &= -F(t,r)g_1(t,r) + g_2(t,r)G(t,r) + L_r(g_2(t,r)). \end{aligned} \tag{4.150}$$

Now we proceed to characterize the spinning fluid Dirac field by employing the following definitions,

$$\begin{aligned} \sigma_\phi &= \widehat{\phi}\gamma_0, \\ \sigma_r &= \widehat{r}\gamma_0, \\ R &= \cosh(\chi) + \sigma_2 \sinh(\chi), \\ \psi &= S(t,r) + T(t,r) i\sigma_3, \\ S_B &= \frac{1}{2}\psi i\sigma_3 \widetilde{\psi}, \\ \rho &= \psi\widetilde{\psi}, \\ v &= \frac{1}{\rho}\psi\gamma_0\widetilde{\psi}. \end{aligned} \tag{4.151}$$

We can now add the new components from spin to the ω's and Ω's. As in equation 4.84 we have

$$\begin{aligned} \omega(a) &= \omega_{before}(a) + \kappa\sigma(t,r)(a \cdot vS_B + \frac{1}{2}a \cdot (v \wedge S_B)), \\ \Omega(a) &= \omega_{before}(a) + \kappa\sigma(t,r)(\underline{\mathrm{h}}(a) \cdot vS_B + \frac{1}{2}\underline{\mathrm{h}}(a) \cdot (v \wedge S_B)). \end{aligned} \tag{4.152}$$

We can now construct the new covariant derivative as described in the previous sections.

The Dirac equation setup

We can start exploring the physical setup by employing the equation derived from spin divergence in Appendix A,

$$\partial_a \wedge \mathcal{T}(a) + v \cdot \mathcal{D}(\sigma S_B) + \sigma S_B \mathcal{D} \cdot v = 0. \tag{4.153}$$

As it has a similar form to the work presented in [107] we will follow their approach. Analysing this equation leads to two possible setups.

Case 1: If we ignore all the gravitational components, i.e. set

$$\begin{aligned} g_1(t,r) &= 1, \quad g_2(t,r) = G(t,r) = F(t,r) = 0, \\ L_t(\sigma(t,r)) &= L_r(\sigma(t,r)) = 0, \\ L_r(S(t,r)) &= S(t,r)\frac{L_r(T(t,r))}{T(t,r)}, \end{aligned} \tag{4.154}$$

and put σ = constant and S/T =constant in space, we find a solution. In this case there will be a non-zero interesting part of the SET from the bivector of $J\bar{\mathrm{h}}(k)$ if χ is non-zero and S/T is time dependent. We can see that in order for $\nabla \cdot J = 0$ we need to set ρ = constant. So this is *a stationary homogeneous configuration*.

Case 2: Alternatively for $\chi = 0$ a cosmological setup works as follows:

$$\begin{aligned} g_1(t,r) &= 1, \quad G(t,r) = 0, \quad F(t,r) = H(t), \quad g_2(t,r) = H(t)r, \\ L_t(\sigma(t,r)) &= 0, \\ L_r(S(t,r)) &= S(t,r)\frac{L_r(T(t,r))}{T(t,r)}. \end{aligned} \tag{4.155}$$

In this case there is no new contribution to the standard SET and the conservation of J gives the standard ρ conservation equation.

We proceed by constructing the Riemann tensor when spin terms are present. Then $\mathcal{R}$ can be written in terms of the new ω's as

$$\mathcal{R}(a \wedge b) = L_a\omega(b) - L_b\omega(a) + \omega(a) \times \omega(b) - \omega(c), \text{ for } c = a \cdot \omega(b) - b \cdot \omega(a) + L_a b - L_b a. \tag{4.156}$$

We set the L's to the forms they take for a radially-symmetric perfect fluid as introduced in [18], i.e.

$$\begin{aligned}
L_t &= l_t \equiv f_1\partial + g_2\partial_r,\\
L_r &= l_r \equiv g_1\partial_r,\\
L_\phi &= l_\phi \equiv \partial_\phi,\\
L_\theta &= l_\theta \equiv \partial_\theta.
\end{aligned} \tag{4.157}$$

In similar fashion we proceed to obtain the Ricci, R and Einstein tensor. Now we construct the SET equations,

$$\mathcal{T}(a) = (\epsilon + P)a\cdot vv - Pa + \rho v(\bar{\mathrm{h}}(k)\wedge v)\cdot(a\wedge v). \tag{4.158}$$

Using this form we write Einstein's equations for the second case, i.e. set

$$\begin{aligned}
\chi &= 0,\\
g_1(t,r) &= 1, \quad G(t,r) = 0, \quad F(t,r) = H(t), \quad g_2(t,r) = rH(t),\\
L_t(\sigma(t,r)) &= 0, \quad L_r(S(t,r)) = S(t,r)\frac{L_r(T(t,r))}{T(t,r)},\\
L_t(1) &= L_r(1) = L_t(0) = L_r(0) = 0,\\
L_r(H(t)r) &= H(t), \quad L_t(H(t)r) = rL_t(H(t)) + rH(t)^2.
\end{aligned} \tag{4.159}$$

By solving the equations we find the following constraints for the L's

$$\begin{aligned}
L_r(\sigma(t,r)) &= -2\frac{L_r(T(t,r))\sigma(t,r)}{T(t,r)},\\
L_t(H(t,r)) &= \frac{1}{32}\sigma(t,r)^2T(t,r)^4\kappa^2 + \frac{1}{16}\sigma(t,r)^2T(t,r)^2S(t,r)^2\kappa^2\\
&\quad +\frac{1}{32}\sigma(t,r)^2S(t,r)^4\kappa^2 - \frac{1}{2}\kappa P(t,r) - \frac{3}{2}H(t)^2,\\
L_t(T(t,r)) &= -\frac{3T(t,r)^2H(t) + 3S(t,r)^2H(t) + 2L_t(S(t,r))S(t,r)}{2T(t,r)}.
\end{aligned} \tag{4.160}$$

When we substitute the above in the initial equations we are only left with an equation for the energy density

$$\epsilon(t,r) = \frac{1}{16\kappa}(\sigma(t,r)^2T(t,r)^4\kappa^2 + 2\sigma(t,r)^2T(t,r)^2S(t,r)^2\kappa^2 + \sigma(t,r)^2S(t,r)^4\kappa^2 + 48H(t)^2). \tag{4.161}$$

This is the standard energy density ($= \frac{3H^2}{8\pi G}$) plus a new part ($= \frac{1}{16}\kappa\sigma^2\rho^2$).

Now we can rewrite some of the above equations. Firstly we can write $S(t,r)$ in terms of ρ as

$$S^2(t,r) = -T(t,r)^2 + \rho(t,r). \tag{4.162}$$

By direct substitution we find

$$l_t(\rho) = -3H(t)\rho(t,r), \text{ as expected.} \tag{4.163}$$

Then substitute in equation 4.161 to find

$$\epsilon(t,r) = \frac{1}{16}\left(\kappa\sigma(t,r)^2\rho(t,r)^2 + \frac{3H(t)^2}{\kappa}\right) \tag{4.164}$$

$$\Rightarrow H^2(t,r) = \frac{\kappa}{48}(16\epsilon(t,r) - \kappa\sigma(t,r)^2\rho(t,r)^2). \tag{4.165}$$

which lets us re-write $L_t(H(t,r))$ as

$$L_t(H(t,r)) = -\frac{\kappa}{16}(-\kappa\sigma(t,r)^2\rho(t,r)^2 + 8\epsilon(t,r) + 8P(t,r)). \tag{4.166}$$

By converting equation 4.161 and making the substitution

$$\frac{dH}{dt} = \frac{L_t(H(t))}{f_1(t,r)} \text{ we find that} \tag{4.167}$$

$$L_t(\epsilon(t,r)) = -3H(t)(\epsilon(t,r) + P(t,r)), \text{ which is what we were hoping for.} \tag{4.168}$$

We can also show that $L_r(\epsilon(t,r))$ vanishes by showing directly that $L_r(\sigma(t,r)\rho(t,r))$ vanishes.

Now we look at spin divergence equation and find that after imposing $g_1 = 1$ and either $g_2 = 0$ or $\sinh(\chi) = 0$ we recover either *case 1* or *case 2*.

As the contraction of $S_B(a)$ vanishes in this setup, the Bianchi equations simplify to

$$\partial_a \wedge [a \cdot \mathcal{D}\overline{\mathcal{R}}(B) - \overline{\mathcal{R}}(a \cdot \mathcal{D}B)] = \kappa\partial_a \mathcal{S}(a) \wedge \overline{\mathcal{R}}(B). \tag{4.169}$$

The $L_r S$ relation implies that we can write

$$L_r(\rho(t,r)) = \frac{2\rho(t,r)L_r(T(t,r))}{T(t,r)}, \text{ and thus reconstruct all the } R\text{'s to be independent on } L\text{'s.} \tag{4.170}$$

By comparing the new form with our previous results the only information we find is that $L_r P(t,r) = 0$.

For a first application we look at a *particular case* where we set the pressure to zero.

The system reduces to the pair of basic equations:

$$\begin{aligned} \frac{dH}{dt} - \frac{1}{2}\kappa\epsilon(t) + 3H(t)^2 &= 0, \\ \frac{d\epsilon(t)}{dt} + 3H(t)\epsilon(t) &= 0, \end{aligned} \tag{4.171}$$

which are solved by

$$\begin{aligned} H(t) &= \frac{2C_1 t + 2C_2}{3C_1 t^2 + 6C_2 t + 6}, \\ \epsilon(t) &= \frac{4C_1}{3\kappa(C_1 t^2 + 2C_2 t + 2)}, \text{ for } C_1 \text{ and } C_2 \text{ constants.} \end{aligned} \tag{4.172}$$

Unfortunately by looking at ϵ it is fairly obvious that the scale factor will be proportional to the 1/3 power of its denominator, so we will not get useful amounts of inflation.

Now we manipulate the above expressions to have the origin of time at the symmetry point and we also need to make sure that $(\kappa\rho\sigma)^2 > 0$.

$$\begin{aligned} \frac{d\epsilon(t)}{dt} &= 0 \Rightarrow t = -\frac{C_2}{C_1} \\ \Rightarrow \tau &= t + \frac{C_2}{C_1}, \text{ or } t = \frac{C_1\tau - C_2}{C_1}. \end{aligned} \tag{4.173}$$

Now we can write

$$C_2 = -\frac{C_1^2}{3\kappa}\left(\frac{3\alpha^2}{C_1^2} + 1\right), \text{ for } \alpha \text{ a suitable constant.} \tag{4.174}$$

We can substitute the above in the expression

$$(\sigma(t)\rho(t))^2 = \frac{48}{\kappa^2}\left(\frac{1}{3}\kappa\epsilon(t) - H(t)^2\right) \text{ to find} \tag{4.175}$$

$$(\sigma(t)\rho(t))^2 = -\frac{192C_1^2(C_1^4 + 6C_1^2\alpha^2 + 9\alpha^4 - 18C_1\kappa^2)}{\left(-9C_1^2\kappa^2\tau^2 + C_1^4 + 6C_1^2\alpha^2 + 9\alpha^4 - 18C_1\kappa^2\right)^2}, \tag{4.176}$$

and thus

$$\begin{aligned} H(t) &= -\frac{6tC_1^2\kappa^2}{-9C_1^2\kappa^2 t^2 + C_1^4 + 6C_1^2\alpha^2 + 9\alpha^4 - 18C_1\kappa^2}, \\ \epsilon(t) &= -\frac{12C_1^2\kappa}{-9C_1^2\kappa^2 t^2 + C_1^4 + 6C_1^2\alpha^2 + 9\alpha^4 - 18C_1\kappa^2}. \end{aligned} \tag{4.177}$$

We will continue our investigation by looking at the *radiation only* case as we would like to compare with [104]'s results.

The system reduces to the pair of basic equations,

$$\begin{aligned} \frac{dH}{dt} - \frac{1}{2}\kappa(\epsilon(t) - P(t)) + 3H(t)^2 &= 0, \\ \frac{d\epsilon(t)}{dt} + 3H(t)(\epsilon(t) + P(t)) &= 0. \end{aligned} \tag{4.178}$$

and thus the governing equation

$$\frac{d^2\epsilon(t)}{dt^2} = -\frac{4}{3}\epsilon^2(t)\kappa + \frac{7}{4\epsilon(t)}\left(\frac{d\epsilon(t)}{dt}\right)^2. \tag{4.179}$$

This is solved by ϵ that satisfies,

$$3\left(32\kappa^{5/2}\sqrt{3C_1\epsilon^{7/2} + 48\kappa\epsilon^3}\right)^{-1} \left(\epsilon\sqrt{C_1\epsilon^{1/2} + 16\kappa}\left(\tanh^{-1}\left(\frac{1}{4}\sqrt{\frac{C_1\epsilon^{1/2} + 16\kappa}{\kappa}}\right)\kappa C_1\epsilon^{1/2} - 4\kappa^{3/2}\sqrt{C_1\epsilon^{1/2} + 16\kappa}\right)\right) + t + C_2 = 0 \tag{4.180}$$

We can differentiate to find

$$\frac{d\epsilon}{dt} = -\frac{1}{3}\sqrt{3C_1\epsilon^{7/2} + 48\kappa\epsilon^3}, \tag{4.181}$$

$$\Rightarrow H(t) = \frac{\sqrt{3C_1\epsilon^{7/2} + 48\kappa\epsilon^3}}{12\epsilon}. \tag{4.182}$$

and also $C_1 = -16\frac{\kappa}{\sqrt{\epsilon_0}}$.

If we substitute $C_2 = 0$ we recover

$$t = \sqrt{\frac{3}{4\kappa\epsilon_0\epsilon}}\left(\epsilon_0^{1/4}\sqrt{\sqrt{\epsilon_0} - \sqrt{\epsilon}} + \sqrt{\epsilon}\tanh^{-1}\left(\sqrt{1 - \sqrt{\frac{\epsilon}{\epsilon_0}}}\right)\right). \tag{4.183}$$

Now we rescale by R, i.e. $\epsilon = \frac{\epsilon_0 R_0^4}{R^4}$, and find

$$t = \frac{\sqrt{3}}{2R_0^2\sqrt{\kappa\epsilon_0}}\left(R_0^2\tanh^{-1}\left(\sqrt{1 - \frac{R_0^2}{R^2}}\right) + R\sqrt{R^2 - R_0^2}\right) \tag{4.184}$$

Around the origin as $R = R_0 + \delta$ for small δ,

$$t \to \sqrt{\frac{6}{R_0\epsilon_0\kappa}}\delta^{1/2} + \sqrt{\frac{6}{R_0^3\epsilon_0\kappa}}\delta^{3/2} + O(\delta^{5/2}), \tag{4.185}$$

so indeed it goes through the origin. On the other hand, as $R \to \infty$

$$t \to R^2 - \frac{R_0^2}{2} + R_0^2\left(\ln(x) - \frac{1}{2}\ln\left(\frac{R_0^2}{4}\right)\right) + O(R^{-2}). \tag{4.186}$$

Now we can use this when looking at inflation. The governing equation becomes

$$\frac{d^2R}{dt^2} = -\frac{R_0^4\epsilon_0\kappa\left(R^2 - 2R_0^2\right)}{3R^5}, \text{ so the inflationary period does not last very long.} \tag{4.187}$$

This leads to an inflation of just $\sqrt{2}$ which agrees with the expression in [104].

In the case for dust by using the above, it can be shown that

$$t = \frac{2^{1/3}R_0}{2\alpha^{2/3}}\left(3\kappa^2t^2 + 4\alpha^2\right)^{1/3}. \tag{4.188}$$

If we differentiate this twice, we find that inflation ends at $t = \frac{2\alpha}{\kappa}$ with a total expansion of $2^{2/3}$.

4.4.1 Corrections for Curved Space Time

As presented in [36], for a cosmological fluid based on ECKS, the Cosmological Principle takes two different forms. The Strong Cosmological Principle (SCP) states that the Lie derivatives of the metric and torsion must vanish. The Weak Cosmological Principle (WCP) relaxes this condition by allowing the Lie derivatives of torsion to be unconstrained.

It has been argued that the Weyssenhoff fluid is incompatible with the SCP [108]. The spin density tensorial terms violate this principle and thus by removing them (and leaving only the scalars) we could construct a viable fluid.

The standard assumption that can be made, as used in [109], [40] and later on in [36], [105], [110], is that "locally macroscopic spin averaging leads to a vanishing expectation value for the spin density, i.e. $\langle S_{ab}\rangle = 0$" [36]. The expectation value for the spin density squared scalar term survives as it is a variance term, i.e. $\langle S^2\rangle \neq 0$. In other words we allow for a spin tensor to have a particular direction on macroscopic scales, but the particles have a random spin distribution.

Our previous formalism holds for flat spacetime but if we want to compare with models for curved space time, such as [105], we need to spin-average before we can construct the ω's.

Understanding scaling

The 'classical' approach is to start from the Friedman equations with torsion and find a conservation law, such as the one presented in [105],

$$\frac{d}{dt}((\epsilon - \kappa s^2)a^3) + (p - \kappa s^2/4)\frac{d}{dt}a^3 = 0. \tag{4.189}$$

In order to relate the energy and spin densities some assumptions are made. Firstly, the number density in a fluid,n, is related to the energy density and pressure by,

$$\frac{dn}{n} = \frac{d\epsilon}{\epsilon + p}. \tag{4.190}$$

This assumption is reasonable for a homogeneous and isotropic fluid such as the one described in [105]. For non-homogeneous cases significant work has to be carried in order to find a new expression that accounts for the new statistical structure explained in [111].

Secondly, from statistical arguments it was first claimed in [109] (and then followed by [105] and [110]) that for a fermion fluid with no spin polarization

$$s^2 = \frac{1}{8}(\hbar c n)^2. \tag{4.191}$$

For a fluid described by a barotropic equation of state $p = w\epsilon$ we reach the scaling

$$s^2 \propto \epsilon^{2/(1+w)} \text{ and thus we find a spin contribution to the energy density } \epsilon_S = -\frac{1}{4}\kappa s^2. \tag{4.192}$$

It is shown in [105] and [110] that this term can drive 'inflation' and there is a corresponding fine-tuned parameter Ω_S for which the density parameter

$$\Omega(a) \equiv 1 + \frac{(\Omega - 1)a^4}{\Omega_R a^2 + \Omega_S}, \tag{4.193}$$

agrees with the estimates from observations.

Using our model we have tried to understand why there is an apparent incompatibility between Brechet et al ([36]) and [105], as we have shown that we recover the structure of the equations in [36].

We returned to the case where we have radiation only, and we required that the energy density takes the form advertised in [105]. We traced back in order to find the spin density (from the ω's) in terms of energy density. We reached a complicated set of differential equations that we solved numerically. It turns out equation 4.192 is not satisfied for our setup as we show in Plot 4.1. For this plot we have scaled our results when imposing 4.192.

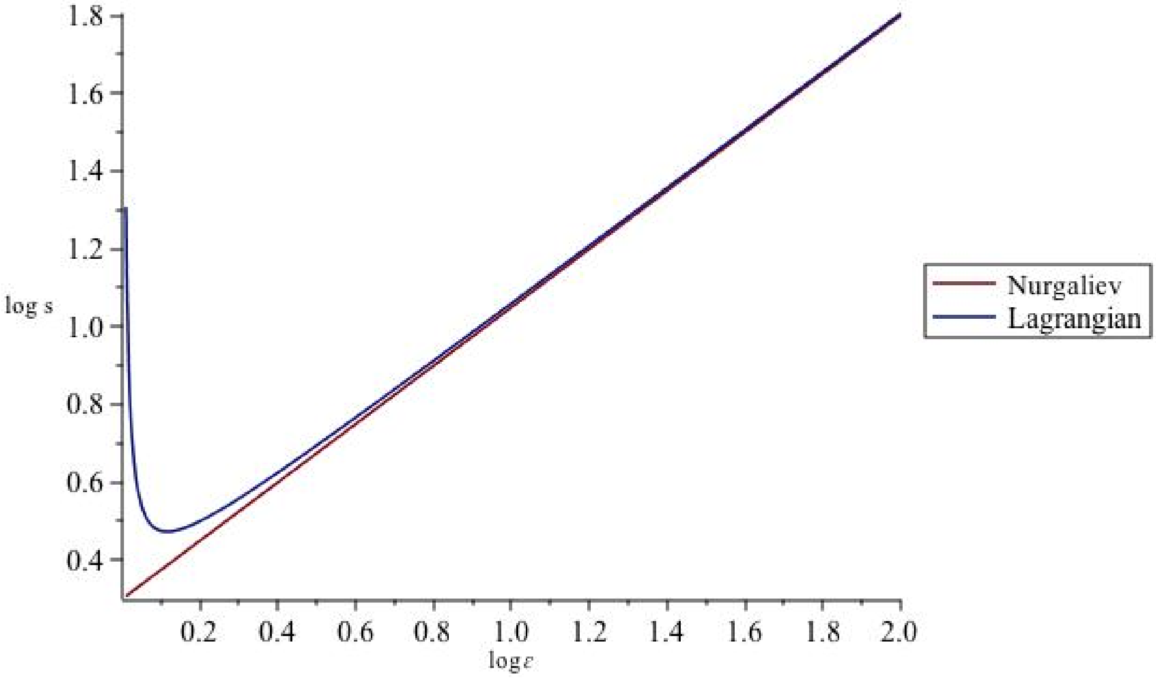

Figure 4.1: Plot comparison in log-log space for the dependency of spin density on energy density

We can observe there is a discrepancy for low ϵ after which the solution quickly ($\log \epsilon > 2.4$) becomes equal within solver tolerance error to the form advertised in [109]. The initial steep decrease is a relic from steep ODE solvers (and thus not a feature). The convergence of the solution can be proved using the interpolator package in Pytorch. In order to better understand this anomaly we will focus in our future research on finding analytic solutions. Although one possible explanation could be that we employ a different model for a spinning fluid, we do recover [36]'s results 4.187. Another option could be that we are not allowed to use equation 4.191 for this setup and we should further investigate the correctness of this method. Furthermore we might have to reconsider the way we constructed the spin averaging in the Lagrangian case.

4.5 Conclusions

In this Chapter we have constructed a new Lagrangian formalism for a spinning fluid. We used the work presented in [79] for a non-spinning fluid and extended it by introducing a Dirac matter field. We started working on this project as we wanted to find an alternative way to introduce matter in more general cosmological setups. We hoped that by employing such a formalism we could take Conformal Gravity in a new direction. We initially investigated using a Lagrangian in the style of [89], but we realised that it would be fairly complicated to extend. Space-Time Algebra proved fruitful in past applications, as exemplified in [18], so we decided to start from scratch.

As STA is not as widely known as we would like, we have tried to present every derivation in detail, or link directly to well known work that was carried in this language, such as [107] and [18]. This chapter is thus a dense, complete representation of the theory that we will employ to introduce matter in our future work. It is fair to say that the simplicity of the method does not transpire at this point, but we hope that once we provide more applications the formalism will become more palatable.

We have shown that we can recover the main result and predictions presented in [36] for a flat space time. As [36] and [43] do not seem to agree on their final predictions regarding the number of e-folds the theory would produce, we have tried to find an explanation. Imposing the energy density advertised in [43] and tracing back to find an equation for the spin density proved to be a complex task and the result not easily interpretable. In our future work we will thoroughly investigate all possible sources of tension to make sure that the current Lagrangian setup works correctly for a curved space-time. Considering spin averaging at this level has not been used (to our knowledge) we might have to rethink how it should be introduced.

CHAPTER 5

CONCLUSIONS AND FUTURE WORK

"*La recherche de la vérité être le but de notre activité; c'est la seule fin qui soit digne d'elle."*(*H.Poincaré, 1885*)

On the path to a 'Unified Theory of Everything' many steps must be taken. Einstein made a significant leap and cemented a new path, Weyl made a few steps forward and a couple back and in their footsteps many others followed. Overall we have made constant progress which has been aided by experimental results. This changed once LIGO detected the first gravitational waves and propelled us significantly closer to our goal. At the time of this significant discovery a new theory was being developed, the extended Weyl Gauge Theory. This thesis is the first study that explores the potential eWGT. Similarly to the infancy of Poincaré Gauge Theory, in our first applications we tried to show that we can recover and extend classical results.

We started with a simplified scenario that simulated Weyl^2 Gravity and we showed that the classical results in Conformal Gravity for vacuum can be recovered. Our first goal was to highlight the importance of understanding gauge transformations in this context as their misuse can lead to thinking gauge artefacts are physical. We continued our research by trying to show that the theory is compatible with astrophysical matter introduced as a perfect fluid. Despite our efforts we could not reach this conclusion, even though we included progressively more general torsion.

Although a disappointing result, we learned a great deal about the way torsion interacts with this setup. When we employed a cosmological-type torsion we found two important cases depending on our initial premise. In the first case we have constructed a form that although present, is untraceable at the level of the matter, whereas in the second case we have found a possible mechanism that could introduce extra pressure (that could be fine-tuned) for inflation. In our future work we hope that we can explore the second setup further and introduce it in a more complex theory that could actually be used in physical scenarios.

On the other hand, when we considered 'real' torsion (that interacts with matter), we found that regardless of the torsion components we always reach the same scenario. Unfortunately, in this case the setup was unphysical. We looked at possible ways to construct this type of torsion as we are hoping to use it in future models that can incorporate matter. We found a new way to couple two torsion components that could simplify more complex setups.

Although a Lagrangian given only in terms of Weyl^2 is not sufficiently complex to accommodate matter, we decided it was worth investigating a pure Riemann^2 Lagrangian. The Weyl action was represented for a certain constraint in terms of the Riemann and Ricci and future work will investigate relaxing this condition.

We found an interesting density profile that does not oscillate regularly in conformal time and could change our understanding of perturbations. One direct implication is that the power spectrum is discrete in this setup. As new libraries for power spectra (e.g. PySpectrum, Pylians) are being developed we could use certain classes to check if our predictions are physical. In future work we should focus on exploring this solution space further.

We also proved that this theory predicts gravitational waves in a de Sitter background. Several authors have discussed gravitational waves in the context of Gauge Theories and we believe the following research is relevant to our future study: forms for torsion compatible with PP-waves [86], exact vacuum solutions to three-dimensional gravity with propagating torsion [87] and waves corresponding to AdS spacetime [88]. In future work we will investigate how our results compare to theirs and work towards developing a wave profile that can be used with LIGO data.

In this study we have not explored the full torsion space. As shown in the Weyl^2 case, it is important that we pursue all the possible scenarios. An analysis as the one carried out in Chapter 2 will be employed for this model in the near future.

In the final chapter we constructed a new Lagrangian formalism for a spinning fluid. Our work is based on the approach introduced in [79] for a non-spinning fluid. We wanted to build a setup that could easily be generalised for extra fields. As current models include matter by introducing a Dirac field, we decided this would be a good first extension.

We have shown that our setup recovers the correct behaviour for a flat space time. We were hoping that we could use our model to explain the apparent disagreement between [36] and [43]. Unfortunately our results were inconclusive and future work needs to investigate the possible sources of tension between the Lagrangian formalism and 'standard' approach in curved space time.

One of the beautiful aspects of science is that the methods we develop in one field can usually be used ubiquitously. Recently great progress has been made in the study of blood flow using Machine Learning techniques. Kihm et al. [112] have managed to classify red blood cell shapes in the flow. Current research is being carried into predicting the interactions between different types of cells in 'real' time. This is particularly important as it could lead to an improved dialysis process. Several attempts have been made to model blood flow using Lagrangians, for example [113], but the predictions are currently not accurate enough. The formalism could be significantly improved by introducing new terms that control the rotation and spin of the particles. Having an easily extensible form will be crucial. Constructing a Lagrangian inspired by the work presented in the final chapter could bring a significant scientific leap and we aim to pursue this project in our future work.

Appendix A

Derivations for the Spinning Fluid Formalism

In this Appendix we present the full derivations for partial results that we have used in Chapter 4.

A.1 Looking at $\dot{\mathcal{S}}(\dot{\mathcal{D}})$

As we do not have a fair understanding of the latter term in the spin conservation equation we will start our analysis from first principles. By direct manipulation,

$$\dot{\mathcal{S}}(\dot{\mathcal{D}}) = a \cdot \dot{\mathcal{D}}\dot{\mathcal{S}}(\partial_a) = a \cdot \mathcal{D}\mathcal{S}(\partial_a) - \mathcal{S}(\partial_a \cdot \mathcal{D}a) = a \cdot \mathcal{D}(\sigma\partial_a \cdot v S_B) - \mathcal{S}(\partial_a \cdot \mathcal{D}a) \quad (A.1)$$

In order to simplify this expression we need three intermediate results,

$$a \cdot \mathcal{D}\partial_a \cdot v = \gamma^\mu \cdot \mathcal{D}(\gamma_\mu \cdot v) = \gamma^\mu \cdot \bar{\mathrm{h}}(\nabla)\gamma_\mu \cdot v = \gamma^\mu \cdot g^\nu \partial_\nu(\gamma_\mu \cdot v) = g^\nu \cdot \partial_\nu v, \quad (A.2)$$

$$\partial_a \mathcal{D}a = \gamma^\mu \cdot \mathcal{D}\gamma_\mu = \gamma^\mu \cdot \bar{\mathrm{h}}(\nabla)\gamma_\mu, \quad (A.3)$$

and

$$w(\gamma^\mu) \cdot \gamma_\mu = w(e_\mu) \cdot e^\mu = w(g_\mu) \cdot g^\mu = \Omega(e_\mu) \cdot g^\mu. \quad (A.4)$$

This leads to

$$\begin{aligned}\dot{\mathcal{S}}(\dot{\mathcal{D}}) &= v\cdot\mathcal{D}(\sigma S_B)+\sigma S_B g^{\nu}\cdot\partial_\nu v-\sigma S_B(\Omega_\mu\cdot g^{\mu})\cdot v\\ &= v\cdot\mathcal{D}(\sigma S_B)+\sigma S_B(g^{\nu}\cdot\partial_\nu v+g^{\nu}\cdot(\Omega_\nu\cdot v))\\ &= v\cdot\mathcal{D}(\sigma S_B)+\sigma S_B\mathcal{D}\cdot v.\end{aligned}\tag{A.5}$$

A.2 Equations for the Spin bivector

Proposition: The following identity holds $v\cdot\mathcal{D}S_B=\frac{1}{2}v\cdot\mathcal{D}(\psi i\sigma_3\widetilde{\psi})$.

We will start our analysis from the identity,

$$\overline{\mathrm{h}}(\partial_a)\langle D_a\psi i\sigma_3\widetilde{\psi}\rangle\wedge v=v\cdot\mathcal{D}S_B+S_B\mathcal{D}\cdot v \tag{A.6}$$

Now we would like to prove that $\overline{\mathrm{h}}(\partial_a)\langle D_a\psi i\sigma_3\widetilde{\psi}\rangle=(v\cdot D\psi)i\gamma_3\widetilde{\psi}$.

$$\overline{\mathrm{h}}(\partial_a)\langle D_a\psi i\sigma_3\widetilde{\psi}\rangle\wedge v=\langle v\cdot D\psi i\sigma_3\widetilde{\psi}\rangle_2=v\cdot\mathcal{D}S_B+S_B\mathcal{D}\cdot v. \tag{A.7}$$

$$\overline{\mathrm{h}}(\partial_a)\langle D_a\psi i\sigma_3\widetilde{\psi}\rangle=\langle v\cdot D\psi i\sigma_3\widetilde{\psi}v\rangle_1=\langle v\cdot D\psi i\sigma_3\widetilde{\psi}\rangle_2\cdot v+\langle v\cdot D\psi i\sigma_3\widetilde{\psi}\rangle v \tag{A.8}$$

Dot with v to find:

$$v\cdot\overline{\mathrm{h}}(\partial_a)\langle D_a\psi i\sigma_3\widetilde{\psi}\rangle=\langle v\cdot D\psi i\sigma_3\widetilde{\psi}\rangle \tag{A.9}$$

Wedge with v to find:

$$\overline{\mathrm{h}}(\partial_a)\langle D_a\psi i\sigma_3\widetilde{\psi}\rangle\wedge v=\Big(\langle v\cdot D\psi i\sigma_3\widetilde{\psi}\rangle_2\cdot v\Big)\wedge v=v\cdot\mathcal{D}S_B+S_B\mathcal{D}\cdot v \tag{A.10}$$

$$\Rightarrow((v\cdot\mathcal{D}S_B)\cdot v)\wedge v=v\cdot\mathcal{D}S_B+S_B\mathcal{D}\cdot v. \tag{A.11}$$

We will try and simplify our maths a bit by writing the above equation as

$$(B\cdot v)\wedge v=B+C. \tag{A.12}$$

Now

$$\frac{1}{4}\left((Bv-vB)v-v(Bv-vB)\right)=\frac{1}{2}(B-vBv)=B+C, \tag{A.13}$$

$$\Rightarrow C=-\frac{1}{2}(B+vBv). \tag{A.14}$$

Thus we can rewrite equation (4.113) as:

$$S_B\mathcal{D}\cdot v = -\frac{1}{2}\left(v\cdot\mathcal{D}S_B + v(v\cdot\mathcal{D}S_B)v\right). \tag{A.15}$$

Now look at

$$v\cdot\mathcal{D}(S_BS_B) = (v\cdot\mathcal{D}S_B)S_B + S_B(v\cdot\mathcal{D}S_B). \tag{A.16}$$

$$S_B^2\mathcal{D}\cdot v = -\frac{1}{2}\left((v\cdot\mathcal{D}S_B)S_B + v(v\cdot\mathcal{D}S_B)S_Bv\right) = -frac12\left(S_B(v\cdot\mathcal{D}S_B) + vS_B(v\cdot\mathcal{D}S_B)v\right), \tag{A.17}$$

$$\Rightarrow 2S_B^2\mathcal{D}\cdot v = -\frac{1}{2}(v\cdot\mathcal{D}S_B^2 + vv\cdot\mathcal{D}S_B^2v) \text{ i.e } v\cdot\mathcal{D}S_B^2 = -2S_B^2\mathcal{D}\cdot v. \tag{A.18}$$

A.2.1 Revisiting the J-constraint

We would like to derive the J-equation in the dynamical variable form in order to make use of H.

By definition for $f(\rho)$, $\partial_J f(\rho) = \partial_J f((\mathcal{J}\cdot\mathcal{J})^{1/2}) = \frac{1}{2}(\mathcal{J}\cdot\mathcal{J})^{-1/2}f'\partial_J(\mathcal{J}\cdot\mathcal{J})$, which leads to

$$\partial_J(\mathcal{J}\cdot\mathcal{J}) = \partial_J(\det{}^2\mathrm{h}\underline{\mathrm{h}}^{-1}(J)\cdot\underline{\mathrm{h}}^{-1}(J)) = 2\det{}^2\mathrm{h}\overline{\mathrm{h}}^{-1}\underline{\mathrm{h}}^{-1}(J), \tag{A.19}$$

and finally

$$\partial_J f(\rho) = \frac{1}{\rho}f'\det{}^2\mathrm{h}\overline{\mathrm{h}}^{-1}\underline{\mathrm{h}}^{-1}(J). \tag{A.20}$$

Varying equation (4.72), we find that

$$\frac{\partial\mathcal{L}}{\partial J} = -\det\mathrm{h}^{-1}\frac{\partial\epsilon}{\partial\rho}\frac{1}{\rho}\det{}^2\mathrm{h}\overline{\mathrm{h}}^{-1}\underline{\mathrm{h}}^{-1}(J) + \nabla\lambda - \mu\nabla\eta + k + k + \sigma\langle(J\cdot\nabla\psi + \frac{1}{2}\Omega(J)\psi)i\sigma_3\widetilde{\psi}\rangle\partial_J\left(\frac{1}{\rho}\right) \tag{A.21}$$

$$= -\det\mathrm{h}^{-1}\frac{\epsilon+P}{\rho}\frac{1}{\rho}\det{}^2\mathrm{h}\overline{\mathrm{h}}^{-1}\underline{\mathrm{h}}^{-1}(J) + \nabla\lambda - \mu\nabla\eta + 2k - \frac{1}{\rho^3}\det{}^2\mathrm{h}\overline{\mathrm{h}}^{-1}\underline{\mathrm{h}}^{-1}(J)\sigma\langle(J\cdot\nabla\psi + \frac{1}{2}\Omega(J)\psi)i\sigma_3\widetilde{\psi}\rangle \tag{A.22}$$

$$= -\det\mathrm{h}\frac{\epsilon+P}{\rho^2}\overline{\mathrm{h}}^{-1}\underline{\mathrm{h}}^{-1}(J) - \frac{1}{\rho^2}\det{}^2\mathrm{h}\overline{\mathrm{h}}^{-1}\underline{\mathrm{h}}^{-1}(J)k\cdot J + \nabla\lambda - \mu\nabla\eta + 2k = 0 \tag{A.23}$$

We can dot this equation with J and since

$$\overline{\mathrm{h}}^{-1}\underline{\mathrm{h}}^{-1}(J)\cdot J = \underline{\mathrm{h}}^{-1}(J)\cdot\underline{\mathrm{h}}^{-1}(J) = \det{}^{-2}\mathrm{h}\rho^2, \tag{A.24}$$

we find that the equation reduces to

$$-\mathrm{det}\underline{\mathrm{h}}^{-1}(\epsilon+P)-k\cdot J+(\nabla\lambda-\mu\nabla\eta)\cdot J+2k\cdot J=0 \text{ i.e. } k\cdot J-\mathrm{det}\underline{\mathrm{h}}^{-1}(\epsilon+P)+J\cdot\nabla\lambda=0. \quad \text{(A.25)}$$

We would like to find the covariant equivalent of equation (4.134) and we can do that by applying $\bar{\mathrm{h}}$,

$$-\frac{\epsilon+P}{\rho^2}\mathcal{J}-\frac{1}{\rho^2}\bar{\mathrm{h}}(k)\cdot\mathcal{J}\mathcal{J}+\mathcal{D}\lambda-\mu\mathcal{D}\eta+2\bar{\mathrm{h}}(k)=0 \quad \text{(A.26)}$$

$$\Leftrightarrow -(\epsilon+P)v-\rho\bar{\mathrm{h}}(k)\cdot vv+\rho(\mathcal{D}\lambda-\mu\mathcal{D}\eta)+2\rho\bar{\mathrm{h}}(k)=0. \quad \text{(A.27)}$$

We can rewrite the equation as

$$\rho(\bar{\mathrm{h}}(\nabla)\lambda-\mu\bar{\mathrm{h}}(\nabla)\eta)=(\epsilon+P)v+\rho\bar{\mathrm{h}}(k)\cdot vv-2\rho\bar{\mathrm{h}}(k), \quad \text{(A.28)}$$

and since $\epsilon+P=\bar{\mathrm{h}}(k)\cdot\mathcal{J}+\mathcal{J}\cdot\bar{\mathrm{h}}(\nabla)\lambda$,

$$\Rightarrow \rho(\bar{\mathrm{h}}(\nabla)\lambda-\mu\bar{\mathrm{h}}(\nabla)\eta)=\mathcal{J}\cdot\bar{\mathrm{h}}(\nabla)\lambda v-2\rho\bar{\mathrm{h}}(k)+2\bar{\mathrm{h}}(k)\cdot\mathcal{J}v, \quad \text{(A.29)}$$

$$\Leftrightarrow \rho(\bar{\mathrm{h}}(\nabla)\lambda\wedge v)\cdot v-\rho\mu\bar{\mathrm{h}}(\nabla)\eta=-2(\rho\bar{\mathrm{h}}(k)\wedge v)\cdot v, \quad \text{(A.30)}$$

$$\Leftrightarrow \rho H(\bar{\mathrm{h}}(\nabla)\lambda-\mu\bar{\mathrm{h}}(\nabla)\eta)=-2H(\rho\bar{\mathrm{h}}(k)). \quad \text{(A.31)}$$

A.3 Identity for Riemann construction

Proposition: $[a\cdot\mathcal{D},b\cdot\mathcal{D}]M=\mathcal{R}(a\wedge b)\times M$.

$$\begin{aligned}&[a\cdot\bar{\mathrm{h}}(\nabla)+w(a)\times][b\cdot\bar{\mathrm{h}}(\nabla)M+w(b)\times M]-[b\cdot\bar{\mathrm{h}}(\nabla)+\omega(b)\times][a\cdot\bar{\mathrm{h}}(\nabla)M+\omega(a)\times M]\\ =\ &(L_aL_b-L_bL_a)M+E_1+E_2, \text{ where} \quad \text{(A.32)}\end{aligned}$$

$$\begin{aligned}E_1 &= w(a)\times(w(b)\times M)-w(b)\times(w(a)\times M),\\ E_2 &= L_a[w(b)\times M]-L_b[w(a)\times M]-w(b)\times L_aM+w(a)\times L_bM. \quad \text{(A.33)}\end{aligned}$$

After direct manipullation,

$$E_1 = [w(a) \times w(b)] \times M, \tag{A.34}$$

and

$$\begin{aligned} E_2 &= L_a w(b) \times M + w(b) \times L(a)M - L_b w(a) \times M - w(a) \times L_b M - w(b) \times L_a M + w(a) \times L_b M \\ &= L_a w(b) \times M - L_b w(a) \times M, \end{aligned} \tag{A.35}$$

so the identity holds.

A.4 Identity for the Ω equation

Proposition: $S[\partial_c \cdot \overline{\mathcal{R}}(B)] \wedge c = -S(\partial_c) \wedge (c \cdot \overline{\mathcal{R}}(B))$.

$$\begin{aligned} c \cdot [\overline{S}(a \wedge b) \cdot \overline{R}(B)] &= \frac{1}{2}\langle c\overline{S}(a \wedge b)\overline{\mathcal{R}}(B) - c\overline{\mathcal{R}}(B)\overline{S}(a \wedge b)\rangle \\ &= \frac{1}{2}\langle c\overline{S}(a \wedge b)\overline{\mathcal{R}}(B) + \overline{S}(a \wedge b)c\overline{\mathcal{R}}(B) - \overline{S}(a \wedge b)c\overline{\mathcal{R}}(B) - c\overline{\mathcal{R}}(B)\overline{S}(a \wedge b)\rangle \\ &= c\overline{S}(a \wedge b)\overline{\mathcal{R}}(B) - c \wedge [\overline{S}(a \wedge b) \cdot \overline{\mathcal{R}}(B)] \\ &= c\overline{S}(a \wedge b)\overline{\mathcal{R}}(B) - c\overline{\mathcal{R}}(B)\overline{S}(a \wedge b) - \overline{S}(a \wedge b)\overline{\mathcal{R}}(B)c + \overline{\mathcal{R}}(B)\overline{S}(a \wedge b)c. \end{aligned} \tag{A.36}$$

Recall the identities

$$\begin{aligned} \overline{S}(a \wedge b) \cdot c &= (a \wedge b) \cdot S(c), \\ \overline{S}(a \wedge b) &= \partial_c[S(c) \cdot (a \wedge b)]. \end{aligned} \tag{A.37}$$

$$\Rightarrow \partial_c[S(c) \cdot (a \wedge b)] \cdot \overline{\mathcal{R}}(B) \wedge \partial_b \wedge \partial_a = \partial_b \wedge \partial_a \wedge (\partial_c[S(c) \cdot (a \wedge b)] \cdot \overline{\mathcal{R}}(B)) = 2S(c) \wedge \partial_c \cdot \overline{\mathcal{R}}(B). \tag{A.38}$$

Finally we can show that

$$\begin{aligned} \partial_c \wedge \partial_b \wedge \partial_a \langle [c \wedge \overline{S}(a \wedge b)] \cdot \overline{\mathcal{R}}(B)\rangle &= \partial_c \wedge \partial_b \wedge \partial_a c \cdot [\overline{S}(a \wedge b) \cdot \overline{\mathcal{R}}(B)] \\ &= \partial_b \wedge \partial_a \wedge [\overline{S}(a \wedge b) \cdot \overline{\mathcal{R}}(B)] = \partial_b \wedge \partial_a \wedge (\partial_c[S(c) \cdot (a \wedge b)] \cdot \overline{\mathcal{R}}(B)) \\ &= 2S(c) \wedge \partial_c \cdot \overline{\mathcal{R}}(B). \end{aligned} \tag{A.39}$$

$$\Rightarrow S[\partial_c \cdot \overline{\mathcal{R}}(B)] \wedge c = -S(\partial_c) \wedge (c \cdot \overline{\mathcal{R}}(B)). \tag{A.40}$$

Bibliography

[1] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis. Modified gravity and cosmology. *Physics Reports*, 513(1):1–189, 2012.

[2] M. Blagojević and F. W. Hehl. *Gauge Theories of Gravitation*. Imperial College Press, Apr 2013.

[3] S. Schlamminger, K. Y. Choi, T. A. Wagner, J. H. Gundlach, and E. G. Adelberger. Test of the Equivalence Principle Using a Rotating Torsion Balance. *Phys. Rev. Lett.*, 100:041101, Jan 2008.

[4] V. W. Hughes, H. G. Robinson, and V. Beltran-Lopez. Upper Limit for the Anisotropy of Inertial Mass from Nuclear Resonance Experiments. *Phys. Rev. Lett.*, 4:342–344, Apr 1960.

[5] I. I. Shapiro. Fourth Test of General Relativity. *Phys. Rev. Lett.*, 13:789–791, Dec 1964.

[6] B. P. Abbott et al. Tests of general relativity with gw150914. *Phys. Rev. Lett.*, 116(22):221101, 2016.

[7] J. M. Cline, S. Jeon, and G. D. Moore. The phantom menaced: Constraints on low-energy effective ghosts. *Phys. Rev. D*, 70:043543, Aug 2004.

[8] R. Utiyama. Invariant Theoretical Interpretation of Interaction. *Phys. Rev.*, 101:1597–1607, Mar 1956.

[9] T. W. B. Kibble. Lorentz invariance and the gravitational field. *Journal of Mathematical Physics*, 2:212–221, 1961.

[10] P. Baekler, F. W. Hehl, and J. M. Nester. Poincaré gauge theory of gravity: Friedman cosmology with even and odd parity modes: Analytic part. *Phys. Rev. D*, 83:024001, Jan 2011.

[11] Yu. N. Obukhov, V. N. Ponomariev, and V. V. Zhytnikov. Quadratic Poincaré gauge theory of gravity: A comparison with the ceneral celativity theory. *General Relativity and Gravitation*, 21:1107–1142, 1989.

[12] E. Anderson, J. Barbour, B. Foster, and N. O'Murchadha. Scale invariant gravity: Geometrodynamics. *Class. Quant. Grav.*, 20:1571, 2003.

[13] A. N. Lasenby and M. P. Hobson. Scale-invariant gauge theories of gravity: Theoretical

foundations. *Journal of Mathematical Physics*, 57:092505, Sep 2016.

[14] P. S. Wesson. Scale Invariant Gravity - a Simple Formulation. *Astronomy and Astrophysics*, 102:45, Sep 1981.

[15] G. Hooft. Local Conformal Symmetry: the Missing Symmetry Component for Space and Time. *International Journal of Modern Physics D*, 24:1543001, 2015.

[16] G. Hooft. A Class of Elementary Particle Models Without any Adjustable Real Parameters. *Found. Phys.*, 41:1829, Sep 2011.

[17] T. Chen, M. Fasiello, E. A. Lim, and A. J. Tolley. Higher derivative theories with constraints: exorcising Ostrogradski's ghost. *Journal of Cosmology and Astroparticle Physics*, 2013(02):042, Feb 2013.

[18] A. Lasenby, C. Doran, and Gull S. Gravity, Gauge Theories and Geometric Algebra. *Phil. Trans. R. Soc. Lond.*, 356:487–582, 1998.

[19] E. Cartan and A. Einstein. *Letters of Absolute Parallelism, 1929-1932*. 1979.

[20] E. Schrodinger. The General Unitary Theory of the Physical Fields. *Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences*, 49:43–58, 1943.

[21] D. J. McCrea. Irreducible decompositions of nonmetricity, torsion, curvature and bianchi identities in metric-affine spacetimes. *Class. Quantum Grav.*, page 553, 1992.

[22] S. Capozziello, G. Lambiase, and C. Stornaiolo. Geometric classification of the torsion tensor of space-time. *Annalen der Physik*, 10:713–727, Aug 2001.

[23] R. T. Hammond. Torsion gravity. *Reports on Progress in Physics*, 65(5):599, 2002.

[24] M. Hohmann, L. Järv, and U. Ualikhanova. Covariant formulation of scalar-torsion gravity. *Phys. Rev. D*, 97:104011, May 2018.

[25] H. H. Soleng. Scalar-tensor theory of gravitation with spin-scalar-torsion coupling. *Classical and Quantum Gravity*, 5(11):1489, 1988.

[26] H. Goenner and F. Muller-Hoissen. Spatially homogeneous and isotropic spaces in theories of gravitation with torsion. *Classical and Quantum Gravity*, 1(6):651, 1984.

[27] R. T. Hammond. Dynamic torsion from a linear lagrangian. *Gen Relat Gravit*, 22:451–456, Apr 1990.

[28] N. J. Poplawski. Torsion as Electromagnetism and Spin. *Int J Theor Phys*, 49:1481–1488, Jul 2010.

[29] W. M. Baker, W. K. Atkins, and W. R. Davis. On the properties of torsions in Riemann-Cartan space-times. *Nuov Cim B*, 44, 1978.

[30] F. W. Hehl, G. D. Kerlick, and J. M. Nester. General Relativity with Spin and Torsion: Foundations and Prospects. *Rev. Mod. Phys*, 48:393–416, 1976.

[31] R. T. Hammond. The necessity of torsion in gravity. *Gen Relat Gravit*, 42:2345–2348, Oct 2010.

[32] A. Challinor, A. Lasenby, C. Doran, and S. Gull. Massive,Non-ghost Solutions for the Dirac Field Coupled Self-consistently to Gravity. *General Relativity and Gravitation*, 29:1527–1544, Dec 1997.

[33] T. Watanabe and M. J. Hayashi. General Relativity with Torsion. *arXiv:gr-qc/0409029*, 2004.

[34] L. Fabbri. Weakly-interacting massive particles in torsionally-gravitating Dirac theory. *International Journal of Modern Physics D*, 22, 2013.

[35] A. Trautman. Spin and Torsion May Avert Gravitational Singularities. *Nature Physical Science*, 242(7-8), Mar 1973.

[36] S. D. Brechet, M. P. Hobson, and A. N. Lasenby. Classical big-bounce cosmology: Dynamical analysis of a homogeneous and irrotational Weyssenhoff fluid. *Classical and Quantum Gravity*, 25(24):245016, 2008.

[37] D. Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara. Progress toward a theory of supergravity. *Phys. Rev. D*, 13:3214, 1976.

[38] S. Capozziello and C. Stornaiolo. Scalar perturbations in a cosmological model with torsion. *Nuovo Cimento B Serie*, 113:879–886, Jul 1998.

[39] R. T. Hammond. Helicity flip cross section from gravity with torsion. *Classical and Quantum Gravity*, 13(7):1691, 1996.

[40] M. Gasperini. Spin-dominated inflation in the Einstein-Cartan theory. *Phys. Rev. Lett.*, 56:2873–2876, Jun 1986.

[41] A. Trautman. Spin and torsion may avert gravitational singularities. *Nature 242 7*, 1973.

[42] G. D. Kerlick. Spin and torsion in general relativity: Foundations, and implications for astrophysics and cosmology. *Phys. Rev. D 12 3003-6*, 1975.

[43] N. J. Poplawski. Cosmology with torsion: An alternative to cosmic inflation. *Physics Letters B*, 694(3):181–185, 2010.

[44] M. I. Wanas and H. A. Hassan. Torsion and Problems of Standard Cosmology. *arXiv:1209.6218*, Nov 2012.

[45] de Andrade. Spin-torsion effects in the Stern-Gerlach experiment. *Nuovo Cimento B 108 947-8*, 1993.

[46] J. Audretsch and C. Lämmerzah. Neutron interference: general theory of the influence of gravity, inertia and space-time torsion. *J. Phys. A: Math. Gen. 16*, 2457, 1983.

[47] Athar H., Peltoniemi J.T., and Smirnov A. Yu. Neutrino spin-flip effects in collapsing stars. *Phys.Rev.D*, 51, 1995.

[48] P. D. Mannheim and D. Kazanas. Exact vacuum solution to conformal Weyl gravity and galactic rotation curves. *Astrophysical Journal, Part 1*, 342:635–638, Jul 1989.

[49] H. Weyl. Gravitation und elektrizitat. *Das Relativitate's prinzip. Fortschritte der Mathem-*

atischen Wissenschaften in Monographien. Vieweg+Teubner Verlag, Wiesbaden, 1923.

[50] H. Weyl. Gravitation and the Electron. *Proc. Nat. Acad. Sci. USA*, 15:323–334, Apr 1929.

[51] R. Bach. Zur weylschen relativitätstheorie und der weylschen erweiterung des krümmungstensorbegriffs. *Mathematische Zeitschrift*, 9(1):110–135, Mar 1921.

[52] P. van Nieuwenhuizen. Supergravity. *Physics Reports*, 68(4):189 – 398, 1981.

[53] L. B. Anderson and J. T. Wheeler. Biconformal supergravity and the AdS/CFT conjecture. *Nuclear Physics B*, 686(1):285–309, 2004.

[54] R. R. Metsaev. 6D conformal gravity. *Journal of Physics A: Mathematical and Theoretical*, 44(17):175402, 2011.

[55] P. D. Mannheim. Making the Case for Conformal Gravity. *Foundations of Physics*, 42:388–420, Mar 2012.

[56] E. A. Ivanov and J. Niederle. $n = 1$ supergravity as a nonlinear realization. *Phys. Rev. D*, 45:4545–4554, Jun 1992.

[57] J. G. O'Brien, S. S. Chaykov, J. Dentico, M. Stulge, B. Stefanski, and R. J. Moss. Recent advancements in conformal gravity. *Journal of Physics: Conference Series*, 845(1):012004, 2017.

[58] P. D. Mannheim and J. G. O'Brien. Galactic rotation curves in conformal gravity. *Journal of Physics: Conference Series*, 437(1):012002, 2013.

[59] P. D. Mannheim. Alternatives to dark matter and dark energy. *Progress in Particle and Nuclear Physics*, 56(2):340–445, 2006.

[60] S. S. McGaugh, F. Lelli, and J. M. Schombert. Radial Acceleration Relation in Rotationally Supported Galaxies. *Phys. Rev. Lett.*, 117:201101, Nov 2016.

[61] P. D. Mannheim and J. G. O'Brien. Impact of a Global Quadratic Potential on Galactic Rotation Curves. *Phys. Rev. Lett.*, 106:121101, Mar 2011.

[62] J. Sultana, D. Kazanas, and J L. Said. Conformal Weyl gravity and perihelion precession. *Phys. Rev. D*, 86:084008, Oct 2012.

[63] K. Horne. Conformal Gravity Rotation Curves with a Conformal Higgs Halo. *Mon. Not. Roy. Astron. Soc.*, 458:4122–4128, Mar 2016.

[64] C. Roberts, K. Horne, A. O. Hodson, and A. D. Leggat. Tests of λCDM and Conformal Gravity using GRB and Quasars as Standard Candles out to $z \sim 8$. *arXiv:1711.10369*, 2017.

[65] P. R. Phillips. Schwarzschild and linear potentials in Mannheim's model of conformal gravity. *Monthly Notices of the Royal Astronomical Society*, 478, May 2018.

[66] Cornelius Lanczos. A remarkable property of the riemann-christoffel tensor in four dimensions. *Annals of Mathematics*, 39(4):842–850, 1938.

[67] D. Y. Kim, A. N. Lasenby, and M. P. Hobson. Spherically-symmetric solutions in general relativity using a tetrad-based approach. *General Relativity and Gravitation*, 50, Mar 2018.

[68] R. Nandra, A. N. Lasenby, and M. P. Hobson. The effect of an expanding universe on massive objects. *Mon. Not. Roy. Astron. Soc.*, 422:2945–2959, May 2012.

[69] P. D. Mannheim. Alternatives to dark matter and dark energy. *Progress in Particle and Nuclear Physics*, 56(2):340–445, 2006.

[70] M. C. Campigotto, A. Diaferio, and L. Fatibene. Conformal gravity: light deflection revisited and the galactic rotation curve failure. *arXiv:1712.03969*, Dec 2017.

[71] Y. Brihaye and Y. Verbin. Spherical structures in conformal gravity and its scalar-tensor extension. *Phys. Rev. D*, 80:124048, Dec 2009.

[72] G. Karananas. Poincaré, Scale and Conformal Symmetries Gauge Perspective and Cosmological Ramifications. 2016.

[73] P. D. Mannheim. Dynamical mass and geodesic motion. *General Relativity and Gravitation*, 25:697–715, Jul 1993.

[74] G. K. Karananas and J. Rubio. On the geometrical interpretation of scale-invariant models of inflation. *Physics Letters B*, 761:223–228, 2016.

[75] G. Stephenson. Quadratic Lagrangians and General Relativity. *Il Nuovo Cimento*, 9:263–269, Jul 1958.

[76] P. W. Higgs. Quadratic Lagrangians and General Relativity. *Il Nuovo Cimento*, 11:816–820, Mar 1959.

[77] P. A. M. Dirac. Long range forces and broken symmetries. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 333(1595):403–418, Jun 1973.

[78] A. N. Lasenby, C. J. L. Doran, and R. Heinke. Analytic solutions to Riemann-squared gravity with background isotropic torsion. *ArXiv e-prints 0509014*, 2005.

[79] C. Doran and A. Lasenby. *Geometric Algebra for Physicists*. Cambridge University Press, 2003.

[80] D. H. Lyth and A. R. Liddle. *The Primordial Density Perturbation: Cosmology, Inflation and the Origin of Structure*. Cambridge University Press, 2009.

[81] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger. Theory of Cosmological Perturbations. *Physics Reports*, 215(5):203–333, 1992.

[82] B. D. Sleeman. Integral representations for solutions of Heun's equation. *Mathematical Proceedings of the Cambridge Philosophical Society*, 65(2):447–459, 1969.

[83] Paszke A., Gross S., and Chintala S. Automatic differentiation in pytorch. *NIPS Workshop*, 2017.

[84] I. D. Saltas and M. Kunz. Anisotropic stress and stability in modified gravity models. *Phys. Rev. D*, 83:064042, Mar 2011.

[85] I. D. Saltas, I. Sawicki, L. Amendola, and M. Kunz. Anisotropic Stress as a Signature of Nonstandard Propagation of Gravitational Waves. *Phys. Rev. Lett.*, 113:191101, Nov 2014.

[86] W. Adamowicz. Plane waves in Gauge Theories of Gravitation. *General Relativity and Gravitation*, 12:677–691, Sep 1980.

[87] M. Blagojević and B. Cvetković. Gravitational waves with torsion in 3D. *Phys. Rev. D*, 90:044006, Aug 2014.

[88] M. Blagojević and B. Cvetković. Siklos waves in Poincaré gauge theory. *Phys. Rev. D*, 92:024047, Jul 2015.

[89] Y. N. Obukhov and V. A. Korotky. The Weyssenhoff fluid in Einstein-Cartan theory. *Classical and Quantum Gravity*, 4(6):1633, 1987.

[90] F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester. General relativity with spin and torsion: Foundations and prospects. *Rev. Mod. Phys.*, 48:393–416, Jul 1976.

[91] W. Kopczynski. A non-singular universe with torsion. *Physics Letters A*, 39:219–220, 1972.

[92] J. Tafel. A non-singular homogeneous universe with torsion. *Phys. Lett 45A 341*, 1973.

[93] J. Weyssenhoff and A. Raabe. Relativistic dynamics of spin-fluids and spin-particles. *Acta Phys. Polon., t. 9, 7*, 1947.

[94] I. S. Nurgaliev and V. N. Ponomariev. Torsion of space-time and cosmology. *Sov. Phys. J. 25 796*, pages 929–931, 1982.

[95] W. Arkuszewski, W. Kopczynski, and V. N. Ponomariev. On the linearized Einstein-Cartan theory. *Annales de l'I.H.P., section A, tome 21*, pages 89–95, 1974.

[96] N. Popławski. Black-hole cosmology, big bounce, and inflation. *ArXiv e-prints*, jan 2018.

[97] M. L. Bedran et al. Stationary cosmological solution with torsion. *Nuovo Cimento B 87 101*, 1985.

[98] L. L. Smalley. Godel cosmology in Riemann-Cartan spacetime with spin density. *Phys. Rev. D 32 3124*, Dec 1985.

[99] M. Szydłowski and A. Krawiec. Cosmological model with macroscopic spin fluid. *Phys. Rev. D*, 70:043510, Aug 2004.

[100] J. R. Ray and L. L. Smalley. Perfect fluids in the einstein-cartan theory. *Phys. Rev. D 26, 2615*, 1982.

[101] A. V. Minkevich and F. J. Karakura. On the relativistic dynamics of spinning matter in space-time with curvature torsion. *J. Phys. A: Math Gen.*, pages 1409–1418, 1983.

[102] R. de Ritis, M. Lavorgna, G. Platania, and C. Stornaiolo. Spin fluid in Einstein-Cartan theory: A variational principle and an extension of the velocity potential representation. *Phys. Rev. D*, 28:713–717, Aug 1983.

[103] J. R. Ray and L. L. Smalley. Spinning fluids in the Einstein-Cartan theory. *Phys. Rev. D*, 27:1383–1385, Mar 1983.

[104] S. D. Brechet, M. P. Hobson, and A. N. Lasenby. Weyssenhoff fluid dynamics in general relativity using a 1 + 3 covariant approach. *Classical and Quantum Gravity*, 24(24):6329, 2007.

[105] N. J. Poplawski. Cosmological consequences of gravity with spin and torsion. *Astronomical Review*, 8(3):108–115, 2013.

[106] R.A Weiss. *Relativistic Thermodynamics*. Exposition Press, 1976.

[107] C. Doran, A. Lasenby, A. Challinor, and S. Gull. Effects of spin-torsion in gauge theory gravity. *Journal of Mathematical Physics*, 39(6):3303–3321, 1998.

[108] M Tsamparlis. Cosmological principle and torsion. *Physics Letters A*, 75:27–28, 1979.

[109] I. Nurgaliev and W. Ponomariev. The earliest evolutionary stages of the universe and space-time torsion. *Physics Letters B*, 130(6):378 – 379, 1983.

[110] G. de Berredo-Peixoto and E. de Freitas. On the torsion effects of a relativistic spin fluid in early cosmology. *Classical and Quantum Gravity*, 26(17):175015, 2009.

[111] P. Schofield and J. R. Henderson. Statistical mechanics of inhomogeneous fluids. 379(1776):231–246, 1982.

[112] A. Kihm, L. Kaestner, C. Wagner, and S. Quint. Classification of red blood cell shapes in flow using outlier tolerant machine learning. *PLOS Computational Biology*, 14, 06 2018.

[113] M. Xenos. An euler-lagrange approach for studying blood flow in an aneurysmal geometry. *Proceedings of the Royal Society of London Series A*, 473, 2017.