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Weibull parameter estimation and goodness-of-fit for glass strength data

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ABSTRACT

Strength data from macroscopically identical glass specimens is commonly described by a two-parameter Weibull distribution, but there is lack of research on the methods used for fitting strength data to the Weibull distribution. This study investigates 4 different methods for fitting data and estimating the parameters of the Weibull distribution namely, good linear unbiased estimators, least squares regression, weighted least squares regression and maximum likelihood estimation. These methods are implemented on fracture surface strength data from 418 annealed soda-lime-silica glass specimens, grouped in 30 nominally identical series, including as-received, naturally aged and artificially aged specimens. The strength data are evaluated based on their goodness of fit. Comparison of conservativeness of strength estimates is also provided. It is found that a weighted least squares regression is the most effective fitting method for the analysis of small samples of glass strength data.

1. Introduction

Glass strength is governed by the condition of its surface, including microscopic flaws on the surface of glass that may be indiscernible to the naked eye. There is a large variation in fracture strength obtained from seemingly identical specimens which are produced, stored and tested destructively under the same conditions. Therefore, destructive testing of several nominally identical glass specimens and the subsequent statistical analysis of their strength data is essential for establishing an accurate design strength, corresponding to a sufficiently low probability of failure. Glass is susceptible to sub-critical crack growth, therefore in order to normalise the effects of glass specimens failing after different load durations, the fracture strength data from the destructive tests is often expressed as a time-equivalent strength. This is achieved by converting the stress history exerted during the destructive test over the time to failure, t_f to an equivalent constant stress, $\sigma_{f,ref}$, for a reference time period, t_{ref}, (60 s is a typical value) as shown in Eq. (1) [1]:

$$\int_0^{t_f} \sigma^n(t)dt = \int_0^{t_{ref}} \sigma^n_{f,ref} dt \quad (1)$$

There are three statistical distributions that have historically been used to describe strength data: Weibull, normal and lognormal [2–5]. The 2-parameter Weibull distribution is often preferred because: (a) it is more accurate in describing glass strength data than a normal distribution [2] and; (b) is always more conservative in the tail of the distribution than a lognormal distribution [3] (this is also verified for the strength data used in the present study as shown in Appendix B). Conservative estimates are more desirable for engineering design applications. As a result the Weibull distribution is the established way of describing glass strength data in both research [5–11] and engineering applications [12–14].

The general equation for the cumulative distribution function (*CDF*) of the Weibull distribution [15] is:

$$P_f(\sigma_{f,60}) = 1 - \exp\left[-\left(\frac{\sigma_{f,60} - \sigma_u}{\theta}\right)^{\beta}\right] \quad (2)$$

where β is the shape parameter, θ is the scale parameter, $\sigma_{f,60}$ is the equivalent fracture stress for a reference time period of 60 s and σ_u is the location parameter.

The location parameter, σ_u, represents the stress level below which the material never fails (i.e. $P_f = 0$). Safety reasons dictate that σ_u is set to 0 as recommended in Trustrum and Jayatilaka [16] for brittle materials. Therefore, Eq. (2) is reduced to a two-parameter Weibull function and the *CDF* can be linearized (Eq. (3)) in the form of $y = bx + c$ by taking the logarithm of each side twice:

$$\ln\left(\ln\left(\frac{1}{1-P_f}\right)\right) = \beta \cdot \ln\sigma - \beta \cdot \ln\theta \quad (3)$$

Hence, the CDF becomes a linear plot of $\ln\left(\ln\left(\frac{1}{1-P_f}\right)\right)$ vs. $\ln\sigma$ as illustrated in Fig. 1, and where the gradient of the distribution is

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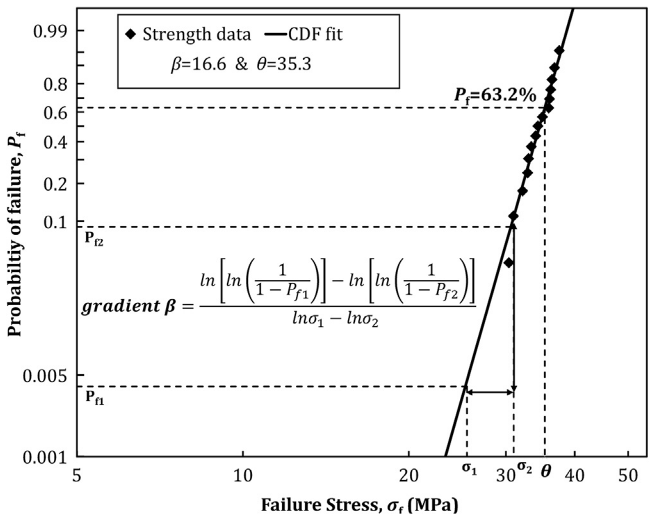

Fig. 1. Cumulative distribution function (CDF) of glass strength data.

equal to the shape parameter, β and the intercept is $-\beta \cdot \ln\theta$. The shape parameter, β, indicates the variability of the data and thus, higher values of β lead to a steeper *CDF* and represent a smaller scatter of strength in the data. The scale parameter, θ, represents the stress level, below which 63.2% of the specimens fail and together with the shape parameter dictates the position of the *CDF* along the horizontal axis.

There are various approaches for estimating the Weibull parameters from a given set of strength data. They can be classified either as manual or computational methods. Manual calculations can be performed by: (a) least square regression (*LR*); (b) weighted least squares regression (*WLR*) and; (c) a linear approach based on good linear unbiased estimators (*GLUEs*); while computational (computer-based) methods are: (a) the maximum likelihood estimation (*MLE*) and; (b) the method of moments estimation (*MME*).

The aim of this study is to review these different estimation methods for Weibull parameters and to propose the most effective method for the statistical analysis of small sized samples of glass strength. To the best of the authors' knowledge, this study is the first to use real glass strength data in order to assess methods for their statistical analysis. Therefore, the observations and conclusions from this study are valuable for researchers and practitioners who have performed destructive tests on a relatively small number of nominally identical glass components and wish to perform a statistical analysis of the strength data. An overview of the existing methods for estimating the Weibull parameters and goodness-of-fit for glass strength data are first reviewed in Section 2. The existing methods (*LR*, *WLR*, *GLUEs* and *MLE*) are then implemented on 30 real data sets, obtained from destructive tests on naturally aged, as-received and artificially aged glass in Section 3. The goodness-of-fit and strength estimate results of each method are presented and discussed in Section 4 and the conclusions are provided in Section 5.

2. Review of Weibull statistics methods

The two principal steps when performing a statistical analysis are: estimating the statistical parameters and evaluating the goodness-of-fit. These are reviewed in this section in the context of a Weibull distribution for glass strength data.

2.1. Parameter estimation

The most commonly used approaches, within the Weibull statistics community, for the estimation of the shape and scale parameters of the Weibull distribution are described below.

2.1.1. Manual calculations methods

Equivalent strength data are ranked in ascending order ($i = 1$ to n) for the manual calculation methods. Equal probabilities of failure, P_f, are assigned to each data point in cumulative form with functions called probability estimators, E_i. The simplest forms of probability estimators are $E = i/n$ or $E = (i-1)/n$ but these estimators eliminate the highest or lowest data point of the sample in the *CDF* graph for $P_f = 1$ or $P_f = 0$ respectively; the highest/lowest strength point are therefore, also eliminated during the estimation of the Weibull parameters so that instead of n specimens, only ($n - 1$) would be considered. Therefore, these estimators are avoided and probability estimators of the following form are preferred instead:

$$E_j = \frac{i - C_j}{n + 1 - 2C_j} \tag{4}$$

where C_j is a constant $0 \le C_j < 1$, i is the index of the ascending order and n is the sample size. The following four probability estimators (E_j, j = 1– 4, [17–19]) are most commonly used in Weibull statistics:

$$E_1 \text{ (mean rank)}: \; C_1 = 0 \to E_1 = \frac{i}{n+1} \tag{4a}$$

$$E_2 \text{(Hazen's)}: \; C_2 = 0.5 \to E_2 = \frac{i - 0.5}{n} \tag{4b}$$

$$E_3 \text{ (median rank)}: \; C_3 = 0.3 \to E_3 = \frac{i - 0.3}{n + 0.4} \tag{4c}$$

$$E_4 \text{ (small sample)}: \; C_4 = 0.375 \to E_3 = \frac{i - 0.375}{n + 0.25} \tag{4d}$$

2.1.1.1. Least Squares Regression (LR). The Weibull parameters are determined in the Least Squares Regression method (*LR*), by minimizing the sum of squared residuals of the x values about Eq. (3):

$$\beta = \frac{n \cdot \sum_{i=1}^{n}[\ln(\sigma_i) \cdot y_i] - \sum_{i=1}^{n}(\ln \sigma_i) \cdot \sum_{i=1}^{n}(y_i)}{n \cdot \sum_{i=1}^{n}[(\ln \sigma_i)^2] - [\sum_{i=1}^{n}(\ln \sigma_i)]^2} \tag{5a}$$

$$-\beta \cdot \ln\theta = \frac{\sum_{i=1}^{n}(y_i) - \beta \times \sum_{i=1}^{n}(\ln \sigma_i)}{n} \tag{5b}$$

However, *LR* implicitly applies the same unit weight to each data point without accounting for the uncertainty of $y = \ln(\ln(\frac{1}{1-E}))$ or E_i. and thus provide biased estimates.

2.1.1.2. Weighted Least Squares Regression (WLR). Weibull parameters with smaller bias than those deriving from *LR*, can be obtained (Eq. (6a) and (6b)) by introducing weight functions based on the uncertainty of y and E within the *LR* method leading to a Weighted Least Squares Regression, *WLR* [20].

$$\beta = \frac{\sum_{i=1}^{n} W_i \cdot \sum_{i=1}^{n}[\ln(\sigma_i) \cdot y_i \cdot W_i] - \sum_{i=1}^{n}[\ln(\sigma_i) \cdot W_i] \cdot \sum_{i=1}^{n}(y_i \cdot W_i)}{\sum_{i=1}^{n} W_i \cdot \sum_{i=1}^{n}[(\ln \sigma_i)^2 \cdot W_i] - [\sum_{i=1}^{n}(\ln \sigma_i) \cdot W_i]^2} \tag{6a}$$

$$-\beta \cdot \ln\theta = \frac{\sum_{i=1}^{n}(y_i \cdot W_i) - \beta \cdot \sum_{i=1}^{n}[\ln(\sigma_i) \cdot W_i]}{\sum_{i=1}^{n} W_i} \tag{6b}$$

where W_i is the weight applied to each data point.

Various weight functions have been proposed over the years [20–22] with Bergman's (Eq. (7a), [20]) and Faucher & Tyson's weight function (Eq. (7b), [21]) being mostly used. Faucher and Tyson's (*F&T*) was found to produce the most accurate estimates for data sets produced with Monte Carlo simulation [22–24]. However, these studies disagree on the choice of estimator used in conjunction with the *F&T* function; the two most accurate probability estimators for small samples (n = 10–20) were found to be the mean and median rank estimators (Eqs. (4a) and (4c), [22–24]).

$$W_i = [(1-P_i)\cdot\ln(1-P_i)]^2 \tag{7a}$$

$$W_i = 3.3\cdot P_i - 27.5\cdot\left[1-(1-P_f)^{0.025}\right] \tag{7b}$$

2.1.1.3. Good Linear Unbiased Estimators (GLUEs). Simple unbiased estimators for the shape and the scale factor known as good linear unbiased estimators (*GLUEs*, Eq. (8a) and (8b) for complete/uncensored samples) were proposed by Bain [25,26]. This method is prescribed in EN 12603:2002 [12] and uses an un-biasing constant, k_n and an integer number, s, to minimize the variance of the *GLUEs* (details given in [25–27]). Probability estimators, E_i, (Eqs. (4a) (4d)) are solely used in this method to assign a probability of failure to each equivalent strength data point whilst plotting the *CDF* (Eq. (3)) and are not considered during the computation of the Weibull parameters; a median rank estimator (Eq. (4c)) is proposed in EN 12603:2002 [12].

$$\beta = \frac{nk_n}{\frac{s}{n-s}\sum_{i=s+1}^{n}(\ln\sigma_i)-\sum_{i=1}^{s}(\ln\sigma_i)} \tag{8a}$$

$$\theta = \exp\left(\frac{1}{n}\sum_{i=1}^{n}(\ln\sigma_i)+0.5772\frac{1}{\beta}\right) \tag{8b}$$

where k_n is an un-biasing constant, values provided in Tables for n = 2–60 [12,25] and s is the largest integer for the product of 0.84·n.

2.1.2. Computational methods

2.1.2.1. Maximum likelihood estimation (MLE). The maximum likelihood estimation (*MLE*) is the method prescribed in ASTM C1239-13 [13] and DIN EN 843–5 [14]. This method computes a set of shape and scale factors that maximizes the likelihood / provides the highest probability of producing the data obtained from the destructive testing. The likelihood function (Eq. (9)) is defined as the product, Π, of function, *f*, which associates the strength data, the shape and the scale parameters. The logarithm of the likelihood function is maximized by differentiating ln*L* over each of the unknown parameters (β, θ) and subsequently setting each of the partial derivatives to 0 (Eq. 10a-b, likelihood equations). ln*L* is preferred in this step to reduce the complexity of calculations (i.e. the logarithm of the product Π(*f*) is now converted in sums Σ*f*). Analytical calculations lead to Eqs. (11a-b) (details are not shown here for brevity but can be found in [28]). A closed form solution of Eq. (11a-b) is not available; therefore, iterative numerical methods (e.g. gradient method) are used to obtain estimates for β and θ. A MATLAB script is used in this study.

$$L = \prod_{i=1}^{n} f(\sigma_i;\beta,\theta) \tag{9}$$

$$\frac{\partial\sum_{i=1}^{n}\ln L}{\partial\beta}=0 \quad and; \quad \frac{\partial\sum_{i=1}^{n}\ln L}{\partial\theta}=0 \tag{10a-b}$$

$$\frac{n}{\beta}-n\frac{\sum_{i=1}^{n}[\sigma_i^{\beta}\cdot\ln(\sigma_i)]}{\sum_{i=1}^{n}\sigma_i^{\beta}}+\sum_{i=1}^{n}\sigma_i=0 \tag{11a}$$

$$\theta=\left(\frac{\sum_{i=1}^{n}\sigma_i^{\beta}}{n}\right)^{1/\beta} \tag{11b}$$

2.1.2.2. Method of moments estimation (MME). Moments are quantitative characteristics that describe a distribution and can be taken about the origin of the distribution (raw moments, μ', Eq. (12a)) or the mean (central moments, μ, Eq. (12b)).

$$\mu'_r = \int_0^{\infty}\sigma^r\cdot P(\sigma)d\sigma \tag{12a}$$

$$\mu_r = \int_0^{\infty}(\sigma-\overline{\mu})^r\cdot P(\sigma)d\sigma \tag{12b}$$

where r is the order of the moment; $P(\sigma)$ is the probability density function; σ is the strength of glass and $\overline{\mu}$ is the mean of the distribution.

For example, the 0th raw moment describes the total probability of failure ($\mu'_0 = 1$) and the 1st raw moment describes the mean of the distribution ($\mu'_1 = \overline{\mu}$). Additionally, the second, third and fourth central moments describe the variance ($\mu_2 = s^2$), the skewness and the kurtosis of the distribution, respectively. For the Weibull distribution, the formula for the theoretical distributional moments is given by Eq. (12c) (analytical calculations shown in [28]).

$$\mu_r = \theta^r\int_0^{\infty}y^{r/\beta}\cdot e^{-y}dy \tag{12c}$$

where $y = \left(\frac{\sigma}{\theta}\right)^{\beta}$.

Moments are used in the method of moments (*MME*) to calculate the shape and the scale factor of the Weibull distribution by extending the known characteristics/moments of the sample to the corresponding characteristics/moments of the population. The number of moments that are needed is defined by the number of unknowns; two moments are therefore, needed for the estimation of the shape, β, and scale, θ, factor. In particular, the known sample moments which can be obtained from strength data (Eq. (13)), are equated to the corresponding raw theoretical distributional moments (Eq. (14a) and (14b), [29]), as shown in Eq. (15a). Subsequently, their members are divided by parts as shown in Eq. (15b) wherein β is the only unknown. β can be computed by numerical methods or tables provided in literature (e.g. in [29]). θ is then easily computed by substituting β in Eq. (14a).

$$\mathrm{M}_r=\frac{1}{n}\cdot\sum_{i=1}^{n}x_i^r \Rightarrow \mathrm{M}_1=\frac{1}{n}\cdot\sum_{i=1}^{n}x_i \quad and; \quad \mathrm{M}_2=\frac{1}{n}\cdot\sum_{i=1}^{n}x_i^2 \tag{13}$$

$$\mu'_1=\overline{\mu}=\theta\cdot\Gamma\left(1+\frac{1}{\beta}\right) \tag{14a}$$

$$\begin{aligned}&\mu'_2=s^2+\mu_1'^2\Rightarrow\\&\mu'_2=\theta^2\cdot\left[\Gamma\left(1+\frac{2}{\beta}\right)-\Gamma^2\left(1+\frac{1}{\beta}\right)\right]+\theta^2\cdot\Gamma^2\left(1+\frac{1}{\beta}\right)\Rightarrow\\&\mu'_2=\theta^2\cdot\Gamma\left(1+\frac{2}{\beta}\right)\end{aligned} \tag{14b}$$

$$\mu\prime_1 = M_1 \quad \& \quad \mu'_2 = M_2 \tag{15a}$$

$$\begin{aligned}&\frac{\mu_1'^2}{\mu'_2}=\frac{M_1^2}{M_2}\Rightarrow\frac{\theta^2\cdot\Gamma^2\left(1+\frac{1}{\beta}\right)}{\theta^2\cdot\Gamma\left(1+\frac{2}{\beta}\right)}=\frac{\left(\frac{1}{n}\cdot\sum_{i=1}^{n}x_i\right)^2}{\frac{1}{n}\cdot\sum_{i=1}^{n}x_i^2}\Rightarrow\\&\frac{\Gamma^2\left(1+\frac{1}{\beta}\right)}{\Gamma\left(1+\frac{2}{\beta}\right)}=\frac{\left(\frac{1}{n}\cdot\sum_{i=1}^{n}x_i\right)^2}{\frac{1}{n}\cdot\sum_{i=1}^{n}x_i^2}\end{aligned} \tag{15b}$$

where $\Gamma(n)=\int_0^{\infty}e^{-x}\cdot x^{n-1}dx$.

2.1.3. Comparison of parameter estimation methods

Previous research involving data generated numerically by Monte Carlo simulations, show contradictory results when comparing different Weibull parameter estimation methods. Computational methods (*MLE* and *MME*) are identified as more accurate in [30–32]. However, manual calculations and *WLS* in particular, produced the smallest standard deviation, *s*, for the shape parameter and was therefore found to be more accurate than computational methods [23]. In other studies, the accuracy of estimation methods was found to vary with the size of the sample, *n* [24,33]; computational methods performed better for medium or large size samples ($n > 52$) whilst manual calculations for small samples ($n < 52$).

More specifically, the following conclusions were reached by previous studies:

(a) for manual methods:
  - *WLR* outperforms *LR* [21,24,33].
  - Faucher and Tyson's (*F&T*) weight function provides the most accurate estimates (for $n \geq 7$ [22–24]).
  - No clear conclusion has been reached on the choice of probability estimators; E_1 (Eq. (4a)) is considered the most conservative estimator and is suggested for engineering purposes in [17]; E_2 (Eq. (4b)) provided the least biased estimations according to [16,32] and; E_3 (Eq. (4c)) produced the smallest coefficient of variation of Weibull parameters in [30].
  - the bias of estimation is a function of sample size, *n*, and linear regression method [24].

(b) for computational methods:
  - Estimates from *MLE* are very similar to *MME* [33,34].
  - *MLE* provides the lowest dispersion i.e. narrower confidence intervals among *LR* methods [24,34].
  - The probability of overestimating strength is high ($P = 60\%$) for *MLE* [32].

A significant gap in the studies to-date is that the European standard CEN12603:200 [12] on Weibull distributed glass strength data prescribes a GLUEs method, the accuracy of which does not appear to have been investigated or compared in any of these studies. Another limitation of the above studies is that Weibull parameters were estimated for "artificial" data sets that were randomly generated with Monte Carlo simulations for a given shape and scale factor (β_{true} and θ_{true}). These data sets were then evaluated based on their accuracy i.e. the discrepancy between the true shape factor that was initially chosen, β_{true}, and the estimated shape factor obtained for a specific estimation method. This is a valid approach for artificially generated data sets, but this evaluation method cannot be used for real data sets since the true value of the shape factor of the population is unknown. Therefore, based on the knowledge that glass strength can be described with a 2-parameter, an alternative procedure that employs a goodness-of-fit test is used to assess the accuracy of each Weibull parameter estimation method. Accuracy in this paper is a measure of the goodness of fit i.e. the higher the goodness-of-fit the more effective the method.

2.2. Goodness-of-fit methods

The methods for estimating the Weibull parameters are based on the null hypothesis (H_0) that the equivalent strength data follow a Weibull distribution. Goodness-of-fit methods are used to evaluate whether H_0 needs to be rejected, which would indicate that the Weibull distribution does not provide a good fit to the data under consideration. A significance level, α, is therefore, chosen prior to the statistical analysis ($\alpha = 0.05$ in this study); α represents the probability of rejecting a good fit i.e. it defines the confidence level that the data do not follow a Weibull distribution. The observed significance level, *p*, that describes the specific data set is then compared to the chosen significance level, α. If:

- $p \leq \alpha$: the data do not follow a Weibull distribution (H_0 rejected);
- $p > \alpha$: there is lack of evidence that data do not follow a Weibull distribution and therefore, H_0 cannot be rejected;

This study uses goodness-of-fit methods based on the empirical distribution function (*EDF*) statistics. *EDF* statistics are used in various goodness-of-fit methods (Kolmogorov-Smirnov, Cramer-von Mises, Chi-squared, Anderson Darling etc., [35]); these methods depend on the distance, *D*, between the *EDF* (Eq. (16)) and the theoretical distribution function (*TDF*) i.e. the assumed *CDF* for the estimated shape and scale factor as shown in Fig. 2. *EDF* statistics are therefore, independent of the choice of probability estimator, *E* (Eq. (4)). *TDF* should follow closely the *EDF* when the assumed distribution is a good fit for the particular data set.

$$F_n = \frac{1}{n}\sum_{i}^{n} 1\{\sigma_i \leqslant \sigma\} \tag{16}$$

where $1\{\sigma_i \leqslant \sigma\}$ is the indicator function that counts the number of observations that are equal to or smaller to a fixed σ which each time represents a particular data point of the set i.e.:

$$1\{\sigma_i \leqslant \sigma\} = \begin{cases} 1 & \sigma_i \leqslant \sigma \\ 0 & \sigma_i > \sigma \end{cases}$$

The Anderson Darling (*AD*) goodness-of-fit is typically used during the assessment of glass strength data [6,7,10]. *AD* belongs to the group of quadratic *EDF* statistics as it is a function of the squared distance, D^2, between the *EDF* and *TDF*. Additionally, this method employs a weight function, w_{AD} (Eq. (17a)) in order to apply more weight to the upper and lower tail of the *CDF*. This is essential for the statistical analysis of glass strength data as strengths at low probabilities of failure (i.e. at the lower tail of *CDF*) are typically used in engineering design. Therefore, the AD goodness-of-fit will be used in this study. An approximation formula commonly used for the observed significance level, p_{AD}, of the *AD* goodness-of-fit is given by Eq. (17b) [36].

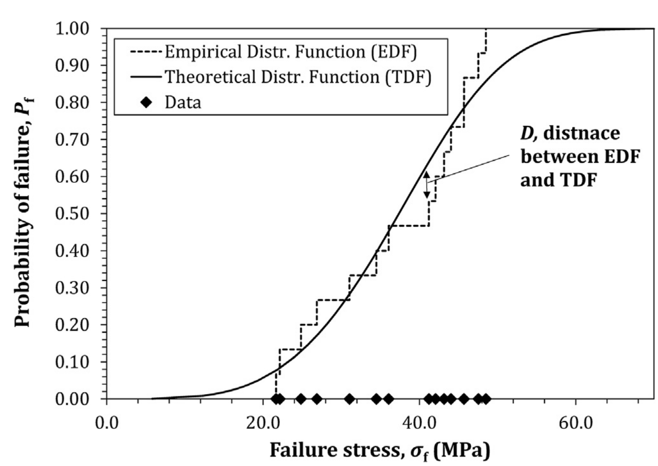

Fig. 2. Empirical distribution function and assumed cumulative distribution function for glass strength data.

$$AD = n\int_0^1 \frac{[F_n(\sigma) - P_f(\sigma)]^2}{w_{AD}} dP_f(\sigma) \Rightarrow$$
$$AD = n\int_0^1 \frac{[F_n(\sigma) - P_f(\sigma)]^2}{P_f(\sigma)\cdot[1-P_f(\sigma)]} dP_f(\sigma) \tag{17a}$$

$$p_{AD} = \frac{1}{1+\exp(-0.1+1.24\cdot\ln(AD^*)+4.48\cdot AD^*)} \tag{17b}$$

where $AD^* = \left(1+\frac{0.2}{\sqrt{n}}\right)\cdot AD^2$ and $AD^2 = -n - \sum_{i=1}^{n}\frac{(2i-1)}{n}\cdot[\ln(P_f(\sigma_i)) + \ln(1-P_f(\sigma_{n+1-i}))]$.

A unimodal Weibull distribution is typically sufficient to describe failures in glass. However, poor goodness of fit could potentially denote bi-modal distributions. Bi-modal distributions can occur due to different underlying causes of failure/flaw morphologies in different specimens in the same series. These data series are more faithfully described by mixed Weibull distributions (Eq. (19)). The estimation of the mixed Weibull parameters is based on graphical approaches [37]. Even though computational methods (*MME* and *MLE*) exist for such cases [38,39], there are 5 unknowns (β_1, β_2, θ_1, θ_2 and p) and they can therefore, be time consuming.

$$P_f(\sigma_{f,60}) = p\cdot\left\{1-\exp\left[-\left(\frac{\sigma_{f,60}}{\theta_1}\right)^{\beta_1}\right]\right\} + q\cdot\left\{1-\exp\left[-\left(\frac{\sigma_{f,60}}{\theta_2}\right)^{\beta_2}\right]\right\} \tag{17c}$$

where: p and q: the mixing weights for the two Weibull distributions ($p + q = 1$).

3. Method

3.1. Glass specimens and destructive tests

30 real data sets (n = 10–18 specimens) of glass surface strength data, were obtained from as-received, naturally aged and artificially aged annealed glass specimens [9,11]. All specimens were soda-lime-silica glass produced by the float process and the series capture a wide range of as-received and aged glass (as evidenced in the broad range of shape and scale parameters in Table A3). Some of these series provide a close correlation to the naturally aged glass of this study while others provide a lower or higher scatter of strength data [11] covering a diverse range of ageing scenarios that are used to formulate an ageing procedure for the assessment of the strength of aged glass in [40]. The specimens are therefore, representative of the glass used in most real world applications in the construction industry. Therefore, the observations and conclusions of this study are valid for the broad range of surface damage scenarios of glass.

The naturally aged glass ($NA_{1\text{-}2}$) used in this study was obtained from a façade in Norfolk, UK which had been exposed to 20 years of natural ageing. The artificially aged glass was either sand abraded ($SA_{1\text{-}24}$) or scratched ($SC_{1\text{-}2}$). Table 1 provides an overview of the series. In total 418 specimens, grouped in 30 series each consisting of between 10 and 18 specimens, were tested destructively. The destructive tests were performed in a coaxial double ring set-up complying with ASTM C1499-3 [41] to obtain glass surface strength data. Specimens failed across a range of loads (250–2217 N) and durations (1–6 s). Strength data were excluded from further consideration for specimens whose origin of failure was located outside the boundaries of the loading ring. Variations in sub-critical crack growth were normalised by converting strength results into equivalent strengths for a constant load for a time period of 60 s (Eq. (1), raw data are available in Table A1, Appendix A).

3.2. Estimation methods

The following estimation methods are implemented in this study on each set of glass strength data to determine its Weibull parameters: (a) *LR*; (b) *WLS* using Faucher & Tyson's (F&T) and Bergman's (B) weight function; (c) *GLUEs* and; (d) *MLE*. In total, 14 combinations of estimation methods, probability estimators and weight functions are used (Table 2). The Anderson Darling goodness-of-fit (p_{AD}) was the main criterion for evaluating the accuracy of each method. The relative conservativeness of strength for each estimation method is also provided as supplementary.

To reduce the number of possible permutations the assessment is divided in the following two steps:

- *Section 4.1:* the performance (goodness-of-fit and conservativeness of strength estimates) of probability estimators (E_1-E_4) is assessed for the *LR*, *WLR-F&T* and *WLR-B* estimation methods. The *GLUEs* and *MLE* are excluded from this assessment, because as explained in Section 2, the probability estimators do not influence the Weibull parameters.
- *Section 4.2:* The best performing probability estimators from Section 4.1 is used to assess the performance of the five different Weibull parameter estimation methods (*LR-E*$_2$, *WLR-F&T-E*$_2$ and *WLR-B-E*$_2$, *GLUEs* and *MLE*).

4. Results and discussion

4.1. Performance of probability estimators

Probability estimators are ranked in ascending order (1–4) in Table 3 for methods *LR*, *WLR-F&T* and *WLR-B* based on their goodness-of-fit, p_{AD}, and strength estimates, $\sigma_{0.001}$. *1* denotes the probability estimator with the highest goodness of fit, p_{AD} and the highest strength estimate of $\sigma_{0.001}$ whilst *4* denotes the probability estimator with the lowest goodness of fit, p_{AD} and the lowest strength estimate of $\sigma_{0.001}$. The performance of each probability estimator is evaluated based on the number of times it scores 1st in terms of goodness of fit or 4th in terms of design strength estimates, providing the best fit to the Weibull distribution or the most conservative design strength respectively. The rationale behind this is that: (a) the Weibull distribution is shown to be more accurate in describing glass strength data than the normal or the lognormal distributions in Appendix B and therefore, the highest

Table 1
Specimens.

Abbr.	Glass type	Processing	Dimensions (mm)	No of data sets	No of specimens
NA	annealed	Naturally aged	150 × 150 × 3	2	31
AR	annealed	As-received	150 × 150 × 3	1	10
SA	annealed	Sand abraded	150 × 150 × 3	25	348
SC	annealed	Scratched	150 × 150 × 3	2	29
Total				**30**	**418**

Table 2
Combinations of estimation methods, estimators and weight functions.

Ref. No	Method	Abbr.	Estimator	Abbr.	Weight Function
1	Least squares regression	*LR*	Mean rank (Eq. (4a))	E_1	–
2			Hazen's (Eq. (4b))	E_2	–
3			Median rank (Eq. (4c))	E_3	–
4			Small sample (Eq. (4d))	E_4	–
5	Weighted least squares regression	*WLR-F&T*	Mean rank (Eq. (4a))	E_1	Faucher and Tyson (Eq. 7b)
6			Hazen's (Eq. (4b))	E_2	
7			Median rank (Eq. (4c))	E_3	
8			Small sample (Eq. (4d))	E_4	
9	Weighted least squares regression	*WLR-B*	Mean rank (Eq. (4a))	E_1	Bergman (Eq. 7a)
10			Hazen's (Eq. (4b))	E_2	
11			Median rank (Eq. (4c))	E_3	
12			Small sample (Eq. (4d))	E_4	
13	Good linear unbiased estimator	*GLUEs*	Hazen's (Eq. (4b))	E_2	–
14	Maximum likelihood estimation	*MLE*	Hazen's (Eq. (4b))	E_2	–

Table 3
Best performing estimator for data sets of different size.

n	Series	LR								WLR-F&T								WLR-B							
		p_{AD}				$\sigma_{0.001}$				p_{AD}				$\sigma_{0.001}$				p_{AD}				$\sigma_{0.001}$			
		E1	E2	E3	E4	E1	E2	E3	E4	E1	E2	E3	E4	E1	E2	E3	E4	E1	E2	E3	E4	E1	E2	E3	E4
10	AR	2	4	1	3	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2
11	SA8	4	2	3	1	4	1	3	2	4	1	3	2	4	1	3	2	2	4	1	3	4	1	3	2
	SA25	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2
12	SA6	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	2	3	1	4	1	3	2
	SA11	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	3	1	2	4	1	3	2
13	NA_b	4	2	3	1	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2
	SA9	1	4	2	3	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2
14	SA2	1	4	2	3	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2
	SA5	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2
	SA7	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2
	SA12	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2
	SA13	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2
	SA16	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	2	3	1	4	1	3	2
	SA18	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2
	SA19	4	3	2	1	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2
	SA22	4	3	1	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2
	SA23	2	4	1	3	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2
	SA24	1	4	2	3	4	1	3	2	4	1	3	2	4	1	3	2	3	4	1	2	4	1	3	2
	SC2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2
15	SA1	1	4	2	3	4	1	3	2	4	1	3	2	4	1	3	2	4	2	3	1	4	1	3	2
	SA3	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2
	SA4	3	4	1	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2
	SA10	4	2	3	1	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2
	SA14	1	4	2	3	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2
	SA15	4	2	3	1	4	1	3	2	4	1	3	2	4	1	3	2	4	3	1	2	4	1	3	2
	SA17	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	2	4	1	3	4	1	3	2
	SA20	4	3	1	2	4	1	3	2	4	1	3	2	4	1	3	2	4	3	2	1	4	1	3	2
	SA21	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2
	SC1	1	4	2	3	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2
18	NA_a	1	4	2	3	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2	4	1	3	2

goodness of fit to the Weibull distribution denotes the most accurate statistical analysis method and; (b) lower strength estimates are always more conservative for engineering applications and are therefore, preferred in design. True values for all methods are available in Appendix A, Tables A2a–c.

The following can be observed for E_1-E_4 in terms of:

- *goodness-of-fit (p_{AD}):* The ranking of probability estimators is consistent for all data sets of *WLR-F&T* and is therefore, independent of the sample sizes considered in this study. Hazen's estimator, E_2, delivers the best goodness-of-fit for *WLR-F&T*, followed by the small sample estimator E_4, the median rank estimator E_3 and finally the mean estimator E_1. The difference in goodness-of-fit between E_2 and E_1 is $12.5\% \leq \Delta p_{AD,E2\text{-}E1}/p_{AD,E2} \leq 51.8\%$. Additionally, Hazen's estimator E_2 provided the best fit for 70% of the *WLR-B* data sets and for 43.3% of the *LR* data sets. This low percentage of effectiveness is not essential in the case of *LR* as its goodness-of-fit is largely poorer with respect to WLR-F&T (Table 4). Therefore, LR will be omitted for the rest of this study.

Table 4
Goodness-of-fit of max LR (LR-E1, LR-E2, LR-E3 and LR-E4) vs. WLR-F&T-E2 methods (highest value for each series shown in bold).

n	Series	WLR-F&T-E2 p_{AD}	max LR p_{AD}
10	AR	**0.666**	0.633
11	SA8	**0.117**	0.102
	SA25	**0.623**	0.577
12	SA6	**0.686**	0.685
	SA11	**0.599**	0.592
13	NA_b	**0.796**	0.796
	SA9	**0.194**	0.188
14	SA2	0.359	**0.360**
	SA5	**0.172**	0.118
	SA7	**0.784**	0.776
	SA12	**0.189**	0.127
	SA13	**0.507**	0.500
	SA16	**0.569**	0.198
	SA18	**0.800**	0.800
	SA19	**0.672**	0.668
	SA22	**0.482**	0.482
	SA23	**0.575**	0.535
	SA24	**0.151**	0.130
	SC2	0.238	**0.244**
15	SA1	**0.489**	0.460
	SA3	**0.641**	0.357
	SA4	**0.729**	0.729
	SA10	**0.212**	0.212
	SA14	**0.728**	0.715
	SA15	**0.324**	0.295
	SA17	**0.173**	0.150
	SA20	**0.148**	0.126
	SA21	**0.580**	0.570
	SC1	**0.178**	0.151
18	NA_a	**0.097**	0.085

- *strength estimates ($\sigma_{0.001}$):* the ranking of probability estimators is identical for all data sets and methods of Table 3 and it is therefore, independent of the sample sizes or the estimation methods considered in this study. Hazen's estimator, E_2, provides the highest strength estimates, followed by the small size estimator, E_4, the median rank estimator, E_3, and finally the mean estimator, E_1. Therefore, the mean estimator E_1 provides the most conservative strength estimates. The difference in strength at low probabilities of failure between E_2 and E_1 is $1.8\% \leq \Delta\sigma_{0.001,E2\text{-}E1}/\sigma_{0.001,E2} \leq 25.6\%$.

Conservative strength estimates are traditionally preferred for engineering purposes. However, the most conservative strength estimates for the data sets used in this study were produced by the probability estimator with the poorest goodness-of-fit. Therefore, these probability estimators are not deemed effective enough and are therefore, not recommended. Overall, E_2 is the best performing estimator and will be subsequently used for the rest of this study.

4.2. Performance of estimation methods for the Weibull parameters

Similarly to Section 4.1, the different estimation methods (*WLR-F&T*, *WLR-B*, *GLUEs* and *MLE*) are ranked in ascending order (Table 5) based on their Anderson Darling goodness of fit, p_{AD}, and their strength estimates, $\sigma_{0,001}$. Again *1* denotes the method with the highest goodness of fit, p_{AD} and the highest strength estimate of $\sigma_{0.001}$ and *4* the lowest goodness of fit, p_{AD} and the lowest strength estimate of $\sigma_{0.001}$. Similarly, the performance of each estimation method is evaluated based on the number of times it scores 1st in terms of goodness of fit or 4th in terms of design strength estimates, providing the best fit to the Weibull distribution or

Table 5
Ranking of estimation methods in terms of p_{AD} and $\sigma_{0.001}$.

n	Series	p_{AD}				$\sigma_{0.001}$			
		WLR-F&T-E2	*WLR-B-E2*	*GLUEs*	*MLE*	*WLR-F&T-E2*	*WLR-B-E2*	*GLUEs*	*MLE*
10	AR	1	2	4	3	3	2	4	1
11	SA8	1	4	3	2	4	1	2	3
	SA25	1	2	4	3	3	4	2	1
12	SA6	1	2	4	3	3	2	4	1
	SA11	1	4	3	2	4	1	2	3
13	NA_b	1	2	3	4	3	4	2	1
	SA9	1	3	4	2	3	4	1	2
14	SA2	1	4	3	2	3	4	2	1
	SA5	4	1	3	2	3	2	4	1
	SA7	1	3	2	4	3	4	2	1
	SA12	2	3	4	1	3	2	4	1
	SA13	1	2	3	4	3	4	2	1
	SA16	3	1	4	2	3	1	4	2
	SA18	1	2	3	4	2	4	3	1
	SA19	1	3	4	2	3	4	2	1
	SA22	1	2	3	4	3	4	2	1
	SA23	1	3	4	2	4	3	2	1
	SA24	1	3	4	2	3	1	4	2
	SC2	1	2	4	3	3	4	2	1
15	SA1	1	2	4	3	4	2	3	1
	SA3	2	1	4	3	4	2	3	1
	SA4	1	3	2	4	3	4	2	1
	SA10	1	2	4	3	3	4	1	2
	SA14	1	2	3	4	3	4	2	1
	SA15	1	2	4	3	3	1	2	4
	SA17	1	3	4	2	3	1	2	4
	SA20	1	2	4	3	4	2	1	3
	SA21	2	3	1	4	4	3	2	1
	SC1	1	2	4	3	4	3	1	2
18	NA_a	1	3	4	2	3	4	2	1

the most conservative design strength respectively. True values are shown in Table A3, Appendix A.

The following can be concluded on the effectiveness of the different estimation methods (Table 5) in terms of their:

- *goodness-of-fit: WLR-F&T-E*$_2$ provides the best fit amongst the rest of the manual and computational methods for 83.3% of the 30 data sets. Similarly, *WLR-B-E*$_2$ is the most effective method for 10% of the data sets whilst *GLUEs* and *MLE* are most effective only for 3.3% of the data sets each. The difference in goodness-of-fit between *WLR-F&T* and *GLUEs* is $0.4\% \leq \Delta p_{AD,\text{WLPR-F\&T-GLUEs}}/p_{AD,E2} \leq 80.3\%$.
- *strength estimates: WLR-B-E*$_2$ provided the most conservative estimates for strengths at low probabilities of failure, $\sigma_{0.001}$ for 46.7% of the data sets, followed by *WLR-F&T-E*$_2$ for 26.7% of the data sets, *GLUEs* for 20% of the data sets and finally *MLE* for 6.7% of the data sets. However, *WLR-B-E*$_2$ provided a lower goodness-of-fit than *WLR-F&T-E*$_2$ and therefore, is considered inferior as its results are less reliable.

Overall, *WLR-F&T-E*$_2$ was found to be the most effective method for the small sized samples investigated in this study because it provided the best goodness-of-fit. *GLUE*s proposed for the statistical analysis of glass in EN 12603 [12], was one of the least effective estimation methods. In particular, the difference between *WLR-F&T-E*$_2$ and *GLUEs* can be as high as 80% for p_{AD} (SA10, Fig. 3a, b) and 65% for $\sigma_{0.001}$ estimates (SC2, Fig. 3c, d) (true values shown in Table A3, Appendix A). *MLE*, proposed for the statistical analysis of glass in ASTM C1239-1 [13] and DIN843-5 [14], was found to perform better than *GLUEs* for the majority of series. However, *WLR-F&T-E*$_2$ is superior to *MLE*. *WLR-F&T-E*$_2$ improved the goodness-of-fit up to 63% (SA10, Fig. 3a, b) and provided up to 73% more conservative strength estimates of $\sigma_{0.001}$ with respect to *MLE* (SC2, Fig. 3c, d). The difference in p_{AD} and $\sigma_{0.001}$ between *WLR-F&T-E*$_2$ and *GLUE*s / *MLE* increases as the shape factor of the Weibull distribution decreases (Table A3, Appendix A). Low shape factors are typical of naturally aged glass. Therefore, in this instance the choice of estimation method becomes even more important.

5. Conclusions

This study reviewed the statistical analysis of glass strength data with a Weibull distribution. The following methods were considered for the estimation of the Weibull parameters: (a) Unweighted least squares regression (*LR*) using 4 probability estimators namely, mean rank (*E*$_1$), Hazen's (*E*$_2$), median rank (*E*$_3$) and small sample (*E*$_4$) estimators; (b) Weighted Least Squares Regression (*WLR*) using Bergman's (*B*) and Faucher and Tyson's (*F&T*) weight functions and 4 probability estimators namely, mean rank (*E*$_1$), Hazen's (*E*$_2$), median rank (*E*$_3$) and small sample (*E*$_4$) estimators; (c) Good Linear Unbiased Estimators (*GLUEs*); (d) Maxi-

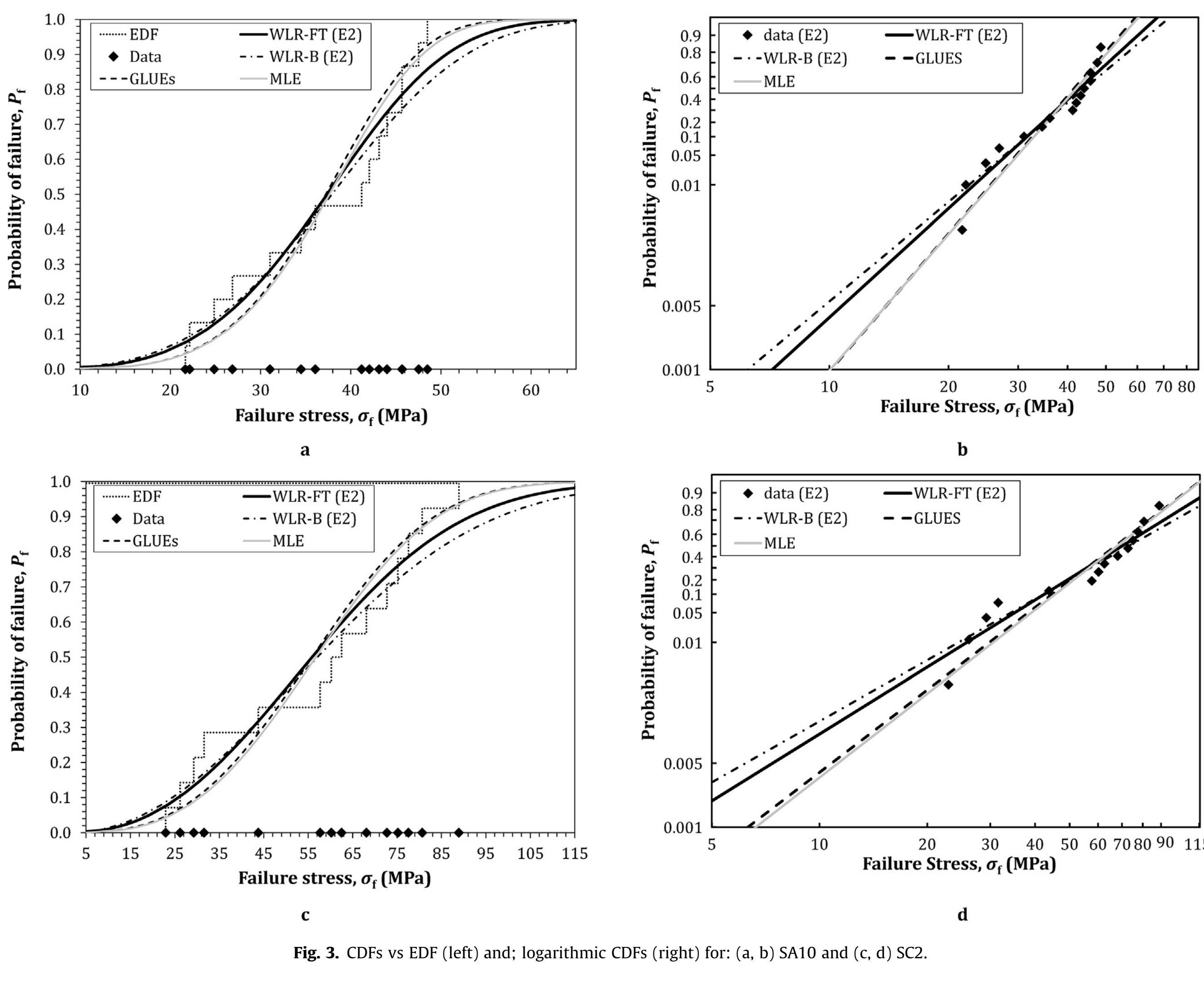

Fig. 3. CDFs vs EDF (left) and; logarithmic CDFs (right) for: (a, b) SA10 and (c, d) SC2.

mum Likelihood Estimation (*MLE*) and; (e) Method of Moments Estimation (*MME*). A total of 14 combinations of estimation methods, probability estimators and weight functions were subsequently implemented for the statistical analysis of small sized data sets of glass strength. The accuracy of these combinations were evaluated based on their goodness-of-fit. The Anderson Darling goodness-of-fit is used in this study because is more sensitive to the lower parts of the *CDF* which correspond to low probabilities of failure that are typically used for engineering design and was therefore, preferred over other goodness-of-fit tests for glass strength.

The analysis of the 30 series of real strength data used in this study shows that glass strength data can be successfully described with a 2-parameter Weibull distribution. This concurs with the established approach adopted by the glass engineering community.

The Weighted Least Squares Regression (*WLR*) in conjunction with Faucher and Tyson's weight function (*F&T*) and Hazen's probability estimator (E_2) were identified as the most accurate estimation method. *WLR-F&T-*E_2 outperformed the Maximum Likelihood Estimation (*MLE*) and the Good Linear Unbiased Estimators (*GLUEs*) prescribed in CEN 12603 [12] and ASTM C1239-13 [13] and DIN843-5 [14] respectively for the analysis of glass strength data. In fact, for the glass strength data considered in this study, *WLR-F&T-*E_2 provided up to 80% and 63% improvement in the goodness-of-fit with respect to the *GLUEs* and *MLE*, respectively, and up to 65% and 73% more conservative estimates of strength with respect to *GLUEs* and *MLE*, respectively.

Therefore, *WLR-F&T-*E_2 is recommended for small sized samples of glass strength data because it firstly produces a better goodness-of-fit than other manual calculations and computational methods, and secondly it is simpler to implement.

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Appendix A

Appendix A comprises the raw glass strength data (Table A1) and also the results from each statistical analysis method (Tables A2a–c and A3) for all series.

The discrepancy noticed in some series between the range of raw strength (Table A1) and the design strength (Tables A2 and A3) is a function of the scatter/uncertainty in the raw strength data which can be expressed in terms of the coefficient of variation, *CV*. The higher the *CV* in the data (e.g. series NA_a), the lower the shape factor, β of the Weibull distribution (and consequently the gradient of the distribution) and eventually, the lower the strength at design level.

Table A1
Raw strength data.

Failure Strength, σf (MPa)															
n	*n* = 10	*n* = 11		*n* = 12		*n* = 13		*n* = 14							
a/a	AR	SA8	SA25	SA6	SA11	NA-b	SA9	SA2	SA5	SA7	SA12	SA13	SA16	SA18	SA19
1	93.24	44.89	14.20	39.22	39.44	29.41	43.10	40.96	38.87	38.54	31.80	35.72	28.54	20.12	26.57
2	94.57	48.03	16.70	41.92	41.77	33.42	43.16	42.09	39.94	40.95	36.09	36.30	32.89	23.22	29.43
3	99.89	49.34	16.96	43.23	42.65	41.69	43.85	42.14	45.81	44.04	36.52	38.64	34.31	25.50	29.60
4	103.31	49.34	17.41	45.51	43.86	43.64	44.83	42.25	47.02	44.44	36.70	38.84	34.47	25.50	30.24
5	105.22	50.35	18.57	47.14	44.38	47.74	45.21	42.44	47.71	45.33	37.14	41.26	35.25	25.58	30.35
6	108.83	50.72	18.93	48.19	44.60	48.69	45.69	42.77	47.85	45.76	37.16	42.07	35.63	26.00	30.80
7	112.90	50.92	19.21	48.36	44.78	50.65	45.84	44.32	48.08	46.31	37.25	42.58	35.75	26.98	30.82
8	115.41	50.98	19.74	48.66	45.84	56.16	46.10	44.81	48.31	47.70	37.33	42.95	35.82	27.65	31.22
9	121.42	51.22	19.76	49.37	45.93	62.19	48.55	45.09	48.57	48.76	37.83	43.22	35.88	28.49	32.09
10	129.66	52.17	19.78	49.51	46.05	62.82	48.65	45.66	49.08	48.76	38.06	44.14	36.31	28.92	32.69
11		55.51	20.97	50.91	46.27	62.98	48.97	45.74	49.31	48.90	38.40	44.24	37.10	29.34	33.00
12				53.58	48.61	66.74	49.06	46.19	51.57	50.17	39.60	44.38	37.39	30.59	33.59
13						75.80	51.61	47.31	51.85	50.47	39.78	45.45	37.93	30.66	33.64
14								48.14	52.88	51.26	40.11	45.67	38.45	32.02	34.90

Failure Strength, σ_f (MPa)															
n	*n* = 14				*n* = 15										*n* = 18
a/a	SA22	SA23	SA24	SC2	SA1	SA3	SA4	SA10	SA14	SA15	SA17	SA20	SA21	SC1	NA-a
1	12.81	13.02	16.16	22.92	44.61	30.96	30.38	21.66	33.22	36.54	16.65	27.84	16.29	18.61	24.12
2	13.18	14.57	17.20	26.16	44.79	35.50	30.96	22.14	33.32	36.62	16.76	30.00	17.95	19.77	24.13
3	13.50	14.98	17.93	29.24	44.92	38.06	32.16	24.85	34.08	37.66	18.05	31.40	20.80	19.91	28.52
4	13.71	15.26	18.09	31.57	45.69	38.74	32.84	26.88	34.87	37.85	18.07	31.79	21.52	19.98	29.18
5	14.35	16.03	18.48	43.76	46.17	39.39	32.98	31.05	35.21	38.08	18.25	31.96	23.92	20.69	29.67
6	14.56	16.14	18.81	57.67	46.26	39.92	33.35	34.49	35.62	38.11	18.30	32.03	24.73	20.74	30.48
7	14.60	16.17	19.18	60.21	46.33	40.16	33.98	36.06	36.20	38.58	18.38	32.25	26.80	21.81	32.98
8	15.23	16.47	19.20	62.53	46.75	40.36	34.30	41.21	36.87	38.59	18.45	32.28	27.00	22.73	35.91
9	15.42	17.49	19.24	68.09	46.80	40.60	34.96	42.06	37.10	38.76	18.64	32.70	27.60	23.83	35.92
10	15.53	17.61	19.33	72.74	46.85	42.20	35.88	43.15	37.46	38.84	18.70	33.99	27.92	24.25	36.38
11	15.62	18.30	19.54	75.19	47.28	42.24	36.00	44.03	37.47	39.13	18.94	34.29	27.95	24.59	37.60
12	15.71	18.35	20.70	77.57	47.72	42.51	36.17	45.68	38.22	39.20	19.03	34.78	27.95	25.20	37.70
13	15.82	19.51	21.51	80.66	48.02	42.65	36.38	45.71	38.97	39.24	19.17	34.95	30.25	25.23	39.71
14	16.10	20.91	22.57	88.94	48.68	43.78	36.76	47.54	39.41	39.29	19.53	35.18	31.06	26.88	49.10
15					49.11	44.25	37.50	48.51	39.48	40.61	20.53	38.48	33.31	30.75	52.43
16															52.46
17															52.61
18															61.72

Table A2a
Results of the LR method for different estimators.

LR																	
n	Series	Shape factor, β				Scale factor, θ				Goodness of fit, p_{AD}				Des. Strength, $\sigma_{0.001}$ (MPa)			
		E1	E2	E3	E4	E1	E2	E3	E4	E1	E2	E3	E4	E1	E2	E3	E4
10	AR	9.2	11.0	10.1	10.4	113.9	113.4	113.6	113.5	0.623	0.532	0.633	0.610	53.6	60.6	57.5	58.6
11	SA8	18.3	22.4	20.4	21.1	51.6	51.5	51.6	51.5	0.071	0.101	0.099	0.102	35.4	37.8	36.8	37.2
	SA25	9.0	10.9	10.0	10.3	19.3	19.2	19.3	19.3	0.268	0.577	0.456	0.505	8.9	10.2	9.7	9.9
12	SA6	11.5	13.8	12.7	13.1	49.1	48.9	49.0	48.9	0.456	0.685	0.622	0.653	26.9	29.6	28.4	28.9
	SA11	18.4	22.1	20.3	21.0	45.7	45.6	45.6	45.6	0.372	0.592	0.530	0.561	31.4	33.4	32.5	32.8
13	NA_b	3.7	4.4	4.1	4.2	58.1	57.6	57.8	57.7	0.683	0.787	0.787	0.796	9.1	12.0	10.6	11.1
	SA9	17.5	20.5	19.1	19.6	47.8	47.7	47.8	47.7	0.188	0.108	0.162	0.145	32.2	34.0	33.3	33.5
14	SA2	20.2	23.5	22.0	22.5	45.4	45.3	45.3	45.3	0.360	0.247	0.335	0.310	32.2	33.7	33.1	33.3
	SA5	11.2	13.3	12.3	12.7	49.7	49.5	49.6	49.5	0.033	0.118	0.072	0.087	26.8	29.5	28.3	28.7
	SA7	12.7	14.9	13.9	14.2	48.3	48.1	48.2	48.2	0.523	0.776	0.708	0.741	28.0	30.3	29.3	29.7
	SA12	17.1	20.6	18.9	19.5	38.5	38.4	38.4	38.4	0.030	0.127	0.072	0.090	25.7	27.4	26.7	26.9
	SA13	12.8	15.0	14.0	14.4	43.4	43.3	43.3	43.3	0.303	0.500	0.439	0.467	25.3	27.3	26.5	26.8
	SA16	13.2	15.9	14.6	15.1	36.7	36.6	36.6	36.6	0.032	0.198	0.092	0.123	21.8	23.7	22.8	23.1
	SA18	8.4	9.9	9.2	9.4	28.7	28.6	28.6	28.6	0.563	0.800	0.743	0.773	12.6	14.2	13.5	13.8
	SA19	14.6	17.3	16.0	16.5	32.4	32.3	32.3	32.3	0.544	0.652	0.659	0.668	20.2	21.7	21.0	21.3
	SA22	13.8	16.2	15.1	15.4	15.2	15.2	15.2	15.2	0.417	0.456	0.482	0.482	9.2	9.9	9.6	9.7
	SA23	8.3	9.7	9.0	9.2	17.7	17.6	17.7	17.7	0.523	0.441	0.535	0.513	7.7	8.6	8.2	8.4
	SA24	11.7	13.8	12.8	13.2	19.9	19.9	19.9	19.9	0.130	0.080	0.118	0.106	11.1	12.0	11.6	11.8
	SC2	2.2	2.6	2.4	2.4	65.8	64.8	65.2	65.0	0.174	0.244	0.232	0.241	2.8	4.4	3.6	3.9
15	SA1	35.3	40.8	38.3	39.1	47.3	47.3	47.3	47.3	0.460	0.313	0.423	0.391	38.9	39.9	39.5	39.6
	SA3	11.1	13.2	12.2	12.5	41.8	41.7	41.7	41.7	0.077	0.357	0.198	0.249	22.4	24.7	23.7	24.0
	SA4	16.3	19.0	17.7	18.2	35.3	35.3	35.3	35.3	0.693	0.665	0.729	0.717	23.2	24.5	23.9	24.1
	SA10	3.6	4.2	3.9	4.0	41.2	40.8	41.0	40.9	0.163	0.208	0.207	0.212	6.0	7.8	7.0	7.3
	SA14	18.1	20.9	19.6	20.0	37.5	37.4	37.5	37.4	0.715	0.616	0.713	0.689	25.6	26.9	26.3	26.5
	SA15	37.3	43.6	40.6	41.6	39.0	39.0	39.0	39.0	0.220	0.294	0.286	0.295	32.4	33.2	32.9	33.0
	SA17	19.2	22.5	20.9	21.4	19.0	18.9	19.0	18.9	0.098	0.150	0.139	0.146	13.2	13.9	13.6	13.7
	SA20	13.4	15.7	14.6	15.0	34.1	34.0	34.1	34.1	0.108	0.111	0.126	0.124	20.3	21.9	21.2	21.5
	SA21	5.1	6.0	5.6	5.7	27.9	27.7	27.7	27.7	0.333	0.570	0.493	0.527	7.3	8.8	8.1	8.3
	SC1	7.2	8.3	7.8	8.0	24.5	24.4	24.4	24.4	0.151	0.098	0.137	0.125	9.4	10.6	10.1	10.3
18	NA_a	3.7	4.2	4.0	4.1	42.5	42.2	42.3	42.3	0.085	0.028	0.054	0.044	6.7	8.3	7.5	7.8

Table A2b
Results of the WLR-F&T method for different estimators.

WLR-F&T																	
n	Series	Shape factor, β				Scale factor, θ				Goodness of fit, p_{AD}				Des. Strength, $\sigma_{0.001}$ (MPa)			
		E1	E2	E3	E4	E1	E2	E3	E4	E1	E2	E3	E4	E1	E2	E3	E4
10	AR	8.2	9.4	8.8	9.0	113.7	112.9	113.2	113.1	0.505	0.666	0.623	0.644	48.8	54.0	51.8	52.6
11	SA8	17.5	20.9	19.2	19.8	51.5	51.3	51.4	51.3	0.066	0.117	0.099	0.107	34.7	36.8	35.9	36.2
	SA25	9.9	11.9	11.0	11.3	19.3	19.2	19.2	19.2	0.449	0.623	0.583	0.604	9.6	10.7	10.3	10.4
12	SA6	12.0	14.0	13.1	13.4	49.0	48.8	48.9	48.9	0.534	0.686	0.654	0.672	27.5	29.8	28.8	29.2
	SA11	18.9	22.3	20.8	21.3	45.6	45.5	45.6	45.5	0.437	0.599	0.563	0.583	31.7	33.4	32.7	32.9
13	NA_b	3.7	4.2	4.0	4.1	58.0	57.5	57.7	57.7	0.687	0.796	0.776	0.788	9.1	11.2	10.3	10.6
	SA9	15.1	16.7	16.0	16.2	47.7	47.6	47.6	47.6	0.135	0.194	0.176	0.184	30.2	31.5	30.9	31.1
14	SA2	17.7	19.5	18.7	19.0	45.3	45.2	45.2	45.2	0.258	0.359	0.328	0.341	30.7	31.7	31.3	31.4
	SA5	12.7	14.8	13.9	14.2	49.4	49.3	49.3	49.3	0.090	0.172	0.141	0.154	28.7	30.9	30.0	30.3
	SA7	13.5	15.3	14.5	14.8	48.3	48.2	48.2	48.2	0.663	0.784	0.761	0.774	28.9	30.7	30.0	30.2
	SA12	19.5	22.7	21.3	21.8	38.3	38.2	38.2	38.2	0.100	0.189	0.159	0.172	26.9	28.2	27.6	27.8
	SA13	13.5	15.1	14.4	14.7	43.4	43.3	43.3	43.3	0.381	0.507	0.473	0.489	25.9	27.4	26.8	27.1
	SA16	16.5	19.9	18.4	18.9	36.5	36.4	36.4	36.4	0.274	0.569	0.468	0.512	24.0	25.8	25.0	25.3
	SA18	8.8	9.9	9.4	9.6	28.6	28.5	28.6	28.5	0.669	0.800	0.774	0.788	13.0	14.2	13.7	13.9
	SA19	14.4	16.2	15.4	15.7	32.3	32.3	32.3	32.3	0.535	0.672	0.641	0.658	20.0	21.1	20.6	20.8
	SA22	13.6	15.1	14.5	14.7	15.2	15.2	15.2	15.2	0.394	0.482	0.462	0.473	9.2	9.6	9.4	9.5
	SA23	7.5	8.4	8.0	8.2	17.7	17.6	17.6	17.6	0.430	0.575	0.537	0.556	7.0	7.7	7.4	7.6
	SA24	10.3	11.5	11.0	11.2	19.8	19.8	19.8	19.8	0.098	0.151	0.134	0.142	10.1	10.9	10.5	10.7
	SC2	2.2	2.4	2.3	2.4	65.6	65.0	65.2	65.1	0.177	0.238	0.220	0.229	2.8	3.8	3.4	3.6
15	SA1	31.4	34.7	33.2	33.8	47.3	47.2	47.3	47.3	0.368	0.489	0.456	0.471	38.0	38.7	38.4	38.5
	SA3	13.4	15.8	14.7	15.1	41.6	41.5	41.6	41.6	0.399	0.641	0.572	0.605	24.9	26.8	26.0	26.3
	SA4	15.7	17.3	16.6	16.9	35.3	35.3	35.3	35.3	0.638	0.729	0.711	0.721	22.8	23.7	23.3	23.4
	SA10	3.6	4.0	3.8	3.9	41.2	41.0	41.1	41.0	0.164	0.212	0.199	0.205	6.1	7.2	6.7	6.9
	SA14	16.8	18.4	17.7	18.0	37.5	37.4	37.5	37.4	0.628	0.728	0.705	0.717	24.9	25.7	25.4	25.5
	SA15	35.8	41.0	38.6	39.4	39.0	38.9	38.9	38.9	0.199	0.324	0.283	0.301	32.1	32.9	32.5	32.7
	SA17	18.4	21.2	19.8	20.3	18.9	18.9	18.9	18.9	0.093	0.173	0.143	0.156	13.0	13.6	13.3	13.4
	SA20	12.1	13.8	13.0	13.3	34.0	33.9	33.9	33.9	0.083	0.148	0.124	0.134	19.2	20.5	20.0	20.2
	SA21	5.5	6.2	5.9	6.0	27.8	27.6	27.7	27.7	0.451	0.580	0.552	0.567	7.8	9.1	8.5	8.7
	SC1	6.2	6.9	6.6	6.7	24.3	24.1	24.2	24.2	0.104	0.178	0.150	0.161	7.9	8.8	8.4	8.6
18	NA_a	3.2	3.4	3.3	3.3	42.1	41.7	41.9	41.8	0.074	0.097	0.091	0.094	4.7	5.4	5.1	5.2

Table A2c
Results of the WLR-B method for different estimators.

WLR-B																	
n	Series	Shape factor, β				Scale factor, θ				Goodness of fit, p_{AD}				Des. Strength, $\sigma_{0.001}$ (MPa)			
		E1	E2	E3	E4	E1	E2	E3	E4	E1	E2	E3	E4	E1	E2	E3	E4
10	AR	8.2	9.4	8.9	9.1	113.1	112.3	112.7	112.5	0.523	0.637	0.619	0.631	48.9	54.1	51.9	52.7
11	SA8	20.0	25.9	23.0	24.0	51.3	51.1	51.2	51.2	0.110	0.061	0.110	0.096	36.3	39.2	37.9	38.4
	SA25	10.1	11.6	11.0	11.2	19.3	19.3	19.3	19.3	0.456	0.607	0.566	0.585	9.8	10.6	10.3	10.4
12	SA6	12.5	14.6	13.7	14.0	49.0	48.9	48.9	48.9	0.596	0.681	0.679	0.686	28.2	30.4	29.5	29.9
	SA11	20.3	24.0	22.4	23.0	45.6	45.5	45.5	45.5	0.538	0.569	0.599	0.596	32.5	34.1	33.5	33.7
13	NA_b	3.7	4.0	3.9	3.9	58.2	57.8	58.0	57.9	0.654	0.779	0.745	0.760	8.9	10.4	9.8	10.0
	SA9	14.4	15.5	15.1	15.3	47.6	47.5	47.6	47.6	0.100	0.143	0.128	0.134	29.5	30.5	30.1	30.2
14	SA2	16.9	18.1	17.6	17.8	45.3	45.2	45.2	45.2	0.187	0.271	0.241	0.253	30.1	30.9	30.6	30.7
	SA5	13.5	15.8	14.8	15.2	49.4	49.3	49.4	49.3	0.129	0.191	0.178	0.185	29.7	31.8	31.0	31.3
	SA7	13.4	14.7	14.2	14.4	48.3	48.3	48.3	48.3	0.630	0.761	0.724	0.740	28.9	30.2	29.7	29.9
	SA12	20.4	23.1	22.0	22.4	38.2	38.2	38.2	38.2	0.129	0.174	0.167	0.172	27.3	28.3	27.9	28.1
	SA13	13.2	14.5	14.0	14.2	43.5	43.5	43.5	43.5	0.322	0.450	0.407	0.425	25.8	27.0	26.5	26.7
	SA16	18.3	21.7	20.2	20.8	36.5	36.4	36.4	36.4	0.462	0.591	0.583	0.594	25.0	26.5	25.9	26.1
	SA18	8.7	9.6	9.3	9.4	28.6	28.5	28.5	28.5	0.669	0.786	0.756	0.770	13.0	13.9	13.5	13.7
	SA19	14.2	15.5	15.0	15.1	32.3	32.2	32.2	32.2	0.508	0.629	0.596	0.611	19.9	20.6	20.3	20.4
	SA22	12.7	13.6	13.3	13.4	15.3	15.3	15.3	15.3	0.267	0.371	0.333	0.348	8.9	9.2	9.1	9.1
	SA23	7.7	8.5	8.2	8.3	17.6	17.5	17.5	17.5	0.463	0.540	0.532	0.539	7.1	7.8	7.5	7.6
	SA24	10.9	12.5	11.8	12.0	19.7	19.6	19.7	19.7	0.123	0.116	0.134	0.130	10.5	11.3	11.0	11.1
	SC2	2.1	2.2	2.2	2.2	67.6	67.3	67.4	67.4	0.111	0.164	0.144	0.152	2.4	3.0	2.8	2.8
15	SA1	31.9	35.1	33.7	34.3	47.2	47.2	47.2	47.2	0.387	0.452	0.446	0.452	38.0	38.8	38.5	38.6
	SA3	14.2	16.2	15.4	15.7	41.7	41.6	41.6	41.6	0.508	0.651	0.620	0.636	25.7	27.1	26.6	26.8
	SA4	14.9	15.8	15.5	15.6	35.4	35.3	35.4	35.4	0.517	0.652	0.607	0.625	22.2	22.9	22.6	22.7
	SA10	3.4	3.6	3.5	3.6	42.0	42.0	42.0	42.0	0.098	0.141	0.125	0.131	5.4	6.2	5.9	6.0
	SA14	16.3	17.5	17.0	17.2	37.5	37.4	37.5	37.4	0.561	0.691	0.650	0.668	24.5	25.2	25.0	25.1
	SA15	39.2	45.3	42.6	43.6	38.9	38.9	38.9	38.9	0.296	0.301	0.330	0.325	32.6	33.4	33.1	33.2
	SA17	20.8	24.7	22.9	23.6	18.9	18.8	18.8	18.8	0.164	0.124	0.168	0.156	13.5	14.2	13.9	14.0
	SA20	12.9	14.5	13.8	14.1	33.9	33.7	33.8	33.8	0.113	0.136	0.137	0.139	19.8	21.0	20.5	20.7
	SA21	5.6	6.2	5.9	6.1	27.9	27.8	27.8	27.8	0.469	0.577	0.553	0.565	8.1	9.2	8.7	8.9
	SC1	6.3	7.0	6.7	6.8	24.1	24.0	24.0	24.0	0.120	0.166	0.154	0.160	8.1	8.9	8.6	8.7
18	NA_a	3.0	3.2	3.1	3.1	41.7	41.3	41.5	41.4	0.049	0.064	0.059	0.061	4.2	4.7	4.5	4.6

Table A3
Results of different estimation methods.

n	Series	Shape factor, β					Scale factor, θ					Goodness of fit, p_{AD}					Design Strength, $\sigma_{0.001}$ (MPa)				
		a*	b*	c*	d*	e*	a*	b*	c*	d*	e*	a*	b*	c*	d*	e*	a*	b*	c*	d*	e*
10	AR	9.2	9.4	9.4	9.0	10.2	0.0	112.9	112.3	115.0	113.6	0.000	0.666	0.637	0.458	0.632	0.0	54.0	54.1	53.6	57.7
11	SA8	18.3	20.9	25.9	20.6	20.7	51.6	51.3	51.1	51.7	51.5	0.071	0.117	0.061	0.077	0.109	35.4	36.8	39.2	36.9	36.9
	SA25	9.0	11.9	11.6	13.1	13.1	19.3	19.2	19.3	19.1	19.2	0.268	0.623	0.607	0.529	0.563	8.9	10.7	10.6	11.3	11.3
12	SA6	11.5	14.0	14.6	13.6	14.7	49.1	48.8	48.9	49.0	48.8	0.456	0.686	0.681	0.669	0.679	26.9	29.8	30.4	29.5	30.5
	SA11	18.4	22.3	24.0	22.3	22.4	45.7	45.5	45.5	45.6	45.6	0.372	0.599	0.569	0.579	0.597	31.4	33.4	34.1	33.5	33.5
13	NA_b	3.7	4.2	4.0	4.5	4.6	58.1	57.5	57.8	57.7	57.5	0.683	0.796	0.779	0.776	0.748	9.1	11.2	10.4	12.4	12.8
	SA9	17.5	16.7	15.5	18.8	18.8	47.8	47.6	47.5	47.9	47.7	0.188	0.194	0.143	0.135	0.174	32.2	31.5	30.5	33.1	33.1
12	SA2	20.2	19.5	18.1	21.0	22.3	45.4	45.2	45.2	45.5	45.3	0.360	0.359	0.271	0.284	0.320	32.2	31.7	30.9	32.7	33.3
	SA5	11.2	14.8	15.8	14.7	16.7	49.7	49.3	49.3	49.4	49.2	0.033	0.172	0.191	0.175	0.176	26.8	30.9	31.8	30.9	32.6
	SA7	12.7	15.3	14.7	15.6	17.0	48.3	48.2	48.3	48.1	48.1	0.523	0.784	0.761	0.781	0.575	28.0	30.7	30.2	30.9	32.0
	SA12	17.1	22.7	23.1	21.4	24.9	38.5	38.2	38.2	38.4	38.2	0.030	0.189	0.174	0.149	0.193	25.7	28.2	28.3	27.8	29.0
	SA13	12.8	15.1	14.5	17.1	17.8	43.4	43.3	43.5	43.1	43.2	0.303	0.507	0.450	0.385	0.361	25.3	27.4	27.0	28.8	29.3
	SA16	13.2	19.9	21.7	19.3	21.2	36.7	36.4	36.4	36.4	36.4	0.032	0.569	0.591	0.523	0.577	21.8	25.8	26.5	25.4	26.3
	SA18	8.4	9.9	9.6	9.7	10.7	28.7	28.5	28.5	28.7	28.5	0.563	0.800	0.786	0.777	0.772	12.6	14.2	13.9	14.1	14.9
	SA19	14.6	16.2	15.5	16.2	17.5	32.4	32.3	32.2	32.4	32.3	0.544	0.672	0.629	0.616	0.646	20.2	21.1	20.6	21.2	21.7
	SA22	13.8	15.1	13.6	17.6	18.0	15.2	15.2	15.3	15.2	15.2	0.417	0.482	0.371	0.309	0.278	9.2	9.6	9.2	10.2	10.4
	SA23	8.3	8.4	8.5	8.5	8.8	17.7	17.6	17.5	17.8	17.7	0.523	0.575	0.540	0.425	0.548	7.7	7.7	7.8	7.9	8.1
	SA24	11.7	11.5	12.5	11.1	12.0	19.9	19.8	19.6	20.1	19.9	0.130	0.151	0.116	0.062	0.138	11.1	10.9	11.3	10.8	11.2
	SC2	2.2	2.4	2.2	3.0	3.0	65.8	65.0	67.3	63.2	64.0	0.174	0.238	0.164	0.098	0.111	2.8	3.8	3.0	6.3	6.6
15	SA1	35.3	34.7	35.1	34.1	37.3	47.3	47.2	47.2	47.4	47.3	0.460	0.489	0.452	0.290	0.450	38.9	38.7	38.8	38.7	39.3
	SA3	11.1	15.8	16.2	16.2	17.0	41.8	41.5	41.6	41.4	41.4	0.077	0.641	0.651	0.583	0.609	22.4	26.8	27.1	27.0	27.6
	SA4	16.3	17.3	15.8	18.9	19.6	35.3	35.3	35.3	35.3	35.3	0.693	0.729	0.652	0.674	0.600	23.2	23.7	22.9	24.5	24.8
	SA10	3.6	4.0	3.6	5.0	5.0	41.2	41.0	42.0	40.1	40.5	0.163	0.212	0.141	0.042	0.076	6.0	7.2	6.2	10.1	10.1
	SA14	18.1	18.4	17.5	18.7	20.9	37.5	37.4	37.4	37.6	37.4	0.715	0.728	0.691	0.661	0.620	25.6	25.7	25.2	26.0	26.9
	SA15	37.3	41.0	45.3	43.8	40.6	39.0	38.9	38.9	39.0	39.0	0.220	0.324	0.301	0.274	0.297	32.4	32.9	33.4	33.3	32.9
	SA17	19.2	21.2	24.7	21.2	20.5	19.0	18.9	18.8	19.0	18.9	0.098	0.173	0.124	0.112	0.145	13.2	13.6	14.2	13.7	13.5
	SA20	13.4	13.8	14.5	14.5	13.7	34.1	33.9	33.7	34.2	34.1	0.108	0.148	0.136	0.093	0.127	20.3	20.5	21.0	21.2	20.5
	SA21	5.1	6.2	6.2	6.3	6.7	27.9	27.6	27.8	27.6	27.6	0.333	0.580	0.577	0.583	0.535	7.3	9.1	9.2	9.2	9.9
	SC1	7.2	6.9	7.0	7.4	7.1	24.5	24.1	24.0	24.6	24.4	0.151	0.178	0.166	0.096	0.163	9.4	8.8	8.9	9.7	9.3
18	NA_a	3.7	3.4	3.2	3.7	3.9	42.5	41.7	41.3	43.3	42.4	0.085	0.097	0.064	0.057	0.072	6.7	5.4	4.7	6.5	7.1

* a = LR-E1, b = WLR-F&T-E2, c = WLR-B-E2, d = GLUES & e = MLE.

Appendix B

In this Appendix, the glass strength data used in the present study are fitted to a normal and lognormal distribution using the Maximum Likelihood estimation method and compared to the one obtained from the Weibull distribution. The aim is to verify that the data from physical tests used in the present study conform to the findings of previous studies [2,3] i.e. that a 2-parameter Weibull distribution is more accurate in describing glass strength data than a normal distribution and is always more conservative in the tail of the distribution than a lognormal distribution (provided in Section 1).

Fig. B1 shows that the Weibull distribution provides a good fit i.e. the observed significance level is larger than the selected significance level ($p_{AD} \geq \alpha$=0.05) for all of the data series. However, the normal and the lognormal distribution fail to provide a good fit ($p_{AD} < \alpha$=0.05) for 7% and 20% of the data series, respectively. Therefore, the Weibull distribution is more accurate in describing

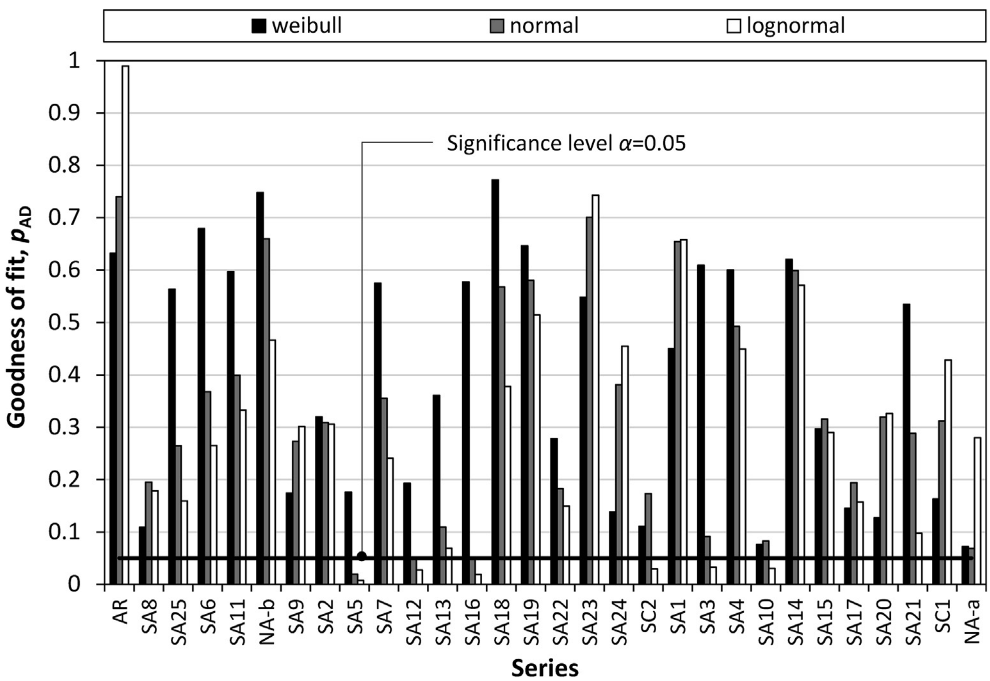

Fig. B1. Goodness of fit for all data series fitted to a Weibull, a normal and a lognormal distribution.

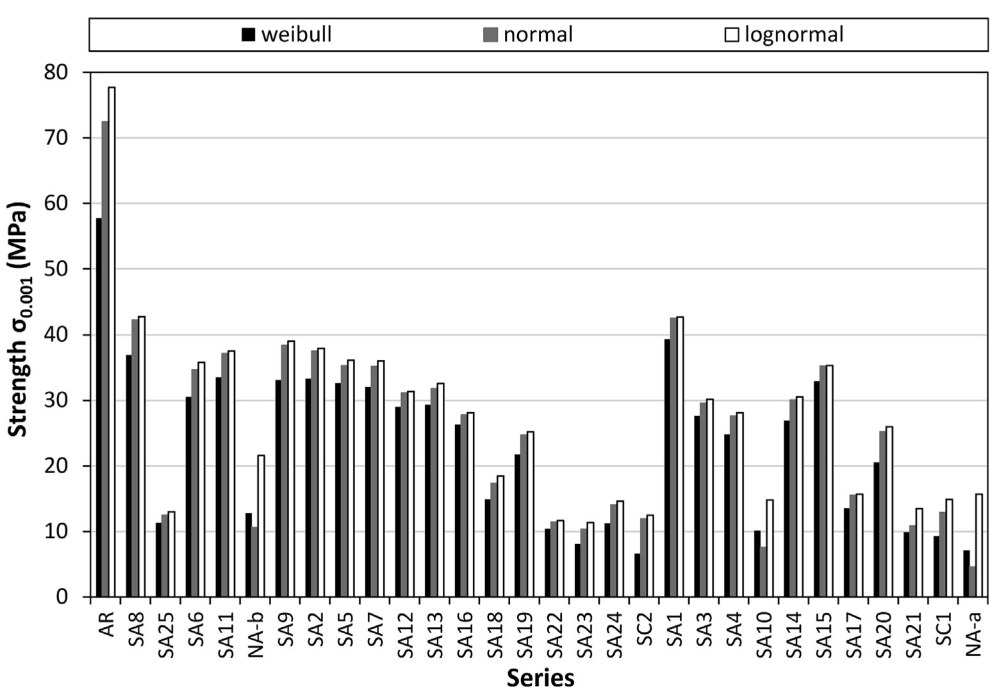

Fig. B2. Design strength for all data series fitted to a Weibull, a normal and a lognormal distribution.

glass strength data than a normal and a lognormal distribution. Additionally, it is found that the Weibull distribution provides the highest goodness-of-fit for 57% of the data series, followed by the lognormal distribution for 27% of the data series and subsequently by the normal distribution for 16% of the data series.

The Weibull distribution also provides the most conservative design strength values at a probability of failure of P_f=0.001 for 90% of the data series whilst the normal distribution provides the most conservative strength estimates for the remaining 10% (Fig. B2). In total, the strength estimated with the Weibull distribution is always more conservative than that estimated with the lognormal.

Therefore, this confirms that the conclusions of studies [2,3] that "a 2 parameter Weibull distribution is more accurate in describing glass strength data than a normal distribution and is always more conservative in the tail of the distribution than a lognormal distribution" are valid for the strength data used in the present study.

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