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# The Eigenvalues of Tridiagonal Sign Matrices are Dense in the Spectra of Periodic Tridiagonal Sign Operators

Raffael Hagger\*

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## Abstract

Chandler-Wilde, Chonchaiya and Lindner conjectured that the set of eigenvalues of finite tridiagonal sign matrices (i.e. plus and minus ones on the first sub- and superdiagonal, zeroes everywhere else) is dense in the set of spectra of periodic tridiagonal sign operators on the usual Hilbert space of square summable bi-infinite sequences. We give a simple proof of this conjecture. As a consequence we get that the set of eigenvalues of tridiagonal sign matrices is dense in the unit disk. In fact, a recent paper further improves this result, showing that this set of eigenvalues is dense in an even larger set.

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*Keywords:* sign matrix, tridiagonal, periodic operator, eigenvalues, spectrum

## 1 Introduction

Let  $n, m \in \mathbb{N}$ . For  $k, l \in \{\pm 1\}^n$  we define the corresponding (finite) tridiagonal matrix

$$A_{fin}^{k,l} := \begin{pmatrix} 0 & l_1 & & \\ k_1 & \ddots & \ddots & \\ & \ddots & \ddots & l_n \\ & & k_n & 0 \end{pmatrix} \in \mathbb{C}^{(n+1) \times (n+1)}.$$

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\*raffael.hagger@tuhh.de



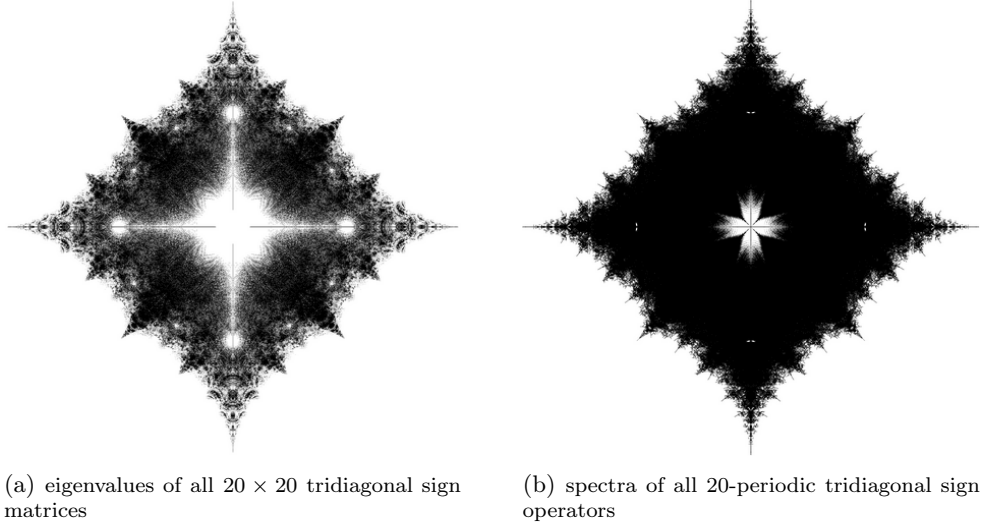


Figure 1

**Theorem 1.**  $\sigma_\infty$  is a dense subset of  $\pi_\infty$ .

First observe that it is enough to consider pairs  $(k, l)$  with  $l = (1, \dots, 1)$  since every  $A_{fin}^{k, l}$  is unitarily equivalent, via a diagonal so-called gauge transform ([12]), to some  $A_{fin}^{\tilde{k}, \tilde{l}}$  with  $\tilde{k} \in \{\pm 1\}^n$  and  $\tilde{l} = (1, \dots, 1)$ . Similarly, every  $A_{per}^{k, l}$  is unitarily equivalent to some  $A_{per}^{\tilde{k}, \tilde{l}}$  with  $\tilde{k} \in \{\pm 1\}^m$  and  $\tilde{l} = (1, \dots, 1)$ . Thus we omit the index  $l$  from now on.

In the proof of Theorem 1 we will use three auxiliary lemmas. The first one is well-known in the theory of block Laurent operators. It generalizes the symbol calculus for Laurent operators to periodic operators. Just observe that an  $m$ -periodic operator can be interpreted as a block Laurent operator with block size  $m \times m$ . Using a block Fourier transform it is then not hard to see that the operator  $A_{per}^k$  is unitarily equivalent to the (generalized) multiplication operator  $M_{a^k} : L^2([0, 2\pi), \mathbb{C}^m) \rightarrow L^2([0, 2\pi), \mathbb{C}^m)$  defined by the matrix valued function

$$a^k : [0, 2\pi) \rightarrow \mathbb{C}^{m \times m}, \quad \varphi \mapsto a^k(\varphi) := \begin{pmatrix} 0 & 1 & & k_m e^{i\varphi} \\ k_1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ e^{-i\varphi} & & k_{m-1} & 0 \end{pmatrix}.$$

The first lemma is then an easy consequence. For a proof and more material on this subject see e.g. [2, Theorem 2.93], [6, Theorem 4.4.9] or [8, Chap. VIII, Theorem 5.1].

**Lemma 1.** Let  $m \in \mathbb{N}$  and  $k \in \{\pm 1\}^m$ . Then we have

$$\text{sp}(A_{per}^k) = \{\lambda \in \mathbb{C} : \det(a^k(\varphi) - \lambda I_m) = 0 \text{ for some } \varphi \in [0, 2\pi)\}.$$

In the case of tridiagonal periodic operators, only the constant term of  $\det(a^k(\varphi) - \lambda I_m)$  (as a polynomial of  $\lambda$ ) depends on  $\varphi$  as the next lemma shows. This leads to the fact that the spectrum of every periodic operator  $A_{per}^k$  can be written as  $p^{-1}([-2, 2])$  for some polynomial  $p$  (cf. Figure 2). For an explicit formula of  $p$  we refer to [10].

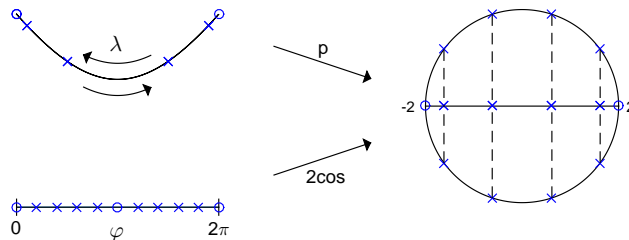


Figure 2: a sketch of the maps involved in the computation of the spectrum of a periodic operator

**Lemma 2.** *Let  $m \in \mathbb{N}$  and  $k \in \{\pm 1\}^m$ . Then there exists a polynomial  $p: \mathbb{C} \rightarrow \mathbb{C}$  of degree  $m$  (depending on  $k$ ) such that*

$$\det(a^k(\varphi) - \lambda I_m) = (-1)^m \left( p(\lambda) - e^{i\varphi} \prod_{j=1}^m k_j - e^{-i\varphi} \right)$$

for all  $\varphi \in [0, 2\pi)$ .

*Proof.* Using the Leibniz formula

$$\det(a^k(\varphi) - \lambda I_m) = \sum_{\tau \in S_m} \text{sign}(\tau) \prod_{i=1}^m a_{i, \tau_i}^k(\varphi),$$

it is easily seen that the only surviving term containing  $e^{i\varphi}$  but not  $e^{-i\varphi}$  is  $(-1)^{m+1} e^{i\varphi} \prod_{j=1}^m k_j$  and similar for  $e^{-i\varphi}$ . Thus the assertion follows. Alternatively one could also apply Laplace's formula twice to get the same result (see [10] for details).  $\square$

The third lemma is a discrete version of Lemma 1. Although the result is well-known (see e.g. [1, Section 2.1]), we include a short proof for the reader's convenience.

**Lemma 3.** *Let  $n, m \in \mathbb{N}$  and  $A, B, C \in \mathbb{C}^{m \times m}$ . Furthermore, denote by  $e^{i\xi_1}, \dots, e^{i\xi_n}$  the  $n$ -th roots of unity, i.e.  $\xi_j := \frac{2j}{n}\pi$  for  $j \in \{1, \dots, n\}$ . Then the following block matrices are unitarily*

equivalent:

$$T_1 := \begin{pmatrix} B & C & & A \\ A & \ddots & \ddots & \\ & \ddots & \ddots & C \\ C & & A & B \end{pmatrix} \in \mathbb{C}^{nm \times nm},$$

$$T_2 := \begin{pmatrix} Ae^{i\xi_1} + B + Ce^{-i\xi_1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & Ae^{i\xi_n} + B + Ce^{-i\xi_n} \end{pmatrix} \in \mathbb{C}^{nm \times nm}$$

$$= \text{diag}(Ae^{i\xi_1} + B + Ce^{-i\xi_1}, \dots, Ae^{i\xi_n} + B + Ce^{-i\xi_n}).$$

*Proof.* With the help of the Kronecker product  $\otimes$ , we can write  $T_1 = P \otimes A + I_n \otimes B + P^* \otimes C$ , where

$$P := \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \in \mathbb{C}^{n \times n}.$$

$P$  is unitarily equivalent to the diagonal matrix

$$D := \begin{pmatrix} e^{i\xi_1} & & \\ & \ddots & \\ & & e^{i\xi_n} \end{pmatrix} \in \mathbb{C}^{n \times n}.$$

Thus  $T_1$  is unitarily equivalent to  $D \otimes A + I_n \otimes B + D^* \otimes C$ , which is exactly  $T_2$ .  $\square$

*Proof.* (Theorem)

That  $\sigma_\infty \subset \pi_\infty$  holds was proven in [4, Theorem 4.1]. So let  $m \in \mathbb{N}$ ,  $k \in \{\pm 1\}^m$  and consider the operator  $A_{per}^k$ . W.l.o.g. we can assume that the number of  $-1$ 's in  $k$  is even because we can always double the period without changing the operator. This implies that we have

$$\det(a^k(\varphi) - \lambda I_m) = (-1)^m (p(\lambda) - 2 \cos(\varphi)) \quad (1)$$

in Lemma 2. Using Lemma 1, we get  $\text{sp}(A_{per}^k) = p^{-1}([-2, 2])$ . This implies, in particular, that the set

$$S_\infty^k := \{\lambda \in \mathbb{C} : \det(a^k(\varphi) - \lambda I_m) = 0 \text{ for some } \varphi \in \pi(\mathbb{Q} \setminus \mathbb{Z})\} = p^{-1}(2 \cos(\pi(\mathbb{Q} \setminus \mathbb{Z})))$$

is dense in  $\text{sp}(A_{per}^k)$ , cf. again Figure 2. Let  $n \in \mathbb{N}$  and  $\xi_j := \frac{2j}{n}\pi$  for  $j \in \{1, \dots, n\}$  as above. We will show that for every  $n \in \mathbb{N}$  there exists a finite matrix  $A_{fin}^l \in \mathbb{C}^{(nm-1) \times (nm-1)}$  such that

$$S_n^k := \bigcup_{j \in \{1, \dots, n-1\} \setminus \{\frac{n}{2}\}} \text{sp}(a^k(\xi_j)) \subset \text{sp}(A_{fin}^l) \subset \sigma_\infty. \quad (2)$$





where  $\sigma \in (0, 1)$ , as considered in [5] or [9] for example, the conclusion fails. This was of course to be expected because a finite matrix

$$\begin{pmatrix} 0 & l_1 & & \\ k_1 & \ddots & \ddots & \\ & \ddots & \ddots & l_n \\ & & k_n & 0 \end{pmatrix} \in \mathbb{C}^{(n+1) \times (n+1)}, k \in \{\pm\sigma\}^n, l \in \{\pm 1\}^n$$

is always similar (diagonal gauge transform, see [12]) to a matrix

$$\begin{pmatrix} 0 & \tilde{l}_1 & & \\ \tilde{k}_1 & \ddots & \ddots & \\ & \ddots & \ddots & \tilde{l}_n \\ & & \tilde{k}_n & 0 \end{pmatrix} \in \mathbb{C}^{(n+1) \times (n+1)}, \tilde{k} \in \{\pm\sqrt{\sigma}\}^n, \tilde{l} \in \{\pm\sqrt{\sigma}\}^n.$$

This is no longer true for periodic operators. For tridiagonal operators on  $\ell^2(\mathbb{Z})$  we can only shift phases to the other side. This remaining freedom, however, can be used to prove Theorem 1 for arbitrary alphabets as long as all elements share the same absolute value. This only needs a small modification of Lemma 3 and a refinement of [4, Theorem 4.1]. The details are left to the reader.

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**Author's address:**

Raffael Hagger  
Institute of Mathematics  
Hamburg University of Technology  
Schwarzenbergstr. 95 E  
D-21073 Hamburg  
GERMANY

`raffael.hagger@tuhh.de`