Hubbert, Simon and Levesley, J. (2019) Convergence of Multilevel Stationary Gaussian Convolution. In: Radu, F. and Kumar, K. and Berre, I. and Nordbotten, J. and Pop, I. (eds.) Numerical Mathematics and Advanced Applications - ENUMATH 2017. Lecture Notes in Computational Science and Engineering 126. Springer, pp. 83-92. ISBN 9783319964140.

Convergence of Multilevel Stationary Gaussian Convolution

Simon Hubbert[1] and Jeremy Levesley[2]

[1] Birkbeck, University of London, Malet Street, London, WC1E 7HX, s.hubbert@bbk.ac.uk

[2] Department of Mathematics, University of Leicester, Leicester, LE1 7RH, jl1@le.ac.uk

Abstract. In this paper we give a short note showing convergence rates for multilevel periodic approximation of smooth functions by multilevel Gaussian convolution. We will use the Gaussian scaling in the convolution at the finest level as a proxy for degrees of freedom d in the model. We will show that, for functions in the native space of the Gaussian, convergence is of the order $d^{-\frac{\ln(d)}{\ln(2)}}$. This paper provides a baseline for what should be expected in discrete convolution, which will be the subject of a follow up paper.

1 Introduction

Approximation by convolution involves selecting a suitable integrable function $K(x)$ (*the convolution kernel*) satisfying $\int_{\mathbb{R}} K(x)dx = 1$. A parameterized family of convolution kernels is generated from K by setting $K_h(x) = h^{-1}K(x/h)$ where $h > 0$. Then, for a given target function f, its convolution approximation $f * K_h$ converges to f, as $h \to 0$. The rate of convergence depends upon the smoothness of f and the polynomial reproduction properties of the underlying convolution kernel. In this paper we consider the approximation of 1-periodic continuous functions by convolution with the Gaussian kernel. In this case it is only possible to reproduce the constant function and so, as we will see, convergence is limited to $\mathcal{O}(h^2)$, regardless of additional smoothness requirements. However, if we employ a multilevel iterative refinement scheme we see that we get very rapid convergence. If the width of the Gaussian is halved at each iteration, then at the nth level of refinement we have essentially a $d = 2^n$ scaling in the Gaussian. For a discrete scheme we would apply a quadrature at $d = 2^n$ equally spaced points, giving d degrees of freedom. If the target function is taken from a certain periodic Sobolev space, whose order prescribes the smoothness of the functions, then we see improved but saturated convergence rates. If we consider functions in the native space of the Gaussian, a space of infinitely differentiable functions, then the order of convergence is $d^{-\frac{\ln(d)}{\ln(2)}}$.

This work is motivated by the desire to prove convergence of the multilevel sparse grid quasi-interpolation introduced by Levesley and Usta [5]. Multilevel sparse grid algorithms using smooth functions were introduced by the author and collaborators [4].

Our focus is on the Gaussian kernel

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$$

and we define $\psi_h(x) = h^{-1}\psi(x/h)$. We note the Fourier transform of ψ is

$$\widehat{\psi}(x) = \int_{-\infty}^{\infty} \psi(y) \exp(-2\pi ixy) dy = \exp(-2\pi^2 x^2).$$

We wish to approximate a continuous 1-periodic function

$$f = \sum_{k\in\mathbb{Z}}^{\infty} f_k e_k, \quad \text{where} \quad e_k(x) = \exp(2\pi ikx), \quad (k \in \mathbb{Z}),$$

by

$$\begin{aligned} f * \psi_h(x) &= \int_{-\infty}^{\infty} f(t)\psi_h(x-t)dt \;=\; \sum_{j=-\infty}^{\infty} \int_j^{j+1} f(t)\psi_h(x-t)dt \\ &= \sum_{j=-\infty}^{\infty} \int_0^1 f(t)\psi_h(x-t-j)dt \;=\; \int_0^1 f(t)\phi_h(x-t)dt, \end{aligned}$$

where

$$\phi_h(x) = \sum_{j=-\infty}^{\infty} \psi_h(x-j).$$

We note that ϕ_h is 1-periodic and a straightforward computation shows that

$$\phi_h(x) = \sum_{k=-\infty}^{\infty} \widehat{\psi}(hk) e_k(x), \tag{1}$$

In view of this and the well-known convolution formula for periodic functions (see e.g. [3]) we have that

$$f * \psi_h = \sum_{k=-\infty}^{\infty} f_k \widehat{\psi}(hk) e_k,$$

so that the error in convolution approximation is

$$E_h f = f - f * \psi_h = \sum_{k=-\infty}^{\infty} f_k (1 - \widehat{\psi}(hk)) e_k. \tag{2}$$

This representation immediately shows that the convolution reproduces the constant, but not any other trigonometric polynomial.

In Section 2 we will examine the convergence of the convolution approximation to target functions taken from certain periodic Sobolev spaces. We will show that no non-constant periodic function can have an approximation order smaller than $\mathcal{O}(h^2)$. In view of this we explore a multilevel iterative refinement, halving h in the convolution approximation at each level. In Section 3 we examine the error for this scheme and show that we can improve upon $\mathcal{O}(h^2)$ for functions with additional but finite smoothness. Specifically, we see that rapid improvements in the accuracy are exhibited in the early iterations but once the number of iterations passes a certain level, relating to the smoothness of the function, the algorithm settles to converge at a polynomial rate. In Section 4 we introduce the *native space* for Gaussian approximation. The native space is a subspace of infinitely smooth functions and, for such functions, we show that the algorithm exhibits rapid improvements in accuracy at every iteration.

2 Convergence of the convolution approximation

The functions we wish to approximate are continuous 1-periodic and taken from a periodic Sobolev space

$$\mathcal{N}_\beta = \left\{ f = \sum_{k=-\infty}^{\infty} f_k e_k : \|f\|_\beta = \left(|f_0|^2 + \sum_{k\in\mathbb{Z}} k^{2\beta}|f_k|^2 \right)^{1/2} < \infty \right\}.$$

The Sobolev embedding theorem [2] ensures that if $\beta > \frac{1}{2}$ then all functions in $\mathcal{N}_\beta$ will be continuous. The following result gives error bounds for Gaussian convolution approximation of such functions.

Proposition 1. *Let $f \in \mathcal{N}_\beta$, where $\beta > \frac{1}{2}$. Then*

$$\|E_h f\|_\infty \leq \begin{cases} C_1 h^2 & \textit{for} \;\; \beta > \frac{5}{2}; \\ h^2\left(C_2\sqrt{\ln\left(\frac{1}{h}\right)} + C_3\right) & \textit{for} \;\; \beta = \frac{5}{2}; \\ C_4 h^{\beta-\frac{1}{2}} & \textit{for} \;\; \frac{1}{2} < \beta < \frac{5}{2}, \end{cases}$$

where C_i $i = 1, 2, 3, 4$, are positive constants independent of h.

Proof. Since $\widehat{\psi}(0) = 1$ and $\widehat{\psi}(-k) = \widehat{\psi}(k)$, $(k \in \mathbb{Z}_+)$, we have, from (2), that

$$\|E_h f\| \leq \sum_{k=1}^{\infty} (1 - \widehat{\psi}(hk))(|f_k| + |f_{-k}|). \tag{3}$$

Suppose that $\beta = \frac{5}{2} + \alpha$, where $\alpha > 0$. Using the elementary bound

$$1 - \exp(-x) \leq x \quad \text{for} \quad x > 0, \tag{4}$$

we have that $1-\widehat{\psi}(hk) \le 2\pi^2h^2k^2$. This yields

$$\begin{aligned}\|E_hf\|_\infty &\le 2\pi^2h^2\left(\sum_{k=1}^{\infty}k^2|f_k|+\sum_{k=1}^{\infty}k^2|f_{-k}|\right)\\ &= 2\pi^2h^2\left(\sum_{k=1}^{\infty}k^{\frac{5}{2}+\alpha}|f_k|\frac{1}{k^{\alpha+\frac{1}{2}}}+\sum_{k=1}^{\infty}k^{\frac{5}{2}+\alpha}|f_{-k}|\frac{1}{k^{\alpha+\frac{1}{2}}}\right).\end{aligned}$$

Applying the Cauchy Schwarz inequality we have

$$\begin{aligned}&\|E_hf\|_\infty \le\\ &2\pi^2h^2\left(\sum_{k=1}^{\infty}\frac{1}{k^{2\alpha+1}}\right)^{\frac{1}{2}}\left[\left(\sum_{k=1}^{\infty}\left(k^2\right)^{\frac{5}{2}+\alpha}|f_k|^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}\left(k^2\right)^{\frac{5}{2}+\alpha}|f_{-k}|^2\right)^{\frac{1}{2}}\right]\\ &\le 2\pi^2h^2C\|f\|_{\frac{5}{2}+\alpha}.\end{aligned}$$

Now assume that $\beta = \frac{5}{2}-\alpha$ where $0 \le \alpha < 2$. In the following development we will partition the right hand side of (3) as follows

$$\|E_hf\| \le \sum_{k=1}^{m_h}(1-\widehat{\psi}(hk))(|f_k|+|f_{-k}|)+\sum_{k=m_h+1}^{\infty}(1-\widehat{\psi}(hk))(|f_k|+|f_{-k}|),$$

where m_h denotes the integer satisfying $m_h \le \frac{1}{h} \le m_h+1$. For $k \le m_h$ we will, as before, use $1-\widehat{\psi}(hk) \le 2\pi^2h^2k^2$ whereas for $k \ge m_h+1$ we use $1-\widehat{\psi}(hk) \le 1$. This leads to

$$\|E_hf\| \le \underbrace{2\pi^2h^2\sum_{k=1}^{m_h}k^2|f_k|+\sum_{k=m_h+1}^{\infty}|f_k|}_{E_+}+\underbrace{2\pi^2h^2\sum_{k=1}^{m_h}k^2|f_{-k}|+\sum_{k=m_h+1}^{\infty}|f_{-k}|}_{E_-}.$$

Focusing on E_+ we can write:

$$E_+ = 2\pi^2h^2\sum_{k=1}^{m_h}k^{\frac{5}{2}-\alpha}|f_k|k^{\alpha-\frac{1}{2}}+\sum_{k=m_h+1}^{\infty}k^{\frac{5}{2}-\alpha}|f_k|\frac{1}{k^{\frac{5}{2}-\alpha}}.$$

Applying the Cauchy-Schwarz inequality to the first sum, in the case where $0<\alpha<2$, we have

$$\begin{aligned}2\pi^2h^2\sum_{k=1}^{m_h}k^{\frac{5}{2}-\alpha}|f_k|k^{\alpha-\frac{1}{2}} &\le 2\pi^2h^2\left(\sum_{k=1}^{m_h}\left(k^2\right)^{\frac{5}{2}-\alpha}|f_k|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{m_h}k^{2\alpha-1}\right)^{\frac{1}{2}}\\ &\le 2\pi^2h^2C\|f\|_{\frac{5}{2}-\alpha}m_h^{\alpha} \le 2\pi^2Ch^{2-\alpha}\|f\|_{\frac{5}{2}-\alpha}.\end{aligned}$$

For $\alpha = 0$ we have

$$2\pi^2 h^2 \sum_{k=1}^{m_h} k^{\frac{5}{2}} |f_k| k^{-\frac{1}{2}} \leq 2\pi^2 h^2 \left(\sum_{k=1}^{m_h} \left(k^2\right)^{\frac{5}{2}} |f_k|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{m_h} \frac{1}{k} \right)^{\frac{1}{2}}$$
$$\leq 2\pi^2 h^2 \|f\|_{\frac{5}{2}} (2\ln(m_h))^{\frac{1}{2}} \leq \sqrt{2}\pi^2 h^2 \left(\ln\left(\frac{1}{h}\right)\right)^{\frac{1}{2}} \|f\|_{\frac{5}{2}}.$$

Applying the Cauchy-Schwarz inequality to the second sum we have

$$\sum_{k=m_h+1}^{\infty} k^{\frac{5}{2}-\alpha} |f_k| \frac{1}{k^{\frac{5}{2}-\alpha}} \leq \left(\sum_{k=m_h+1}^{\infty} \left(k^2\right)^{\frac{5}{2}-\alpha} |f_k|^2 \right)^{\frac{1}{2}} \left(\sum_{k=m_h+1}^{\infty} \frac{1}{k^{5-2\alpha}} \right)^{\frac{1}{2}}$$
$$\leq \|f\|_{\frac{5}{2}-\alpha} \mathcal{C} \left(\frac{1}{(m_h+1)^{4-2\alpha}} \right)^{\frac{1}{2}} \leq \mathcal{C} \|f\|_{\frac{5}{2}-\alpha} h^{2-\alpha}.$$

These two bounds allow us to deduce that $E_+ \leq \mathcal{C}h^{2-\alpha}$ for $0 < \alpha < 2$ and $E_+ \leq h^2(A\sqrt{\ln(1/h)} + B)$. The equivalent bound for E_-, the contribution involving Fourier coefficients of negative index, follows in the same fashion. Hence we conclude that

$$\|E_h f\| \leq Ch^{2-\alpha} \|f\|_{\frac{5}{2}-\alpha} = Ch^{\beta-\frac{1}{2}} \|f\|_\beta \quad \text{where} \quad \frac{1}{2} < \beta < \frac{5}{2},$$

and

$$\|E_h f\| \leq \|f\|_{\frac{5}{2}} h^2((A\sqrt{\ln(1/h)} + B).$$

3 Iterative refinement

The result from the previous section suggests that we cannot improve convergence by imposing additional smoothness conditions, beyond $f \in \mathcal{N}_{\frac{5}{2}}$. In view of this we now consider using a multilevel iterative refinement where, at each level, we halve h in the convolution approximation. In this setting the multilevel error in convolution is defined by

$$M_1 f = f - f * \psi_h, \quad \text{and} \quad M_j f := M_{j-1} f - M_{j-1} f * \psi_{\frac{h}{2^{j-1}}}, \quad (j \geq 2). \quad (5)$$

Our analysis here will follow in the same fashion as in the previous section. We begin by noting that if $f = \sum_{k\in\mathbb{Z}} f_k e_k$ is a continuous 1-periodic function then the Fourier series representation of the multilevel error is given by

$$M_j f = \sum_{k=-\infty}^{\infty} b_j(k) \widehat{f}(k) e_k, \quad j = 1, 2, \cdots, \quad (6)$$

where, for $k \in \mathbb{Z}$ and

$$b_j(k) = \prod_{\ell=1}^{j} \left(1 - \widehat{\psi}\left(\frac{hk}{2^{\ell-1}} \right) \right) \quad j = 1, 2, \ldots. \quad (7)$$

For a given $f \in \mathcal{N}_\beta$, the following result quantifies the accuracy of the multilevel convolution approximation after j steps of iterative refinement.

Proposition 2. *Let $f \in \mathcal{N}_\beta$, $\beta > \frac{1}{2}$ and $0 < h < \frac{1}{2\pi}$. Then the multilevel error after the j^{th} iterative refinement satisfies:*

$$\|M_j f\|_\infty \leq \begin{cases} C_1 \left(\frac{2\pi h}{2^{\frac{j}{2}}}\right)^{2j} & \text{if } j < \frac{2\beta-1}{4}; \\ \left(\frac{2\pi h}{2^{\frac{j}{2}}}\right)^{2j} \left(C_2 \sqrt{\ln\left(\frac{2^{\frac{j}{2}}}{2\pi h}\right)} + C_3\right) & \text{if } j = \frac{2\beta-1}{4}; \\ C_4 \left(2\pi h\right)^{\beta-\frac{1}{2}} \left(\frac{1}{2^j}\right)^{\frac{2\beta-1}{4}} & \text{if } j > \frac{2\beta-1}{4}, \end{cases}$$

where C_i $i = 1, 2, 3, 4,$ are positive constants independent of h.

Proof. The proof follows the same pattern as that of Proposition 1 and so rather than repeat the details we briefly outline the steps. We begin by noting that, since $\widehat{\psi}(0) = 1$ and $\widehat{\psi}(-k) = \widehat{\psi}(k)$, $(k \in \mathbb{Z}_+)$, we have

$$\|M_j f\|_\infty \leq \sum_{k=1}^{\infty} b_j(k)(|f_k| + |f_{-k}|). \tag{8}$$

To reach the bound for the case where $j < \frac{2\beta-1}{4}$ (or equivalently $\beta > 2j+1$) we note that (4) allows us to deduce that

$$b_j(k) \leq \prod_{\ell=1}^{j} \frac{2\pi^2(2k)^2h^2}{2^{2\ell}} = \prod_{\ell=1}^{j} \frac{2^3\pi^2k^2h^2}{2^{2\ell}} = \left(\frac{2\pi h}{2^{\frac{j}{2}}}\right)^{2j} k^{2j}. \tag{9}$$

Using this we replicate the steps from Proposition 1 (associated to the $\beta > \frac{5}{2}$ case) to reach the required bound. For the remaining cases captured by $j \geq \frac{2\beta-1}{4}$ (or equivalently $\beta \leq 2j + \frac{1}{2}$) we follow Proposition 1 again and define m_j to be the integer satisfying

$$m_j \leq \frac{2^{\frac{j}{2}}}{2\pi h} \leq m_j + 1.$$

and, using this, the error expression is split into a finite sum (including the first m_j terms) and the remaining infinite sum. For bounding purposes we employ (9) for the finite sum and $b_j(k) \leq 1$ for the infinite sum. Once again by mimicking the steps from Proposition 1 (associated to the $\beta \leq \frac{5}{2}$ case) one can establish the stated bounds.

We comment here that for functions of finite smoothness the speed at which the multilevel iterative refinement converges is restricted by the smoothness of the function. The error is reduced significantly in the early iterations, when $j < \frac{2\beta-1}{4}$ but beyond this point the error decays asymptotically at a polynomial rate.

4 Native space for Gaussian approximation

It is more natural to approximate functions from the so called *native space* for Gaussians:

$$\mathcal{N}_\psi = \left\{ f : \|f\|_\psi = \left(\sum_{k=-\infty}^{\infty} \widehat{\psi}(k)^{-1} |f_k|^2 \right)^{1/2} < \infty \right\}.$$

If we approximate such very smooth functions we have the following result.

Theorem 3. *Let $f \in \mathcal{N}_\psi$ and $0 < h < 1$. Then*

$$\|M_j f\|_\infty \le C \left(\frac{4h^2 j}{e} \right)^j \left(\frac{1}{2^j} \right)^j \|f\|_\psi.$$

Proof. As in Proposition 2 we can use (9) to develop the multilevel error as follows

$$\begin{aligned}
\|M_j f\|_\infty &\le \sum_{k=1}^{\infty} b_j(k)(|f_k| + |f_{-k}|) \\
&\le \left(\frac{4\pi^2 h^2}{2^j} \right)^j \sum_{k=1}^{\infty} k^{2j} \left(|f_k| + |f_{-k}| \right) \\
&= \left(\frac{4\pi^2 h^2}{2^j} \right)^j \sum_{k=1}^{\infty} \sqrt{\widehat{\psi}(k)} k^{2j} \left(\frac{|f_k|}{\sqrt{\widehat{\psi}(k)}} + \frac{|f_{-k}|}{\sqrt{\widehat{\psi}(k)}} \right) \qquad (10) \\
&\le \left(\frac{4\pi^2 h^2}{2^j} \right)^j \sqrt{\sum_{k=1}^{\infty} \widehat{\psi}(k) k^{4j}} \left(\sqrt{\sum_{k=1}^{\infty} \frac{|f_k|^2}{\widehat{\psi}(k)}} + \sqrt{\sum_{k=1}^{\infty} \frac{|f_k|^2}{\widehat{\psi}(k)}} \right) \\
&\le 2 \left(4\pi^2 h^2 \right)^j \left(\frac{1}{2^j} \right)^j \sqrt{\sum_{k=1}^{\infty} \widehat{\psi}(k) k^{4j}} \|f\|_\psi.
\end{aligned}$$

Concerning the infinite sum appearing in the bound above, we observe that for each $j = 1, 2, \ldots$ the non-negative function $x \mapsto \widehat{\psi}(x) x^{4j}$ is increasing for $0 \le x \le \frac{\sqrt{j}}{\pi}$ and decreasing for $x \ge \frac{\sqrt{j}}{\pi}$. Let m_j denote the integer satisfying $m_j \le \frac{\sqrt{j}}{\pi} \le m_j + 1$, then we can write

$$\begin{aligned}
&\sum_{k=1}^{\infty} \widehat{\psi}(k) k^{4j} = \sum_{k=1}^{m_j} \widehat{\psi}(k) k^{4j} + \sum_{k=m_j+1}^{\infty} \widehat{\psi}(k) k^{4j} \\
&\le \int_1^{m_j} \widehat{\psi}(x) x^{4j} dx + \widehat{\psi}(m_j)(m_j)^{4j} + \widehat{\psi}(m_j+1)(m_j+1)^{4j} + \int_{m_j+1}^{\infty} \widehat{\psi}(x) x^{4j} dx \\
&\le 2\widehat{\psi} \left(\frac{\sqrt{j}}{\pi} \right) \left(\frac{\sqrt{j}}{\pi} \right)^{4j} + \int_0^{\infty} \widehat{\psi}(x) x^{4j} dx = 2 \left(\frac{j}{\pi^2 e} \right)^{2j} + \int_0^{\infty} e^{-2\pi^2 x^2} x^{4j} dx.
\end{aligned}$$

Examining the integral we have

$$\int_0^\infty e^{-2\pi^2 x^2} x^{4j} dx = \int_0^\infty e^{-s} \left(\frac{s}{2\pi}\right)^{2j} \frac{ds}{2\sqrt{2s}}$$
$$= \frac{1}{\sqrt{2}(2\pi)(2\pi^2)^{2j}} \int_0^\infty e^{-s} s^{2j-\frac{1}{2}} ds = \frac{\Gamma\left(2j+\frac{1}{2}\right)}{\sqrt{2}(2\pi)(2\pi^2)^{2j}}.$$

Using Stirling's formula for the Gamma function [1] Formula 6.1.39 we have

$$\Gamma\left(2j+\frac{1}{2}\right) \le C\sqrt{2\pi}\left(\frac{2j}{e}\right)^{2j}.$$

Substituting this into the bound above we see that

$$\sum_{k=1}^{\infty} \widehat{\psi}(k) k^{4j} \le C\left(\frac{j}{e\pi^2}\right)^{2j}.$$

Taking the square root and substituting into (10) we conclude that

$$\|M_j f\|_\infty \le 2\left(4\pi^2 h^2\right)^j \left(\frac{1}{2^j}\right)^j \left(\frac{j}{e\pi^2}\right)^{2j} \|f\|_\psi$$
$$\le C\left(\frac{4h^2 j}{e}\right)^j \left(\frac{1}{2^j}\right)^j \|f\|_\psi.$$

We remark that, setting $d = 2^j$, we get a convergence rate of $\mathcal{O}\left(\frac{4h^2\alpha \ln d}{ed}\right)^{\alpha \ln d}$, where $\alpha = 1/\ln(2)$ which is faster than any polynomial.

References

1. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Dover Publications, New York (1964).
2. R. A. Adams, Sobolev Spaces, *Academic Press*, New York, 1975.
3. R. Bracewell, The Fourier Transform and Its Applications (2nd ed.), McGrawHill, 1986.
4. E. H. Georgoulis, J. Levesley and F. Subhan, Multilevel sparse kernel-based interpolation, SIAM Journal of Scientific Computing **35** (2013), 815–831.
5. F. Usta and J. Levesley, Multilevel quasi-interpolation on a sparse grid with the Gaussian, *Numerical Algorithms* , 2017.