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Aequationes Mathematicae

Around a problem of Nicole Brillouët-Belluot, II

Janusz Morawiec

Abstract. For every $\alpha \in \mathbb{R}$ we determine all increasing bijections $f\colon (0,+\infty) \to (0,+\infty)$ such that $f(1) \neq 1$ and $f(x)f^{-1}(x) = x^{\alpha}$ for every $x \in (0,+\infty)$.

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In this note we answer Problem 3.8 posed in my article [4]. More precisely, given $\alpha \in \mathbb{R}$ we determine all increasing bijections $f\colon (0,+\infty) \to (0,+\infty)$ such that $f(1) \neq 1$ and

$$f(x)f^{-1}(x) = x^{\alpha} \qquad \text{for every } x \in (0,+\infty). \tag{1.1}$$

Theorem 1. (i) *If* (1.1) *admits a solution f which is an increasing bijection from $(0,+\infty)$ onto $(0,+\infty)$, then we have $\alpha > 2$. Let $\beta \in (1,+\infty)$ be the unique number satisfying $\beta + \frac{1}{\beta} = \alpha$.*

If $f(1) = c \neq 1$ and $f_0 = f|_{[\min\{1,c\},\max\{1,c\}]}$, then f_0 is continuous, $f_0(c) = c^{\alpha}$ and

$$\left(\frac{y}{x}\right)^{\frac{1}{\beta}} \leq \frac{f_0(y)}{f_0(x)} \leq \left(\frac{y}{x}\right)^{\beta} \tag{1.2}$$

for all $x, y \in [\min\{1,c\},\max\{1,c\}]$ with $x < y$.

(ii) *Conversely, let $c \in (0,1) \cup (1,+\infty)$ and let $f_0\colon [\min\{1,c\},\max\{1,c\}] \to \mathbb{R}$ be a continuous function such that $f_0(1) = c$, $f_0(c) = c^{\alpha}$ and* (1.2) *holds for all $x, y \in [\min\{1,c\},\max\{1,c\}]$ with $x < y$. Then f_0 can be uniquely extended to an increasing bijection $f\colon (0,+\infty) \to (0,+\infty)$ satisfying* (1.1)*.*

Moreover:

(a) *if $c \in (0,1)$, then $f(x) < \min\{x^{\beta}, x^{\frac{1}{\beta}}\}$ for every $x \in (0,+\infty)$;*
(b) *if $c \in (1,+\infty)$, then $f(x) > \max\{x^{\beta}, x^{\frac{1}{\beta}}\}$ for every $x \in (0,+\infty)$.*

Proof. (i) The case $f(1) = 1$ was treated in [4]. Then we have $\alpha > 0$ by Lemma 3.2 of [4].

Fix an increasing bijective solution $f\colon (0,+\infty) \to (0,+\infty)$ of (1.1) such that $f(1) \neq 1$.

Put $c = f(1)$ and $f_0 = f|_{[\min\{1,c\},\max\{1,c\}]}$. Then f_0 is continuous, $f_0(1) = c$ and $f_0(c) = f^2(1) = [f(1)]^\alpha = c^\alpha$.

Fix $x, y \in [\min\{1,c\},\max\{1,c\}]$ with $x < y$. Since all the iterates of f are increasing, by Lemma 2.3 of [4] we have

$$\frac{[f_0(x)]^{a_n}}{x^{a_{n-1}}} < \frac{[f_0(y)]^{a_n}}{y^{a_{n-1}}} \qquad \text{and} \qquad \frac{x^{a_{n+1}}}{[f_0(x)]^{a_n}} < \frac{y^{a_{n+1}}}{[f_0(y)]^{a_n}} \qquad \text{for every } n \in \mathbb{N}.$$

Hence

$$\left(\frac{y}{x}\right)^{\frac{a_{n-1}}{a_n}} \le \frac{f_0(y)}{f_0(x)} \le \left(\frac{y}{x}\right)^{\frac{a_{n+1}}{a_n}} \qquad \text{for every } n \in \mathbb{N}. \tag{1.3}$$

In particular, $a_{n-1} < a_{n+1}$ for every $n \in \mathbb{N}$. Finally, using Lemma 3.3 of [4] we get $\alpha > 2$ and passing with n to $+\infty$ in (1.3) we obtain (1.2).

(ii) Define a function $g_0\colon [\min\{0,\log c\},\max\{0,\log c\}] \to \mathbb{R}$ by setting $g_0(x) = \log f_0(e^x)$. Then g_0 is continuous, $g_0(0) = \log c$, $g_0(\log c) = \alpha \log c$ and $\frac{1}{\beta} \le \frac{g_0(y)-g_0(x)}{y-x} \le \beta$ for all $x, y \in [\min\{0,\log c\},\max\{0,\log c\}]$ with $x < y$.

Since $0 < \frac{1}{\beta} < 1 < \beta$, we conclude from Theorem 2 of [5] that g_0 can be uniquely extended to a continuous function $g\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$g^2(x) = \alpha g(x) - x \qquad \text{for every } x \in \mathbb{R}, \tag{1.4}$$

and this unique extension g satisfies

$$g(x) < \min\left\{\beta x, \frac{1}{\beta}x\right\} \qquad \text{for every } x \in \mathbb{R}, \text{ if } c \in (0,1),$$

$$g(x) > \max\left\{\beta x, \frac{1}{\beta}x\right\} \qquad \text{for every } x \in \mathbb{R}, \text{ if } c \in (1,+\infty),$$

$$\frac{1}{\beta}(y-x) \le g(y) - g(x) \le \beta(y-x) \qquad \text{for all } x, y \in \mathbb{R} \text{ with } x < y.$$

Consequently, g is strictly increasing and maps $\mathbb{R}$ onto $\mathbb{R}$. This jointly with (1.4) implies $g(x) + g^{-1}(x) = \alpha x$ for every $x \in \mathbb{R}$.

Now define a function $f\colon (0,+\infty) \to (0,+\infty)$ by setting $f(x) = e^{g(\log x)}$. Then f is an increasing bijection satisfying (1.1) such that (a) and (b) hold. The proof of the theorem is complete. □

Remark 2. An answer to the problem posed during The Forty-ninth International Symposium on Functional Equations by Nicole Brillouët-Belluot (see [2]) as well as the result presented during The Fiftieth International Symposium on Functional Equations by Zoltán Boros (see [1]) can be derived from Theorem 10 of [5] in the same way as above (involving Lemmas 2.1–2.5 of [3] in the case of the problem of Nicole Brillouët-Belluot).

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