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Decomposition of two functions in the orthogonality equation

RADOSŁAW ŁUKASIK AND PAWEŁ WÓJCIK

Abstract. The aim of this paper is to solve the orthogonality equation with two unknown functions. This problem was posed by J. Chmieliński during two international conferences.

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1. Introduction

Let \mathcal{H}, \mathcal{K} be Hilbert spaces over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $\langle \cdot | \cdot \rangle$ denote the inner product and $\|\cdot\|$ the norm associated with it. We shall not distinguish between the symbols used for \mathcal{H} and \mathcal{K} . The Banach space of all bounded linear operators from \mathcal{H} to \mathcal{K} is denoted by $\mathcal{L}(\mathcal{H}; \mathcal{K})$. It is known that $h: \mathcal{H} \rightarrow \mathcal{K}$ is a solution of the orthogonality equation:

$$\forall_{x,y \in \mathcal{H}} \quad \langle h(x)|h(y) \rangle = \langle x|y \rangle$$

if and only if h is a linear isometry (or equivalently $h \in \mathcal{L}(\mathcal{H}; \mathcal{K})$ and $h^*h = I_{\mathcal{H}}$).

The following considerations have been inspired by the talks of J. Chmieliński during the 15th ICFEI and CUTS (see [3, p. 95] and [4, p. 145]). Namely, we will solve the generalized orthogonality equation:

$$\forall_{x,y \in \mathcal{H}} \quad \langle f(x)|g(y) \rangle = \langle x|y \rangle, \quad (1)$$

with two unknown functions $f, g: \mathcal{H} \rightarrow \mathcal{K}$. The paper [1] also deals with Eq. (1) and similar topics. We need the following lemma for further investigations.

Lemma 1. *Let $f, g: \mathcal{H} \rightarrow \mathcal{K}$ satisfy Eq. (1) and $\overline{\text{Lin}f(\mathcal{H})} = \mathcal{K}$. Then g is linear.*

Proof. Fix $y \in \text{Lin}f(\mathcal{H})$. Then $y = \sum_{k=1}^n \alpha_k f(x_k)$ for some $\alpha_1, \dots, \alpha_n \in \mathbb{K}$, $x_1, \dots, x_n \in \mathcal{H}$. Thus we have

$$\begin{aligned}
 \langle y|g(u + \beta w) \rangle &= \left\langle \sum_{k=1}^n \alpha_k f(x_k)|g(u + \beta w) \right\rangle = \sum_{k=1}^n \alpha_k \langle f(x_k)|g(u + \beta w) \rangle \\
 &= \sum_{k=1}^n \alpha_k \langle x_k|u + \beta w \rangle = \sum_{k=1}^n \alpha_k (\langle x_k|u \rangle + \bar{\beta} \langle x_k|w \rangle) \\
 &= \sum_{k=1}^n \alpha_k (\langle f(x_k)|g(u) \rangle + \bar{\beta} \langle f(x_k)|g(w) \rangle) \\
 &= \sum_{k=1}^n \langle \alpha_k f(x_k)|g(u) + \beta g(w) \rangle \\
 &= \left\langle \sum_{k=1}^n \alpha_k f(x_k)|g(u) + \beta g(w) \right\rangle \\
 &= \langle y|g(u) + \beta g(w) \rangle, \quad u, w \in \mathcal{H}, \quad \beta \in \mathbb{K}.
 \end{aligned}$$

The set $\text{Lin}f(\mathcal{H})$ is dense in \mathcal{K} , whence $g(u + \beta w) = g(u) + \beta g(w)$ for $u, v \in \mathcal{H}$, $\beta \in \mathbb{K}$. This means that g is linear. □

Some well-known facts will be useful for further considerations. Let $T \in \mathcal{L}(\mathcal{K}; \mathcal{H})$. Then there exists a unique $T^* \in \mathcal{L}(\mathcal{H}; \mathcal{K})$ such that

$$\forall x \in \mathcal{K} \quad \forall y \in \mathcal{H} \quad \langle Tx|y \rangle_{\mathcal{H}} = \langle x|T^*y \rangle_{\mathcal{K}}.$$

Moreover,

$$\text{if } \forall x \in \mathcal{H} \quad \vartheta \|x\| \leq \|Tx\|, \quad \text{then } T(\mathcal{H}) \text{ is closed.} \tag{2}$$

It is also known that

$$T \text{ is invertible if and only if } T^* \text{ is invertible.} \tag{3}$$

Lemma 2. *Let $T_1, T_2: \mathcal{H} \rightarrow \mathcal{K}$ be linear maps and satisfy*

$$\forall x, y \in \mathcal{H} \quad \langle T_1(x)|T_2(y) \rangle = \langle x|y \rangle. \tag{4}$$

Assume that $T_1(\mathcal{H})$ is dense. Then T_2 is continuous, i.e., $T_2 \in \mathcal{L}(\mathcal{H}; \mathcal{K})$.

Proof. Fix a sequence $(x_n)_{n=1,2,\dots}$ such that $x_n \in \mathcal{H}$ and $x_n \rightarrow x_o$ for some $x_o \in \mathcal{H}$. Suppose that $T_2x_n \rightarrow z$ for some $z \in \mathcal{H}$. It suffices to show that $T_2x_o = z$ and apply The Closed Graph Theorem. Fix $y \in \mathcal{H}$ and notice that $\langle x_n - x_o|y \rangle \rightarrow 0$. On the other hand $\langle x_n - x_o|y \rangle \stackrel{(4)}{=} \langle T_2x_n - T_2x_o|T_1y \rangle \rightarrow \langle z - T_2x_o|T_1y \rangle$. Thus we get

$$\forall y \in \mathcal{H} \quad \langle z - T_2x_o|T_1y \rangle = 0.$$

Since $T_1(\mathcal{H})$ is dense in \mathcal{K} , it follows that $z - T_2x_o = 0$. We have shown that the graph of T_2 is closed. □

Lemma 3. *Let $T, S \in \mathcal{L}(\mathcal{H}; \mathcal{K})$ satisfy the equation*

$$\forall x, y \in \mathcal{H} \quad \langle T(x)|S(y) \rangle = \langle x|y \rangle. \tag{5}$$

Then T is invertible if and only if S is invertible.

Proof. Assume that T is invertible. Then $\langle x|y \rangle = \langle Tx|Sy \rangle = \langle x|T^*Sy \rangle$ for all $x, y \in \mathcal{H}$. Thus we have $I_{\mathcal{H}} = T^*S$. It follows from Lemma (3) that T^* is invertible. Therefore $(T^*)^{-1} = S$, hence S is invertible. The proof of the reverse is the same. \square

2. Main result

In this section, we solve functional Eq. (1). Now we can state and prove the main result of the paper.

Theorem 4. *Let $f, g: \mathcal{H} \rightarrow \mathcal{K}$ satisfy Eq. (1) if and only if there exist suitable closed subspaces $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 \subset \mathcal{K}$ such that $\mathcal{K} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3$, $\mathcal{M}_k \perp \mathcal{M}_j$ (for $k \neq j$) and f, g can be written as the following decomposition*

$$f = A + \varphi, \quad g = (A^*)^{-1} + \gamma,$$

for some invertible $A \in \mathcal{L}(\mathcal{H}; \mathcal{M}_1)$ and for some mappings $\varphi: \mathcal{H} \rightarrow \mathcal{M}_2, \gamma: \mathcal{H} \rightarrow \mathcal{M}_3$.

Proof. The implication “ \Leftarrow ” is immediate. We start with proving “ \Rightarrow ”. It is clear that $\mathcal{K} = \overline{\text{Lin}f(\mathcal{H})} \oplus \overline{\text{Lin}f(\mathcal{H})}^\perp$, whence there are two mappings $T_1: \mathcal{H} \rightarrow \overline{\text{Lin}f(\mathcal{H})}$ and $\varphi_1: \mathcal{H} \rightarrow \overline{\text{Lin}f(\mathcal{H})}^\perp$ such that $g(x) = T_1(x) + \varphi_1(x)$ for all $x \in \mathcal{H}$. It is easy to check that f, T_1 satisfy

$$\forall_{x,y \in \mathcal{H}} \quad \langle f(x)|T_1(y) \rangle = \langle x|y \rangle. \tag{6}$$

Indeed,

$$\begin{aligned} \langle x|y \rangle &= \langle f(x)|g(y) \rangle = \langle f(x)|T_1(y) + \varphi_1(y) \rangle = \langle f(x)|T_1(y) \rangle + \langle f(x)|\varphi_1(y) \rangle \\ &= \langle f(x)|T_1(y) \rangle + 0 = \langle f(x)|T_1(y) \rangle. \end{aligned}$$

We have shown that f, T_1 satisfy (6). Moreover, the set $\text{Lin}f(\mathcal{H})$ is dense in $\overline{\text{Lin}f(\mathcal{H})}$. It follows from 1 that T_1 is linear.

There exists the closed subspace $\mathcal{M} \subset \overline{\text{Lin}f(\mathcal{H})}$ such that $\overline{T_1(\mathcal{H})} \perp \mathcal{M}$ and $\overline{\text{Lin}f(\mathcal{H})} = \overline{T_1(\mathcal{H})} \oplus \mathcal{M}$. Therefore there are two mappings $T_2: \mathcal{H} \rightarrow \overline{T_1(\mathcal{H})}$ and $\varphi_2: \mathcal{H} \rightarrow \mathcal{M}$ such that $f(x) = T_2(x) + \varphi_2(x)$ for all $x \in \mathcal{H}$. In a similar way we prove that T_2, T_1 satisfy

$$\forall_{x,y \in \mathcal{H}} \quad \langle T_2(x)|T_1(y) \rangle = \langle x|y \rangle. \tag{7}$$

Now, we can consider the linear operators $T_2: \mathcal{H} \rightarrow \overline{T_1(\mathcal{H})}$ and $T_1: \mathcal{H} \rightarrow \overline{T_1(\mathcal{H})}$ instead of $T_1: \mathcal{H} \rightarrow \overline{\text{Lin}f(\mathcal{H})}$. By Lemma 1 and (7), T_2 is linear. By Lemma 2 and (7), T_2 is continuous, i.e., $T_2 \in \mathcal{L}(\mathcal{H}; \overline{T_1(\mathcal{H})})$.

There is the closed subspace $\mathcal{N} \subset \overline{T_1(\mathcal{H})}$ such that $\overline{T_1(\mathcal{H})} = \overline{T_2(\mathcal{H})} \oplus \mathcal{N}$ and $T_2(\mathcal{H}) \perp \mathcal{N}$. Hence there are two mappings $T_3: \mathcal{H} \rightarrow \overline{T_2(\mathcal{H})}$ and $\varphi_3: \mathcal{H} \rightarrow \mathcal{N}$ such

that $T_1(x) = T_3(x) + \varphi_3(x)$ for all $x \in \mathcal{H}$. In a similar way we prove that T_3 is linear and T_2, T_3 satisfy

$$\forall_{x,y \in \mathcal{H}} \quad \langle T_2(x)|T_3(y) \rangle = \langle x|y \rangle. \tag{8}$$

Now, we consider the linear operator $T_2 \in \mathcal{L}(\mathcal{H}; \overline{T_2(\mathcal{H})})$ (instead of $T_2 \in \mathcal{L}(\mathcal{H}; \overline{T_1(\mathcal{H})})$). Let us consider also the linear operator $T_3: \mathcal{H} \rightarrow \overline{T_2(\mathcal{H})}$. Applying again Lemma 2 and (8) we can say that T_3 is continuous, i.e., $T_3 \in \mathcal{L}(\mathcal{H}; \overline{T_2(\mathcal{H})})$. Now, we get

$$\|x\|^2 = \langle x|x \rangle \stackrel{(8)}{=} \langle T_2x|T_3x \rangle \leq \|T_2x\| \cdot \|T_3x\| \leq \|T_2x\| \cdot \|T_3\| \cdot \|x\|$$

for all $x \in \mathcal{H}$. It follows from the above inequality that

$$\forall_{y \in \mathcal{H}} \quad \frac{1}{\|T_3\|} \cdot \|x\| \leq \|T_2x\|. \tag{9}$$

Thus $T_2(\mathcal{H})$ is closed (see 2). Thus we obtain that $T_2(\mathcal{H}) = \overline{T_2(\mathcal{H})}$ and $T_2, T_3 \in \mathcal{L}(\mathcal{H}; T_2(\mathcal{H}))$. It follows from (9) that T_2 is injective, hence T_2 is invertible. Therefore $T_3 \in \mathcal{L}(\mathcal{H}; T_2(\mathcal{H}))$ is also invertible by Theorem 3 and (8).

Finally, we define $\mathcal{M}_1 := T_2(\mathcal{H})$, $\mathcal{M}_2 := \mathcal{M}$ and $\mathcal{M}_3 := \mathcal{N} \oplus \overline{\text{Lin}f(\mathcal{H})}^\perp$. Next, we define $\varphi: \mathcal{H} \rightarrow \mathcal{M}_2, \gamma: \mathcal{H} \rightarrow \mathcal{M}_3$ by $\varphi := \varphi_2, \gamma := \varphi_3 + \varphi_1$. Moreover, we define $A \in \mathcal{L}(\mathcal{H}; \mathcal{M}_1)$ by $A := T_2$. Clearly A is invertible and $(A^*)^{-1} = T_3$. Thus we get $f = A + \varphi$ and $g = (A^*)^{-1} + \gamma$ and $\mathcal{K} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3$, where $\mathcal{M}_k \perp \mathcal{M}_j$ for $k \neq j$. □

Corollary 5. *Suppose that $\dim \mathcal{H} < \infty$. Let $f, g: \mathcal{H} \rightarrow \mathcal{H}$ satisfy Eq. (1). Then $f, g \in \mathcal{L}(\mathcal{H}; \mathcal{H})$ and $f = (g^*)^{-1}$.*

Proof. By the above theorem we have a decomposition $f = A + \varphi, g = (A^*)^{-1} + \gamma$ for some invertible $A \in \mathcal{L}(\mathcal{H}; \mathcal{M}_1)$ and some mappings $\varphi: \mathcal{H} \rightarrow \mathcal{M}_2, \gamma: \mathcal{H} \rightarrow \mathcal{M}_3$ (where $\mathcal{H} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3$ and $\mathcal{M}_k \perp \mathcal{M}_j$). Since $A \in \mathcal{L}(\mathcal{H}; \mathcal{M}_1)$ is invertible and $\dim \mathcal{H} < \infty$, it follows that $\mathcal{M}_1 = \mathcal{H}$ and $\mathcal{M}_2 = \{0\} = \mathcal{M}_3$. So $\varphi = 0 = \gamma$. □

Corollary 6. *Let $f: \mathcal{H} \rightarrow \mathcal{K}$ satisfy the equation*

$$\forall_{x,y \in \mathcal{H}} \quad \langle f(x)|f(y) \rangle = \langle x|y \rangle. \tag{10}$$

Then f is a linear isometry.

Proof. By Theorem 4 we have a decomposition $f = A + \varphi, f = (A^*)^{-1} + \gamma$ for some $A \in \mathcal{L}(\mathcal{H}; \mathcal{M}_1)$ and $\varphi: \mathcal{H} \rightarrow \mathcal{M}_2, \gamma: \mathcal{H} \rightarrow \mathcal{M}_3$. Thus we get $A + \varphi = (A^*)^{-1} + \gamma$. Since $\mathcal{M}_2 \perp \mathcal{M}_3$, we get $\varphi = 0 = \gamma$. Hence $f = A$. Then f is a linear mapping and by (10), f is a linear isometry. □

Similar investigations have been carried out in the manuscript [2]. Namely, Eq. (1) and its stability, as well as the approximate orthogonality preserving property are considered in [2].

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