On Classical Representations of Convex Descriptions *

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It is demonstrated that if V^* is not a vector lattice, where V is a base norm Banach space, then there is no commutative observable providing a classical representation for V. This observation generalizes a similar result of Busch and Lahti, obtained for V – the trace class of operators on a separable complex Hilbert space.

1. Introduction

Recently the interest in the general properties of and conditions for phase-space representations of quantum theories has grown. Since the pioneering work of Wigner and others the classical (phase-space) representation of standard quantum mechanics reached a remarkable level of mathematical elegance and clarity. This representation can be extended over the class of physical theories admitting a description in terms of statistical dualities (see [1–3]), which includes the standard (von Neumann's) quantum mechanics as a special case. It is natural then to look for these properties of classical representations of the standard quantum mechanics which could be lifted up to the more general level of description. One of such properties, demonstrated by Busch and Lahti ([4], Th. 2.1.2, see also [1], Prop. 2.3) says that no commutative semi-spectral measure is informationally complete.

The mathematics employed below can be found in the known monographs of Alfsen [5] and Asimow and Ellis [6] as well as in the Marburg Institute lecture volume [7].

2. The General Framework

Let V be a base norm Banach space with base S. We will see S as representing the set of states of a physical system. The extreme boundary ∂S of S represents pure states, whereas the other elements of S correspond to mixed states. The model (description) of a physical system resulting from V will be called *classical* if the dual space V^* is a Banach lattice in the natural ordering. With some abuse of terminology we will apply the term "classical" directly to V or to its base S. If V is classical, then the base of the second dual space V^{**} is a Bauer simplex. S is a weak* dense subset of the base of V^{**}, hence our definition identifies classical descriptions as those having the unique decomposability property, see [8, 9]. Nevertheless a classical (according to our definition) S does not need to be a simplex.

The Banach dual space V^* of a base norm Banach space V is known to be an order unit Banach space. The order interval [o, e] of V^*, with o the origin and e the order unit of V^*, is here interpreted as the set of effects (elementary observables) related to the physical system in question. Another interpretation assumes V^* to represent the set of expectation functionals of a specific class of real-valued observables (sharp observables, observables represented by bounded self-adjoint operators in the Hilbertian model). These interpretations are not equivalent, the spectral theory of Alfsen and Shultz [10] provides conditions which lead to a resolution of elements of V^* into integrals of extreme effects.

If V^* is a lattice, the extreme boundary ∂[o, e] of the order interval $[o, e] \subset V^*$ is a σ-complete Boolean lattice in the natural ordering. This fact supports our use of the term "classical" in this case, because ∂[o, e] is an analogue of the "quantum logic".

3. Observables

A map $B: V_1 \mapsto V_2$ will be called a (generalized) *observable* if (i) V_1 and V_2 are base norm Banach

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spaces with bases S_1, S_2, respectively, (ii) B is linear, (iii) $B(S_1) \subset S_2$, and (iv) V_2^* is a Banach lattice. The notion of observable is a special case of a more general notion of coarse-graining [11]. To clarify the definition of observable we should imagine S_1 as a model of the set of states of a physical system and S_2 (or the base of V_2^{**}) as a generalization of the set of all probability measures on the value-space (the space of possible outcomes of single measurements) of a physical observable.

Any observable B: $V_1 \mapsto V_2$ is a positive norm-continuous map, hence its dual B^*: $V_2^* \mapsto V_1^*$ is positive and weak* to weak* continuous. B^* restricted to $\partial\,[o_2, e_2]_2$ = the extreme boundary of the set of effects of V_2, defines a $[o_1, e_1]_1$-valued measure on the Boolean σ-lattice $\partial\,[o_2, e_2]_2$, which will be called the *abstract semi-spectral resolution* of the observable B. In order to get in contact with the semi-spectral resolution of Hilbert-space operators we should represent the order unit Banach lattice V_2^* as $C(X)$ with X = the extreme boundary of the base of V_2^{**} (the Stone-Kakutani-Krein-Yosida theorem). Then B^* defines a $[o_1, e_1]_1$-valued measure on X.

If an observable B: $V_1 \mapsto V_2$ is injective, we will call it *informationally complete*. In this case B^* is surjective (comp. [5] Th. II.5.9). Hence, for an informationally complete B, the fact that Lin $\partial\,[o_2, e_2]_2$ is weak* dense in V_2^* (a consequence of the Banach-Alaoglu and the Krein-Milman theorems) implies that Lin $B^*(\partial\,[o_2, e_2]_2)$ is weak* dense in V_1^*. This observation restates a result of Singer and Stulpe ([1], Th. 3.6) obtained in a somewhat different context.

4. Classical Representations

Because of its injectivity an informationally complete observable B: $V_1 \mapsto V_2$ provides an alternative description of the physical system represented by V_1. Moreover, the resulting description is classical, so any informationally complete observable can be seen as a *classical representation* of the system in question. The equivalence of the notion of informationally complete observable and the one of classical representation was for the first time realized by Prugovečki (see references in [3], compare [1] Th. 3.5, [12]).

An observable B: $V_1 \mapsto V_2$ will be called *commutative* if (i) $B^*(V_2^*)$ is a Banach lattice with respect to the order and norm inherited from V_1^*, and (ii) B^* is a lattice homomorphism. It is well known that a C^*-algebra with unit is commutative if its self-adjoint part is a Banach lattice with the order-unit norm (see e.g. [6] Cor. 2.4, p. 230). This fact explains our definition of commutative observable.

Let us recall that if B: $V_1 \mapsto V_2$ is a classical representation of V_1 then B^* has to be surjective. If B would be then commutative, V_1^* should be a Banach lattice. Thus we conclude with the general version of the result of Busch and Lahti:

If a base norm Banach space V is not classical (i.e. if V^* is not a Banach lattice) then no classical representation of V could be provided by a commutative observable.

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