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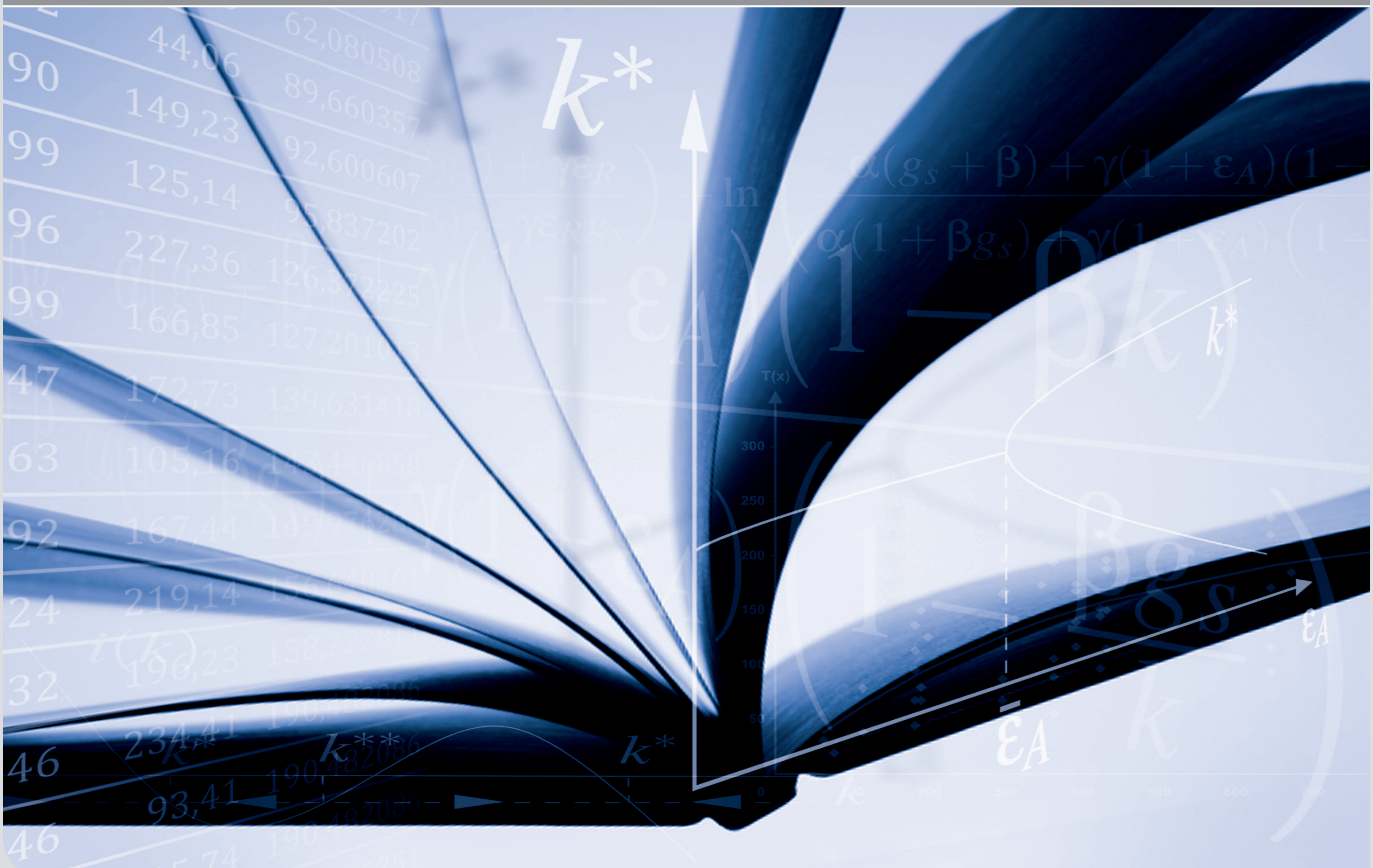

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Resource Allocation by Frugal Majority Rule*

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Abstract

We propose a model of 'frugal aggregation' in which the evaluation of social welfare must be based on information about agents' top choices plus general qualitative background conditions on preferences. The former is elicited individually, while the latter is not. We apply this model to problems of public budget allocation, relying on the specific assumption of separable and convex preferences.

We propose and analyze a particularly aggregation rule called 'Frugal Majority Rule.' It is defined in terms of a suitably localized net majority relation. This relation is shown to be consistent, i.e. acyclic and decisive; its maxima minimize the sum of the natural resource distances to the individual tops. As a consequence of this result, we argue that the Condorcet and Borda perspectives – which conflict in the standard, ordinal setting – converge here. The second main result provides a crisp algorithmic characterization that renders the Frugal Majority Rule analytically tractable and efficiently computable.

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1 Introduction

Many economic and political decisions involve collective decisions to allocate resources under a budget constraint. Examples are the allocation of public goods, the redistribution across classes of beneficiaries, the allocation of tax burden, the choice of intertemporal expenditure streams, or the macro-allocation between expenditure, tax receipts and net debt. Standard approaches to preference aggregation and voting assume ordinal or even cardinal preference information as their input. Their application to public resource allocation problems poses substantial difficulties for a variety of reasons.

First, the number of alternatives is typically large (with discrete inputs), or even infinite (with continuous inputs). This numerical 'complexity' is aggravated when the number of alternative uses of the public resource is large. In the first instance, this means that it may be practically difficult – if not entirely unrealistic – to elicit an agent's entire preference ordering as an input to the aggregation, even setting aside additional issues of active strategic 'manipulation.' Not only is the determination of a complete ordering over a rich set of alternatives a cognitively demanding task (even allowing for short-cuts such as the restriction to reasonably flexible functional forms), in the context of collective decision-making and voting, there is a serious motivational issue due to the limited influence of any single agent on the final decision.

Secondly, the operation of common aggregation rules such as the Borda count, the Copeland or Kemeny rules is unclear, intransparent or even unsound; on an infinite domain, these rules are in fact not necessarily well-defined.[1]

Third, except for the one-dimensional case with two public goods and single-peaked preferences (Black [1948], Arrow [1951/63]), one obtains generic impossibility results under almost every reasonable domain restriction (Kalai et al. [1979], Le Breton and Weymark [2011]) just as in spatial voting models (Plott [1967]). In particular, in higher dimensions there is no hope to generally find a Condorcet winner even if all agents have well-behaved preferences. Finally, the indeterminacy of majority voting is generic and can be severe; for example, generically every alternative can be the outcome of a dynamic (non-strategic) majority vote for an appropriate agenda (McKelvey [1979]).

In this paper, we tackle these difficulties by drastically shrinking the informational base of the aggregation procedure; specifically, (i) we elicit from each agent her top-ranked alternative only, and (ii) we rely on appropriate background information that is assumed to be satisfied by all agent's true complete preference rankings. We shall refer to this as the '*frugal* model of aggregation.'[2]

We will propose a particular frugal aggregation rule called 'Frugal Majority Rule' (FMR), defend its normative rationale and analyze its basic properties. Throughout, we take a non-strategic point of view. To further justify its potential application in practice, the present contribution needs to be complemented by an analysis of its strategic properties which we leave to future work; below, we briefly comment on some indications of why the FMR may

[1]But see, e.g., Feld and Grofman [1988] for an extension of Borda count to an n-dimensional issue space.

[2]Our use of the term 'frugal' alludes to the notion of 'fast and frugal heuristics' made popular in cognitive psychology by G. Gigerenzer and his co-authors, see, e.g., Gigerenzer and Goldstein [1996]. We intend to connote both the cognitive economy at the individual level, and the simplicity and efficacy of the aggregation procedure itself.

perform quite well from a strategic angle. Anticipating the brief discussion below, for now we note that the 'tops-only' requirement (i) above can be heuristically viewed as a incentive compatibility requirement of sorts which says that non-top preference information cannot be reliably used when agents choose their input strategically.[3]

To understand the role of the background information (ii) in the frugal aggregation model generally, consider the limiting, special case of *no* information, i.e., *all* that is known about an agent is her top. In this case, that is: relative to this informational state, there is a clearcut answer what the anonymously welfare-*optimal* social choice is: namely that chosen by the most agents ('plurality rule'); see Goodin and List [2006] for a formal treatment. It is widely agreed that this choice may be quite poor when evaluated on the basis of the complete ordinal ranking of alternatives, a much richer informational base, see e.g., Laslier [2012]. But this simply means that a lot of valuable information may be lost when the social evaluation/social evaluator needs to rely exclusively on agents' top choices.[4]

Yet, in resource allocation problems, one can do better, as it is sensible in many cases to rely on richer background information such as preference convexity. Consider the special case of two goods which leads to a one-dimensional aggregation problem on the 'budget line,' so that convexity is simply single-peakedness over feasible allocations. With single-peaked preferences, the choice of the median voter, i.e. the median of the tops, has strong credentials as welfare-optimal on Arrowian grounds (Arrow [1951/63]). But choosing the median of the tops defines a frugal aggregation rule, so, from this perspective, the restriction to a frugal informational basis arguably involves no loss at all!

Note that the argument for plurality rule is an argument from ignorance, while the argument for the median top in one dimension is one from partial knowledge. Our goal in this paper is to extend these two lines of argument to the case of multiple goods in order to determine the 'frugally best' alternative(s). To achieve this, we will assume that the social evaluator's background information consists of convexity plus separability of preferences. This assumption turns out to work particularly well since it strikes just the right balance between ignorance and knowledge.[5]

The social evaluator's total information thus consists of this background information, the individual tops, *and nothing else*. Ethically, the social evaluator is committed to give all agent's interests equal consideration. In most of the paper, feasible sets are discrete, with a fixed number of resource units to be allocated, possibly subject to upper and lower bounds; the extension to the continuous case is described towards the end. Under these assumptions, the FMR can be motivated from three different perspectives –broadly Condercetian, Borda-like and 'imputed-utilitarian.' We begin by developing the Condorcetian perspective which is based on an appropriate pairwise 'frugal betterness' relation. So, given the social evaluator's

[3]Indeed, incentive compatibility ('strategy-proofness') implies the tops-onliness property on a large class of preference domains, see Chatterji and Sen [2011].

[4]The loss of information under plurality rule is especially severe in situations in which all agents have different tops, rendering all tops plurality winners with a count of one. Such profiles would be common in budget allocation problems in which the number of alternatives is typically large relative to the number of agents.

[5]The precise condition of *'separable convexity'* corresponds roughly to, but is weaker than, the existence of an additively separable and concave utility representation, see Section 3.

total information and equal-treatment commitment, when is some feasible allocation x to be deemed 'frugally better' than some other allocation y? For some tops, the evaluator can deduce a preference of x over y, for others a preference of y over x, while for still others the comparison remains ambiguous. If one of the first two groups is in absolute majority, this arguably settles the question, but if not, one needs to deal with the ambiguous cases. This can be done by appeal to a 'principle of insufficient reason' which treats the absence of comparison as equivalent to indifference. By consequence, 'frugally betterness' is then determined by the relative (or 'net') majority of those agents whose preference among x and y can be deduced unambiguously.

While this is on the right track the argument needs refinement since net majorities among arbitrary pairs of alternatives may be cyclical and preclude the existence of any best element. Hence, we argue that a more circumspect application of this principle should govern comparisons of 'local' pairs only, so that frugal betterness is properly identified with the localized net majority relation. The first main result of the paper (Theorem 1) shows that this relation is indeed always acyclic and yields a well-behaved plateau of frugal optima; moreover, these optima are exactly those allocations that minimize the sum of (l_1-)distances to the tops.[6]

On the basis of this characterization, we also show that frugal majority winners can be motivated from a Borda-inspired outlook; the conflict between the Condorcet and Borda philosophies that shapes much of ordinal preference aggregation theory is thus rendered moot. Relatedly, one can give the frugal majority set a utilitarian-type justification by imputing to each agent a cardinal, interpersonally comparable utility function that is concave and separable. These characterizations also imply that the FMR is immune to the famous 'no-show paradox' variants of which plague Condorcet consistent choice rules under complete ordinal preference (see especially Moulin [1988]).

We provide a complementary second characterization result (Theorem 2) of FMR as equivalent to a *uniform quota rule* which significantly clarifies the effective operation of the FMR and delivers a fast and crisp, spreadsheet-implementable algorithm to compute it. This characterization starts with the observation that, in the two-goods case, the allocation associated with the (one-dimensional) median of the tops is given by the vector of coordinate-wise medians of agents demands. How can this generalized to multiple goods? The vector of coordinate-wise medians will not work in general, since it will not generally add up to the budget constraint. But this can be fixed by aggregating coordinate-wise demands based on a uniform quota q^* possibly different from $\frac{1}{2}$ so that exactly q^* agents' demands can be satisfied, and q^* is chosen endogenously based on the profile to satisfy the budget constraint as exactly as possible. Theorem 2 is analytically quite powerful. Among other things, it helps clarifying the size of the set of FMR winners and the conditions under which this set is (essentially) unique.

The Frugal Majority Rule is motivated and defended here as characterizing anonymous welfare optima under frugal information conditions. Can these optima be achieved, at least to a satisfactory degree, and if so, how? And, more specifically, can the FMR itself be employed as a viable voting mechanism?

[6]The FMR can thus be viewed as an instance of the 'median rule' known from general aggregation theory (see, e.g., Barthélémy and Monjardet [1981], Nehring and Pivato [2018]).

To address this question adequately, one will need to analyze the incentive properties of the mechanism and their influence on plausible voting equilibria. Obviously, one cannot expect the FMR to be fully strategy-proof, since it is known that, on the domain of resource allocation problems with convex and separable preferences only dictatorial choice functions are strategy-proof.[7] At the same time, there are promising indications that the FMR may have favorable incentive properties; let us mention two. First, since it chooses the median of the tops FMR is known to be strategy-proof in the baseline case of allocating two goods.[8] Second, the general analysis of strategy-proof social choice has taught quite robustly (Chatterji and Sen [2011]) that incentive compatibility *requires* exclusive reliance on preference-tops; further, in those domains for which strategy-proof mechanisms do exist, they have majoritarian character. These two feature are shared by the FMR; the problem is 'just' that the FMR operates on an impossibility domain so that full strategy-proofness is an impossible standard to meet. Nevertheless, one can show that FMR has interesting and valuable partial strategy-proofness properties, see Nehring and Puppe [2019b].[9]

Relation to the Literature

To the best of our knowledge, this is the first frugal model of its kind, in the context of resource allocation or elsewhere. Of course, the broader issue of informational parsimony is widely recognized as significant; this is testified, for instance, by the great popularity of approval voting in recent voting theory and, to some extent, also in practice, see the Handbook of Approval Voting (Laslier and Sanver [2010]). In approval voting, agents classify alternatives dichotomously in 'good' or 'bad,' but this furnishes all the preference information that is used; background information on the shared structure of preferences as in the present frugal model plays no role.

There are a few strands of literature specifically devoted to the welfare and/or voting aspects of public budgeting. A first strand addresses Arrovian preference aggregation on economic domains, see Kalai et al. [1979] for a central contribution and Le Breton and Weymark [2011] for an extensive survey. The conclusion from this literature is that all standard aggregation procedures violate at least some fundamental desiderata even under reasonable domain restrictions. Among the preference domains that have been shown to yield impossibility results are the domains of all separable preferences and the domain of all convex preferences, respectively, and a straightforward application of the general arguments shows that the same conclusion is also obtained if the conditions of separability and convexity are imposed jointly.

With an eye towards possibility results – specifically with respect to characterizing the uncovered and Banks sets – Dutta et al. [2005] study public budgeting from a majoritarian perspective. They assume linear preferences and uncover an interesting richness of facets

[7]Zhou [1991] proves this for the domain of all convex preferences; for separably convex preferences, this follows from Nehring and Puppe [2007, 2010].

[8]In this regard, it contrasts rather starkly to, for example, the *mean rule* that chooses the average of the individual tops and that is an obvious alternative as a frugal choice rule; the mean rule and its properties are discussed in more detail in Nehring and Puppe [2019a].

[9]Beyond partial strategy-proofness, one would want to determine plausible equilibrium voting outcomes under FMR as a game form. A preliminary study of this question has been undertaken in the dissertation Lindner [2011] based on Nehring and Puppe [2008].

already with three goods and three agents. By contrast, in the frugal model, an arbitrary number of goods and agents is handled quite easily and transparently for a much larger class of preferences. In line with their broad motivation, we have some hope that the frugal model might be useful also from a more positive rather than normative angle as an 'institution-free' modeling tool in the study of political economy questions.

There is also a sizeable and multi-faceted literature, largely outside economics, on 'participatory budgeting' which concerns voting among a range of (local) public projects. On the one hand, the political science literature focuses on questions of institutional design, see Shah [2007] and the references therein. On the other hand, the computer science literature tends to emphasize combinatorial and algorithmic aspects arising from indivisibilities, see, e.g., Shapiro and Talmon [2017] and Faliszewski and Talmon [2019]. Normatively, these contributions focus on issues of participation and fairness, frequently assuming cardinal information, see, e.g., Aziz et al. [2017]. By contrast, we consider the allocation of expenditure shares and pose the Arrovian-style question of what constitutes a welfare optimum assuming a frugal informational base.

The remainder of the paper is organized as follows. The next section introduces the general frugal aggregation model. We define the notion of frugal majority rule with the principle of insufficient reason at its heart, and we discuss two well-known and well-studied special cases. The first is the case of an unrestricted domain for which frugal majority rule amounts to *plurality rule*; indeed, without any background restriction on individual preferences the principle of insufficient reason forces to treat all non-top alternatives symmetrically. The second is the one-dimensional case of two public goods in which frugal majority rule with single-peaked (i.e. convex) preferences amounts to standard median voting on the line. In particular, in this case the social optimum is given by the median alternative (or by a set of median alternatives in the case of an even number of agents). While in this case the winning alternative(s) are the same as in the standard median voter model, the frugal aggregation model uses much less information (and in fact does in general not induce the same ranking of non-top alternatives).

Section 3 discusses the application of the frugal aggregation model to the multi-dimensional collective resource allocation problem with separably convex preferences. Separable convexity is a multi-dimensional version of single-peakedness and reduces to single-peakedness in the one-dimensional case. Our first main result, Theorem 1, shows that under this background assumption on preferences the set of frugal majority winners is always non-empty, box-convex and coincides with the allocations that minimize the sum of the box-distances (mathematically, l_1-distances) to the individual tops. We also demonstrate that the particular combination of separability and convexity of preferences represents a distinguished compromise between specificity and flexibility in our present context. Indeed, a result akin to Theorem 1 would fail both for the larger class of convex but not necessarily separable preferences, and for the smaller class of Euclidean preferences (see Section 3.1). In Section 3.2, we show that the frugal majority winners coincide with a natural version of 'frugal Borda winners' (Proposition 1); we also demonstrate that the frugal majority winners admit an interpretation as the utilitarian maximizers with respect to a natural class of imputed cardinal utility functions (Proposition 2).

Section 4 provides a simple first-order characterization of the set of frugal majority winners in terms of an endogenous quota (Theorem 2). Besides the efficient computability of the set of frugal majority winners, this has a number of other corollaries. In Section 4.1, we observe that frugal majority rule respects coordinate-wise unanimity; moreover, it satisfies an appropriate condition of 'frugal Pareto efficiency' (Proposition 3). In Section 4.2, we use Theorem 2 to show that every pair of frugal majority winners differ in each coordinate by at most one unit ('essential uniqueness') whenever the profile of tops is connected in each coordinate (Proposition 4). Finally, in Section 4.3 we show that frugal majority rule avoids the no-show paradox in the strong sense that, for each agent, the frugal majority set under own participation dominates the corresponding set without own participation.

In Section 5 we observe that much of the analysis carries over without difficulty to the case of alternatives in continuous space, i.e. to budget hyperplanes in $\mathbb{R}^L$. Moreover, we indicate how to accommodate non-finite, e.g. continuous, distributions of agents; with an atomless distribution of agents, the frugal majority winner is in fact always unique. Section 6 concludes.

2 Frugal aggregation

In this section, we introduce and propose a general framework, the *frugal aggregation model*, which we will apply later to address the specific problem of collective resource allocation. The frugal aggregation model assumes that individual preferences are only partially elicited. Specifically, we consider a maximally sparse message space of one single alternative for each individual (the respective top alternative). Moreover, the collective choice mechanism is assessed under the background assumption that the non-elicited preferences of individuals come from a common and known domain of *admissible* complete preference orderings on a set of alternatives. Henceforth, a *frugal aggregation problem* is given by a pair $(X, \mathcal{D})$, where X is a universal set of alternatives and $\mathcal{D}$ is the associated domain of admissible preference orderings on X. We assume that all admissible preference orderings $\succcurlyeq \in \mathcal{D}$ have a unique top element, denoted by $\tau(\succcurlyeq) \in X$.

In our exposition, we envisage a social evaluator who has knowledge about individuals' top alternatives and the general admissibility restriction on preferences but who is either ignorant about the true individual preferences among non-top alternatives, or refrains from using that information. Observe that the preference information that can be inferred from a specific top alternative crucially depends on the general admissibility restriction; indeed, by submitting the top alternative $\theta \in X$ an individual 'reveals' that her complete preference ordering must come from the set $\mathcal{D}_\theta$, where, for all $\theta \in X$,

$$\mathcal{D}_\theta := \{\succcurlyeq \in \mathcal{D} \ : \ \tau(\succcurlyeq) = \theta\}.$$

For all $\theta \in X$, and all distinct $x, y \in X$, let

$$x >_\theta^{\mathcal{D}} y \ :\Leftrightarrow \ \text{for all } \ \succcurlyeq \in \mathcal{D}_\theta, \ x \succ y.$$

Thus, the partial order $>_\theta^{\mathcal{D}}$ describes exactly the strict preference information that the social evaluator can infer from an agent's top alternative θ given the background assumption that all preferences come from the common domain $\mathcal{D}$.[10] Observe that we allow for the possibility that there is in fact no preference ordering in $\mathcal{D}$ with top θ, in which case $x >_\theta^{\mathcal{D}} y$ holds trivially for all pairs $x, y \in X$; on the other hand, if $\mathcal{D}_\theta \neq \emptyset$ then $>_\theta^{\mathcal{D}}$ is a strict partial order, in particular asymmetric.

2.1 Frugal majority rule and the principle of insufficient reason

Suppose now that each individual i submits an alternative $\theta_i \in X$ representing his or her most preferred alternative from X and denote by $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ the profile of tops, where we allow for variable population size $n \in \mathbb{N}$. For every $x \in X$ denote by $\#_{\boldsymbol{\theta}}(x)$ the *mass of* x *under* $\boldsymbol{\theta}$, i.e. the number of individuals i such that $\theta_i = x$. Moreover, denote by $\operatorname{supp} \boldsymbol{\theta} := \{x \in X : \#_{\boldsymbol{\theta}}(x) > 0\}$ the *support* of $\boldsymbol{\theta}$. Finally, for all subsets $Y \subseteq X$, denote by $\#_{\boldsymbol{\theta}}(Y) := \#\{i : \theta_i \in Y\} = \sum_{x \in Y} \#\boldsymbol{\theta}(x)$ the popular support of Y.

Consider any pair $x, y \in X$ of alternatives, and denote by $\mathrm{B}_{\boldsymbol{\theta}}^{\mathcal{D}}$ the *social betterness* relation given the profile $\boldsymbol{\theta}$ in the frugal aggregation problem $(X, \mathcal{D})$. In the special case in which the support of $\boldsymbol{\theta}$ is concentrated on the pair x, y, i.e. if $\operatorname{supp} \boldsymbol{\theta} = \{x, y\}$, then by standard arguments (using the conditions of anonymity, neutrality and positive responsiveness, cf. May [1952]) one obtains

$$x \, \mathrm{B}_{\boldsymbol{\theta}}^{\mathcal{D}} \, y \text{ if and only if a (weak) majority votes for } x.$$

But $\operatorname{supp} \boldsymbol{\theta} = \{x, y\}$ is clearly a very special case, and the key question becomes how to count other votes in the comparison between x and y. The fundamental idea underlying the frugal aggregation model is to treat missing information *symmetrically*, i.e. to treat the absence of unambiguous comparison as indifference. Thus, in our approach the principle of insufficient reason is embodied in the *net majority criterion* to which we turn now.

For all pairs of distinct alternatives $x, y \in X$, let

$$\rangle x, y \langle^{\mathcal{D}} := \{\theta \in X : x >_\theta^{\mathcal{D}} y\}.$$

Thus, $\rangle x, y \langle^{\mathcal{D}}$ the set of alternatives such that all agents with top in $\rangle x, y \langle^{\mathcal{D}}$ are known to strictly prefer x to y, and we refer to this set as the *set supporting* x *over* y. If the popular support of $\rangle x, y \langle^{\mathcal{D}}$ is at least as large as the popular support of $\rangle y, x \langle^{\mathcal{D}}$, we say that x is the *(weak) net majority winner against* y under $\boldsymbol{\theta}$, and denote this by $x \, \mathrm{NM}_{\boldsymbol{\theta}}^{\mathcal{D}} \, y$. Formally, for all $x, y \in X$,

$$x \, \mathrm{NM}_{\boldsymbol{\theta}}^{\mathcal{D}} \, y \ :\Leftrightarrow \ \#_{\boldsymbol{\theta}} \left(\rangle x, y \langle^{\mathcal{D}} \right) \geq \#_{\boldsymbol{\theta}} \left(\rangle y, x \langle^{\mathcal{D}} \right).$$

The binary relation $\mathrm{NM}_{\boldsymbol{\theta}}^{\mathcal{D}}$ will also be referred to as the *net majority tournament* (given $\boldsymbol{\theta}$). For agents with top outside the union of the sets $\rangle x, y \langle^{\mathcal{D}}$ and $\rangle y, x \langle^{\mathcal{D}}$, the background assumption of preference admissibility is compatible with indifference between x and y, or with either strict ranking among x and y. For future reference, denote this set, for all distinct $x, y \in X$, by

$$x \bowtie^{\mathcal{D}} y \ := \ X \setminus \left(\rangle x, y \langle^{\mathcal{D}} \cup \rangle y, x \langle^{\mathcal{D}} \right).$$

[10] While the social evaluator could also infer indifference judgements in principle, it will become clear presently that it is useful to concentrate on *strict* preference information.

The frugal aggregation approach refrains from eliciting individual preferences beyond the information about the tops and admissibility. By consequence, the appeal to the principle of insufficient reason implies to treat the agents with top alternative in $x \bowtie^{\mathcal{D}} y$ symmetrically, i.e. as indifferent between x and y. Our provisional answer to the question of social betterness among pairs of alternatives within the frugal aggregation model is thus to identify the relation $\mathrm{B}^{\mathcal{D}}_{\boldsymbol{\theta}}$ with the net majority tournament:

Normative claim (provisional) *For all distinct* $x, y \in X$*,*

$$x \, \mathrm{B}^{\mathcal{D}}_{\boldsymbol{\theta}} \, y \;\Leftrightarrow\; x \, \mathrm{NM}^{\mathcal{D}}_{\boldsymbol{\theta}} \, y.$$

Accordingly, we (provisionally) define the set $C(X; \boldsymbol{\theta})$ of 'social welfare optima' given the profile $\boldsymbol{\theta}$ as the maximal elements in X with respect to $\mathrm{B}^{\mathcal{D}}_{\boldsymbol{\theta}}$, i.e.

$$C(X; \boldsymbol{\theta}) \;:=\; \{x \in X \;:\; x \, \mathrm{B}^{\mathcal{D}}_{\boldsymbol{\theta}} \, y \;\text{ for all } y \in X\} \tag{2.1}$$

(observe the completeness of the binary relations $\mathrm{NM}^{\mathcal{D}}_{\boldsymbol{\theta}}$ and $\mathrm{B}^{\mathcal{D}}_{\boldsymbol{\theta}}$). This proposal of course raises the question whether $C(X; \boldsymbol{\theta})$ is always non-empty. Before we analyze this problem in detail, let us consider two simple and instructive special cases that will help to understand the underlying motivation and the difference between the frugal aggregation model and ordinal preference aggregation on restricted domains.

2.2 Two special cases

2.2.1 Frugal aggregation on the unrestricted domain: the plurality rule

Suppose that admissibility does not convey any additional information, i.e. consider the special case $\mathcal{D} = \mathcal{U}$ where $\mathcal{U}$ is the unrestricted domain of all weak preference orderings with a unique top. Evidently, in this case we have for all $x, y \in X$, $\rangle x, y \langle^{\mathcal{U}} = \{x\}$, and hence

$$x \, \mathrm{NM}^{\mathcal{U}}_{\boldsymbol{\theta}} \, y \;\Leftrightarrow\; \#_{\boldsymbol{\theta}}(x) \geq \#_{\boldsymbol{\theta}}(y),$$

i.e. the maximal elements of the net majority tournament are simply the plurality winners: the alternatives that are named by most agents as their respective top choice. The indifference principle is justified since, from an impartial welfare perspective, there is simply no basis to favor an alternative x over any other alternative y given a vote for $\theta \notin \{x, y\}$. Clearly, the net majority tournament $\mathrm{NM}^{\mathcal{U}}_{\boldsymbol{\theta}}$ and hence also the social betterness relation $\mathrm{B}^{\mathcal{U}}_{\boldsymbol{\theta}}$ are transitive in this case; consequently, the set $C(X; \boldsymbol{\theta})$ of social welfare optima is non-empty for every profile (and coincides with the plurality winners, as noted).

2.2.2 Frugal aggregation on the line: median voting

As another special case, consider the case of a one-dimensional space of alternatives X linearly ordered from left to right by $>$ à la Downs [1957]. Here, it is natural to require that individual preferences be *single-peaked*. Denote by $\mathcal{D}_{\mathrm{sp}}$ the set of all weak orderings on X that are single-peaked with a unique top alternative.[11] Consider two distinct alternatives $x, y \in X$, say such

[11] Formally, a preference ordering $\succcurlyeq$ with top alternative θ is single-peaked if, for all distinct $x, y \in X$, $x \succ y$ whenever $y < x \leq \theta$ or $\theta \geq x > y$.

that $x < y$. Any top θ not in the interior of the interval $[x, y]$ induces a strict preference either for x (if $\theta \leq x$), or for y (if $\theta \geq y$); on the other hand, a top strictly between x and y is compatible with any preference between x and y, i.e. $x \bowtie^{\mathcal{D}_{\mathrm{sp}}} y = \{w \in X : x < w < y\}$. In particular, we obtain

$$x \, \mathrm{NM}_{\boldsymbol{\theta}}^{\mathcal{D}_{\mathrm{sp}}} y \;\Leftrightarrow\; \#_{\boldsymbol{\theta}}(\{w \in X : w \leq x\}) \geq \#_{\boldsymbol{\theta}}(\{w \in X : w \geq y\}).$$

As is easily verified the net majority tournament is transitive in this case. Moreover, the social welfare optima $C(X; \boldsymbol{\theta})$ according to (2.1) are given by the median alternative(s).[12]

The case of the unrestricted domain in 2.2.1 clearly demonstrates the difference between the frugal aggregation model and standard ordinal preference aggregation. Indeed, the principle of insufficient reason necessarily yields plurality rule *provided* that one refrains to use any non-top preference information.[13] The criticism frequently raised by social choice theorists against plurality rule (see, e.g., Laslier [2012]) can thus be traced back to its complete neglect of *any* non-top preference information.

By contrast, the outcome of the aggregation in the case of single-peaked preferences 2.2.2 is identical under the proposed frugal model and, say, simple majority voting with ordinal, single-peaked preferences. However, despite this superficial similarity in the case of single-peaked preferences on a one-dimensional space, the frugal aggregation model is fundamentally different from ordinal preference aggregation. As a first indication, observe that if all agents' tops are strictly between the alternatives x and y, then the frugal aggregation model necessarily declares them as indifferent (again, by the appeal to the principle of insufficient reason), whereas any ranking between x and y is possible under simple majority rule with single-peaked preferences using ordinal (but not top-induced) preference information. Moreover, other aggregation rules can give fundamentally different results. For instance, it is well known that the Borda rule need not select the Condorcet winner even if all preferences are single-peaked; by contrast, we will argue that in the frugal model the appropriate formulation of Borda rule yields exactly the net majority winners as outcome. Intuitively, the reason is that symmetric treatment of non-available preference information corresponds to applying Borda rule to *metric* individual preferences that admit an ordinal utility representation in terms of the negative distance to the top alternative; and for this class of ordinal preferences, Borda's and Condorcet's aggregation methods indeed give the same result.

The main task of the present study is to generalize Example 2.2.2 to higher dimensions and to show that in the frugal aggregation model, Borda's and Condorcet's methods coincide generally on multi-dimensional resource allocation problems.

3 Resource allocation with separably convex preferences

In the remainder of this paper we study the frugal aggregation model in the specific context of the following simple but fundamental resource allocation problem. A group of agents (a

[12]More precisely, the unique median top if the number of agents is odd, and all alternatives between the two middle tops if the number agents is even.

[13]For a related derivation of plurality rule, see Goodin and List [2006].

'society') has to collectively decide on how to allocate a fixed budget $Q \geq 0$ to a number L of public goods. Throughout we assume fixed prices, thus the problem is fully determined by specifying the expenditure shares. Furthermore, we assume that expenditure shares are measured in discrete amounts of money and that all individuals have monotone preferences. Expenditure x^ℓ on public good ℓ may be bounded from below and above, so that feasibility requires $x^\ell \in [q^\ell_-, q^\ell_+]$ for some integers q^ℓ_-, q^ℓ_+ where we allow that $q^\ell_- = -\infty$ and/or $q^\ell_+ = \infty$. Together, these assumptions allow us to model the allocation problem as the choice of an element of the following $(L-1)$-dimensional polytope

$$X := \left\{ x \in \mathbb{Z}^L \ : \ \sum_{\ell=1}^{L} x^\ell = Q \text{ and } x^\ell \in [q^\ell_-, q^\ell_+] \text{ for all } \ell = 1, ..., L \right\}, \tag{3.1}$$

where $\mathbb{Z}$ is the set of integers and $x = (x^1, ..., x^L)$. The space X is referred to as the set of *feasible allocations*, or alternatively, as a *resource agenda*.

Collective choice mechanisms on sets of alternatives of the form (3.1) have been addressed in the literature under various, economically meaningful preference restrictions, see e.g., Kalai et al. [1979], and Le Breton and Weymark [2011] for an extensive survey. Notwithstanding the many impossibility results that have been established in this literature within the framework of standard Arrovian preference aggregation, we will now demonstrate that a natural variant of separability plus convexity generates a strong possibility result in the frugal aggregation model. The domain of all 'separably convex' preferences on a resource agenda of the form (3.1) contains in particular all preferences that can be represented by an additively separable and concave utility function.

Definition (Separable Convexity) *For any allocation $x \in X$ denote by $x_{(kj)}$ the allocation that results from x by transferring one unit of money from good j to good k, i.e. $x^k_{(kj)} = x^k + 1$, $x^j_{(kj)} = x^j - 1$ and $x^\ell_{(kj)} = x^\ell$ for all $\ell \neq k, j$. Say that a preference order $\succcurlyeq$ on X is* separably convex *if $x \succ x_{(kj)}$ implies $y \succ y_{(kj)}$ for all k, j, x, y such that $y^k \geq x^k$ and $y^j \leq x^j$.*

Separable convexity contains two special cases: (i) 'linear' convexity (i.e. single-peakedness) and (ii) separability. Case (i) is given by the additional condition that $x^\ell = y^\ell$ for all $\ell \neq k, j$ (see the left panel in Fig. 1), while case (ii) is given by the additional condition that $y^k = x^k$ and $y^j = x^j$. Separable convexity integrates these two requirements but is somewhat stronger than the logical conjunction of linear convexity and separability. To see this, note that separability is vacuous for $L = 3$ due to the budget constraint on the domain of feasible allocations X over which $\succcurlyeq$ is defined. The right panel in Fig. 1 shows the general case for $L = 3$, $k = 3$ and $j = 2$, combining convexity and separability.

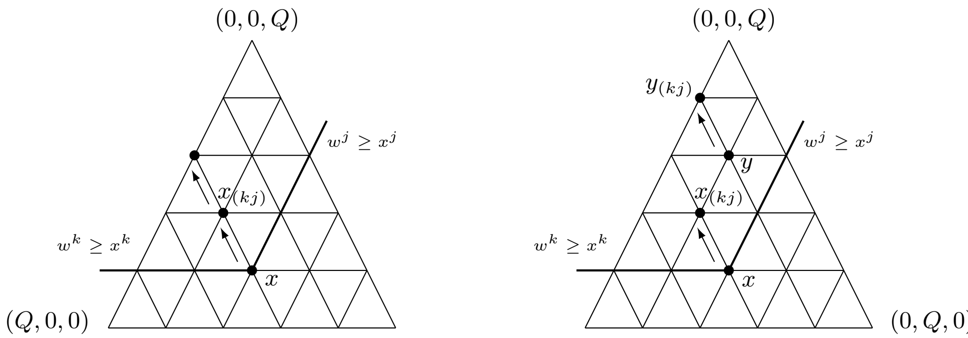

Fig. 1: Separable convexity

For all $x, y \in \mathbb{Z}^L$, let

$$[x, y] \ := \ \Big\{ w \in \mathbb{Z}^L \ : \ \text{for all } \ell = 1, ..., L, \ x^\ell \leq w^\ell \leq y^\ell \text{ or } y^\ell \leq w^\ell \leq x^\ell \Big\} .$$

We will refer to $[x, y]$ as the *box spanned by x and y* and to the elements of $[x, y]$ as the (not necessarily feasible) allocations *between x and y*. We say that a subset $Y \subseteq X$ is *box-convex* if $Y = [x, y] \cap X$ for some pair $x, y \in \mathbb{Z}^L$. As is easily verified, a subset $Y \subseteq X$ is box-convex if and only if $([w, z] \cap X) \subseteq Y$ for all $w, z \in Y$, i.e. if and only if Y contains with any two feasible allocations all feasible allocations between them.

Consider any separably convex preference $\succcurlyeq$ with top θ and two distinct feasible allocations x and y such that $x \in [\theta, y]$, i.e. such that x is between the top θ and y. Then, there exists a sequence of unit transfers from y to x that are strictly preferred at each step (by separable convexity) so that $x \succ y$ by transitivity. In other words, for all distinct $x, y \in X$, and all separably convex preferences with top θ,

$$x \in [\theta, y] \ \Rightarrow \ x \succ y.$$

Conversely, suppose that $x \notin [\theta, y]$, then there clearly exist separably convex preference orders with top θ such that $y \succcurlyeq x$ (see the proof of Lemma 3.1 in the Appendix for a concrete construction). Thus, for the domain $\mathcal{D}^*$ of all separably convex preferences on X and every $\theta \in X$, we obtain

$$x >_\theta^{\mathcal{D}^*} y \ \Leftrightarrow \ x \in [\theta, y].$$

For every domain $\mathcal{D}$, let $\tau(\mathcal{D}) := \{\tau(\succcurlyeq) : \succcurlyeq \in \mathcal{D}\}$. We will say that $\mathcal{D}$ is a *rich* subdomain of $\mathcal{D}'$ if $\mathcal{D} \subseteq \mathcal{D}'$ and, for all $\theta \in \tau(\mathcal{D})$, $>_\theta^{\mathcal{D}} = >_\theta^{\mathcal{D}'}$. Summarizing, we have the following result.

Fact 3.1 *Let $\mathcal{D}$ be a rich subdomain of separably convex preferences on X. Then, for all distinct $x, y \in X$,*

$$\rangle x, y \rangle^{\mathcal{D}} \ = \ \{\theta \in \tau(\mathcal{D}) \ : \ x \in [\theta, y]\} .$$

Two important examples of rich subdomains of separably convex preferences are (i) the domain $\mathcal{D}_{\text{add}}$ of all preference orders on X that can be represented by an additively separable utility function of the form

$$u(x) \;=\; u(x^1,...,x^L) \;=\sum_{\ell=1}^{L} u^\ell(x^\ell), \tag{3.2}$$

where the $u^\ell : \mathbb{R} \to \mathbb{R}$ are strictly increasing and concave for all $\ell = 1,...,L$, and (ii) the domain $\mathcal{D}_{\text{lin}}$ of all linear preferences on X that can be represented by a linear utility function of the form

$$u(x) \;=\; u(x^1,...,x^L) \;=\sum_{\ell=1}^{L} a^\ell \cdot x^\ell,$$

with pairwise different and strictly positive coefficients $\{a^1,...,a^L\}$.[14] Observe that linear preferences have a natural interpretation in terms of expected utility theory if the agenda X is viewed as a set of probabilistic lotteries over a set of L deterministic outcomes; in that case, the a^ℓ correspond to the von-Neumann-Morgenstern utilities and the requirement that they be pairwise distinct simply means that the preference over deterministic outcomes displays no indifferences.

Lemma 3.1 *The domains $\mathcal{D}_{\text{add}}$ and $\mathcal{D}_{\text{lin}}$ are rich subdomains of separably convex preference orders on X. Moreover, the domain $\mathcal{D}_{\text{qlin}}$ of all quasi-linear preferences representable as in (3.2) with $u^1(x^1) = x^1$ also forms a rich subdomain of separably convex preferences.*

(Proof in Appendix.)

Note that every allocation $\theta \in X$ can be the top of a preference order in $\mathcal{D}_{\text{add}}$ and even of a preference order in $\mathcal{D}_{\text{qlin}}$. By contrast, linear preferences in the domain $\mathcal{D}_{\text{lin}}$ can have their respective top only at an extreme point of X.

Examples 1 and 2 above also involve special cases of rich separably convex domains. The universal domain (on L alternatives) is obtained by considering $(X, \mathcal{D}^*)$ with the agenda $X = \{x \in \mathbb{Z}^L_+ : \sum_{\ell=1}^{L} x^\ell = 1\} = \{(1,0,...,0),(0,1,0,...,0),...,(0,...,0,1)\}$. The line considered in Example 2 is obtained by setting $L = 2$; in fact, the condition of separable convexity corresponds exactly to single-peakedness here, with the separability part vacuously satisfied.

In the following, we fix a rich domain $\mathcal{D}$ of separably convex preferences and will often simplify notation by writing "$\rangle x,y\langle$" instead of "$\rangle x,y\langle^{\mathcal{D}}$," "$\mathrm{B}_{\boldsymbol{\theta}}$" instead of "$\mathrm{B}^{\mathcal{D}}_{\boldsymbol{\theta}}$" and "$\mathrm{NM}_{\boldsymbol{\theta}}$" instead of "$\mathrm{NM}^{\mathcal{D}}_{\boldsymbol{\theta}}$" etc., whenever no confusion can arise.

It follows from Example 2 above that the set of maximal elements of the net majority tournament in the frugal resource allocation problem $(X, \mathcal{D})$ is non-empty for any profile $\boldsymbol{\theta}$ if $L = 2$. The following example shows that this does not generalize to the case $L > 2$.

Example 3 (Non-existence of a non-localized frugal majority winner) Suppose that X is given as in (3.1) above with $L = 3$, $Q = 3$ and $[q^\ell_-, q^\ell_+] \supseteq [0,3]$ for all $\ell = 1,2,3$. Consider

[14] The coefficients need to be distinct in order to ensure that every order in $\mathcal{D}_{\text{lin}}$ has a unique top on X.

the following profile $\boldsymbol{\theta}$ with seven agents (see Figure 2): $\theta_1 = (1,1,1)$, $\theta_2 = \theta_3 = (3,0,0)$, $\theta_4 = \theta_5 = (0,3,0)$, and $\theta_6 = \theta_7 = (0,0,3)$, and fix any rich domain of separably convex preferences. Using Fact 3.1, it is easily verified that $(1,1,1) >_{\theta_i} (0,1,2)$ for $i = 1,2,3$ and $(0,1,2) >_{\theta_i} (1,1,1)$ for $i = 6,7$, while any ranking between $(1,1,1)$ and $(0,1,2)$ is compatible with admissibility for agents $i = 4,5$. Thus,

$$(1,1,1)\,\widehat{\mathrm{NM}}_{\boldsymbol{\theta}}\,(0,1,2), \tag{3.3}$$

where $\widehat{\mathrm{NM}}_{\boldsymbol{\theta}}$ is the asymmetric ('strict') part of $\mathrm{NM}_{\boldsymbol{\theta}}$. Moreover, by Fact 3.1, we have $(0,1,2) >_{\theta_i} (0,0,3)$ for $i = 1,4,5$ while any ranking between $(0,1,2)$ and $(0,0,3)$ is compatible with admissibility for agents $i = 2.3$; hence, notwithstanding the fact $(0,0,3) >_{\theta_i} (0,1,2)$ for $i = 6,7$, we obtain

$$(0,1,2)\,\widehat{\mathrm{NM}}_{\boldsymbol{\theta}}\,(0,0,3). \tag{3.4}$$

Finally, again using Fact 3.1, we obtain

$$(0,0,3)\,\widehat{\mathrm{NM}}_{\boldsymbol{\theta}}\,(1,1,1) \tag{3.5}$$

since agents $i = 6,7$ have their top at $(0,0,3)$ while only agent $i = 1$ has her top at $(1,1,1)$ and the ranking between these two allocations is not determined by admissibility for the other agents $i = 2,3,4,5$. Combining (3.3), (3.4) and (3.5) we thus obtain that both $(1,1,1)$ and $(0,0,3)$ are contained in a $\widehat{\mathrm{NM}}_{\boldsymbol{\theta}}$-cycle. By a completely symmetric argument, also the allocations $(3,0,0)$ and $(0,3,0)$ are part of a $\widehat{\mathrm{NM}}_{\boldsymbol{\theta}}$-cycle. This easily implies that the set of (unrestricted) net majority winners is in fact empty for the profile $\boldsymbol{\theta}$.

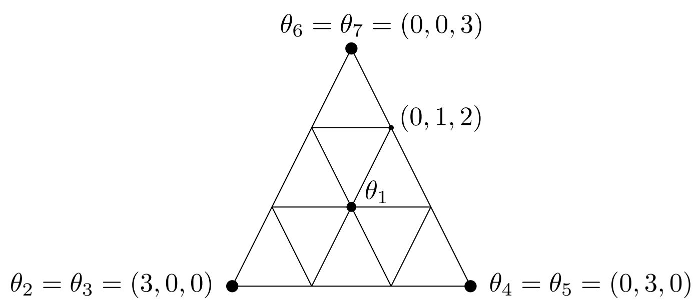

Fig. 2: Non-existence of a non-localized net majority winner

The last example demonstrates that an overly simplistic application of the indifference principle may lead to a cyclic net majority tournament, and hence does not deliver a coherent account of 'frugal social betterness.' To provide the diagnosis of the underlying difficulty, consider the following binary comparisons between x and y of agents with top θ.

(a) $x = (2,1,0)$, $y = (1,2,0)$, $\theta = (1,1,1)$.
(b) $x = (3,0,0)$, $y = (1,1,1)$, $\theta = (0,0,3)$.
(c) $x = (2,1,0)$, $y = (1,2,0)$, $\theta = (0,1,2)$.

In each case, the top θ neither supports x nor y since any preference between x and y is compatible with a top θ under separable convexity. However, the indifference principle is not equally plausible in all cases. Consider case (a) first; here we have full symmetry of x and y vis-á-vis the top θ, say under permutation of coordinates, and the appeal to the indifference principle appears to be safe. On the other hand, in case (b) the indifference between x and y given θ is more problematic; in the pairwise comparison an impartial evaluator might come up with favoring y because it is closer to θ (in the natural 'resource metric' defined below). We will not settle this and avoid an appeal to the indifference principle in this case. Finally, in (c) we still do not have full symmetry as in (a), but less of an asymmetry as compared to (b); in particular, x and y are equidistant from θ. We will appeal to the indifference principle in this case.

One important difference between cases (a) and (c) on the one hand and case (b) on the other is that in two former cases x and y are adjacent, i.e. they differ in the allocation of one unit of expenditure only. This suggests to refine the indifference principle and to restrict judgements of social betterness to a subset of *well-decidable* binary comparisons via a *comparison graph* Γ, i.e. to consider $\mathrm{B}_{\boldsymbol{\theta}} = \mathrm{NM}_{\boldsymbol{\theta}} \cap \Gamma$ for an appropriate graph Γ on X. In our present context, the natural choice of the comparison graph is given by the set of all neighbors, i.e. all pairs of allocations that result from each other by transferring one unit from one coordinate to another.

Formally, say that two allocations $x, y \in X$ are *neighbors* if they differ only by the allocation of one monetary unit, i.e. if $\sum_{\ell=1}^{L} |x^\ell - y^\ell| = 2$. Denote by Γ_{res} the graph that results from connecting all neighbors in X by an edge. (Observe that all our figures in fact depict this graph.) As is easily seen, two distinct allocations x and y are neighbors if and only if $[x, y] = \{x, y\}$.

For all $x, y \in X$, denote by

$$d(x, y) := \frac{1}{2} \sum_{\ell=1}^{L} |x^\ell - y^\ell|.$$

the natural 'resource' metric on X. The normalization ensures that neighbors have distance one, i.e. that a transfer of one unit of expenditure from one public good to another yields an allocation with unit distance from the original allocation. Also observe that, for all $x, y \in X$,

$$[x, y] \;=\; \{w \in X \;:\; d(x, y) = d(x, w) + d(w, y)\}, \tag{3.6}$$

i.e. the allocations between two other allocations are precisely those that are geodesically between them with respect to the natural 'resource' metric d; in other words, the allocations between x and y are precisely those that lie on some shortest path connecting x and y in the graph Γ_{res}.

The following figure depicts the sets $\rangle x, y\rangle$, $\rangle y, x\rangle$ and $x \bowtie y$ for two neighbors x and y. It shows that, for fixed $x \in X$, the 'agnostic' region $x \bowtie y$ is inclusion minimal if and only if y is a neighbor of x. Moreover, it illustrates the following *equidistance property* that enables the appeal to the indifference principle for adjacent alternatives in our present context. For all $x, y, \theta \in X$,

$$[d(x, y) = 1 \text{ and } \theta \in x \bowtie y] \;\Rightarrow\; d(\theta, x) = d(\theta, y), \tag{3.7}$$

which follows easily from Fact 3.1 using (3.6).

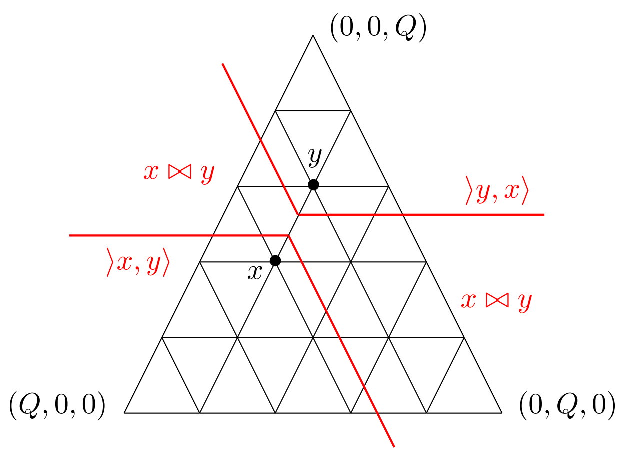

Fig. 3: The sets $\rangle x, y\rangle$, $\rangle y, x\rangle$ and $x \bowtie y$ for adjacent x and y

By 'localizing' the net majority tournament, i.e. by restricting it to the graph Γ_{res}, we obtain the central solution concept advocated here, the *set of frugal majority winners*, henceforth simply the *frugal majority set*, formally defined by

$$\text{FM}(\boldsymbol{\theta}) \;:=\; \{x \in X \;:\; \text{for all} \;\; w \in X \text{ with } \; w\Gamma_{\text{res}}x,\; x\,\text{NM}_{\boldsymbol{\theta}}\,w\}.$$

Thus, our claim is that the indifference principle is justified if applied to adjacent alternatives due to (3.7), and that therefore a coherent account of frugal betterness is obtained by applying the net majority criterion *locally* to neighbors:

Normative claim (refined) *For a rich subdomain of separably convex preferences, frugal betterness is given by the restriction of the net majority relation restricted to neighbors, i.e.,*

$$\text{B}_{\boldsymbol{\theta}} \;=\; \text{NM}_{\boldsymbol{\theta}} \cap \Gamma_{\text{res}}\;.$$

In Example 3, the allocation $(1, 1, 1)$ is the unique frugal majority winner. In the following, we will show that the frugal majority set is always non-empty; indeed, the net majority tournament restricted to Γ_{res} is acyclic and decisive, as follows.

Consider a weak tournament (i.e. a complete binary relation) R' on X and its restriction R to a connected comparison graph Γ, i.e. $R = R' \cap \Gamma$. Denote by P the asymmetric part of R, and by R^* and P^* the transitive closures of R and P, respectively. The relation R is *acyclic* if P displays no cycles, i.e. if P^* is irreflexive. Say that $x \in X$ is a *local optimum (with respect to* Γ*)* if xRy for all $y \in X$ such that $x\Gamma y$, and say that R is *decisive (on* X*)*, if for some $x \in X$, xR^*y for all $y \in X$. Finally, say that a subset $Y \subseteq X$ is *connected* if any pair of elements of Y is connected by a Γ-path that stays in Y. The following fact is easily verified.

Fact 3.2 *If R is acyclic, there exists a local optimum. Moreover, R is decisive if and only if the set of local optima is non-empty and connected.*

As our first main result, we will now show that, for all profiles $\boldsymbol{\theta}$, the restricted tournament $\mathrm{NM}_{\boldsymbol{\theta}} \cap \Gamma_{\mathrm{res}}$ is both acyclic and decisive; in particular, the frugal majority set is non-empty and connected. Moreover, the frugal majority set will be shown to consist exactly of the allocations that minimize the aggregate graph distance with respect to the graph Γ_{res}, as follows. For all profiles $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ and all $x \in X$, denote by

$$\Delta_{\boldsymbol{\theta}}(x) := \sum_{i=1}^{n} d(x, \theta_i)$$

the *aggregate distance* of the feasible allocation x given the profile $\boldsymbol{\theta}$. The aggregate distance is a natural way to quantify the 'overall remoteness' of an allocation to the set of individual top alternatives. The collective choice rule that selects, for any profile, the allocations that minimize the aggregate distance represents the well-known *median rule* (Barthélémy and Monjardet [1981], Barthélémy and Janowitz [1991], McMorris et al. [2000], Nehring and Pivato [2018]) applied to the present context of resource allocation. Accordingly, we will refer to an allocation as a *median allocation* if it solves

$$\arg\min_{x \in X} \Delta_{\boldsymbol{\theta}}(x) \;=\; \arg\min_{x \in X} \sum_{i=1}^{n} d(x, \theta_i),$$

and we denote the set of median allocations for a given profile θ by $\mathrm{Med}(\boldsymbol{\theta})$. Evidently, $\mathrm{Med}(\boldsymbol{\theta})$ need not be a singleton but it is always non-empty since the median allocations are obtained as the solution of a minimization problem on a finite set. Note that, if $L = 2$, the set of median allocations coincides with the standard median(s). In particular, if there is an odd number of individuals, the set $\mathrm{Med}(\boldsymbol{\theta})$ consists of the unique median top.

Theorem 1 *Consider a frugal aggregation problem* $(X, \mathcal{D})$ *where* X *is a resource agenda and* $\mathcal{D}$ *a rich subdomain of separably convex preferences. For all profiles* $\boldsymbol{\theta}$*, the restricted net majority tournament* $\mathrm{NM}_{\boldsymbol{\theta}} \cap \Gamma_{\mathrm{res}}$ *is acyclic and decisive, in particular* $\mathrm{FM}(\boldsymbol{\theta})$ *is non-empty. Moreover,* $\mathrm{FM}(\boldsymbol{\theta})$ *is box-convex and coincides with* $\mathrm{Med}(\boldsymbol{\theta})$*.*

(Proof in Appendix.)

Illustration: The frugal majority set with three voters

The following figure illustrates the frugal majority set in the case of three voters.[15] First note that, evidently, for two voters with tops θ and θ', respectively, the frugal majority set is given by the interval $[\theta, \theta']$. Indeed, every allocation $x \in [\theta, \theta']$ lies on a shortest path between θ and θ', and therefore has aggregate distance equal to $d(\theta, \theta')$ by (3.6), while every allocation outside $[\theta, \theta']$ has strictly larger distance.

Now consider the case of three agents with distinct tops θ, θ' and θ'', respectively. In Figure 4, we fix the two tops θ' and θ'' in generic position, and describe how the frugal majority set changes when θ moves clockwise 'around' the interval $[\theta', \theta'']$ (with the frugal majority winners marked in red in each case; the depicted shapes of $\mathrm{FM}(\boldsymbol{\theta})$ can be verified by computing aggregate distances and using Theorem 1).

[15] The website http://www.frugalmajority.de provides an online application to compute and visualize the frugal majority for any number of agents for $L = 3$ and $Q \leq 15$.

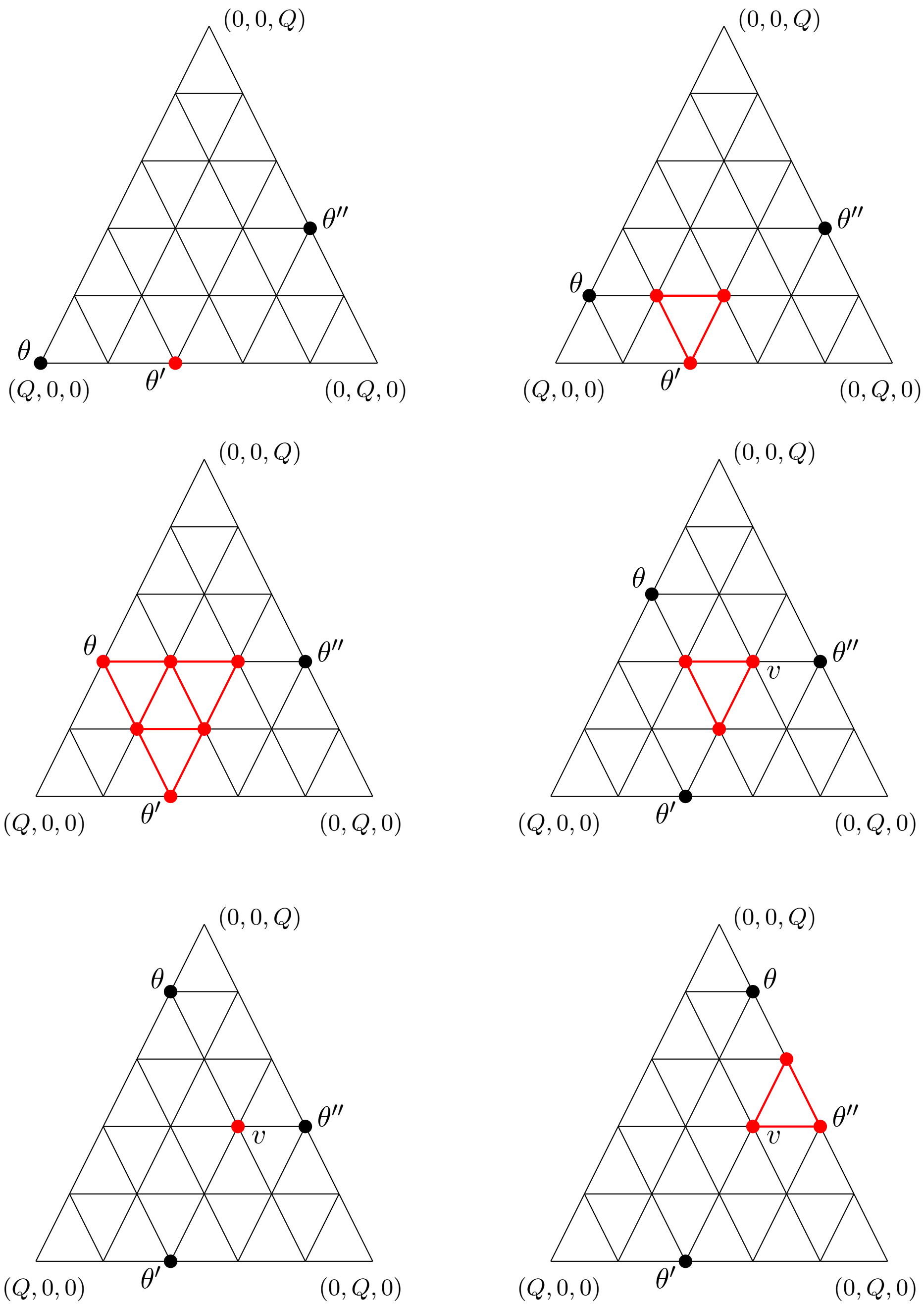

Fig. 4: The frugal majority set with three agents

We note that while the frugal majority set with three voters is always a triangle (possibly consisting of a single allocation, cf. Fig. 4), it can take on a variety of other shapes if there are more than three agents.

3.1 On the scope of Theorem 1: When is it valid (and why)?

The assumption of separable convexity provides a distinguished compromise between the use of exogenous structure (given through the graph Γ_{res}) and flexibility in terms of the size of the domain of preferences left admissible. Indeed, a result akin to Theorem 1 fails to hold both on the larger domain of convex, but not necessarily separably convex, preferences, and on the (much) smaller domain of Euclidean preferences frequently used in models of spatial voting (Austen-Smith and Banks [1999]).

First, consider the domain of all *convex* (but not necessarily separably convex) preferences on X, i.e. the set of all preferences that can be represented by a quasi-concave utility function, and let us denote it by $\mathcal{D}_{\text{conv}}$. As is easily seen, for each pair of neighbors $x, y \in X$ the supporting set $\rangle x, y\rangle^{\mathcal{D}_{\text{conv}}}$ is given by the set of all tops θ such that x is on the (Euclidean) straight line connecting θ and y (see Figure 5 which depicts the sets $\rangle x, y\rangle^{\mathcal{D}_{\text{conv}}}$ and $\rangle y, x\rangle^{\mathcal{D}_{\text{conv}}}$ in red).

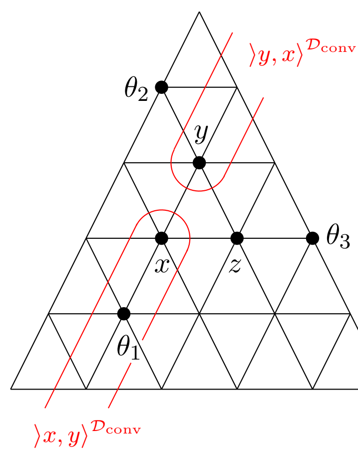

Fig. 5: Cyclic frugal majorities with convex preferences

Evidently, frugal majority rule under the domain assumption $\mathcal{D}_{\text{conv}}$ does not coincide with the median rule, i.e. Theorem 1 fails to hold for this domain. In fact, the restricted net majority tournament is in general not even acyclic. To see this, consider the binary comparisons among the adjacent alternatives x, y, z under the three-agent profile $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ depicted in Fig. 5. Agent 1 with top θ_1 supports x against y but not against z, and no other alternative in $\{x, y, z\}$ against any other alternative in $\{x, y, z\}$; similarly, the only discernible binary support of agent 2 among the alternatives in $\{x, y, z\}$ is for y against z, and the only discernible binary support of agent 3 among these alternatives is for z against x. Thus, frugal majority rule yields the cycle $x\, \widehat{\text{NM}}_{\boldsymbol{\theta}}^{\mathcal{D}_{\text{conv}}} y$, $y\, \widehat{\text{NM}}_{\boldsymbol{\theta}}^{\mathcal{D}_{\text{conv}}} z$, $z\, \widehat{\text{NM}}_{\boldsymbol{\theta}}^{\mathcal{D}_{\text{conv}}} x$. Observe that the frugal majority set is nevertheless non-empty in Fig. 5 as it contains the three tops θ_1, θ_2 and θ_3; it is however neither convex nor even connected.

The previous example shows that Theorem 1 may fail on 'too large' domains. But the result may also fail for 'small' domains. Specifically, consider the domain $\mathcal{D}_{\text{Euclid}}$ of all Euclidean preferences, i.e. the class of preferences that can be represented by the negative Euclidean distance to some allocation $\theta \in X$; this domain has been widely used in the literature on spatial voting (see, e.g., Austen-Smith and Banks [1999]). Note that within the domain $\mathcal{D}_{\text{Euclid}}$ an individual top reveals the entire preference ordering; indeed, for each $\theta \in X$, the order $>_\theta^{\mathcal{D}}$ coincides with the asymmetric part of the unique preference ordering in $\mathcal{D}_{\text{Euclid}}$ with top θ. For all pairs $x, y \in X$ of neighbors, the set $x \bowtie^{\mathcal{D}_{\text{Euclid}}} y$ is given by the straight line normal to the (Euclidean) segment connecting x and y, and the sets $\rangle x, y\rangle^{\mathcal{D}_{\text{Euclid}}}$ and $\rangle y, x\rangle^{\mathcal{D}_{\text{Euclid}}}$ are on the two respective sides of this line (see Figure 6). Local net majority rule under the domain of Euclidean preferences is in general not acyclic. This can be verified using the same example as in Fig. 5 above. Remarkably, however, under the domain $\mathcal{D}_{\text{Euclid}}$, we obtain the reverse cycle $y\,\widehat{\text{NM}}_{\boldsymbol{\theta}}^{\mathcal{D}_{\text{Euclid}}}x$, $z\,\widehat{\text{NM}}_{\boldsymbol{\theta}}^{\mathcal{D}_{\text{Euclid}}}y$, $x\,\widehat{\text{NM}}_{\boldsymbol{\theta}}^{\mathcal{D}_{\text{Euclid}}}z$. Indeed, now both θ_2 and θ_3 support y over x, both θ_1 and θ_3 support z over y, and both θ_1 and θ_2 support x over z (see Fig. 6).

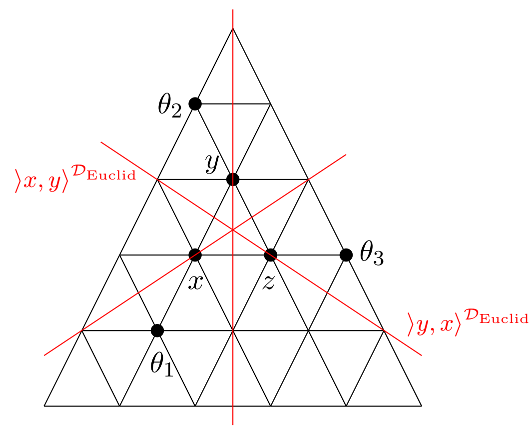

Fig. 6: Cyclic frugal majorities with Euclidean preferences

3.2 On the interpretation of the frugal majority set: Condorcet meets Borda

By relying on an appropriate notion of (local) majorities, our approach to frugal social optimality may appear to have a strong 'Condorcetian' flavor. However, as we argue in this subsection, the claim of the frugal majority set as the appropriate concept of social welfare optimum in the context of resource allocation does not rest on a specific Condorcetian philosophy. First, we note in Subsection 3.2.1 that the frugal majority set arises naturally also from the perspective of formulating an appropriate 'frugal' version of Borda rule. Moreover, in Subsection 3.2.2, we show that the frugal majority set represents the utilitarian optimum with respect to a particularly natural class of imputed utility functions, the 'goal satisfaction' functions.

3.2.1 Frugal majority rule as frugal Borda rule

How can one apply the Borda rule under the informational constraints of the frugal aggregation model? One immediate problem is that each agent is characterized by an *incomplete* ordering, namely by the partial order $>_{\boldsymbol{\theta}}^{\mathcal{D}^*}$, and that the Borda rule is defined for complete orderings only. But the representation of the frugal majority set in terms of the sum of the (negative) distances in the natural resource metric suggests a simple solution in the present context. Given a top $\theta \in X$, define the *score* $s_\theta(x) \leq 0$ of an alternative $x \in X$ as follows. A *chain (with respect to* $>_\theta$*)* is any subset of X that is totally ordered by the partial order $>_\theta$. For each $x \in X$, let $\tilde{s}_\theta(x)$ be the maximal cardinality of a chain $Y \ni x$ that has x at its bottom (i.e. $y >_\theta x$ for all $y \in Y \setminus \{x\}$), and let $s_\theta(x) := -\tilde{s}_\theta(x) + 1$, so that θ itself uniquely receives the highest score $s_\theta(\theta) = 0$.

For every profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ and every alternative x, let

$$\mathrm{FB}(\boldsymbol{\theta}) := \arg\max_{x \in X} \sum_{i=1}^{n} s_{\theta_i}(x)$$

denote the set of *frugal Borda winners.*[16] The following result follows easily from Fact 3.1 and (3.6) above.

Proposition 1 *Consider the frugal aggregation problem* $(X, \mathcal{D})$ *where* $\mathcal{D}$ *is a rich domain of separably convex preferences. For all* $\theta \in X$ *and all* $x \in X$*, we have* $s_\theta(x) = -d(x, \theta)$ *for the scores derived from the partial order* $>_\theta^{\mathcal{D}}$*. Thus, in particular, for all profiles* $\boldsymbol{\theta}$*,*

$$\mathrm{FB}(\boldsymbol{\theta}) \; = \; \mathrm{Med}(\boldsymbol{\theta}) \; = \; \mathrm{FM}(\boldsymbol{\theta}).$$

3.2.2 The frugal majority set as 'imputed' utilitarian solution

We now show that the frugal majority set also coincides with the utilitarian maxima with respect to a natural class of 'imputed' convex and separable cardinal utility functions. Specifically, for each agent with top θ consider the *goal satisfaction* function $v_\theta : \mathbb{Z}^L \to \mathbb{R}$ defined by

$$v_\theta(x) := \sum_{\ell=1}^{L} \min\{x^\ell, \theta^\ell\}, \tag{3.8}$$

for all $x \in \mathbb{Z}^L$. Evidently, for every $\theta \in X$, $v_\theta(\cdot)$ is an additively separable function which is (weakly) increasing and concave in each component. The term $\min\{x^\ell, \theta^\ell\}$ measures the extent to which the 'goal' θ^ℓ is satisfied in coordinate ℓ by the allocation x. Note that with monotone preferences oversatisfaction of the goal in the sense that $x^\ell > \theta^\ell$ does not hurt *per se*; but for a feasible allocation $x \in X$, oversatisfaction in one coordinate is necessarily accompanied by undersatisfaction of the goal in some other coordinate due to the budget constraint incorporated in the resource agenda X. Thus, the term

$$\sum_{\ell=1}^{L} |x^\ell - \theta^\ell|_+,$$

[16] For a different approach to extending the Borda rule for partial orders, see Cullinan et al. [2014].

where $|x^\ell - \theta^\ell|_+ := \max\{x^\ell - \theta^\ell, 0\}$, can be interpreted as the potential 'waste' of resources at the allocation x from the point of view of an agent with top allocation θ. Noting that, for all $\theta, x \in X$,

$$\sum_{\ell=1}^{L} |x^\ell - \theta^\ell|_+ \;=\; \sum_{\ell=1}^{L} |\theta^\ell - x^\ell|_+ \;=\; d(x,\theta)/2,$$

we obtain for all $x \in X$ and all $\theta_i \in X$,

$$v_{\theta_i}(x) \;=\; \sum_{\ell=1}^{L} x^\ell - \sum_{\ell=1}^{L} |x^\ell - \theta_i^\ell|_+ \;=\; Q - \sum_{\ell=1}^{L} |x^\ell - \theta_i^\ell|_+ \quad (3.9)$$

$$= \; Q - d(x, \theta_i)/2. \quad (3.10)$$

Equation (3.9) states that, up to a constant, goal satisfaction simply measures aggregate (potential) waste, and (3.10) implies that minimizing aggregate distance of $x \in X$ for a profile $\boldsymbol{\theta}$ of tops in X amounts to maximizing the aggregate goal satisfaction $v_{\boldsymbol{\theta}}(\cdot)$ defined by

$$v_{\boldsymbol{\theta}}(x) \;:=\; \sum_{i=1}^{n} v_{\theta_i}(x).$$

Thus, for all profiles $\boldsymbol{\theta}$ in X, the frugal majority set coincides with the utilitarian maximizers of the individual goal satisfaction functions, i.e. we have the following result.

Proposition 2 *Consider the frugal aggregation problem $(X, \mathcal{D})$ where $\mathcal{D}$ is a rich domain of separably convex preferences. Then for all profiles $\boldsymbol{\theta}$,*

$$\mathrm{FM}(\boldsymbol{\theta}) = \mathrm{Med}(\boldsymbol{\theta}) = \arg\max_{x \in X} v_{\boldsymbol{\theta}}(x).$$

Within the class of additively separable and concave utility functions, the goal satisfaction functions v_{θ_i} arise naturally from the perspective of the frugal aggregation model, by the following heuristic argument. Consider a (cardinal) differentiable utility function $u(x^1, ..., x^L) = \sum_\ell u^\ell(x^\ell$ with monotone and concave component functions u^ℓ. In the optimum $\theta \in X$ among all feasible allocations in X, the marginal rates of substitution must all be equal to unity because allocations are defined in terms of expenditure (neglecting any integer problems for simplicity); that is, for the marginal utilities, we obtain $\partial u^\ell(\theta^\ell)/\partial x^\ell = \partial u^k(\theta^k)/\partial x^k$ for all ℓ, k. By the concavity of the component functions, marginal utility is higher below than above the optimum, i.e. for all ℓ, k, all $r < \theta^\ell$ and $\theta^k < s$ we have

$$\frac{\partial u^\ell(r)}{\partial x^\ell} \;\geq\; \frac{\partial u^k(s)}{\partial x^k} \;.$$

Since the only available information in the frugal model is the top θ, an application of the principle of insufficient reason suggests treating all marginals below the top equal to each other, and likewise all marginals above the top. Setting the marginal utilities below the top equal to α and those above the top equal to $\beta < \alpha$, these are affinely equivalent on the feasible set X to the goal-satisfaction utilities defined in (3.8), which correspond in fact to the special case of $\alpha = 1$ and $\beta = 0$. Note that this argument forces the imputed utilities to be non-differentiable but allows them to be strictly monotone.

4 A simple characterization in terms of an endogenous quota: The frugal majority winners as balanced medians

In this section, we provide a simple and powerful characterization of the frugal majority set that allows one to compute it very efficiently and to immediately derive a number of its basic properties.

In the following fix a profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ with n voters and denote, for every $\ell = 1, ..., L$ and every $k = 1, ..., n$, by $\theta^\ell_{[k]} \in X$ the k-th smallest vote in coordinate ℓ, that is, the vector $(\theta^\ell_{[1]}, \theta^\ell_{[2]}, ..., \theta^\ell_{[n]})$ results from the values $\theta^\ell_1, \theta^\ell_2, ..., \theta^\ell_n$ simply by re-arranging the latter in ascending order so that $\theta^\ell_{[1]} \leq \theta^\ell_{[2]} \leq ... \leq \theta^\ell_{[n]}$ (possibly with some equalities). Denote by $Q_{[k]} := \sum_{\ell=1}^{L} \theta^\ell_{[k]}$, and let $k^*(\boldsymbol{\theta})$ be the largest $k = 1, ..., n$ such that $Q_{[k]} \leq Q$. Finally, say that the profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ is *unanimous* if $\theta_1 = \theta_2 = ... = \theta_n$. Note that for a unanimous profile one has $k^*(\boldsymbol{\theta}) = n$ since, evidently, $\theta^\ell_{[1]} = \theta^\ell_{[2]} = ... = \theta^\ell_{[n]} = \theta^\ell_i$ for all $i = 1, ..., n$ and all $\ell = 1, ..., L$. Also observe that $k^*(\boldsymbol{\theta}) < n$ for all non-unanimous profiles.

Theorem 2 *Consider the frugal aggregation problem $(X, \mathcal{D})$ where $\mathcal{D}$ is a rich domain of separably convex preferences on the resource agenda X. For every non-unanimous profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ and every $x \in X$ the following are equivalent.*

a) $x \in \mathrm{FM}(\boldsymbol{\theta})$,

b) *x maximizes aggregate goal satisfaction* $v_{\boldsymbol{\theta}}(\cdot)$,

c) *for all* $\ell = 1, ..., L$,

$$\theta^\ell_{[k^*(\boldsymbol{\theta})]} \leq x^\ell \leq \theta^\ell_{[k^*(\boldsymbol{\theta})+1]}. \tag{4.1}$$

(Proof in Appendix.)

Condition (4.1) means that $q^*(\boldsymbol{\theta}) := k^*(\boldsymbol{\theta})/n$ is the *'endogenous' (i.e. profile-dependent) quota* of voters who can be satisfied in all coordinates (of course, these have to be different sets of voters in different coordinates).

Example 4 As a simple example illustrating the endogenous quota interpretation of the characterization in Theorem 2c), consider the case $L = 3$, $Q = 10$, and a profile $\boldsymbol{\theta}$ with four voters such that $\theta_1 = (5, 0, 5)$, $\theta_2 = (0, 2, 8)$, $\theta_3 = (2, 6, 2)$ and $\theta_4 = (4, 3, 3)$, say. For the corresponding matrices (θ^ℓ_i) and $(\theta^\ell_{[k]} | \, Q_{[k]})$ with $\ell = 1, ..., L$ and $i, k = 1, ..., n$ we thus obtain

$$(\theta^\ell_i) = \begin{pmatrix} 5 & 0 & 5 \\ 0 & 2 & 8 \\ 2 & 6 & 2 \\ 4 & 3 & 3 \end{pmatrix} \quad \text{and} \quad (\theta^\ell_{[k]} | \, Q_{[k]}) = \begin{pmatrix} 0 & 0 & 2 & | & 2 \\ 2 & 2 & 3 & | & 7 \\ 4 & 3 & 5 & | & 12 \\ 5 & 6 & 8 & | & 19 \end{pmatrix}.$$

Since $Q_{[2]} = 7 < 10 \; (= Q) < 12 = Q_{[3]}$, we obtain $k^*(\boldsymbol{\theta}) = 2$, and thus an endogenous quota of $q^*(\boldsymbol{\theta}) = 0.5$; in accordance with (4.1), the frugal majority set is given by

$$\mathrm{FM}(\boldsymbol{\theta}) = \{(2, 3, 5), (3, 2, 5), (3, 3, 4), (4, 2, 4), (4, 3, 3)\}.$$

Now suppose that voter 4 changes her vote to $\tilde{\theta}_4 = (3, 2, 5)$ while the other voters keep their position. If we denote the resulting profile by $\tilde{\boldsymbol{\theta}}$, we obtain

$$(\tilde{\theta}_i^\ell) \;=\; \begin{pmatrix} 5 & 0 & 5 \\ 0 & 2 & 8 \\ 2 & 6 & 2 \\ 3 & 2 & 5 \end{pmatrix} \quad \text{and} \quad (\tilde{\theta}_{[k]}^\ell \,|\, \tilde{Q}_{[k]}) \;=\; \begin{pmatrix} 0 & 0 & 2 & | & 2 \\ 2 & 2 & 5 & | & 9 \\ 3 & 2 & 5 & | & 10 \\ 5 & 6 & 8 & | & 19 \end{pmatrix}.$$

Now, since $\tilde{Q}_{[3]} = 3 + 2 + 5 = 10$ $(= Q)$, we obtain $k^*(\tilde{\boldsymbol{\theta}}) = 3$, hence an endogenous quota of $q^*(\tilde{\boldsymbol{\theta}}) = 0.75$. Moreover, since $\tilde{Q}_{[k^*(\tilde{\boldsymbol{\theta}})]} = Q$ there is a unique net majority winner, and indeed $\text{FM}(\tilde{\boldsymbol{\theta}}) = \{(3, 2, 5)\}$.

The previous example suggests to view the frugal majority set as the allocations corresponding to a *balanced median* that maximizes, uniformly across all coordinates, the fraction of agents who can be given their preferred amount of each public good or more. **cut?**

Using Theorem 2c) the complexity of computing the set of majority winners can be determined as follows. First, one needs to sort n numbers L times. The best sorting algorithms are known to be of order $n \cdot \log n$, hence this yields $L \cdot n \cdot \log n$ computational steps.[17] In addition, one needs to sum L numbers at most n times and compare their sum to the fixed quantity Q. Thus, the determination of the box $\prod_\ell [\theta^\ell_{[k^*(\boldsymbol{\theta})]}, \theta^\ell_{[k^*(\boldsymbol{\theta})+1]}]$ involves in total a computational complexity of at most

$$L \cdot n \cdot \log n \;+\; L \cdot n \;+\; n$$

single steps. The frugal majority set corresponding to the profile $\boldsymbol{\theta}$ results from intersecting this box with the resource agenda X.

The simple characterization provided by Theorem 2 allows one to derive a number of further important properties of the frugal majority set to which we turn now.

4.1 Coordinate-wise unanimity and the frugal Pareto criterion

By Theorem 2c), for any profile $\boldsymbol{\theta}$, the frugal majority set is given by $X \cap \prod_{\ell=1}^L [\theta^\ell_{k^*(\boldsymbol{\theta})}, \theta^\ell_{k^*(\boldsymbol{\theta})+1}]$. This immediately implies that frugal majority rule respects *coordinate-wise unanimity* in the sense that, for all $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$,

$$\text{FM}(\boldsymbol{\theta}) \;\subseteq\; \prod_{\ell=1}^{L} [\min_i \theta_i^\ell, \max_i \theta_i^\ell]. \tag{4.2}$$

Moreover, every frugal majority winner is Pareto efficient with respect to the induced partial orders. Formally, say that an allocation $x \in X$ is *frugally Pareto efficient* given a profile of tops $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ if there does not exist any other feasible allocation y such that $y >_{\theta_i} x$ for all $i = 1, ..n$.

Proposition 3 *Consider the frugal aggregation problem $(X, \mathcal{D})$ where $\mathcal{D}$ is a rich domain of separably convex preferences. Then every frugal majority winner is frugally Pareto efficient.*

[17] See, for instance, Knuth [1998]. On finite domains, there exist deterministic sorting algorithms with complexity of order $n \cdot \log^2 n$, see Han [2004].

(Proof in Appendix.)

Note that frugal Pareto efficiency does clearly not imply ex-post efficiency with respect to the true underlying complete orders. A simple example is the case of two agents with tops $\theta_1 = (0,1,2)$ and $\theta_2 = (1,2,0)$, in which case the frugal majority set is given by $[(0,1,2),(1,2,0)] = \{(0,1,2),(1,1,1),(0,2,1),(1,2,0)\}$. But separable convexity is evidently compatible with the strict preferences $(1,1,1) \succ_i (0,2,1)$ for $i = 1,2$ where $\succcurlyeq_i$ denotes the underlying complete preference order of agent i; thus, the frugal majority winner $(0,2,1)$ may not be ex-post Pareto efficient in general. But, of course, requiring ex-post Pareto efficiency with respect to *every* possible underlying complete preference order is inappropriate under the informational assumptions of the frugal aggregation model.[18] Note, however, that in the two special cases considered in Section 2.2 above (plurality rule and median voting in the line, respectively) frugal majority rule does achieve ex-post efficiency.

4.2 Essential uniqueness

As many other solution concepts in social choice theory, also the frugal majority set does frequently not deliver a unique outcome. Nevertheless, as already noted, the size of frugal majority set is bounded by the size of the box $\prod_{\ell=1}^{L}[\theta^{\ell}_{k^*(\boldsymbol{\theta})}, \theta^{\ell}_{k^*(\boldsymbol{\theta})+1}]$. In particular, the 'denser' the support of a profile, the smaller its frugal majority set. Specifically, we have the following result. Say that a subset $Y \subseteq X$ is *essentially unique* if

$$\max_{x,y \in Y,\ \ell=1,...,L} |x^{\ell} - y^{\ell}| \leq 1.$$

Thus, a subset of X is essentially unique if every two of its elements differ in each coordinate by at most one unit. Also, say that the support of a profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ is *coordinate-wise connected* if, for each $\ell = 1, ..., L$, the set $\{\theta_i^{\ell}\}_{i=1,...,n}$ forms an interval in $\mathbb{Z}$, i.e. $\{\theta_1^{\ell}, ..., \theta_n^{\ell}\} = [\min_i \theta_i^{\ell}, \max_i \theta_i^{\ell}]$. The following is an immediate corollary of the characterization in Theorem 2c).

Proposition 4 *The frugal majority set* $\mathrm{FM}(\boldsymbol{\theta})$ *is essentially unique whenever* $\boldsymbol{\theta}$ *is coordinate-wise connected.*

Figure 7 depicts the set $\mathrm{FM}(\boldsymbol{\theta})$ for a profile of five agents. Observe that while the support of $\boldsymbol{\theta}$ is not connected in the usual sense (because different agents' tops cannot be connected by a Γ_{res}-path within the support of $\boldsymbol{\theta}$), the depicted profile is nevertheless coordinate-wise connected in the sense defined above.

[18]Moreover, it is well-known that ex-post efficiency is generally not achievable in a large class of related models, see, for instance, Benoît and Kornhauser [2010].

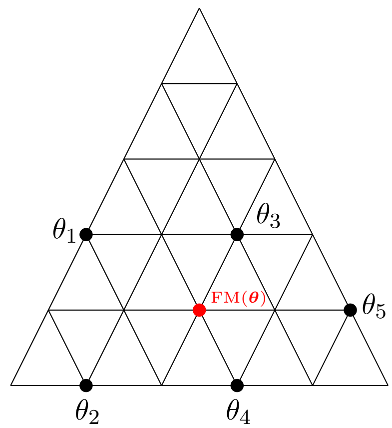

Fig. 7: Essential uniqueness under coordinate-wise connectedness

If one considers the population preferences as a statistical sample resulting from independent draws from an underlying continuous distribution with connected support, the expected gap $[\theta^{\ell}_{k^*(\boldsymbol{\theta})}, \theta^{\ell}_{k^*(\boldsymbol{\theta})+1}]$ would shrink roughly in inverse proportion to the number of agents n. Hence the diameter of the frugal majority set FM($\boldsymbol{\theta}$) would likewise shrink in inverse proportion to n. So, heuristically, in such situations one would expect the frugal majority set to shrink quite rapidly with the number of agents.

4.3 Frugal majority rule avoids the no-show paradox

The scoring rule representation of the frugal majority winners as the set of allocations that minimize the aggregate distance function implies that the corresponding choice rule avoids the famous 'no-show' paradox (Brams and Fishburn, 1983; Moulin 1988). For instance, by voting for the most preferred allocation among the frugal majority winners that would result without her vote, an individual can guarantee this allocation to be the unique new frugal majority winner.

More generally, we have the following result which shows that it can never be harmful for an agent to participate in collective decision according to frugal majority rule. For every profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ and every agent $h \notin \{1, ..., n\}$, denote by $\boldsymbol{\theta} \sqcup \theta_h$ the profile $(\theta_1, ..., \theta_n, \theta_h)$. Moreover, denote by $\text{FM}(\boldsymbol{\theta})^h$ the set of all allocations $x \in \text{FM}(\boldsymbol{\theta})$ such that $\text{FM}(\boldsymbol{\theta}) \cap [x, \theta_h] = \{x\}$. Thus, $\text{FM}(\boldsymbol{\theta})^h$ is the subset of *undominated* allocations in FM($\boldsymbol{\theta}$) from the perspective of an agent with top θ_h (and separably convex preferences). The following result shows that by participating in the collective decision according to frugal majority rule and submitting the top θ_h and agent is always better off in the sense that (i) the resulting frugal majority winners contain all undominated allocations among the former majority winners, and (ii) every new majority winner (if any) dominates one of these.

Proposition 5 *Consider any profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ and any agent $h \notin \{1, ..., n\}$ with top θ_h*

and preference $\succcurlyeq_h$*. Then,*

$$\mathrm{FM}(\boldsymbol{\theta})^h \subseteq \mathrm{FM}(\boldsymbol{\theta} \sqcup \theta_h) \subseteq \bigcup_{x \in \mathrm{FM}(\boldsymbol{\theta})^h} [x, \theta_h]. \tag{4.3}$$

(Proof in Appendix.)

Figure 8 illustrates Proposition 5. On the left-hand side, the frugal majority set $\mathrm{FM}(\boldsymbol{\theta})$ (without agent h's participation) is marked in red. The right-hand side depicts the frugal majority set with participation of agent h whose top is at the upper vertex of the red triangle representing $\mathrm{FM}(\boldsymbol{\theta} \sqcup \theta_h)$; the subset $\mathrm{FM}(\boldsymbol{\theta})^h$ of the undominated elements of $\mathrm{FM}(\boldsymbol{\theta})$ is encircled by the black oval. Indeed, from h's perspective, the two allocations discarded by h's participation, θ_1 and θ_2, are strictly worse than their right and left neighbor, respectively; and each of the three frugal majority winners gained by h's participation are strictly preferred by h to at least one element of $\mathrm{FM}(\boldsymbol{\theta})^h$.

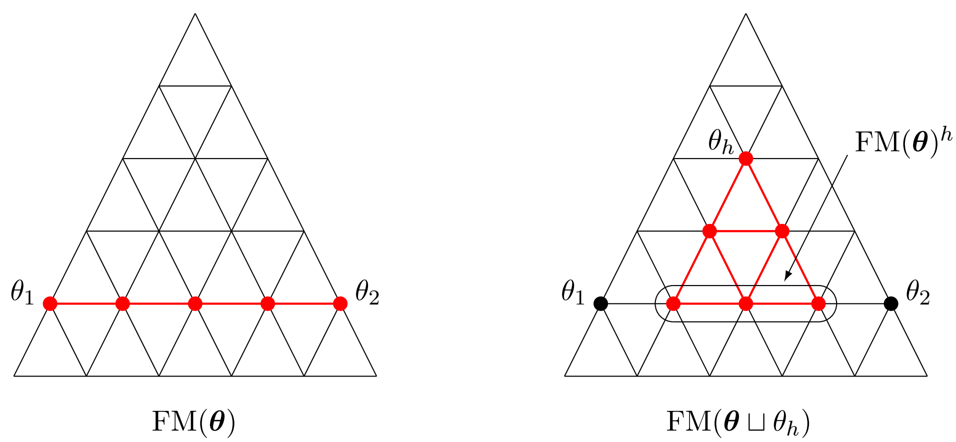

Fig. 8: Additional participation of agent h

5 Extension to the continuous case

Our analysis carries over to the case of a continuous resource agenda $X \subset \mathbb{R}^L$, as follows. A preference order $\succcurlyeq$ on X is *separably convex* if the following condition is satisfied: whenever a marginal transfer from good j to good k at allocation $x \in X$ makes an agent worse off (keeping the allocation fixed otherwise), then so does the same transfer at any allocation that has at most the amount x^j of good j and at least the amount x^k of good k. Given a profile of tops $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ on X, define the symmetric binary relation $\Gamma_{\boldsymbol{\theta}}$ on X by $x \Gamma_{\boldsymbol{\theta}} y$ if (i) x and y differ in exactly two coordinates, i.e. $x \neq y$ and for some distinct $j, k \in \{1, ..., L\}$, $x^\ell = y^\ell$ for all $\ell \neq j, k$, and (ii) for no $i = 1, ..., n$, $\min\{x^j, y^j\} < \theta_i^j < \max\{x^j, y^j\}$ or $\min\{x^k, y^k\} < \theta_i^k < \max\{x^k, y^k\}$. Thus, if $x \Gamma_{\boldsymbol{\theta}} y$ then x and y are 'neighbors' in the sense that they differ only in two coordinates and no top lies strictly between them in these two coordinates; geometrically, condition (ii) means that no top lies in the 'stripe' between x^j and y^j parallel to the j-axis, and no top lies in the 'stripe' between x^j and y^j parallel to the j-axis, see Fig. 9. Observe that for every profile with finite support, and for all j, k, this condition is satisfied whenever x and y are sufficiently close to each other.

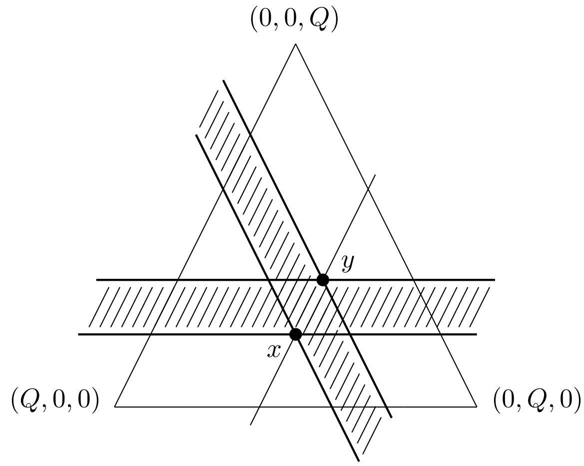

Fig. 9: The binary relation $\Gamma_{\boldsymbol{\theta}}$

Define the *local net majority relation* as $\widetilde{\text{NM}}_{\boldsymbol{\theta}} := \text{NM}_{\boldsymbol{\theta}} \cap \Gamma_{\boldsymbol{\theta}}$ and the *frugal majority set* by

$$\text{FM}(\boldsymbol{\theta}) \;:=\; \{x \in X \;:\; \text{for all } \; w \in X,\; x \,\widetilde{\text{NM}}_{\boldsymbol{\theta}}\, w\}$$

(for simplicity, we do not distinguish between the frugal majority set in the continuous and discrete cases).

Theorem 1′ *For every rich subdomain of separably convex preferences and all profiles* $\boldsymbol{\theta}$*, the local net majority relation* $\widetilde{\text{NM}}_{\boldsymbol{\theta}}$ *is acyclic and decisive, in particular* $\text{FM}(\boldsymbol{\theta})$ *is non-empty. Moreover,* $\text{FM}(\boldsymbol{\theta})$ *is box-convex and coincides with* $\text{Med}(\boldsymbol{\theta})$ *(the latter set being defined exactly as in the discrete case).*

(Proof in Appendix.)

We note that characterization of the frugal majority set provided in Theorem 2 also continues to hold without change. Indeed, the endogenous quota characterization in Theorem 2c) can be used to define the frugal majority set also for atomless distributions of agent's tops; and indeed, it always yields a unique solution in that case, as follows. For each $\ell = 1, ..., L$, and all $t \in [0, 1]$, denote by $\xi^\ell(t)$ the cumulative distribution of the tops in coordinate ℓ, i.e. $\xi^\ell(t) = r$ if and only if the fraction t of agents' tops has at most the amount r in coordinate ℓ; evidently, $\xi^\ell(\cdot)$ is an increasing function for all ℓ. Let $Q(t) := \sum_{\ell=1}^{L} \xi^\ell(t)$ which is clearly also increasing. If the underlying distribution $\boldsymbol{\theta}$ of tops is atomless, $Q(\cdot)$ is in fact *strictly* increasing and continuous on $[0, 1]$ with $Q(0) < Q < Q(1)$. By the intermediate value theorem, there exists exactly one $t^* \in (0,1)$ such that $Q(t^*) = Q$; then

$$\mathrm{FM}(\boldsymbol{\theta}) \;=\; \{(\xi^1(t^*), ..., \xi^L(t^*))\},$$

i.e. the allocation $(\xi^1(t^*), ..., \xi^L(t^*))$ is the unique frugal majority winner.

6 Conclusion

We propose and advocate a solution to the problem of collective resource allocation, the frugal majority set. In this paper, we have provided a definition via local comparisons involving unit transfers of expenditure. In the case of separably convex preferences we have presented two fundamental characterizations, one via the minimization of the aggregate l_1-distances to the agent's tops, the other in terms of 'balanced median' allocations that maximize the endogenous quota of coordinate-wise satisfaction. We have derived a number of basic and attractive properties of the frugal majority set, in particular we have demonstrated that it naturally arises not only from the Condorcetian perspective of pairwise (local) majority comparisons but also from a scoring rule point of view á la Borda and from a utilitarian standpoint. Finally, we have indicated how to apply our solution concept in the continuous case.

A number of important issue remain open; we mention two. First, it is desirable to have a more solid foundation of frugal majority rule through an axiomatic characterization that would enable a comparison of the frugal majority rule with possible contenders such as, for instance, the mean rule that takes as solution simply the coordinate-wise average of the agents' most preferred allocations. We provide such an axiomatic normative foundation of frugal majority rule in a companion paper (see Nehring and Puppe [2019a]).

Secondly, in the present paper we have not studied the incentive problem that may arise from the fact that not only the agents' complete preference order but also their respective tops may be private information. This is addressed in a another paper where we show that, while in general not fully strategy-proof, frugal majority rule is robust against manipulation in a number of ways if all agents have separably convex preferences (see Nehring and Puppe [2019b]).

References

K. J. Arrow. *Social Choice and Individual Values.* Wiley, New York, 1951/63.

D. Austen-Smith and J. Banks. *Positive Political Theory I: Collective Preference.* Michigan University Press, Ann Arbor, 1999.

Haris Aziz, Barton Lee, and Nimrod Talmon. Proportionally representative participatory budgeting: Axioms and algorithms. *(preprint)*, 2017.

J.-P. Barthélémy and M. F. Janowitz. A formal theory of consensus. *SIAM J. Discrete Math.*, 4(3):305–322, 1991.

Jean-Pierre Barthélémy and Bernard Monjardet. The median procedure in cluster analysis and social choice theory. *Math. Social Sci.*, 1(3):235–267, 1981.

Jean-Pierre Benoît and Lewis Kornhauser. Only a dictatorship is efficient. *Games and Economic Behavior*, 70:261–270, 2010.

Duncan S. Black. On the rationale of group decision-making. *J. Political Economy*, 56:23–34, 1948.

Shurojit Chatterji and Arunava Sen. Tops-only domains. *Economic Theory*, 46:255–282, 2011.

John Cullinan, Samuel Hsiao, and David Polett. A Borda count for partially ordered ballots. *Social Choice and Welfare*, 42:913–926, 2014.

Anthony Downs. *An Economic Theory of Democracy.* Harper, New York, 1957.

Bhaskar Dutta, Matthew Jackson, and Michel LeBreton. The banks set and the uncovered set in budget allocation problems. In David Austen-Smith and John Duggan, editors, *Social Choice and Strategic Decisions. Studies in Choice and Welfare*, pages 163–199. Springer, Berlin, Heidelberg, 2005.

Piotr Faliszewski and Nimrod Talmon. A framework for approval-based budgeting methods. *(preprint)*, 2019.

Scott L. Feld and Bernard Grofman. The Borda count in n-dimensional issue space. *Public Choice*, 59:167–176, 1988.

Gerd Gigerenzer and Daniel G. Goldstein. Reasoning the fast and frugal way: Models of bounded rationality. *Psychological Review*, 104:650–669, 1996.

Robert Goodin and Christian List. A conditional defense of plurality rule: Generalizing May's theorem in a restricted informational environment. *American Journal of Political Science*, 50:940–949, 2006.

Yijie Han. Deterministic sorting in $o(n \log \log n)$ time and linear space. *Journal of Algorithms*, 50:96–105, 2004.

Ehud Kalai, Eitan Muller, and Mark A. Satterthwaite. Social welfare functions when preferences are convex, strictly monotonic, and continuous. *Public Choice*, 34:87–97, 1979.

Donald Knuth. *The Art of Computer Programming, Volume 3: Sorting and Searching.* Addison-Wesley, Reading, Massachusetts, 1998.

Jean-Francois Laslier. And the loser is ... plurality voting. In Dan S. Felsenthal and Moshé Machover, editors, *Electoral Systems. Studies in Choice and Welfare*, pages 327–351. Springer, Berlin, Heidelberg, 2012.

Jean-Francois Laslier and Remzi Sanver. *Handbook of Approval Voting.* Springer, Berlin, 2010.

Michel Le Breton and John A. Weymark. Arrovian social choice theory on economic domains. In Kenneth J. Arrow, Amartya Sen, and Kotaro Suzumura, editors, *Handbook of Social Choice and Welfare, Volume 2*, pages 191–299. North-Holland, Amsterdam, 2011.

Tobias Lindner. *Zur Manipulierbarkeit der Allokation öffentlicher Güter – Theoretische Analyse und Simulationsergebnisse*. Ph.D. Dissertation, Karlsruhe Institute of Technology, 2011.

Kenneth May. A set of independent, necessary, and sufficient conditions for simple majority decision. *Econometrica*, 20:680–684, 1952.

Richard McKelvey. General conditions for global intransitivities in formal voting models. *Econometrica*, 47 (5):1085–1112, 1979.

F.R. McMorris, H.M. Mulder, and R.C. Powers. The median function on median graphs and semilattices. *Discrete Applied Mathematics*, 101:221–230, 2000.

Hervé Moulin. Condorcet's principle implies the no show paradox. *Journal of Economic Theory*, 45:53–64, 1988.

Klaus Nehring and Marcus Pivato. The median rule in judgement aggregation. *(preprint)*, 2018.

Klaus Nehring and Clemens Puppe. The structure of strategy-proof social choice I: General characterization and possibility results on median spaces. *J.Econ.Theory*, 135:269–305, 2007.

Klaus Nehring and Clemens Puppe. Allocating public goods via the midpoint rule. *(preprint)*, 2008.

Klaus Nehring and Clemens Puppe. Abstract arrowian aggregation. *J.Econ.Theory*, 145:467–494, 2010.

Klaus Nehring and Clemens Puppe. Normative foundations of frugal majority rule in resource allocation. *(preprint)*, 2019a.

Klaus Nehring and Clemens Puppe. Frugal majority rule is robust against strategic manipulation. *(preprint)*, 2019b.

Charles Plott. A notion of equilibrium and its possibility under majority rule. *American Economic Review*, 57:787–806, 1967.

Anwar Shah. *Participatory Budgeting*. The World Bank, Washington D.C., 2007.

Ehud Shapiro and Nimrod Talmon. A Condorcet-optimal participatory budgeting algorithm. *(preprint)*, 2017.

Lin Zhou. Impossibility of strategy-proof mechanisms for economies with pure public goods. *Review of Economic Studies*, 58:107–119, 1991.

Appendix: Proofs

Proof of Fact 3.1 in text.

Proof of Lemma 3.1. First, we show that any preference $\succcurlyeq \in \mathcal{D}_{\text{add}}$ is separably convex; since $\mathcal{D}_{\text{lin}} \subset \mathcal{D}_{\text{qlin}} \subset \mathcal{D}_{\text{add}}$ the same conclusion holds for these domains as well. Thus, let $\succcurlyeq$ be represented as in (3.2) by an additively separable utility function $u(x) = \sum_\ell u^\ell(x^\ell)$ with strictly increasing and concave component functions $u^\ell : \mathbb{R} \to \mathbb{R}$. In fact, the separable convexity follows from the concavity alone, no monotonicity condition on the functions u^ℓ is needed. Indeed, suppose that $x \succ x_{(kj)}$ and $y^j \leq x^j$ as well as $y^k \geq x^k$; since the allocations x and $x_{(kj)}$ differ only in coordinates j and k, we have $u(x) > u(x_{(kj)})$ if and only if $u^j(x^j) - u^j(x^j-1) > u^k(x^k+1) - u^k(x^k)$. By the concavity of $u^k(\cdot)$ and $u^j(\cdot)$ we obtain

$$\left[u^j(y^j) - u^j(y^j-1)\right] \geq \left[u^j(x^j) - u^j(x^j-1)\right] > \left[(u^k(x^k+1) - u^k(x^k)\right] \geq \left[u^k(y^k+1) - u^k(y^k)\right],$$

and hence $u(y) > u(y_{(kj)})$ as desired.

Next, we show that $\mathcal{D}_{\text{add}}$ is indeed a *rich* domain of separably convex preferences. Thus, consider $x, y, \theta \in X$ such that $x \notin [\theta, y]$. We will show that there exists $\succcurlyeq \in \mathcal{D}_{\text{add}}$ with top θ such that $y \succ x$. This in fact not only shows that $\mathcal{D}_{\text{add}}$ is a rich domain of separably convex preferences but, in addition, that every allocation can be the top of a preference order in $\mathcal{D}_{\text{add}}$. This is trivial if $\theta = y$; thus, assume henceforth that $\theta \neq y$. In the following, we explicitly construct appropriate strictly increasing and strictly concave functions $u^\ell : X^\ell \to \mathbb{R}$ for $\ell = 1, ..., L$, where X^ℓ is the projection of X to coordinate ℓ. First observe that it is clearly possible, for any given $\theta^\ell \in X^\ell$ and any $\epsilon > 0$, to slightly 'perturb' the identity function $f(x^\ell) = x^\ell$ to a strictly concave and strictly increasing function $\tilde{f}$ such that the difference $\tilde{f}(\theta^\ell) - \theta^\ell$ is strictly larger that $\tilde{f}(w^\ell) - w^\ell$ for all $w^\ell \in X^\ell \setminus \{\theta^\ell\}$, and such that the absolute values $|\tilde{f}(w^\ell) - w^\ell| < \epsilon$ for all $w^\ell \in X^\ell$. Note that if all utility functions u^ℓ arise from such perturbations, we obtain in particular

$$\sum_{\ell=1}^{L}(u^\ell(\theta^\ell) - \theta^\ell) > \sum_{\ell=1}^{L}(u^\ell(w^\ell) - w^\ell) \tag{A.1}$$

for all $w \in X \setminus \{\theta\}$ (note that every $w \in X \setminus \{\theta\}$ differs from θ in at least one coordinate, hence the inequality in (A.1) is indeed strict). Since $\sum_{\ell=1}^{L} \theta^\ell = \sum_{\ell=1}^{L} w^\ell = Q$, this implies $\sum_{\ell=1}^{L} u^\ell(\theta^\ell) > \sum_{\ell=1}^{L} u^\ell(w^\ell)$ for all $w \in X \setminus \{\theta\}$, i.e. θ is the unique top of the preference ordering represented by the utility function $u = \sum_\ell u^\ell$.

Now let $x \notin [\theta, y]$ and assume with loss of generality that x, y, θ are pairwise distinct. Since $x \notin [\theta, y]$ there exists a coordinate $j = 1, ..., L$ such that $x^j \notin [\theta^j, y^j]$. Thus, either $(x^j < \theta^j \ \& \ x^j < y^j)$ or $(x^j > \theta^j \ \& \ x^j > y^j)$. Consider the first case. It is possible to choose, for any position of θ^j and y^j, a strictly increasing and strictly concave function $u^j : X^l \to \mathbb{R}$ such that

$$u^j(\theta^j) - \theta^j \geq u^j(y^j) - y^j \geq \delta > 0 \geq u^j(x^j) - x^j, \tag{A.2}$$

where the first inequality in (A.2) is strict whenever $\theta^j \neq y^j$, and such that the difference $u^j(\theta^j) - \theta^j$ is in fact strictly larger than $u^j(w^j) - w^j$ for all $w^j \in X^j \setminus \{\theta^j\}$. Figure A.1 depicts the two cases $\theta^j < y^j$ (left) and $y^j < \theta^j$ (right).

Now choose all other functions u^ℓ strictly increasing and strictly concave such that $u^\ell(\theta^\ell) - \theta^\ell$ is strictly larger than $u^\ell(w^\ell) - w^\ell$ for all $w^\ell \in X^\ell \setminus \{\theta^\ell\}$, and such that $|u^\ell(w^\ell) - w^\ell| < \delta/[2(L-1)]$ for all $w^\ell \in X^\ell$, as described above. Let $\succcurlyeq$ be the preference order represented by $u = \sum_{\ell=1}^{L} u^\ell$. As argued above, θ is the top alternative of $\succcurlyeq$. Moreover, we have

$$u^j(y^j) - y^j \ + \ \sum_{\ell \neq j}(u^\ell(y^\ell) - y^\ell) \ > \ \delta/2 \ > \ u^j(x^j) - x^j \ + \ \sum_{\ell \neq j}(u^\ell(x^\ell) - x^\ell),$$

i.e. $u(y) > u(x)$, and hence $y \succ x$ as desired. The argument in the case $x^j > \theta^j \ \& \ x^j > y^j$ is completely symmetric.

An inspection of the preceding argument shows that, if $L \geq 2$, it is indeed possible to choose $u^1(\cdot)$ to be the identity, i.e. to choose the desired utility function $u(\cdot)$ to be quasi-linear.

Finally, consider the domain $\mathcal{D}_{\text{lin}}$ of all linear preferences on X. Observe first that only corner allocations can be the top of a linear preference order on X. To show that the domain $\mathcal{D}_{\text{lin}}$ also forms a rich subdomain of separably convex preferences, we need to show that for every corner allocation $\theta \in X$ and any pair $x, y \in X$ such that $x \notin [\theta, y]$ one can find an element $\succcurlyeq \in \mathcal{D}_{\text{lin}}$ with top θ and $u(y) > u(x)$. This is obvious if $L = 2$, i.e. in the case of a line; thus, assume $L \geq 3$. Without loss of generality assume that $\theta = (Q, 0, ..., 0)$. We distinguish three cases.

Case 1. If $y^1 > x^1$, choose $a^1 = 1$ and all other a^ℓ pairwise distinct such that $0 < a^\ell < \varepsilon$ for all $\ell \neq 1$. If ε is sufficiently small, we obtain

$$\sum_{\ell=1}^{L} a^\ell \cdot y^\ell \ > \ \sum_{\ell=1}^{L} a^\ell \cdot x^\ell \tag{A.3}$$

i.e. $u(y) > u(x)$ for the linear utility function represented by the a^ℓ; moreover, since $a^1 > a^\ell$ for all $\ell \neq 1$, $\theta = (Q, 0, ..., 0)$ is indeed the top of the corresponding linear preference.

Case 2. If $y^1 = x^1$, then there exists $k > 1$ such that $y^k > x^k$. Choose $a^1 = 1$, $a^k = 1 - \varepsilon$, and all other a^ℓ pairwise distinct such that $0 < a^\ell < \varepsilon$ for all $\ell \notin \{1, k\}$. If ε is sufficiently small the coefficients $\{a^1, ..., a^L\}$ represent a linear preference with top $\theta = (Q, 0, ..., 0)$ and $y \succ x$ as desired.

Case 3. Finally, consider the case $y^1 < x^1$. Let $L^- := \{\ell : y^\ell < x^\ell\}$ and $L^+ := \{\ell : y^\ell > x^\ell\}$. By the feasibility of x and y, we have

$$\sum_{\ell \in L^-} (x^\ell - y^\ell) \ = \ \sum_{\ell \in L^+} (y^\ell - x^\ell) \tag{A.4}$$

with each summand in this equation being strictly positive by construction. By the assumption of the present case, we have $1 \in L^-$, and since $x \notin [\theta, y]$ there exists $k \neq 1$ such that $k \in L^-$. This implies by (A.4)

$$(x^1 - y^1) \ < \ \sum_{\ell \in L^+} (y^\ell - x^\ell). \tag{A.5}$$

Now choose $a^1 = 1$, and all other a^ℓ pairwise distinct such that $1 - \varepsilon < a^\ell < 1$ for $\ell \in L^+$ and $0 < a^\ell < \varepsilon$ for $\ell \in L \setminus (\{1\} \cup L^+)$. For ε sufficiently small, we obtain by (A.5) that $y \succ x$ for the linear preference represented by the coefficients $\{a^1, ..., a^L\}$ as in (A.3). Furthermore, since a^1 is the uniquely largest coefficient, $\theta = (Q, 0, ..., 0)$ is indeed the top of this preference. □

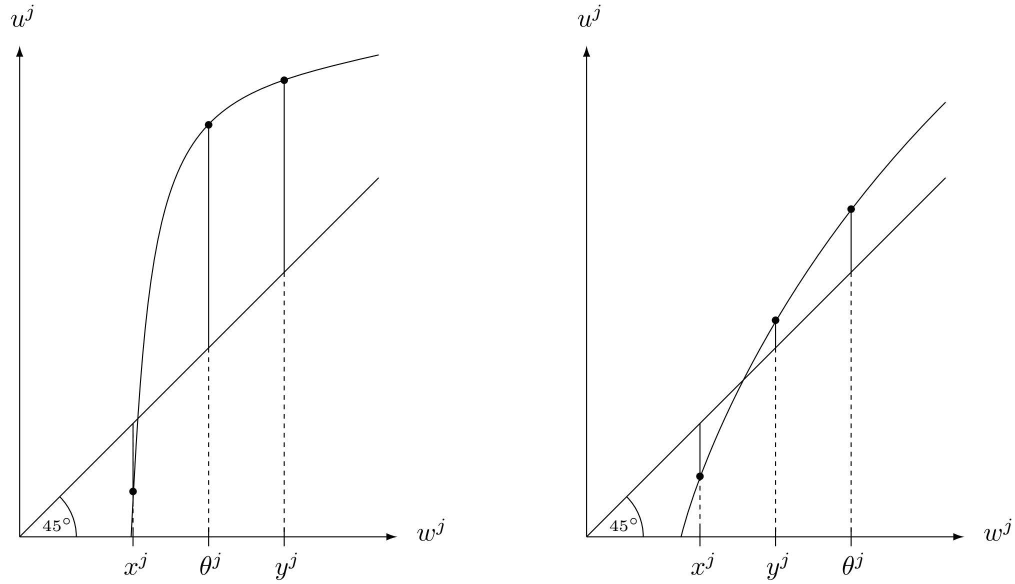

Figure A.1: Construction of u^j if $x^j < \theta^j$ and $x^j < y^j$

Proof of Fact 3.2 follows easily from the adopted definitions.

The next result shows that the restricted net majority tournament coincides with the ranking induced by the median rule, i.e. with the ranking of neighbors according to their aggregate distance.

Lemma A.1 *For any profile $\boldsymbol{\theta}$ and any two neighbors x and y,*

$$\#_{\boldsymbol{\theta}}(\rangle x, y\rangle) - \#_{\boldsymbol{\theta}}(\rangle y, x\rangle) = \Delta_{\boldsymbol{\theta}}(x) - \Delta_{\boldsymbol{\theta}}(y).$$

In particular, $x \, \mathrm{NM}_{\boldsymbol{\theta}} \, y$ if and only if $\Delta_{\boldsymbol{\theta}}(x) \leq \Delta_{\boldsymbol{\theta}}(y)$.

Proof. Combining Fact 3.1 and (3.6), we obtain that, for all $w \in \rangle x, y\rangle$, $d(w,x) - d(w,y) = -1$, for all $w \in \rangle y, x\rangle$, $d(w,x) - d(w,y) = 1$, and for all other $w \in X$, $d(w,x) - d(w,y) = 0$. From this the result immediate. □

As an immediate corollary of Lemma A.1, we obtain the acyclicity of the restricted net majority tournament and the inclusion $\mathrm{Med}(\boldsymbol{\theta}) \subseteq \mathrm{FM}(\boldsymbol{\theta})$. Note moreover, that a neighbor y of a median allocation x is itself a median allocation if and only if $x \, \mathrm{NM}_{\boldsymbol{\theta}} \, y$ and $y \, \mathrm{NM}_{\boldsymbol{\theta}} \, x$.

Fact A.1 *Let $f : X \to \mathbb{R}$ be a separable function with $f(x) = \sum_{\ell=1}^{L} f^{\ell}(x^{\ell})$ such that all functions $f^{\ell}(\cdot)$ are concave. Then, any local optimum of f on X is also a global optimum of f on X, i.e. if $f(x) \geq f(w)$ for all neighbors $w \in X$ of x, then $f(x) \geq f(w)$ for all $w \in X$. Moreover, the set of optima is box-convex, i.e. every point on a shortest path between two optima is also an optimum.*

Proof. As in the first part of the proof of Lemma 3.1, the stated conditions imply that f represents a separably convex order on X (recall that no monotonicity condition on f or the

f^ℓ is required for this conclusion). This implies that f must be constant along any shortest path connecting two local optima. Indeed, suppose by way of contradiction that f is not constant along some shortest path connecting two local optima x and z. Then there exist two neighbors along that path, say y and $y_{(kj)}$, such that $f(y) < f(y_{(kj)})$. Since y and $y_{(kj)}$ are on a shortest path connecting x and z, we have $y_{(kj)} \in [y, z]$ or $y_{(kj)} \in [y, x]$. Without loss of generality, assume the former; then, by the separable convexity, we obtain $f(z) < f(y_{(kj)})$ contradicting the assumption that z is a local optimum. From this, all other assertions in Fact A.1 follow at once. □

Proof of Theorem 1. Since the negative of the aggregate distance function $-\Delta_{\boldsymbol{\theta}}(\cdot)$ is the sum of the separable and concave functions $-d(\cdot, \theta_i)$ it is itself separable and concave. Hence, by Fact A.1, each of its local optima is a global optimum. This implies $\text{FM}(\boldsymbol{\theta}) \subseteq \text{Med}(\boldsymbol{\theta})$; using Lemma A.1, we thus obtain $\text{FM}(\boldsymbol{\theta}) = \text{Med}(\boldsymbol{\theta})$ for all profiles $\boldsymbol{\theta}$. From this the box-convexity of $\text{FM}(\boldsymbol{\theta})$ follows using Fact A.1 again. This completes the proof of Theorem 1. □

Proof of Proposition 1 in text.

Proof of Proposition 2 in text.

Proof of Theorem 2. The equivalence of (i) and (ii) follows from Proposition 2. Thus, consider statement (iii). The idea of the following proof is to show that (4.1) is equivalent to x being a local maximum of aggregate goal satisfaction; this implies the equivalence of (ii) and (iii) by Fact A.1.

We first introduce some notation. For a fixed profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n) \in X^n$, each $\ell = 1, ..., L$ and $r \in \mathbb{Z}$, denote by $v^\ell(r) := \sum_{i=1}^n \min\{r, \theta_i^\ell\}$ so that for the aggregate goal satisfaction function $v(\cdot)$ we have $v(x) = \sum_\ell v^\ell(x^\ell)$. Moreover, let

$$\begin{aligned} \nabla_- v^\ell(r) &:= v^\ell(r) - v^\ell(r-1), \\ \nabla_+ v^\ell(r) &:= v^\ell(r+1) - v^\ell(r). \end{aligned}$$

By construction, we obtain

$$\begin{aligned} \nabla_- v^\ell(r) &= \#\{i : \theta_i^\ell \geq r\}, \\ \nabla_+ v^\ell(r) &= \#\{i : \theta_i^\ell \geq r+1\}. \end{aligned} \tag{A.6}$$

By definition of $\theta_{[k]}^\ell$, we have $\#\{i : \theta_i^\ell \geq r\} \geq (n-k+1)$ whenever $r \leq \theta_{[k]}^\ell$, and hence by (A.3),

$$r \leq \theta_{[k]}^\ell \;\Rightarrow\; \nabla_- v^\ell(r) \geq (n-k+1). \tag{A.7}$$

Similarly, we have $\#\{i : \theta_i^\ell \geq r+1\} \leq (n-k)$ whenever $r \geq \theta_{[k]}^\ell$, hence, again by (A.3),

$$r \geq \theta_{[k]}^\ell \;\Rightarrow\; \nabla_+ v^\ell(r) \leq (n-k). \tag{A.8}$$

Now consider any $x \in X$ satisfying (4.1), i.e. for all $\ell = 1, ..., L$, $\theta_{[k^*(\boldsymbol{\theta})]}^\ell \leq x^\ell \leq \theta_{[k^*(\boldsymbol{\theta})+1]}^\ell$. We will show that x is a local maximizer of aggregate goal satisfaction v. By Fact A.1, x is then also a global optimum, hence a frugal majority winner by Proposition 2. Thus, consider any neighbor y of x. Without loss of generality, assume that $y = x_{(21)}$, i.e.$y^1 = x^1 - 1$, $y^2 = x^2 + 1$, and $y^\ell = x^\ell$ for all $\ell = 3, ..., L$. We have $x^1 \leq \theta^\ell_{[k^*(\boldsymbol{\theta})+1]}$ and $x^2 \geq \theta^\ell_{[k^*(\boldsymbol{\theta})]}$, therefore, using (A.4) and (A.5),

$$\begin{aligned} v(x) - v(y) &= \nabla_- v^1(x^1) - \nabla_+ v^2(x^2) \\ &\geq n - (k^*(\boldsymbol{\theta}) + 1) + 1 - (n - k^*(\boldsymbol{\theta})) \\ &= 0. \end{aligned}$$

This proves that every $x \in X$ satisfying (4.1) is indeed a maximizer of aggregate goal satisfaction.

Conversely, consider $x \in X$ that violates (4.1). There are two (not mutually exclusive) cases.

Case 1. For some coordinate h, $x^h < \theta^h_{[k^*(\boldsymbol{\theta})]}$. In this case, there must exist some other coordinate j such that $x^j > \theta^j_{[k^*(\boldsymbol{\theta})]}$. Consider the neighbor y of x such that $y^h = x^h + 1$, $y^j = x^j - 1$, and $y^\ell = x^\ell$ for all coordinates $\ell \neq h, \ell$, i.e. $y = x_{(hj)}$. By the same arguments as above, we obtain using (A.3),

$$r < \theta^\ell_{[k]} \;\Rightarrow\; \nabla_+ v^\ell(r) \geq (n - k + 1) \tag{A.9}$$

and

$$r > \theta^\ell_{[k]} \;\Rightarrow\; \nabla_- v^\ell(r) \leq (n - k). \tag{A.10}$$

Therefore,

$$\begin{aligned} v(y) - v(x) &= \nabla_+ v^h(x^h) - \nabla_- v^j(x^j) \\ &\geq n - k^*(\boldsymbol{\theta}) + 1 - (n - k^*(\boldsymbol{\theta})) \\ &= 1, \end{aligned}$$

hence x is not a maximizer of aggregate goal satisfaction.

Case 2. For some coordinate h, $x^h > \theta^h_{[k^*(\boldsymbol{\theta})+1]}$. In this case, there must exist some other coordinate ℓ such that $x^\ell < \theta^\ell_{[k^*(\boldsymbol{\theta})+1]}$. Consider the neighbor y of x such that $y^h = x^h - 1$, $y^\ell = x^\ell + 1$, and $y^\ell = x^\ell$ for all coordinates $\ell \neq h, \ell$, i.e. $y = x_{(jh)}$. By (A.6) and (A.7), we obtain

$$\begin{aligned} v(y) - v(x) &= \nabla_+ v^j(x^j) - \nabla_- v^h(x^h) \\ &\geq n - (k^*(\boldsymbol{\theta}) + 1) + 1 - (n - (k^*(\boldsymbol{\theta}) + 1)) \\ &= 1, \end{aligned}$$

hence x is not a maximizer of aggregate goal satisfaction in this case either. This completes the proof of Theorem 2. □

Proof of Proposition 3. By contraposition, suppose that $y >_{\theta_i} x$ for all i. By Fact 3.1, this implies $y \in [\theta_i, x]$ for all i. In particular, all individuals necessarily strictly prefer any neighbor of x in direction of y to x itself by separable convexity. But since x thus looses at least one local majority comparison (even by a unanimous vote), it cannot be a frugal majority winner. This completes the proof of Proposition 3. □

Proof of Proposition 4 follows at once from Theorem 2.

Proof of Proposition 5. The idea of the proof is to use the characterization in Theorem 2c) in order to show that by participating an agent moves the interval $[\theta^\ell_{[k^*(\cdot)]}, \theta^\ell_{[k^*(\cdot)+1]}]$ 'closer' to her top in all coordinates ℓ simultaneously.

The statement of Proposition 5 is easily verified for a unanimous profile, thus assume in the following that $\boldsymbol{\theta}$ is non-unanimous. Then there exists $k^*(\boldsymbol{\theta}) < n$ such that $Q_{[k^*(\boldsymbol{\theta})]} \leq Q$ and $Q_{[k^*(\boldsymbol{\theta})+1]} > Q$ as in Theorem 2c). Now consider the additional participation of agent h with top θ_h. We use the following notation: the profile $\boldsymbol{\theta} \sqcup \theta_h$ will also be denoted $\tilde{\boldsymbol{\theta}}$; for each ℓ, $\tilde{\theta}^\ell_{[1]} \leq ... \leq \tilde{\theta}^\ell_{[n+1]}$ are the $n+1$ ordered values among $\{\theta^\ell_1, ..., \theta^\ell_n, \theta^\ell_h\}$, and

$$\tilde{Q}_{[k]} \ := \sum_{\ell=1}^{L} \tilde{\theta}^\ell_{[k]}$$

Since, for each ℓ, both the values $\theta^\ell_{[k]}$ and $\tilde{\theta}^\ell_{[k]}$ are weakly increasing in k, we obtain, for all $k \leq n$,

$$\tilde{Q}_{[k]} \leq Q_{[k]}.$$

Moreover, by the addition of agent h, we have $\theta^\ell_{[k-1]} \leq \tilde{\theta}^\ell_{[k]}$ for all ℓ and $k \leq n$, and hence

$$\tilde{Q}_{[k+1]} \geq Q_{[k]}.$$

In particular, we obtain $\tilde{Q}_{[k^*(\boldsymbol{\theta})]} \leq Q_{[k^*(\boldsymbol{\theta})]} \leq Q$, and $\tilde{Q}_{[k^*(\boldsymbol{\theta})+2]} \geq Q_{[k^*(\boldsymbol{\theta})+1]} > Q$. Thus, there are only two cases, either (i) $k^*(\tilde{\boldsymbol{\theta}}) = k^*(\boldsymbol{\theta})$, or (ii) $k^*(\tilde{\boldsymbol{\theta}}) = k^*(\boldsymbol{\theta}) + 1$.

In either case, it follows immediately from the definitions that, for all ℓ, the interval $[\tilde{\theta}^\ell_{[k^*(\tilde{\boldsymbol{\theta}})]}, \tilde{\theta}^\ell_{[k^*(\tilde{\boldsymbol{\theta}})+1]}]$ is 'closer' to θ^ℓ_h than the interval $[\theta^\ell_{[k^*(\boldsymbol{\theta})]}, \theta^\ell_{[k^*(\boldsymbol{\theta})+1]}]$ in the sense that both

$$\tilde{\theta}^\ell_{[k^*(\tilde{\boldsymbol{\theta}})]} \ \in \ \left[\theta^\ell_{[k^*(\boldsymbol{\theta})]}, \theta^\ell_h\right],$$

and

$$\tilde{\theta}^\ell_{[k^*(\tilde{\boldsymbol{\theta}})+1]} \ \in \ \left[\theta^\ell_{[k^*(\boldsymbol{\theta})+1]}, \theta^\ell_h\right].$$

This implies the two inclusions stated in (4.3) and completes the proof of Proposition 5. □

Proof of Theorem 1′. Fact 3.1 continues to hold in the continuous case since we still have $x >^{\mathcal{D}^*}_\theta y \Leftrightarrow x \in [\theta, y]$ for any top θ where $\mathcal{D}^*$ is the domain of all separably convex preference orders on X. We also have, for all profiles $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ and every pair x, y with $x\Gamma_{\boldsymbol{\theta}} y$,

$$\#_{\boldsymbol{\theta}}(\rangle x, y\rangle) - \#_{\boldsymbol{\theta}}(\rangle y, x\rangle) = \Delta_{\boldsymbol{\theta}}(x) - \Delta_{\boldsymbol{\theta}}(y). \tag{A.11}$$

as asserted by Lemma A.1 in the discrete case. In the continuous case, (A.11) can be derived as follows. Assume without loss of generality that $x^j > y^j$, $x^k < y^k$, $x^\ell = y^\ell$ for all $\ell \neq j, k$, and consider any top θ_i. By condition (ii) in the definition of $\Gamma_{\boldsymbol{\theta}}$, there are four cases (corresponding to the four non-shaded regions in Fig. 9 above):

(a) $\theta_i^j \geq x^j$ and $\theta_i^k \leq x^k$,
(b) $\theta_i^j \geq x^j$ and $\theta_i^k \geq y^k$,
(c) $\theta_i^j \leq y^j$ and $\theta_i^k \leq x^k$, or
(d) $\theta_i^j \leq y^j$ and $\theta_i^k \geq y^k$.

In case (a), we have $\theta_i \in \rangle x, y \rangle$ and hence $d(\theta_i, y) = d(\theta_i, x) + d(x, y)$; in case (d) we have $\theta_i \in \rangle y, x \rangle$ and hence $d(\theta_i, x) = d(\theta_i, y) + d(x, y)$. In cases (b) and (c), we have $\theta_i \in x \bowtie y$, and since $|x^j - y^j| = |x^k - y^k|$ by the feasibility of x and y, we obtain $d(\theta_i, x) = d(\theta_i, y)$ in either of these two cases. Thus, for all supporters of x over y the distance of their top to y is by $d(x, y)$ larger than the distance of their top to x; for all supporters of y over x the distance of their top to x is by $d(x, y)$ larger than the distance of their top to y; and for all other the distance of their top to x is the same as the distance of their top to y. This immediately implies (A.11).

The rest of the proof follows from straightforward adaption of the arguments given in the proof of Theorem 1. In particular, Fact A.1 generalizes in a straightforward manner in the continuous case. $\square$

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