



Existence and multiplicity of homoclinic solutions for a second-order Hamiltonian system

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Abstract. In this paper, we find new conditions to ensure the existence of one nontrivial homoclinic solution and also infinitely many homoclinic solutions for the second order Hamiltonian system

$$\ddot{u} - a(t)|u|^{p-2}u + \nabla W(t, u) = 0, \quad t \in \mathbb{R},$$

where $p > 2$, $a \in C(\mathbb{R}, \mathbb{R})$ with $\inf_{t \in \mathbb{R}} a(t) > 0$ and $\int_{\mathbb{R}} \left(\frac{1}{a(t)}\right)^{2/(p-2)} dt < +\infty$, and $W(t, x)$ is, as $|x| \rightarrow \infty$, superquadratic or subquadratic with certain hypotheses different from those used in previous related studies. Our approach is variational and we use the Cerami condition instead of the Palais–Smale one for deformation arguments.

Keywords: homoclinic solutions, Hamiltonian systems, variational methods, weighted L^p space.

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1 Introduction

Consider the second order Hamiltonian system

$$\ddot{u}(t) - a(t)|u(t)|^{p-2}u(t) + \nabla W(t, u(t)) = 0, \quad (HS)$$

where $p > 2$, $t \in \mathbb{R}$, $u \in \mathbb{R}^N$, $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and $\nabla W(t, x)$ denotes the gradient of $W(t, x)$ with respect to x . As usual, we say that a solution u of (HS) is homoclinic (to 0) if $u(t) \rightarrow 0$ as $|t| \rightarrow \infty$. Furthermore, if $u \neq 0$, then u is called a nontrivial homoclinic solution.

Homoclinic orbits of nonlinear differential equations have long been studied in the dynamical systems literature, generally in a setting involving perturbations and using a Melnikov function. The existence of many homoclinic orbits is a classical problem and the first multiplicity results go back to Poincaré [19] and Melnikov [17]. They proved, by means of perturbation techniques, that the system possesses infinitely many homoclinic orbits in the case of $N = 1$ when the potential depends periodically on time.

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If $p = 2$, system (HS) reduces to

$$\ddot{u}(t) - a(t)u(t) + \nabla W(t, u(t)) = 0. \quad (1.1)$$

During the past twenty years a large quantity of papers has been devoted to the use of variational methods to seek homoclinic motions of (1.1), see [2,3,5–9,11,15,16,18,20,22–27,30–33,35] and the references therein. The case where $a(t)$ and $W(t, x)$ are either periodic in t or independent of t were studied in [2, 3, 7, 9, 11, 15, 20, 22, 32]. The existence of one homoclinic solution can be obtained by going to the limit of periodic solutions of approximating problems on expanding interval; in this argument the variational method can be applied to solve the approximated problems as well as to obtain a good estimates for their solutions, see [2, 3, 11, 20]. Problem (1.1) without periodicity assumption on both a and W was considered in [5, 6, 8, 16, 18, 23–27, 30–32, 35]. Applying a symmetric mountain pass theorem, Omana and Willem [18] proved the existence of infinitely many homoclinic orbits of (1.1) provided that $a(t) \rightarrow +\infty$ as $|t| \rightarrow \infty$ and $W(t, x)$, besides other technical assumptions, satisfies the growth conditions $W(t, x)/|x|^2 \rightarrow +\infty$ (resp. 0) as $|x| \rightarrow +\infty$ (resp. $|x| \rightarrow 0$).

Nevertheless, to our knowledge, results obtained on system (HS) are considerably less, see [6, 24, 25]. Salvatore [24] constructed the existence and multiplicity results of system (HS) by applying a compact embedding between suitable weighted Sobolev spaces.

Theorem 1.1 (see Theorem 1.3 in [24]). *Assume that the following conditions are satisfied:*

(W_1) $a(t)$ is a continuous, positive function on \mathbb{R} such that for all $t \in \mathbb{R}$

$$a(t) \geq \gamma|t|^\alpha \quad \text{with } \alpha > \frac{p-2}{2}, \quad \gamma > 0;$$

(W'_1) there exists a constant $\mu > p$ such that

$$0 < \mu W(t, x) \leq (\nabla W(t, x), x), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N \setminus \{0\};$$

(W_2) $\nabla W(t, x) = o(|x|^{p-1})$ as $x \rightarrow 0$ uniformly in t ;

(W_3) there exists $\overline{W} \in C(\mathbb{R}^N, \mathbb{R}^+)$ such that

$$|W(t, x)| + |\nabla W(t, x)| \leq \overline{W}(x), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

Then there exists a nontrivial homoclinic solution of system (HS). Moreover, if $W(t, x)$ is even in x , i.e., $W(t, -x) = W(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$, then there exists an unbounded sequence of homoclinic solutions of system (HS).

Observe that condition (W'_1) characterizes the potential W as superquadratic at infinity, that is,

(W_1) $W(t, x)/|x|^p \rightarrow +\infty$ as $|x| \rightarrow \infty$ for a.e. $t \in \mathbb{R}$;

and is important in the argument for showing particularly the boundedness of Palais-Smale sequences. This kind of technical condition was first introduced by Ambrosetti and Rabinowitz [1], and often appears as necessary to solve superlinear differential equations such as elliptic problems, Hamiltonian systems and wave equations.

Chen and Tang [6] generalized Theorem 1.1 by relaxing the conditions imposed on $W(t, x)$. They proved the same conclusion by using the mountain pass theorem and the symmetric mountain pass theorem.

Theorem 1.2 (see Theorems 1.1 and 1.3 in [6]). Assume that (V_1) holds and $W(t, x)$ satisfies the following:

(H_1) $W(t, x) = W_1(t, x) - W_2(t, x)$, $W_1, W_2 \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and there is $R > 0$ such that $|\nabla W(t, x)|/a(t) = o(|x|^{p-1})$ as $x \rightarrow 0$ uniformly in $t \in (-\infty, -R] \cup [R, +\infty)$.

(H_2) There is a constant $\mu > p$ such that $0 < \mu W_1(t, x) \leq (\nabla W_1(t, x), x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N \setminus \{0\}$.

(H_3) $W_2(t, 0) \equiv 0$ and there is a constant $\varrho \in (p, \mu)$ such that $W_2(t, x) \geq 0$ and $(\nabla W_2(t, x), x) \leq \varrho W_2(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$.

Then system (HS) has a nontrivial homoclinic solution. Moreover, if $W(t, x)$ is even in x , then system (HS) has an unbounded sequence of homoclinic solutions.

Theorem 1.3 (see Theorems 1.2 and 1.4 in [6]). The conclusion of Theorem 1.2 is valid if we replace assumption (H_1) with

(H'_1) $W(t, x) = W_1(t, x) - W_2(t, x)$, $W_1, W_2 \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and $|\nabla W(t, x)|/a(t) = o(|x|^{p-1})$ as $x \rightarrow 0$ uniformly in $t \in \mathbb{R}$.

and assumption (H_3) with

(H'_3) $W_2(t, 0) \equiv 0$ and there is a constant $\varrho \in (p, \mu)$ such that $(\nabla W_2(t, x), x) \leq \varrho W_2(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$.

Although [6] improved Theorem 1.1 by relaxing conditions (W'_1) and (W_2) and removing (W_3) , it still requires the potential W satisfies:

$$\exists \varrho > p \text{ such that } (\nabla W(t, x), x) \geq \varrho W(t, x) \text{ for all } (t, x) \in \mathbb{R} \times \mathbb{R}^N \quad (1.2)$$

(see (H_2) and (H_3) (or (H'_3))). Hence it is somewhat restrictive and eliminates the superquadratic potentials, for example,

Ex 1.

$$W(t, x) = \begin{cases} |x|^q \ln |x| - |x|^2 + \frac{p}{p+1}, & |x| > 1, \\ -|x|^{p+1}/(p+1), & |x| \leq 1, \end{cases}$$

where $q > p$;

Ex 2. $W(t, x) = g(t)|x|^p \ln(1 + |x|^2)$, where $g : \mathbb{R} \rightarrow \mathbb{R}^+$ is a continuous bounded function with $\inf_{t \in \mathbb{R}} g(t) > 0$;

Ex 3. $W(t, x) = g(t) (|x|^\mu + (\mu - p)|x|^{\mu-\varepsilon} \sin^2(|x|^\varepsilon/\varepsilon))$, where $\mu > p$, $\varepsilon \in (0, \mu - p)$ and $g : \mathbb{R} \rightarrow \mathbb{R}^+$ is a continuous bounded function with $\inf_{t \in \mathbb{R}} g(t) > 0$;

and the subquadratic potentials, for example,

Ex 4. $W(t, x) = |x|^2 + b(t)|x|^\gamma$ for all $t \in \mathbb{R}$ and $|x| \leq \delta$, where $\gamma \in (1, 2)$ and $b \in L^{2/(2-\gamma)}(\mathbb{R}, \mathbb{R}^+)$ with $\text{meas} \{t \in \mathbb{R} : b(t) > 0\} > 0$;

Ex 5. $W(t, x) = \frac{1}{\gamma} b(t)|x|^\gamma + \frac{1}{s} c(t)|x|^s$ for all $t \in \mathbb{R}$ and $|x| \leq \delta$, where $\gamma, s \in (1, 2)$, $b \in L^{2/(2-\gamma)}(\mathbb{R}, \mathbb{R}^+)$ and $c \in L^{2/(2-s)}(\mathbb{R}, \mathbb{R}^+)$ with $\text{meas} \{t \in \mathbb{R} : c(t) > 0\} > 0$.

Motivated by the works mentioned above, the main goal of this paper is to find new conditions to guarantee the existence of homoclinic solutions of problem (HS). We are particularly interested in the cases where $a(t)$ satisfies:

$$(V) \ a \in C(\mathbb{R}, \mathbb{R}) \text{ with } a_0 := \inf_{t \in \mathbb{R}} a(t) > 0 \text{ and } \int_{\mathbb{R}} \left(\frac{1}{a(t)} \right)^{\frac{2}{p-2}} dt < +\infty,$$

and $W(t, x)$ satisfies conditions which are more general than (W'_1) . Typical examples, which match our setting but not satisfying Theorems 1.1–1.3, are Examples 1–5.

Remark 1.4. Assumption (V) is weaker than (V_1) . There are functions a which match our setting but not satisfying (V_1) . For example, let

$$a(t) = \begin{cases} \left(1 - n^2|t - n| + e^{-\frac{t^2}{2}}\right)^{(2-p)/2}, & |t - n| \leq \frac{1}{n^2} \ (n \in \mathbb{Z}, \ |n| \geq 2), \\ e^{(p-2)t^2/4}, & \text{elsewhere.} \end{cases}$$

We first handle the **superquadratic** case. Assume furthermore the following hypotheses:

(W₄) There exist $\mu > p$ and $L > 1$ such that

$$\mu W(t, x) \leq (\nabla W(t, x), x), \quad \forall t \in \mathbb{R}, \ |x| \geq L, \quad (1.3)$$

and

$$\inf_{t \in \mathbb{R}, |x|=L} W(t, x) > 0.$$

(W₅) For any $0 < \alpha < \beta$,

$$C_\alpha^\beta := \inf \left\{ \frac{\mathcal{W}(t, x)}{|x|^p} \mid t \in \mathbb{R}, \alpha \leq |x| < \beta \right\} > 0,$$

where $\mathcal{W}(t, x) := \frac{1}{p}(\nabla W(t, x), x) - W(t, x)$.

(W₆) There exist $a > 0$, $L_1 > 0$ and $\sigma \in (0, p - 1)$ such that

$$(\nabla W(t, x), x) \leq a \mathcal{W}(t, x) |x|^{p-\sigma}, \quad \forall t \in \mathbb{R}, \ |x| \geq L_1.$$

(W₇) $W(t, 0) \equiv 0$ for all $t \in \mathbb{R}$, and there is $\theta \geq 1$ such that

$$\theta W(t, x) \geq W(t, sx), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N, \ s \in [0, 1].$$

Theorem 1.5. Assume that (V) holds and $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ satisfies (W₂)–(W₄). Then problem (HS) possesses at least one nontrivial homoclinic solution. Moreover, if $W(t, x)$ is even in x , then problem (HS) possesses an unbounded sequence of homoclinic solutions (u_k) such that

$$\int_{\mathbb{R}} \left[\frac{1}{2} |\dot{u}_k|^2 + \frac{1}{p} a(t) |u_k|^p - W(t, u_k) \right] dt \rightarrow +\infty \quad \text{as } k \rightarrow \infty.$$

Remark 1.6. The potential W in Theorem 1.5 allows to be sign-changing. Example 1 verifies (W₂)–(W₄) if $p < \mu < \min \{q, p + 1\}$. One can check this fact by noting that

$$(\nabla W(t, x), x) - \mu W(t, x) = |x|^q [(q - \mu) \ln |x| + 1] + (\mu - 2) |x|^2 - \frac{\mu p}{p + 1}$$

for all $t \in \mathbb{R}$ and $|x| > 1$.

Theorem 1.7. Assume that (V) holds and $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ satisfies (W_1) – (W_3) and (W_5) – (W_6) . Then problem (HS) possesses at least one nontrivial homoclinic solution. Moreover, if $W(t, x)$ is even in x , then problem (HS) possesses an unbounded sequence of homoclinic solutions (u_k) such that

$$\int_{\mathbb{R}} \left[\frac{1}{2} |\dot{u}_k|^2 + \frac{1}{p} a(t) |u_k|^p - W(t, u_k) \right] dt \rightarrow +\infty \quad \text{as } k \rightarrow \infty.$$

Remark 1.8. Condition (W'_1) implies the ones (W_1) and (W_6) . Indeed, assuming (W'_1) holds, it is clear that (W_1) is satisfied. Setting $L_1 \geq 1$ so large that

$$\frac{1}{\mu} < \frac{1}{p} - \frac{1}{|x|^{p-\sigma}} \quad \text{whenever } |x| \geq L_1.$$

Then, for such $|x|$,

$$W(t, x) \leq \left(\frac{1}{p} - \frac{1}{|x|^{p-\sigma}} \right) (\nabla W(t, x), x),$$

and hence

$$(\nabla W(t, x), x) \leq |x|^{p-\sigma} \left[\frac{1}{p} (\nabla W(t, x), x) - W(t, x) \right] = |x|^{p-\sigma} \mathcal{W}(t, x).$$

Remark 1.9. The functions of Examples 2–3 verify the conditions (W_1) – (W_3) and (W_5) – (W_6) . One can check this fact for Example 2 by noting that

$$\nabla W(t, x) = g(t) \left[p|x|^{p-2}x \ln(1 + |x|^2) + \frac{2|x|^p x}{1 + |x|^2} \right], \quad \mathcal{W}(t, x) = g(t) |x|^p \frac{2|x|^2}{p(1 + |x|^2)},$$

and for Example 3 by noting that

$$\begin{aligned} \nabla W(t, x) &= \mu|x|^{\mu-2}x + (\mu - p)|x|^{\mu-\varepsilon} \left[(\mu - \varepsilon)|x|^{-2}x \sin^2 \left(\frac{|x|^\varepsilon}{\varepsilon} \right) + |x|^{\varepsilon-2}x \sin \left(\frac{2|x|^\varepsilon}{\varepsilon} \right) \right], \\ \frac{\mathcal{W}(t, x)}{|x|^{\mu-\varepsilon}} &= \frac{\mu - p}{p} g(t) \left[|x|^\varepsilon \left(1 + \sin \left(\frac{2|x|^\varepsilon}{\varepsilon} \right) \right) + (\mu - p - \varepsilon) \sin^2 \left(\frac{|x|^\varepsilon}{\varepsilon} \right) \right]. \end{aligned}$$

Theorem 1.10. Assume that (V) holds and $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ satisfies (W_1) – (W_3) and (W_7) . Then problem (HS) possesses at least one nontrivial homoclinic solution. Moreover, if $W(t, x)$ is even in x , then problem (HS) possesses an unbounded sequence of homoclinic solutions (u_k) such that

$$\int_{\mathbb{R}} \left[\frac{1}{2} |\dot{u}_k|^2 + \frac{1}{p} a(t) |u_k|^p - W(t, u_k) \right] dt \rightarrow +\infty \quad \text{as } k \rightarrow \infty.$$

Remark 1.11. We mention that the monotonicity condition like (W_7) was used in Jeanjean [12] to obtain one positive solution for a semilinear problem in \mathbb{R}^N , in [14] to get infinitely many solutions for quasilinear elliptic problems setting on a bounded domain, and in [10] to compute the critical points of the energy functional and obtain nontrivial solutions via Morse theory. It turns out that if for fixed $(t, x) \in \mathbb{R} \times \mathbb{R}^N \setminus \{0\}$,

$$s \mapsto \frac{(\nabla W(t, sx), x)}{s^{p-1}} \quad \text{is increasing in } s \in (0, 1],$$

then (W_7) is satisfied.

Remark 1.12. Hypotheses (W_2) and (W_5) (or (W_7)) yield that

$$W(t, x) \geq 0, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N. \quad (1.4)$$

In fact, it follows from (W_5) (or (W_7)) that

$$\mathcal{W}(t, x) = \frac{1}{p}(\nabla W(t, x), x) - W(t, x) \geq 0, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

Hence, for $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ and $s > 0$, we have

$$\frac{d}{ds} \left(\frac{W(t, sx)}{s^p} \right) = \frac{(\nabla W(t, sx), sx) - pW(t, sx)}{s^{p+1}} \geq 0. \quad (1.5)$$

Besides, (W_2) implies that

$$\lim_{s \rightarrow 0^+} W(t, sx)/s^p = 0,$$

which, jointly with (1.5), shows that (1.4) holds.

Next we consider the **subquadratic** case. Assume that:

(W_8) $W \in C^1(\mathbb{R} \times B_\delta(0), \mathbb{R})$, $W(t, -x) = W(t, x)$ for all $(t, x) \in \mathbb{R} \times B_\delta(0)$, where $B_\delta(0)$ is the ball in \mathbb{R}^N centered at 0 with radius $\delta > 0$.

(W_9) $W(t, 0) \equiv 0$, and there exist constants $a_1 > 0$, $\gamma \in (1, 2)$ and a function $b_1 \in L^{\frac{2}{2-\gamma}}(\mathbb{R}, \mathbb{R}^+)$ such that

$$|\nabla W(t, x)| \leq a_1|x| + b_1(t)|x|^{\gamma-1}, \quad \forall t \in \mathbb{R}, |x| \leq \delta.$$

(W_{10}) There exist $t_0 \in \mathbb{R}$, two sequences $\{\delta_n\}$, $\{M_n\}$ and constants $a_2, d > 0$ such that $\delta_n > 0$, $M_n > 0$ and

$$\begin{aligned} \lim_{n \rightarrow \infty} \delta_n &= 0, & \lim_{n \rightarrow \infty} M_n &= +\infty, \\ |x|^{-2}W(t, x) &\geq M_n \quad \text{for } |t - t_0| \leq d \text{ and } |x| = \delta_n, \\ |x|^{-2}W(t, x) &\geq -a_2 \quad \text{for } |t - t_0| \leq d \text{ and } |x| \leq \delta. \end{aligned}$$

Theorem 1.13. *Suppose that (V) and (W_8) – (W_{10}) are satisfied. Then problem (HS) possesses infinitely many nontrivial homoclinic solutions (u_k) such that $\max_{t \in \mathbb{R}} |u_k(t)| \rightarrow 0$ as $k \rightarrow \infty$.*

Remark 1.14. Theorem 1.13 improves [24, Theorem 1.2]. The functions of Examples 4–5 satisfy (W_8) – (W_{10}) but do not satisfy the results in [6, 24, 25]. It is trivial for Example 4. To check this fact for Example 5, note that

$$\begin{aligned} |\nabla W(t, x)| &\leq b(t)|x|^{\gamma-1} + \frac{2-s}{2-\gamma} \left(c(t)|x|^{\frac{2-s}{2-\gamma}(\gamma-1)} \right)^{\frac{2-\gamma}{2-s}} + \frac{s-\gamma}{2-\gamma} \left(|x|^{\frac{s-\gamma}{2-\gamma}} \right)^{\frac{2-\gamma}{s-\gamma}} \\ &\leq \left(b(t) + \frac{2-s}{2-\gamma} c(t)^{\frac{2-\gamma}{2-s}} \right) |x|^{\gamma-1} + \frac{s-\gamma}{2-\gamma} |x| \end{aligned}$$

for all $t \in \mathbb{R}$ and $|x| \leq \delta$.

The paper is organized as follows. After presenting some preliminaries, we prove the above existence and multiplicity results for the superquadratic and subquadratic cases in turn.

Notation. Throughout the paper we denote by c, c_i the various positive constants which may vary from line to line and are not essential to the problem.

2 Preliminaries

We shall construct the variational setting under condition (V). For a nonnegative measurable function a and a real number $s > 1$, define the weighted Lebesgue space

$$L_a^s = L^s(\mathbb{R}, \mathbb{R}^N; a) = \left\{ u : \mathbb{R} \rightarrow \mathbb{R}^N \text{ is measurable} \mid \int_{\mathbb{R}} a(t)|u(t)|^s dt < +\infty \right\}$$

and associated with it the norm

$$\|u\|_{a,s} = \left(\int_{\mathbb{R}} a(t)|u(t)|^s dt \right)^{1/s}.$$

We define, for any $r \in [1, +\infty]$,

$$L^r = L^r(\mathbb{R}, \mathbb{R}^N), \quad H^1 = H^1(\mathbb{R}, \mathbb{R}^N)$$

with the usual norms

$$\|u\|_r = \left(\int_{\mathbb{R}} |u(t)|^r dt \right)^{1/r}, \quad \|u\|_{\infty} = \sup_{t \in \mathbb{R}} |u(t)|, \quad \|u\|_{H^1} = \left(\int_{\mathbb{R}} (|\dot{u}|^2 + |u|^2) dt \right)^{1/2}.$$

Let $E := H^1 \cap L_a^p$, where $a(t)$ is the function introduced in (V). It is easy to check that E is a reflexive Banach space under the norm

$$\|u\| = \|\dot{u}\|_2 + \|u\|_{a,p} = \left(\int_{\mathbb{R}} |\dot{u}|^2 dt \right)^{1/2} + \left(\int_{\mathbb{R}} a(t)|u|^p dt \right)^{1/p}.$$

Observing

$$\begin{aligned} \int_{\mathbb{R}} |u(t)|^2 dt &= \int_{\mathbb{R}} a(t)^{-\frac{2}{p}} \cdot a(t)^{\frac{2}{p}} |u|^2 dt \\ &\leq \left(\int_{\mathbb{R}} \left(\frac{1}{a(t)} \right)^{\frac{2}{p-2}} dt \right)^{\frac{p-2}{p}} \cdot \left(\int_{\mathbb{R}} a(t)|u|^p dt \right)^{\frac{2}{p}} \\ &\leq \|a(t)^{-1}\|_{2/(p-2)}^{2/p} \left(\int_{\mathbb{R}} a(t)|u|^p dt \right)^{\frac{2}{p}}, \end{aligned} \tag{2.1}$$

we have

$$\begin{aligned} \int_{\mathbb{R}} (|\dot{u}|^2 + |u|^2) dt &\leq \int_{\mathbb{R}} |\dot{u}|^2 dt + \|a(t)^{-1}\|_{2/(p-2)}^{2/p} \left(\int_{\mathbb{R}} a(t)|u|^p dt \right)^{\frac{2}{p}} \\ &\leq c \left[\int_{\mathbb{R}} |\dot{u}|^2 dt + \left(\int_{\mathbb{R}} a(t)|u|^p dt \right)^{\frac{2}{p}} \right] \\ &\leq c\|u\|^2, \end{aligned}$$

which implies that E is continuously embedded into H^1 . So E is continuously embedded into L^r for $2 \leq r \leq +\infty$, and hence, for each $r \in [2, +\infty]$, there is $\tau_r > 0$ such that

$$\|u\|_r \leq \tau_r \|u\|, \quad \forall u \in E. \tag{2.2}$$

Furthermore, we have the following lemma.

Lemma 2.1. *If assumption (V) is satisfied, then the embedding $E \hookrightarrow L^r$ is compact for $2 \leq r \leq +\infty$.*

Proof. We adapt an argument in Ding [8]. Let $K \subset E$ be a bounded set. Then there is $C_0 > 0$ such that

$$\|u\| \leq C_0, \quad \forall u \in K. \quad (2.3)$$

We shall show that K is precompact in L^r for $2 \leq r \leq +\infty$.

Since (V) implies that

$$\int_{|t| \geq R} \left(\frac{1}{a(t)} \right)^{\frac{2}{p-2}} dt \rightarrow 0 \quad \text{as } R \rightarrow +\infty, \quad (2.4)$$

for any $\varepsilon > 0$, we take $R_0 > 0$ large enough such that

$$\left[\int_{|t| \geq R} \left(\frac{1}{a(t)} \right)^{\frac{2}{p-2}} dt \right]^{\frac{p}{p-2}} < \frac{\varepsilon}{8C_0^2}, \quad \forall R \geq R_0. \quad (2.5)$$

Noting the embedding $E \hookrightarrow H^1$ is continuous, K is bounded in H^1 . Applying the Sobolev compact embedding theorem, $H^1((-R_0, R_0), \mathbb{R}^N)$ is compactly embedded in $L^r((-R_0, R_0), \mathbb{R}^N)$ for all $1 \leq r \leq +\infty$. Thus, there are $u_1, u_2, \dots, u_m \in K$ such that for any $u \in K$, there is u_i ($1 \leq i \leq m$) such that

$$\int_{|t| \leq R_0} |u - u_i|^2 dt < \frac{\varepsilon}{2}.$$

Hence, using Hölder's inequality, (2.5) and (2.3), we obtain

$$\begin{aligned} \int_{\mathbb{R}} |u - u_i|^2 dt &\leq \int_{|t| \leq R_0} |u - u_i|^2 dt + \int_{|t| > R_0} |u - u_i|^2 dt \\ &\leq \frac{\varepsilon}{2} + \left[\int_{|t| > R_0} \left(\frac{1}{a(t)} \right)^{\frac{2}{p-2}} dt \right]^{\frac{p-2}{p}} \left(\int_{|t| > R_0} a(t) |u - u_i|^p dt \right)^{\frac{2}{p}} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{8C_0^2} \|u - u_i\|^2 \\ &< \varepsilon. \end{aligned}$$

The above arguments yield that K has a finite ε -net and so is precompact in L^2 .

For any $n \in \mathbb{N}$, $t \in \mathbb{R}$ and $u \in E$, one has

$$u(t) = \int_t^{t+1} [-\dot{u}(s)(t+1-s)^{n+1} + u(s)(n+1)(t+1-s)^n] ds,$$

which implies that

$$|u(t)| \leq \frac{1}{\sqrt{2n+3}} \left(\int_t^{t+1} |\dot{u}|^2 ds \right)^{1/2} + \frac{n+1}{\sqrt{2n+1}} \left(\int_t^{t+1} |u|^2 ds \right)^{1/2}$$

by the Hölder inequality. Particularly, for any $R > 0$ and $u, v \in K$, we obtain

$$\begin{aligned} |u(t) - v(t)| &\leq \frac{1}{\sqrt{2n+3}} \left(\int_{|s| \geq R} |\dot{u} - \dot{v}|^2 ds \right)^{1/2} + \frac{n+1}{\sqrt{2n+1}} \left(\int_{|s| \geq R} |u - v|^2 ds \right)^{1/2} \\ &\leq \frac{1}{\sqrt{2n+3}} \|u - v\| + \frac{n+1}{\sqrt{2n+1}} \left[\int_{|s| \geq R} \left(\frac{1}{a(t)} \right)^{\frac{2}{p-2}} ds \right]^{\frac{p-2}{2p}} \left(\int_{|s| \geq R} a(t) |u - v|^p ds \right)^{\frac{1}{p}} \\ &\leq \frac{2C_0}{\sqrt{2n+3}} + \frac{2C_0(n+1)}{\sqrt{2n+1}} \left[\int_{|s| \geq R} \left(\frac{1}{a(t)} \right)^{\frac{2}{p-2}} ds \right]^{\frac{p-2}{2p}}, \quad \forall |t| \geq R. \end{aligned}$$

For any $\varepsilon > 0$, first choosing n sufficiently large such that

$$\frac{2C_0}{\sqrt{2n+3}} < \frac{\varepsilon}{2},$$

and then R_1 large enough satisfying

$$\frac{2C_0(n+1)}{\sqrt{2n+1}} \left[\int_{|s| \geq R_1} \left(\frac{1}{a(t)} \right)^{\frac{2}{p-2}} ds \right]^{\frac{p-2}{2p}} < \frac{\varepsilon}{2}$$

by (2.4). It follows that

$$\sup_{|t| \geq R_1} |u(t) - v(t)| < \varepsilon, \quad \forall u, v \in K. \quad (2.6)$$

Again, using the Sobolev compact embedding theorem, there are $u_1, u_2, \dots, u_m \in K$ such that for any $u \in K$, there is u_i ($1 \leq i \leq m$) such that

$$\max_{|t| \leq R_1} |u(t) - u_i(t)| < \varepsilon,$$

which, together with (2.6), shows that

$$\|u - u_i\|_\infty < \varepsilon.$$

Thus, K is precompact in L^∞ .

Now for any $r \in (2, +\infty)$, since

$$\int_{\mathbb{R}} |u|^r dt \leq \|u\|_\infty^{r-2} \int_{\mathbb{R}} |u|^2 dt, \quad \forall u \in K,$$

we see immediately that K is precompact in L^r . \square

Lemma 2.2.

(i) For $u \in E$, there holds

$$\frac{1}{2} \int_{\mathbb{R}} |\dot{u}|^2 dt + \frac{1}{p} \int_{\mathbb{R}} a(t) |u|^p dt \leq c(\|u\|^2 + \|u\|^p).$$

(ii) Given $\alpha, \beta > 0$, there is $c > 0$ such that for every $u \in E$, there holds

$$\alpha \int_{\mathbb{R}} |\dot{u}|^2 dt + \beta \int_{\mathbb{R}} a(t) |u|^p dt \geq \begin{cases} c\|u\|^p, & \text{if } \|u\| \leq 1, \\ c\|u\|^2, & \text{if } \|u\| \geq 1. \end{cases}$$

Proof. The conclusion follows easily from the definition of $\|\cdot\|$. \square

3 The superquadratic case

By assumptions (V) and (W₂), the energy functional associated to problem (HS) on E given by

$$I(u) = \int_{\mathbb{R}} \left(\frac{1}{2} |\dot{u}|^2 + \frac{1}{p} a(t) |u|^p \right) dt - \int_{\mathbb{R}} W(t, u) dt$$

is of C^1 -class, and

$$\langle I'(u), v \rangle = \int_{\mathbb{R}} [(\dot{u}, \dot{v}) - a(t)|u|^{p-2}(u, v)] dt - \int_{\mathbb{R}} (\nabla W(t, u), v) dt$$

for all $u, v \in E$. It is routine to show that any nontrivial critical point of I is a classical solution of system (HS) with $u(\pm\infty) = 0$.

To find the critical points of I , we shall show that I satisfies the Cerami condition, i.e., $(u_n) \subset E$ has a convergent subsequence whenever $\{I(u_n)\}$ is bounded and $(1 + \|u_n\|)\|I'(u_n)\| \rightarrow 0$ as $n \rightarrow \infty$. Such a sequence is then called a Cerami sequence.

Lemma 3.1. *Let (V) and (W₂)–(W₄) be satisfied. Then I satisfies the Cerami condition.*

Proof. Let (u_n) be a Cerami sequence, i.e.,

$$\sup_n |I(u_n)| < c \quad \text{and} \quad \|I'(u_n)\|(1 + \|u_n\|) \xrightarrow{n} 0. \quad (3.1)$$

We show that (u_n) is bounded. Arguing indirectly, assume that $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$. We consider $w_n := u_n / \|u_n\|$. Then, up to a subsequence, we get

$$w_n \rightharpoonup w \quad \text{in } E, \quad w_n \rightarrow w \quad \text{in } L^r \quad (2 \leq r \leq +\infty), \quad w_n(t) \rightarrow w(t) \quad \text{a.e. } t \in \mathbb{R}. \quad (3.2)$$

Case 1. $w \equiv 0$ in E . From (W₂), for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) \in (0, 1)$ such that

$$|\nabla W(t, x)| \leq \varepsilon |x|^{p-1}, \quad \forall t \in \mathbb{R}, |x| < \delta, \quad (3.3)$$

and

$$|W(t, x)| \leq \varepsilon |x|^p, \quad \forall t \in \mathbb{R}, |x| \leq \delta. \quad (3.4)$$

Combining this with (W₃), we obtain

$$|(\nabla W(t, x), x) - \mu W(t, x)| \leq (\mu + 1) \left(\varepsilon + \delta^{-p} \max_{\delta \leq |x| \leq L} \overline{W}(x) \right) |x|^p, \quad \forall t \in \mathbb{R}, |x| \leq L,$$

and then, using (W₄),

$$(\nabla W(t, x), x) - \mu W(t, x) \geq -c|x|^p, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

Hence we obtain

$$\begin{aligned} o(1) &= \frac{1}{\|u_n\|^p} \left(I(u_n) - \frac{1}{\mu} \langle I'(u_n), u_n \rangle \right) \\ &= \left(\frac{1}{2} - \frac{1}{\mu} \right) \frac{1}{\|u_n\|^p} \int_{\mathbb{R}} |\dot{u}_n|^2 dt + \left(\frac{1}{p} - \frac{1}{\mu} \right) \int_{\mathbb{R}} a(t) |w_n|^p dt \\ &\quad + \frac{1}{\|u_n\|^p} \int_{\mathbb{R}} \left[\frac{1}{\mu} (\nabla W(t, u_n), u_n) - W(t, u_n) \right] dt \\ &\geq o(1) + \left(\frac{1}{p} - \frac{1}{\mu} \right) \int_{\mathbb{R}} a(t) |w_n|^p dt - c \int_{\mathbb{R}} |w_n|^p dt, \end{aligned}$$

which implies that

$$\int_{\mathbb{R}} a(t) |w_n|^p dt \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (3.5)$$

by the second limit of (3.2). Here, and in what follows, $o(1)$ denotes a quantity which goes to zero as $n \rightarrow \infty$. On the other hand, (W_4) implies

$$W(t, x) \geq c_1 |x|^\mu, \quad \forall t \in \mathbb{R}, \quad |x| \geq L, \quad (3.6)$$

where $c_1 = L^{-\mu} \inf_{t \in \mathbb{R}, |x|=L} W(t, x) > 0$. Thus

$$W(t, x) \geq 0, \quad \forall t \in \mathbb{R}, \quad |x| \geq L, \quad (3.7)$$

and

$$\frac{W(t, x)}{|x|^p} \rightarrow +\infty \quad \text{as } |x| \rightarrow \infty \quad \text{uniformly for } t \in \mathbb{R}. \quad (3.8)$$

It follows from (3.3), (3.4) and (W_3) that

$$\begin{aligned} & |(\nabla W(t, x), x) - pW(t, x)| \\ & \leq (p+1) \left(\varepsilon + \delta^{-p} \max_{\delta \leq |x| \leq L} \overline{W}(x) \right) L^{p-2} |x|^2 \leq c|x|^2, \quad \forall t \in \mathbb{R}, \quad |x| \leq L, \end{aligned}$$

and, using (3.7) and (1.3),

$$(\nabla W(t, x), x) - pW(t, x) \geq -c|x|^2, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

Therefore,

$$\begin{aligned} o(1) &= \frac{1}{\|u_n\|^2} \left(I(u_n) - \frac{1}{p} \langle I'(u_n), u_n \rangle \right) \\ &= \left(\frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}} |\dot{w}_n|^2 dt + \frac{1}{\|u_n\|^2} \int_{\mathbb{R}} \left[\frac{1}{p} (\nabla W(t, u_n), u_n) - W(t, u_n) \right] dt \\ &\geq \left(\frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}} |\dot{w}_n|^2 dt - c \int_{\mathbb{R}} |w_n|^2 dt, \end{aligned}$$

which yields that

$$\int_{\mathbb{R}} |\dot{w}_n|^2 dt \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This, jointly with (3.5), contradicts the fact $\|w_n\| = 1$.

Case 2. $w \neq 0$ in E . Taking $\Theta := \{t \in \mathbb{R} : w(t) \neq 0\}$, then the set Θ has positive measure. For $t \in \Theta$, we have $|u_n(t)| \rightarrow \infty$, and then, using (3.8),

$$\frac{W(t, u_n(t))}{|u_n(t)|^p} |w_n(t)|^p \rightarrow +\infty \quad \text{as } n \rightarrow \infty.$$

It follows from the Fatou lemma (see [34]) that

$$\int_{\Theta} \frac{W(t, u_n)}{|u_n|^p} |w_n|^p dt \rightarrow +\infty \quad \text{as } n \rightarrow \infty. \quad (3.9)$$

Moreover, it follows from (3.7) and (W_2) – (W_3) that

$$W(t, x) \geq -c|x|^p, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

So, by (2.2),

$$\int_{\mathbb{R} \setminus \Theta} \frac{W(t, u_n)}{\|u_n\|^p} dt \geq - \int_{\mathbb{R} \setminus \Theta} \frac{c|u_n|^p}{\|u_n\|^p} dt \geq -c \int_{\mathbb{R}} |w_n|^p dt \geq -c\tau_p^p, \quad \forall n \in \mathbb{N}. \quad (3.10)$$

Consequently, using (3.10), (3.9) and the first inequality of (3.1),

$$\begin{aligned} \frac{1}{p} \int_{\mathbb{R}} a(t) |w_n|^p dt + o(1) &= \int_{\mathbb{R}} \frac{W(t, u_n)}{\|u_n\|^p} dt \\ &= \left(\int_{\Theta} + \int_{\mathbb{R} \setminus \Theta} \right) \frac{W(t, u_n)}{|u_n|^p} |w_n|^p dt \\ &\rightarrow +\infty \quad \text{as } n \rightarrow \infty, \end{aligned}$$

a contradiction again. This completes the proof of the boundedness of (u_n) .

Passing to a subsequence, $u_n \rightharpoonup u$ weakly in E , $u_n \rightarrow u$ in L^2 and $u_n(t) \rightarrow u(t)$ for a.e. $t \in \mathbb{R}$. The boundedness of (u_n) implies that

$$\|u_n\|_{\infty}, \quad \|u\|_{\infty} \leq M, \quad \forall n \in \mathbb{N}$$

for some $M > 1$. Thus, using (3.3) and (W_3) ,

$$\int_{\mathbb{R}} |\nabla W(t, u_n) - \nabla W(t, u)|^2 dt \leq \int_{\mathbb{R}} c(|u_n| + |u|)^2 dt \leq 2c(\|u_n\|_2^2 + \|u\|_2^2), \quad (3.11)$$

where $c_2 := \varepsilon M^{p-2} + \delta^{-1} \max_{\delta \leq |x| \leq M} \bar{W}(x)$. It is easy to see that there holds

$$(|x|^{p-2}x - |y|^{p-2}y)(x - y) \geq c|x - y|^p, \quad \forall x, y \in \mathbb{R}^N, \quad (3.12)$$

and therefore by (3.11) and the fact $u_n \rightarrow u$ in L^2 we obtain

$$\begin{aligned} o(1) &= \langle I'(u_n) - I'(u), u_n - u \rangle \\ &= \int_{\mathbb{R}} |\dot{u}_n - \dot{u}|^2 dt + \int_{\mathbb{R}} a(t)[|u_n|^{p-2}u_n - |u|^{p-2}u](u_n - u) dt \\ &\quad - \int_{\mathbb{R}} [\nabla W(t, u_n) - \nabla W(t, u)](u_n - u) dt \\ &\geq \int_{\mathbb{R}} |\dot{u}_n - \dot{u}|^2 dt + c \int_{\mathbb{R}} a(t)|u_n - u|^p dt - \|\nabla W(t, u_n) - \nabla W(t, u)\|_2 \|u_n - u\|_2 \\ &\geq \int_{\mathbb{R}} |\dot{u}_n - \dot{u}|^2 dt + c \int_{\mathbb{R}} a(t)|u_n - u|^p dt + o(1), \end{aligned}$$

which yields that $u_n \rightarrow u$ in E . This completes the proof. \square

Lemma 3.2. *Let (V) , (W_2) – (W_3) and (W_5) – (W_6) be satisfied. Then I satisfies the (C) condition.*

Proof. Set (u_n) be a Cerami sequence. We verify that (u_n) is bounded. Assuming the contrary, $\|u_n\| \rightarrow \infty$, $w_n := u_n/\|u_n\| \rightharpoonup w$ in E and $w_n(t) \rightarrow w(t)$ for a.e. $t \in \mathbb{R}$ after passing to a subsequence. We claim that

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}} \frac{(\nabla W(t, u_n), u_n)}{\|u_n\|^p} dt < 1. \quad (3.13)$$

Indeed,

$$c \geq I(u_n) - \frac{1}{p} \langle I'(u_n), u_n \rangle = \left(\frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}} |\dot{u}_n|^2 dt + \int_{\mathbb{R}} \mathcal{W}(t, u_n) dt \geq \int_{\mathbb{R}} \mathcal{W}(t, u_n) dt, \quad \forall n.$$

Taking $\Omega_n(\alpha, \beta) := \{t \in \mathbb{R} : \alpha \leq |u_n(t)| < \beta\}$ for $0 \leq \alpha < \beta$, we obtain

$$\begin{aligned} c &\geq \int_{\mathbb{R}} \mathcal{W}(t, u_n) dt \\ &= \int_{\Omega_n(0, \alpha)} \mathcal{W}(t, u_n) dt + \int_{\Omega_n(\alpha, \beta)} \mathcal{W}(t, u_n) dt + \int_{\Omega_n(\beta, +\infty)} \mathcal{W}(t, u_n) dt \end{aligned} \quad (3.14)$$

for all n . By (W_2) , for any $\varepsilon > 0$ ($< 1/3$), there exists $a_\varepsilon > 0$ such that

$$|\nabla W(t, x)| \leq (\varepsilon/\tau_p^p)|x|^{p-1}, \quad \forall t \in \mathbb{R}, \quad |x| \leq a_\varepsilon,$$

which implies that

$$\int_{\Omega_n(0, a_\varepsilon)} \frac{|\nabla W(t, u_n)|}{|u_n|^{p-1}} |w_n|^p dt \leq \int_{\Omega_n(0, a_\varepsilon)} \frac{\varepsilon}{\tau_p^p} |w_n|^p dt \leq \frac{\varepsilon}{\tau_p^p} \|w_n\|_p^p < \varepsilon, \quad \forall n. \quad (3.15)$$

Since $\sigma > 0$, using (W_6) , (3.14) and (2.2), we can take $b_\varepsilon \geq L_1$ so large that

$$\begin{aligned} \int_{\Omega_n(b_\varepsilon, +\infty)} \frac{(\nabla W(t, u_n), u_n)}{\|u_n\|^p} dt &\leq \int_{\Omega_n(b_\varepsilon, +\infty)} \frac{a|w_n|^p \mathcal{W}(t, u_n)}{|u_n|^\sigma} dt \\ &\leq \frac{a\|w_n\|_\infty^p}{b_\varepsilon^\sigma} \int_{\Omega_n(b_\varepsilon, +\infty)} \mathcal{W}(t, u_n) dt \\ &\leq \frac{c}{b_\varepsilon^\sigma} \\ &< \varepsilon \end{aligned} \quad (3.16)$$

for all n . It follows from (W_5) that $\mathcal{W}(t, u_n) \geq C_{a_\varepsilon}^{b_\varepsilon} |u_n|^p$ for $t \in \Omega_n(a_\varepsilon, b_\varepsilon)$. Since $C_{a_\varepsilon}^{b_\varepsilon} > 0$, we have

$$\int_{\Omega_n(a_\varepsilon, b_\varepsilon)} |w_n|^p dt = \frac{1}{\|u_n\|^p} \int_{\Omega_n(a_\varepsilon, b_\varepsilon)} |u_n|^p dt \leq \frac{1}{C_{a_\varepsilon}^{b_\varepsilon} \|u_n\|^p} \int_{\Omega_n(a_\varepsilon, b_\varepsilon)} \mathcal{W}(t, u_n) dt \leq \frac{c}{C_{a_\varepsilon}^{b_\varepsilon} \|u_n\|^p} \xrightarrow{n} 0,$$

and then, using (W_3) ,

$$\int_{\Omega_n(a_\varepsilon, b_\varepsilon)} \frac{|\nabla W(t, u_n)|}{|u_n|^{p-1}} |w_n|^p dt \leq a_\varepsilon^{1-p} \max_{|x| \in [a_\varepsilon, b_\varepsilon]} \bar{W}(x) \int_{\Omega_n(a_\varepsilon, b_\varepsilon)} |w_n|^p dt \xrightarrow{n} 0. \quad (3.17)$$

Therefore, a combination of (3.15)–(3.17) shows that

$$\int_{\mathbb{R}} \frac{(\nabla W(t, u_n), u_n)}{\|u_n\|^p} dt \leq \int_{\mathbb{R}} \frac{|\nabla W(t, u_n)|}{|u_n|^{p-1}} |w_n|^p dt \leq 3\varepsilon < 1 \quad \text{for } n \text{ sufficiently large,}$$

and consequently (3.13) holds.

Now, noting

$$\langle I'(u_n), u_n \rangle / \|u_n\|^p = o(1),$$

it follows that

$$o(1) = \int_{\mathbb{R}} a(t) |w_n|^p dt - \int_{\mathbb{R}} \frac{(\nabla W(t, u_n), u_n)}{\|u_n\|^p} dt,$$

which, jointly with (3.13), shows that

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}} a(t) |w_n|^p dt < 1. \quad (3.18)$$

On the other hand, by (W_5) , we have

$$\begin{aligned} o(1) &= \frac{1}{\|u_n\|^2} \left[I'(u_n) - \frac{1}{p} \langle I'(u_n), u_n \rangle \right] \\ &= \left(\frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}} |\dot{w}_n|^2 dt + \frac{1}{\|u_n\|^2} \int_{\mathbb{R}} \mathcal{W}(t, u_n) dt \\ &\geq \left(\frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}} |\dot{w}_n|^2 dt, \end{aligned}$$

which yields that

$$\int_{\mathbb{R}} |\dot{w}_n|^2 dt = o(1).$$

This, jointly with (3.18), produces a contradiction since $\|w_n\| = 1$. Thus (u_n) is bounded in E , and hence contains a subsequence, relabeled (u_n) which converges to some $u \in E$ weakly in E and strongly in L^2 . Arguing as in the latter part of the proof of Lemma 3.1, we conclude that the (C) condition is satisfied. \square

Lemma 3.3. *Let (V) , (W_1) – (W_3) and (W_7) be satisfied. Then I satisfies the (C) condition.*

Proof. As in the proof of Lemma 3.1, it suffices to consider the case $w = 0$ and $w \neq 0$.

If $w = 0$, inspired by [12], we choose a sequence $(s_n) \subset \mathbb{R}$ such that

$$I(s_n u_n) = \max_{s \in [0,1]} I(s u_n).$$

For any $m \geq 1$ and $\bar{w}_n := \sqrt{m} w_n$, we have $\bar{w}_n \rightharpoonup 0$ in E and $\bar{w}_n \rightarrow 0$ in L^∞ . Combining this with (V) and (3.4), we have, for sufficiently large n ,

$$\int_{\mathbb{R}} \mathcal{W}(t, \bar{w}_n) dt \leq \varepsilon \int_{\mathbb{R}} |\bar{w}_n|^p dt \leq \frac{\varepsilon}{a_0} \int_{\mathbb{R}} a(t) |\bar{w}_n|^p dt,$$

and then, using Lemma 2.2 (ii),

$$\begin{aligned} I(s_n u_n) &\geq I(\bar{w}_n) \\ &\geq \int_{\mathbb{R}} \left[\frac{1}{2} |\dot{\bar{w}}_n|^2 + \left(\frac{1}{p} - \frac{\varepsilon}{a_0} \right) a(t) |\bar{w}_n|^p \right] dt \\ &\geq c \|\bar{w}_n\|^2 \\ &\geq cm \end{aligned}$$

which implies that

$$\lim_{n \rightarrow \infty} I(s_n u_n) = +\infty \tag{3.19}$$

by the arbitrariness of m . Observing $I(0) = 0$ and $\{I(u_n)\}$ is bounded, one sees that for n large enough, $s_n \in (0, 1)$ and

$$\langle I'(s_n u_n), s_n u_n \rangle = s_n \frac{d}{ds} \Big|_{s=s_n} I(s u_n) = 0.$$

Combining this with (W_7) , we obtain

$$\begin{aligned}
I(s_n u_n) &= I(s_n u_n) - \frac{1}{p} \langle I'(s_n u_n), s_n u_n \rangle \\
&= \left(\frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}} s_n^2 |\dot{u}_n|^2 dt + \int_{\mathbb{R}} \left[\frac{1}{p} (\nabla W(t, s_n u_n), s_n u_n) - W(t, s_n u_n) \right] dt \\
&\leq \left(\frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}} |\dot{u}_n|^2 dt + \theta \int_{\mathbb{R}} \mathcal{W}(t, u_n) dt \\
&\leq \theta \left[I(u_n) - \frac{1}{p} \langle I'(u_n), u_n \rangle \right] \\
&< +\infty,
\end{aligned}$$

a contradiction with (3.19).

If $w \neq 0$, the proof follows the same lines as that of Lemma 3.1, and therefore is omitted. \square

We shall apply the mountain pass theorem (see [21, Theorem 2.2]) and the symmetric mountain pass theorem (see [21, Theorem 9.12]) to prove our results. In the linking theorem, it is usually supposed that the functional Φ satisfies the stronger Palais–Smale condition. Nevertheless, the Cerami condition is sufficient for the deformation lemma (see [4]), and therefore for the linking theorem to hold.

Proposition 3.4. *Let E be a real Banach space and $\Phi \in C^1(E, \mathbb{R})$ satisfying the Cerami condition (C). Suppose that $\Phi(0) = 0$ and:*

- (i) *there exist $\rho, \alpha > 0$ such that $\Phi|_{\partial B_\rho(0)} \geq \alpha$;*
- (ii) *there is an $e \in E \setminus \overline{B_\rho(0)}$ such that $\Phi(e) \leq 0$.*

Then Φ possesses a critical value $c \geq \alpha$.

Proposition 3.5. *Let E be an infinite dimensional Banach space and $\Phi \in C^1(E, \mathbb{R})$ be even, satisfy the Cerami condition (C) and $\Phi(0) = 0$. If $E = Y \oplus Z$, where Y is finite-dimensional, and Φ satisfies:*

- (i) *there are constants $\rho, \alpha > 0$ such that $\Phi|_{\partial B_\rho \cap Z} \geq \alpha$;*
- (ii) *for each finite dimensional subspace $\tilde{E} \subset E$, there exists an $r = r(\tilde{E})$ such that $\Phi \leq 0$ on $\tilde{E} \setminus B_r(0)$.*

Then Φ possesses an unbounded sequence of critical values.

Lemma 3.6. *Let (V) and (W_2) be satisfied. Then there exist constants $\alpha, \rho > 0$ such that $I(u)|_{\|u\|=\rho} \geq \alpha$.*

Proof. It follows from (V), (3.4) and (2.2) that, for $u \in E$ with $\|u\| \leq \delta/\tau_\infty$,

$$\begin{aligned}
I(u) &\geq \int_{\mathbb{R}} \left(\frac{1}{2} |\dot{u}|^2 + \frac{1}{p} a(t) |u|^p \right) dt - \varepsilon \int_{\mathbb{R}} |u|^p dt \\
&\geq \frac{1}{2} \int_{\mathbb{R}} |\dot{u}|^2 dt + \left(\frac{1}{p} - \frac{\varepsilon}{a_0} \right) \int_{\mathbb{R}} a(t) |u|^p dt.
\end{aligned}$$

Thus the desired result follows when $\varepsilon > 0$ sufficiently small. \square

Lemma 3.7. *Let (V) and (W₂)–(W₄) be satisfied. Then, for any finite dimensional subspace $\tilde{E} \subset E$, there holds*

$$I(u) \rightarrow -\infty, \quad \|u\| \rightarrow \infty, \quad u \in \tilde{E}.$$

Proof. The equivalence of the norms on the finite dimensional space \tilde{E} implies there exists $C_0 = C_0(\tilde{E}) > 0$ such that

$$\|u\|_\mu \geq C_0 \|u\|, \quad \forall u \in \tilde{E}. \quad (3.20)$$

Combining (3.6) with (W₃) and (3.4), we obtain

$$W(t, x) \geq c_1 |x|^\mu - c_3 |x|^p, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N, \quad (3.21)$$

where $c_3 = c_1 L^{\mu-p} + \varepsilon + \delta^{-p} \max_{|x| \in [\delta, L]} \bar{W}(x)$. Consequently, using (3.21), (3.20) and (2.2), we obtain,

$$\begin{aligned} I(u) &\leq \frac{1}{2} \|u\|^2 + \frac{1}{p} \|u\|^p - \int_{\mathbb{R}} W(t, u) dt \\ &\leq \frac{1}{2} \|u\|^2 + \frac{1}{p} \|u\|^p - c_1 \|u\|_\mu^\mu + c_3 \|u\|_p^p \\ &\leq \frac{1}{2} \|u\|^2 + \left(\frac{1}{p} + c_3 \tau_p^p \right) \|u\|^p - c_1 C_0^\mu \|u\|^\mu \\ &\rightarrow -\infty \quad \text{as } \|u\| \rightarrow \infty. \end{aligned} \quad \square$$

Lemma 3.8. *Let (V) and (W₁) be satisfied and*

$$W(t, x) \geq 0, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N. \quad (3.22)$$

Then, for any finite dimensional subspace $\tilde{E} \subset E$, there holds

$$I(u) \rightarrow -\infty, \quad \|u\| \rightarrow \infty, \quad u \in \tilde{E}.$$

Proof. We claim that

$$\int_{\mathbb{R}} \frac{W(t, u)}{\|u\|^p} dt \rightarrow +\infty \quad \|u\| \rightarrow \infty, \quad u \in \tilde{E}. \quad (3.23)$$

If (3.23) is true, then there is $L_2 > 0$ such that

$$\int_{\mathbb{R}} W(t, u) dt \geq \|u\|^p, \quad \|u\| \geq L_2,$$

so that

$$I(u) \leq \frac{1}{2} \|u\|^2 + \frac{1}{p} \|u\|^p - \int_{\mathbb{R}} W(t, u) dt \leq \frac{1}{2} \|u\|^2 - \frac{p-1}{p} \|u\|^p \rightarrow -\infty \quad \text{as } \|u\| \rightarrow \infty.$$

Now we turn to showing that (3.23) holds. By contradiction, we assume that for some $(u_n) \subset \tilde{E}$ with $\|u_n\| \rightarrow \infty$, there is $c_4 > 0$ such that

$$\sup_n \int_{\mathbb{R}} \frac{W(t, u_n)}{\|u_n\|^p} dt \leq c_4. \quad (3.24)$$

Taking $v_n = u_n / \|u_n\|$, then $\|v_n\| = 1$. Noting $\dim \tilde{E} < +\infty$, there exists $v_0 \in \tilde{E} \setminus \{0\}$ such that

$$v_n \rightarrow v_0 \quad \text{in } \tilde{E} \quad \text{and} \quad v_n(t) \rightarrow v_0(t) \quad \text{a.e. } t \in \mathbb{R},$$

after passing to a subsequence. Let

$$\Omega_1 = \{t \in \mathbb{R} : W(t, x) / |x|^p \rightarrow +\infty \text{ as } |x| \rightarrow \infty\}$$

and $\Omega_2 = \{t \in \mathbb{R} : v_0(t) \neq 0\}$. We have $\Omega_1 \cap \Omega_2 \neq \emptyset$, and for $t \in \Omega_1 \cap \Omega_2$,

$$\frac{W(t, u_n(t))}{|u_n(t)|^p} |v_n(t)|^p \rightarrow +\infty \quad \text{as } n \rightarrow \infty.$$

Consequently, using (3.22) and Fatou's lemma (see [34]),

$$\int_{\mathbb{R}} \frac{W(t, u_n)}{\|u_n\|^p} dt = \left(\int_{\Omega_1 \cap \Omega_2} + \int_{\mathbb{R} \setminus (\Omega_1 \cap \Omega_2)} \right) \frac{W(t, u_n)}{\|u_n\|^p} dt \geq \int_{\Omega_1 \cap \Omega_2} \frac{W(t, u_n)}{|u_n|^p} |v_n|^p dt \xrightarrow{n} +\infty,$$

a contradiction with (3.24). \square

Particularly, we have the following results.

Lemma 3.9. *Let (V) and (W₂)–(W₄) be satisfied. Then there exists $e \in E$ with $\|e\| > \rho$ such that $I(e) < 0$.*

Lemma 3.10. *Let (V), (W₁) and (3.22) be satisfied. Then there exists $e \in E$ with $\|e\| > \rho$ such that $I(e) < 0$.*

Proof of Theorem 1.5. (Existence) Lemma 3.1 shows that I satisfies the Cerami condition. It follows from Lemmas 3.6 and 3.9 that the mountain pass geometry is satisfied. Consequently, in virtue of Proposition 3.4, I admits at least one nontrivial critical point.

(Multiplicity) Suppose that W is even in x , then I is even, $I(0) = 0$, and satisfies the conditions of Proposition 3.5 by Lemmas 3.1, 3.6 and 3.7. Therefore, I has an unbounded sequence of critical values $c_k = I(u_k)$. Obviously,

$$\int_{\mathbb{R}} (|\dot{u}_k|^2 + a(t)|u_k|^p) dt = \int_{\mathbb{R}} (\nabla W(t, u_k), u_k) dt, \quad \forall k \in \mathbb{N}. \quad (3.25)$$

Hence we obtain, by (W₃), (3.4) and (2.1),

$$\left| \int_{|u_k| \leq L} W(t, u_k) dt \right| \leq c \int_{\mathbb{R}} |u_k|^2 dt \leq c \left(\int_{\mathbb{R}} a(t) |u_k|^p dt \right)^{2/p},$$

and then, using (3.25) and (3.7),

$$\begin{aligned} c_k &= I(u_k) \\ &= \frac{1}{2} \int_{\mathbb{R}} |\dot{u}_k|^2 dt + \frac{1}{p} \int_{\mathbb{R}} a(t) |u_k|^p dt - \int_{|u_k| \geq L} W(t, u_k) dt - \int_{|u_k| \leq L} W(t, u_k) dt \\ &\leq \frac{1}{2} \int_{\mathbb{R}} |\dot{u}_k|^2 dt + \frac{1}{p} \int_{\mathbb{R}} a(t) |u_k|^p dt + c \left(\int_{\mathbb{R}} a(t) |u_k|^p dt \right)^{2/p}. \end{aligned}$$

Since $c_k \rightarrow +\infty$ as $k \rightarrow \infty$, it follows that (u_k) is unbounded in E . \square

Proof of Theorem 1.7. Since (W₅), jointly with (W₂), implies (3.22) (see Remark 1.12), the proof follows the same lines as that of Theorem 1.5 with Lemmas 3.1, 3.7 and 3.9 replaced by Lemmas 3.2, 3.8 and 3.10, respectively. \square

Proof of Theorem 1.10. Because (W₇), together with (W₂), yields (3.22) (see Remark 1.12), the proof follows the same lines as that of Theorem 1.5 with Lemmas 3.1, 3.7 and 3.9 replaced by Lemmas 3.3, 3.8 and 3.10, respectively. \square

4 The subquadratic case

Inspired by [28], we shall extend W to an appropriate $\tilde{W} \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and introduce the following Hamiltonian systems

$$ii - a(t)|u|^{p-2}u + \nabla \tilde{W}(t, u) = 0, \quad \forall t \in \mathbb{R}. \quad (4.1)$$

Then, applying variational methods, we show that system (4.1) possesses a sequence of homoclinic solutions, which converges to zero in L^∞ norm, and consequently, obtain infinitely many solutions for the original problem (HS).

Let $\chi \in C^\infty(\mathbb{R}, [0, 1])$ be a function satisfying

$$\chi(s) = \begin{cases} 0 & \text{if } s \leq \delta/4, \\ 1 & \text{if } s \geq \delta/2, \end{cases} \quad (4.2)$$

and $0 < \chi'(s) \leq 8/\delta$ for $t \in (\delta/4, \delta/2)$. We define a function $\tilde{W} : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ by:

$$\tilde{W}(t, x) = (1 - \chi(|x|))W(t, x) + \chi(|x|)|x|^2.$$

Then the following lemma holds.

Lemma 4.1. *Assume that (W_8) – (W_{10}) hold. Then \tilde{W} possesses the following properties:*

(C₁) $\tilde{W} \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $\tilde{W}(t, -x) = \tilde{W}(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$.

(C₂) $\tilde{W}(t, 0) \equiv 0$, and there exist constants $a_1 > 0$, $\gamma \in (1, 2)$ and a function $b_1 \in L^{\frac{2}{2-\gamma}}(\mathbb{R}, \mathbb{R}^+)$ such that

$$|\nabla \tilde{W}(t, x)| \leq a_1|x| + b_1(t)|x|^{\gamma-1}, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

(C₃) There exist $t_0 \in \mathbb{R}$, two sequences $\{\delta_n\}$, $\{M_n\}$ and constants $a_2, \delta, d > 0$ such that $\delta_n > 0$, $M_n > 0$ and

$$\begin{aligned} \lim_{n \rightarrow \infty} \delta_n &= 0, & \lim_{n \rightarrow \infty} M_n &= +\infty, \\ |x|^{-2}\tilde{W}(t, x) &\geq M_n \quad \text{for } |t - t_0| \leq d \text{ and } |x| = \delta_n, \\ |x|^{-2}\tilde{W}(t, x) &\geq -a_2 \quad \text{for } |t - t_0| \leq d \text{ and } |x| \leq \delta. \end{aligned}$$

Proof. By definition, it is clear that

$$\tilde{W}(t, x) = W(t, x), \quad \forall t \in \mathbb{R}, \quad |x| \leq \delta/4,$$

and

$$\tilde{W}(t, x) = |x|^2, \quad \forall t \in \mathbb{R}, \quad |x| \geq \delta/2.$$

Then $\tilde{W}(t, 0) = W(t, 0) \equiv 0$ and \tilde{W} satisfies (C₃) by (W₁₀). From (W₉), we get

$$|W(t, x)| \leq \frac{a_1}{2}|x|^2 + \frac{1}{\gamma}b_1(t)|x|^\gamma, \quad \forall t \in \mathbb{R}, \quad |x| \leq \delta. \quad (4.3)$$

Note that

$$\nabla \tilde{W}(t, x) = (1 - \chi(|x|))\nabla W(t, x) + \chi'(|x|)(|x| - W(t, x)/|x|)x + 2\chi(|x|)x,$$

which, together with (4.3), (4.2) and (W₉), implies that

$$|\nabla \tilde{W}(t, x)| \leq c(a_1|x| + b_1(t)|x|^{\gamma-1}), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

i.e., (C₂) holds. □

Now we define the variational functional Φ associated to system (4.1) by:

$$\Phi(u) = \int_{\mathbb{R}} \left(\frac{1}{2} |\dot{u}|^2 + \frac{1}{p} a(t) |u|^p \right) dt - \psi(u), \quad \text{where } \psi(u) = \int_{\mathbb{R}} \tilde{W}(t, u) dt.$$

It follows from (C₂) that

$$\tilde{W}(t, x) \leq \frac{a_1}{2} |x|^2 + \frac{1}{\gamma} b_1(t) |x|^\gamma, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N, \quad (4.4)$$

and then, using the Hölder inequality,

$$\int_{\mathbb{R}} \tilde{W}(t, u) dt \leq c \int_{\mathbb{R}} (|u|^2 + b_1(t) |u|^\gamma) dt \leq c \left(\|u\|_2^2 + \|b_1\|_{\frac{2}{2-\gamma}} \|u\|_2^2 \right) < +\infty.$$

Thus Φ is well defined. In addition, we have the following lemma.

Lemma 4.2. *Let (V) and (C₂) be satisfied. Then $\Phi \in C^1(E, \mathbb{R})$ and*

$$\langle \Phi'(u), v \rangle = \int_{\mathbb{R}} [(\dot{u}, \dot{v}) + a(t) |u|^{p-2} (u, v) - (\nabla \tilde{W}(t, u), v)] dt$$

for all $u, v \in E$. The critical point u of Φ is a homoclinic orbit of problem (4.1) with $u(\pm\infty) = 0$.

Proof. For $\Phi \in C^1(E, \mathbb{R})$, it suffices to show it for the functional $\psi(u) = \int_{\mathbb{R}} \tilde{W}(t, u) dt$. It follows from (C₂) and the Young inequality that, for $u, v \in E$ and $s \in [0, 1]$,

$$\begin{aligned} |(\nabla \tilde{W}(t, u + sv), v)| &\leq a_1 |u + sv| |v| + b_1(t) |u + sv|^{\gamma-1} |v| \\ &\leq c \left(|u| |v| + |v|^2 + b_1(t) |u|^{\gamma-1} |v| + b_1(t) |v|^\gamma \right) \\ &\leq c \left[|u|^2 + |v|^2 + b_1(t)^{2/(2-\gamma)} + \left(|u|^{\gamma-1} |v| \right)^{2/\gamma} + |v|^2 \right] \\ &\leq c \left(|u|^2 + |v|^2 + b_1(t)^{2/(2-\gamma)} \right) \in L^1, \end{aligned}$$

which implies that

$$\begin{aligned} \langle \psi'(u), v \rangle &= \lim_{s \rightarrow 0} \frac{\psi(u + sv) - \psi(u)}{s} \\ &= \lim_{s \rightarrow 0} \int_{\mathbb{R}} \frac{\tilde{W}(t, u + sv) - \tilde{W}(t, u)}{s} dt \\ &= \lim_{s \rightarrow 0} \int_{\mathbb{R}} (\nabla \tilde{W}(t, u + \theta sv), v) dt \\ &= \int_{\mathbb{R}} (\nabla \tilde{W}(t, u), v) dt \end{aligned}$$

by the mean value theorem and Lebesgue dominated convergence theorem.

To show the continuity of $\psi'(u)$ in u , we suppose that $u_n, u \in E$ and $u_n \xrightarrow{n} u$ in E . Lemma 2.1 implies that $u_n \rightarrow u$ in L^2 . According to [29, Lemma A.1], there exists a subsequence, still denote by (u_n) , and $g \in L^2$ such that $u_n(t) \rightarrow u(t)$ for a.e. $t \in \mathbb{R}$ and

$$|u_n|, |u| \leq g(t), \quad \forall n \in \mathbb{N}.$$

Combining this with (C_2) and the Young inequality, we obtain

$$\begin{aligned} |\nabla \tilde{W}(t, u_n) - \nabla \tilde{W}(t, u)|^2 &\leq c[(|u_n| + |u|) + b_1(t)(|u_n|^{\gamma-1} + |u|^{\gamma-1})]^2 \\ &\leq c[(|u_n|^2 + |u|^2) + b_1^2(t)(|u_n|^{2(\gamma-1)} + |u|^{2(\gamma-1)})] \\ &\leq c \left[(|u_n|^2 + |u|^2) + (2 - \gamma)b_1(t)^{2/(2-\gamma)} + (\gamma - 1)(|u_n|^2 + |u|^2) \right] \\ &\leq c \left(g^2(t) + b_1(t)^{2/(2-\gamma)} \right) \in L^1, \end{aligned}$$

which yields that

$$\int_{\mathbb{R}} |\nabla \tilde{W}(t, u_n) - \nabla \tilde{W}(t, u)|^2 dt \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

by the dominated convergence theorem. Therefore,

$$\begin{aligned} \|\psi'(u_n) - \psi'(u)\|_{E^*} &= \sup_{\|v\|=1} \left| \int_{\mathbb{R}} (\nabla \tilde{W}(t, u_n) - \nabla \tilde{W}(t, u), v) dt \right| \\ &\leq \sup_{\|v\|=1} \left(\int_{\mathbb{R}} |\nabla \tilde{W}(t, u_n) - \nabla \tilde{W}(t, u)|^2 dt \right)^{1/2} \|v\|_2 \\ &= o(1). \end{aligned}$$

This completes the proof. \square

We shall make use of the new version of symmetric mountain pass lemma of Kajikiya (see [13]) to prove Theorem 1.13. Let E be a Banach space and

$$\Gamma := \{A \subset E \setminus \{0\} : A \text{ is closed and symmetric with respect to the origin}\}.$$

For $A \in \Gamma$, the genus $\gamma(A)$ of A is defined as being the least positive integer k such that there is an odd mapping $h \in C(A, \mathbb{R}^k) \setminus \{0\}$. If k does not exist, we set $\gamma(A) = +\infty$. Furthermore, by definition, $\gamma(\emptyset) = 0$.

In the sequel, we only recall the properties of the genus that will be need throughout the paper. See [21] for more information on this subject.

Proposition 4.3. *Let $A, B \in \Gamma$, then (i)–(iii) below hold.*

- (i) *There is an odd continuous mapping from A to B , then $\gamma(A) \leq \gamma(B)$.*
- (ii) *If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*
- (iii) *The n -dimensional sphere S^n has a genus of $n + 1$ by the Borsuk–Ulam theorem.*

Proposition 4.4. *Let E be an infinite dimensional Banach space and $\Phi \in C^1(E, \mathbb{R})$ be even, $\Phi(0) = 0$ and satisfies the following conditions:*

- (i) *Φ is bounded from below and satisfies the Palais–Smale condition (PS), i.e., $(u_n) \subset E$ has a convergent subsequence whenever $\{\Phi(u_n)\}$ is bounded and $\Phi'(u_n) \rightarrow 0$ as $n \rightarrow \infty$.*
- (ii) *For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma$ such that $\gamma(A_k) = k$ and $\sup_{u \in A_k} \Phi(u) < 0$.*

Then either (1) or (2) holds.

- (1) *There exists a sequence $\{u_k\}$ such that $\Phi'(u_k) = 0$, $\Phi(u_k) < 0$ and $\{u_k\}$ converges to zero.*

(2) There exist two sequences $\{u_k\}$ and $\{v_k\}$ such that $\Phi'(u_k) = 0$, $\Phi(u_k) = 0$, $u_k \neq 0$, $\lim_{k \rightarrow \infty} u_k = 0$, $\Phi'(v_k) = 0$, $\Phi(v_k) < 0$, $\lim_{k \rightarrow \infty} \Phi(v_k) = 0$ and $\{v_k\}$ converges to a non-zero limit.

Remark 4.5. From Proposition 4.4, we deduce a sequence of critical points $\{u_k\}$ such that $\Phi(u_k) \leq 0$, $u_k \neq 0$ and $\lim_{k \rightarrow \infty} u_k = 0$.

Proof of Theorem 1.13. According to Lemma 4.2 and the evenness of $\tilde{W}(t, \cdot)$, we know that $\Phi \in C^1(E, \mathbb{R})$ and $\Phi(-u) = \Phi(u)$. It remains to verify conditions (i) and (ii) of Proposition 4.4.

Verification of (i). By (4.4), (2.1) and Hölder's inequality, we obtain

$$\begin{aligned} \Phi(u) &\geq \int_{\mathbb{R}} \left(\frac{1}{2} |\dot{u}|^2 + \frac{1}{p} a(t) |u|^p \right) dt - c \int_{\mathbb{R}} (|u|^2 + b_1(t) |u|^\gamma) dt \\ &\geq c \|u\|_2^p - c \left(\|u\|_2^2 + \|b_1\|_{\frac{2}{2-\gamma}} \|u\|_2^\gamma \right) \end{aligned} \quad (4.5)$$

for all $u \in E$, which implies that Φ is bounded from below.

Let $(u_n) \subset E$ be a (PS)-sequence of Φ , i.e., $\{\Phi(u_n)\}$ is bounded and $\Phi'(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Since (4.5) implies that

$$c \geq \Phi(u_n) \geq c \|u_n\|_2^p - c \left(\|u_n\|_2^2 + \|b_1\|_{\frac{2}{2-\gamma}} \|u_n\|_2^\gamma \right),$$

it follows that $\{\|u_n\|_2\}$ and $\{\frac{1}{2} \|\dot{u}_n\|_2^2 + \frac{1}{p} \|u_n\|_{a,p}^p\}$ are bounded. Thus (u_n) is bounded in E .

Up to a subsequence, we assume that $u_n \rightharpoonup u$ in E , $u_n \rightarrow u$ in L^2 and $u_n(t) \rightarrow u(t)$ a.e. $t \in \mathbb{R}$. Hence we obtain, by (C_2) and the Hölder inequality,

$$\begin{aligned} &\int_{\mathbb{R}} (\nabla \tilde{W}(t, u_n) - \nabla \tilde{W}(t, u), u_n - u) dt \\ &\leq c \int_{\mathbb{R}} \left[(|u_n| + |u|) |u_n - u| + b_1(t) (|u_n|^{\gamma-1} + |u|^{\gamma-1}) |u_n - u| \right] dt \\ &\leq c (\|u_n\|_2 + \|u\|_2) \|u_n - u\|_2 + c \left(\int_{\mathbb{R}} |b_1(t)|^{\frac{2}{2-\gamma}} dt \right)^{\frac{2-\gamma}{2}} \left(\int_{\mathbb{R}} |u_n|^{\frac{2(\gamma-1)}{\gamma}} |u_n - u|^{\frac{2}{\gamma}} dt \right)^{\frac{\gamma}{2}} \\ &\quad + c \left(\int_{\mathbb{R}} |b_1(t)|^{\frac{2}{2-\gamma}} dt \right)^{\frac{2-\gamma}{2}} \left(\int_{\mathbb{R}} |u|^{\frac{2(\gamma-1)}{\gamma}} |u_n - u|^{\frac{2}{\gamma}} dt \right)^{\frac{\gamma}{2}} \\ &\leq o(1) + c \|b_1\|_{\frac{2}{2-\gamma}} (\|u_n\|_2^{\gamma-1} + \|u\|_2^{\gamma-1}) \|u_n - u\|_2 \\ &= o(1). \end{aligned} \quad (4.6)$$

Observe that

$$\begin{aligned} o(1) &= \langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle \\ &= \|\dot{u}_n - \dot{u}\|_2^2 + \int_{\mathbb{R}} a(t) [|u_n|^{p-2} u_n - |u|^{p-2} u] (u_n - u) dt \\ &\quad - \int_{\mathbb{R}} (\nabla \tilde{W}(t, u_n) - \nabla \tilde{W}(t, u), u_n - u) dt. \end{aligned}$$

Consequently, using (4.6) and (3.12), we conclude that $u_n \rightarrow u$ in E .

Verification of (ii). We prove that for arbitrary $k \in \mathbb{N}$ there is an $A_k \in \Gamma$ such that $\gamma(A_k) = k$ and $\sup_{u \in A_k} \Phi(u) < 0$. We adapt an argument in [13]. For simplicity, we assume that $t_0 = 0$ in (C_3) . Divide $[-d, d]$ equally into k closed subintervals and denote them by F_i with $1 \leq i \leq k$.

Setting $d_1 = 2d/k$, then $|F_i| = d_1$, where $|F_i|$ is the Lebesgue measure of the set F_i . For $1 \leq i \leq k$, set t_i be the center of F_i and J_i be the closed interval centered at t_i with $|J_i| = d_1/2$. Choose a function $\eta \in C_0^\infty(\mathbb{R}, \mathbb{R}^N)$ such that $|\eta(t)| = 1$ for $t \in [-d_1/4, d_1/4]$, $\eta(t) = 0$ for $t \in \mathbb{R} \setminus [-d_1/2, d_1/2]$ and $|\eta(t)| \leq 1$ for $t \in \mathbb{R}$. Define

$$\eta_i(t) = \eta(t - t_i), \quad t \in \mathbb{R}, \quad 1 \leq i \leq k.$$

It follows that

$$|\eta_i(t)| \leq 1 \quad (t \in \mathbb{R}), \quad |\eta_i(t)| = 1 \quad (t \in J_i),$$

and

$$\text{supp } \eta_i \subset F_i, \quad \text{supp } \eta_i \cap \text{supp } \eta_j = \emptyset \quad (i \neq j). \quad (4.7)$$

Let

$$V_k = \left\{ (s_1, s_2, \dots, s_k) \in \mathbb{R}^k : \max_{1 \leq i \leq k} |s_i| = 1 \right\} \quad (4.8)$$

and

$$W_k = \left\{ \sum_{i=1}^k s_i \eta_i(t) : (s_1, s_2, \dots, s_k) \in V_k \right\}.$$

Noticing V_k is homeomorphic to the unit sphere in \mathbb{R}^k by an odd mapping, one has $\gamma(V_k) = k$. Furthermore, $\gamma(W_k) = \gamma(V_k) = k$ because the mapping $(s_1, s_2, \dots, s_k) \mapsto \sum_{i=1}^k s_i \eta_i(t)$ is odd and homeomorphic. Since W_k is compact, there exists $C_k > 0$ such that

$$\|u\| \leq C_k, \quad \forall u \in W_k. \quad (4.9)$$

For $u = \sum_{i=1}^k s_i \eta_i(t) \in W_k$ and the sequence $\{\delta_n\}$ given in (C₃), by (4.8) and (4.7), we obtain

$$\begin{aligned} \int_{\mathbb{R}} \tilde{W} \left(t, \delta_n \sum_{i=1}^k s_i \eta_i(t) \right) dt &= \sum_{i=1}^k \int_{F_i} \tilde{W}(t, \delta_n s_i \eta_i(t)) dt \\ &= \int_{J_{i_0}} \tilde{W}(t, \delta_n s_{i_0} \eta_{i_0}(t)) dt + \int_{F_{i_0} \setminus J_{i_0}} \tilde{W}(t, \delta_n s_{i_0} \eta_{i_0}(t)) dt \\ &\quad + \sum_{i \neq i_0} \int_{F_i} \tilde{W}(t, \delta_n s_i \eta_i(t)) dt, \end{aligned} \quad (4.10)$$

where $i_0 \in [1, k]$ satisfying $|s_{i_0}| = 1$. It follows from (C₃) and the fact $|\delta_n s_{i_0} \eta_{i_0}(t)| = \delta_n$ for $t \in J_{i_0}$ that

$$\begin{aligned} \int_{F_{i_0} \setminus J_{i_0}} \tilde{W}(t, \delta_n s_{i_0} \eta_{i_0}(t)) dt + \sum_{i \neq i_0} \int_{F_i} \tilde{W}(t, \delta_n s_i \eta_i(t)) dt &\geq -a_2 \delta_n^2 \left| \bigcup_{i=1}^k F_i \right| \\ &= -a_2 \delta_n^2 (2d), \end{aligned} \quad (4.11)$$

and

$$\int_{J_{i_0}} \tilde{W}(t, \delta_n s_{i_0} \eta_{i_0}(t)) dt \geq M_n \delta_n^2 |J_{i_0}| = \frac{d_1}{2} M_n \delta_n^2. \quad (4.12)$$

Hence, using (4.9)–(4.12), we obtain

$$\begin{aligned}\Phi(\delta_n u) &\leq \frac{1}{2} \|\delta_n u\|^2 + \frac{1}{p} \|\delta_n u\|^p - \int_{\mathbb{R}} \tilde{W} \left(t, \delta_n \sum_{i=1}^k s_i \eta_i(t) \right) dt \\ &\leq \delta_n^2 \left(\frac{C_k^2}{2} + \delta_n^{p-2} \frac{C_k^p}{p} + 2da_2 - \frac{d_1}{2} M_n \right).\end{aligned}$$

As $\delta_n \rightarrow 0^+$ and $M_n \rightarrow +\infty$ ($n \rightarrow \infty$), we choose $n_0 > 0$ large enough such that the right side of the last inequality is negative. Now, letting

$$A_k = \delta_{n_0} W_k,$$

We have

$$\gamma(A_k) = \gamma(W_k) = k \quad \text{and} \quad \sup_{u \in A_k} \Phi(u) < 0.$$

Consequently, by Proposition 4.4, Φ has infinitely many nontrivial solutions (u_k) such that $u_k \rightarrow 0$ in E as $k \rightarrow \infty$. By (2.2), $u_k \rightarrow 0$ in L^∞ . Hence, for k large, they are homoclinic solutions of (HS). This completes the proof. \square

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