Electronic Journal of Qualitative Theory of Differential Equations
2019, No. **20**, 1–14; https://doi.org/10.14232/ejqtde.2019.1.20 www.math.u-szeged.hu/ejqtde/

New multiple positive solutions for elliptic equations with singularity and critical growth

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Received 2 January 2019, appeared 15 March 2019
Communicated by Dimitri Mugnai

Abstract. In this note, the existence of multiple positive solutions is established for a semilinear elliptic equation $-\Delta u = \frac{\lambda}{u^\gamma} + u^{2^*-1}$, $x \in \Omega$, $u = 0, x \in \partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, $2^* = \frac{2N}{N-2}$, $\gamma \in (0,1)$ and $\lambda > 0$ is a real parameter. We show by the variational methods and perturbation functional that the problem has at least two positive solutions $w_0(x)$ and $w_1(x)$ with $w_0(x) < w_1(x)$ in Ω.

Keywords: semilinear elliptic equations, critical growth, singularity, positive solution.

2010 Mathematics Subject Classification: 35A15, 35B09, 35B33, 35J75.

1 Introduction

The singular bounded value problem of the type

$$\begin{cases} -\Delta u = \lambda f(x) u^{-\gamma} + \mu g(x) u^{p-1}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 3)$, $\gamma \in (0,1)$ and f, g satisfying some certain conditions, was extensively investigated. Such problem describes naturally several physical phenomena, therefore, only the positive solutions are relevant in most cases.

Singular elliptic problems have been intensively studied in the last decades. For example, in the case when $\mu = 0$, the existence or uniqueness of positive solutions to problem (1.1) has been studied extensively (see [6, 7, 12, 13, 18, 24] and the references therein).

For the case of $\mu > 0$. When $1 < p < 2^*$, Sun, Wu and Long [21] established two positive solutions to problem (1.1) by using the Nehari manifold provided $\lambda > 0$ is enough small. For singular elliptic problems with subcritical growth, please see [2–5, 8, 9, 19] and the references therein. For the case of critical growth, there are many interesting results, see [10, 11, 15, 20, 22, 23]. In particular, Yang [23] considered the problem

$$\begin{cases} -\Delta u = \lambda u^{-\gamma} + u^{2^*-1}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

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The author firstly proved that problem (1.2) has a positive local minimizer solution u_λ for $\lambda > 0$ enough small. After that, with the helps of the sub-supersolutions and variational arguments, and a second positive solution v_λ was obtained with $u_\lambda < v_\lambda$ in Ω. In additional, in problem (1.2), if u is replaced by $\lambda^{\frac{1}{1+\gamma}} v$, problem (1.2) reduces to

$$\begin{cases} -\Delta v = v^{-\gamma} + \mu v^{2^*-1}, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases}$$

here $\mu = \lambda^{\frac{2^*-2}{1+\gamma}}$. By using the Nehari manifold, Sun and Wu [20] proved that there was an exact μ_* such that the problem has two positive solutions for all $\mu \in (0, \mu_*)$ and no solution for $\mu > \mu_*$. In the case when $0 < \gamma \leq 1$, by using the variational method, Hirano, Saccon and Shioji showed the existence of two positive solutions for problem (1.2) with $\lambda > 0$ small enough, see [10].

Thus, observing the all above studies, it is natural to ask whether problem (1.2) has multiple positive solutions by other methods? We shall give a positive answer to this question, the main technical approaches are based on the variational and perturbation functional. Now, the main result can be stated as follows.

Theorem 1.1. *Assume that $\gamma \in (0, 1)$. Then there exists $\lambda_* > 0$, such that for any $\lambda \in (0, \lambda_*)$, problem (1.2) has at least two positive solutions $w_0(x)$ and $w_1(x)$ with $w_0(x) < w_1(x)$ in Ω.*

Remark 1.2. Compared with [23], with the help of a perturbation functional, we give a simple and direct method to obtain the size relation of the two positive solutions.

Throughout this paper, we make use of the following notations:

- the space $H_0^1(\Omega)$ is equipped with the norm $\|u\|^2 = \int_\Omega |\nabla u|^2 dx$, which is equivalent to the usual norm. The norm in $L^p(\Omega)$ is denoted by $|u|_p^p = \int_\Omega |u|^p dx$;
- $C, C_1, C_2, \ldots$ denote various positive constants, which may vary from line to line;
- we denote by B_r (respectively, ∂B_r) the closed ball (respectively, the sphere) of center zero and radius r, i.e., $B_r = \{u \in H_0^1(\Omega) : \|u\| \leq r\}$, $\partial B_r = \{u \in H_0^1(\Omega) : \|u\| = r\}$;
- $u = u^+ + u^-$, $u^\pm = \pm \max\{\pm u, 0\}$;
- let S be the best Sobolev constant, i.e.,

$$S := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{(\int_\Omega |u|^{2^*})^{\frac{2}{2^*}}}.$$

2 Existence of the first positive solution of problem (1.2)

We define the energy functional of problem (1.2) by

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{1-\gamma}\int_\Omega (u^+)^{1-\gamma} dx - \frac{1}{2^*}\int_\Omega (u^+)^{2^*} dx, \qquad \forall u \in H_0^1(\Omega).$$

In general, a function u is called a positive solution of problem (1.2) if $u \in H_0^1(\Omega)$ and for all $v \in H_0^1(\Omega)$ it holds

$$\int_\Omega (\nabla u, \nabla v) dx - \lambda \int_\Omega u^{-\gamma} v dx - \int_\Omega u^{2^*-1} v dx = 0.$$

From [23] and [10], we obtain the following result.

Theorem 2.1. *For $0 < \gamma < 1$, there exists $\Lambda_0 > 0$ such that problem (1.2) has a positive solution $w_0 \in L^\infty(\Omega) \cap C^\infty(\Omega)$ with $I(w_0) < 0$ when $\lambda \in (0, \Lambda_0)$.*

3 Existence of a second positive solution of problem (1.2)

Up to now, we get that problem (1.2) has a positive solution w_0. Next we will prove that there is another positive solution for problem (1.2) by a translation argument. For $\alpha > 0$, we define a C^1 functional $J_\alpha : H_0^1(\Omega) \to \mathbb{R}$ by

$$\begin{aligned} J_\alpha(v) = &\frac{1}{2}\|v\|^2 - \frac{1}{2^*}\int_\Omega [(v^+ + w_0)^{2^*} - w_0^{2^*} - 2^* w_0^{2^*-1} v^+] dx \\ &- \frac{\lambda}{1-\gamma}\int_\Omega \left[(v^+ + w_0 + \alpha)^{1-\gamma} - (w_0+\alpha)^{1-\gamma} - (1-\gamma)\frac{v^+}{w_0^\gamma}\right] dx, \end{aligned}$$

for $v \in H_0^1(\Omega)$. Now, we show that the functional J_α satisfies the mountain-pass lemma.

Lemma 3.1. *There exist $r, \rho > 0$ such that J_α satisfies the following conditions for any $\lambda > 0$,*

(i) *$J_\alpha(u) > \rho$ for any $u \in \partial B_r$;*

(ii) *there exists $\zeta \in H_0^1(\Omega)$ with $\|\zeta\| > r$ such that $J_\alpha(\zeta) < 0$.*

Proof. (i) For $u \in H_0^1(\Omega)$ with $u^+ \neq 0$, by the mean value theorem and the Lebesgue dominated convergence theorem, one has

$$\begin{aligned} \lim_{t\to 0^+} \frac{J_\alpha(tu)}{t} &= -\frac{1}{2^*}\lim_{t\to 0^+}\int_\Omega \frac{(tu^+ + w_0)^{2^*} - w_0^{2^*} - 2^* w_0^{2^*-1} tu^+}{t} dx \\ &\quad - \lambda \lim_{t\to 0^+}\int_\Omega \frac{(tu^+ + w_0 + \alpha)^{1-\gamma} - (w_0+\alpha)^{1-\gamma} - (1-\gamma) w_0^{-\gamma} tu^+}{(1-\gamma)t} dx \\ &= -\lim_{t\to 0^+}\int_\Omega [(\eta t u^+ + w_0)^{2^*-1} - w_0^{2^*-1}] u^+ dx \\ &\quad - \lambda \lim_{t\to 0^+}\int_\Omega [(\xi t u^+ + w_0 + \alpha)^{-\gamma} - w_0^{-\gamma}] u^+ dx \\ &= \lambda \int_\Omega \left[\frac{u^+}{w_0^\gamma} - \frac{u^+}{(w_0+\alpha)^\gamma}\right] dx \\ &> 0, \end{aligned}$$

which implies that there exist $\rho, r > 0$ such that $J_\alpha|_{\|u\|=r} \geq \rho > 0$ for each $\lambda > 0$.

(ii) For $a, b \geq 0$, there holds

$$(a+b)^{2^*} \geq a^{2^*} + b^{2^*} - 2^* a^{2^*-1} b.$$

Therefore, for $u \in H_0^1(\Omega), u^+ \neq 0$ and $t > 0$, one has

$$\begin{aligned} J_\alpha(tu) &\leq \frac{t^2}{2}\|u\|^2 - \frac{t^{2^*}}{2^*}\int_\Omega (u^+)^{2^*} dx + t\lambda \int_\Omega \frac{u^+}{w_0^\gamma} dx \\ &\to -\infty \end{aligned}$$

as $t \to +\infty$. Therefore we can easily find $\zeta \in H_0^1(\Omega)$ with $\|\zeta\| > r$, such that $J_\alpha(\zeta) < 0$. The proof is complete. □

Lemma 3.2. *The functional J_α satisfies the $(PS)_c$ condition with $c < \frac{1}{N}S^{\frac{N}{2}} - D\lambda$, where $D = D(|\Omega|, N, S, \gamma, |w_0|_\infty)$.*

Proof. Let $\{v_n\} \subset H_0^1(\Omega)$ be a $(PS)_c$ sequence for J_α, namely,

$$J_\alpha(v_n) \to c, \qquad J'_\alpha(v_n) \to 0, \quad \text{as } n \to \infty. \tag{3.1}$$

Set

$$\begin{aligned} H_1(v_n) = & \int_\Omega [(v_n^+ + w_0)^{2^*-1}(v_n + w_0) - w_0^{2^*-1}(v_n + w_0)]dx \\ & - \int_\Omega [(v_n^+ + w_0)^{2^*} - w_0^{2^*} - 2^* w_0^{2^*-1} v_n^+]dx, \end{aligned}$$

and

$$\begin{aligned} H_2(v_n) = \lambda & \int_\Omega \left[\frac{v_n + w_0}{(v_n^+ + w_0 + \alpha)^\gamma} - \frac{v_n + w_0}{w_0^\gamma} \right] dx \\ & - \frac{2^*\lambda}{1-\gamma} \int_\Omega \left[(v_n^+ + w_0 + \alpha)^{1-\gamma} - (w_0 + \alpha)^{1-\gamma} - (1-\gamma)\frac{v_n^+}{w_0^\gamma} \right] dx. \end{aligned}$$

Then

$$\begin{aligned} H_1(v_n) = & \int_{v_n \geq 0} [(v_n^+ + w_0)^{2^*} - w_0^{2^*-1} v_n^+ - w_0^{2^*}]dx \\ & - \int_{v_n \geq 0} [(v_n^+ + w_0)^{2^*} - w_0^{2^*} - 2^* w_0^{2^*-1} v_n^+]dx \\ & = (2^* - 1) \int_\Omega w_0^{2^*-1} v_n^+ dx \\ & \geq 0, \end{aligned}$$

and

$$\begin{aligned} H_2(v_n) \geq & -\lambda \int_\Omega \frac{|v_n| + w_0}{(v_n^+ + w_0 + \alpha)^\gamma} dx - \lambda \int_\Omega \frac{|v_n| + w_0}{w_0^\gamma} dx - \frac{2^*\lambda}{1-\gamma} \int_\Omega (v_n^+)^{1-\gamma} dx \\ \geq & -\lambda \int_\Omega \frac{|v_n| + w_0}{w_0^\gamma} dx - \lambda \int_\Omega \frac{|v_n| + w_0}{w_0^\gamma} dx - \frac{2^*\lambda}{1-\gamma} \int_\Omega |v_n|^{1-\gamma} dx \\ = & -2\lambda \int_\Omega \frac{|v_n|}{w_0^\gamma} dx - 2\lambda \int_\Omega w_0^{1-\gamma} dx - \frac{2^*\lambda}{1-\gamma} \int_\Omega |v_n|^{1-\gamma} dx. \end{aligned}$$

It follows from (3.1) that

$$\begin{aligned} 2^*c + o(\|v_n\|) \geq & \; 2^* J_\alpha(v_n) - \langle J'_\alpha(v_n), v_n + w_0 \rangle \\ = & \; \frac{2}{N-2}\|v_n\|^2 - \int_\Omega (\nabla w_0, \nabla v_n) dx + H_1(v_n) + H_2(v_n) \\ = & \; \frac{N}{N-2}\|v_n\|^2 - \int_\Omega (\lambda w_0^{-\gamma} + w_0^{2^*-1}) v_n dx + H_1(v_n) + H_2(v_n) \\ \geq & \; \frac{2}{N-2}\|v_n\|^2 - \int_\Omega w_0^{2^*-1}|v_n| dx - 3\lambda \int_\Omega \frac{|v_n|}{w_0^\gamma} dx \\ & - 2\lambda \int_\Omega w_0^{1-\gamma} dx - \frac{2^*\lambda}{1-\gamma} \int_\Omega |v_n|^{1-\gamma} dx \\ \geq & \; \frac{2}{N-2}\|v_n\|^2 - C_1\|v_n\| - \frac{2^*\lambda}{1-\gamma}|\Omega|^{\frac{2^*\gamma-1}{2^*}} S^{-\frac{1-\gamma}{2}} \|v_n\|^{1-\gamma} - C_2\|w_0\|^{1-\gamma}, \end{aligned}$$

which implies that $\{v_n\}$ is bounded in $H_0^1(\Omega)$. Moreover, by using the concentration compactness principle (see [16, 17]), there exist a subsequence, say $\{v_n\}$ and $v_* \in H_0^1(\Omega)$ such that

$$\int_\Omega |\nabla v_n|^2 dx \rightharpoonup d\mu \geq \int_\Omega |\nabla v_*|^2 dx + \sum_{j\in K} \mu_j \delta_{x_j},$$
$$\int_\Omega (v_n^+)^{2^*} dx \rightharpoonup d\eta = \int_\Omega (v_*^+)^{2^*} dx + \sum_{j\in K} \eta_j \delta_{x_j},$$

where K is an at most countable index set, δ_{x_j} is the Dirac mass at x_j, and $x_j \in \Omega$ is in the support of μ, η. Moreover, there holds

$$\mu_j \geq S\eta_j^{\frac{2}{2^*}} \quad \text{for all } j \in K. \tag{3.2}$$

For $\varepsilon > 0$, let $\psi_{\varepsilon,j}(x)$ be a smooth cut-off function centered at x_j such that $0 \leq \psi_{\varepsilon,j}(x) \leq 1$,

$$\psi_{\varepsilon,j}(x) = 1 \ \text{ in } B\left(x_j, \varepsilon/2\right), \quad \psi_{\varepsilon,j}(x) = 0 \ \text{ in } \Omega\backslash B(x_j,\varepsilon), \ \ |\nabla \psi_{\varepsilon,j}(x)| \leq \frac{2}{\varepsilon}.$$

Since $\psi_{\varepsilon,j} v_n$ is bounded in $H_0^1(\Omega)$, according to (3.1), there holds

$$\begin{aligned}
0 &= \lim_{\varepsilon\to 0}\lim_{n\to\infty} \langle J_\alpha'(v_n), v_n\psi_{\varepsilon,j}\rangle \\
&= \lim_{\varepsilon\to 0}\lim_{n\to\infty} \left\{ \int_\Omega \nabla v_n \nabla (v_n\psi_{\varepsilon,j}) dx \right. \\
&\quad \left. - \int_\Omega [(v_n^+ + w_0)^{2^*-1} v_n\psi_{\varepsilon,j} - w_0^{2^*-1} v_n \psi_{\varepsilon,j}] dx \right\} \\
&\quad - \lambda \lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_\Omega \left[\frac{v_n\psi_{\varepsilon,j}}{(v_n^+ + w_0 + \alpha)^\gamma} - \frac{v_n\psi_{\varepsilon,j}}{w_0^\gamma} \right] dx \\
&= \lim_{\varepsilon\to 0}\lim_{n\to\infty} \left\{ \int_\Omega \nabla v_n \nabla (v_n\psi_{\varepsilon,j}) dx - \int_\Omega (v_n^+)^{2^*} \psi_{\varepsilon,j} dx \right. \\
&\quad \left. - \int_\Omega [(v_n^+ + w_0)^{2^*-1} v_n\psi_{\varepsilon,j} - (v_n^+)^{2^*-1} v_n\psi_{\varepsilon,j} - w_0^{2^*-1} v_n\psi_{\varepsilon,j}] dx \right\} \\
&\quad - \lambda \lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_\Omega \left[\frac{v_n\psi_{\varepsilon,j}}{(v_n^+ + w_0 + \alpha)^\gamma} - \frac{v_n\psi_{\varepsilon,j}}{w_0^\gamma} \right] dx.
\end{aligned} \tag{3.3}$$

Note that $\{v_n\}$ is bounded in $H_0^1(\Omega)$. Then

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty} \left| \int_\Omega [(v_n^+ + w_0)^{2^*-1} v_n\psi_{\varepsilon,j} - (v_n^+)^{2^*-1} v_n\psi_{\varepsilon,j}] dx \right| \\
&\qquad \leq \lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_\Omega [(v_n^+ + w_0)^{2^*-1} + (v_n^+)^{2^*-1}] |v_n| \psi_{\varepsilon,j} dx \\
&\qquad \leq \lim_{\varepsilon\to 0}\lim_{n\to\infty} C \int_{B(x_j,\varepsilon)} |v_n| dx \\
&\qquad = 0.
\end{aligned}$$

Similarly, one has

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_\Omega \left[\frac{v_n\psi_{\varepsilon,j}}{(v_n^+ + w_0 + \alpha)^\gamma} - \frac{v_n\psi_{\varepsilon,j}}{w_0^\gamma} \right] dx = 0,$$

and

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega w_0^{2^*-1}v_n\psi_{\varepsilon,j}dx=0,\quad \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega v_n\nabla v_n\nabla\psi_{\varepsilon,j}dx=0.$$

From the above information, by (3.3), we have

$$\eta_j=\mu_j,$$

this combining (3.2) we deduce that

$$\eta_j\ge S^{\frac{N}{2}}\quad\text{or}\quad \eta_j=0.$$

Next we show that $\eta_j\ge S^{\frac{N}{2}}$ is impossible. By contradiction, there exists some $j_0\in J$ such that $\eta_{j_0}\ge S^{\frac{N}{2}}$. By (3.1), there holds

$$\begin{aligned}
c=&\lim_{n\to\infty}\left\{J_\alpha(v_n)-\frac{1}{2}\langle J'_\alpha(v_n),v_n+w_0\rangle\right\}\\
=&-\frac{1}{2}\int_\Omega(\nabla v_n,\nabla w_0)dx+\frac{1}{N}\int_\Omega(|v_n^++w_0|^{2^*}-w_0^{2^*})dx+\frac{1}{2}\int_\Omega w_0^{2^*-1}v_n^+dx\\
&-\frac{\lambda}{1-\gamma}\int_\Omega\left[(v_n^++w_0+\alpha)^{1-\gamma}-(w_0+\alpha)^{1-\gamma}-(1-\gamma)\frac{v_n^+}{w_0^\gamma}\right]dx\\
&+\frac{\lambda}{2}\int_\Omega\left[\frac{v_n+w_0}{(v_n^++w_0+\alpha)^\gamma}-\frac{v_n+w_0}{w_0^\gamma}\right]dx+o(1/n)\\
\ge&\frac{1}{N}\int_\Omega(v_n^+)^{2^*}dx+\frac{1}{2}\int_\Omega w_0^{2^*-1}v_n^+dx-\frac{1}{2}\left(\int_\Omega w_0^{2^*-1}v_ndx+\lambda\int_\Omega\frac{v_n}{w_0^\gamma}dx\right)\\
&-\frac{\lambda}{1-\gamma}\int_\Omega\left[(v_n^++w_0+\alpha)^{1-\gamma}-(w_0+\alpha)^{1-\gamma}-(1-\gamma)\frac{v_n^+}{w_0^\gamma}\right]dx\\
&+\frac{\lambda}{2}\int_\Omega\left[\frac{v_n+w_0}{(v_n^++w_0+\alpha)^\gamma}-\frac{v_n+w_0}{w_0^\gamma}\right]dx+o(1/n)\\
\ge&\frac{1}{N}\int_\Omega(v_n^+)^{2^*}dx+H_3(v_n)+o(1/n),
\end{aligned}$$

where

$$\begin{aligned}
H_3(v_n)\triangleq&\frac{\lambda}{2}\int_\Omega\left[\frac{v_n+w_0}{(v_n^++w_0+\alpha)^\gamma}-\frac{v_n+w_0}{w_0^\gamma}\right]dx-\frac{\lambda}{2}\int_\Omega\frac{v_n}{w_0^\gamma}dx+\lambda\int_\Omega\frac{v_n^+}{w_0^\gamma}dx\\
&-\frac{\lambda}{1-\gamma}\int_\Omega\left[(v_n^++w_0+\alpha)^{1-\gamma}-(w_0+\alpha)^{1-\gamma}\right]dx\\
=&\frac{\lambda}{2}\int_\Omega\left[\frac{v_n^++v_n^-+w_0}{(v_n^++w_0+\alpha)^\gamma}-w_0^{1-\gamma}\right]dx-\lambda\int_\Omega\frac{v_n}{w_0^\gamma}dx+\lambda\int_\Omega\frac{v_n^+}{w_0^\gamma}dx\\
&-\frac{\lambda}{1-\gamma}\int_\Omega\left[(v_n^++w_0+\alpha)^{1-\gamma}-(w_0+\alpha)^{1-\gamma}\right]dx\\
=&\frac{\lambda}{2}\int_\Omega\left[\frac{v_n^++v_n^-+w_0}{(v_n^++w_0+\alpha)^\gamma}+\frac{v_n^-}{(v_n^++w_0+\alpha)^\gamma}-w_0^{1-\gamma}\right]dx-\lambda\int_\Omega\frac{v_n^-}{w_0^\gamma}dx\\
&-\frac{\lambda}{1-\gamma}\int_\Omega\left[(v_n^++w_0+\alpha)^{1-\gamma}-(w_0+\alpha)^{1-\gamma}\right]dx\\
\ge&\frac{\lambda}{2}\int_\Omega\left[\frac{v_n^++v_n^-+w_0}{(v_n^++w_0+\alpha)^\gamma}-w_0^{1-\gamma}\right]dx+\frac{\lambda}{2}\int_\Omega\frac{v_n^-}{w_0^\gamma}dx-\lambda\int_\Omega\frac{v_n^-}{w_0^\gamma}dx\\
&-\frac{\lambda}{1-\gamma}\int_\Omega\left[(v_n^++w_0+\alpha)^{1-\gamma}-(w_0+\alpha)^{1-\gamma}\right]dx
\end{aligned}$$

$$
\begin{aligned}
&\geq -\left(\frac{1}{1-\gamma}-\frac{1}{2}\right)\lambda\int_\Omega (v_n^+ + w_0+\alpha)^{1-\gamma}dx\\
&\quad +\left(\frac{1}{1-\gamma}-\frac{1}{2}\right)\lambda\int_\Omega (w_0+\alpha)^{1-\gamma}dx-\frac{\lambda}{2}\int_\Omega \frac{\alpha}{(v_n^+ + w_0+\alpha)^{\gamma}}dx\\
&\geq -\left(\frac{1}{1-\gamma}-\frac{1}{2}\right)\lambda\int_\Omega (v_n^+ + w_0+\alpha)^{1-\gamma}dx-\frac{\lambda}{1-\gamma}\int_\Omega (w_0+\alpha)^{1-\gamma}dx\\
&\quad -\frac{\lambda}{2}\int_\Omega (v_n^+ + w_0+\alpha)^{1-\gamma}dx\\
&= -\frac{\lambda}{1-\gamma}\int_\Omega (v_n^+)^{1-\gamma}dx-\frac{2\lambda}{1-\gamma}\int_\Omega (w_0+\alpha)^{1-\gamma}dx\\
&\geq -\frac{\lambda}{1-\gamma}\int_\Omega (v_n^+)^{1-\gamma}dx-\frac{2\lambda}{1-\gamma}(|\Omega|+|\Omega||w_0|_\infty^{1-\gamma}).
\end{aligned}
$$

Then, by the Sobolev inequality and Young inequality, we have

$$
\begin{aligned}
c &\geq \frac{1}{N}\int_\Omega (v_n^+)^{2^*}dx + H_3(v_n)+o(1/n)\\
&\geq \frac{1}{N}\left(\int_\Omega (v_*^+)^{2^*}dx+\sum_{j\in J}\eta_j\right)+H_3(v_n)+o(1/n)\\
&\geq \frac{1}{N}\eta_{j_0}+\int_\Omega (v_*^+)^{2^*}dx-\frac{\lambda}{1-\gamma}|\Omega|^{\frac{2^*+\gamma-1}{2^*}}\left(\int_\Omega (v_*^+)^{2^*}dx\right)^{\frac{1-\gamma}{2^*}}-A\lambda+o(1/n)\\
&\geq \frac{1}{N}S^{\frac{N}{2}}-A_1\lambda^{\frac{2^*}{2^*+\gamma-1}}-A\lambda+o(1/n)\\
&\geq \frac{1}{N}S^{\frac{N}{2}}-A_1\lambda-A\lambda+o(1/n)\\
&= \frac{1}{N}S^{\frac{N}{2}}-D\lambda+o(1/n),
\end{aligned}
$$

where $A=\frac{2}{1-\gamma}(|\Omega|+|\Omega||w_0|_\infty^{1-\gamma})$, $A_1=A_1(N,\gamma,S,|\Omega|)$, $D=A_1+A$. Therefore, we get $\frac{1}{N}S^{\frac{N}{2}}-D\lambda\leq c<\frac{1}{N}S^{\frac{N}{2}}-D\lambda$, which contradicts to the assumption. It implies that K is empty, so $\int_\Omega (v_n^+)^{2^*}dx\to\int_\Omega (v_*^+)^{2^*}dx$ as $n\to\infty$. Recalling that for $p\geq 2$, there holds

$$
\left||x|^{p-1}x-|y|^{p-1}y\right|\leq C_p|x-y|(|x|+|y|)^{p-1},\qquad C_p>0.
$$

As a result, there holds

$$
\begin{aligned}
0&\leq\left|\int_\Omega [(v_n^+ + w_0)^{2^*}-(v_*^+ + w_0)^{2^*}]dx\right|\\
&\leq C_p\int_\Omega |v_n^+ - v_*^+|(v_n^+ + v_*^+ + 2w_0)^{2^*-1}dx\\
&\leq C_p|v_n^+ - v_*^+|_{2^*}|v_n^+ + v_*^+ + 2w_0|_{2^*}^{2^*-1}\\
&\leq C|v_n^+ - v_*^+|_{2^*}\\
&\to 0,
\end{aligned}
$$

which implies that $\int_\Omega (v_n^+ + w_0)^{2^*}dx\to\int_\Omega (v_*^+ + w_0)^{2^*}dx$ as $n\to\infty$. Note that $J'_\alpha(v_n)\to 0$, it follows

$$
\begin{aligned}
\int_\Omega (\nabla v_*,\nabla\phi)dx &= \int_\Omega [(v_*^+ + w_0)^{2^*-1}\phi - w_0^{2^*-1}\phi]dx\\
&\quad +\lambda\int_\Omega\left[\frac{\phi}{(v_*^+ + w_0+\alpha)^{\gamma}}-\frac{\phi}{w_0^{\gamma}}\right]dx
\end{aligned}
\tag{3.4}
$$

for each $\phi \in H_0^1(\Omega)$. Taking the test function $\phi = v_* + w_0$ in (3.4), we have

$$\begin{aligned}\int_\Omega (\nabla v_*, \nabla(v_* + w_0))dx = &\int_\Omega [(v_*^+ + w_0)^{2^*} - w_0^{2^*-1}(v_*^+ + w_0)]dx \\ &+ \lambda \int_\Omega \left[\frac{v_* + w_0}{(v_*^+ + w_0 + \alpha)^\gamma} - \frac{v_* + w_0}{w_0^\gamma}\right] dx.\end{aligned} \tag{3.5}$$

According to $\langle J'_\alpha(v_n), v_n + w_0\rangle \to 0$, one has

$$\begin{aligned}\int_\Omega (\nabla v_n, \nabla(v_n + w_0))dx = &\int_\Omega [(v_n^+ + w_0)^{2^*-1}(v_n + w_0) - w_0^{2^*-1}(v_n + w_0)]dx \\ &+ \lambda \int_\Omega \left[\frac{v_n + w_0}{(v_n^+ + w_0 + \alpha)^\gamma} - \frac{v_n + w_0}{w_0^\gamma}\right] dx + o(1/n).\end{aligned}$$

Consequently,

$$\begin{aligned}\int_\Omega (\nabla v_*, \nabla w_0)dx + \|v_n\|^2 = &\int_\Omega [(v_*^+ + w_0)^{2^*} - w_0^{2^*-1}(v_*^+ + w_0)]dx \\ &+ \lambda \int_\Omega \left[\frac{v_* + w_0}{(v_*^+ + w_0 + \alpha)^\gamma} - \frac{v_* + w_0}{w_0^\gamma}\right] dx + o(1/n).\end{aligned} \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\|v_n\|^2 \to \|v_*\|^2, \quad \text{as } n \to \infty.$$

Hence, we have $v_n \to v_*$ in $H_0^1(\Omega)$. The proof is complete. □

It is known that the function

$$U_\varepsilon(x) = \frac{[N(N-2)\varepsilon^2]^{\frac{N-2}{4}}}{(\varepsilon^2 + |x|^2)^{\frac{N-2}{2}}}, \quad x \in \mathbb{R}^N, \ \ \varepsilon > 0$$

satisfies

$$-\Delta U_\varepsilon = U_\varepsilon^{2^*-1} \quad \text{in } \mathbb{R}^N.$$

We choose a function $\varphi \in C_0^\infty(\Omega)$ such that $0 \le \varphi(x) \le 1$ in Ω, $\varphi(x) = 1$ near $x = 0$ and it is radially symmetric. Let $u_\varepsilon(x) = \varphi(x)U_\varepsilon(x)$. Moreover, from [10], there exist two constants $m, M > 0$ such that $m \le w_0(x) \le M$ for each $x \in \text{supp}\varphi$, and

$$\begin{cases} \|u_\varepsilon\|^2 \le S^{\frac{N}{2}} + O(\varepsilon^{N-2}) + O(\varepsilon^N), \\ \int_\Omega |u_\varepsilon|^{2^*}dx = S^{\frac{N}{2}} - O(\varepsilon^N). \end{cases}$$

Moreover, one has

$$\begin{cases} \int_\Omega w_0 u_\varepsilon^{2^*-1}dx = C\varepsilon^{\frac{N-2}{2}} + O(\varepsilon^{\frac{N+2}{2}}), \\ \int_\Omega \frac{u_\varepsilon}{w_0^\gamma}dx = O(\varepsilon^{\frac{N-2}{2}}). \end{cases} \tag{3.7}$$

Lemma 3.3. *For every $0 < \alpha < 1$ and $\lambda > 0$ small, there holds*

$$\sup_{t \ge 0} J_\alpha(tu_\varepsilon) < \frac{1}{N}S^{\frac{N}{2}} - D\lambda,$$

where D is defined by Lemma 3.2.

Proof. For $t \geq 0$, there holds

$$\begin{aligned} J_\alpha(tu_\varepsilon) &= \frac{t^2}{2}\|u_\varepsilon\|^2 - \frac{1}{2^*}\int_\Omega [(tu_\varepsilon + w_0)^{2^*} - w_0^{2^*} - 2^* w_0^{2^*-1} tu_\varepsilon] dx \\ &\quad - \frac{\lambda}{1-\gamma}\int_\Omega \left[(tu_\varepsilon + w_0)^{1-\gamma} - w_0^{1-\gamma} - (1-\gamma)\frac{tu_\varepsilon}{w_0^\gamma}\right] dx \\ &\leq \frac{t^2}{2}\|u_\varepsilon\|^2 - \frac{t^{2^*}}{2^*}\int_\Omega u_\varepsilon^{2^*} dx - t^{2^*-1}\int_\Omega w_0 u_\varepsilon^{2^*-1} dx + t\lambda \int_\Omega \frac{u_\varepsilon}{w_0^\gamma} dx, \end{aligned}$$

where the following inequality is used

$$(a+b)^{2^*} \geq a^{2^*} + b^{2^*} - 2^* a^{2^*-1} b - 2^* a b^{2^*-1}, \quad \forall a, b > 0.$$

Set

$$g(t) = \frac{t^2}{2}\|u_\varepsilon\|^2 - \frac{t^{2^*}}{2^*}\int_\Omega u_\varepsilon^{2^*} dx - t^{2^*-1}\int_\Omega w_0 u_\varepsilon^{2^*-1} dx + t\lambda \int_\Omega \frac{u_\varepsilon}{w_0^\gamma} dx.$$

As $\lim_{t\to+\infty} g(tu_\varepsilon) = -\infty$, similar to the paper [14], we can prove that there exist $t_\varepsilon > 0$ and positive constants t_0, t_1 which are independent of ε, λ, such that $\sup_{t\geq 0} g(t) = g(t_\varepsilon)$ and

$$0 < t_0 \leq t_\varepsilon \leq t_1 < \infty. \tag{3.8}$$

Therefore, from (3.7) and (3.8), there holds

$$\begin{aligned} \sup_{t\geq 0} g(t) &\leq \sup_{t\geq 0}\left\{\frac{t^2}{2}\|u_\varepsilon\|^2 - \frac{t^{2^*}}{2^*}\int_\Omega u_\varepsilon^{2^*} dx\right\} - t_0^{2^*-1}\int_\Omega w_0 u_\varepsilon^{2^*-1} dx + t_1\lambda \int_\Omega \frac{u_\varepsilon}{w_0^\gamma} dx \\ &\leq \frac{1}{N} S^{\frac{N}{2}} + C_1 \varepsilon^{N-2} - C_2 \varepsilon^{\frac{N-2}{2}} + C_3 \lambda \varepsilon^{\frac{N-2}{2}}, \end{aligned}$$

where $C_i > 0$ (independent of ε, λ), $i = 1, 2, 3$. Therefore, let $\varepsilon = \lambda^{\frac{1}{N-2}}$, $\lambda < \Lambda_1 = \left(\frac{C_2}{C_1+C_3+D}\right)^2$, then

$$\begin{aligned} C_1 \varepsilon^{N-2} - C_2 \varepsilon^{\frac{N-2}{2}} + C_3 \lambda \varepsilon^{\frac{N-2}{2}} &= C_1 \lambda + C_3 \lambda^{\frac{3}{2}} - C_2 \lambda^{\frac{1}{2}} \\ &\leq \lambda\left(C_1 + C_3 - C_2 \lambda^{\frac{-1}{2}}\right) \\ &< -D\lambda. \end{aligned}$$

From the above information, it holds that

$$\begin{aligned} \sup_{t\geq 0} g(t) &\leq \frac{1}{N} S^{\frac{N}{2}} + C_1 \varepsilon^{N-2} - C_2 \varepsilon^{\frac{N-2}{2}} + C_3 \lambda \varepsilon^{\frac{N-2}{2}} \\ &< \frac{1}{N} S^{\frac{N}{2}} - D\lambda, \end{aligned}$$

which implies that

$$\sup_{t\geq 0} J_\alpha(tu_\varepsilon) < \frac{1}{N} S^{\frac{N}{2}} - D\lambda$$

for any $0 < \lambda < \Lambda_1$. The proof is complete. □

Lemma 3.4. *For given $0 < \alpha < 1$ and $\lambda > 0$ is sufficiently small, there exists $v_\alpha \in H_0^1(\Omega)$ such that $J'_\alpha(v_\alpha) = 0$ and $J_\alpha(v_\alpha) > 0$.*

Proof. Let $\lambda^* = \min\left\{\Lambda_0, \Lambda_1, \frac{S^{\frac{N}{2}}}{ND}, 1\right\}$ and $0 < \lambda < \lambda^*$. By Lemma 3.1, the functional J_α satisfies the geometry of the mountain-pass lemma. Applying the mountain-pass lemma [1], there exists a sequence $\{v_n\} \subset H_0^1(\Omega)$, such that

$$J_\alpha(v_n) \to c > \rho \quad \text{and} \quad J'_\alpha(v_n) \to 0, \tag{3.9}$$

where

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\alpha(\gamma(t)), \qquad \Gamma = \left\{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = \zeta\right\}.$$

By Lemma 3.2 and Lemma 3.3, $\{v_n\} \subset H_0^1(\Omega)$ has a convergent subsequence, say $\{v_n\}$, we may assume that $v_n \to v_\alpha$ in $H_0^1(\Omega)$ as $n \to \infty$. Hence, from (3.9), it holds

$$J_\alpha(v_\alpha) = \lim_{n\to\infty} J_\alpha(v_n) = c > 0.$$

Furthermore, we have $J'_\alpha(v_\alpha) = 0$. The proof is complete. □

Now, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $0 < \lambda < \lambda^*$, where λ^* is defined by Lemma 3.4. Since v_α is a critical for J_α, for each $\phi \in H_0^1(\Omega)$, one has

$$\begin{aligned}\int_\Omega (\nabla v_\alpha, \nabla\phi)dx = &\int_\Omega [(v_\alpha^+ + w_0)^{2^*-1}\phi - w_0^{2^*-1}\phi]dx \\ &+ \lambda \int_\Omega \left[\frac{\phi}{(v_\alpha^+ + w_0 + \alpha)^\gamma} - \frac{\phi}{w_0^\gamma}\right] dx,\end{aligned} \tag{3.10}$$

which implies that v_α satisfies the following equation

$$\begin{cases} -\Delta v_\alpha = (v_\alpha^+ + w_0)^{2^*-1} - w_0^{2^*-1} + \dfrac{\lambda}{(v_\alpha^+ + w_0 + \alpha)^\gamma} - \dfrac{\lambda}{w_0^\gamma}, & \text{in } \Omega, \\ v_\alpha = 0, & \text{on } \partial\Omega. \end{cases}$$

Moreover, we can easily prove that $\{v_\alpha\}$ is bounded in $H_0^1(\Omega)$, thus there exist a subsequence, still denoted by $\{v_\alpha\}$, and $v_0 \in H_0^1(\Omega)$ such that

$$\begin{cases} v_\alpha \rightharpoonup v_0, & \text{weakly in } H_0^1(\Omega), \\ v_\alpha \to v_0, & \text{strongly in } L^p(\Omega)\ (1 \le p < 2^*), \\ v_\alpha(x) \to v_0(x), & \text{a.e. in } \Omega, \end{cases}$$

as $\alpha \to 0$. Note that w_0 fulfilling

$$\begin{cases} -\Delta w_0 = w_0^{2^*-1} + \dfrac{\lambda}{w_0^\gamma}, & \text{in } \Omega, \\ w_0 = 0, & \text{on } \partial\Omega. \end{cases}$$

Since $0 < w_0 \le M$, from the above information, we have

$$\begin{aligned} -\Delta(v_\alpha + w_0) &= (v_\alpha^+ + w_0)^{2^*-1} + \frac{\lambda}{(v_\alpha^+ + w_0 + \alpha)^\gamma} \\ &\ge (v_\alpha^+)^{2^*-1} + \frac{\lambda}{(v_\alpha^+ + M + \alpha)^\gamma} \\ &\ge \min\left\{1, \frac{\lambda}{(M+2)^\gamma}\right\}. \end{aligned} \tag{3.11}$$

Let e be a positive solution of the following problem

$$\begin{cases} -\Delta u = 1, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Then for every $\Omega_0 \subset\subset \Omega$, there exists $e_0 > 0$ such that $e|_{\Omega_0} \geq e_0$. So, by (3.11) and comparison principle, we get

$$v_\alpha + w_0 \geq \min\left\{1, \frac{\lambda}{(M+2)^\gamma}\right\} e > 0.$$

In particular, we have

$$v_\alpha + w_0 \geq \min\left\{1, \frac{\lambda}{(M+2)^\gamma}\right\} e_0 > 0, \quad \text{in } \Omega_0.$$

Hence, it is similar to the paper [14] that for every $\phi \in H_0^1(\Omega)$, passing the limit as $\alpha \to$ in (3.10), there holds

$$\int_\Omega (\nabla v_0, \nabla\phi)dx = \int_\Omega [(v_0^+ + w_0)^{2^*-1}\phi - w_0^{2^*-1}\phi]dx + \lambda \int_\Omega \left[\frac{\phi}{(v_0^+ + w_0)^\gamma} - \frac{\phi}{w_0^\gamma}\right] dx. \tag{3.12}$$

Note that

$$\begin{aligned}
\int_\Omega \left[\frac{v_0^-}{(v_0^+ + w_0)^\gamma} - \frac{v_0^-}{w_0^\gamma}\right] dx &= \left(\int_{v_0 \geq 0} + \int_{v_0 < 0}\right) \int_\Omega \left[\frac{v_0^-}{(v_0^+ + w_0)^\gamma} - \frac{v_0^-}{w_0^\gamma}\right] dx \\
&= \int_{v_0<0} \left[\frac{v_0^-}{(v_0^+ + w_0)^\gamma} - \frac{v_0^-}{w_0^\gamma}\right] dx \\
&= \int_{v_0<0} \left[\frac{v_0^-}{w_0^\gamma} - \frac{v_0^-}{w_0^\gamma}\right] dx \\
&= 0.
\end{aligned}$$

Therefore, taking the test $\phi = v_0^-$ in (3.12), then

$$\begin{aligned}
\|v_0^-\|^2 &= \int_\Omega [(v_0^+ + w_0)^{2^*-1}v_0^- - w_0^{2^*-1}v_0^-]dx + \lambda \int_\Omega \left[\frac{v_0^-}{(v_0^+ + w_0)^\gamma} - \frac{v_0^-}{w_0^\gamma}\right] dx \\
&= 0,
\end{aligned}$$

which implies that $\|v_0^-\| = 0$. Consequently, $v_0 \geq 0$ a.e. in Ω. Moreover, from (3.12), v_0 is a positive solution of the following problem

$$\begin{cases} -\Delta v = (v^+ + w_0)^{2^*-1} - w_0^{2^*-1} + \dfrac{\lambda}{(v^+ + w_0)^\gamma} - \dfrac{\lambda}{w_0^\gamma}, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases}$$

Similar to Lemma 3.2, we can deduce that $v_\alpha \to v_0$ in $H_0^1(\Omega)$ and

$$J_0(v_0) = \lim_{\alpha \to 0} J_\alpha(v_\alpha) > 0,$$

where

$$\begin{aligned}
J_0(v) = \frac{1}{2}\|v\|^2 - \frac{1}{2^*} \int_\Omega [(v^+ + w_0)^{2^*} - w_0^{2^*} - 2^* w_0^{2^*-1} v^+]dx \\
- \frac{\lambda}{1-\gamma} \int_\Omega \left[(v^+ + w_0)^{1-\gamma} - w_0^{1-\gamma} - (1-\gamma)\frac{v^+}{w_0^\gamma}\right] dx.
\end{aligned}$$

Set $w_1 = v_0 + w_0$, we have

$$\begin{aligned}
&\int_\Omega (\nabla w_1, \nabla \phi)dx - \int_\Omega w_1^{2^*-1}\phi dx - \lambda \int_\Omega \frac{\phi}{w_1^\gamma}dx \\
&= \int_\Omega (\nabla w_0, \nabla \phi)dx - \int_\Omega w_0^{2^*-1}\phi dx - \lambda \int_\Omega \frac{\phi}{w_0^\gamma}dx \\
&\quad + \int_\Omega (\nabla v_0, \nabla \phi)dx - \int_\Omega [(v_0 + w_0)^{2^*-1}\phi - w_0^{2^*-1}\phi]dx \\
&\quad - \lambda \int_\Omega \left[\frac{\phi}{(v_0+w_0)^\gamma} - \frac{\phi}{w_0^\gamma}\right]dx \\
&= 0,
\end{aligned}$$

which implies that w_1 is a positive solution of problem (1.2). Moreover, there holds

$$\begin{aligned}
I(w_1) &= \frac{1}{2}\|w_1\|^2 - \frac{1}{2^*}\int_\Omega w_1^{2^*}dx - \frac{\lambda}{1-\gamma}\int_\Omega w_1^{1-\gamma}dx \\
&= \frac{1}{2}\|w_0\|^2 - \frac{1}{2^*}\int_\Omega w_0^{2^*}dx - \frac{\lambda}{1-\gamma}\int_\Omega w_0^{1-\gamma}dx + J_0(v_0) \\
&\quad + \int_\Omega (\nabla w_0, \nabla v_0)dx - \int_\Omega w_0^{2^*-1}v_0 dx - \lambda \int_\Omega \frac{v_0}{w_0^\gamma}dx \\
&= I(w_0) + J_0(v_0) \\
&> I(w_0).
\end{aligned}$$

Consequently, $I(w_1) > I(w_0)$, it suggests that $w_1 \neq w_0$. Recalling that $w_1 = v_0 + w_0$, we can deduce that $v_0 > 0$ and $w_1 > w_0$. The proof is complete. □

Acknowledgements

The authors express their gratitude to the reviewer for careful reading and helpful suggestions which led to an improvement of the original manuscript. This work was supported the National Natural Science Foundation of China (11661021,11861021), Young Science and Technology Scholars of Guizhou Provincial Department of Education (KY[2016]163), Scientific Research Fund of Sichuan Provincial Education Department (18ZA0471), Fundamental Research Funds of China West Normal University (16E014, 18B015) and Innovative Research Team of China West Normal University (CXTD2018-8).

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