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# Existence of solutions of nonlinear third-order two-point boundary value problems

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**Abstract.** We study various two-point boundary value problems for the equation $x''' = f(t, x, x', x'')$. Using barrier strips type conditions, we give sufficient conditions guaranteeing positive or non-negative, monotone, convex or concave $C^3[0, 1]$-solutions.

**Keywords:** third-order differential equation, boundary value problem, existence, positive or non-negative, monotone, convex or concave solutions, sign conditions.

**2010 Mathematics Subject Classification:** 34B15, 34B18.

## 1 Introduction

In this paper, we are concerned with boundary value problems (BVPs) for the differential equation

$$x''' = f(t, x, x', x''), \qquad t \in (0, 1), \tag{1.1}$$

with boundary conditions either

$$x''(0) = A, \qquad x'(0) = B, \qquad x(1) = C, \tag{1.2}$$
$$x''(0) = A, \qquad x'(0) = B, \qquad x(0) = C, \tag{1.3}$$
$$x''(0) = A, \qquad x'(1) = B, \qquad x(1) = C, \tag{1.4}$$
$$x''(0) = A, \qquad x'(1) = B, \qquad x(0) = C, \tag{1.5}$$

or

$$x''(0) = A, \qquad x(0) = B, \qquad x(1) = C, \tag{1.6}$$

where $f : [0, 1] \times D_x \times D_p \times D_q \to R$, and $D_x, D_p, D_q \subseteq R$.

We study the existence of $C^3[0, 1]$-solutions to the above problems which do not change their sign, are monotone and do not change their curvature.

Third-order differential equations arise in a large number of physical and technological processes, see, for example, M. Aïboudi and B. Brighi [1], J. R. Graef et al. [9], Z. Zhang [33]

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for facts and references. Recently, various third-order BVPs have received much attention and a lot of research has been done in this area. Here, we cite sources devoted to two-point BVPs.

Two-point BVPs for equations of the form

$$x''' = f(t, x), \qquad t \in (0, 1),$$

have been studied by A. Cabada [3], H. Li et al. [17], S. Li [18] (the problem may be singular at $t = 0$ and/or $t = 1$), Zh. Liu et al. [20] (with singularities at $t = 0, t = 1$ and/or $x = 0$), Z. Liu et al. [21–23], X. Lin and Z. Zhao [24], D. O'Regan [27] (the problem is singular at $x = 0$), S. Smirnov [28], Q. Yao and Y. Feng [32]. The boundary conditions in these works are as follows:

$$x^{(i)}(0) - x^{(i)}(1) = \lambda_i, \qquad \lambda_i \in R,\ i = 0, 1, 2, \quad \text{in [3]},$$

$$x(0) = x'(0) = x'(1) = 0, \quad \text{in [17,24,32]}, \tag{1.7}$$

in [18,21] they are

$$x(0) = x'(0) = x''(1) = 0, \tag{1.8}$$

$$x(0) = x'(0) = x(1) = 0, \quad \text{in [28]},$$

$$x(0) = x'(0), \qquad \alpha x'(1) + \beta x''(1) = \lambda, \qquad \lambda > 0, \qquad \alpha, \beta \geq 0, \quad \text{in [20]},$$

in [22] they are (1.6) with $A = B = C = 0$,

$$x(0) = x(1) = x''(1) = 0, \quad \text{in [23]},$$

and in [27] they are either (1.2)(with $A = B = 0$), (1.5) or (1.6) (with $A = 0$).

Two-point BVPs for equations of the form

$$x''' = f(t, x, x'), \qquad t \in (0, 1),$$

have been studied by Y. Feng [7], the boundary conditions in this work are

$$x(1) = x'(0) = x'(1) = 0,$$

Y. Feng and S. Liu [8] (with boundary conditions (1.7)), D. O'Regan [27] (with (1.5)).

Y. Feng [6] and R. Ma and Y. Lu [25] have considered, respectively, BVPs for the equations

$$f(t, x, x', x''') = 0 \quad \text{and} \quad x''' + Mx'' + f(t, x) = 0, \qquad t \in (0, 1),$$

with (1.7).

The solvability of BVPs for the equation

$$x''' = f(t, x, x', x''), \qquad t \in (0, 1),$$

has been investigated by G. Chen [4], Z. Du et al. [5], J. Graef et al. [9], A. Granas et al. [10], M. Grossinho et al. [11,12], B. Hopkins and N. Kosmatov [13], Y. Li and Y. Li [19], F. Minhós [26], J. Wang [29] and Z. Weili [31]. In [13,19], the boundary conditions are

$$x(0) = x'(1) = x''(1) = 0,$$

in fact, in [13] the following ones

$$x(0) = x'(0) = x''(1) = 0$$

are also considered. The boundary conditions [10] are (1.7), these in [12, 26, 31] include more general linear ones, and in [4, 5, 11, 29] they are nonlinear.

M. Aïboudi and B. Brighi [1] and B. Brighi [2] have considered the equation

$$x''' + xx'' + g(x') = 0, \qquad t \in [0, \infty),$$

with boundary conditions similar to (1.3), and Z. Zhang [33] and Z. Zhang and J. Wang [34] have studied the BVP

$$n(\pm x'')^{n-1} x''' + \lambda x x'' - x' g(x') = 0, \qquad t \in [0, \infty),\ \lambda > 0,$$

$$x(0) = 0, \qquad x'(0) = 1, \qquad x'(+\infty) = 0.$$

Along with the existence results of one, two or more solutions given in the mentioned sources, nonexistence results can be found in [20, 26, 33], and uniqueness ones in [1, 2, 6, 31]. Positive or non-negative solutions are guaranteed in [6–8, 13, 18–23, 25, 32–34], negative or nonpositive in [6, 8, 32], monotone ones in [8, 21, 32–34], and convex and/or concave solutions have been established in [2, 33, 34].

In the works mentioned above, the main nonlinearity is a Carathéodory function on unbounded set, see [3, 13], or is defined and continuous on a set such that each dependent variable changes in a left- and/or a right-unbounded set, see [1–13, 17–34]. The results are obtained by using the upper and lower solutions technique [3–8, 11, 12, 17, 25, 26, 29, 31, 32], Nagumo type growth conditions [5, 11, 12, 19, 26, 31], Lipschitz conditions [1, 2, 9], Green's functions [17, 18, 20, 22–24], maximum principles [3, 6, 7], assumptions that the main nonlinearity does not change its sign [18–23, 27] or is monotone with respect to some of the variables [5, 17, 24].

We do not use the above tools. The imposed condition in this paper allows the main nonlinearity to be defined on a bounded set, to be continuous on a suitable subset of its domain and to change its sign. So, our results rely on the following hypotheses.

**($H_1$)** There are constants $F_i, L_i, i = 1, 2$, and a sufficiently small $\sigma > 0$ such that

$$F_2 + \sigma \le F_1 \le A \le L_1 \le L_2 - \sigma, \qquad [F_2, L_2] \subseteq D_q,$$

$$f(t, x, p, q) \le 0 \quad \text{for } (t, x, p, q) \in [0, 1] \times D_x \times D_p \times [L_1, L_2], \tag{1.9}$$

$$f(t, x, p, q) \ge 0 \quad \text{for } (t, x, p, q) \in [0, 1] \times D_x \times D_p \times [F_2, F_1]. \tag{1.10}$$

Besides, we will say that for some of the BVPs (1.1),(1.$k$), $k = 2, 3, 4, 5, 6$ ($k = \overline{2, 6}$ for short), the condition **($H_2$)** holds for constants $m_i \le M_i$, $i = \overline{0, 2}$, (these constants will be specified later for each problem) if:

**($H_2$)** $[m_0 - \sigma, M_0 + \sigma] \subseteq D_x, [m_1 - \sigma, M_1 + \sigma] \subseteq D_p, [m_2 - \sigma, M_2 + \sigma] \subseteq D_q$, where $\sigma$ is as in **($H_1$)**, and $f(t, x, p, q)$ is continuous on $[0, 1] \times J$, where $J = [m_0 - \sigma, M_0 + \sigma] \times [m_1 - \sigma, M_1 + \sigma] \times [m_2 - \sigma, M_2 + \sigma]$.

Such type of conditions have been used for studying the solvability of various problems for first and second order differential equations, see P. Kelevedjiev and N. Popivanov [14] and R. Ma et al. [16] for results and references. Here we adapt this approach for the considered problems developing ideas partially announced in P. Kelevedjiev et al. [15] on the BVP (1.1), (1.8). **(H$_1$)** ensures priori bounds for $x''(t), x'(t)$ and $x(t)$, in this order, for each eventual solution $x(t) \in C^3[0,1]$ to the families of BVPs for

$$x''' = \lambda f(t, x, x', x''), \qquad t \in (0,1), \tag*{(1.1)$_\lambda$}$$

with one of the boundary conditions (1.$k$), $k = \overline{2,6}$, and **(H$_2$)** gives the bounds for $x'''(t)$. The priori bounds are needed for application of the global existence theorem from Section 2, and the auxiliary results which guarantee them are given in Section 3. The results for problems (1.1), (1.$k$), $k = \overline{2,5}$, are in Section 4, and these for (1.1), (1.6) in Section 5.

## 2 Global existence theorem

Let $E$ be a Banach space, $Y$ be its convex subset, and $U \subset Y$ be open in $Y$. The compact map $F : \overline{U} \to Y$ is called admissible if it is fixed point free on $\partial U$. By $L_{\partial U}(\overline{U}, Y)$ we denote the set of all admissible maps of $\overline{U}$ into $Y$.

A map $F \in L_{\partial U}(\overline{U}, Y)$ is called essential if every map $G \in L_{\partial U}(\overline{U}, Y)$ with the property $G/\partial U = F/\partial U$ has a fixed point in $U$. Clearly, every essential map has a fixed point in $U$.

**Theorem 2.1** ([10, Chapter I, Theorem 2.2]). *Let $p \in U$ be fixed and $F \in L_{\partial U}(\overline{U}, Y)$ be the constant map $F(x) = p$ for $x \in \overline{U}$. Then $F$ is essential.*

**Theorem 2.2** ([10, Chapter I, Theorem 2.6]). *Suppose:*

*(i)* $F, G : \overline{U} \to Y$ *are compact maps.*

*(ii)* $G \in L_{\partial U}(\overline{U}, Y)$ *is essential.*

*(iii)* $H(x,\lambda), \lambda \in [0,1]$, *is a compact homotopy joining $F$ and $G$, i.e.*

$$H(x,1) = F(x) \quad \textit{and} \quad H(x,0) = G(x).$$

*(iv)* $H(x,\lambda), \lambda \in [0,1]$, *is fixed point free on $\partial U$.*

*Then $H(x,\lambda), \lambda \in [0,1]$, has at least one fixed point in $U$ and in particular there is a $x_0 \in U$ such that $x_0 = F(x_0)$.*

Consider the BVP

$$x''' + a(t)x'' + b(t)x' + c(t)x = f(t, x, x', x''), \qquad t \in (0,1), \tag{2.1}$$

$$V_i(x) = r_i, \qquad i = 1,2,3, \tag{2.2}$$

where $a, b, c \in C([0,1], R), f : [0,1] \times D_x \times D_p \times D_q \to R$,

$$V_i(x) = \sum_{j=0}^{2} [a_{ij}x^{(j)}(0) + b_{ij}x^{(j)}(1)], \qquad i = 1,2,3,$$

with constants $a_{ij}$ and $b_{ij}$ such that $\sum_{j=0}^{2}(a_{ij}^2 + b_{ij}^2) > 0, i = 1,2,3$, and $r_i \in R$, $i = 1,2,3$.

Besides, for $\lambda \in [0,1]$ consider the family of BVPs for

$$x''' + a(t)x'' + b(t)x' + c(t)x = g(t, x, x', x'', \lambda), \qquad t \in (0,1), \tag{2.1$_\lambda$}$$

with boundary conditions (2.2), where the scalar function $g$ is defined on $[0,1] \times D_x \times D_p \times D_q \times [0,1]$, and $a, b, c$ are as above.

Finally, let $BC$ be the set of functions satisfying boundary conditions (2.2), $C^3_{BC}[0,1] = C^3[0,1] \cap BC$, $BC_0$ be the set of functions satisfying the homogeneous boundary conditions $V_i(x) = 0, i = 1,2,3$, and $C^3_{BC_0}[0,1] = C^3[0,1] \cap BC_0$.

We are now ready to state our basic existence result which is a variant of [10, Chapter I, Theorem 5.1 and Chapter V, Theorem 1.2].

**Theorem 2.3.** *Suppose:*

*(i) Problem* $(2.1)_0$, (2.2) *has a unique solution* $x_0 \in C^3[0,1]$.

*(ii) Problems* (2.1), (2.2) *and* $(2.1)_1$, (2.2) *are equivalent.*

*(iii) The map* $\mathbf{L}_h : C^3_{BC_0}[0,1] \to C[0,1]$ *is one-to-one: here,*

$$\mathbf{L}_h x = x''' + a(t)x'' + b(t)x' + c(t)x.$$

*(iv) Each solution* $x \in C^3[0,1]$ *to family* $(2.1)_\lambda$, (2.2) *satisfies the bounds*

$$m_i \le x^{(i)} \le M_i \quad \text{for } t \in [0,1],\ i = \overline{0,3},$$

*where the constants* $-\infty < m_i, M_i < \infty,\ i = \overline{0,3}$, *are independent of* $\lambda$ *and* $x$.

*(v) There is a sufficiently small* $\sigma > 0$ *such that*

$$[m_0 - \sigma, M_0 + \sigma] \subseteq D_x, \qquad [m_1 - \sigma, M_1 + \sigma] \subseteq D_p, \qquad [m_2 - \sigma, M_2 + \sigma] \subseteq D_q,$$

*and* $g(t, x, p, q, \lambda)$ *is continuous for* $(t, x, p, q, \lambda) \in [0,1] \times J \times [0,1]$; $m_i, M_i, i = \overline{0,3}$, *are as in (iv).*

*Then boundary value problem* (2.1), (2.2) *has at least one solution in* $C^3[0,1]$.

*Proof.* For a start, introduce the set

$$\overline{U} = \left\{ x \in C^3_{BC}[0,1] : m_i - \sigma \le x^{(i)} \le M_i + \sigma,\ i = \overline{0,3},\ \text{on } [0,1] \right\}$$

and define the maps

$$j : C^3_{BC}[0,1] \to C^2[0,1] \qquad \text{by } jx = x,$$
$$\mathbf{L} : C^3_{BC}[0,1] \to C[0,1] \qquad \text{by } \mathbf{L}x = x''' + a(t)x'' + b(t)x' + c(t)x,$$

and for $\lambda \in [0,1]$

$$\Phi_\lambda : C^2[0,1] \to C[0,1] \quad \text{by } \Phi_\lambda x = g(t, x, x', x'', \lambda),\ x \in j(\overline{U}).$$

Our first task is to establish that $\mathbf{L}^{-1} : C[0,1] \to C^3_{BC}[0,1]$ exists and is continuous. Therefore, we use (iii) which implies that for each $y \in C[0,1]$ the BVP

$$x''' + a(t)x'' + b(t)x' + c(t)x = y(t),$$
$$V_i(x) = 0, \qquad i = 1,2,3,$$

has a unique $C^3[0,1]$-solution of the form

$$x(t) = C_1^* x_1(t) + C_2^* x_2(t) + C_3^* x_3(t) + \eta(t),$$

where $x_i(t), i = 1,2,3$, are linearly independent solutions to the homogeneous equation

$$x''' + a(t)x'' + b(t)x' + c(t)x = 0, \tag{2.3}$$

$\eta(t)$ is a solution to the inhomogeneous equation, and $(C_1^*, C_2^*, C_3^*)$ is the unique solution to the system

$$C_1 V_i(x_1) + C_2 V_i(x_2) + C_3 V_i(x_3) = -V_i(\eta), \qquad i = 1,2,3.$$

The last means that $\det[V_i(x_j)] \neq 0$ and so the system

$$C_1 V_i(x_1) + C_2 V_i(x_2) + C_3 V_i(x_3) = r_i, \qquad i = 1,2,3,$$

also has a unique solution $(\overline{C}_1, \overline{C}_2, \overline{C}_3)$. Then,

$$l(t) = \overline{C}_1 x_1(t) + \overline{C}_2 x_2(t) + \overline{C}_3 x_3(t)$$

is the unique $C^3[0,1]$-solution to the homogeneous equation (2.3) satisfying the inhomogeneous boundary conditions

$$V_i(x) = r_i, i = 1,2,3.$$

As a result, conclude that $\mathbf{L}^{-1}$ exists and $\mathbf{L}^{-1}y = \mathbf{L}_h^{-1}y + l$ for each $y \in C[0,1]$. To show that $\mathbf{L}^{-1}$ is continuous observe that $\mathbf{L}_h$ is bounded because

$$\begin{aligned}
\|\mathbf{L}_h x\|_{C[0,1]} &\le \|x'''\|_{C[0,1]} + S_2\|x''\|_{C[0,1]} + S_1\|x'\|_{C[0,1]} + S_0\|x\|_{C[0,1]} \\
&\le \|x\|_{C^3[0,1]} + S_2\|x\|_{C^3[0,1]} + S_1\|x\|_{C^3[0,1]} + S_0\|x\|_{C^3[0,1]} \\
&\le (1 + S_2 + S_1 + S_0)\|x\|_{C^3[0,1]},
\end{aligned}$$

where $S_2 = \max_{[0,1]} |a(t)|, S_1 = \max_{[0,1]} |b(t)|, S_0 = \max_{[0,1]} |c(t)|$. Thus, the linear map $\mathbf{L}_h$ is continuous. Then, $\mathbf{L}_h^{-1}$ is continuous and so $\mathbf{L}^{-1}$ is also continuous.

Now, introduce the homotopy $H_\lambda : \overline{U} \times [0,1] \to C^3_{BC}[0,1]$ defined by $H_\lambda = \mathbf{L}^{-1}\Phi_\lambda j$. The map $j$ is a completely continuous embedding and $U$ is a bounded set, hence the set $j(\overline{U})$ is compact. The set $\Phi_\lambda(j(\overline{U})), \lambda \in [0,1]$, is also compact since the map $\Phi_\lambda$ is continuous on $j(\overline{U})$ in view of (v). Finally, because of the continuity of $\mathbf{L}^{-1}$ proved above, the set $\mathbf{L}^{-1}(\Phi_\lambda(j(\overline{U}))), \lambda \in [0,1]$, is compact. Thus, the homotopy is compact. For its fixed points we have

$$x = \mathbf{L}^{-1}\Phi\lambda j x$$

and

$$\mathbf{L}x = \Phi\lambda j x$$

which means that the fixed points of $H_\lambda$ are precisely the solutions of family $(2.1)_\lambda$, (2.2) and in view of (iv) we conclude that the homotopy is fixed point free on the boundary of $U$. Using (i), we see that $H_0 = x_0, x_0 \in U$, is essential by Theorem 2.1. Then, $H_1$ is also essential by Theorem 2.2 and so it has a fixed point, that is, $(2.1)_\lambda$, (2.2) has a solution in $C^3[0,1]$ when $\lambda = 1$, and, by (ii), problem (2.1), (2.2) has a solution in $C^3[0,1]$. □

## 3 Auxiliary results

The results stated in this part guarantee the bounds from (iv) of Theorem 2.3.

**Lemma 3.1.** *Let* $(\mathbf{H}_1)$ *hold. Then every solution* $x \in C^3[0,1]$ *to a BVP for* $(1.1)_\lambda$ *with one of the boundary conditions (1.k),* $k = \overline{2,6}$*, satisfies the bounds*

$$F_1 \leq x''(t) \leq L_1 \quad on\ [0,1].$$

*Proof.* Assume on the contrary that $x''(t) > L_1$ for some $t \in (0,1]$. Then, the continuity of $x''(t)$ on $[0,1]$ together with $x''(0) \leq L_1$ implies that the set

$$S_- = \{t \in [0,1] : L_1 < x''(t) \leq L_2\}$$

is not empty and there is a $\gamma \in S_-$ such that

$$x'''(\gamma) > 0.$$

On the other hand, since $x(t)$ is a $C^3[0,1]$-solution to $(1.1)_\lambda$, we have in particular

$$x'''(\gamma) = \lambda f(\gamma, x(\gamma), x'(\gamma), x''(\gamma)).$$

Now, from $(\gamma, x(\gamma), x'(\gamma), x''(\gamma)) \in S_- \times R^2 \times (L_1, L_2]$ and (1.9) it follows

$$x'''(\gamma) \leq 0,$$

a contradiction. Thus,

$$x''(t) \leq L_1 \quad \text{for } t \in [0,1].$$

In an analogous way, using (1.10), we can prove that

$$F_1 \leq x''(t) \quad \text{for } t \in [0,1].$$

□

**Lemma 3.2.** *Let* $(\mathbf{H}_1)$ *hold. Then every solution* $x \in C^3[0,1]$ *to a BVP for* $(1.1)_\lambda$ *with one of the boundary conditions* (1.$k$), $k = \overline{2,5}$*, satisfies the bounds*

$$\begin{aligned} |x(t)| &\leq |A| + |B| + \max\{|F_1|, |L_1|\}, && t \in [0,1], \\ |x'(t)| &\leq |B| + \max\{|F_1|, |L_1|\}, && t \in [0,1]. \end{aligned} \tag{3.1}$$

*Proof.* Let firstly the solution satisfies $x'(0) = B$. Then, by the mean value theorem, for each $t \in (0,1]$ there is a $\xi \in (0,t)$ such that

$$x'(t) - x'(0) = x''(\xi)t$$

from where, using Lemma 3.1, derive (3.1). If $x'(1) = B$, we obtain similarly that for each $t \in [0,1)$ there is a $\eta \in (t,1)$ with the property

$$x'(1) - x'(t) = x''(\eta)(1-t),$$

which implies (3.1).

Using again the mean value theorem and (3.1), we get the bound for $|x(t)|$ in both cases $x(1) = C$ and $x(0) = C$. □

**Lemma 3.3.** *Let $A, B \le 0, C \ge 0$ and* **($H_1$)** *hold with $L_1 \le 0$. Then each solution $x \in C^3[0,1]$ to* (1.1)$_\lambda$, (1.2) *satisfies the bounds*

$$\begin{aligned} C \le x(t) &\le C - B - F_1, && t \in [0,1], \\ B + F_1 \le x'(t) &\le B, && t \in [0,1]. \end{aligned} \tag{3.2}$$

*Proof.* From Lemma 3.1 we know that

$$F_1 \le x''(t) \le L_1 \le 0 \quad \text{on } [0,1].$$

Then, for $t \in (0,1]$ we get

$$\int_0^t F_1 ds \le \int_0^t x''(s) ds \le \int_0^t L_1 ds,$$

which yields consecutively $F_1 t \le x'(t) - B \le L_1 t, t \in [0,1]$, and $F_1 \le x'(t) - B \le 0$, $t \in [0,1]$, from where (3.2) follows. Similarly, integrating (3.2) from $t \in [0,1)$ to 1 we get

$$(B + F_1)(1-t) \le x(1) - x(t) \le B(1-t), \qquad t \in [0,1],$$

which implies the bounds for $x(t)$. □

Using similar arguments to those in the proof of Lemma 3.3, we can also show that the following three auxiliary results are held.

**Lemma 3.4.** *Let $A, B, C \ge 0$ and* **($H_1$)** *hold with $F_1 \ge 0$. Then each solution $x \in C^3[0,1]$ to* (1.1)$_\lambda$, (1.3) *satisfies the bounds*

$$\begin{aligned} C \le x(t) &\le B + C + L_1, && t \in [0,1], \\ B \le x'(t) &\le B + L_1, && t \in [0,1]. \end{aligned}$$

**Lemma 3.5.** *Let $A, C \ge 0, B \le 0$ and* **($H_1$)** *hold with $F_1 \ge 0$. Then each solution $x \in C^3[0,1]$ to* (1.1)$_\lambda$, (1.4) *satisfies the bounds*

$$\begin{aligned} C \le x(t) &\le C - B + L_1, && t \in [0,1], \\ B - L_1 \le x'(t) &\le B, && t \in [0,1]. \end{aligned}$$

**Lemma 3.6.** *Let $A \le 0$, $B, C \ge 0$ and* **($H_1$)** *hold with $L_1 \le 0$. Then each solution $x \in C^3[0,1]$ to* (1.1)$_\lambda$, (1.5) *satisfies the bounds*

$$\begin{aligned} C \le x(t) &\le B + C - F_1, && t \in [0,1], \\ B \le x'(t) &\le B - F_1, && t \in [0,1]. \end{aligned}$$

**Lemma 3.7.** *Let* **($H_1$)** *hold. Then each solution $x \in C^3[0,1]$ to* (1.1)$_\lambda$, (1.6) *satisfies the bounds*

$$\begin{aligned} |x(t)| &\le |B| + |C - B| + \max\{|F_1|, |L_1|\}, && t \in [0,1], \\ |x'(t)| &\le |C - B| + \max\{|F_1|, |L_1|\}, && t \in [0,1]. \end{aligned}$$

*Proof.* It is clear, there is a $\mu \in (0,1)$ with the property $x'(\mu) = C - B$. Then, for each $t \in [0, \mu)$ there is a $\xi \in (t, \mu)$ such that

$$x'(\mu) - x'(t) = x''(\xi)(\mu - t),$$

which yields

$$|x'(t)| \le |C-B| + \max\{|F_1|, |L_1|\}, \qquad t \in [0,\mu].$$

Similarly establish that the same bound is valid for $t \in [\mu, 1]$. Using again the mean value theorem, we obtain that for each $t \in (0,1]$ and some $\eta \in (0,t)$ we have

$$x(t) - x(0) = x'(\eta)t.$$

This together with the obtained bound for $|x'(t)|$ gives the bound for $|x(t)|$. □

**Lemma 3.8.** *Let $A \le 0, B, C \ge 0$ and* **($H_1$)** *hold with $L_1 \le 0$. Then each solution $x \in C^3[0,1]$ to* (1.1)$_\lambda$, (1.6) *satisfies the bounds*

$$\begin{aligned} \min\{B,C\} \le x(t) \le B + |C-B| + |F_1|, \qquad & t \in [0,1], \\ C - B + F_1 \le x'(t) \le C - B - F_1, \qquad & t \in [0,1]. \end{aligned}$$

*Proof.* By Lemma 3.1, $F_1 \le x''(t) \le L_1$ on $[0,1]$. Clearly, $x'(\mu) = C - B$ for some $\mu \in (0,1)$. Then,

$$\int_t^\mu F_1 ds \le \int_t^\mu x''(s) ds \le \int_t^\mu L_1 ds, \qquad t \in [0,\mu),$$

gives

$$C - B \le x'(t) \le C - B - F_1, \qquad t \in [0,\mu],$$

and

$$\int_\mu^t F_1 ds \le \int_\mu^t x''(s) ds \le \int_\mu^t L_1 ds, \qquad t \in (\mu,1],$$

implies

$$C - B + F_1 \le x'(t) \le C - B, \qquad t \in [\mu,1].$$

As a result,

$$C - B + F_1 \le x'(t) \le C - B - F_1, \qquad t \in [0,1].$$

Using Lemma 3.7, conclude

$$|x(t)| \le B + |C-B| + |F_1| \quad \text{for } t \in [0,1].$$

But, $x(t)$ is concave on $[0,1]$ because $x''(t) \le L_1 \le 0$ for $t \in [0,1]$. This fact together with $B, C \ge 0$ means that $x(t) \ge \min\{B,C\}$ on $[0,1]$, which completes the proof. □

## 4 Problems (1.1), (1.2)–(1.5)

**Theorem 4.1.** *Let* **($H_1$)** *hold and* **($H_2$)** *hold for*

$$\begin{aligned} M_0 = |A| + |B| + \max\{|F_1|, |L_1|\}, \qquad & m_0 = -M_0, \\ M_1 = |B| + \max\{|F_1|, |L_1|\}, \qquad & m_1 = -M_1, m_2 = F_1, M_2 = L_1. \end{aligned}$$

*Then each BVP for equation* (1.1) *with one of the boundary conditions* (1.$k$), $k = \overline{2,5}$, *has at least one solution in* $C^3[0,1]$.

*Proof.* We will show that each BVP for (1.1)$_\lambda$, $\lambda \in [0,1]$, with one of the boundary conditions (1.$k$), $k = \overline{2,5}$, satisfies all hypotheses of Theorem 2.3. It is not hard to check that (i) holds for each BVP for (1.1)$_0$ with one of the boundary conditions (1.$k$), $k = \overline{2,5}$. Obviously, each BVP for (1.1) is equivalent to the BVP for (1.1)$_1$ with the same boundary conditions, that is, (ii) is satisfied. Because now $\mathbf{L}_h = x'''$, (iii) also holds. Further, for each solution $x(t) \in C^3[0,1]$ to a BVP for (1.1)$_\lambda$, $\lambda \in [0,1]$, with one of the boundary conditions (1.$k$), $k = \overline{2,5}$, we have

$$m_i \le x^{(i)}(t) \le M_i, \quad t \in [0,1], \ i = 0,1, \quad \text{by Lemma 3.2},$$

$$m_2 \le x''(t) \le M_2, \quad t \in [0,1], \quad \text{by Lemma 3.1}.$$

Because of the continuity of $f$ on $[0,1] \times J$ there are constants $m_3$ and $M_3$ such that

$$m_3 \le \lambda f(t,x,p,q) \le M_3 \quad \text{for } \lambda \in [0,1] \text{ and } (t,x,p,q) \in [0,1] \times J.$$

Since $(x(t), x'(t), x''(t)) \in J$ for $t \in [0,1]$, the equation (1.1)$_\lambda$ implies

$$m_3 \le x'''(t) \le M_3, \qquad t \in [0,1].$$

Hence, (iv) also holds. Finally, (v) follows from the continuity of $f$ on the set $J$. So, we can apply Theorem 2.3 to conclude that the assertion is true. □

The following results guarantee $C^3[0,1]$-solutions with important properties.

**Theorem 4.2.** *Let $A \le 0, B < 0, C > 0$ $(B = C = 0)$. Suppose* **(H$_1$)** *holds with $L_1 \le 0$ and* **(H$_2$)** *holds for*

$$m_0 = C, M_0 = C - B - F_1, m_1 = B + F_1, M_1 = B, m_2 = F_1, M_2 = L_1.$$

*Then BVP (1.1), (1.2) has at least one positive, decreasing (non-negative, non-increasing), concave solution in $C^3[0,1]$.*

*Proof.* Following the proof of Theorem 4.1, we establish that (1.1), (1.2) has a solution $x(t) \in C^3[0,1]$. Now, the bounds

$$m_0 \le x^{(i)}(t) \le M_0, \qquad t \in [0,1], \ i = 0,1,2,$$

follow from Lemmas 3.3 and 3.1. These lemmas imply in particular $x(t) \ge C > 0$, $x'(t) \le B < 0$ $(x(t) \ge 0,\ x'(t) \le 0)$ and $x''(t) \le L_1 \le 0$ for $t \in [0,1]$, which yields the assertion. □

**Theorem 4.3.** *Let $A \ge 0,\ B > 0,\ C > 0,\ (B = C = 0)$. Suppose* **(H$_1$)** *holds with $F_1 \ge 0$ and* **(H$_2$)** *holds for*

$$m_0 = C, \quad M_0 = B + C + L_1, \quad m_1 = B, \quad M_1 = B + L_1, \quad m_2 = F_1, \quad M_2 = L_1.$$

*Then BVP (1.1), (1.3) has at least one positive, increasing (non-negative, non-decreasing), convex solution in $C^3[0,1]$.*

*Proof.* Using Lemmas 3.4 and 3.1, as in the proof of Theorem 4.1 we establish that the considered problem has a solution $x(t) \in C^3[0,1]$. Now, for $t \in [0,1]$ we have $x(t) \ge C > 0$, $x'(t) \ge B > 0$ $(x(t) \ge 0,\ x'(t) \ge 0)$, by Lemma 3.4, and $x''(t) \ge F_1 \ge 0$, by Lemma 3.1, from where it follows that $x(t)$ has the desired properties. □

**Theorem 4.4.** *Let* $A \geq 0,\ B < 0,\ C > 0,\ (B = C = 0)$. *Suppose* $\mathbf{(H_1)}$ *holds with* $F_1 \geq 0$ *and* $\mathbf{(H_2)}$ *holds for*

$$m_0 = C, \quad M_0 = C - B + L_1, \quad m_1 = B - L_1, \quad M_1 = B, \quad m_2 = F_1, \quad M_2 = L_1.$$

*Then BVP* (1.1), (1.4) *has at least one positive, decreasing (non-negative, non-increasing), convex solution in* $C^3[0,1]$.

*Proof.* Following again the proof of Theorem 4.1 and using Lemmas 3.5 and 3.1, we establish that there is a solution $x(t) \in C^3[0,1]$ to (1.1), (1.4). In fact, from Lemma 3.5 we know that $x(t) \geq C > 0,\ x'(t) \leq B < 0\,(x(t) \geq 0,\ x'(t) \leq 0), t \in [0,1]$, and from Lemma 3.1 have $x''(t) \geq F_1 \geq 0, t \in [0,1]$, which completes the proof. □

**Theorem 4.5.** *Let* $A \leq 0, B > 0, C > 0\,(B = C = 0)$. *Suppose* $\mathbf{(H_1)}$ *holds with* $L_1 \leq 0$ *and* $\mathbf{(H_2)}$ *holds for*

$$m_0 = C, \qquad M_0 = B + C - F_1, \qquad m_1 = B, \qquad M_1 = B - F_1, \qquad m_2 = F_1, \qquad M_2 = L_1.$$

*Then BVP* (1.1), (1.5) *has at least one positive, increasing (non-negative, non-decreasing), concave solution in* $C^3[0,1]$.

*Proof.* Following again the proof of Theorem 4.1 and using Lemmas 3.6 and 3.1, we establish that (1.1), (1.5) has a solution $x(t) \in C^3[0,1]$. From these lemmas we know that $x(t) \geq C > 0$, $x'(t) \geq B > 0\,(x(t) \geq 0, x'(t) \geq 0)$ and $x''(t) \leq L_1 \leq 0$ for $t \in [0,1]$, which completes the proof. □

We will illustrate the application of the obtained results.

**Example 4.6.** Consider the BVPs for equations of the form

$$x'''(t) = P_n(x''), \qquad t \in (0,1), \tag{4.1}$$

with one of the boundary conditions $(1.k)$, $k = \overline{2,5}$, where the polynomial $P_n(q)$, $n \geq 2$, has simple zeros $q_1$ and $q_2$ such that $q_1 > A > q_2$.

Fix some $\theta > 0$ with the properties $q_1 - \theta \geq A \geq q_2 + \theta$ and

$$P_n(q) \neq 0 \quad \text{on } (q_i - \theta, q_i + \theta) \setminus q_i,\ i = 1,2.$$

Consider the case

$$P_n(q) < 0 \quad \text{for } q \in (q_1, q_1 + \theta] \quad \text{and} \quad P_n(q) > 0 \quad \text{for } q \in [q_2 - \theta, q_2);$$

the other cases for the sign of $P_n(q)$ around the zeros can be studied by analogy. In this case, if we choose, for example, $F_2 = q_2 - \theta, F_1 = q_2, L_1 = q_1, L_2 = q_1 + \theta$ and $\sigma = \theta/2$, $\mathbf{(H_1)}$ and $\mathbf{(H_2)}$ hold and so each BVP for (4.1) with one of the boundary conditions $(1.k)$, $k = \overline{2,5}$, has a solution in $C^3[0,1]$ by Theorem 4.1.

**Example 4.7.** Consider the BVP

$$x'''(t) = \frac{t(2 - x'')\sqrt{625 - x'^2}}{\sqrt{900 - x^2}\sqrt{100 - x''^2}}, \quad t \in (0,1),$$
$$x''(0) = 3, \quad x'(1) = -1, \quad x(1) = 2.$$

It is not hard to see that if, for example, $F_2 = 0, F_1 = 1, L_1 = 4, L_2 = 5$ and $\sigma = 0.1$ this problem has a positive, decreasing, convex solution in $C^3[0,1]$ by Theorem 4.4; notice, here $J$ is bounded.

**Example 4.8.** Consider the BVP

$$x'''(t) = (x'' + 5)(x'' - 1)\sqrt{400 - x'^2}, \quad t \in (0,1),$$
$$x''(0) = -4, \quad x'(1) = 1, \quad x(0) = 2.$$

The assumptions of Theorem 4.5 are satisfied for $F_2 = -7, F_1 = -6$, $L_1 = -2$, $L_2 = -1$ and $\sigma = 0.1$, for example. Thus, the considered problem has a positive, increasing, concave solution in $C^3[0,1]$ by Theorem 4.5.

## 5 Problem (1.1), (1.6)

**Theorem 5.1.** *Let* **(H$_1$)** *hold and* **(H$_2$)** *hold for*

$$M_0 = |B| + |C - B| + \max\{|F_1|, |L_1|\}, \qquad m_0 = -M_0,$$
$$M_1 = |C - B| + \max\{|F_1|, |L_1|\}, \qquad m_1 = -M_1, m_2 = F_1, M_2 = L_1.$$

*Then BVP* (1.1), (1.6) *has at least one solution in* $C^3[0,1]$.

*Proof.* As in the proof of Theorem 4.1, we check that family $(1.1)_\lambda$, (1.6) and BVP (1.1), (1.6) satisfy all hypotheses of Theorem 2.3 and so the assertion is true. Moreover, now each $C^3[0,1]$-solution $x(t)$ to $(1.1)_\lambda$, (1.6) satisfies the bounds

$$m_0 \le x(t) \le M_0 \quad \text{on } [0,1], \quad \text{by Lemma 3.7},$$
$$m_1 \le x'(t) \le M_1 \quad \text{on } [0,1], \quad \text{by Lemma 3.7},$$
$$m_2 \le x''(t) \le M_2 \quad \text{on } [0,1], \quad \text{by Lemma 3.1}.$$

□

**Theorem 5.2.** *Let* $A \le 0$, $B, C > 0\,(B, C = 0)$. *Suppose* **(H$_1$)** *holds with* $L_1 \le 0$, *and* **(H$_2$)** *holds for*

$$m_0 = \min\{B, C\}, \qquad M_0 = B + |C - B| - F_1,$$
$$m_1 = C - B + F_1, \qquad M_1 = C - B - F_1, \qquad m_2 = F_1, \qquad M_2 = L_1.$$

*Then BVP* (1.1), (1.6) *has at least one positive (non-negative), concave solution in* $C^3[0,1]$.

*Proof.* Following the proof of Theorem 4.1 and using Lemmas 3.8 and 3.1, we establish that there is a solution $x(t) \in C^3[0,1]$ to (1.1), (1.6). In fact, from Lemmas 3.8 and 3.1 we know that $x(t) \ge \min\{B, C\} > 0$ $(x(t) \ge 0)$ and $x''(t) \le L_1 \le 0$ for $t \in [0,1]$, which completes the proof. □

**Corollary 5.3.** *Let* $A \le 0, C > B > 0$. *Suppose* **(H$_1$)** *holds with* $L_1 \le 0$ *and* $F_1 > B - C$ $(F_1 = B - C)$, *and* **(H$_2$)** *holds for* $m_i, M_i, i = 0, 1, 2$, *as in Theorem* 5.2. *Then BVP* (1.1), (1.6) *has at least one positive, increasing (non-decreasing), concave solution in* $C^3[0,1]$.

*Proof.* By Theorem 5.2, (1.1), (1.6) has a positive, concave solution $x(t) \in C^3[0,1]$. Moreover, Lemma 3.8 implies $x'(t) \ge C - B + F_1 > 0\,(x'(t) \ge 0)$ for $t \in [0,1]$, which completes the proof. □

**Corollary 5.4.** *Let* $A \le 0, B = C > 0\,(B = C = 0)$. *Suppose* **(H$_1$)** *holds with* $L_1 \le 0$, *and* **(H$_2$)** *holds for* $m_i, M_i, i = 0, 1, 2$, *as in Theorem* 5.2. *Then BVP* (1.1), (1.6) *has at least one positive (non-negative), concave solution* $x(t) \in C^3[0,1]$ *for which there is a* $\mu \in (0,1)$ *with the property* $x(\mu) = \max_{[0,1]} x(t)$.

*Proof.* A positive (non-negative), concave solution $x(t) \in C^3[0,1]$ exists by Theorem 5.2. By the mean value theorem there is a $\mu \in (0,1)$ such that $x'(\mu) = C - B = 0$, which yields the assertion. □

**Example 5.5.** Consider the BVP

$$x'''(t) = -(x'' + 3)\sqrt{900 - x^2}, \qquad t \in (0,1),$$
$$x''(0) = -4, \qquad x(0) = 1, \qquad x(1) = 9.$$

The assumptions of Corollary 5.3 are satisfied for $F_2 = -7, F_1 = -6$, $L_1 = -1$, $L_2 = -2$ and $\sigma = 0.1$, for example. Thus, the considered problem has a positive, increasing, concave solution in $C^3[0,1]$.

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