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Infinitely many solutions for fractional Kirchhoff–Sobolev–Hardy critical problems

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Abstract. We investigate a class of critical stationary Kirchhoff fractional p-Laplacian problems in presence of a Hardy potential. By using a suitable version of the symmetric mountain-pass lemma due to Kajikiya, we obtain the existence of a sequence of infinitely many arbitrarily small solutions converging to zero.

Keywords: fractional p-Laplacian, Kirchhoff coefficient, Hardy potentials, critical Sobolev exponent, variational methods.

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1 Introduction

In this paper we consider the following fractional problem

$$\begin{cases} M([u]_{s,p}^p)(-\Delta)_p^s u - \gamma \dfrac{|u|^{p-2}u}{|x|^{sp}} = \lambda w(x)|u|^{q-2}u + \dfrac{|u|^{p_s^*(\alpha)-2}u}{|x|^\alpha}, & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $0 < s < 1 < p < \infty$, $0 \le \alpha < sp < N$, $1 < q < p$, $p_s^*(\alpha) = \frac{p(N-\alpha)}{N-sp} \le p_s^*(0) = p_s^*$ is the critical Hardy–Sobolev exponent, γ and λ are real parameters, w is a positive weight whose assumption will be introduced in the sequel and $\Omega \subseteq \mathbb{R}^N$ is a general open set. Naturally, the condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$ disappears when $\Omega = \mathbb{R}^N$.

Here $(-\Delta)_p^s$ denotes the fractional p-Laplace operator which, up to normalization factors, may be defined by the Riesz potential as

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} dy, \quad x \in \mathbb{R}^N,$$

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along any $u \in C_0^\infty(\mathbb{R}^N)$, where $B_\varepsilon(x) = \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$. See [11,23] and the references therein for further details on the fractional Sobolev space $W^{s,p}(\Omega)$ and some recent results on the fractional p-Laplacian.

Problem (1.1) is fairly delicate due to the intrinsic lack of compactness, which arise from the Hardy term and the nonlinearity with critical exponent $p_s^*(\alpha)$. For this reason, we strongly need that the Kirchhoff coefficient M is non–degenerate, namely $M(t) > 0$ for any $t \geq 0$. Hence, along the paper, we suppose that *the Kirchhoff function* $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is continuous and satisfies*

(M_1) $\inf_{t\in\mathbb{R}_0^+} M(t) = a > 0$;

(M_2) *there exists* $\theta \in [1, p_s^*(\alpha)/p)$*, such that* $M(t)t \leq \theta \mathscr{M}(t)$ *for all* $t \in \mathbb{R}_0^+$*, where* $\mathscr{M}(t) = \int_0^t M(\tau)d\tau$.

Concerning the *positive* weight w, we assume that

(w) $w(x)|x|^{\frac{q\alpha}{p_s^*(\alpha)}} \in L^r(\mathbb{R}^N)$*, with* $r = \frac{p_s^*(\alpha)}{p_s^*(\alpha)-q}$.

Condition (w) is necessary, since it guaranties that the embedding $Z(\Omega) \hookrightarrow L^q(\Omega, w)$ is compact, even when Ω is the entire space $\mathbb{R}^N$. Indeed, the natural solution space for problem (1.1) is the fractional density space $Z(\Omega)$, that is the closure of $C_0^\infty(\Omega)$ with respect to the norm $[\,\cdot\,]_{s,p}$, given by

$$[u]_{s,p} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{1/p}.$$

Thus, by arguing similarly to Lemma 4.1 of [15], we have that the embedding $Z(\Omega) \hookrightarrow L^q(\Omega, w)$ is compact with

$$\|u\|_{q,w} \leq C_w [u]_{s,p} \quad \text{for any } u \in Z(\Omega), \tag{1.2}$$

where the weighted norm is set by

$$\|u\|_{q,w} = \left(\int_\Omega w(x)|u(x)|^q dx \right)^{1/q}$$

and $C_w = H_\alpha^{-1/p} \big(\int_{\mathbb{R}^N} w^r(x)|x|^{\frac{q\alpha}{p_s^*(\alpha)-q}} dx \big)^{1/qr}$ is a positive constant. Here $H_\alpha = H(N,p,s,\alpha)$ denotes the best fractional critical Hardy–Sobolev constant, given by

$$H_\alpha = \inf_{u\in Z(\Omega)\setminus\{0\}} \frac{[u]_{s,p}^p}{\|u\|_{H_\alpha}^p}, \qquad \|u\|_{H_\alpha}^{p_s^*(\alpha)} = \int_\Omega |u(x)|^{p_s^*(\alpha)} \frac{dx}{|x|^\alpha}. \tag{1.3}$$

Of course number H_α is well–defined and strictly positive for any $\alpha \in [0, ps]$, since Lemma 2.1 of [15]. We observe that when $\alpha = 0$ then H_0 coincides with the critical Sobolev constant, while when $\alpha = sp$ then H_{sp} is the true critical Hardy constant. In order to simplify the notation, throughout the paper we denote the true fractional Hardy constant and the true fractional Hardy norm with $H = H_{sp}$ and $\|\cdot\|_H = \|\cdot\|_{H_{sp}}$, in (1.3) when $\alpha = sp$.

When $s = 1$ and $p = 2$, our problem (1.1) is related to the celebrated Kirchhoff equation

$$\rho\, u_{tt} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L |u_x|^2 dx \right) u_{xx} = 0, \tag{1.4}$$

proposed by Kirchhoff [21] in 1883 as a nonlinear generalization of D'Alembert's wave equation for free vibrations of elastic strings. This model describes a vibrating string, taking into account the changes in the length of the string during vibrations. In the equation (1.4) $u = u(x,t)$ is the transverse string displacement at the space coordinate x and time t, L is the length of the string, h is the area of the cross section, E is Young's modulus of the material, ρ is the mass density, and P_0 is the initial tension. The early studies devoted to the Kirchhoff model were given by Bernstein [6], Lions [22] and Pohozaev [26].

In the nonlocal setting, Fiscella and Valdinoci [17] proposed a stationary Kirchhoff variational model in smooth bounded domains of $\mathbb{R}^N$, which takes into account the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string, given by Caffarelli et al. in [8]. In other words, the problem studied in [17] is the fractional version of the Kirchhoff equation (1.4). Starting from [17], a great attention has been devoted to the study of fractional Kirchhoff problems; see for example [1–3, 9, 13–16, 24, 27].

The true local version of problem (1.1), namely when $M \equiv 1$ and $s = 1$, given by

$$\begin{cases} -\Delta_p u - \gamma \dfrac{|u|^{p-2}u}{|x|^p} = \lambda w(x)|u|^{q-2}u + \dfrac{|u|^{p^*(\alpha)-2}u}{|x|^\alpha}, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

has been widely studied in [10, 12, 18, 19]. In these works, the authors proved the existence of infinitely many solutions of (1.5), when the parameter λ is controlled by a suitable threshold depending on the following Sobolev–Hardy constant

$$S_\gamma = \inf_{W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \left(|\nabla u(x)|^p - \gamma \frac{|u(x)|^p}{|x|^p} \right) dx}{\left(\displaystyle\int_\Omega \frac{|u(x)|^{p^*(\alpha)}}{|x|^\alpha} dx \right)^{\frac{p}{p^*(\alpha)}}}.$$

In order to overcome the lack of compactness, due to the presence of two Hardy potentials in (1.5), they exploit a concentration compactness principle, applied to the combined norm $\int_\Omega \left(|\nabla u|^p - \gamma \frac{|u|^p}{|x|^p} \right) dx$ and to the critical norm $\int_\Omega \frac{|u|^{p^*(\alpha)}}{|x|^\alpha} dx$. Because of the bi–nonlocal nature of the problem (1.1), the same approach of [10, 12, 18, 19] can not work in our case. Indeed, due to the presence of a Kirchhoff coefficient M, for which the equation in (1.1) is no longer a pointwise identity, we have difficulties in considering a combined norm. Since Ω could be unbounded, we can not apply a concentration compactness argument because of the nonlocal nature of $(-\Delta)_s^p$, as well explained in Section 2.3 of [25]. For these reasons, we use a tricky analysis of the energy functional which allows us to handle the two Hardy potentials in (1.1); see Sections 2 and 3.

Thus, we get the next multiplicity result for (1.1), which involves the main geometrical parameter $\kappa_\sigma = \kappa(\sigma)$ defined by

$$\kappa_\sigma = \frac{a(\sigma - \theta p)}{\theta(\sigma - p)}, \tag{1.6}$$

for any $\sigma \in (p\theta, p_s^*(\alpha))$. A parameter similar to (1.6) already appeared in [9]. Clearly $\kappa_\sigma \le a$, since $\theta \ge 1$ and $p\theta \le \sigma$. When $\theta = 1$ in (M_2), we observe that parameter $\kappa_\sigma = a$ does not depend by the choice of σ. As shown in Section 2 of [9], the situation $\theta = 1$ holds true in other cases, besides the obvious one $M \equiv a$.

Now, we are ready to state the main result of the present paper.

Theorem 1.1. *Let $N > ps > \alpha \geq 0$, $q \in (1, p)$, with $s \in (0,1)$ and $p \in (1, \infty)$. Assume that M and w satisfy assumptions (M_1)–(M_2) and (w).*

Then, for any $\sigma \in (p\theta, p_s^*(\alpha))$ and for any $\gamma \in (-\infty, \kappa_\sigma H)$, there exists $\bar{\lambda} = \bar{\lambda}(\sigma, \gamma) > 0$ such that for any $\lambda \in (0, \bar{\lambda})$ problem (1.1) admits a sequence of solutions $\{u_n\}_n$ in $Z(\Omega)$ with the energy functional $\mathcal{J}_{\gamma,\lambda}(u_n) < 0$, $\mathcal{J}_{\gamma,\lambda}(u_n) \to 0$ and $\{u_n\}_n$ converges to zero as $n \to \infty$.

The proof of Theorem 1.1 is obtained by applying suitable variational methods and consists of several steps. In Section 2 we study the compactness property of the Euler-Lagrange functional associated with (1.1). After that, in Section 3, we introduce a truncated functional which allows us to apply the symmetric mountain pass lemma in [20]. Finally, we prove that the critical points of the truncated functional are indeed solutions of the original problem (1.1).

2 The Palais–Smale condition

Throughout the paper we assume that $N > ps > \alpha \geq 0$, $s \in (0,1)$, $p \in (1, \infty)$, $q \in (1, p)$, (M_1)–(M_2) and (w), without further mentioning.

According to the variational nature, (weak) solutions of (1.1) correspond to critical points of the Euler–Lagrange functional $\mathcal{J}_{\gamma,\lambda} : Z(\Omega) \to \mathbb{R}$, defined by

$$\mathcal{J}_{\gamma,\lambda}(u) = \frac{1}{p}\mathscr{M}([u]_{s,p}^p) - \frac{\gamma}{p}\|u\|_H^p - \frac{\lambda}{q}\|u\|_{q,w}^q - \frac{1}{p_s^*(\alpha)}\|u\|_{H_\alpha}^{p_s^*(\alpha)}.$$

Note that $\mathcal{J}_{\gamma,\lambda}$ is a $C^1(Z(\Omega))$ functional and for any u, $\varphi \in Z(\Omega)$

$$\langle \mathcal{J}'_{\gamma,\lambda}(u), \varphi \rangle = M([u]_{s,p}^p)\langle u, \varphi \rangle_{s,p} - \gamma \langle u, \varphi \rangle_H - \lambda \langle u, \varphi \rangle_{q,w} - \langle u, \varphi \rangle_{H_\alpha}, \tag{2.1}$$

where

$$\langle u, \varphi \rangle_{s,p} = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}[u(x) - u(y)] \cdot [\varphi(x) - \varphi(y)]}{|x - y|^{N+sp}} dxdy,$$

$$\langle u, \varphi \rangle_{q,w} = \int_\Omega w(x)|u(x)|^{q-2}u(x)\varphi(x)dx,$$

$$\langle u, \varphi \rangle_H = \int_\Omega |u(x)|^{p-2}u(x)\varphi(x)\frac{dx}{|x|^{sp}}, \quad \langle u, \varphi \rangle_{H_\alpha} = \int_\Omega |u(x)|^{p_s^*(\alpha)-2}u(x)\varphi(x)\frac{dx}{|x|^\alpha}.$$

Now, we discuss the compactness property for the functional $\mathcal{J}_{\gamma,\lambda}$, given by the Palais–Smale condition. We recall that $\{u_n\}_n \subset Z(\Omega)$ is a Palais–Smale sequence for $\mathcal{J}_{\gamma,\lambda}$ at level $c \in \mathbb{R}$ if

$$\mathcal{J}_{\gamma,\lambda}(u_n) \to c \quad \text{and} \quad \mathcal{J}'_{\gamma,\lambda}(u_n) \to 0 \quad \text{in } (Z(\Omega))' \text{ as } n \to \infty. \tag{2.2}$$

We say that $\mathcal{J}_{\gamma,\lambda}$ satisfies the Palais–Smale condition at level c if any Palais–Smale sequence $\{u_n\}_n$ at level c admits a convergent subsequence in $Z(\Omega)$.

Lemma 2.1. *Let $c < 0$.*

Then, for any $\sigma \in (p\theta, p_s^*(\alpha))$ and any $\gamma \in (-\infty, \kappa_\sigma H)$ there exists $\lambda_0 = \lambda_0(\sigma, \gamma) > 0$ such that for any $\lambda \in (0, \lambda_0)$, the functional $\mathcal{J}_{\gamma,\lambda}$ satisfies the Palais–Smale condition at level c.

Proof. Fix $\sigma \in (p\theta, p_s^*(\alpha))$ and $\gamma \in (-\infty, \kappa_\sigma H)$. Since $\gamma < \kappa_\sigma H \leq a\,H$, there exists a number $\widetilde{c} \in [0,1)$ such that $\gamma^+ = \widetilde{c}\,a\,H$. Thus, let us consider $\lambda_0 = \lambda_0(\sigma, \gamma) > 0$ sufficiently small such that

$$\left(\frac{1}{\sigma} - \frac{1}{p_s^*(\alpha)}\right)^{-\frac{p_s^*(\alpha)}{p_s^*(\alpha)-q}} \left[\lambda_0 \left(\frac{1}{q} - \frac{1}{\sigma}\right) \|w\|_r\right]^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-q}} < [(1 - \widetilde{c})a\,H_\alpha]^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-p}} \tag{2.3}$$

where $q < p < p_s^*(\alpha)$, a is set in (M_1), while H_α is given in (1.3).

Fix $\lambda \in (0, \lambda_0)$. Let $\{u_n\}_n$ be a $(PS)_c$ sequence in $Z(\Omega)$. We first show that $\{u_n\}_n$ is bounded. By using the assumptions (M_1) and (M_2), and the inequalities (1.2) and (1.3), we get

$$\begin{aligned}
\mathcal{J}_{\gamma,\lambda}(u_n) - \frac{1}{\sigma}\langle \mathcal{J}'_{\gamma,\lambda}(u_n), u_n\rangle \geq{}& \left(\frac{1}{p\theta} - \frac{1}{\sigma}\right) M([u_n]_{s,p}^p)[u_n]_{s,p}^p - \frac{\gamma^+}{H}\left(\frac{1}{p} - \frac{1}{\sigma}\right)[u_n]_{s,p}^p \\
& - \lambda\left(\frac{1}{q} - \frac{1}{\sigma}\right) C_w^q [u_n]_{s,p}^q - \left(\frac{1}{p_s^*(\alpha)} - \frac{1}{\sigma}\right)\|u_n\|_{H_\alpha}^{p_s^*(\alpha)} \\
\geq{}& \nu [u_n]_{s,p}^p - \lambda\left(\frac{1}{q} - \frac{1}{\sigma}\right) C_w^q [u_n]_{s,p}^q \\
& - \left(\frac{1}{p_s^*(\alpha)} - \frac{1}{\sigma}\right)\|u_n\|_{H_\alpha}^{p_s^*(\alpha)},
\end{aligned} \tag{2.4}$$

where

$$\nu = \left(\frac{1}{p\theta} - \frac{1}{\sigma}\right) a - \frac{\gamma^+}{H}\left(\frac{1}{p} - \frac{1}{\sigma}\right) > 0 \tag{2.5}$$

in view of (1.6) and the fact that $\sigma > p\theta \geq p$ and $\gamma \in (-\infty, \kappa_\sigma H)$. Thus, by (2.2) there exists $\beta > 0$ such that as $n \to \infty$

$$c + \beta [u_n]_{s,p}^q + o(1) \geq \nu [u_n]_{s,p}^p,$$

which implies at once that $\{u_n\}_n$ is bounded in $Z(\Omega)$, being $q < p$.

Therefore, using arguments similar to Lemma 4.1 of [15], there exists a subsequence, still denoted by $\{u_n\}_n$, and a function $u \in Z(\Omega)$ such that

$$\begin{aligned}
&u_n \rightharpoonup u \quad \text{in } Z(\Omega), && [u_n]_{s,p} \to d, \\
&u_n \rightharpoonup u \quad \text{in } L^p(\Omega, |x|^{-sp}), && \|u_n - u\|_H \to \iota, \\
&u_n \rightharpoonup u \quad \text{in } L^{p_s^*(\alpha)}(\Omega, |x|^{-\alpha}), && \|u_n - u\|_{H_\alpha} \to \ell, \\
&u_n \to u \quad \text{in } L^q(\Omega, w), && u_n \to u \text{ a.e. in } \Omega
\end{aligned} \tag{2.6}$$

as $n \to \infty$.

Furthermore, as shown in the proof of Lemma 2.4 of [9], by (2.6) the sequence $\{\mathcal{U}_n\}_n$, defined in $\mathbb{R}^{2N} \setminus \operatorname{Diag}\mathbb{R}^{2N}$ by

$$(x, y) \mapsto \mathcal{U}_n(x, y) = \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x - y|^{\frac{N+sp}{p'}}},$$

is bounded in $L^{p'}(\mathbb{R}^{2N})$ as well as $\mathcal{U}_n \to \mathcal{U}$ a.e. in $\mathbb{R}^{2N}$, where

$$\mathcal{U}(x, y) = \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{\frac{N+sp}{p'}}}.$$

Thus, up to a subsequence, we get $\mathcal{U}_n \rightharpoonup \mathcal{U}$ in $L^{p'}(\mathbb{R}^{2N})$, and so as $n \to \infty$

$$\langle u_n, \varphi\rangle_{s,p} \to \langle u, \varphi\rangle_{s,p} \tag{2.7}$$

for any $\varphi \in Z(\Omega)$, since $|\varphi(x) - \varphi(y)| \cdot |x - y|^{-\frac{N+sp}{p}} \in L^p(\mathbb{R}^{2N})$. Similarly, (2.6) and Proposition A.8 of [4] imply that $|u_n|^{p-2}u_n \rightharpoonup |u|^{p-2}u$ in $L^{p'}(\Omega, |x|^{-sp})$ and $|u_n|^{p_s^*(\alpha)-2}u_n \rightharpoonup |u|^{p_s^*(\alpha)-2}u$ in $L^{p_s^*(\alpha)'}(\Omega, |x|^{-\alpha})$, from which as $n \to \infty$

$$\langle u_n, \varphi\rangle_H \to \langle u, \varphi\rangle_H, \qquad \langle u_n, \varphi\rangle_{H_\alpha} \to \langle u, \varphi\rangle_{H_\alpha}, \tag{2.8}$$

for any $\varphi \in Z(\Omega)$.

Thanks to (2.6), by using Hölder inequality it results

$$\lim_{n\to\infty} \int_\Omega w(x)|u_n(x)|^{q-2}u_n(x)(u_n(x) - u(x))dx = 0. \tag{2.9}$$

Consequently, from (2.2), (2.6)–(2.9) we deduce that, as $n \to \infty$

$$\begin{aligned} o(1) = \langle \mathcal{J}'_{\gamma,\lambda}(u_n), u_n - u\rangle &= M([u_n]^p_{s,p})[u_n]^p_{s,p} - M([u_n]^p_{s,p})\langle u_n, u\rangle_{s,p} \\ &- \gamma \int_\Omega |u_n(x)|^{p-2}u_n(x)(u_n(x) - u(x))\frac{dx}{|x|^{sp}} \\ &- \lambda \int_\Omega w(x)|u_n(x)|^{q-2}(u_n(x) - u(x))dx \\ &- \int_\Omega |u_n(x)|^{p^*_s(\alpha)-2}u_n(x)(u_n(x) - u(x))\frac{dx}{|x|^\alpha} \\ = M([u_n]^p_{s,p})([u_n]^p_{s,p} &- [u]^p_{s,p}) - \gamma(\|u_n\|^p_H - \|u\|^p_H) \\ &- \|u_n\|^{p^*_s(\alpha)}_{H_\alpha} + \|u\|^{p^*_s(\alpha)}_{H_\alpha} + o(1). \end{aligned} \tag{2.10}$$

Furthermore, by using (2.6) and the celebrated Brézis and Lieb Lemma in [7], we have

$$\begin{aligned} \|u_n\|^p_H &= \|u_n - u\|^p_H + \|u\|^p_H + o(1), \\ \|u_n\|^{p^*_s(\alpha)}_{H_\alpha} &= \|u_n - u\|^{p^*_s(\alpha)}_{H_\alpha} + \|u\|^{p^*_s(\alpha)}_{H_\alpha} + o(1), \end{aligned} \tag{2.11}$$

as $n \to \infty$. By applying again the Brézis and Lieb Lemma [7] to

$$\frac{(u_n - u)(x) - (u_n - u)(y)}{|x - y|^{\frac{N+sp}{p}}} \in L^p(\mathbb{R}^{2N})$$

we can see that

$$[u_n]^p_{s,p} = [u_n - u]^p_{s,p} + [u]^p_{s,p} + o(1) \quad \text{as } n \to \infty. \tag{2.12}$$

Therefore, combining (2.6), the continuity of M and relations (2.10)–(2.12), we have proved the crucial formula

$$M(d^p) \lim_{n\to\infty}[u_n - u]^p_{s,p} = \gamma \lim_{n\to\infty} \|u_n - u\|^p_H + \lim_{n\to\infty} \|u_n - u\|^{p^*_s(\alpha)}_{H_\alpha} = \gamma\imath^p + \ell^{p^*_s(\alpha)}. \tag{2.13}$$

Now, let us rewrite the formula (2.13) as

$$(1 - \widetilde{c})M(d^p) \lim_{n\to\infty}[u_n - u]^p_{s,p} + \widetilde{c}M(d^p) \lim_{n\to\infty}[u_n - u]^p_{s,p} = \gamma\imath^p + \ell^{p^*_s(\alpha)},$$

with $\widetilde{c} \in [0, 1)$ fixed at the beginning of the proof. By (M_1) and (1.3), we have

$$\begin{aligned} (1 - \widetilde{c})a\, H_\alpha\ell^p + \widetilde{c}a\, H\imath^p &\le (1 - \widetilde{c})M(d^p) \lim_{n\to\infty}[u_n - u]^p_{s,p} + \widetilde{c}M(d^p) \lim_{n\to\infty}[u_n - u]^p_{s,p} \\ &\le \gamma^+\imath^p + \ell^{p^*_s(\alpha)}. \end{aligned}$$

Therefore, since $\gamma^+ = \widetilde{c}\, a\, H$, we obtain

$$\ell^{p^*_s(\alpha)} \ge (1 - \widetilde{c})a\, H_\alpha\ell^p,$$

from which, assuming by contradiction that $\ell > 0$, we get

$$\ell^{p^*_s(\alpha)} \ge [(1 - \widetilde{c})a\, H_\alpha]^{\frac{p^*_s(\alpha)}{p^*_s(\alpha)-p}}. \tag{2.14}$$

Exploiting (2.4) and (2.5), taking the limit as $n \to \infty$, and by using (2.2), (2.6), (2.10), assumption (w), Hölder inequality and Young inequality, we can infer

$$\begin{aligned} c &\geq \left(\frac{1}{\sigma} - \frac{1}{p_s^*(\alpha)}\right)\left(\ell^{p_s^*(\alpha)} + \|u\|_{H_\alpha}^{p_s^*(\alpha)}\right) - \lambda\left(\frac{1}{q} - \frac{1}{\sigma}\right)\|u\|_{q,w}^q \\ &\geq \left(\frac{1}{\sigma} - \frac{1}{p_s^*(\alpha)}\right)\left(\ell^{p_s^*(\alpha)} + \|u\|_{H_\alpha}^{p_s^*(\alpha)}\right) - \lambda\left(\frac{1}{q} - \frac{1}{\sigma}\right)\|w\|_r\|u\|_{H_\alpha}^q \\ &\geq \left(\frac{1}{\sigma} - \frac{1}{p_s^*(\alpha)}\right)\left(\ell^{p_s^*(\alpha)} + \|u\|_{H_\alpha}^{p_s^*(\alpha)}\right) - \left(\frac{1}{\sigma} - \frac{1}{p_s^*(\alpha)}\right)\|u\|_{H_\alpha}^{p_s^*(\alpha)} \\ &\quad - \left(\frac{1}{\sigma} - \frac{1}{p_s^*(\alpha)}\right)^{-\frac{q}{p_s^*(\alpha)-q}}\left[\lambda\left(\frac{1}{q} - \frac{1}{\sigma}\right)\|w\|_r\right]^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-q}}. \end{aligned}$$

Finally, by (2.14) we get

$$\begin{aligned} 0 > c &\geq \left(\frac{1}{\sigma} - \frac{1}{p_s^*(\alpha)}\right)\left[(1-\tilde{c})a\,H_\alpha\right]^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-p}} \\ &\quad - \left(\frac{1}{\sigma} - \frac{1}{p_s^*(\alpha)}\right)^{-\frac{q}{p_s^*(\alpha)-q}}\left[\lambda\left(\frac{1}{q} - \frac{1}{\sigma}\right)\|w\|_r\right]^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-q}} > 0, \end{aligned}$$

where the last inequality follows from (2.3). This is impossible, so $\ell = 0$.

Now, let us assume by contradiction that $\imath > 0$. Then, from (M_1), (1.3) and (2.13) we have

$$\begin{aligned} M(d^p)\lim_{n\to\infty}[u_n - u]_{s,p}^p &= \gamma\lim_{n\to\infty}\|u_n - u\|_H^p \\ &< a\,H\lim_{n\to\infty}\|u_n - u\|_H^p \leq M(d^p)\lim_{n\to\infty}[u_n - u]_{s,p}^p, \end{aligned}$$

which gives a contradiction. Therefore, $\imath = 0$ and by using again (M_1) and (2.13) it follows that $u_n \to u$ in $Z(\Omega)$ as $n \to \infty$, as claimed. □

3 The truncated functional

In this section we prove that problem (1.1) admits a sequence of solutions which goes to zero. Firstly, we recall the definition of genus and some its fundamental properties; see [29] for more details.

Let E be a Banach space and A a subset of E. We say that A is symmetric if $u \in A$ implies that $-u \in A$. For a closed symmetric set A which does not contain the origin, we define the genus $\mu(A)$ of A as the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$. If there does not exist such a k, we put $\mu(A) = \infty$. Moreover, we set $\mu(\emptyset) = 0$.

Let us denote by Σ_k the family of closed symmetric subsets A of E such that $0 \notin A$ and $\mu(A) \geq k$. Then we have the following result.

Proposition 3.1. *Let A and B be closed symmetric subsets of E which do not contain the origin. Then we have*

(i) If there exists an odd continuous mapping from A to B, then $\mu(A) \leq \mu(B)$.

(ii) If there is an odd homeomorphism from A onto B, then $\mu(A) = \mu(B)$.

(iii) If $\mu(B) < \infty$, then $\mu(A \setminus B) \geq \mu(A) - \mu(B)$.

(*iv*) *The n-dimensional sphere* $\mathbb{S}^n$ *has a genus of* $n+1$ *by the Borsuk–Ulam Theorem.*

(*v*) *If A is compact, then* $\mu(A) < \infty$ *and there exist* $\delta > 0$ *and a closed and symmetric neighborhood* $N_\delta(A) = \{x \in E : \|x - A\| \leq \delta\}$ *of A such that* $\mu(N_\delta(A)) = \mu(A)$.

Now, we state the following variant of symmetric mountain pass lemma due to Kajikija [20].

Lemma 3.2. *Let E be an infinite-dimensional Banach space and let* $I \in C^1(E, \mathbb{R})$ *be a functional satisfying the conditions below:*

(h_1) *$I(u)$ is even, bounded from below, $I(0) = 0$ and $I(u)$ satisfies the local Palais–Smale condition; that is, for some $c^* > 0$, in the case when every sequence $\{u_n\}_n$ in E satisfying $I(u_n) \to c < c^*$ and $I'(u_n) \to 0$ in E^* has a convergent subsequence;*

(h_2) *For each $n \in \mathbb{N}$, there exists an $A_n \in \Sigma_n$ such that $\sup_{u \in A_n} I(u) < 0$.*

Then either (i) or (ii) below holds.

(*i*) *There exists a sequence $\{u_n\}_n$ such that $I'(u_n) = 0$, $I(u_n) < 0$ and $\{u_n\}_n$ converges to zero.*

(*ii*) *There exist two sequences $\{u_n\}_n$ and $\{v_n\}_n$ such that $I'(u_n) = 0$, $I(u_n) = 0$, $u_n \neq 0$, $\lim_{n\to\infty} u_n = 0$, $I'(v_n) = 0$, $I(v_n) < 0$, $\lim_{n\to\infty} I(v_n) = 0$ and $\{v_n\}_n$ converges to a non-zero limit.*

Remark 3.3. It is worth to point out that in [20] the functional I verifies the Palais–Smale condition in global. Anyway, a careful analysis of the proof of Theorem 1 in [20], allows us to deduce that the result in [20] holds again if I satisfies the local Palais–Smale condition with the critical levels below zero.

Let us note that the functional $\mathcal{J}_{\gamma,\lambda}$ is not bounded from below in $Z(\Omega)$. Indeed, assumption (M_1) implies that $M(t) > 0$ for any $t \in \mathbb{R}^+_0$ and consequently by (M_2) we have $\frac{M(t)}{\mathscr{M}(t)} \leq \frac{\theta}{t}$. Thus, integrating on $[1, t]$, with $t > 1$, we get

$$\mathscr{M}(t) \leq \mathscr{M}(1)t^\theta \quad \text{for any } t \geq 1.$$

From this, by using (1.2) and (1.3), for any $u \in Z(\Omega)$ we have

$$\begin{aligned}\mathcal{J}_{\gamma,\lambda}(tu) \leq t^{p\theta}\frac{\mathscr{M}(1)}{p}[u]^{p\theta}_{s,p} - t^p\frac{\gamma}{p}\|u\|^p_H - t^q\frac{\lambda}{q}\|u\|^q_{q,w} \\ - t^{p^*_s(\alpha)}\frac{1}{p^*_s(\alpha)}\|u\|^{p^*_s(\alpha)}_{H_\alpha} \to -\infty \quad \text{as } t \to \infty.\end{aligned}$$

Now, fix $\gamma \in (-\infty, aH)$ and $\lambda > 0$ and let us consider the function

$$\mathcal{Q}_{\gamma,\lambda}(t) = \frac{1}{p}\left(a - \frac{\gamma^+}{H}\right)t^p - \frac{\lambda C_w}{q}t^q - \frac{1}{p^*_s(\alpha)H_\alpha}t^{p^*_s(\alpha)}.$$

Choose $R_1 > 0$ such that

$$\frac{1}{p}\left(a - \frac{\gamma^+}{H}\right)R_1^p > \frac{1}{p^*_s(\alpha)H_\alpha}R_1^{p^*_s(\alpha)} \tag{3.1}$$

and define

$$\lambda^* = \frac{C_w}{2qR_1^q}\left[\left(a - \frac{\gamma^+}{H}\right)R_1^p - \frac{1}{p^*_s(\alpha)H_\alpha}R_1^{p^*_s(\alpha)}\right] \tag{3.2}$$

such that $\mathcal{Q}_{\gamma,\lambda^*}(R_1) > 0$. Let us set

$$R_0 = \max\{t \in (0, R_1) : \mathcal{Q}_{\gamma,\lambda^*}(t) \le 0\}. \tag{3.3}$$

Taking in mind the fact that $\mathcal{Q}_{\gamma,\lambda}(t) \le 0$ for t near zero, since $q < p < p_s^*(\alpha)$, and $\mathcal{Q}_{\gamma,\lambda^*}(R_1) > 0$, we can infer that $\mathcal{Q}_{\gamma,\lambda^*}(R_0) = 0$.

Choose $\phi \in C_0^\infty([0,\infty))$ such that $0 \le \phi(t) \le 1$, $\phi(t) = 1$ for $t \in [0, R_0]$ and $\phi(t) = 0$ for $t \in [R_1, \infty)$. Thus, we consider the truncated functional

$$\widetilde{\mathcal{J}}_{\gamma,\lambda}(u) = \frac{1}{p}\mathscr{M}([u]_{s,p}^p) - \frac{\gamma}{p}\|u\|_H^p - \frac{\lambda}{q}\|u\|_{w,q}^q - \frac{\phi([u]_{s,p})}{p_s^*(\alpha)}\|u\|_{H_\alpha}^{p_s^*(\alpha)}.$$

It immediately follows that $\widetilde{\mathcal{J}}_{\gamma,\lambda}(u) \to \infty$ as $[u]_{s,p} \to \infty$, by (M_1), since $\gamma \in (-\infty, aH)$ and $q < p$. Hence, $\widetilde{\mathcal{J}}_{\gamma,\lambda}$ is coercive and bounded from below. Now, we prove a local Palais–Smale result for the truncated functional $\widetilde{\mathcal{J}}_{\gamma,\lambda}$.

Lemma 3.4. *For any $\gamma \in (-\infty, aH)$, there exists $\bar{\lambda} > 0$ such that, for any $\lambda \in (0, \bar{\lambda})$*

(i) if $\widetilde{\mathcal{J}}_{\gamma,\lambda}(u) \le 0$ then $[u]_{s,p} \le R_0$, and for any v in a small neighborhood of u we have $\mathcal{J}_{\gamma,\lambda}(v) = \widetilde{\mathcal{J}}_{\gamma,\lambda}(v)$;

(ii) $\widetilde{\mathcal{J}}_{\gamma,\lambda}$ satisfies a local Palais–Smale condition for $c < 0$.

Proof. Let us choose $\bar{\lambda}$ sufficiently small such that $\bar{\lambda} \le \min\{\lambda_0, \lambda^*\}$, where λ_0 is defined in Lemma 2.1 and λ^* in (3.2). Fix $\lambda < \bar{\lambda}$.

(i) Let us assume that $\widetilde{\mathcal{J}}_{\gamma,\lambda}(u) \le 0$.

If $[u]_{s,p} \ge R_1$, then by using (M_1), (1.2), (1.3), the definition of $\phi(t)$ and the fact that $\lambda < \lambda^*$, we obtain

$$\widetilde{\mathcal{J}}_{\gamma,\lambda}(u) \ge \frac{1}{p}\left(a - \frac{\gamma^+}{H}\right)[u]_{s,p}^p - \frac{\lambda^* C_w}{q} t^q [u]_{s,p}^q > 0,$$

where the last inequality follows from $q < p$ and $\mathcal{Q}_{\gamma,\lambda^*}(R_1) > 0$. Thus we get a contradiction because of $0 \ge \widetilde{\mathcal{J}}_{\gamma,\lambda}(u) > 0$.

When $[u]_{s,p} < R_1$, by using (M_1), (1.2), (1.3), $\lambda < \lambda^*$, the definition of $\phi(t)$, we can infer

$$0 \ge \widetilde{\mathcal{J}}_{\gamma,\lambda}(u) \ge \mathcal{Q}_{\gamma,\lambda}([u]_{s,p}) \ge \mathcal{Q}_{\gamma,\lambda^*}([u]_{s,p}).$$

From the definition of R_0 we deduce that $[u]_{s,p} \le R_0$. Moreover, for any $u \in B_{\frac{R_0}{2}}(0)$ we have that $\mathcal{J}_{\gamma,\lambda}(u) = \widetilde{\mathcal{J}}_{\gamma,\lambda}(u)$.

(ii) Being $\widetilde{\mathcal{J}}_{\gamma,\lambda}$ a coercive functional, every Palais–Smale sequence for $\widetilde{\mathcal{J}}_{\gamma,\lambda}$ is bounded. Thus, since $\lambda < \lambda_0$, by Lemma 2.1 we deduce a local Palais–Smale condition for $\mathcal{J}_{\gamma,\lambda} \equiv \widetilde{\mathcal{J}}_{\gamma,\lambda}$ at any level $c < 0$. ☐

Taking into account that $Z(\Omega)$ is reflexive and separable (see Appendix A in [28]), we can find a sequence $\{\varphi_n\}_n \subset Z(\Omega)$ such that $Z(\Omega) = \overline{\text{span}\{\varphi_n : n \in \mathbb{N}\}}$. For any $n \in \mathbb{N}$ we can set $X_n = \text{span}\{\varphi_n\}$ and $Y_n = \oplus_{i=1}^n X_i$.

Lemma 3.5. *For any $\gamma \in (-\infty, aH)$, $\lambda > 0$ and $k \in \mathbb{N}$, there exists $\varepsilon = \varepsilon(\gamma, \lambda, k) > 0$ such that*

$$\mu(\widetilde{\mathcal{J}}_{\gamma,\lambda}^{-\varepsilon}) \ge k,$$

where $\widetilde{\mathcal{J}}_{\gamma,\lambda}^{-\varepsilon} = \{u \in Z(\Omega) : \widetilde{\mathcal{J}}_{\gamma,\lambda}(u) \le -\varepsilon\}$.

Proof. Fix $\gamma \in (-\infty, aH)$, $\lambda > 0$ and $k \in \mathbb{N}$. Since Y_k is finite dimensional, there exist two positive constants $c_1(k)$ and $c_2(k)$ such that for any $u \in Y_k$

$$c_1(k)[u]_{s,p}^p \le \|u\|_H^p \quad \text{and} \quad c_2(k)[u]_{s,p}^q \le \|u\|_{q,w}^q. \tag{3.4}$$

By using (3.4), for any $u \in Y_k$ such that $[u]_{s,p} \le R_0$, we can infer

$$\widetilde{\mathcal{J}}_{\gamma,\lambda}(u) = \mathcal{J}_{\gamma,\lambda}(u) \le \frac{M^*}{p}[u]_{s,p}^p + \frac{\gamma^-}{p}c_1(k)[u]_{s,p}^p - \frac{\lambda}{q}c_2(k)[u]_{s,p}^q, \tag{3.5}$$

with $M^* = \max_{\tau\in[0,R_0]} M(\tau) < \infty$, by continuity of M. Now, let ϱ be a positive constant such that

$$\varrho < \min\left\{R_0, \left[\frac{\lambda c_2(k)p}{q(M^* + \gamma^- c_1(k))}\right]^{\frac{1}{p-q}}\right\}. \tag{3.6}$$

Then, for any $u \in Y_k$ such that $[u]_{s,p} = \varrho$, by (3.5) we get

$$\widetilde{\mathcal{J}}_{\gamma,\lambda}(u) \le \varrho^q \left[\frac{M^* + \gamma^- c_1(k)}{p}\varrho^{p-q} - \frac{\lambda c_2(k)}{q}\right] < 0, \tag{3.7}$$

where the last inequality follows from (3.6). Hence we can find a constant $\varepsilon = \varepsilon(\gamma, \lambda, k) > 0$ such that $\widetilde{\mathcal{J}}_{\gamma,\lambda}(u) \le -\varepsilon$ for any $u \in Y_k$ such that $[u]_{s,p} = \varrho$. As a consequence

$$\{u \in Y_k : [u]_{s,p} = \varrho\} \subset \{u \in Z(\Omega) : \widetilde{\mathcal{J}}_{\gamma,\lambda}(u) \le -\varepsilon\} \setminus \{0\}.$$

By using (ii) and (iv) of Proposition 3.1 we have the thesis. □

For any $c \in \mathbb{R}$ and any $k \in \mathbb{N}$, let us define the set

$$K_c = \{u \in Z(\Omega) : \widetilde{\mathcal{J}}'_{\gamma,\lambda}(u) = 0 \text{ and } \widetilde{\mathcal{J}}_{\gamma,\lambda}(u) = c\}$$

and the number

$$c_k = \inf_{A\in\Sigma_k} \sup_{u\in A} \widetilde{\mathcal{J}}_{\gamma,\lambda}(u). \tag{3.8}$$

Lemma 3.6. *For any $\gamma \in (-\infty, aH)$, $\lambda > 0$ and $k \in \mathbb{N}$, we have that $c_k < 0$.*

Proof. Fix $\gamma \in (-\infty, aH)$, $\lambda > 0$ and $k \in \mathbb{N}$. Then, by using Lemma 3.5 we can find a positive constant ε such that $\mu(\widetilde{\mathcal{J}}_{\gamma,\lambda}^{-\varepsilon}) \ge k$. Moreover, $\widetilde{\mathcal{J}}_{\gamma,\lambda}^{-\varepsilon} \in \Sigma_k$ since $\widetilde{\mathcal{J}}_{\gamma,\lambda}$ is a continuous and even functional. Taking into account that $\widetilde{\mathcal{J}}_{\gamma,\lambda}(0) = 0$, we have $0 \notin \widetilde{\mathcal{J}}_{\gamma,\lambda}^{-\varepsilon}$ and $\sup_{u\in\widetilde{\mathcal{J}}_{\gamma,\lambda}^{-\varepsilon}} \widetilde{\mathcal{J}}_{\gamma,\lambda}(u) \le -\varepsilon$. Therefore, recalling that $\widetilde{\mathcal{J}}_{\gamma,\lambda}$ is bounded from below, we get

$$-\infty < c_k = \inf_{A\in\Sigma_k} \sup_{u\in A} \widetilde{\mathcal{J}}_{\gamma,\lambda}(u) \le \sup_{u\in\widetilde{\mathcal{J}}_{\gamma,\lambda}^{-\varepsilon}} \widetilde{\mathcal{J}}_{\gamma,\lambda}(u) \le -\varepsilon < 0.$$ □

Lemma 3.7. *Let $\gamma \in (-\infty, aH)$ and $\lambda \in (0, \bar{\lambda})$, where $\bar{\lambda}$ is given by Lemma 3.4. Then all c_k are critical values for $\widetilde{\mathcal{J}}_{\gamma,\lambda}$ and $c_k \to 0$ as $k \to \infty$.*

Proof. Fix $\gamma \in (-\infty, aH)$ and $\lambda > 0$. It is easy to see that $c_k \le c_{k+1}$ for all $k \in \mathbb{N}$. By Lemma 3.6 it follows that $c_k < 0$, so we can assume that $c_k \to \bar{c} \le 0$. Since $\widetilde{\mathcal{J}}_{\gamma,\lambda}$ satisfies the Palais–Smale condition at level c_k by Lemma 3.4, we can argue as in [29] to see that all c_k are critical value of $\widetilde{\mathcal{J}}_{\gamma,\lambda}$.

Now, we prove that $\bar{c} = 0$. We argue by contradiction, and we suppose that $\bar{c} < 0$. In view of Lemma 3.4, we know that $K_{\bar{c}}$ is compact, so, by applying part (v) of Proposition 3.1 we can deduce that $\mu(K_{\bar{c}}) = k_0 < \infty$ and there exists $\delta > 0$ such that $\mu(K_{\bar{c}}) = \mu(N_\delta(K_{\bar{c}})) = k_0$. By Theorem 3.4 of [5], there exists $\varepsilon \in (0, \bar{c})$ and an odd homeomorphism $\eta : Z(\Omega) \to Z(\Omega)$ such that

$$\eta(\widetilde{\mathcal{J}}_{\gamma,\lambda}^{\bar{c}+\varepsilon} \setminus N_\delta(K_{\bar{c}})) \subset \widetilde{\mathcal{J}}_{\gamma,\lambda}^{\bar{c}-\varepsilon}.$$

Now, taking into account that c_k is increasing and $c_k \to \bar{c}$, we can find $k \in \mathbb{N}$ such that $c_k > \bar{c} - \varepsilon$ and $c_{k+k_0} \leq \bar{c}$. Take $A \in \Sigma_{k+k_0}$ such that $\sup_{u \in A} \widetilde{\mathcal{J}}_{\gamma,\lambda}(u) < \bar{c} + \varepsilon$. By using part (iii) of Proposition 3.1, we obtain

$$\mu(\overline{A \setminus N_\delta(K_{\bar{c}})}) \geq \mu(A) - \mu(N_\delta(K_{\bar{c}})) \quad \text{and} \quad \mu(\eta(\overline{A \setminus N_\delta(K_{\bar{c}})})) \geq k, \tag{3.9}$$

from which $\eta(\overline{A \setminus N_\delta(K_{\bar{c}})}) \in \Sigma_k$. Thus

$$\sup_{u \in \eta(\overline{A \setminus N_\delta(K_{\bar{c}})})} \widetilde{\mathcal{J}}_{\gamma,\lambda}(u) \geq c_k > \bar{c} - \varepsilon. \tag{3.10}$$

However, in view of (3.7) and (3.9) we can see that

$$\eta(\overline{A \setminus N_\delta(K_{\bar{c}})}) \subset \eta(\widetilde{\mathcal{J}}_{\gamma,\lambda}^{\bar{c}+\varepsilon} \setminus N_\delta(K_{\bar{c}})) \subset \widetilde{\mathcal{J}}_{\gamma,\lambda}^{\bar{c}},$$

which gives a contradiction in virtue of (3.10). Therefore, $\bar{c} = 0$ and $c_k \to 0$. □

Proof of Theorem 1.1. Let $\sigma \in (p\theta, p_s^*(\alpha))$, $\gamma \in (-\infty, \kappa_\sigma H)$ and $\lambda \in (0, \bar{\lambda})$. Since $\kappa_\sigma \leq a$, putting together Lemma 3.4, Lemma 3.5, Lemma 3.6 and Lemma 3.7, we can see that $\widetilde{\mathcal{J}}_{\gamma,\lambda}$ verifies all the assumptions of Lemma 3.2. Therefore, the thesis follows by point (i) of Lemma 3.4. □

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