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Capacity of permutations*

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Abstract

Permutations of $[n] = \{1, 2, \dots, n\}$ may be represented geometrically as bargraphs with column heights in $[n]$. We define the notion of capacity of a permutation to be the amount of water that the corresponding bargraph would hold if the region above it could retain water assuming the usual rules of fluid flow. Let $C(n)$ be the sum of the capacities of all permutations of $[n]$. We obtain, in a unique manner, all permutations of length $n+1$ from those of length n, which yields a recursion for $C(n+1)$ in terms of $C(n)$ that we can subsequently solve. Finally, we consider permutations that have a single dam (i.e., a single area of water containment) and compute the total number and capacity of all such permutations of a given length. We also provide bijective proofs of these formulas and an asymptotic estimate is found for the average capacity as n increases without bound.

Keywords: permutation statistic, generating functions, asymptotics

MSC: 05A05, 05A15, 05A16

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1. Introduction

A permutation of $[n]$ is an ordering of the elements of $[n]$. In recent years, a variety of different statistics on permutations have been studied in the literature; see, for example, [1–3, 6–12, 14, 15, 17, 18]. In order to describe our new statistic, we represent a permutation of $[n]$ as a bargraph with column heights in $[n]$. The height of the i-th column of the bargraph equals the size of the i-th letter of the permutation. We define the *capacity* of a permutation to be the amount of water the representing bargraph would retain if water is poured onto it from above and allowed to escape in any direction (if needed) subject to the usual rules of fluid flow. It is thus a measure of the area in the plane where the water would be retained. See [16] where the capacity statistic is considered on compositions and finite set partitions, represented geometrically as bargraphs, and also [4, 5] for further related results.

The organization of this paper is as follows. In the next section, we find an explicit formula for the sum of the capacities of all permutations of length n. In the third section, we consider the situation in which the retained water is restricted to a single area, i.e., to a single subsequence of consecutive entries, and refer to such permutations as having one dam. We then prove an analogous formula for the total capacity taken over all one-dam permutations of length n as well as an explicit formula for the total number of such permutations by considering a refinement according to the width of the dam. Some asymptotic estimates as n approaches infinity are also found for these quantities, and in the final section, bijective proofs are provided.

Illustrated below in Figure 1 is the capacity of the permutation 526134 of [6].

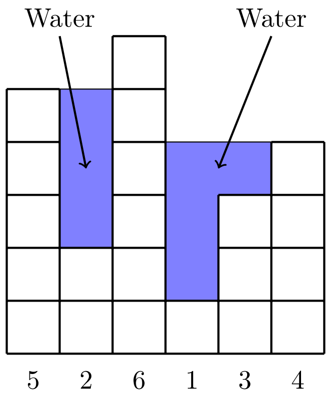

Figure 1: Permutation 526134 of [6] with capacity 7

2. Total capacity of permutations

Let $C(n)$ be the total capacity of all permutations of $[n]$. We employ a direct counting approach in order to obtain a recurrence for $C(n+1)$. This involves the following procedure. Consider an arbitrary permutation of $[n]$; from this, we obtain a unique permutation of $[n+1]$ via a simple two-step process:

- We raise the permutation of $[n]$ by adding one to each element in the original permutation. This produces a permutation of the elements of $[n+1] \setminus \{1\}$ as illustrated below in Figure 2.

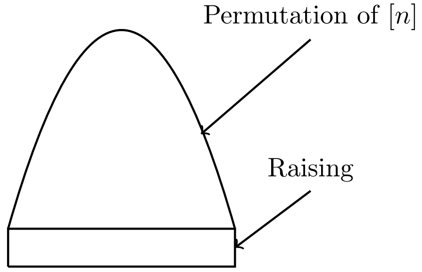

Figure 2: Raising a permutation of $[n]$ by one

- To convert this to an arbitrary permutation of $[n+1]$, we insert the element 1 within the raised permutation in any one of $n+1$ possible positions as shown in Figure 3.

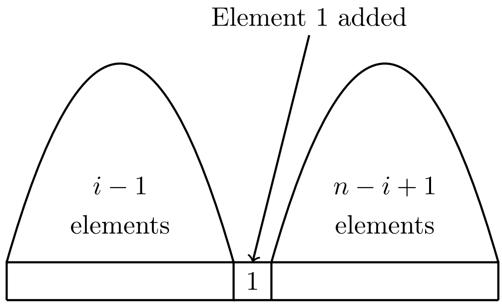

Figure 3: Element 1 added in the i-th position, $1 \leq i \leq n+1$

We denote the set of all permutations of $[n]$ by $\mathcal{S}_n$. Note that each member of $\mathcal{S}_{n+1}$ arises uniquely upon applying the above procedure to $\mathcal{S}_n$.

If the element 1 is added in either the first or the last position, there is no change to the capacity of the original permutation. In general, we will consider adding the 1 in all other positions i, where $2 \leq i \leq n$, and determine what addition this makes to the capacity of the member of $\mathcal{S}_n$ from which it arose. Note that the two-step procedure above is seen to leave the capacity of the precursor permutation unchanged, except for the additional capacity above the added element 1.

So our method will consist of calculating (see Figure 3) how many times the unchanged original capacity is to be counted, and secondly what is the total additional contribution above the 1 over all the possible original permutations of $[n]$.

So let us consider our general case where the 1 is added in the i-th position. Let r denote the maximum element to the left of 1, where $i \leq r \leq n+1$. First, consider the case $i \leq r \leq n$, which is illustrated in Figure 4. Then $n+1$ must occur to the right of the 1 and hence the additional capacity above the 1 is $r-1$. For each maximum r, the set of numbers to the left of 1 can be chosen, and then permuted, in $\binom{r-2}{i-2}(i-1)!$ ways, while the remaining numbers to the right of 1 can be permuted in $(n-i+1)!$ ways.

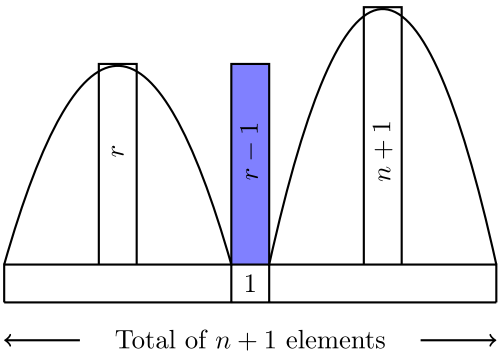

Figure 4: Additional capacity above the element 1, $i \leq r \leq n$

Thus, the total additional capacity is

$$\sum_{i=2}^{n}\sum_{r=i}^{n}\binom{r-2}{i-2}(i-1)!(n-i+1)!(r-1). \tag{2.1}$$

Now let us consider the case $r = n+1$. The sketch for this case is in Figure 5.

Here, by the pigeonhole principle, we have $n-i+2 \leq s \leq n$, and by a similar argument as for equation (2.1), the total additional capacity in this case is

$$\sum_{i=2}^{n}\sum_{s=n-i+2}^{n}\binom{s-2}{n-i}(n-i+1)!(i-1)!(s-1). \tag{2.2}$$

Expression (2.2) is equivalent to (2.1), which can also be realized by applying the reversal operation.

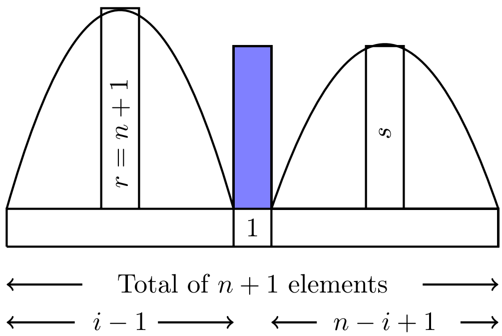

Figure 5: Additional capacity above the element 1

Thus, the total additional capacity over all permutations is

$$\begin{aligned}
&2\sum_{i=2}^{n}(i-1)!(n-i+1)!\sum_{r=i}^{n}\binom{r-2}{i-2}(r-1) \\
&= 2\sum_{i=2}^{n}(i-1)(i-1)!(n-i+1)!\binom{n}{i} \\
&= 2n!\sum_{i=2}^{n}\frac{(i-1)(n-i+1)}{i} \\
&= 2n!\sum_{i=1}^{n}\left(-i+(n+2)-\frac{n+1}{i}\right) \\
&= 2n!\left(\binom{n}{2}+2n-(n+1)H_n\right),
\end{aligned}$$

where H_n is the n-th Harmonic number $\sum_{i=1}^{n}\frac{1}{i}$.

So the recursion is

$$C(n+1) = (n+1)C(n) + 2n!\left(\binom{n}{2}+2n-(n+1)H_n\right), \quad n \geq 1,$$

with $C(1) = 0$.

We solve this first order linear recursion and obtain the following result.

Theorem 2.1. *The total capacity $C(n)$ over all permutations of $[n]$ is*

$$C(n) = \frac{n!}{2}\left(n(n+7)-4(n+1)H_n\right).$$

The values of $C(n)$ for $1 \le n \le 12$ are

$$0, 0, 2, \mathbf{28}, 312, 3384, 37872, 446688, 5595840, 74617920, 1058711040, 15958667520.$$

To illustrate, we list all the permutations of length 4 and their respective capacities in the table below. Note that the total is indeed 28, shown in bold in the list above.

Permutation	1234	1243	1324	1342	1423	1432
Capacity	0	0	1	0	1	0
Permutation	2134	2143	2314	2341	2413	2431
Capacity	1	1	2	0	2	0
Permutation	3124	3142	3214	3241	3412	3421
Capacity	3	2	3	1	1	0
Permutation	4123	4132	4213	4231	4312	4321
Capacity	3	2	3	1	1	0

Using the asymptotic expansion of H_n, we obtain the following estimate.

Corollary 2.2. *The average capacity for permutations of* $[n]$ *is*

$$\frac{1}{2}\left(n(n+7) - 4(n+1)H_n\right) = \frac{n^2}{2} - 2n\ln n + \left(\frac{7}{2} - 2\gamma\right)n - 2\ln n + O(1)$$

as $n \to \infty$*, where* γ *is Euler's constant.*

3. Total capacity in the one-dam situation

For permutations of $[n]$, we have computed the total capacity $C(n)$. We now determine the total capacity of permutations having exactly one dam defined as follows.

A permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ of $[n]$ is said to have exactly *one dam* if there exists only a single connected area of water containment. More precisely, we define the one-dam situation as that in which all of the water retained by a permutation σ is contained within a subsequence of σ of the form $r\sigma_i\sigma_{i+1}\cdots\sigma_j s$, where $2 \le r, s \le n$ and $\sigma_i, \sigma_{i+1}, \ldots, \sigma_j < \min\{r, s\}$. Moreover, the contribution of each σ_ℓ for $1 \le \ell < i$ or $j < \ell \le n$ towards the capacity is zero.

For example, the permutation $\sigma = 463152$ of [6] has only one dam, with $r = 6$ and $s = 5$, whereas the permutation in Figure 1 above has two. We give, in Figure 6 below, a symbolic sketch of a generic permutation having a single dam.

Let us define the *dam width* p of a one-dam permutation as the number of letters p that actually contribute to the capacity, i.e., the aforementioned

$$\sigma_i\sigma_{i+1}\cdots\sigma_j \text{ has } j - i + 1 = p.$$

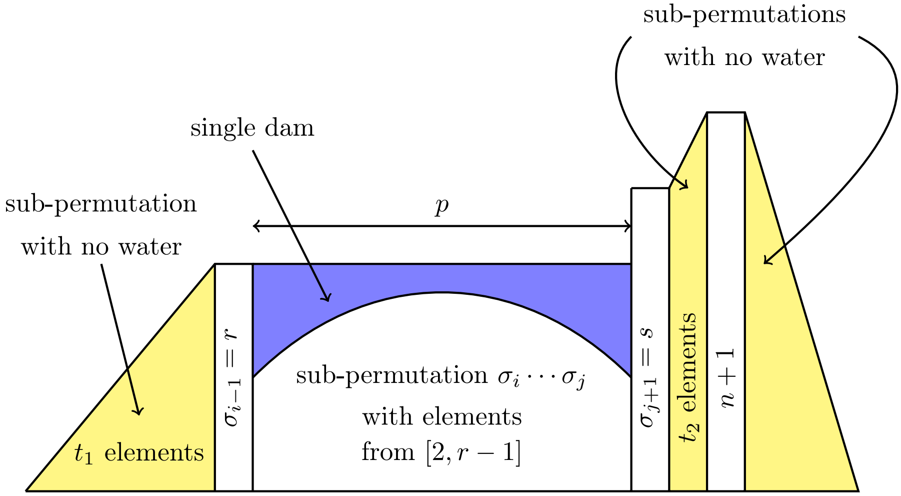

Figure 6: Permutation with one dam only, after raising but before adding 1

Let $C_1(n, p)$ be the *total* capacity taken over all permutations of $[n]$ with one dam of width p. Now let us obtain *all* one-dam permutations of $[n+1]$ of width $p+1$ from all possible precursors in $\mathcal{S}_n$. Each one-dam member of $\mathcal{S}_{n+1}$ of width $p+1$ can be obtained *in a unique way* from a certain subset of $\mathcal{S}_n$ by the following modified two-step procedure:

- Raising such permutations by one,
- Adding 1 to these permutations in every possible way that results in a one-dam permutation of $[n+1]$.

Let us first write a recursion for $C_1(n+1, 1)$. We consider the following cases:
First case, where we add the element 1 to any raised unimodal permutation at all points other than the ends.
Second case, where there is a single dam of width one both before and after adding the element 1 to either end of a raised permutation.

So for the first case, we fix a raised unimodal permutation. The total contribution of adding the element 1 in any of the specified positions is

$$1 + 2 + 3 + \cdots + (n-1) = \binom{n}{2}.$$

There are precisely

$$\binom{n-1}{0} + \binom{n-1}{1} + \cdots \binom{n-1}{n-1} = 2^{n-1}$$

unimodal permutations of length n. Hence, the contribution towards $C_1(n+1,1)$ is $2^{n-1}\binom{n}{2}$.

For the second case, the contribution is seen to be $2C_1(n,1)$. Combining the prior two cases, we have the recurrence

$$C_1(n+1,1) = 2C_1(n,1) + \binom{n}{2}2^{n-1}, \quad n \geq 1,$$

with the initial condition $C_1(1,1) = 0$, which yields the following result.

Proposition 3.1. *The total capacity of all one-dam permutations of* $[n]$ *with dam width 1 is*

$$C_1(n,1) = \frac{2^n\, n}{24}(n-1)(n-2).$$

We now write a recurrence for $C_1(n+1,p+1)$ where $p \geq 1$. For this, note that obtaining all one-dam permutations of length $n+1$ having width $p+1$ entails either

i) Adding 1 to any of the permutations counted in $C_1(n,p)$ (after first raising them) in any of the $p+1$ positions available inside the dam, or

ii) Adding 1 to either end of a permutation counted by $C_1(n,p+1)$ (after raising).

Now for case i) above, let r be the left bound of the dam in a one-dam permutation and s be the right bound. Assume for now that $r < s$ where $s \leq n$. (The case $s = n+1$ must be considered separately.) The width of the dam is p. Let there be t_1 increasing parts to the left of r and t_2+1 increasing parts to the right of s of which the last part must be $n+1$.

We note the following restrictions:

$$1 \leq p \leq n-3,$$

$$0 \leq t_1 \leq (r-1)-(p+1) = r-2-p,$$

$$0 \leq t_2 \leq n-s.$$

After raising and inserting the 1, we see that

$$p+1 < r \leq n-1$$

(because all $p+1$ elements of the new wider dam must be $< r$).

When we add 1 to the dam (in any of the $p+1$ possible positions), the additional capacity above the 1 is $r-1$. There are $\binom{r-2-p}{t_1}$ and $\binom{n-s}{t_2}$ ways to choose t_1 and t_2 elements, respectively, to form the increasing sequences. There are $\binom{r-2}{p}p!$ ways to choose and order the p elements in the dam prior to inserting 1. Thus, the additional contribution for permutations enumerated by $C_1(n,p)$ with given parameters r and s as stated is

$$\sum_{t_1=0}^{r-2-p}\sum_{t_2=0}^{n-s}\binom{r-2}{p}p!(r-1)(p+1)\binom{r-2-p}{t_1}\binom{n-s}{t_2}$$

$$= \binom{r-2}{p}(r-1)(p+1)!2^{n-s}\sum_{t_1=0}^{r-2-p}\binom{r-2-p}{t_1}$$
$$= \binom{r-2}{p}(r-1)(p+1)!2^{n-s}\,2^{r-2-p}. \tag{3.1}$$

Summing (3.1) over all possible values of s yields

$$\sum_{s=r+1}^{n}\binom{r-2}{p}(r-1)(p+1)!2^{n-s+r-2-p}$$
$$= (p+1)!\binom{r-2}{p}(r-1)2^{n-2-p}\left(1-2^{r-n}\right). \tag{3.2}$$

Finally, summing (3.2) over all possible values of r, the total additional capacity is

$$\sum_{r=p+2}^{n-1}(p+1)!\binom{r-2}{p}(r-1)2^{n-2-p}\left(1-2^{r-n}\right)$$
$$= (p+1)!2^{n-2-p}\sum_{r=p+2}^{n-1}\binom{r-2}{p}(r-1)\left(1-2^{r-n}\right). \tag{3.3}$$

Now for the case $s = n+1$, the restrictions are

$$1 \le p \le n-2,$$
$$0 \le t_1 \le (r-1)-(p+1) = r-2-p.$$

Considering all possible values of r and t_1, the additional contribution for permutations in the case $s = n+1$ is

$$\sum_{r=p+2}^{n}\sum_{t_1=0}^{r-2-p}\binom{r-2}{p}p!(r-1)(p+1)\binom{r-2-p}{t_1}$$
$$= \sum_{r=p+2}^{n}\binom{r-2}{p}(r-1)(p+1)!2^{r-2-p}. \tag{3.4}$$

Finding the total capacity requires taking into account the cases when $r > s$ and exploiting the obvious symmetry (i.e., multiplying by 2). Thus, by (3.3) and (3.4), the total additional capacity in case i) above is

$$(p+1)!2^{n-1-p}\sum_{r=p+2}^{n-1}\binom{r-2}{p}(r-1)\left(1-2^{r-n}\right)$$
$$+\sum_{r=p+2}^{n}\binom{r-2}{p}(r-1)(p+1)!2^{r-1-p}$$

$$= (p+1)!2^{n-1-p} \sum_{r=p+2}^{n} \binom{r-2}{p}(r-1) = (p+1)(p+1)!2^{n-1-p}\binom{n}{p+2}, \quad (3.5)$$

where we have made use of [13, p. 174] to obtain the last equality.

The original total capacity from i) is

$$(p+1)C_1(n,p). \quad (3.6)$$

Case ii) leads to a contribution towards $C_1(n+1,p+1)$ of

$$2C_1(n,p+1). \quad (3.7)$$

So adding (3.5), (3.6) and (3.7), we have the recurrence:

$$\begin{aligned} C_1(n+1,p+1) &= 2C_1(n,p+1) + (p+1)C_1(n,p) \\ &\quad + (p+1)(p+1)!2^{n-1-p}\binom{n}{p+2}. \end{aligned} \quad (3.8)$$

We have the following explicit formula for $C_1(n,p)$.

Theorem 3.2. *The total capacity of all one-dam permutations of* $[n]$ *with dam width* p *is*

$$C_1(n,p) = \frac{p}{p+2}2^{n-2-p}\frac{n!}{(n-2-p)!},$$

for $1 \le p \le n-2$.

Proof. We prove the result for a given $n \ge 3$ and all $p \in [n-2]$ by induction on n. The $n = 3$ case is clear since $C_1(3,1) = 2$. If $n \ge 3$ and $p \ge 1$, then the formula for $C_1(n+1,p+1)$ follows from (3.8) and the induction hypothesis, upon considering separately the cases when $p \le n-3$ and $p = n-2$. By Proposition 3.1, the formula holds for $p = 1$ and all $n \ge 3$, which fully establishes the $n+1$ case and completes the induction. □

Remark 3.3. From Theorem 3.2, we obtain the generating function

$$\sum_{n \ge p+2} C_1(n,p)\, x^n = \frac{p(p+1)!x^{p+2}}{(1-2x)^{p+3}}, \quad p \ge 1.$$

Below is an array of values for $C_1(n,p)$ for small n and p:

$$[C_1(n,p)]_{n\ge 3, p\ge 1} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 16 & 12 & 0 & 0 & 0 & 0 \\ 80 & 120 & 72 & 0 & 0 & 0 \\ 320 & 720 & 864 & 480 & 0 & 0 \\ 1120 & 3360 & 6048 & 6720 & 3600 & 0 \\ 3584 & 13440 & 32256 & 53760 & 57600 & 30240 \end{pmatrix}.$$

Corollary 3.4. *The total capacity of one-dam permutations of* $[n]$ *is*

$$C_1(n) = \sum_{p=1}^{n-2} \frac{p}{p+2} 2^{n-2-p} \frac{n!}{(n-2-p)!}.$$

The values of $C_1(n)$ for $1 \leq n \leq 12$ are

$$0, 0, 2, 28, 272, 2384, 20848, 190880, 1871808, 19832448, 227360256, 2814303232.$$

4. Total number of one-dam permutations

In this section, we find the number of permutations of $[n]$ that have exactly one dam. Let $N(n, p)$ be the number of one-dam permutations of size n with width p. In order to obtain a recursion for $N(n+1, p+1)$ in terms of $N(n, p)$, we apply the same two-step procedure as before. We again consider separately the cases $p = 1$ and $p > 1$.

4.1. Case where $p = 1$

First, we add 1 at all points other than the ends to a raised unimodal permutation; then the contribution to the number of permutations is $(n-1)2^{n-1}$.

Next, we add 1 to the ends of a one-dam permutation, which yields a contribution of $2N(n, 1)$. Combining the prior cases gives

$$N(n+1, 1) = (n-1)2^{n-1} + 2N(n, 1), \quad n \geq 1,$$

with initial condition $N(1, 1) = 0$.

Solving this first order linear recursion gives the following result.

Proposition 4.1. *The number of one-dam permutations of* $[n]$ *with dam width* 1 *is*

$$N(n, 1) = 2^{n-3}(n-1)(n-2).$$

4.2. Case where $p > 1$

First, we add 1 to a permutation counted in $N(n, p)$ in any of the $p+1$ positions within the dam, which gives a contribution of $(p+1)N(n, p)$. Otherwise, add the 1 to either end of a permutation counted by $N(n, p+1)$.

Thus, the recursion (3.8) is replaced by

$$N(n+1, p+1) = (p+1)N(n, p) + 2N(n, p+1). \tag{4.1}$$

One then has the following explicit formula for $N(n, p)$.

Theorem 4.2. *The number of one-dam permutations of* $[n]$ *with dam width* p *is*

$$N(n,p) = \frac{1}{p+1} 2^{n-1-p} \frac{(n-1)!}{(n-2-p)!},$$

for $1 \leq p \leq n-2$.

Proof. This is shown by induction on n as before using (4.1) and Proposition 4.1. □

Remark 4.3. From Theorem 4.2, we obtain the generating function

$$\sum_{n \geq p+2} N(n,p)\, x^n = \frac{2p! x^{p+2}}{(1-2x)^{p+2}}, \quad p \geq 1.$$

Below are the values for $N(n,p)$ for small n and p:

$$[N(n,p)]_{n\geq 3, p \geq 1} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 12 & 4 & 0 & 0 & 0 & 0 \\ 48 & 32 & 12 & 0 & 0 & 0 \\ 160 & 160 & 120 & 48 & 0 & 0 \\ 480 & 640 & 720 & 576 & 240 & 0 \\ 1344 & 2240 & 3360 & 4032 & 3360 & 1440 \end{pmatrix}.$$

Corollary 4.4. *The number of permutations of* $[n]$ *with one dam is*

$$N(n) = \sum_{p=1}^{n-2} \frac{1}{p+1} 2^{n-1-p} \frac{(n-1)!}{(n-2-p)!}.$$

The values of $N(n)$ for $1 \leq n \leq 12$ are

$$0, 0, 2, 16, 92, 488, 2656, 15776, 105696, 806592, 6974592, 67573504.$$

5. Asymptotics for $C_1(n)$ and $N(n)$

5.1. Asymptotics for $C_1(n)$

In order to find the asymptotic average capacity for one-dam permutations of $[n]$, we need asymptotic estimates of the quantities $C_1(n)$ and $N(n)$ in Corollaries 3.4 and 4.4.

We first find the maximum value of $C_1(n,p)$ over p for a fixed n. For this, we compute the ratio $C_1(n,p+1)/C_1(n,p)$ and determine where it is greater than or less than one.

By the formula $C_1(n,p) = \frac{p}{p+2} 2^{n-2-p} \frac{n!}{(n-2-p)!}$ from Theorem 3.2, we have

$$\frac{C_1(n,p+1)}{C_1(n,p)} = \frac{(n-2-p)(p+1)(p+2)}{2p(p+3)}.$$

Since $\frac{(p+1)(p+2)}{p(p+3)} > 1$, the ratio $\frac{C_1(n,p+1)}{C_1(n,p)}$ exceeds 1 if $p \le n-4$. Comparing directly $C_1(n,n-3) = \frac{2(n-3)n!}{n-1}$ and $C_1(n,n-2) = (n-2)(n-1)!$, we have $C_1(n,n-3) > C_1(n,n-2)$ if $n \ge 4$, which we will assume. Thus, the size of the largest term is given by $C_1(n,n-3)$.

We represent the general term $C_1(n,p)$ for $p \le n-3$ by $C(n,n-3-j)$, where j runs from 0 to $n-4$. Thus, the ratio of the general term $C_1(n,p)$ to the maximum term $C_1(n,n-3)$ is

$$\frac{C_1(n,n-3-j)}{C_1(n,n-3)} = \frac{2^j(3+j-n)(n-1)}{(1+j-n)(j+1)!(n-3)}.$$

At this stage, the final term where $p = n-2$ is omitted and will be reintroduced later.

Summing over all possible values of j yields

$$\sum_{j=0}^{n-4} \frac{2^j(3+j-n)(n-1)}{(1+j-n)(j+1)!(n-3)} = \frac{n-1}{n-3} \sum_{j=0}^{n-4} \frac{2^j(3+j-n)}{(1+j-n)(j+1)!}. \tag{5.1}$$

To estimate this sum, we perform a series expansion on the summand

$$\frac{2^j(3+j-n)}{(1+j-n)(j+1)!} = \frac{2^j}{(1+j)!} - \frac{2^{1+j}}{(1+j)!n} + O\left(\frac{1}{n^2}\right).$$

We shall replace the original summand by $\frac{2^j}{(1+j)!} - \frac{2^{1+j}}{(1+j)!n}$.

Thus, consider the sum

$$\sum_{j=0}^{n-4} \left(\frac{2^j}{(1+j)!} - \frac{2^{1+j}}{(1+j)!n} \right). \tag{5.2}$$

The terms

$$\sum_{j=0}^{n-4} \frac{2^{1+j}}{(1+j)!n}$$

may be ignored as they only make a small contribution for large n since

$$\frac{1}{n} \sum_{j=0}^{n-4} \frac{2^{1+j}}{(1+j)!} < \frac{e^2}{n}.$$

Therefore, the sum in (5.2) can be approximated by an infinite sum

$$\sum_{j=0}^{\infty} \frac{2^j}{(1+j)!},$$

since the terms for $j \ge n-3$ are exponentially small. Thus, the sum in (5.2) equals

$$\sum_{j=0}^{\infty} \frac{2^j}{(1+j)!} + O\left(\frac{1}{n}\right) = \frac{e^2-1}{2} + O\left(\frac{1}{n}\right).$$

Finally, we include the factor $\frac{n-1}{n-3}$ from equation (5.1) above that was left out, multiply by the largest term $C(n, n-3) = \frac{2(n-3)n!}{n-1}$ and then add the missing last term when $p = n-2$ to obtain

$$(e^2-1)n!\left(1+O\left(\frac{1}{n}\right)\right) + n!\left(1+O\left(\frac{1}{n}\right)\right),$$

which yields the following result.

Theorem 5.1. *As $n \to \infty$, the asymptotic expression for $C_1(n)$, the total capacity of all one-dam permutations of $[n]$, is given by*

$$C_1(n) = e^2\, n!\left(1+O\left(\frac{1}{n}\right)\right).$$

5.2. Asymptotics for $N(n)$

One can also find an asymptotic expression for the number of permutations of $[n]$ with one dam, following the method used for $C_1(n)$. By Theorem 4.2, the ratio of $N(n, p+1)$ to $N(n, p)$ simplifies to $\frac{(n-2-p)(1+p)}{2(2+p)}$. Since $\frac{2}{3} \le \frac{1+p}{2+p} < 1$, we have $\frac{N(n,p+1)}{N(n,p)} \ge 1$ if $1 \le p \le n-5$ and $\frac{N(n,p+1)}{N(n,p)} < 1$ if $p = n-3$ or $n-4$. (Note that there is equality in the inequality $\frac{N(n,p+1)}{N(n,p)} \ge 1$ if and only if $n = 6$ and $p = 1$.) Thus, the maximum value of $N(n, p)$ for $1 \le p \le n-2$ where $n \ge 5$ is given by

$$N(n, n-4) = \frac{4(n-1)!}{n-3}.$$

This time however there are two cases to add at the end, namely, when $p = n-2$ and $p = n-3$. We consider the ratio $\frac{N(n,n-4-j)}{N(n,n-4)}$ of the general term to the largest term for $0 \le j \le n-5$ and sum over j to get

$$\sum_{j=0}^{n-5} \frac{N(n, n-4-j)}{N(n, n-4)} = \sum_{j=0}^{n-5} \frac{2^{1+j}(n-3)}{(n-3-j)(j+2)!}.$$

Similar to before, we have

$$\sum_{j=0}^{n-5} \frac{2^{1+j}(n-3)}{(n-3-j)(j+2)!} = \sum_{j=0}^{n-5}\left(\frac{2^{1+j}}{(2+j)!} + \frac{2^{1+j}j}{(2+j)!n}\right) + O\left(\frac{1}{n}\right).$$

We approximate this last sum, ignoring the second part, by the infinite sum

$$\sum_{j=0}^{\infty} \frac{2^{1+j}}{(2+j)!} = \frac{e^2-3}{2}.$$

Multiplying by the largest term and adding the two missing terms for $p = n - 2$ and $p = n - 3$, we have

$$\left(\frac{e^2-3}{2}\frac{4(n-1)!}{n} + \frac{4(n-1)!}{n} + \frac{2(n-1)!}{n}\right)\left(1 + O\left(\frac{1}{n}\right)\right),$$

which yields the following result.

Theorem 5.2. *As $n \to \infty$, the asymptotic expression for $N(n)$, the number of permutations of $[n]$ with one dam, is given by*

$$N(n) = \frac{2e^2(n-1)!}{n}\left(1 + O\left(\frac{1}{n}\right)\right).$$

Finally, dividing the result of Theorem 5.1 by that of Theorem 5.2 yields the following estimate.

Theorem 5.3. *As $n \to \infty$, the average capacity for the permutations of $[n]$ with one dam is*

$$\frac{C_1(n)}{N(n)} = \frac{n^2}{2}\left(1 + O\left(\frac{1}{n}\right)\right).$$

6. Combinatorial proofs

In this section, we provide bijective proofs of Theorems 3.2 and 4.2 above. Since our combinatorial proof of the former makes use of ideas from the latter, we first argue the latter.

6.1. Combinatorial proof of Theorem 4.2.

Equivalently, we show $N(n,p) = 2^{n-1-p}p!\binom{n-1}{p+1}$. To do so, first let $S = \{s_1 < s_2 < \cdots < s_{p+1}\}$ be an arbitrary subset of $[n-1]$ of size $p+1$. We reorder the elements $s_1, s_2, \ldots, s_p$ according to an arbitrary permutation α of $[p]$ as $s_{\alpha(1)}, s_{\alpha(2)}, \ldots, s_{\alpha(p)}$, which we will denote by α^*. Next, we assign to each member of $[n] - S$ either a or b. From this configuration enumerated by $2^{n-1-p}p!\binom{n-1}{p+1}$, we create a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ of $[n]$ having a single dam $\pi_{i-1}\pi_i\cdots\pi_j\pi_{j+1}$ of width p such that the section $\pi_i\cdots\pi_j$ is a permutation of $\{s_1, \ldots, s_p\}$ and $s_{p+1} = \min\{\pi_{i-1}, \pi_{j+1}\}$. In creating π, we will first form the subsequence Q of π comprising the elements of $S \cup [s_{p+1}+1, n]$; note that Q must consist of consecutive letters of π.

Consider the sequence $c = c_1c_2\cdots c_\ell$ of letters in $\{a, b\}$ assigned to the elements $s_{p+1}+1, s_{p+1}+2, \ldots, n$, where $\ell = n - s_{p+1}$. If $c = a^\ell$ or $c = b^\ell$, then let Q be given by $Q = n(n-1)\cdots(s_{p+1}+1)\alpha^*s_{p+1}$ or $Q = s_{p+1}\alpha^*(s_{p+1}+1)\cdots(n-1)n$, respectively. If $c = b^{\ell-1}a$ or $c = a^{\ell-1}b$, then let $Q = (s_{p+1}+1)\cdots(n-1)n\alpha^*s_{p+1}$ or $Q = s_{p+1}\alpha^*n(n-1)\cdots(s_{p+1}+1)$. So assume c starts with b^ta or a^tb, where

$1 \leq t \leq \ell - 2$. We consider cases based on the final letter c_ℓ to define Q. First assume $c_\ell = a$. If c starts with $b^t a$ for some $1 \leq t \leq \ell - 2$, then let

$$Q = (s_{p+1} + t + 1)\beta' n \beta''(s_{p+1} + t) \cdots (s_{p+1} + 1)\alpha^* s_{p+1},$$

where β' is increasing, β'' is decreasing and $\beta' \cup \beta'' = [s_{p+1} + t + 2, n - 1]$, with membership in the string β' or β'' dependent on whether a or b is assigned to the element in question. If c starts with $a^t b$, then let

$$Q = (s_{p+1} + 1) \cdots (s_{p+1} + t)\beta' n \beta''(s_{p+1} + t + 1)\alpha^* s_{p+1},$$

where β' and β'' are as before. Now assume $c_\ell = b$. If c starts with $b^t a$, then let Q be obtained by reversing the Q from the corresponding case above when $c_\ell = a$. Likewise, if c starts with $a^t b$, then reverse Q from the corresponding case when $c_\ell = a$.

Finally, if $x \in [s_{p+1}] - S$, then either place x before Q if x is assigned a or after Q if x is assigned b such that any elements of $[s_{p+1}] - S$ before (after) Q occur in increasing (decreasing) order. Let π be the permutation of $[n]$ obtained by applying the operations described above. One may verify that π contains a single dam of width p and that the procedure above is reversible.

6.2. Proof of Theorem 3.2.

Let $\mathcal{N}(n,p)$ denote the set of permutations enumerated by $N(n,p)$. To compute the sum of the capacities of all members of $\mathcal{N}(n,p)$, it is enough to consider the contribution from the first letter of each dam, by symmetry, and multiply the result by p. Let $\lambda \in \mathcal{N}(n,p)$ be formed in the manner described above from an ordered triple (S, α, d), where S and α are as before with $S = \{s_1 < s_2 < \cdots < s_{p+1}\}$ and d is a binary sequence in $\{a, b\}$ of length $n - 1 - p$. Let λ' be the member of $\mathcal{N}(n,p)$ obtained from the triple $(S', \gamma\alpha, d)$, where γ denotes the complement operation (i.e., $\gamma(i) = p + 1 - i$ for all $i \in [p]$) and $S' = \{s_{p+1}\} \cup \{s_{p+1} - s_i : 1 \leq i \leq p\}$. Note that $\lambda = \lambda'$ if and only if $p = 1$, s_2 is even and $s_1 = \frac{s_2}{2}$, which is permitted. Taken together, λ and λ' contribute s_{p+1} towards the total capacity of all members of $\mathcal{N}(n,p)$ for all λ (considering only the contribution of the first position within a dam). So we must replace $\binom{n-1}{p+1}$ as the enumerator of S with the sum $\sum_{r=p+1}^{n-1} \binom{r-1}{p} r = \binom{n}{p+2}(p+1)$, where r denotes s_{p+1}; this identity is shown below bijectively. Upon considering separately the cases when $\lambda = \lambda'$ and $\lambda \neq \lambda'$, it is seen that the contribution of each λ is counted twice (note that if $p > 1$, then $\lambda \neq \lambda'$ for all λ with the mapping $\lambda \mapsto \lambda'$ an involution for all p). Thus, multiplying by p, the total capacity of all members of $\mathcal{N}(n,p)$ is given by

$$\frac{1}{2}\left(2^{n-1-p} p \cdot p! \binom{n}{p+2}(p+1)\right) = \frac{p}{p+2} 2^{n-2-p} \frac{n!}{(n-2-p)!},$$

as desired.

For completeness, we provide a bijective proof of the identity

$$\sum_{r=p+1}^{n-1} \binom{r-1}{p} r = \binom{n}{p+2}(p+1), \quad 1 \leq p \leq n-2, \tag{6.1}$$

used above, since the authors were unable to find such a proof in the literature. Note that the right side of (6.1) clearly counts members of the set $\mathcal{A}$ consisting of "marked" subsets of $[n]$ of size $p+2$ wherein one of the elements, not the largest, is marked. To complete the proof, we construct another set $\mathcal{B}$ enumerated by the left side of (6.1) as well as a bijection between the sets $\mathcal{B}$ and $\mathcal{A}$. Given $p+1 \leq r \leq n-1$, let $\mathcal{B}_r$ denote the set of configurations wherein the members of $[r]$ are written in a row, exactly $p+1$ numbers are circled, among them r itself, and a dot is placed directly prior to some member of $[r]$. Let $\mathcal{B} = \bigcup_{r=p+1}^{n-1} \mathcal{B}_r$. To define a bijection from $\mathcal{B}$ to $\mathcal{A}$, renumber the elements to the right of the dot where the dot now receives a number (the number assigned the position of the dot will become the marked element of $A \in \mathcal{A}$). Note that the element r becomes $r+1$ and thus the largest element of A.

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