Muḥyī al-Dīn al-Maghribī's Measurements of Mars at the Maragha Observatory

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Abstract: Muḥyī al-Dīn al-Maghribī (d. 1283 AD) carried out a systematic observational programme at the Maragha observatory in northwestern Iran in order to provide new measurements of solar, lunar, and planetary parameters, as he explains in his treatise *Talkhīṣ al-majisṭī* (*Compendium of the* Almagest). His project produces a new and consistent set of parameters. On the basis of his four documented observations of Mars, carried out in 1264, 1266, 1270, and 1271 AD, he measured the unprecedented values for the radius of the epicycle, the longitude of the apogee, and the mean motion in longitude of the planet and also confirmed that Ptolemy's value for its eccentricity was correct for his time. This paper presents a detailed, critical account of Muḥyī al-Dīn's measurements. Using a criterion described below, we compare the accuracy of his values for the structural parameters of Mars with that of other historically important values known for these parameters from medieval Middle Eastern astronomy from the early eighth to the late fifteenth century. Muḥyī al-Dīn attained a higher degree of precision in his theory of Mars established at Maragha than the majority of his predecessors; the results were also more accurate than those established in his earlier *zīj* written in Damascus in 1258 AD and used in the official astronomical tables produced at the Maragha observatory, the *Īlkhānī zīj*.

Keywords: Positional Astronomy, Spherical Astronomy, Medieval Astronomy, Islamic Astronomy, Middle East, Mars, Orbital Elments, Muḥyī al-Dīn al-Maghribī, The Maragha Observatory, *Īlkhānī Zīj*, *Talkhīṣ al-Majisṭī*.

In the late Islamic period (post-1050 AD), astronomers turned their attention to testing the accuracy of the astronomical tables compiled in the classical period of medieval astronomy in the Middle East (*ca*. 800-1050 AD). This led to a revision of the fundamental parameter values, theories, and ideas adopted in those works via the execution of new measurements of the parameters at the great observatories founded in Beijing, Maragha, and Samarqand. This paper deals with some of these activities, and presents a critical case study of the only surviving account of the measurement of the structural parameters (eccentricity, radius of the epicycle, and longitude of the apogee) and the mean motion of Mars carried out by Muḥyī al-Dīn al-Maghribī (d. 1283 AD) at the Maragha observatory. The historical data put forward in this paper evidently provide conclusive evidence that the accuracy of Ptolemaic planetary theories was significantly improved over the course of the medieval Islamic period. Although these data have been available to modern scholars since the middle of the past century (or even earlier), it appears that they have not attracted the attention they deserve; so much so, in fact, that the achievements of the medieval Middle Eastern astronomers have often been underestimated: "Muslim astronomers, in spite of much boasting, restricted themselves by and large to the most elementary parts of Greek astronomy: refinements in the parameters of the solar motion, and increased accuracy in the determination of the obliquity of the ecliptic and the constant of precession".[1]

The paper is organized as follows. In the first three parts of Section 1, we briefly review the history of the Maragha observatory, discussing the observational activities carried out and the important works written there, paying particular attention to Muḥyī al-Dīn's al-Maghribī's life and works as well as his *Talkhīṣ al-majisṭī*, which is the main focus of the paper. In the last two parts of this section, we outline Ptolemy's planetary models and Muḥyī al-Dīn's exposition of them. In Section 2, Ptolemy's iterative method of the determination of the orbital elements (eccentricity and direction of the apsidal line) of the superior planets is explained. Section 3 deals with Muḥyī al-Dīn's observations of Mars and the spherical astronomy method he used to derive the ecliptical coordinates of a celestial body from the observational data (meridian altitude and time of the meridian transit). In Section 4, we explain in detail Muḥyī al-Dīn's measurement of the orbital elements, mean motion in longitude, and radius of the epicycle of Mars. In Section 5, we first discuss a hitherto unexamined aspect of his activities in the field of observational astronomy at the Maragha observatory; he appears to have conducted a project to test the accuracy of his predecessors' theories through his observations, as reflected in his provisional use of Ibn al-Fahhād's (*ca.* 1170 AD) value for the mean motion in longitude of Mars in his measurement of the orbital

1. Neugebauer 1975, Vol. 1, p. 145.

elements of the planet. We then propose an unbiased criterion for evaluating the accuracy of the historical values for the orbital elements of the superior planets, which provides us with a quantitative and comparative view of the precision of the values measured for these parameters in the medieval Islamic period. Finally, we compare the accuracy of the theories of Mars established at the Maragha observatory —that is, Muḥyī al-Dīn's theory and the one adopted in the *Īlkhānī zīj*.

As Neugebauer emphasized over half a century ago,[2] any research on a particular subject, treatise, or scholar in ancient and medieval astronomy should include a broad-ranging programme to ensure that all relevant matters are systematically analysed. This is the approach applied to the research in the present study.

1. Introduction

1.1. The Maragha observatory

The Marāgha observatory was built in 1259 AD by Hülegü (d. 1265 AD), the founder of the Īlkhānīd dynasty of Iran.[3] For the sixty or so years of its active life, the observatory represented the pinnacle of Islamic astronomic achievement. The influence of this astronomical academy on the later Islamic observatories founded in Samarqand in the early 15th century, in Istanbul in the 1570s, and by Jai Singh II (1688-1743 AD) in India in the early 18th century can hardly be overestimated. The intellectual atmosphere and the availability of financial resources were conducive to the pursuit of serious, proper research.

During the first two decades of the life of the observatory, two *zījes* were written: Naṣīr al-Dīn al-Ṭūsī's (d. 1274 AD) *Īlkhānī zīj* in Persian and Muḥyī al-Dīn al-Maghribī's (d. June 1283) *Adwār al-anwār* in Arabic. According to the colophon of MS. T of the *Īlkhānī zīj*, this work was completed around the end of

2. Neugebauer 1945/1946, p. 45.

3. Sayılı [1960] 1988, pp. 187-223; some necessary corrections to Sayılı's historical arguments have already been given in Mozaffari and Zotti 2013. It appears that astronomical observations were carried out at Maragha a century before the foundation of the observatory there; in his treatise on the stereographic projection of the celestial sphere (the fundamental basis of the astrolabe), Ibn al-Ṣalāḥ al-Hamadhānī (d. 1153 AD) states that in Maragha, he found a magnitude of 23;35° for the obliquity of the ecliptic (see Lorch 2000, p. 401; Mozaffari and Zotti 2013, p. 51, note 10).

Rajab 670 H/end of February 1272.[4] The majority of the underlying parameter values adopted in it are either Ptolemaic or are borrowed from earlier *zījes*, but some are not known in any earlier texts and appear to be the results of the observational programme of the main staff of the observatory in the 1260s; for example, the solar theory in this work is identical to Ibn Yūnus's (d. 1007 AD) as established in his *Ḥākimī zīj*, except for the value adopted for the longitude of the solar apogee,[5] which, according to Quṭb al-Dīn al-Shīrāzī, an early member of the observatory, was the result of the *new observations* (*raṣad jadīd*) performed at the Maragha Observatory.[6] Another example is the non-Ptolemaic star table in the *Īlkhānī zīj*, in which the ecliptical coordinates of 16 bright stars are tabulated along with the updated longitudes from the star tables of Ibn Yūnus and the *Mumtaḥan zīj*.[7] Of particular interest to the present study is the unprecedented value adopted for the radius of the epicycle of Mars in the *Īlkhānī zīj* (see 5.2.2(B)).

Shams al-Munajjim (Shams al-Dīn) Muḥammad b. ʻAlī Khwāja al-Wābkanawī (1254–after 1316 AD), the outstanding figure of the second period of the Maragha observatory (after 1283 AD), conducted a project to test the accuracy of the theoretical data derived from the *Īlkhānī zīj* and *Adwār al-anwār*, the two products of the first period of the observational activities at Maragha. These data were the times of the synodic phenomena, such as the planetary conjunctions and eclipses, and the longitudes at which they occur, which he checked against his own observations apparently carried out over a long period of 40 years. The first documented observation in his *al-Zīj al-muḥaqqaq al-sulṭānī* (*The verified zīj for the sultan*) is a simple measurement of the lunar altitude from Maragha early on the night of 3 December 1272.[8] He also observed the annular solar eclipse of 30 January 1283,[9] the conjunction between Jupiter and Saturn in 1276 AD, and the triple conjunctions between these two planets in 1305-1306 AD.[10] Furthermore, in the instructions to Ibn al-Fahhād's *ʿAlāʾī zīj* he made for Gregory Chioniades,

4. *Īlkhānī zīj*, T: Suppl. P: f. 31v.

5. See Mozaffari 2018, pp. 229-232.

6. al-Shīrāzī, *Tuḥfa*, f. 38v; *Ikhtiyārāt*, f. 50v.

7. This table has recently been analyzed; see Mozaffari 2016a. For the two star tables in the *Mumtaḥan zīj* and their relation to Ibn al-Aʿlam (d. 985 AD), see Mozaffari 2016-2017.

8. Wābkanawī, *Zīj*, T: f. 89v, Y: f. 155r, P: 135r.

9. See Mozaffari 2013a. 5MCSE, no. 07820.

10. Wābkanawī, *Zīj*, T: ff. 2r-v, 134v-135r, Y: ff. 2v-3r, 235v-236r, P: ff. 2v-3r, 205r-v.

Wābkanawī computes the parameters of the solar eclipse of 5 July 1293 and the lunar counterpart of 30 May 1295 from this work for the longitude of Tabriz (north-western Iran), and Chioniades incorporates these data into its Greek translation.[11] These worked examples were to replace the ones given by Ibn al-Fahhād.[12] In the *Revised Canons* —which was, perhaps, a part of a preliminary version of the *Muḥaqqaq zīj*— Chioniades explains the computation of the parameters of the lunar eclipse of 30 May 1295 and the solar eclipse of 28 October 1296 on the basis of the *Īlkhānī zīj* from the "oral teaching of Shams Bukhārī".[13] Wābkanawī's experiments over a period of 40 years (as reflected in his dated observations) led him to severely criticize the *Īlkhānī zīj* for its obvious errors in the planetary ephemerides, the timing and magnitudes of the eclipses, and the times of the planetary conjunctions computed from this official work produced at the Maragha observatory. Instead, he preferred to refer to Muḥyī al-Dīn's observational programme as the "Īlkhānid observations" (*raṣad-i Īlkhānī*). The most notorious error occurred in the case of the 1305-1306 AD triple conjunctions of Saturn with Jupiter, of which the *Īlkhānī zīj* could predict only the first; in contrast, all three could be predicted on the basis of Muḥyī al-Dīn's last planetary theory established at Maragha. This apparently had a decisive influence on Wābkanawī's final conclusion.

1.2. Muḥyī al-Dīn al-Maghribī

Little is known about Muḥyī al-Dīn except that, according to Ibn al-Fuwaṭī, the librarian of the Maragha observatory, his full name was Muḥyī al-Dīn Abu al-Shukr Yaḥyā b. Muḥammad b. Abī al-Shukr b. Ḥamīd al-Tūnisī (of Tunis) al-Maghribī (of the Maghrib). He had learned Islamic jurisprudence (*fiqh*), according to the *Mālikī* school, in his native city. He spent some years in the service of al-Sulṭān al-Malik al-Nāṣir Yūsuf b. al-ʿAzīz b. Ghāzī b. al-Malik al-Nāṣir Yūsuf b. Ayyūb (reign: 1237-2 October 1260 AD) in Damascus, before the king was

11. Pingree 1985-1986, Vol. 1, Chapters 32-36: pp. 131-169. 5MCSE, no. 07846. 5MCLE, no. 07961.

12. In *ʿAlāʾī zīj* I.35 and I.36: pp. 30-35, Ibn al-Fahhād explains how to compute the parameters of the solar eclipse of 11 April 1176 (5MCSE, no. 07555) and the lunar eclipse of 25 April 1176 (5MCLE, no. 07660). On their accuracy, see Mozaffari and Steele 2015, pp. 347-348, note 17.

13. Pingree 1985-1986, Vol. 1, Chapters XVII-XXII: pp. 307-333. 5MCSE, no. 07853.

killed by the Mongols; at that point, Muḥyī al-Dīn was sent to the Maragha observatory. Other than a short migration to Baghdad in the latter part of the 1270s, he lived in Maragha until his death in Rabīʿ I 682 H/June 1283 AD.[14] He taught students at the Maragha observatory,[15] and wrote some 26 works on mathematics, astronomy, and astrology.[16]

Muḥyī al-Dīn wrote his first *zīj*, the *Tāj al-azyāj wa-ghunyat al-muḥtāj* (*The crown of the zījes, sufficient for the needy*), in Syria *ca.* 1258 AD for the longitude of Damascus. Interestingly, in this work he presents non-Ptolemaic values for the orbital elements of Jupiter, Mars, and Mercury together with a set of new values for the solar, lunar, and planetary mean motions in longitude and anomaly;[17] this *zīj* also contains the most precise value measured for the rate of precession throughout the ancient and medieval periods (1°/72 years).[18] Shortly after his arrival at Maragha, Muḥyī al-Dīn prepared his second *zīj*, the *ʿUmdat al-ḥāsib wa-ghunyat al-ṭālib* (*Mainstay of the astronomer, sufficient for the student*), preserved in a unique manuscript (Cairo, Egyptian National Library, MM 188), *ca.* 1262 AD.[19] This *zīj* is a collection of various materials taken from different sources, so it is difficult to establish which parts are originally from his own independent work. Moreover, more than 40% of the folios are missing from the codex. Indeed, its preserved form has very few characteristics in common with Muḥyī al-Dīn's observational activities in Syria and Maragha and with his other two *zījes* written on the basis of their results, and in fact if his name was not mentioned on its first folio, there would be no indication that he is the author. This work appears to have been formulated during a course of astronomy at the Maragha observatory and was compiled by one of

14. Ibn al-Fuwaṭī, Vol. 5, p. 117. A quotation from Mālik b. Anas can be found in Muḥyī al-Dīn's *ʿUmda*, f. 24r (above the table of the anomaly of Saturn).

15. Among them, Ibn al-Fuwaṭī (Vol. 1, 146-147) mentions ʿIzz al-Dīn al-Ḥasan b. al-Shaykh Muḥammad b. al-Shaykh al-Ḥasan al-Wāsiṭī al-ʿaṭṭār Shaykh Dār Sūsīyān.

16. See Suter 1900, p. 155; Brockelmann 1937-1942, Vol. 1, p. 626; 1943-1949, S1, p. 868; Sarton 1927-1948, Vol. 2, Part 2, pp. 1015-1016; Sezgin 1978, p. 292; Rosenfeld and Ihsanoglu 2003, p. 226. Some of his mathematical works were studied; e.g., see Hogendijk 1993. Tekeli's short entry on al-Maghribī in *DSB* (Gillispie *et al.*, Vol. 9, p. 555) covers only his mathematical works. See, also, M. Comes' entry in *BEA* (Hockey *et al.* 2007, pp. 548-549).

17. See Dorce 2002-2003; 2003.

18. See Mozaffari 2017, p. 6.

19. On the basis of the date of a Ptolemaic star table found in it, for the end of 630 Y/9 January 1262; al-Maghribī, *ʿUmda*, f. 137r.

Muḥyī al-Dīn's students there. In the prologue,[20] we are told that "our master, the learned, the model of the lords of teaching, the prince of the geometricians, Yaḥyā b. Muḥammad b. Abī al-Shukr al-Maghribī, said: 'a group of the friends trying to learn the science of mathematics asked me to lay down a *zīj* for them in order to obtain the ephemerides of the planets, so that its understanding might be easy for a student, its sources might be close at hand for a practitioner, and that it might be a benefit for a beginner and a reminiscence for a professional'". It was in essence a tutorial work for learning mathematical astronomy and the basic structure of astronomical tables, and how to work with them. The ease of understanding and the access to the necessary sources, defined as its main goals, may explain why it is an "essential" collection of previous works. According to Benno van Dalen's thorough but unpublished analysis, the work contains scarcely any parameter values that are either unprecedented or related to his Damascus *zīj* composed four years earlier. In fact, the values for the mean motions and orbital elements underlying its tables are mostly taken from earlier *zījes*: the solar mean motion and equation tables are based on Ibn Yūnus's solar theory; the lunar mean motion in longitude derives from al-Battānī (d. 929 AD), and in anomaly from Ibn Yūnus; and in the case of the planets, the *ʿUmda* is completely dependent upon Ibn al-Fahhād's *ʿAlāʾī zīj*. The equation tables are also closely related to the *Īlkhānī zīj*; in the case of Saturn and Jupiter, we are told that they are based on the Īlkhānīd parameter values [derived] from the observations by al-Ṭūsī. Of course, Ptolemy's values for the orbital elements of the planets underpin all of equation tables, and the only difference is in their format (they are displaced, asymmetrical, and always-additive). However, it is very difficult to determine exactly whether they were later used to prepare the *Īlkhānī zīj* or whether they were taken from the *Īlkhānī zīj* and inserted into this codex. Of course, there are some differences, for example, in the case of the table of the epicyclic equation of Mars, which is not based on the new parameter value adopted in the *Īlkhānī zīj*. However, the similarities between the *ʿUmda* and *Īlkhānī zīj* are persuasive enough to conclude that, in accordance with van Dalen's hypothesis, the *ʿUmda* proves to be a side project of working on the *Īlkhānī zīj*. Support for this hypothesis comes from the common values for the stellar coordinates in Ibn Yūnus's star table as preserved in the *ʿUmda* and the *Īlkhānī zīj*, which cannot be found in other source of Ibn Yūnus's star table.[21] If this is indeed

20. al-Maghribī, *ʿUmda*, f. 1v.
21. See Mozaffari 2016a, p. 300.

the case, then the *Īlkhānī zīj* evolved from a tutorial work by Muḥyī al-Dīn, and it is very surprising that his significant contribution to this *zīj* was totally ignored by (or was unknown to) al-Ṭūsī.

Muḥyī al-Dīn successfully performed an extensive systematic observational programme at the Maragha observatory from 1262 to 1274 AD, independently of any other astronomical activity conducted there. He explains in detail his observations and measurements of the solar, lunar, and planetary parameters in his *Talkhīṣ al-majisṭī* (*Compendium of the* Almagest). The parameter values resulting from these activities were incorporated into his third *zīj*, *Adwār al-anwār mada 'l-duhūr wa-'l-akwār* (*Everlasting cycles of lights*, also called as *Zīj-i kabīr*, *Great zīj*).[22] Muḥyī al-Dīn's systematic observations at the Maragha observatory established him as an outstanding figure, so much so that his contemporaries and immediate successors referred to him using unique honorific titles that reflected his skill in astronomical observations. For instance, Ibn al-Fuwaṭī calls him *al-muhandis al-raṣadī*, the "geometrician of the observations".[23] His observational programme is referred to as the "Īlkhānid observations" or the "new Īlkhānid observations" (*al-raṣad al-jadīd al-Īlkhānī*).[24] His fame was so widespread that his astrological doctrines were widely trusted (nine of his treatises are on astrology). A prime example of this is the interpretation of the appearance of the comet C/1402 D1 on the basis of his astrological doctrines, which triggered a major war in the Middle East at the turn of the 15th century.[25] None of Muḥyī al-Dīn's new parameter values were used in the *Īlkhānī zīj*.[26] It is also noteworthy

22. Kamālī, *Ashrafī zīj*, F: ff. 231v, 232r, G: f. 248v; al-Kāshī, *Khāqānī zīj*, IO: f. 104r. Kāshī refers to Muḥyī al-Dīn as the sage/wise (ḥakīm).

23. Ibn al-Fuwaṭī, Vol. 5, p. 117.

24. In Mozaffari and Zotti 2013, all the indications of these terms, as found in the works written either during the lifetime of the Maragha observatory or afterwards, are introduced.

25. See Mozaffari 2016b.

26. Except, perhaps, for his value of 23;30° for the obliquity of the ecliptic, resulting from the measurements of the maximum and minimum annual solar noon-altitudes performed on three successive days after the two dates of 12 June and 7 December 1264; Muḥyī al-Dīn's altitude values (76;9,30° and 29;9,30°) strictly result in the value of 23.5° for the obliquity of the ecliptic. 9 December 1264 was four days prior to the day of the winter solstice (13 December 1264). It is surprising that the decrease in the solar meridian altitude in the days after 9 December should have escaped Muḥyī al-Dīn's attention. In the *Īlkhānī zīj* (C: p. 203, T: f. 102v, P: f. 59v, M1: f. 104v, M2: f. 89v), al-Ṭūsī remarks that "on the basis of our observations, the obliquity of the ecliptic **exceeds 23;30° by a small amount**, and we estimated it to be 23;30°" (the emphasis is added). Also, in his *Risāla*

that his astronomy does not show any influence from Western (Andalusian and Maghribī) Islamic astronomy; the only incidental indication in the *Talkhīṣ* of his Andalusian astronomical background is an error of +1/2 day in Meton's and Euktemon's value for the length of the solar year, which he quotes from *Almagest* III.1 in *Talkhīṣ* III.4, and which can be traced back to some Maghribī/Andalusian MSS of the *Almagest*.[27] In contrast, some traces of the influence of the *Tāj al-azyāj* on the Western Islamic *zījes* have been brought to light by Julio Samsó.[28]

The *Talkhīṣ* is preserved in a unique copy (Leiden, Universiteitsbibliotheek, Orientalis 110) in al-Maghribī's handwriting. According to the table of contents given on f. 2r, the treatise consists of ten books (*maqāla*). They discuss plane and spherical trigonometry (books I and II), spherical astronomy (III), solar theory (IV), lunar theory (V), lunar parallax and theory of eclipses (VI), stellar astronomy (VII), planetary theory in longitude (VIII), retrograde motion and latitude of the planets (IX), and the stereographic projection of the celestial sphere on the plane tangential to its north pole (X). The manuscript is incomplete and corrupt; when the author finishes his computations of Mars, the reader would expect him to commence the computations related to the inferior planets. It is probable that the author never managed to write any more of the treatise. The last two books are also missing from this copy, but the contents of the last book would probably have been adapted from (possibly, a brief survey of) his treatise on the astrolabe, which deals with the same topic. Muḥyī al-Dīn dedicated the *Talkhīṣ* to Ṣadr al-

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fī kayfiyyat al-irṣād (The treatise on how to make [astronomical] observations) (P: f. 7v, N: f. 41r), al-ʿUrḍī states that the same value was known due to the continuous observations in Maragha.

27. Toomer [1984] 1998, p. 139; *Arabic Almagest*, Isḥāq-Thābit: S: f. 30r, PN: f. 33r, Pa1: f. 45v, Pa2: f. 45v, TN: f. 40v, Ḥajjāj: LO: f. 60v, LE: f. 36r; Neugebauer 1975, Vol. 2, p. 601. Meton's and Euktemon's value for the length of the solar year is equal to 365 1/4 + 1/76 days, but Muḥyī al-Dīn has… 1/76.5 days. This erroneous value can be found in the 14th-century Andalusian MS. PN and the Maghribī MS. TN of the Arabic *Almagest*, but the other MSS consulted of both Arabic translations of the *Almagest* have the correct value. al-Ṭūsī also has this faulty value in his *Taḥrīr al-majisṭī* (P1: p. 87, P2: f. 24v, P3: f. 41r). Both al-Maghribī's *Talkhīṣ* and al-Ṭūsī's *Taḥrīr* are based upon Isḥāq-Thābit's translation, and among the MSS we have consulted for the present study only the two abovementioned MSS of this translation share the error in question. Although it is extremely difficult to determine conclusively where and when this error entered the text for the first time, it appears to have originated somewhere in the Eastern Islamic world prior to *ca*. 1260, and then passed into the Western Muslim realm, perhaps, via media of a prototype to which a group of the Andalusian/Maghribī MSS of the *Almagest* belong, including our MSS. PN and TN.

28. See Samsó 1998, pp. 94-97; 2001, pp. 168, 170, 172, 173.

Dīn Abū al-Ḥasan ʿAlī b. Muḥammad b. Muḥammad b. al-Ḥasan al-Ṭūsī,[29] the son of Naṣīr al-Dīn al-Ṭūsī, who was appointed director of the observatory after the death of his father.[30] As described in the *Talkhīṣ*, Muḥyī al-Dīn's period of observations at the Maragha observatory extended from 7 March 1262 AD (lunar eclipse) to 12 August 1274 AD (the meridian transit of Jupiter).

A manuscript of the *Adwār al-anwār* which was apparently written under Muḥyī al-Dīn's supervision is preserved in Iran (Mashhad, Holy Shrine Library, no. 332), in which our author records the dates of the completion of its explanatory parts (canons) and of its tables, respectively, as Dhu al-qaʿda 674 H (April-May 1276)[31] and Rajab 675 H (December 1276-January 1277).[32] Thus, he seems to have finished his observations between 12 August 1274 and April-May 1276, and was busy with the construction of the tables between April and December 1276. In the prologue of this copy of the *Adwār* he also mentions that he began to write it after completing a (now lost) treatise entitled *Manāzil al-ajrām al-ʿulwiyya* (*The mansions of the upper bodies*). Thus, we can safely assume that Muḥyī al-Dīn wrote the *Talkhīṣ* after the *Adwār*, i.e., after 1276 AD. According to Ibn al-Fuwaṭī,[33] Muḥyī al-Dīn left the observatory and spent a while in the service of al-Ṣāḥib Sharaf al-Dīn b. al-Ṣāḥib Shams al-Dīn in Baghdad. The date of his departure is not given, but it very likely occurred after he finished writing the *Adwār*. Ibn al-Fuwaṭī's statements give the strong impression that his departure was due to certain difficulties in Maragha after the death of al-Ṭūsī, because he states that after Muḥyī al-Dīn's return to Maragha, he was honoured and supplied with substantial stipends and honoraria. It is quite probable that he wrote the *Talkhīṣ* after his return to Maragha from Baghdad, when the observatory was under the direction of Ṣadr al-Dīn, and its dedication to Ṣadr al-Dīn was regarded as an act of thanksgiving.

The *Talkhīṣ*, its characteristics, and its place in the history of Islamic observational astronomy were already described in three papers by George Saliba in the 1980s, and the solar and lunar theories in it have been thoroughly studied by the present author.[34] It resembles the analogous treatises belonging to the category of

29. al-Maghribī, *Talkhīṣ*, f. 2r.

30. Sayılı 1960 [1988], p. 205.

31. al-Maghribī, *Adwār*, M: f. 55v.

32. al-Maghribī, *Adwār*, M: f. 124v.

33. Ibn al-Fuwaṭī, Vol. 5, p. 117.

34. The contents of the treatise were described in Saliba 1983. The computations related to the eccentricity of the Sun and of Jupiter were addressed in two critical studies by G. Saliba (1985,

the "*Almagest* revision/rewritten/exposition…" only in the title and subject. These treatises, sometimes accompanied by criticisms of Ptolemy, constituted a genre with its own particular characteristics and played a pivotal role in medieval Islamic theoretical astronomy. However, despite its name, the *Talkhīṣ* is neither a rewriting, nor an abridgement of the contents of the *Almagest*; a significant feature that sets it apart from other similar treatises is the abundance of its observational records, whereas not a single report of an observation can be found in the others. In this respect, the *Talkhīṣ* has no counterpart in the medieval astronomical corpus of the Middle East. Like Ptolemy in the *Almagest*, Muḥyī al-Dīn explains how he systematically established his parameter values, starting from the measurement of the latitude of Maragha, the length of the solar year, the solar mean daily motion in longitude, up to the planetary parameters. There is no denying that the treatise left its mark on other developments in late Islamic astronomy; indeed, it led Taqī al-Dīn Muḥammad b. Maʿrūf (1526-1585 AD), the director of the short-lived observatory at Istanbul, the owner of its surviving manuscript, to register the reports of his observations carried out in Istanbul about two centuries later.[35]

Broadly speaking, each section of the *Talkhīṣ* consists of the following subsections:

(*a*) The narrative subsection, divided into the two major parts: first, a summary of pertinent passages from the *Almagest* with a focus on certain specific topics, and second the reports of Muḥyī al-Dīn's observations carried out at the Maragha observatory;
(*b*) The observational subsection: to present his measurements, Muḥyī al-Dīn first puts forward the data derived from his dated observations, which are arranged chronologically;
(*c*) The computational subsection: he then explains the computational procedure used to determine a parameter. Finally, the calculations which apply the observational data are described in detail. The result obtained is either established as a new value or is verified as an already existing one for that parameter; and

1986). For al-Maghribī's solar theory, see Mozaffari 2013b, pp. 318, 330; 2018, esp. pp. 229, 235. For his lunar measurements, see Mozaffari 2014a.

35. On them, see Mozaffari and Steele 2015.

(*d*) The cosmographical subsection, which appears only at the end of three books, maybe the least important of the treatise; this includes the configuration of the spheres of the Sun, Moon, and planets, according to Ptolemy's *Planetary Hypotheses*, added apparently only for the sake of completeness.

These materials allow us to trace Muḥyī al-Dīn's line of investigation in great detail, to identify his mistakes and their effects on the final results, and to recognize the probable circular arguments, and so on, which are very useful for making a comprehensive critical study.

1.3. The history of the unique MS of the Talkhīṣ al-majisṭī

The history of the surviving manuscript of the *Talkhīṣ* can be retrieved from a number of the statements regarding ownership and marginal annotations left on some of the folios. The earliest of them appears to be the one written in the upper left corner of f. 1r, which belongs to a certain ʿAbd al-ʿAzīz ʿAlī b. Muḥammad al-Munajjim, and which dates from 20 Ramaḍān 746/14 January 1346. The next one is a passage written on the same folio, opposite the abovementioned remark, in which we are told that the manuscript is an autograph;[36] the owner's name, place, and time are barely legible: ʿAbd al-Bāqī Kamāl al-Tūkānī in the city of Kāshān (central Iran)… in the year 757 H/1356 AD (?).

With a lacuna of about two centuries, the MS found its way to Cairo some time before 1544 AD, and was then transmitted to the library of the short-lived observatory in Istanbul where it was placed at the disposal of the two outstanding astronomers working there until 1585 AD at the latest. One of these astronomers was ʿAlī al-Riyāḍī (ʿAlī the mathematician), a scribe who seems to have lived and worked in Cairo about 950-951 H/1543-1544 AD. To the best of our knowledge, he is unknown in the modern literature, but besides the Leiden MS of *Talkhīṣ*, some information about him can be found in a codex preserved at the Rare Book and Manuscript Library at University of Pennsylvania, LJS 412, which is the collection of the following four treatises on astronomical instruments:

36. The survival of an autograph from 13th-century Maragha is not unusual —a good number of manuscripts written in the hand of Quṭb al-Dīn al-Shīrāzī survive (on them, see Pourjavady and Schmidtke 2007).

(i) *Al-ʿAmal bi-'l-kura* of Qusṭā b. Lūqā (ff. 1v-10v, copied on 10 Ṣafar 950/15 May 1543);

Two treatises by Ḥabash b. ʿAbd-Allāh al-Ḥāsib:

(ii) *Risāla fī kayfiyyat al-irṣād wa aʿmāl dhāt al-ḥalaq* (*How to make* [*astronomical*] *observations and the applications of the armillary sphere*) (ff. 11r-12r) and
(iii) *Al-ʿamal bi-'l-asṭurlāb al-kurī wa ʿajāʾibi-hī* (*The application of the spherical astrolabe and its marvels*) (ff. 12v-13v);[37]

and

(iv) *Tasṭīḥ al-asṭurlāb* of Muḥyī al-Dīn al-Maghribī (ff. 14r-18v, copied in mid-Ṣafar 950).

The latter is called *Fī rasm al-asṭurlāb*, and is ascribed to ʿAlī al-Riyāḍī himself on f. 14r, but this cannot be considered a demonstrated case of plagiarism, since some folios between 13v and 14r, containing the initial part of the treatise, are missing from this MS.[38] In it, the scribe introduces himself as ʿAlī al-Riyāḍī *al-munajjim fī bāb/ʿatabat al-salṭanat al-Qāhira/al-qāhira* (on the front cover, ff. 1v, 10v, and 18v). The last word may refer to Cairo (*al-Qāhira*) or may literally mean "the victorious" (*al-qāhira*). In the first case, the sentence would mean "ʿAlī the mathematician, the astronomer in the royal place/threshold in Cairo" and in the latter case, "ʿAlī the mathematician, the astronomer in the victorious royal place/threshold", and thus he could have worked from Istanbul.[39] In the Leiden MS of *Talkhīṣ*, he claims to be its owner, first at the end of a comment on f. 58v:[40] *al-*

37. I have checked this against another copy preserved in Tehran, Library of Mahdawī, no. 503, ff. 95r-100r. Both treatises (ii) and (iii) are wholly different from Ḥabash's treatise on the structure and applications of the astronomical globe, titled *Fī maʿrifat al-kura wa-'l-ʿamal bihā*; for its edition and study, see Lorch and Kunitzsch 1985; also, for the other instrument proposed by Ḥabash, a universal plate for timekeeping by the luminous clock stars, see Charette and Schmidl 2001.

38. This treatise was edited by the present author in his M.Sc. thesis in 2006.

39. I am grateful to an anonymous referee for bringing this point to my attention.

40. In this comment, it is said that on the basis of the value of 365;14,30 days that Muḥyī al-Dīn derived for the length of the solar year, the length of the true solar month is equal to 30;26,12,30 days.

muḥarrir ʿAlī al-Riyāḍī *ṣāḥib al-kitāb*, "The scribe, ʿAlī al-Riyāḍī, the possessor of the book", as well as in the ownership statement in the end of the treatise, on f. 137r, in the same typical phrase that can be found in the Pennsylvania MS; below the latter passage, we are told that the MS was given to a certain ʿUqlidis Najm al-Dīn, otherwise unknown, on 11 Ṣafar 951/4 May 1544.

The two comments written by *al-Muḥarrir* al-Riyāḍī (on f. 67r) or simply *al-Muḥarrir* (on f. 137r) can also be found in the Leiden MS written in a hand different from the one used in two abovementioned samples from ʿAlī al-Riyāḍī, but they appear to have been left by ʿAlī al-Riyāḍī himself. It was by no means extraordinary that scribes were able to write in different handwriting styles and in various types of Islamic calligraphy; ʿAlī al-Riyāḍī exhibits the variety of his handwritings in the Pennsylvania codex; as can be seen clearly on f. 10v of this MS, the body of the treatise and the closing versified statement which contains the year and name of the scribe, are written in two distinctly different hands. On f. 137r of the Leiden MS the ownership statement of al-Riyāḍī is seen at the bottom and a comment by *al-Muḥarrir* on the left side.[41]

The other commentator is a person named al-Riyāḍī who is not, of course, ʿAlī al-Riyāḍī himself. This person signs his remarks with a special signature of his own, and his comments can be found on ff. 18r, 19r, 31r, 43v, 58r, and 64r. He is quite probably Dāwūd al-Riyāḍī of Thessalonica, whom A. Ben-Zaken identifies as David Ben-Shushan, a Jewish scholar. He was a colleague of Taqī al-Dīn Muḥammad b. al-Maʿruf and observed the total lunar eclipse of 26/27 September 1577 from Thessalonica; he supplied corroborating data to Taqī al-Dīn, who could not observe it due to the cloudy weather in Istanbul at the time.[42]

The last and most famous commentator is Taqī al-Dīn, whose only comment is found on f. 50v[43] in which he criticizes Muḥyī al-Dīn's method of deriving the ecliptical coordinates from the horizontal ones (see below, Section 3). His ownership statement is found on f. 1r, which is similar to the one on the title folio of the Leiden MS of Ibn Yūnus's *Ḥākimī zīj*.[44]

41. The comment reads: the end of 600 Y [the epoch of Maragha *zījes*] is equal to 23 Rabīʿ I 629 [according to the astronomical reckoning of the Hijra calendar]; from it to the beginning of the year 841 H, when the observations at Samarqand were made, there are 211 complete Arabic years.

42. See Mozaffari and Steele 2015. On David Ben-Shushan, see Ben-Zaken 2010, esp. pp. 21-24; 2011, pp. 132-134.

43. See below, note 81.

44. See King 2004-2005, Vol. 1, p. 284.

1.4. Ptolemy's Planetary Models

According to Ptolemaic planetary models,[45] a planet revolves on an epicycle whose centre C rotates in the direction of increasing longitude on an eccentric deferent with centre O removed from the Earth T by an eccentricity $OT = e$ (see Figure 1), but its motion is uniform with respect to the equant point E removed from the Earth in the same direction by an eccentricity $ET = 2e$. The extension of TE defines the apsidal line with A as the apogee and Π as the perigee, the spatial direction of which is sidereally fixed. The mean motion in longitude with the daily rate of ω_λ is measured with respect to the vernal equinox. The first inequality of the planet is related to the ecliptic, in the sense that its apparent angular velocity is different in the various longitudes it occupies throughout the ecliptic. This can be explained with the use of the eccentric deferent, and so the inequality can be quantified by the difference between the mean and true longitude of C, respectively as measured from E and T; thus, $\angle ECT$ defines the first correction named the "equation of centre" c_1 which is a double-variable function of e and the mean eccentric anomaly $\bar{\varkappa} = \bar{\lambda} - \lambda_A$ (the daily mean motion in the eccentric anomaly is then equal to $\omega_\lambda - \psi_d$, where ψ_d is the daily precessional motion). The second inequality of the apparent motion of the planet is related to the Sun, and the epicycle is used in order to account for the resulting synodic phenomena. The epicyclic anomaly is uniform, with a daily rate of ω_a, with respect to the line drawn from E to C; the extension of this line intersects the outer rim of the epicycle at the mean epicyclic apogee A_p, and the extension of line TC defines the true epicyclic apogee A_p'; the angular difference between the mean and true epicyclic anomaly respectively, $\bar{\alpha} = A_pCP$ and $\alpha = A_p'CP$ is, indeed, equal to c_1. The planet's motion on the epicycle causes its true longitude, the ecliptic point which the extension of line TP points to, and is different from the longitude of C; the difference between the two is the second correction, called the epicyclic equation c_2, which is a triple-variable function of e, $\bar{\varkappa}$ and $\bar{\alpha}$. For the superior planets, the direction of the epicycle radius points to the mean Sun, i.e., $\omega_\lambda + \omega_a = \omega_\odot$, but for the inferior planets, the vector EC holds it, i.e., $\omega_\lambda = \omega_\odot$.

45. For Ptolemy's models and their properties which, due to page constraints, cannot be explained in detail here, see: Pedersen 1974, chapters 9 and 10; Neugebauer 1975, Vol. 1, pp. 145-190; Duke 2005.

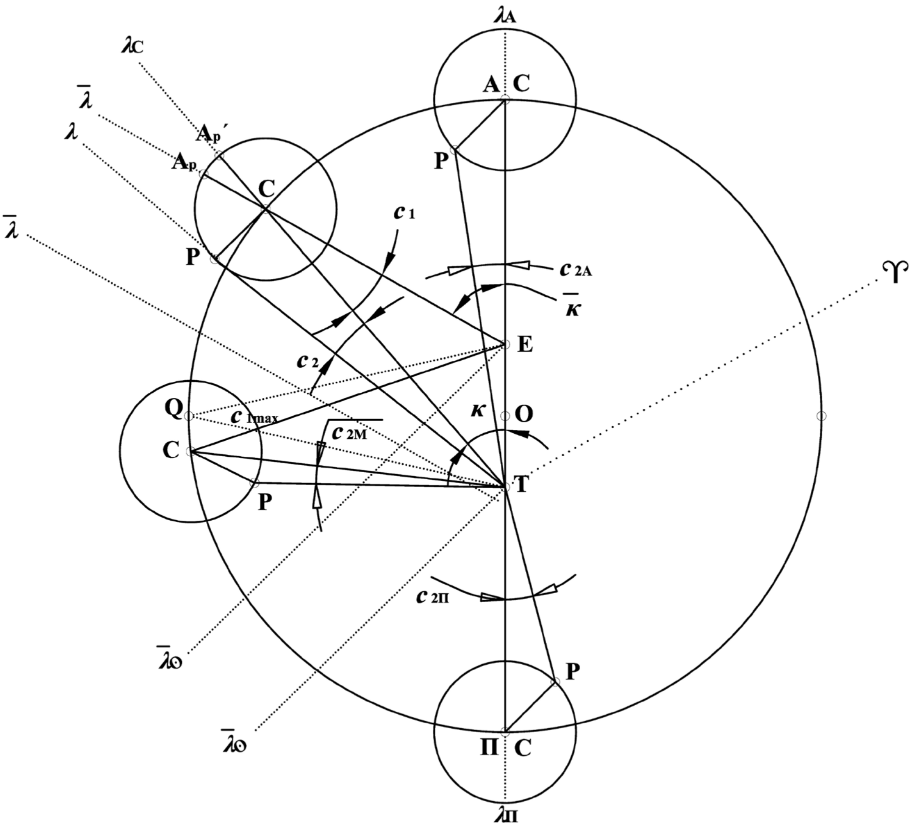


Figure 1. Ptolemy's model of the superior planets and Venus.

This model works well for the superior planets and Venus. For Mercury, Ptolemy constructs a complicated model with a movable deferent, like that of the Moon. In it, the equant point E is far from the Earth T by a single eccentricity e_1, and the centre O of the deferent is movable on the circumference of a small circle of radius e_2, whose centre O' is located at a distance e_2 from E, and so is removed from the Earth by $e_1 + e_2$ (Figure 2). The deferent centre moves uniformly with the angular velocity ω_λ in the direction of decreasing longitude, and the epicycle centre also moves uniformly at the same speed with respect to E, but in the direction of increasing longitude. By this model, Mercury is located at twice the least distance from the Earth, which, of course, unlike the other planets, does not occur at $\bar{\varkappa} = 180°$, but at $\bar{\varkappa} \approx 120°$ (by deploying the *Almagest* values $e_1 = e_2 = 3$, if the radius of the deferent $R = OC = 60$), as displayed in Figure 2. It is evident that in this model, the total eccentricity of the planet varies from a maximum of $e_1 + 2e_2$, when the epicycle centre lies at the apogee, to a minimum of e_1, when the deferent centre coincides with the equant point and the epicycle centre is located on the counter-apogee point A' at a distance $R - e_1$ from the Earth.

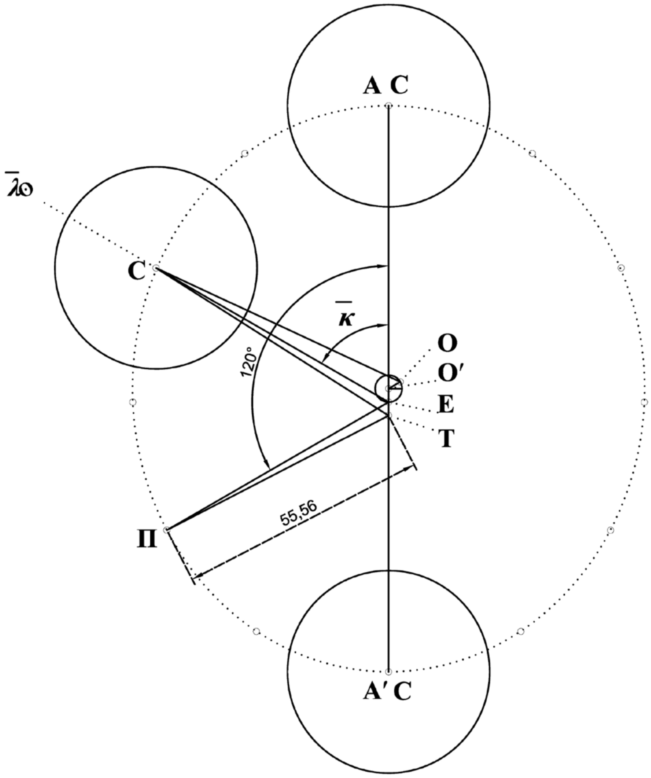


FIGURE 2. Ptolemy's model for Mercury.

In the case of the Moon, Bīrūnī presents an excellent geometrical proof that the trajectory of its epicycle centre has an ellipse-like shape, but does *not* actually describe a perfect ellipse;[46] this proof may also hold for Mercury. A century later, Ibn al-Zarqālluh considered an elliptical deferent for Mercury in order to construct an

46. Bīrūnī, *al-Qānūn* VII.7.1: 1954-1956, Vol. 2, pp. 794-795. It was elucidated in detail in MOZAFFARI 2014b. For an extensive treatment of the path of Mercury's epicycle centre, see Hartner 1955, pp. 109-117.

equatorium for finding the longitude of the planet geometrically, and almost four centuries later, Kāshī deployed it in his own invented device, the *Ṭabaq al-manāṭiq* ("Plate of zones") as described in his treatise *Nuzha al-ḥadāʾiq* (*A fruit-garden stroll*) completed in 818 H/1415 AD.[47] But, somewhat surprisingly, neither of them considered that the oval shape of the path of the epicycle centre is more evident in the case of the Moon, since neither in Ibn al-Zarqālluh's geometrical method for finding the lunar distances nor in the lunar plate of al-Kāshī's instrument was such a smart and useful auxiliary consideration taken into account.[48] Of course, over a decade later, when Kāshī was a fellow at the Samarqand observatory, he appended 10 "afterthought" passages (*Ilḥāqāt al-ʿashara*, completed on 15 Shaʿbān 829/22 June 1426) to his treatise, the first two of which are on the ellipsoidal shapes of the paths of the centres of the epicycles of the Moon and Mercury; Kāshī emphatically asserts that from the geometrical proofs and computational procedures, he obtained that the shapes of both trajectories are neither ellipsoidal (*ihlīlajī*) to be drawn by a compass, nor any [of the known conic] section[s] to be drawn by a "Perfect Compass" (*farjār al-tāmm*),[49] but are similar/close to an ellipsoid; of course, no proof is given there, and he only puts forward an approximate method for drawing the ellipsoidal deferents.[50] Quite probably, it was because of the unique influence of Bīrūnī's *al-Qānūn* on the later medieval Middle Eastern astronomers that the oval trajectories of the epicycle centres of the Moon and Mercury were schematically represented in some cosmographical works of the Maragha school, most notably, in al-Shīrāzī's three well-known treatises, in which he mentions that the path of the centre of the lunar epicycle is ellipsoidal (*ihlīlajī bayḍī*), but not an ellipse (*qatʿ nāqiṣ*, "deficient [conic] section"); of course, he gives no geometrical proof, and does not identify his source. In the case of Mercury, he also refers solely to the el-

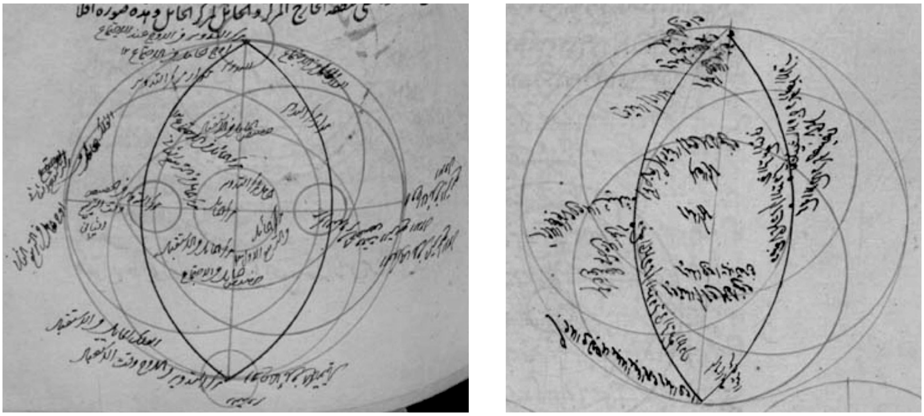
47. Kāshī, *Nuzha*, esp. pp. 266-267; Hartner 1955, pp. 118-122; Samsó and Mielgo 1994; Kennedy 1950/52, Part 2, pp. 46-47 and 49-50; for Kāshī's treatise and instrument, esp. see Kennedy 1960.

48. See Puig 1989; Kennedy 1950/52, Part 1, pp. 182-183. Like Ibn al-Zarqālluh, Georg von Peurbach (1423-1461), in his *Theoricae Novae Planetarvm*, maintains the oval trajectory of the epicycle centre only in the case of Mercury, without referring to the existence of a similar situation for the Moon; see Hartner 1955, pp. 130, 132-134; Aiton 1987, p. 26 (his figure, captioned "Figure 16", is on p. 27).

49. A specialized compass for drawing the conic sections; e.g., see Sezgin and Neubauer 2010, Vol. 3, pp. 149-151.

50. Kāshī, *Nuzha*, pp. 289-291. The fact that Kāshī wrote the appendices at Samarqand is explicitly stated on p. 299. See also Kennedy 1960, pp. 170-174.

lipsoidal shape of the trajectory as drawn in the pertinent figure (see Figures 3(a) and 3(b)).[51] Similarly, in his preliminary cosmographical work, *Lubāb* I.5,7, Kāshī briefly notes the ellipsoidal paths of the epicycle centres of the Moon and Mercury.[52]



(a) (b)

FIGURE 3. (a) The ellipsoidal shape of the trajectory of the lunar epicycle centre and (b) that of Mercury from al-Shīrāzī's *Nihāya al-idrāk*.

For the derivation of the fundamental orbital elements of the superior planets, e and λ_A which define the spatial coordination of the geocentric orbit, a triple opposition of the planet to the mean Sun is needed for each superior planet (see Section 2); the size of the epicycle can be computed from the angle subtending the epicycle radius as seen from the Earth, which can be measured by only one further observation, for which Muḥyī al-Dīn makes use of his observations of the near appulses of the superior planets to the star Regulus (α Leo; see Section 3).

51. Shīrāzī, *Ikhtīyārāt*, ff. 66r-v, 84v; *Nihāya*, P1: ff. 34r, 41v, P2: ff. 82r-v, 103v; *Tuḥfa*, ff. 51r, 61v. Hartner (1971, pp. 282-287) notes a mistake he made due to a confusion between Peurbach's consideration of the ellipsoidal figures described by the apogee and counter-apogee of Mercury, which results from the motion of the centre of the deferent of the planet on a small circle according to the Ptolemaic model, and Quṭb al-Dīn's model of trepidation accounting for the change in the obliquity of the ecliptic. But, surprisingly, Hartner says nothing of the resemblance of Quṭb al-Dīn and Peurbach's figures of the ellipsoidal path of the centre of the epicycle of Mercury, which is much more evident.

52. Kāshī, *Lubāb*, N: ff. 7r, 9r-v, P1: pp. 187, 191-192, P2: ff. 8r, 10v. Note that the *Lubāb* was written before the *Nuzha* and, of course, its appendices. It seems that Kāshī was already aware of the ellipsoidal shape of the trajectories, but it took him some time to consider the matter in depth and to reach his final opinion, as presented in the *Nuzha*.

In the case of the inner planets, the observations of two symmetrical maximum elongations are necessary, at least, for the determination of the direction of the apsidal line, and those of two other maximum elongations when the mean Sun is located at either one of the apses, are required for the measurement of the deferent eccentricity and the epicycle radius. If a further test of bisection of the eccentricity is intended, another maximum elongation of the planet is observed when the centre of the epicycle is in an eccentric anomaly of 90° further from the apogee.

Following the tradition established by Ptolemy's *Almagest*/*Handy Tables* for the reduction of his elegant planetary theories to numerical tables, the medieval astronomers usually tabulated the mean motions $\bar{\alpha}$ and $\bar{\varkappa}$ or $\bar{\lambda}$ which are the linear functions of time. The true longitude of a planet is computed as follows:

$$\lambda_C = \bar{\lambda} \mp c_1 \quad \varkappa = \bar{\varkappa} \mp c_1 \quad (-: 0 \leq \bar{\varkappa} \leq 180°; +: 180° \leq \bar{\varkappa} \leq 360°)$$
$$\alpha = \bar{\alpha} \pm c_1 \quad (+: 0 \leq \bar{\alpha} \leq 180°; -: 180° \leq \bar{\alpha} \leq 360°)$$
$$\lambda = \lambda_C \pm c_2 = \bar{\varkappa} + \lambda_A \mp c_1 \pm c_2 \; (+: 0 \leq \alpha \leq 180°; -: 180° \leq \alpha \leq 360°)$$

Ptolemy uses the following approximate interpolation scheme to reduce the calculation of c_2 to a multiplication and an addition:

$$c_2 = c_{2M} + (c_{2M} - c_{2A}) \times f(\bar{\varkappa}) \quad \text{if } 0 \leq \bar{\varkappa} \leq \bar{\varkappa}_M$$
$$c_2 = c_{2M} + (c_{2\Pi} - c_{2M}) \times f(\bar{\varkappa}) \quad \text{if } \bar{\varkappa}_M \leq \bar{\varkappa} \leq 180°$$

in which c_{2M} is the epicyclic equation in the mean distance ($TC = OC = R$), where the eccentric anomaly is equal to $\bar{\varkappa}_M$, $c_{2\Pi}$ at the least distance, and c_{2A} at the greatest distance, and f is an interpolation function. In the equation tables of Ptolemaic type each of the components of the equations, i.e., c_1, c_{2M}, f, and (c_{2M} - c_{2A}) together with ($c_{2\Pi}$ - c_{2M}) are tabulated in a separate column/table, and due to the symmetry of the underlying trigonometrical functions a double-entry table/column is sufficient for each; for half of the arguments, the entries are subtractive or additive, but the opposite is true for the other half. Medieval Islamic astronomers devised various simple methods for arranging the equations in the displaced asymmetrical equation tables in which the entries are always-additive, so that a practitioner did not need to worry much about the addition-subtraction procedure. Muḥyī al-Dīn's equation tables in both the *Adwār* and the *Tāj* are similar to Ptolemy's, which seems to have been a customary procedure among astronomers with a series of non-Ptolemaic underlying quantities, but the famous *zījes* in which Ptolemy's parameter values were adopted very often deployed these "conventional equation tables", as called by Kamālī who reviews these methods in his *Ashrafī zīj* (some of these methods will be briefly explained later in the apparatus to Tables 6(a)-6(c)).[53]

1.5. A Review of the Introductory Chapters on the Planetary Models in the Talkhīṣ

In VII.4, Muḥyī al-Dīn begins by briefly stating that the apparent anomalies we see in the motion of each of the five planets can be defined by the simple uniform circular motion, which is proper to the heavenly objects, as we have said in the case of the Sun. Attaining this hypothesis/model (*aṣl*) [as that of the Sun for the planets] is of great value and benefit, but it is difficult, because of the weakness of [an observer's] vision and his failure to perceive their motions at the times of observations. For this reason, if there is a deviation in their motions, the error becomes apparent later: quite soon, if the period [at which the motion is measured] is short and less soon, if it is long. And for the same reason, a significant doubt is raised in the sizes of their two inequalities/anomalies, which are combined, but different in size and period: The one is related to the Sun (visibility-invisibility and station-retrogradation), and the other to the ecliptic position (the variation in speed and the motion in latitude). These situations are not specific to a single longitude, but are transformed to all degrees of the ecliptic in the sense [of the zodiacal signs]. It is difficult to isolate one of the inequalities from the other without expending more effort (*tasāhul*) or making more exact observations, especially the observations of the first visibility/invisibility or in the days of the occurrence of the station and retrogradation; the start of these phenomena cannot be fixed, since the motions of the planets cannot be perceived for several days before or after their occurrence. Also, the errors occur in the starting times of the visibilities owing to the differences in the atmospheric situation or in the observer's vision. Then, Muḥyī al-Dīn proceeds to explain why the observation of the planets with stars involves difficult computations and also

53. Kamālī, *Ashrafī zīj* III.5: F: ff. 48r-49r, G: ff. 52r-53v. On the methods of displacements in the medieval planetary equation tables, see, e.g., Samsó and Millás 1998, pp. 270-272; Brummelen 1998; van Dalen 2004a; Chabás and Goldstein 2013; Chabás 2014, pp. 34-39; and the other references mentioned therein.

guesswork: this is mainly because the circle passing through them does not always form right angles with the ecliptic, and also because of the optical illusion (well-known today) which causes the two objects to appear closer to each other when culminating than near the horizon. These matters are in reality an abridgement of the first paragraph of *Almagest* IX.2, which is followed by Ptolemy's last paragraph in the same chapter; in the latter, Ptolemy recommends two types of observations as being the most reliable: the observations of the planetary appulses to the fixed stars and of the conjunctions with the Moon, and those made by an armillary sphere.[54] Thereafter, recapitulating *Almagest* IX.5,[55] our author explains that [for the inequality relative to the Sun,] the time from the maximum to the mean velocity is greater than the period from the mean to the minimum speed, a situation which cannot be accounted for by the eccentric hypothesis alone, as is the case with the Sun, since the opposite occurs. It can thus be explained by the epicyclic hypothesis if it is assumed that the planet moves on its circumference from the west to the east [i.e., in the direction of increasing longitude], so that the greatest motion takes place at the epicyclic apogee and the minimum speed at the epicyclic perigee, in contrast to the lunar model. For the inequality related to the ecliptic, the eccentric model is deployed; our author combines its explanation with the introduction of the equant point in the following manner: it is conceptualized that [the centre of] the epicycle sphere rotates on the circumference [read: equator] of the sphere of the eccentric from the west to the east neither uniformly with respect to the [eccentric] deferent centre, nor uniformly with respect to that of the universe, but with respect to the centre of another sphere whose centre is eccentric from that of the deferent towards the apogee by the same amount of the deferent eccentricity, which is called the sphere of the equant.[56] For all the planets except for Mercury, the centre of the deferent sphere carrying the epicycle is the midway of the centre of the universe and that of the equant sphere, which is to make a true agreement between the various observed motions of the planets and the hypothesis/ model established for them [i.e., epicycle-eccentric]. In a very curious statement, he then adds that "Ptolemy knew it by means of artifice (*ḥīla*) rather than by means of proof, [but]

54. Toomer [1984] 1998, pp. 420-421, 423.

55. Toomer [1984] 1998, pp. 442-3.

56. For Ptolemy: "the eccentric which produces the anomaly", distinguished from "the eccentric that carries the epicycle", i.e., the deferent.

we offer a proof for it later, God willing".[57] The preserved incomplete MS of the *Talkhīṣ* does not contain any such proof, and, as we shall see presently, Ptolemy's iterative procedure is strictly applied to the computation of the eccentricities of the superior planets. Nevertheless, since one of the missing last two books of the treatise is on the retrograde motions and latitudes of the planets, it is tempting to speculate that the mentioned proof (if it was indeed included in the treatise) might have had something to do with the variation of the sizes of the arcs of retrograde motion of the planets, where the bisection of the eccentricity appears to account for their different sizes.[58] It is noteworthy that in his *Nihāya* II.8,[59] Shīrāzī attempts to give an observational proof that in the case of the three outer planets, the centre of the deferent should be located between the equant and the Earth, since the difference in size of the arcs of the retrograde motions of the planets can be explained on this basis.[60] It appears somewhat strange that such a philosophically-minded astronomer as al-Shīrāzī should on the one hand attempt to construct alternative models to eliminate the physical/philosophical difficulties with the equant point in Ptolemaic models[61] by superimposing it on complicated systems of "cycle on epicycle, orb on orb"[62] and, on the other hand accept its validity in an observational aspect. Moreover, Shīrāzī's proof is circular, simply because the first step in his derivation of the deferent eccentricity of the outer planets from their retrograde arcs consists in the use of Apollonius's theorem for the determination of their stationary points; according to it, one needs to calculate the true angular velocity of a planet on the epicycle and that of its epicycle centre —neither of which can, of course, be determined prior to an early derivation of the eccentricity of the equant.[63] Shīrāzī's *Nihāya* was written in the early 1280s,[64] quite probably after Muḥyī al-Dīn's *Talkhīṣ*, but due to

57. Al-Maghribī, *Talkhīṣ*, f. 117v.

58. It should be noted that the relevant section in the *Adwār* (II.6: M: ff. 19r-v, CB: ff. 18r-v) also gives no information about Muḥyī al-Dīn's proof.

59. Shīrāzī, *Nihāya*, P1: ff. 37v-38r, P2: ff. 93r-94v.

60. The text in question is edited and translated in Gamini and Masoumi 2013.

61. Note that there is still equant motion in Shīrāzī's models produced by the eccentricity and the small epicycle.

62. The phrase is taken from Herschel 1851, p. 266.

63. It is probable that Shīrāzī himself recognized the circular reasoning in his proof, and so did not include it in his two later works, *Ikhtiyārāt* and *Tuḥfa*.

64. According to Niazi (2014, pp. 85-86, 98), Shīrāzī's three works were written in the first part of the 1280s; first, *Nihāya*, next, *Ikhtiyārāt*, and then *Tuḥfa*.

the corruption of the only surviving MS of the latter work, we cannot, unfortunately, compare the two proofs given for Ptolemy's bisecting the equant eccentricity at Maragha and examine whether Shīrāzī borrowed his proof from Muḥyī al-Dīn without acknowledging him —as he did in the case of the model of the superior planets and Venus and the idea of placing Venus above the Sun, which he took from his other elder contemporary, al-ʿUrḍī.

At the end of vii.4, Muḥyī al-Dīn deals with how to derive one of the inequalities independently of the other; by this, he in fact means how to isolate the mean motions in longitude from those in anomaly, as he explains that the inequality related to the ecliptic is the amount that the centre of the epicycle travels on the circumference of the ecliptic at a known time, and the inequality related to the Sun is the amount the planet travels on the circumference of the epicycle at that time. Its procedure is as follows: The appulse of a planet to a star is very accurately observed at a particular time and its distance from the mean Sun is known. Then, such an appulse is observed once again under the condition that the distance of the planet from the mean Sun is equal to the corresponding distance in the first appulse. At both times, the planet should be in the same direction both with respect to the Sun and the epicyclic apogee, so that it returns to the first epicyclic position, and thus travels the complete cycles in the period between them. And it is known that the planet travels the complete cycles on the ecliptic plus an amount which is the motion of the fixed stars between the two times. In order to derive the mean daily motions, the cycles of the planet's motion throughout the ecliptic are converted into degrees, and each of them is divided into the time interval counted in days. It is known that the sum of the two mean daily motions in anomaly and in longitude is equal to the solar mean daily motion in the case of the superior planets, but for the inferior ones, the mean motion in longitude is equal to the solar mean daily motion. The motion of the fixed star in the period in question is also known: The elongations of the planet from the mean Sun at both times are equal, which requires the motion of the fixed star to be equal to that of the mean Sun minus the complete cycles travelled in the period in question, which is also equal to the [angular] distance between the centres of the epicycles between the two times.

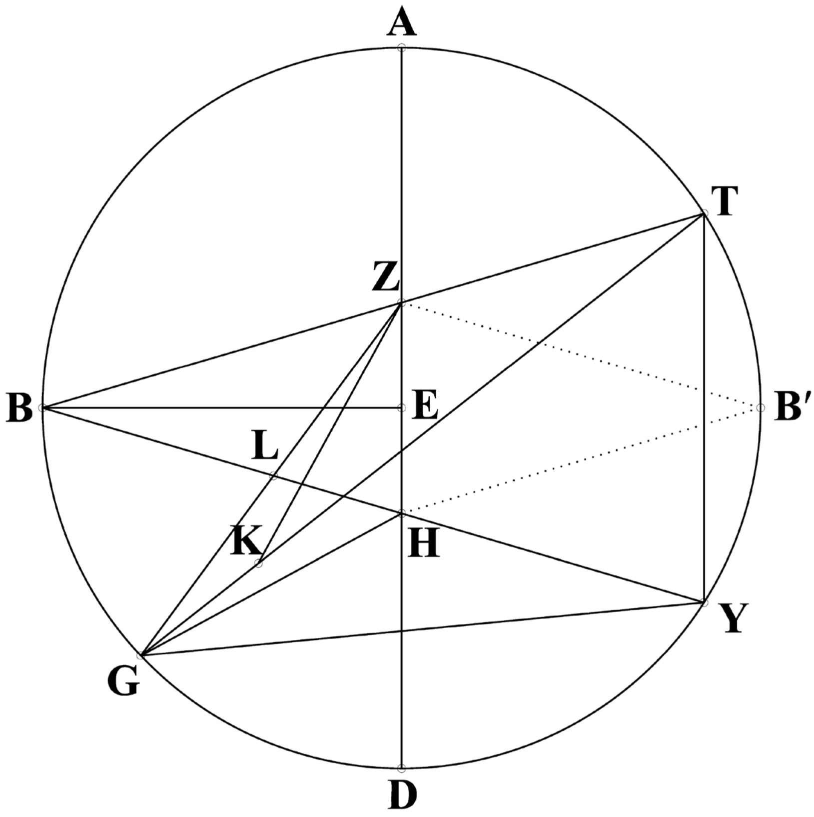


FIGURE 4(a). Lemma 1 (redrawn based on *Talkhīṣ*, f. 120v)

○ $ZE = EH = e$ and $BE \perp AD$; so $BZ = BH$; $\angle ZGH$ is an angle subtending the equant eccentricity ZH from an arbitrary point G on the deferent falling between B and the perigee D. It is wanted to prove that $\angle BHG > \angle BZG$.
● Proof: BZ is extended to T, and BH to Y; $BZT = BHY$, and so $ZT = HY$. We join lines GT, GY and TY; $\angle BTG = \angle BYG$ and $\angle BTY = \angle TYB$; thus, $\angle TYG$ [$= \angle TYB + \angle BYG$] $> \angle YTG$ [$= \angle YTB - \angle BTG$]. Thus, $TG > GY$. We cut off TK to be equal to YG and join ZK; $\Delta ZTK = \Delta HYG$; hence, $\angle TZK = \angle YHG$; thus, $\angle GHB = \angle KZB$; $\angle KZB > \angle GZB$; therefore, $\angle GHB > \angle GZB$. Q.E.D.
○ $\angle ZBH$ is the maximum angle subtending the equant eccentricity from a point on the deferent.
● Proof: from lemma 1, above: $\angle GHB > \angle GZB$; $\angle L$ is equal in triangles ZBL and GHL; therefore, $\angle ZBH > \angle ZGH$. Similarly, it can be demonstrated that $\angle B$ is greater than any other angle subtending the equant eccentricity from a vertex on the cir-

cumference of the deferent, except the equal angle symmetrical to it with respect to diameter *AD* of the deferent ($\angle B'$, not drawn in the original).

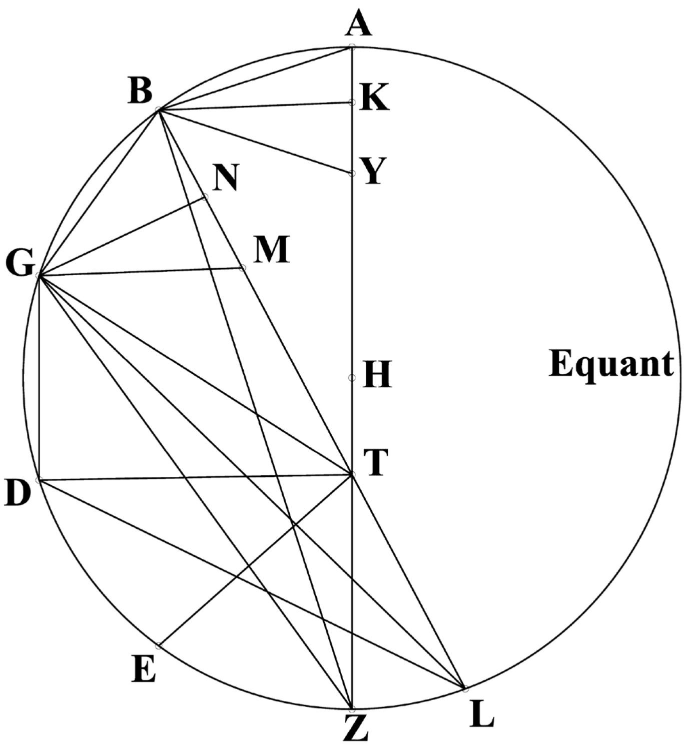


Figure 4(b). Lemmas 2 (redrawn based on *Talkhīṣ*, f. 121v): If the epicycle centre cuts off equal arcs from the circumference of the equant circle at centre *H*, it cuts off the different[-in-size] angles at the centre of the universe [*T*], the least of which is closer to the apogee, and the greatest of which is near the perigee.

○ Arc *AB* = arc *BG* = arc *GD* = arc *DE* = arc *EZ* (the epicycle centre has cut off these equal arcs in the equal times). We join *BT*, *GT*, *DT*, and *ET*. I say: the angles at *T* are successively being increased, so that $\angle ATB$ is the least and $\angle ETZ$ is the greatest of them […].

● Proof: We join lines *AB*, *BG*, *GD*, *BZ*, and *GZ*, cuts off *ZY* to be equal to *ZG*, and join *BY*. Since $\angle AZB = \angle BZG$ and $BY = BG = BA$, ΔABY is isosceles. Also, since $GT + TZ > GZ = ZY$ [$= TZ + TY$], $GT > TY$. *TK* is made equal to *TG*. We join *BK*; $BK < AB = BY$, because it comes from the apex of the isosceles triangle to its base. Thus, $BK < BG$; therefore, $\angle KTB < \angle BTG$.

○ […] Also, I say: $\angle BTG < \angle GTD$.

● Proof: We extend BT to L, join GL and DL, and cut off LM being equal to LD. We join GM; $GM = GD = GB$. Also, sine $DT + TL > DL = LM$ [$= TL + TM$], $TD > TM$. TN is made equal to TD, and we join GN; $GN < GD$, as demonstrated. Therefore, $\angle NTG < \angle GTD$. And analogously it would be demonstrated that $\angle GTD < \angle DTE$, if we extend DT to the circumference [of the equant circle] and proceed according to what mentioned earlier. This is so in the case of the remaining angles. Q.E.D.

In VII.5, the Ptolemaic model of the superior planets and Venus and that of Mercury are explained, and in VII.6, it is proved that in the equal mean eccentric anomalies on the two sides of the apsidal line, the equation of the ecliptic anomaly (i.e., the equation of centre) and the greatest true elongation of the planet from the mean position are identical (corresponding to *Almagest* IX.6).[65] In VII.7, our author posits three lemmas (*muqaddama*, "preliminary") about the planetary motions; the first two concern the variation in the apparent motion of a planet in longitude in comparison with its mean motion, and the fact that its true motion increases from the apogee to the perigee and decreases in the inverse direction (Figures 4(a)-4(b)), analogous to his lemmas in the case of the Sun in *Talkhīṣ* IV.1.[66] A corollary of the first lemma is presented to prove that the equation of centre attains its maximum amount when the line dropped from the circumference of the deferent to its centre is perpendicular to the apsidal line ($c_{1\max} = \angle EQT$ in Figure 1). Although they have simple geometrical structures, these lemmas cannot be addressed in the *Almagest*. The third comprises three parts, each one of which in fact corresponds to one of the lemmas Ptolemy sets forward in *Almagest* X.6;[67] their purpose is to prove that in the case of the superior planets, the line joining the epicycle centre and the body of the planet is parallel to the line from the Earth pointing toward the mean Sun; thus, when the planet is at the true epicyclic apogee, it is in conjunction with the mean Sun, and while it is in the true epicyclic perigee, it is in opposition to the mean Sun.[68]

65. Toomer [1984] 1998, pp. pp. 443-448.

66. Al-Maghribī, *Talkhīṣ*, ff. 51r-53r.

67. Toomer [1984] 1998, pp. 480-484.

68. In his *al-Qānūn* X.3.2 (1954-1956, Vol. 3, pp. 1186-1187), Bīrūnī presents some unjustified objections to Ptolemy in this aspect, which reveal his extraordinary difficulties in unders-

In the introductory chapter viii.1, on only one page does Muḥyī al-Dīn note that for the superior planets, the equant eccentricity and the longitudes of the apogees can be determined by the observations of three mean oppositions, and he explains why the mean conjunctions or quadratures (as used for the derivation of the lunar eccentricity) cannot be used for this purpose: simply because in the first situation the planet is invisible, and the second requires the true longitude of the epicycle centre, which cannot be known prior to the derivation of the eccentricity.

2. Orbital elements: Eccentricity and direction of apsidal line

Ptolemy's iterative method for determining the orbital elements of the superior planets in *Almagest* x.7 and xi.1,5[69] is basically the same general three-point method that was used for quantifying the solar model and computing the radius of the lunar epicycle. For the outer planets, the three points are determined by the planets' triple oppositions to the mean Sun, when they are located at the perigees of their epicycles, and thus the lines of sight from the Earth to the planets point to their epicycle centres. Of course, Ptolemy's planetary models are essentially more complicated than the simple Hipparchan lunar model, on the basis of which the size of the lunar epicycle is determined, as well as his

tanding that (i) the line drawn from the centre of the solar eccentric to the mean Sun, a point on the circumference of the Sun's eccentric, can be replaced by a line parallel to it but drawn from the Earth, so that this second line also points toward the mean Sun; in this concept, the mean Sun would be a fictitious spot circulating uniformly around the ecliptic; and that (ii) as the epicycle radius joining the epicycle centre and the body of the planet is parallel to the line connecting the centre of the solar eccentric and the mean Sun, it is also parallel to the abovementioned second line joining the Earth and the imaginary mean Sun rotating uniformly around the ecliptic. Ptolemy's demonstration of this simple lemma is merely the fact that from $\bar{\lambda}_{\odot} = \bar{\lambda} + \bar{\alpha}$, one can simply derive $\bar{\lambda}_{\odot} = \lambda_C + \alpha$. In Bīrūnī's view, the line pointing toward the mean Sun should necessarily be drawn from the centre of the solar eccentric, and also when the planet is in the true epicyclic apses, the mean Sun does not align with it, except when the epicycle centre is located at the planet's eccentric apogee/perigee: nonetheless, on the condition that the Sun's and the planet's apsidal lines coincide with each other, which itself adds another difficulty, since it does not in fact occur in the case of any of the five planets.

69. Toomer [1984] 1998, pp. 484-499, 507-519, 525-537.

solar model. Consequently, a complicated mathematical procedure is required to measure the fundamental elements of the outer planets. The difficulty arises from the principle of the equant motion in the Ptolemaic model described earlier. In Figure 5, the three continuous circles represent the ecliptic with centre *N*, on which the differences in true longitude between the mean oppositions are marked, the deferent with centre *O*, which carries the epicycle, and the equant circle with centre *T*, on which the arcs representing the differences in mean longitude between the mean oppositions are displayed. The epicycle is illustrated three times, with the planet at *P* in opposition to the mean Sun (not shown). We have arcs Q_1Q_2 and Q_2Q_3 showing the differences in true longitude $\Delta\lambda_{12}$ and $\Delta\lambda_{23}$ respectively, which are measured from the observations of a trio of mean oppositions, and arcs D_1D_2 and D_2D_3 displaying the differences in mean longitude, respectively, $\Delta\bar{\lambda}_{12}$ and $\Delta\bar{\lambda}_{23}$, which are computed from an already available and reliable value for the planet's mean motion in longitude and the known intervals of time between the triple observations. A ready and straightforward application of the three-point method requires knowledge of arcs C_1C_2 and C_2C_3, which are the angular distances between the positions occupied by the epicycle centre at the times of the three mean oppositions, but they are not known (note that points *C* are defined by the projection of the mean positions *D* on the equant circle onto the eccentric deferent). Ptolemy's solution to this problem consists of the following iterative procedure. In the first step, it is "falsely" assumed that the true positions of the planet on the ecliptic are given by the lines drawn from *N* to points *D*, not passing through points *P* and *C*, and then the differences in longitude between the thus marked ecliptic points Q'_1, Q'_2, and Q'_3 are, respectively, equal to $\Delta\lambda_{12}$ and $\Delta\lambda_{23}$. Then, the double eccentricity $2e = TN$ and the direction of the apsidal line $A\Pi$ with respect to points *D* are computed. Then, in the second step, from the value computed for *e*, angles η and ζ in Figure 5 are derived, the differences of which give the small correction angles ε. The lengthy step 3 is the iteration procedure: taking the values of ε, the better estimates $\Delta\lambda_{12}'$ and $\Delta\lambda_{23}'$ can be calculated for arcs $Q'_1Q'_2$ and $Q'_2Q'_3$ (based on the configuration shown in Figure 5: $\Delta\lambda_{12}' = \Delta\lambda_{12} + \varepsilon_1 + \varepsilon_2$ and $\Delta\lambda_{23}' = \Delta\lambda_{23} - \varepsilon_2 - \varepsilon_3$), and then new values for *e* and angles *DTA* and/or $DT\Pi$. This procedure is repeated as often as required until no new values are derived for $\Delta\lambda'$, *e*, and the equant arcs *DA* and/or $D\Pi$. These steps will be clarified further later in the explanation of Muḥyī al-Dīn's account of his measurements in Section 4.1.

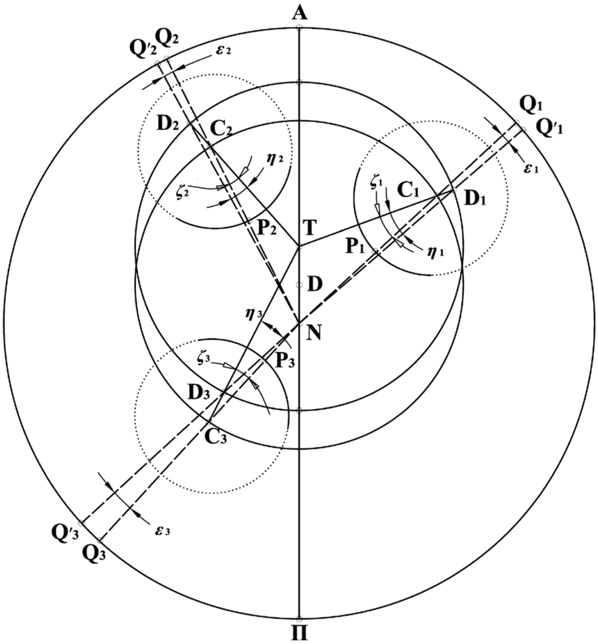


Figure 5. Ptolemy's iterative method for the derivation of the orbital elements of the superior planets.

3. Muḥyī al-Dīn's observations of Mars

In the majority of his recorded observations (28 out of 35), Muḥyī al-Dīn measured the meridian altitude of a celestial object and the time of its meridian transit as counted from the specific moments when a reference celestial body whose celestial and ecliptical coordinates are already known is located in a special position with respect to the horizon; e.g., the meridian transit of the Sun (i.e., true noon) or of a bright clock star or the Sun being located at a low altitude a short while before sunset or after sunrise. Then, the celestial and ecliptical coordinates of that celestial object can be computed from the already known coordinates of the reference body and the observational data.

Muḥyī al-Dīn's observations of Mars are presented as follows. The procedure for deriving the ecliptical coordinates from the observational data and the time of the opposition of the planet to the mean Sun is given only in the first observation, which will be explained in detail later in this section.

1. [70] We released *the minute* (*al-daqīqa*) from its [i.e., Mars'] meridian transit to the meridian transit of *ʿAyn al-thawr* [i.e., Aldebaran, α Tau]. It was in service for 33 *minutes*. Its [i.e., Mars'] maximum altitude in the circle of the meridian was equal to 72;2°. It was on the night of Wednesday, the first [day] of *Bahman-māh* [i.e., the month of *Bahman*, the 11th month] in the year 633 Yazdigird. *Dāʾir al-daqīqa* [i.e., the duration measured by *the minute* as expressed in terms of the revolution of the celestial sphere, time-degree]: 8;16°.

   [Calculation of the ecliptical coordinates of Mars:] The [right] ascension of the transit degree of Aldebaran: 148;56°. We subtract the *dāʾir* from it. The remainder is equal to the [right] ascension of the transit degree [of Mars]: 140;40°. The transit degree of Mars: Taurus 23;5° [= 53;5°]. The [maximum] altitude of the transit degree of Mars [on the meridian of Maragha]: 74;15°. The difference between the two altitudes [i.e., the meridian altitudes of Mars and its transit degree]: 0;47°. Its sine: 0;49,13. The complement of the declination of the ascendant [i.e., the acute angle between the ecliptic and the meridian]: 75;21,38°. Its sine: 58;3,8. We multiply it by the other sine value, and divide the resultant into 60; the result amounts to 0;47,33. Its corresponding arc: 0;45° which is equal to the latitude of Mars in the north direction of the ecliptic. Its complement: 89;15°. Its sine: 59;59,41,32. The complement of the difference between the two altitudes: 89;13°. Its sine: 59;59,39,50. We divide it into the other sine value, and divide the resultant into 60, which results in 59;59,58,18. Its corresponding arc: 89;46°. Its complement: 0;14°. We add it to the degree of transit of Mars; the longitude of Mars is derived as Taurus 23;19° [= 59;19°].

   [Derivation of the time of the mean opposition of the planet to the mean Sun:] The daily motion (*buht li-yawm*) of Mars: 0;22°. At midday on Wednesday, the longitude of Mars: 53;8° and the mean longitude of the Sun: 232;3,44°. The difference between the two [i.e., the angular distance in longitude of either the Sun or Mars from a point on the ecliptic that is diametrically opposed to the other object at the time: 53;8° + 180° - 222;3,44° =]: 1;4,16°. We multiply it by 24, and divide the resultant into the adjusted motion (*al-buht al-muʿaddal*) [i.e., the relative angular velocity of Mars with respect to the Sun], which is equal to 1;21,8° [= the mean daily motion of the Sun, 0;59,8° + the daily motion of the planet given as 0;22°], which

70. al-Maghribī, *Talkhīṣ*, f. 132v.

results in 19 hours after midday. Thus, the date of the mean opposition 632 [years] 301 [days] 19 [hours]; at the time, the mean longitude of the Sun: 232;50,31° and the longitude of Mars: 52;50,31°.

2. [71] We released the clepsydra at the end of the daylight on Sunday, 15 *Isfandhārmadh-māh* [i.e., the month of *Isfandhārmadh* or *Isfand*, the 12th month] in the year 635 [Yazdigird, when] the altitude of the Sun was 3° west. It was in service for 7 [turns] 18 [minutes] until its [i.e., Mars'] meridian transit. Its [i.e., Mars'] maximum altitude was equal to 79;42°. […] *Dāʾir al-mankām*: 112;36,22°.
3. [72] We released the clepsydra in the night on Thursday, the 20th [day] of *Urdībihisht-māh* [i.e., the month of Urdībihisht, the 2nd month] in the year 640 [Yazdigird] from the meridian transit of *Qalb al-asad* [i.e., Regulus, α Leo]. It was in service for 1 [*turn*] 32 [*minutes*] 30 [*seconds*] until its [i.e., Mars'] meridian transit. Its maximum altitude in the circle of the meridian was equal to 62;33°. *Dāʾir al-mankām*: 23;33,25°.

The meridian altitudes in the first three observations were measured with the aid of the mural quadrant of the observatory, called the "high copper quadrant", and the durations, by a clepsydra, called *mankām*. The mural quadrant is described in al-ʿUrḍī's *Fī kayfiyyat al-irṣād* (*How to make* [*astronomical*] *observations*).[73] Muḥyī al-Dīn appears to have been so interested in the instrument that he composed a poem in its praise during the observations of 1265-1266 AD, and a certain astronomer Majd al-Dīn Abū Muḥammad al-Ḥasan b. Ibrāhīm b. Yūsūf al-BaʿAlbakī (modern: Baalbek in Lebanon) had the poem engraved on the quadrant.[74] He called the clepsydra *mankām* (or *minkām*), a corruption of the term *bankām* (or *binkām*) which comes from (in fact, is the Arabicized form of) the Persian term *pangān*. This Persian term appears to have been converted into Arabic in two different forms in the early Islamic period: *fankān* (or *finkān*) which

71. al-Maghribī, *Talkhīṣ*, f. 132v.

72. al-Maghribī, *Talkhīṣ*, f. 133r.

73. Al-ʿUrḍī, *Fī kayfiyyat al-irṣād*, P: ff. 2v-4r, N: ff. 38r-39v; Seemann 1929, pp. 28-32. For a reconstruction of it, see Sezgin and Neubauer 2010, Vol. 2, p. 38.

74. Ibn al-Fuwaṭī, Vol. 4, pp. 413-414; the poem reads:

أنا ربع دائرة الفلک / طوبى لمن مثلي ملک / بي تدرک الأوقات حقّاً / ويقيناً دون شک

A tentative translation of this goes as follows: "I am a quadrant of the circle of orb. / Good for everyone such me as an angel! / By me the times are known truly / and securely, without any doubt".

was used to mean "clock", in general, and *bankām* which meant "water-clock", in particular, as can be understood from the work of al-Jazarī, the 12th century Islamic engineer and craftsman, on mechanical devices.[75] Hardly anything is known about the clepsydra Muḥyī al-Dīn used in his observations at Maragha. It appears that two mechanisms were embedded in it in order to measure hours and minutes independently of each other: one called by the general name of the clepsydra, i.e., *al-mankām*, which was used to measure *hours*, and the other called *the minute* (*al-daqīqa*), which was used to count *minutes*. This is suggested by the first observation of Mars, in which the time interval is less than one hour, and indeed Muḥyī al-Dīn's statements make it clear that only *the minute* was used to measure it. The resulting time-degree is called *dāʾir al-daqīqa* (cf. *dāʾir al-mankām*).

Muḥyī al-Dīn's observational data obtained from the first three observations and the values he computed for the celestial and ecliptical coordinates of Mars from them are presented in Table 1 for comparison with the modern data, which are presented in bold: Cols. **I** and **II** indicate, respectively, the numbers assigned to the observations and their dates in the Yazdigird era, the Julian calendar, and Julian Day Number. The Yazdigird era, 16 June 632, is used with the Egyptian/Persian year consisting of 12 months of 30 days plus five epagomenal days which, in the early Islamic period, were placed after the eighth month, but in the late Islamic period were transferred to the end of the year. Col. **III** gives the origins of time, from which the times of the meridian transits of Mars were measured, and Col. **IV**, the corresponding mean local times; Col. **V** presents Muḥyī al-Dīn's and true modern values for the meridian altitudes of Mars and their differences; Col. **VI** contains the true times of the meridian transits of Mars; Col. **VII** gives his values for the times of the meridian transits of Mars measured from the origins mentioned in Col. **III**, and their differences with respect to the true time intervals (i.e., the differences between Cols. **IV** and **VI**). Cols. **VIII** and **IX** give, respectively, his values for the declination and right ascension of Mars in comparison with the corresponding modern values at the time; note that he does not calculate the declinations, since they were not necessary in his computations; we have derived them simply from $RA = h_{max} + \phi - 90°$ with the values of h_{max} in Col. **V** and $\phi = 37;20,30°$, and have rounded the results to the nearest 0.5′). Cols. **X** and **XI** indicate, respectively, his computed values for the latitude and longitude of Mars together with the corresponding modern values at the time and the differences between them.

75. Al-Jazarī 1973, p. 17.

The longitudes computed from this trio of observations of Mars are used to compute the times of the oppositions of the planets to the mean Sun, and then to determine the orbital elements of the planet and its mean motion in anomaly, as shall be explained below (see Sections 4.1 and 4.2). The fourth, and last, recorded observation of Mars is its near appulse to the star Regulus (α Leo), which is used in order to determine the size of its epicycles (see Section 4.3):

4. [76] I say that the circle of latitude had aligned the centres of Mars and Regulus when 14;30 hours was elapsed since midday on Tuesday, the tenth of *Diy-māh* [i.e., the month of *Diy*, the 10th month] in the year 639 Yazdigird, when nearly the one-third of [the ecliptic sign] Virgo rose. [Mars] was in the north of Regulus as much as 40 arc-minutes. The longitude of Regulus and of Mars was 4 [ecliptic signs] 19;5°2 [= 139;52°].

The time of this observation is given as 14;30^h after true noon on 14 October 1270. Mars rose on 15 October at 0:34 MLT. At 2:30 MLT, the apparent longitude of Mars was about **139;42,30**° and that of Regulus, **139;40,0**° (both at an altitude of **22**°), whereas Muḥyī al-Dīn found them to be in conjunction with each other; consequently, he appears to have committed an error of ~ +2.5′ in the estimation of the distance in longitude between Jupiter and Regulus. The apparent conjunction took place between 0:30 MLT and 1:0 MLT. The difference between the apparent latitudes of Mars (**+1;37**°) and Regulus (**+0;27**°) was ~ **70′**, and thus his error is ~ -30′. It is noteworthy that Muḥyī al-Dīn's style of describing the near appulse of the planets to Regulus as "the circle of latitude aligned…" is fundamentally different from the common standard terminology adopted in Islamic astronomy; e.g., the roots *q-r-n*, "to be in conjunction", *k-s-f*, "to occult", or *l-ṣ-q*, "to associate/to be close".

In *Talkhīṣ* III.11,[77] Muḥyī al-Dīn explains in detail how to compute the ecliptical coordinates of a celestial body from its meridian altitude and time of its meridian transit or from its altitude and azimuth. For a better view of his observational-computational method to determine the longitude and latitude, instead of reading them off directly from an armillary sphere, we explain the procedure with a worked-out example related to Mars-1. Assume that a heavenly object passes

76. Al-Maghribī, *Talkhīṣ*, f. 135r

77. Al-Maghribī, *Talkhīṣ*, ff. 49r-50v.

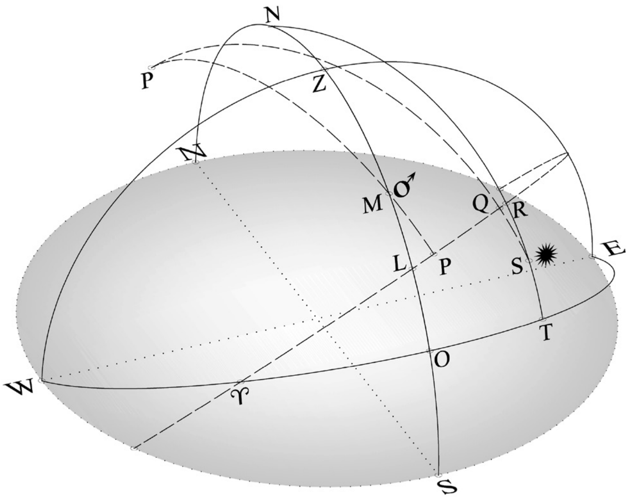


FIGURE 6(a). The derivation of the celestial and ecliptical coordinates of a heavenly object from its maximum altitude measured at the time of its meridian transit and the span of time passed from, or remained until, the meridian transit of a reference celestial object with known celestial/ecliptical coordinates.

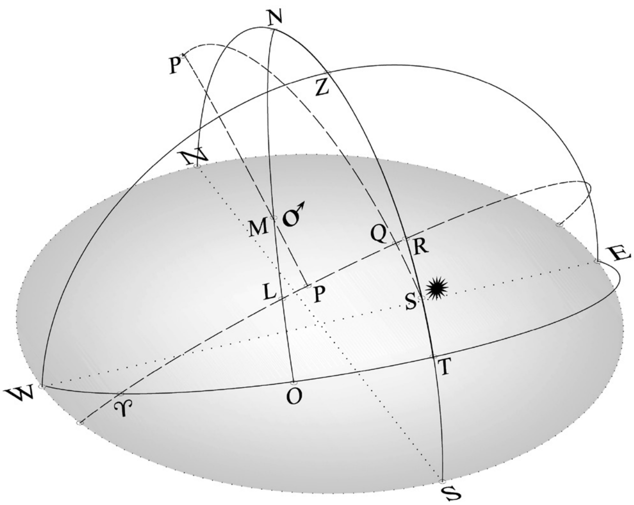


FIGURE 6(b). The derivation of the eclipical coordinates from the horizontal ones (3D reconstruction from the analemma drawn on f. 50r).

across the meridian at a given time t_1 (Mars in our example). See Figure 6(a). The great circle *ETOW* is the celestial equator with *N* as the north celestial pole, and *RQPL* stands for the ecliptic, and *P* is its northern pole. The body is located at *M*. Arc ♈*P* displays its longitude λ, and arc *MP* is its latitude β. Arc ♈*O* shows its right ascension *RA*, and arc *MO* is its declination δ. Arc *MS* is the observed meridian altitude h_{max}. Arc *OS* is the complement of the geographical latitude of place (= 90° - ϕ, where ϕ = 37;20,30°, modern: **37;23,46**°). Point *L* stands for the transit degree λ_m (i.e., polar longitude) of the object, which is also called the mid-heaven at time t_1; arc ♈*O* displays its right ascension, arc *LO*: its declination $\delta_m = \delta(\lambda_m)$, and arc *LS*: its altitude h_m. Arc *ML* is called the difference in altitude, $\Delta h = h_{max} - h_m$. At time t_2, the reference body (the star Aldebaran, α Tau) *S* is located on the meridian, as shown in Figure 6(b): Arc ♈*Q* shows its longitude λ^*, arc *QS*: its latitude β^*, arc ♈*T*: its right ascension *RA**, and arc *TS*: its declination δ^*. Point *R* is the mid-heaven at t_2. The period between the meridian transits of the two objects $\Delta t = t_2 - t_1$ is expressed in terms of the revolution of the celestial sphere (time-degree, 1 hour = 15°), which corresponds to arc *TO*. Thus,

$$RA = RA^* \pm \Delta t \tag{1}$$

In our example, it is obvious that the sign is negative. The declination δ can be readily computed from

$$\delta = h_{max} - \phi + 90°, \tag{2}$$

although our author does not need to do so.

In order to derive the ecliptical coordinates, it is necessary to derive λ_m and then to solve the spherical triangle *MLP*. *RA* in (1) is also the right ascension of the mid-heaven *L* at t_1. Hence, it can either be derived from the table of right ascension[78] or can be directly computed, e.g., from

$$\tan \lambda_m = \tan RA / \cos \varepsilon, \tag{3a}$$

and its declination δ_m either from the table of the declination[79] or from

78. Table of the right ascension $RA(\lambda)$: Al-Maghribī, *Talkhīṣ*, ff. 34v-35r.

79. Table of the first and second declinations $\delta(\lambda)$: Al-Maghribī, *Talkhīṣ*, f. 32r.

$$\sin \delta_m = \sin \lambda_m \sin \varepsilon, \text{ and}$$
$$h_m = \delta_m - \phi + 90°, \tag{4}$$

where ε is the obliquity of the ecliptic (23;30°). In triangle MLP, arc $ML = \Delta h$ is known and also, angle $P = 90°$. We want to calculate the acute angle $MLP = \gamma$ between the ecliptic and the meridian, which has been already explained in *Talkhīṣ* III.6:[80]

$$\cos \gamma = \cos \text{RA} \sin \varepsilon. \tag{5}$$

(Note that knowing angle γ, λ_m can also be derived from

$$\sin \lambda_m = \sin \text{RA} / \sin \gamma, \tag{3b}$$

as our author mentions in *Talkhīṣ* III.6).

Thus, we can compute the latitude β = arc MP and the difference in longitude $\Delta\lambda = |\lambda - \lambda_m|$ = arc PL, from which the longitude can be known as well:

$$\sin \beta = \sin \Delta h \sin \gamma \tag{6}$$
$$\cos \Delta\lambda = \cos \Delta h / \cos \beta, \text{ and thus, } \lambda = \lambda_m \pm \Delta\lambda. \tag{7}$$

In our example (see below), the sign is positive. The conditions are given as follows:

□ *Latitude*: (A1) If the object and the ecliptic are in the southern region of the horizon: If $h_{max} > h_m \rightarrow \beta > 0$ and if $h_{max} < h_m \rightarrow \beta < 0$. (A2) If the body and the ecliptic are in the northern zone of the horizon, the opposite is correct. (B1, B2) If the body and the ecliptic are in the two opposite directions with respect to the prime vertical, the body is in the same direction with respect to the ecliptic as the zenith.
□ *Longitude*: If the latitude of the body and the ascendant/horoscope (i.e., the point of the ecliptic located on the eastern horizon, whose right ascension is more than that of the mid-heaven by 90°) are in the same direction [with respect to the ecliptic], then $\Delta\lambda$ should be added to λ_m, but if they are in the opposite directions, $\Delta\lambda$ should be subtracted from λ_m.

80. Al-Maghribī, *Talkhīṣ*, ff. 40v-42r.

Finally, our author adds that we have demonstrated this issue by many examples in the cases of the fixed stars and other objects, as will be mentioned later, if God wishes.

■ **Example** Mars-1: The date of observation is 4/5 November 1264 (Table 1, no. 1), whereas the reference body is Aldebaran whose coordinates were finally established with the aid of an observation made some four years later on 17 September 1268.

The input data are:

(1) *RA**: Muḥyī al-Dīn takes the right ascension of Aldebaran as equal to 148;56° as counted from the winter solstice point, or the head of Capricorn (i.e., 58;56° from the vernal equinox point, or the head of Aries) at the time of observation Mars-1. The coordinates of the star he measured in 1268 AD are λ = 60;0°, β = -5;29°, and *RA* = 148;59,49°. The precessional motion in longitude in the interval of time of 1412 days between the two observations amounts to about 3.5′ ≈ 4.0′. So, for the time of observation Mars-1: λ ≈ 59;56°, and hence, *RA* ≈ 148;56,35°.

Observational data:

(2) Δt = 33 minutes of *Pangān* = 8;16° (Table 5); and
(3) h_{max} = 72;2°.

Therefore,

(1) *RA* = 148;56° - 8;16° = 140;40° counted from the head of Capricorn;
(2) [δ = 19;22,30°];
(3) λ_m = 53;5°;
(4) [δ_m = 18;35°]; h_m = 71;15°; Δh = 0;47°;
(5) γ = 75;21,38°;
(6) β = +0;45°; and
(7) $\Delta\lambda$ = 0;14°; λ = 53;19°.

Then, in a separate unnumbered chapter in *Talkhīṣ* III.11, Muḥyī al-Dīn explains the general case of how to derive the ecliptical coordinates of a celestial object at a given time from the ascendant/horoscope and its horizontal coordinates, i.e., the altitude and azimuth. With regard to Figure 6(b) (redrawn from f. 50r), arc *AB* is a quadrant of the horizon circle, arc *AC*: a part of the ecliptic and *D* its north pole, *A*: the pole of the great circle *DCB*, so that *AY* = *AC* = 90°, *E*: the zenith, i.e., the pole of arc *AB* (*EH* = 90°), and *Z*: a heavenly body; we draw arc *EZH* of a great circle; hence, *ZH* is its observed altitude h, *ZT*, its latitude β, and *T* marks its ecliptic longitude. The ecliptical coordinates can be known by the following procedure. We draw arc *AZY* of a great circle. We have:

$$\frac{\sin EZ}{\sin ZY} = \frac{\sin EH}{\sin HB} \rightarrow \sin ZY = \cos h \sin HB$$

AH is called the azimuth counted from the ascendant/horoscope, *al-samt min al-ṭāliʿ* (if the celestial object is not on the prime vertical), or the ortive amplitude of the ascendant/horoscope, *saʿat mashriq al-ṭāliʿ* (if the celestial object is on the prime vertical, and thus its azimuth is equal to zero). Accordingly, *HB* is called the complement (*tamām*) of *AH*. From the above, we know arc *AZ* = 90° - arc *ZY*. → sin *YB* = sin h / sin *AZ*

$$\frac{\sin AZ}{\sin ZH} = \frac{\sin AY}{\sin YB} \rightarrow \sin YB = \sin h \,/\, \sin AZ$$

Arc *YB* is called the arc of experiment/observation (*qaws al-iʿtibār*). Arc *CB* is already known, because it is the complement of the altitude of the northern pole of the ecliptic, *D*. Thus, arc *YC* = arc *YB* - arc *CB*.

Also,

$$\frac{\sin AZ}{\sin ZT} = \frac{\sin AY}{\sin YC} \rightarrow \sin \beta = \sin AZ \sin YC$$

So, arc *ZD* = 90° - β.

$$\frac{\sin DZ}{\sin ZY} = \frac{\sin DT}{\sin TC} \rightarrow \sin TC = \sin ZY \,/\, \cos \beta$$

Arc *TC* is called the arc of difference (*qaws al-ikhtilāf*), indicating the difference in longitude between the ecliptic position *T* of the celestial body under scrutiny and point *C*, which is 90° in longitude behind the ascendant/horoscope *A*. Therefore, the longitude of the heavenly object is known.

The conditions below are put forward in order to determine the direction of the celestial body with respect to the ecliptic and how to add *qaws al-iʿtibār YB* to the longitude of point *C* in order to derive the longitude of the object:

> □ *Latitude*: (A1) If the body and the ecliptic are in the southern zone of the horizon: If $YB > CB$, the latitude is northern; if $YB < CB$, the latitude is southern; and if $YB = CB$, the object has no latitude. (A2) If they are in the northern region of the horizon, the situation would be reversed. (B1, B2) If the body and the ecliptic are in two opposite directions with respect to the prime vertical, the body is in the same direction with respect to the ecliptic as the zenith.
> □ *Longitude*: If the latitude of the object and the ascendant/horoscope are in the same direction [with respect to the ecliptic], we add arc *TC* to the longitude of point *C*. If they are in two opposite directions, we subtract *TC* from the longitude of point *C*.

In the right margin opposite the longitude conditions (f. 50v), Taqī al-Dīn Muḥammad b. Maʿrūf leaves this comment:

> Here there is a great detail. And these sentences are neither rejected, nor are believed, except for a particular condition. So, it should be treated with caution. —Released by the poor Taqī al-Dīn.[81]

In his *Sidra muntaha*… (or *Shāhanshāhiyya Zīj*) II.4.5 (topic 53),[82] which bears a separate title "On the explanation of observational operations by the altitude-azimuthal instrument", Taqī al-Dīn explains the above-mentioned computational procedure in order to convert the horizontal coordinates to the ecliptical ones, from which it would be clarified that his critical annotation above only refers to considering the various conditions possibly encountered in adding or subtracting the arc of difference *TC* to/from the longitude of point *C* (e.g., in the case

81. هنا تفصيل كبير. وهذه الجمل لاتطرّد ولاتصدّق، إلّا فى جزوىٍ من جزويّاتها. فليتنقّظ لذلک. حرّرها الفقير تقي الدّين

82. Taqī al-Dīn, *Sidra*, K: ff. 26r-27r, N: ff. 33v-34v, V: ff. 31v-32r. Note that each Book (*maqāla*) in this work is divided into Sections (*bābs*) and every Section into chapters (*faṣls*), but the smallest subdivisions consist of the various topics discussed in a Book (called "principle", *aṣl*, in Book I and "proposition", *qaḍiyya*, in Book III), which are numbered continuously; for example, Book II, Section 4, Chapter 5 contains nine topics, 51-59.

of the circumpolar stars). In a marginal note in the end of his account, Taqī al-Dīn again refers to Muḥyī al-Dīn:

> Sheikh Muḥyī al-Dīn b. Abī [al-]Shukr —May God have mercy on him— wished fervently to deal with it, but his words were confused, as I saw them in a manuscript in his hand.[83]

Both these comments seems harsh, since there is absolutely no confusion in Muḥyī al-Dīn's consideration of conditions: In the case of latitude, they are complete, and those he explains for longitude are also (at the very least) sufficient for handling a transformation of the horizontal coordinates to the ecliptic ones for any place located in a northern latitude between ε and $90° - \varepsilon$ (ε: the obliquity of the ecliptic).

In an unnumbered chapter appended to III.11, Muḥyī al-Dīn explains how to compute the declination from the altitude and azimuth. The proof is as presented above, if in Figure 6(b) we take circle *DCB* as the meridian, *AC*: the celestial equator, *D*: the northern celestial pole, then arc *ZT* becomes the declination of the celestial body, arc *CT*: its distance from the mid-heaven circle (*dāʾirat wasaṭ al-samāʾ*; i.e., the meridian) in terms of the equinoctial time-degree (*bi-azminat muʿaddal al-nahār*), and arc *AT*: its true ascension for the given azimuth *AH* (*al-maṭāliʿ al-ḥaqīqīya li-l-samt al-mafrūḍ*). The computational procedure is as mentioned earlier; if we take arc *HB* as the complement of the observed azimuth *AH*, the declination is known.

Table 2 illustrates how Muḥyī al-Dīn derives the times of the oppositions of Mars to the mean Sun from his measured durations and computed longitudes in the first three observations of the planet. Cols. **I** and **II** display, respectively, the dates of the observations and the ecliptical longitudes of Mars at the times of its meridian passages (identical to Cols. **II** and **IX** of Table 1, nos. 1-3). Col. **III** indicates the values he gives for the true daily motion of the planet in longitude (*al-buht*); our author does not explain how he derived these values, but it seems fair to assume that he observed the meridian transit of the planet at least on the two consecutive nights around each mean opposition, and then derived its longitude from the observed time and the meridian altitude by his method explained above; thus, the difference between the two longitudes is equal to the daily motion of the

83. الشيخ محيى الدّين بن أبى شكر -رحمه الله تعالى- حام فيه حول الحمى ولكن اضطربت عبارته فيه رأيته فى نسخة بخطه

planet in longitude. Col. **IV** gives the longitude of the planet at noon either on the day preceding the observation or on the day of the observation, which are computed from Cols. **II** and **III**. As seems clear, our author took midnight as the time for all three of his observations,[84] since he either added half of the true daily motion to, or subtracted it from, the longitude of the planet in Col. **II** in order to derive the longitude of the planet at noon. Col. **V** presents the mean longitude of the Sun at noon, as computed from his solar theory. Col. **VI** gives Muḥyī al-Dīn's values for times of the mean oppositions as computed from the data in Cols. **III-V** and the mean solar daily motion ($\omega_{\odot}$ = 0;59,8$^{\circ/d}$). For example, as we have already seen in the observation report of Mars-1, at midnight on 5 November 1264, the longitude of the planet λ = 53;19°; with the true daily motion v = 0;22°, at noon on the same day, its longitude λ = 53;19° - 0;22°/2 = 53;8° and the solar mean longitude $\bar{\lambda}_{\odot}$= 232;3,44°. Consequently, a distance of $\Delta\lambda$ = 1;4,16° must be travelled partly by the mean Sun and partly by the planet until they are placed in opposition to each other. Then, the time needed for it is equal to $\Delta\lambda/(v + \omega_{\odot})$ = 1;4,16°/(0;22$^{\circ/d}$ + 0;59,8$^{\circ/d}$) ≈ 19;1 hours; our author has the value 19;0 hours (a trivial error of -1 minute, which has no undesirable consequences). In Col. **VII**, our author's times of the mean oppositions are converted to the mean local time (MLT) of Maragha for comparison with the modern times marked in bold. Of course, it should be noted that the opposition of a superior planet to an "imaginary" spot of the ecliptic, the mean Sun, is in reality an "invisible" phenomenon, and the times of its occurrence are computed solely on the basis of a crude and simple linear method. Thus, the derived times do not appear to be so relevant to (or comparable with) those computed on the basis of modern theories and on highly precise data. Nevertheless, it is sufficient to say that there is a negative systematic error in Muḥyī al-Dīn's times for the opposition of Mars to the mean Sun (μ = -8.3 hours). For the longitudes of the planet at the derived times of the mean oppositions, our author first simply calculates the mean solar longitudes at these times from his solar theory, and then takes the longitudes diametrically opposed to them as those of the planet, as will be shown in the following section.

84. As the true modern times given in Col. VI of Table 1 indicate, all the meridian transits had taken place close enough to midnight: the differences did not exceed 18 minutes.

4. Muḥyī al-Dīn's measurements of Mars

4.1. Orbital elements[85]

Muḥyī al-Dīn gives the times of the mean oppositions in the Yazdigird era (see Table 1, Col. **II** and Table 2, Col. **VI**) and the true longitude of Mars and the mean longitude of the Sun at them as follows:

	t	λ	$\bar{\lambda}_{\odot}$	
1	$632^y\ 301^d\ 19;\ 0^h$	52;50,31°	232;50,31°	
2	$634^y\ 344^d\ 22;27^h$	94;53,26	274;53,26	(1)
3	$639^y\ 50^d\ 1;15^h$	163; 2,53	343; 2,53	

Then, the time intervals between the mean oppositions and the mean and true motion of Mars during these time intervals are derived as follows:

	Δt	$\Delta\lambda$	$\Delta\bar{\lambda}$	
$1 \rightarrow 2$	$2^y\ 43^d\ 3;27^h$	42;2,55°	45;11, 4°	(2)
$2 \rightarrow 3$	$4^y\ 70^d\ 2;48^h$	68;9,27	81;53,36	

The values for $\Delta\bar{\lambda}$ seem to have been computed from a mean daily motion in longitude of about 0;31,26,39,52° which is nearly equal to the value underlying the mean motion tables of the planet in al-Maghribī's ʿ*Umda* which is, in turn, borrowed from Ibn al-Fahhād's ʿ*Alāʾī zīj*.

The value of $\Delta\bar{\lambda}_{12}$ is given as 45;11,**3**° on f. 133r, but as 45;11,**4**° on f. 133v, which is used in the calculations. The first is in accordance with the abovementioned value for the mean daily motion. I do not know how likely it is that Muḥyī al-Dīn first wrote his computed value in *Abjad* numerals, but misread the third sexagesimal digit (د/4 → ج/3) when beginning his calculations. However,

85. al-Maghribī, *Talkhīṣ* VIII.11: ff. 133v-134v, 136r-136v.

such small deviations would hardly cause undesirable consequences in the subsequent derivation.

Step 1: Initial estimations

Muḥyī al-Dīn draws the orbital configuration of the planet at the times of the triple mean oppositions on f. 133v, which we reproduce in Figure 7, and explains that arc *KL* of the ecliptic does not correspond to arc *EZ* of the equant circle, but exceeds its ecliptic projection, arc *RX*, by the sum of arcs *KR* and *XL* (ε_1 and ε_2 in Figure 5); and arc *LM* of the ecliptic is lesser than arc $X\Theta$, the ecliptic projection of arc *ZH* of the equant circle, by the sum of arcs *XL* and $M\Theta$ (ε_2 and ε_3 in Figure 5). Then, he draws the equant circle in the two separate figures on f. 134r, redrawn in Figure 8(a) and 8(b), with *A*, *B*, and *G* representing the planet's mean positions (equivalent to *E*, *Z*, and *H* in Figure 7) and *D* the centre of the Earth (not to be confused with *D* in Figure 7, standing for the centre of the eccentric deferent). We have:

$$\text{arc } AB = \Delta\bar{\lambda}_{12} \quad \angle ADB = \Delta\lambda_{12}$$
$$\text{arc } BG = \Delta\bar{\lambda}_{23} \quad \angle BDG = \Delta\lambda_{23}$$

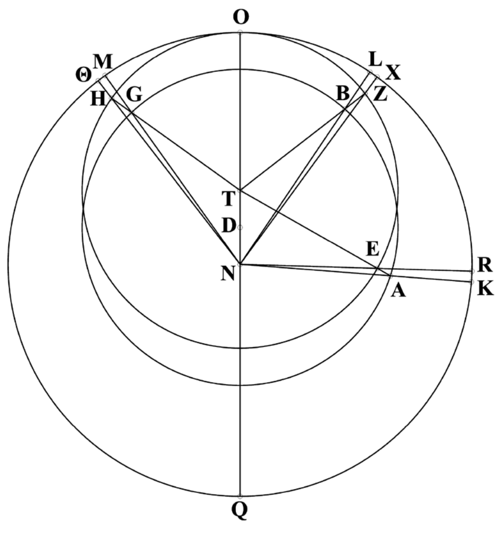
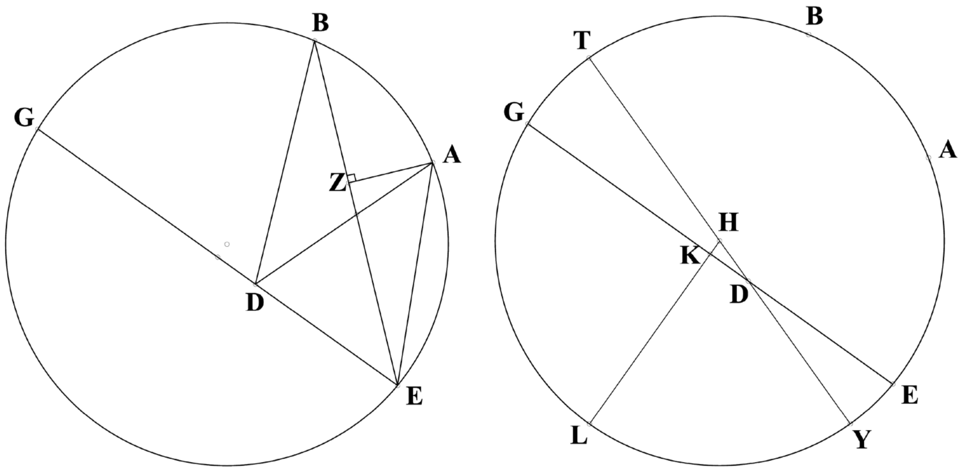


Figure 7. Derivation of the orbital elements of Mars.



(a) (b)

FIGURE 8

Then,

$$\angle BEG = \tfrac{1}{2}\,\text{arc } BG = 40;56,48°$$
$$\angle DBE = \angle BDG - \angle BEG = 27;12,39°$$
$$\text{Sin}(\angle BDE) = \text{Sin}(\angle BDG) = 55;41,33,2\boldsymbol{1} \qquad [\ldots,23]$$
$$\text{Sin}(\angle DBE) = 27;26,9,2\boldsymbol{6} \qquad [\ldots,28]$$

If we take DE = 60 arbitrary units, then we can compute the length of BE in terms of it:

$$BE = DE\,\frac{\text{Sin}\,(\angle BDE)}{\text{Sin}\,(\angle DBE)} = 121;47,41.$$

Also,

$$\text{arc } ABG = \text{arc } AB + \text{arc } BG = 127;4,40°$$
$$\angle AEG = \tfrac{1}{2}\,\text{arc } ABG = 63;32,20°$$
$$\angle ADG = \angle ADB + \angle BDG = 110;12,22°$$
$$\angle EAD = \angle ADG - \angle AEG = 46;40,2°$$
$$\text{Sin}(\angle ADG) = \text{Sin}(\angle ADE) = [56;18,26,32] \qquad \text{(omitted in the text)}$$
$$\text{Sin}(\angle EAD) = 43;38,34,\boldsymbol{10} \qquad [\ldots,9]$$

from which

$$AE = DE\,\frac{\mathrm{Sin}\,(\angle ADE)}{\mathrm{Sin}\,(\angle EAD)} = 77;24,40$$ (text: ***27***;…; scribal error: کز → عز)

in terms of the same arbitrary units of which DE = 60. Also,

$\angle AEB$ = ½ arc AB = 22;35,32°
$\angle EAZ$ = 67;24,28°
Sin($\angle AEB$) = 23;3,0,43 (text: …4***8***; scribal error: مح → مج)
Sin($\angle EAZ$) = 55;23,45
$AZ = AE$ Sin($\angle AEB$)/R = 29;44,2***1*** (text: ***1***9;…; scribal error: 20,…] (یط → کط]
AZ^2 = 14,44;2***5***,***4***,***55***,***21*** […;24,5,26,40]
$EZ = AE$ Sin($\angle EAZ$)/R = 71;28,15 (text: ***2***1;…; scribal error: کا → عا)
$ZB = BE - EZ$ = 50;19,26
ZB^2 = 42,12;29,37,39[,16]
$AB^2 = AZ^2 + ZB^2$ = 56,56;5***4***,4***2***,***34***,***37*** […;53,43,5,56]

and so

AB = 58;27,15,***40*** […,11]

in terms of the same arbitrary units. We now want to convert the lengths computed until now into the trigonometrical norm that the radius of a circle is taken as 60. In doing so, we can compute the length of AB from its corresponding arc:

AB = Crd(arc AB) = 46;6,1,***3***6 […,26]

The error in this value appears to be scribal (کو → لو), but our author deploys the same value, as can be seen in the calculation of the length of AE below. The two values above provide us with a scale for transforming the lengths from our arbitrary units to trigonometrical units:

DE = 60 × 46;6,1,***3***6/58;27,15,4***0*** = 47;19,10
AE = 77;24,40 × 46;6,1,***3***6/58;27,15,4***0***
= 59,28;4***1***,***11***,5***1***,2***8***/58;27,15,4***0*** = 61;3,3

It is clear that the small errors in the two values for the length of *AB* have no negative effect on the values computed for the lengths of *DE* and *AE*.

Then,

arc AE = $\mathrm{Crd}^{-1}(AE)$ = 61;9,44°
arc EAG = arc AE + arc AB + arc BG = 188;14,24°, and thus
arc GE = 171;45,36°

Now, we are able to compute the lengths on *EG*:

½ GE = Sin(½ arc GE) = 59;50,42,
and hence GE = 119;41,24
$GD = GE - DE$ = 72;22,14
$GD \times DE$ = 57,***6***;***6***,***43*** [...,4;32,4]

Unlike the earlier steps, al-Maghribī makes a rather surprising error in this multiplication, which causes a lesser value to be derived for the eccentricity. Segment *GAE* is greater than a semi-circle, and thus its centre lies inside it. With regard to Figure 8(b), he takes the centre to be *H*, and connects *DH* and extends it to points *T* and *Y*. *T* is the position of the apogee and *Y* is the perigee. We have:

$4e^2 = DH^2 = R^2 - GD \times DE$ = 2,5***3***;***53***,***17***, [...,55;27,56]
and thus, $2e = DH$ = 13;1***1***,***12*** [...;14,47]

The perpendicular *HKL* is drawn to bisect *GE* at *K* and arc *GYE* at *L*. The first estimates of the directions of the apsidal line with respect to the mean positions are computed as follows:

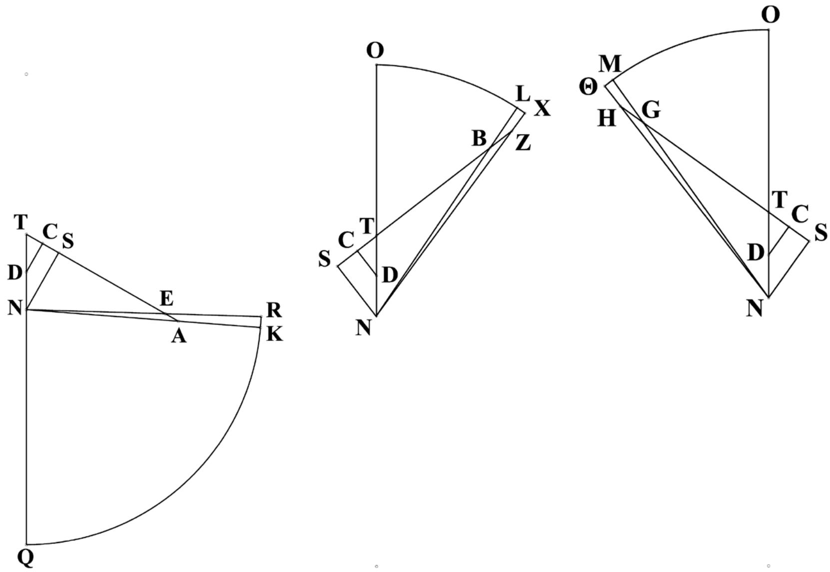
EK = ½ GE = 59;50,42
$DK = EK - DE$ = 12;31,32
arc YL = $\sin^{-1}(DK/DH)$
= $\sin^{-1}$(0;56,59,31) = 71;***46***,4***9***° [...;0,41]
arc GL = ½ arc GE = 85;52,48°
arc GY = arc YL + arc GL = 15***7***;***39***,***37***° [156;53,29]

And thus,

		in Figure 5
arc GT = 2***2;20,23***°	[23; 6,31]	$=\angle D_3TA$
arc BT = arc BG - arc GT = 5***9;33,13***°	[58;47, 5]	$=\angle D_2TA$
arc AT = arc BT + arc AB = 10***4;44,17***°	[103;58, 9]	$=\angle D_1TA$
arc AY = 180° - arc AT = 7***5;15,43***°	[76; 1,51]	$=\angle D_1T\Pi$

The latest value is given in the text as …;13,4***5*** (scribal error: يج مه → يه مج), but our author actually applies 75;15,43° to the later computations. He takes the above values as the angular distances of the epicycle centre from the apsidal line.

Step 2: Derivation of the small angles η, ζ, and ε



(a) (b) (c)

Figure 9

$\angle NTS$ in Figures 9(a)-9(c), redrawn on the basis of Muḥyī al-Dīn's figures on f. 135r, corresponds, respectively, to arcs AY, BT, and GT, derived above. With e = 6;35,36, Muḥyī al-Dīn first computes the lengths of the projections of e and $2e$ onto the lines connecting the epicycle centre with the equant point (TC and TS) and the lengths of the lines DC and NS dropped, respectively, from the centre of the deferent and of the Earth perpendicular to the radius of the equant circle:

∠*NTS*	Sin	Cos	*DC*	*NS* = 2*DC*	*TC* = *CS*	*TS* = 2*TC*
7***5***;***1***5,***43***°	58; ***1***,***33***	15;***15,50***	6;2***2***,3***5***	12;***45***,1***0***	1;***40,38***	3;***21,16***
[76; 1,51]	[58;13,32]	[14;29, 2]	[6;25,38]	[12;51,17]	[1;35;56]	[3;11,52]
5***9***;***33,13***	51;***43,24***[1]	3***0***;***24,14***[2]	5;3***6***, ***2***	11;1***2***, ***4***	3;2***0,28***	6;***40***,5***6***
[58;47, 5]	[51;18,49]	[31; 5,43]	[5;39,52]	[11;19,43]	[3;25,57]	[6;51,54]
2***2***;***20,23***	2***2***;***48,21***	55;***29,49***	2;3***0,22***	5; ***0,44***	6; 5,***55***	12;11,***50***[3]
[23; 6,31]	[23;32,55]	[55;11, ***9***]	[2;35,58]	[5;11,56]	[6; 5,30]	[12;11, 1]

1. It should be …,34, but because of the error in the value computed for the length of *DC* (which should be around 5;41,3 by applying either of the sine values) it cannot be known whether the third sexagesimal digit is simply a scribal error (34/كد/24 → لد).
2. Text: 3***2***;… (obviously, a scribal error: لب → ل).
3. Text: …;***1***,… (a scribal error: ا → يا).

Our author then computes the distances of the mean positions on the equant circle from the points marked by the perpendiculars drawn from the Earth *N* to its radii.

ES = 60 - *TS* = 56;***38,44***	[56;48, 8]
ZS = 60 + *TS* = 66;***40***,5***6***	[66;51,54]
HS = 60 + *TS* = 72; ***1,50***	[72;11, 1]

The angles at the epicycle centres subtending the perpendicular lines drawn from the centre of the deferent to the radii of the equant circle are derived from Sin^{-1}(*DC*) as follows.

∠*DAC* = 6; ***6***, ***3***°[1]	→	∠*ADC* = 83;5***3,57***°
[6; 8,58]		[83;51, 2]
∠*DBC* = 5;2***1,21***		∠*BDC* = 84;3***8,39***
[5;25, 2]		[84;34,58]
∠*DGC* = 2;2***3,38***		∠*GDC* = 87;3***6,22***
[2;28,59]		[87;31, 1]

1. Text: …;5,… (a scribal error: ه → و).

Then,

AC	= Sin(∠*ADC*)	= 59;39,***37***	[...;...,17]
BC	= Sin(∠*BDC*)	= 59;4***4,17***	[...;43,55]
GC	= Sin(∠*GDC*)	= 59;56,***52***	[...;...,37]

and

AS	= *AC* - *CS*	= 57;***58,59***	[58; 3,21]
BS	= *BC* + *CS*	= 63; ***4,45***	[...; 9,52]
GS	= *GC* + *CS*	= 66; 2,***47***	[...;..., 7]

Thus, the small angles can be computed as follows:

		η	ζ	ε
NS/ES	*NS/AS*	∠*NES* = tan⁻¹(*NS/SE*)	∠*NAS* = tan⁻¹(*NS/AS*)	∠*KNR* = arc *KR*
0;13,3***0,28*** [0;13,34,42]	0;13,1***1,47*** [0;13,17, 7]	12;4***1,15***° [12;44,27]	12;2***4,15***°[1] [12;29, 0]	0;1***7***, ***0***° [0;15,27]
NS/ZS	*NS/BS*	∠*NZS* = tan⁻¹(*NS/ZS*)	∠*NBS* = tan⁻¹(*NS/BS*)	∠*XNL* = arc *XL*
0;10, ***4,43*** [0;10, 9,56]	0;10,***39,15*** [0;10,45,40]	9;3***2***, ***7***° [9;36, 6]	10; ***4***, ***8***° [10; 9,28]	0;3***2***, ***1***° [0;33,22]
NS/HS	*NS/GS*	∠*NHS* = tan⁻¹(*NS/HS*)	∠*MGS* = tan⁻¹(*NS/AS*)	∠*MNΘ* = arc *MΘ*
0; 4,1***0,30*** [0; 4,19,17]	0; 4,***33,12*** [0; 4,43,25]	***3;58,49***° [4; 6,54]	4;2***0,23***° [4;29,41]	0;2***1,34***° [0;22,47]

1. Text: ...;...,1***2***, but a scribal error (يه → يب), as understood from the value given for ε.

Step 3: Iterative procedure

With regard to Figure 7, Muḥyī al-Dīn notes again that the sum of the small arcs *KR* and *XL* (i.e., respectively, ε_1 and ε_2 in our general notions) should be subtracted from arc *KL* ($\Delta\lambda_{12}$), and the sum of the small arcs *XL* and *MΘ* (i.e., ε_3) should be added to *LM* ($\Delta\lambda_{23}$):

$$\Delta\lambda_{12}' = \Delta\lambda_{12} - \varepsilon_1 - \varepsilon_2$$
$$\Delta\lambda_{23}' = \Delta\lambda_{23} + \varepsilon_2 + \varepsilon_3$$

He then repeats the whole procedure with the above quantities and derives a new set of values for the eccentricity, the angular distances of the planet's mean positions on the equant circle from the apsidal line (arcs *AY*, *BT*, and *GT* in Figure 8(b), corresponding to $\angle D_1T\Pi$, $\angle D_2TA$, and $\angle D_3TA$ in Figure 5), and the small angles ε. As our author states, it is repeated for the other three times, although he presents only the results, as summarized in the next tabulation. He notes that in the iteration process for the third time, it is $-\varepsilon_1 + \varepsilon_2 = 0;28,21°$ that must be subtracted from $\Delta\lambda_{12}$, which is, indeed, correct;[86] for this reason, we indicate ε_1 resulting from the second round of computation by a negative sign.

	e	$\angle D_1T\Pi$	$\angle D_2TA$	$\angle D_3TA$
1	13;11,12	75;15,43°	59;33,13°	22;20,23°
	[13;14,47]	[76; 1,51]	[58;47, 5]	[23; 6,31]
2	11;53,46	87;14,53	47;34, 3	34;19,33
	[11;50,56]	[87;24,39]	[47;24,17]	[34;29,19]
3	11;58,24	83;44,20	51; 4,36	30;49, 0
	[11;59,29]	[83;39,24]	[51; 9,32]	[30;44, 4]

86. Consider Figures 5 and 7; the equant circle intersects with the eccentric deferent at the two points with the longitudes of *(i)* $\lambda_A - 90° - q = \lambda_\Pi + 90° - q$ and *(ii)* $\lambda_A + 90° + q = \lambda_\Pi - 90° + q$, where λ_A and λ_Π indicate the longitudes, respectively, of the apogee and perigee and $q = \tan^{-1}(e/2R)$. These two points together with the apogee and perigee of the equant circle mark the four segments of it, the two in the vicinity of the apogee and the two in the perigeal region: **S1**: from *(i)* to the apogee, **S2**: from the apogee to *(ii)*, **S3**: from *(ii)* to the perigee, and **S4**: from there to *(i)*. Every time the computational procedure is repeated, $\angle D_3TA$, $\angle D_2TA$, and $\angle D_1T\Pi$ derived at the end of Step 1 show in which segment of the equant circle the planet's mean position falls at the time of each mean opposition. In the first round of computation, Muḥyī al-Dīn derives $e \approx 6.6$, from which $q \approx 3;9°$; hence, the points on the equant circle located within ~ 93;9° from the apogee are in either **S1** or **S2**, and those located within ~ 86;51° from the perigee are placed in either **S3** or **S4**. From $\angle D_1T\Pi$ = 75;…° derived in the first round of computation, it is known that in the first mean opposition, Mars is in **S4**, but in the second round, $\angle D1T\Pi$ = 87;…° results, which means the planet's mean position shifts to **S1**. However, the planet's mean position in the first mean opposition is located again in **S4** in the later round of calculation.

	e	$\angle D_1T\Pi$	$\angle D_2TA$	$\angle D_3TA$
4	11;5***8,48, 6***	——	——	31;***25,27***
	[11;57, 0,55]	[84;37,23]	[50;11,33]	[31;42, 3]
5	11;58,***48, 6***	——	——	31;***25,27***
	[11;58,32,55]	[84;14,55]	[50;34, 1]	[31;19,35]

	ε_1	ε_2	ε_3	$\Delta\lambda_{12}'$	$\Delta\lambda_{13}'$
1	0;1***7***, ***0***°	0;3***2***, ***1***°	0;2***1,34***°	41;1***3,54***°	69; ***3***, ***2***°
	[0;15,27]	[0;33,22]	[0;22,47]	[41;14, 6]	[69; 5,36]
2	-0; 0, ***3***	0;28,***24***	0;2***5,11***	41;34,***34***	69; ***3***, ***2***
	[0; 0, 0]	[0;28,10]	[0;24,35]	[41;34,45]	[69; 2,12]
3	0; 4, ***3***	0;2***8,44***	0;2***3,51***	41;***30***, ***8***	69; ***2***, ***2***
	[0; 4,35]	[0;29, 1]	[0;24,37]	[41;29,19]	[69; 3, 5]
4	——	——	——	——	——
	[0; 2,45]	[0;29, 2]	[0;23,40]	[41;31,8]	[69; 2, 9]
5	——	——	——	——	——
	[0; 3,40]	[0;29, 2]	[0;23,40]	——	——

In the fifth time, al-Maghribī says, the same quantities are achieved as in the fourth time, for which he only gives e and $\angle D_3TA$. Our recomputed values in the above tabulations originate from the re-computed value e = 6;37,23,30 for the first estimation of the eccentricity, which inevitably makes substantial changes in all the subsequent steps of the derivation. If we continue the computation for the two additional times, we obtain $2e$ = 11;58,22 and 11;58,35.

Muḥyī al-Dīn reasonably rounds the finalized value for the eccentricity to e = 6 which is the same Ptolemaic value, and then in an unnumbered chapter appended to viii.11, proceeds to calculate the mean longitude and anomaly and the longitude of the apogee for the time of the third opposition. These values are used in viii.13 for the derivation of the mean motions and epoch positions (see below, 4.2). With regard to Figure 10 (f. 136v): $\angle OTG = \angle D_3TA$ = 31;25,27° and $e = TD = DN$ = 6; we have:

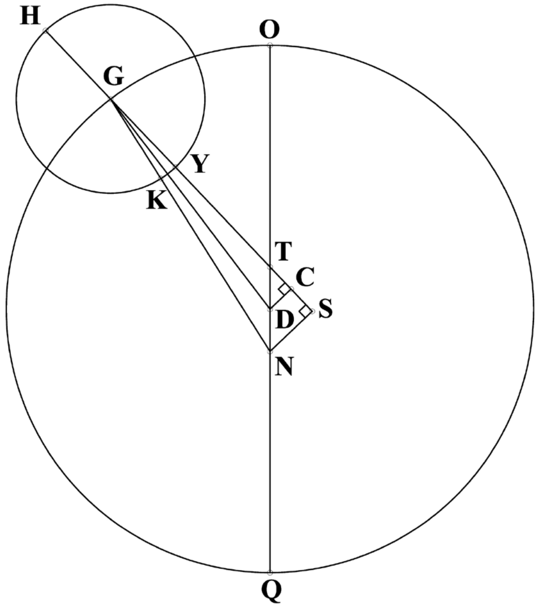


FIGURE 10. Derivation of the epoch positions: Mars.

$DC = e \sin(\angle OTG) = 6 \times 0;31,16,56 = 3;7,41,36$

$TC = CS = e \cos(\angle OTG) = 6 \times 0;51,1\boldsymbol{2},\boldsymbol{0}$ [...,11,59]

$= 5;7,12$

$NS = 2DC = 6;15,23,12$

$\angle DGC = \mathrm{Sin}^{-1}(DC) = 2;59,19°$

$\angle GDC = 87;0;41°$

$GC = \mathrm{Sin}(\angle GDC) = 59;55,6$

$GS = GC + CS = 65;2,18$

$\angle NGS = \tan^{-1}(NS/GS) = \tan^{-1}(0;5,46,18) = 5;29,4\boldsymbol{0}°$ [...,41]

$\angle ONG = \angle OTG - \angle NGS = 25;55,4\boldsymbol{7}°$ [...,46]

The true longitude of Mars at the time of the third opposition was λ_3 = 163;2,53°; thus, the longitude of apogee, mean anomaly, and mean longitude of the planet at that time are:

$\lambda_{A3} = \lambda_3 - \angle ONG = 137;7,\boldsymbol{6}°$ [..., 7]

$\bar{\alpha}_3 = 180° - \angle NGS = 174;30,\boldsymbol{20}°$ [...,19]

$\bar{\lambda}_3 = \lambda_{A3} + \angle OTG = 168;32,3\boldsymbol{3}°$ [...,34]

4.2. Mean motion in longitude and the radixes[87]

Muḥyī al-Dīn compares the mean longitudes of the superior planets in any of his third mean oppositions with the corresponding data derived by Ptolemy for his own third mean oppositions observed over one millennium earlier. The intervals of time between them are thus too long to allow accurate values for the mean daily motion in longitude of the outer planets to be derived. In the tabulation below, t_3 stands for the time of Muḥyī al-Dīn's third mean opposition that he converts into the Nabonassar era (he notes that the interval of time between the Nabonassar and Yazdigird eras is equal to $1379^y\ 90^d$) and the column headed $\bar{\lambda}_3$ contains the values he has already derived for the mean longitude; analogously, t_{3P} denotes the times of Ptolemy's third mean oppositions and $\bar{\lambda}_{3P}$, his mean longitudes. For Mars:

t_{3Z}	t_{3P}	$\bar{\lambda}_3$	$\bar{\lambda}_3$
$2018^y\ 140^d\ 1;15^h$	$885^y\ 312^d\ 10^h$	168;32,33°	251;9°[88]

Thus,

$$\Delta t = 1132^y\ 192^d\ 15;15^h = 41[3]372^d\ 15;15^h = 1,54,49,32;38,7,30^d$$
$$\Delta\bar{\lambda} = 277;23,33°$$

The planet also rotates 601 times on the ecliptic in this time interval, and hence

$$\omega_\lambda = 0;31,26,39,44,40,4\boldsymbol{8}° \qquad [\ldots,47]$$

The mean motions $\Delta\bar{\lambda}_{03}$ of the planet during the time intervals Δt_{03} from the epoch, t_0 = the end of 600 Y, to the moment of the third mean opposition are computed from the values ω_λ derived above and the amounts $\Delta\lambda_{A03}$ of the progress in longitudes of the apogees, from the precessional rate of $1°/66^y$. These are then subtracted, respectively, from $\bar{\lambda}_3$ and λ_{A3} to produce the epoch mean positions $\bar{\lambda}_0$ and λ_{A0}:

87. al-Maghribī, *Talkhīṣ* VIII.13: ff. 135v, 137r.
88. *Almagest* X.7: Toomer [1984] 1998, p. 498.

Δt_{03}	$\Delta\bar{\lambda}_{03}$	$\bar{\lambda}_{0}$	$\Delta\lambda_{A03}$	λ_{A0}
39^{y} 50^{d} $1;15^{h}$	286;24,3***0***°	242; 8, ***3***°	0;35,35°	136;31,31°
	[...,31]	[...,2]		

4.3. Radius of the epicycle[89]

As we have already seen in Section 3, Muḥyī al-Dīn observed an appulse of Mars to Regulus, which he uses to determine the size of the epicycle of the planet. For Mars, our author obtained a new value, but in the cases of Saturn and Jupiter his derivations only confirmed the values Ptolemy had derived over 11 centuries earlier.

According to Muḥyī al-Dīn, a conjunction between Mars and Regulus occurred at 2:30 MLT on 15 October 1270, for which the orbital positions below are given:

For Sun:	$\lambda_{\odot}$	= 210;31,47°	
For Mars:	$\bar{\lambda}$	= 98; 4,55	
	$\bar{\alpha}$	= 112;26,52	
	λ_{A}	= 137; ***7***, ***6***	[137; 6,46]
	$\bar{\kappa}$	= 320;57,***49***	[320;58,11]
	$\lambda = \lambda_{Regulus}$	= 139;52	

Our author again uses the value for λ_{A} he derived for the time of the observation Mars-3, while between this time and the observation of the planet's appulse to Regulus, there is an interval of 134 days. However, it only produces a difference of 0;0,20° in λ_{A} between the two times. So, we ignore the effect of this deviation, which does not cause any noticeable difference in the final result. With regard to Figure 11, we have:

89. al-Maghribī, *Talkhīṣ* VIII.12: ff. 135r-v.

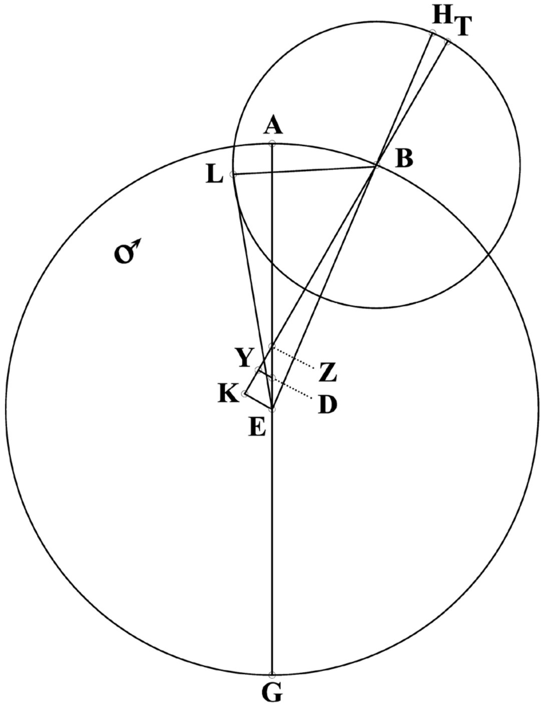


Figure 11. Derivation of the epicycle radius: Mars.

[1] $\angle Z = 360 - \bar{\varkappa} = 39;2,11°$
$\mathrm{Sin}\angle Z = 37;47,20$
$\mathrm{Cos}\angle Z = 46;36,17$

[2] $DY = DZ\ \mathrm{Sin}\angle Z\ /\ R = 3;46,44$
$YK = YZ = DZ\ \mathrm{Cos}\angle Z\ /\ R = 4;39,38$, where $DZ = e$,
$EK = 2\ DY = 7;3\boldsymbol{2},2\boldsymbol{4}$ [...;33,28]

$\angle DBY = \mathrm{Sin}^{-1}(DY) = 3;36,39°$
$\angle BDY = 90° - \angle DBY = 86;23,21°$

$BY = \mathrm{Sin}\angle BDY = 59;52,51$
$BK = BY + YK = 64;32,29$

The equation of centre is computed as follows, instead of extracting from the table:[90]

[3] $\angle EBK = c_{1}(\bar{\kappa}) = \mathrm{Tan}^{-1}(EK \times R / BK) = \mathrm{Tan}^{-1}(7;\boldsymbol{0},34) = 6;\boldsymbol{39},4\boldsymbol{8}°$ [7;1,34] [6;40,44°]

$\mathrm{Sin}\angle EBK = 6;5\boldsymbol{7},4\boldsymbol{3},\boldsymbol{33}$ [...,35] [6;58,41,50]

[4] $BE = EK \times R / \mathrm{Sin}\angle EBK = 64;58,5\boldsymbol{0}$ [64;58,57]

[5] $\kappa = \angle BEA = \angle Z - \angle EBK = 32;2\boldsymbol{2},2\boldsymbol{3}$ [32;21,27]

[6] $\angle AEL = \lambda - \lambda_{A} = 2;44,54°$

[7] $\angle BEL = \angle BEA + \angle AEL = 35;\boldsymbol{7},\boldsymbol{17}°$ [35; 6,21]

$\mathrm{Sin}\angle BEL = 34;31,7$ [34;30,19]

$\angle HBL = \mathrm{arc}\ HL = \alpha = \bar{\alpha} - c_{1} = 105;4\boldsymbol{7},\boldsymbol{4}$ [...;46, 8]

[8] $\angle L = \angle HBL - \angle BEL = 70;39,47$

$\mathrm{Sin}\angle L = 56;36,5\boldsymbol{5},\boldsymbol{8}$ [...54,56]

[9] $r = BL = BE \times \mathrm{Sin}\angle BEL / \mathrm{Sin}\angle L$

$= 37,23;2,19/56;36,55,8$

$= 39;3\boldsymbol{7},\boldsymbol{30}$ [39;36,17]

Our author appears to have rounded the result to the upper half-minute. As can be seen, all his computational errors, beginning with a strange error in doubling the length of *DY* in [2], altogether lead to a minor error of about -0;1 in the final result. At any rate, a non-Ptolemaic value results for *r* which corresponds to a maximum epicyclic equation of 41;20° [41;19°] at mean distance of Mars from the Earth.

90. The Table in the *Adwār* has c1(320°) = 6;49° and c1(321°) = 6;40°, from which c1(320;57,49°) = 6;40,20°. It can plausibly be assumed that at the time when Muḥyī al-Dīn went on to compute the radii of the epicycles of the superior planets, he had the tables of the equation of centre constructed for Jupiter and Saturn, but not for Mars.

5. Discussion and Conclusion

5.1. *Muḥyī al-Dīn's test of the available theories*

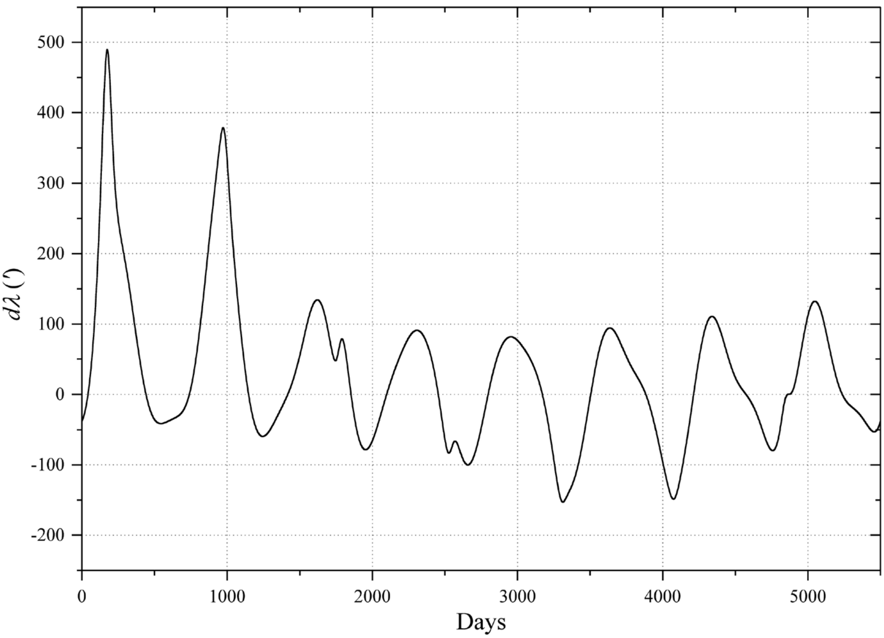
As we have seen, in order to compute Mars' mean motions in longitude between his trio of their oppositions to the mean Sun, Muḥyī al-Dīn used Ibn al-Fahhād's value. Also, in the cases of the other two superior planets, he used the already known values for their mean daily motions in longitude: Ibn Yūnus's value in the case of Jupiter, and his own value established earlier in his *Tāj al-azyāj* for Saturn. Muḥyī al-Dīn probably had access to Ibn Yūnus's *zīj* while working in Syria, and it is possible that he was responsible for introducing it at the Maragha observatory, since, to the best of our knowledge, no trace of Ibn Yūnus's *zīj* is found in the Eastern Islamic lands until that time; more specifically, no reference to it can be found either in al-Khāzinī's *On Experimental Astronomy* (*Kayfiyyat al-iʿtibār*)[91]—in which the two most influential Middle-Eastern works, the *Mumtaḥan zīj* and al-Battānī's *Sābiʾ zīj*, are mentioned—, or in Ibn al-Fahhād's highly informative evaluation of the deficiencies and errors in his Islamic predecessors' works as put forward in the prologue of his *ʿAlāʾī zīj*.[92] Muḥyī al-Dīn appears to have become familiar with Ibn al-Fahhad's work after his arrival at Maragha; at that time, the *ʿAlāʾī zīj* enjoyed a wide dissemination, to the far eastern reaches of the Islamic lands in India[93] as well as in its southernmost region, in the Yemen.[94]

91. This treatise comes as an introduction to his Sanjarī *zīj* and is preserved in a unique copy in the Vatican MS of his *zīj*. In it, Khāzinī deals in depth with the principal features of observational astronomy and describes reasonable ways to reconcile the available theories and the observational data in the context of Ptolemaic models from a coherent methodological point of view. He explicitly refers to the two most influential Middle Eastern works, the *Maʾmūnī* (i.e., *Mumtaḥan*) *zīj* and al-Battānī's *zīj* (al-Khāzinī, II.4: V: f. 8r). This work, which is a major contribution to medieval observational astronomy, is under study by the present author.

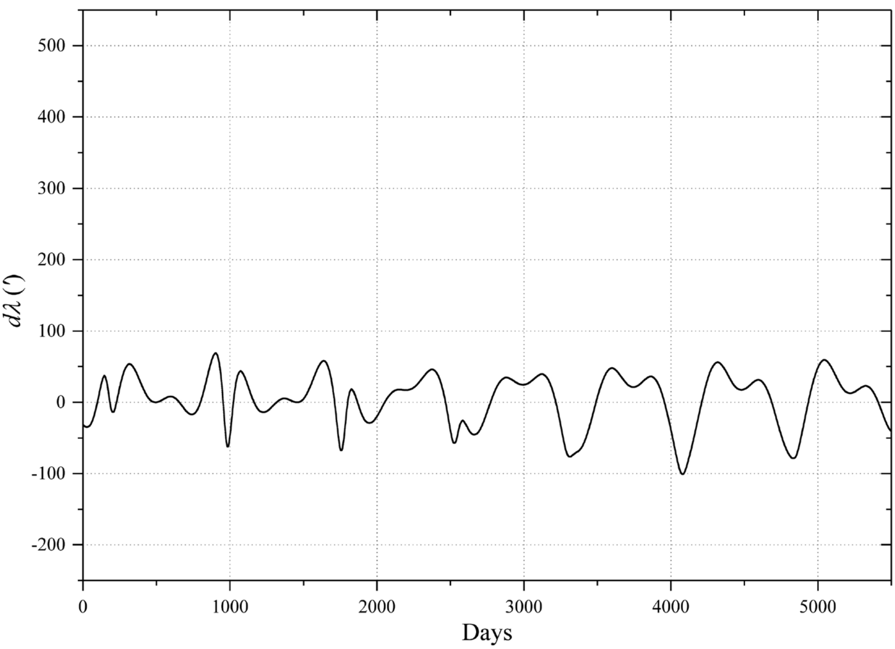
92. Al-Fahhād, *Zīj*, pp. 3-5.

93. E.g., in Maḥmūd b. ʿUmar's *Nāṣirī zīj*, the earliest *zīj* written in India, ***ca***. the mid-13th century; see van Dalen 2004a.

94. E.g., in Maḥmūd b. Abī Bakr al-Fārisī's *Zīj al-mumtaḥan al-Muẓaffarī*; Al-Fārisī, C: f. 57r; see, also, Kennedy 1956, p. 132, no. 54; van Dalen 2004a, p. 829.


500
400
300
200
100
0
-100
-200
$d\lambda$ (′)
0
1000
2000
3000
4000
5000
Days

(a)


500
400
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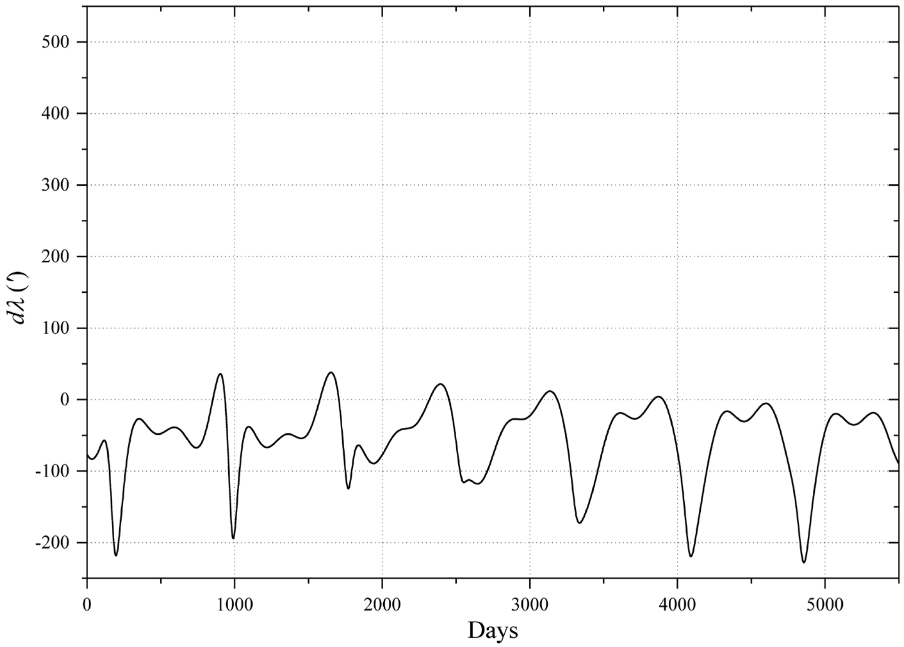
100
0
-100
-200
$d\lambda$ (′)
0
1000
2000
3000
4000
5000
Days

(b)



(c)

Figure 12. The errors in the longitude of Mars at midnight (0:0 LT) computed on the basis of (a) Ibn Yūnus's *Ḥākimī zīj*, (b) Ibn al-Fahhād's *ʿAlāʾī zīj*, and (c) Muḥyī al-Dīn's *Tāj al-azyāj* for the longitude of Maragha in the period of 5,500 days beginning with 1-1-1260.

It is worth exploring why he used Ibn al-Fahhād's value, especially because he had already measured an unprecedented set of the planetary parameter values during his residence and activities in Syria, as recorded in his *Tāj al-azyāj*. One might have been expected him to rely on, and use, these parameters as the provisional estimates for a renewed attempt to solve the complicated problem of measuring the planetary orbital elements at the Maragha observatory. One reason for doing so may have been to do with the accuracy achieved in the ephemerides of the three superior planets as computed on the basis of the three works, Ibn Yūnus's *Ḥākimī zīj*, Ibn al-Fahhād's *ʿAlāʾī zīj*, and our author's *Tāj*, in the sense that Muḥyī al-Dīn might have been interested in establishing which of these three works calculated theoretical positions that were in closest agreement with the longitudes of each superior planet as measured from his observational data. In order to test this hypothesis in the case of Mars, we have recomputed the longitudes of the planet day-by-day over a period of 5,500 days starting from 1-1-1260 (thus extending to 21-1-1275), which effectively covers the period of Muḥyī al-Dīn's activities at Maragha, and compared them with the ones derived from modern theories. The statistical results for the errors $d\lambda$ in longitude, including the mean error together with standard deviation σ, limits of the errors, and Mean Absolute Error, *MAE* (i.e., the average of the absolute values of errors) are summarized in Table 3 and the errors are plotted against time in Figures 12(a)-12(c). In Table 4, the dates and times of the seven oppositions of Mars to the true Sun that occurred in the period in question are given, together with the errors in the longitudes at midnight on the same days calculated from the three works under scrutiny; the true oppositions correlated with Muḥyī al-Dīn's trio of mean oppositions (see Table 2) are highlighted in grey (note that the times of *true* and *mean* oppositions differ from each other by a few days). The oppositions of the superior planets provide astronomers with the best opportunities to observe them, not least because around the true oppositions the superior planets travel through the middle part of their arcs of retrograde motion, culminate around midnight, and reach their maximum magnitude; above all, the measurements of their orbital elements are on the whole dependent upon the positional data measured for the moments of the mean oppositions. We also need to assess whether (and if so, how) the errors committed by Muḥyī al-Dīn in the measurement of the midnight longitudes from the complicated process of observations and computations might distort the truth about the accuracy of the astronomical tables available to him at the time. For this purpose, in Table 5 we present separately the longitudes computed from the three *zījes* mentioned above (Cols. **II**-**IV**, headed "Theoretical Longitudes") in comparison with both the ones measured by Muḥyī al-Dīn from his observational data (in *italics* in Col. **V**) and the modern ones (in **bold** in Col. **VI**) for the moment of any of his triple observations, i.e., midnight on the dates of the observations as given in Col. **I** (see Table 1, Col. **II**). The longitudinal errors $d\lambda$ are given in the last four columns: Col. **VII** indicates the errors in Muḥyī al-Dīn's measured longitudes (= Col. **V** minus Col. **VI**; NB. the values in Cols. **V**-**VII** have already been given in Table 1, Col. **IX**), which we refer to by $d\lambda_{M}$; in the three remaining columns (**VIII**-**X**), the two sets of error values are given: those in *italics* stand for the differences between the theoretical longitudes in Cols. **II**-**IV** and Muḥyī al-Dīn's longitudes in Col. **V** (referred to as $d\lambda'$), and those in **bold** indicate the differences between Cols. **II**-**IV** and the modern values in Col. **VI** ($d\lambda$). It is evident that $d\lambda' = d\lambda - d\lambda_{M}$. Muḥyī al-Dīn must have made his decision about the accuracy of the *zījes* at his disposal on the basis of the comparison between the theoretical values and his own measured values, i.e., the errors $d\lambda'$; thus, with the aid of Table 5, we can determine whether his errors $d\lambda_M$ led him to any wrong conclusions in this regard.

Clearly, Ibn al-Fahhād's values for the longitude of Mars are far superior to the ones derived from either the *Ḥākimī zīj* or the *Tāj al-azyāj*. The range of Ibn Yūnus's errors is remarkably wide. However, the errors are diminished or vanish at some specific points; an example is Mars' true opposition no. 7, for which Ibn Yūnus's *zīj* gives by mere coincidence a more accurate longitude than either Ibn al-Fahhād or the *Tāj al-azyāj*. Ibn al-Fahhād's errors are more or less evenly distributed, while Muḥyī al-Dīn's errors show a tendency towards a negative shift. In general, Ibn al-Fahhād's results are notably more accurate than Muḥyī al-Dīn's; so his value for its mean daily motion is the best choice for the derivation of the fundamental parameters of the planet, and is in fact the value that Muḥyī al-Dīn used.

When we compare the theoretical longitudes with Muḥyī al-Dīn's measured values in a *historically* sound way in Table 5, we find that his measured longitudes of Mars in his trio of observations have a negative error of less than a single degree: $-1° < d\lambda_M < 0°$ (Table 5, Col. **VII**); thus, if a longitude value of this planet computed on the basis of one of the three historical theories in the *Ḥākimī zīj*, the *ʿAlāʾī zīj*, or the *Tāj al-azyāj*, had a negative error with respect to a modern (correct) value (i.e., $d\lambda < 0$), Muḥyī al-Dīn would have found for it a lesser error $d\lambda'$ in absolute value than it had in reality (remember: $d\lambda' = d\lambda - d\lambda_M$), and *vice versa*. Eight of the nine values for the error $d\lambda$ in the case of Mars (the **bold** numbers in Cols. **VIII**-**X** in Table 5) are negative, and so the errors $d\lambda'$ which Muḥyī al-Dīn would have considered (the numbers in *italics* in Cols. **VIII**-**X** in Table 5) become less than $d\lambda$ in absolute value. A consequence of this situation is that Muḥyī al-Dīn would have assumed the size of the errors in Ibn al-Fahhād's theory of Mars at the time of his first and third observations of the planet to be less than half what they were in reality. Only Ibn Yūnus's value for the time of Muḥyī al-Dīn's first observation has a positive error $d\lambda$; so Muḥyī al-Dīn would have found a greater error for it than it really had. Nevertheless, the distortion arising from Muḥyī al-Dīn's own errors $d\lambda_M$ would not have led him to a wrong conclusion in this case, for two obvious reasons: first, the undisputable superiority of Ibn al-Fahhād's theory of Mars over both Ibn Yūnus's and Muḥyī al-Dīn's earlier theory established in the *Tāj al-azyāj*; and second, the fact that nearly all the theoretical longitudes have errors of the same sign and are thus all affected by Muḥyī al-Dīn's errors $d\lambda_{\mathrm{M}}$ in the same way.

By this stage in the discussion, an important point should be sufficiently clear: simultaneously with his extensive observational project, Muḥyī al-Dīn seems also to have been engaged in a parallel project testing the outcomes of the available theories against his empirical data for the purpose of exploiting the best possibilities for his last round of planetary measurements.

5.2. The accuracy of Muḥyī al-Dīn's values for the orbital elements of Mars in the context of medieval Islamic astronomy

The first known attempt to systematically measure the orbital elements of the superior planets was made by Ibn al-Aʿlam (d. 985 AD) at Baghdad. It was followed (after a gap of about three centuries) by Muḥyī al-Dīn at Maragha and by Jamāl al-Dīn al-Zaydī (*fl. ca.* 1260 AD) at the contemporary Mongolian observatory founded in Beijing, and two centuries later by the astronomers working at the Samarqand observatory. This situation in essence marks a difference between the two periods of the astronomical activities in medieval Islam. In its early period, the emphasis was placed mainly on deriving the basic solar and lunar parameters and fundamental parameters such as the rate of precession and the obliquity of the ecliptic; having convincingly solved the related issues, the ground was then prepared for addressing more important problems such as re-quantifying the Ptolemaic planetary models, which, as we have seen in the preceding sections, involved a huge number of observations and required considerable mathematical skill. The most important of these procedures was a re-determination of the direction of the planetary lines of nodes and their orbital inclinations which, as we have explained elsewhere,[95] began at Maragha in the case of the inferior planets and continued at the Samarqand observatory in the case of the superior planets. In this section, we first explain a modern criterion[96] for analysing the accuracy of the eccentricity, the radius of the epicycle, and the longitude of the apogee of Mars, and then classify the known historical values for these parameters.

95. See Mozaffari 2016c.

96. Already briefly set forth in Mozaffari 2014c.

5.2.1. The Derivation of the Geocentric Orbital Elements from their Heliocentric Counterparts

As we have seen earlier (Section 1.4), the structural parameters defining the geocentric orbit of a planet are the eccentricity and the longitude of the apogee (or the perigee). The Ptolemaic (geocentric) eccentricity of a superior planet is in fact the sum of the two vectors: the planet's heliocentric eccentricity projected onto the Earth's orbit, and the Earth's eccentricity; their orientations determine the direction of the geocentric apsidal line.[97] The geocentric orbital elements thus depend upon the two heliocentric eccentricities, the inclination of the planet's orbit, and the angle between the heliocentric apsidal lines of the planet and the Earth. Since these heliocentric parameters change with the passage of time, it might be expected that new values for the Ptolemaic orbital elements would have been obtained during the two millennia elapsing since Ptolemy. After the combination of the two heliocentric eccentricities (i.e., those of the planet and Earth), the epicycle may now be considered to represent the Earth's zero-eccentricity (i.e., circular) orbit with respect to the planet's geocentric orbit. The radius of epicycle is thus, in the case of a superior planet, the ratio of the semi-major axis of the Earth's elliptical orbit to that of the planet. The changes in the semi-major axes of the superior planets are negligible in this period, but better approximations for the size of epicycle may be expected as well.

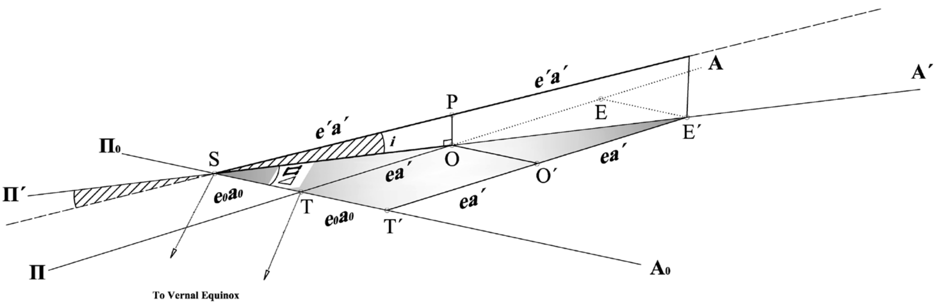


Figure 13. A schematic view of the transformation of the geocentric to the heliocentric orbital elements.

97. The transformation of the heliocentric to the geocentric orbital elements/model for the examination of the ancient and medieval astronomical parameter values have attracted little attention in the modern literature; only some tentative considerations of this problem can be found, e.g., in Neugebauer 1975, Vol. 1, esp. pp. 146-8, 208 (for a major error committed by Neugebauer, see Mozaffari 2017, p. 5, note 3); Swerdlow and Neugebauer 1984, esp. pp. 291-297, 369-371; Aaboe 2001, pp. 154-170.

In the very simplest manner, Figure 13 illustrates how the two vectors of the heliocentric eccentricities of the Earth and a superior planet are combined to produce the planet's geocentric eccentricity and the geocentric direction of its apsidal line. In it, e is the eccentricity, a: the semi-major axis, Π: the longitude of the perihelion/perigee; prime (′) stands for the heliocentric orbital elements of the planet and the subscript zero, for those of the Earth. S represents the Sun; T is the centre of the elliptical orbit of the Earth, T' its empty focus, and hence $A_0\Pi_0$ is the heliocentric apsidal line of the Earth passing through S, T and T'. O is the centre of the elliptical orbit of a superior planet, E' its empty focus, and thus $A'\Pi'$ is the heliocentric apsidal line of the planet, which is inclined from the Earth's orbital plane (i.e., the ecliptic) under an angle $i. \angle OST = \Delta\Pi$ is the angle between the apsidal lines of the planet and the Earth ($= \Pi - \Pi'$). In order to derive the geocentric eccentricity of the planet, i.e., $TO = OE$ (or $T'O' = O'E'$) $= ea'$, the vector $ST = TT' = e_0 a_0$ should be combined with $SO = OE' = e'a' \cos i$. Needless to say, T, O, and E are the same in Figure 1. It should be mentioned that the modern concept of *eccentricity* stands for a dimensionless "ratio" of the semi-minor to the semi-major axis of the orbit (thus, we take e as the geocentric eccentricity), but in ancient and medieval astronomy it is a "length" (taken here as ea') and displays how far the centre of the orbit is from that of the Earth. Figure 14 (drawn to scale) displays the change in the eccentricity and in the direction of the apsidal lines for Mars from the beginning of the Common Era to 2000 AD.

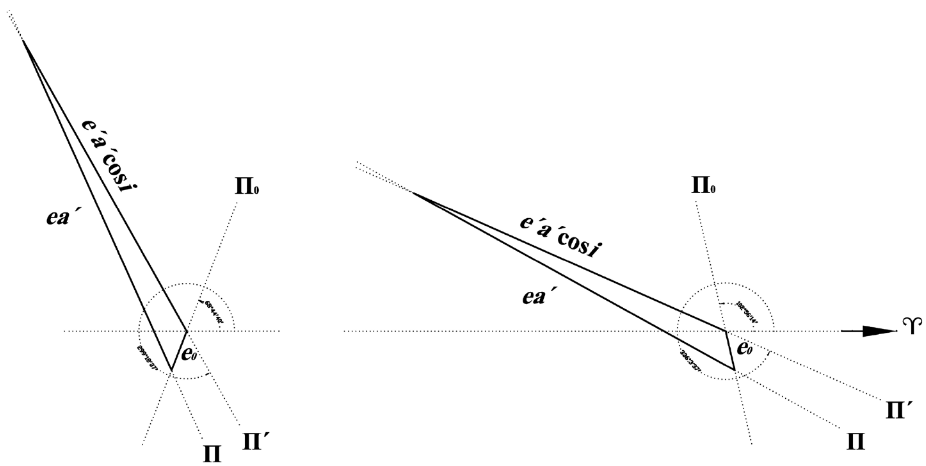


FIGURE 14. The correlation between the heliocentric and geocentric orbital elements of Mars, from 0 AD (Left) to 2000 AD (Right), drawn to scale.

Using the formulae for the heliocentric orbital elements which refer to the true dynamical ecliptic and equinox of date, in Simon *et al.* 1994, pp. 678-679, we derived the following formulae in the form of polynomials for the orbital elements of Mars:

$$e = 0.1003208470 + 0.0005488268775 \cdot t - 0.000004824391555 \cdot t^{2}$$
$$\lambda_{A} = 151.0439377° + 18.48370826°\, t + 0.02190523593°\, t^{2} + 0.0003283502669°\, t^{3} - 0.00007324357930°\, t^{4} - 0.000001033919189°\, t^{5} \qquad (3)$$

in which $t = (\text{JD} - 2451545.0)/\ 365250$ is the time measured in thousands of Julian years from 1 January 2000 (JDN 2451545.0). e should be multiplied by 60, according to the Ptolemaic norm. Figures 15 and 16 show the graphs of e and λ_{A} respectively. Mars' geocentric eccentricity changes by +0.003 ($R = 60$) in a Julian century of 36525 days unvarying (the coefficient of t in the above formula for e multiplied by 60×10^{-1}), from 5.95 in 0 AD to 6.02 in 2000 AD, which is the least secular change in the eccentricity among the superior planets, in comparison with the critical rates of changes in the case of Jupiter (~ +0.01 in a Julian century) and Saturn (~ -0.02 in a Julian century). In the same period, λ_{A} of Mars increased from ~ 114.16° to ~ 151.04° with an annual rate of ~ $66.5''^{/y}$ (or 1° in about 54.1 years; the coefficient of t in the above formula for e multiplied by 10^{-3}).

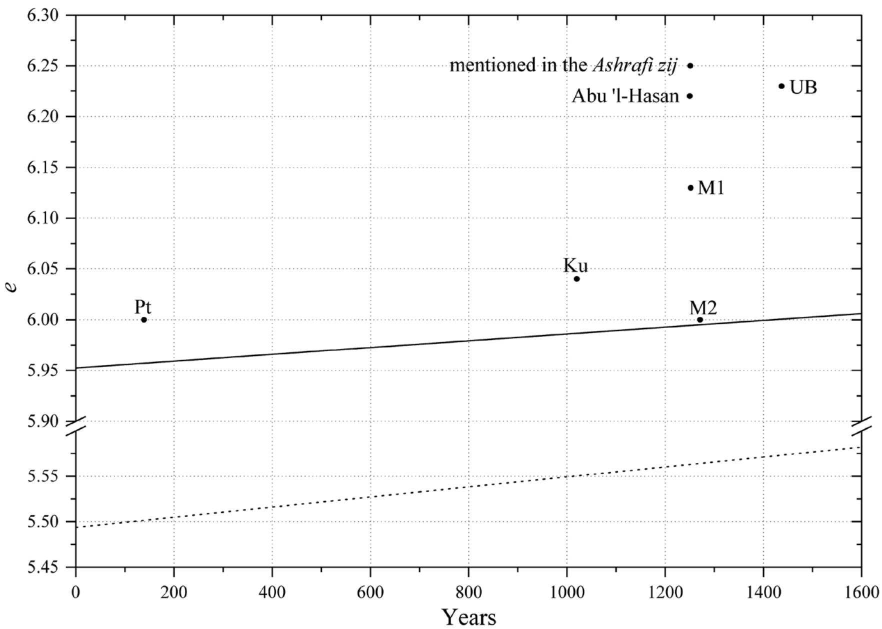


Figure 15. The geocentric (solid line) and heliocentric (dotted) eccentricity of Mars from the beginning of the Common Era to 1600 AD, according to Ptolemaic norm $R = 60$, along with the historical values from Table 6(a). M1 indicates Muḥyī al-Dīn's value in the *Tāj al-azyāj*, and M2 shows his value as derived at the Maragha observatory (respectively, nos. **4a** and **4b** in Table 6(a)).

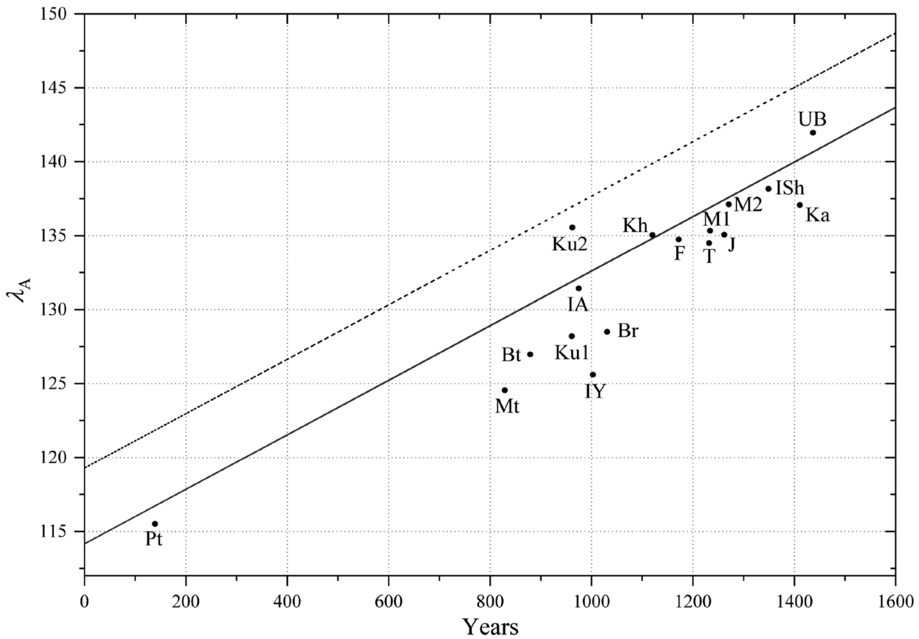


Figure 16. The geocentric (solid line) and heliocentric (dotted) longitude of apogee of Mars from the beginning of the Common Era to 1600 AD along with the historical values from Table 6(c). Ku1 and Ku2 stand for the two values Kūshyār mentions in the *Jāmiʿ zīj* (no. 7 in Table 6(c)). M1 indicates Muḥyī al-Dīn's epoch value in the *Tāj al-azyāj*, and M2 displays his value as derived at the Maragha observatory (respectively, nos. **11a** and **11b** in Table 6(c)).

5.2.2. Analysis of the accuracy of the medieval Islamic values for the orbital elements of Mars

No unprecedented values are available for the orbital elements of the superior planets from the classical period of medieval Islamic astronomy (before 1050 AD). Bīrūnī declares his disapproval of his Islamic predecessors' custom of neither mentioning their observational activities nor explaining how to derive the planetary parameters in the same manner that Ptolemy described his own; so at the beginning of the eleventh century, it seems that no account of such measurements from the early Islamic period was available to him.[98] In what follows, the values for the orbital elements of Mars from Ptolemy through the medieval Islamic period are summarized and compared with the modern values at the relevant times, computed according to the formulae (3), in Tables 6(a) for the eccentricity, 6(b) for the radius of the epicycle, and 6(c) for the longitude of apogee. The values for the eccentricity and the longitude of apogee are shown in Figures 15 and 16 respectively, along with the graph of the geocentric eccentricity and longitude of apogee of Mars.

(A) Eccentricity:

First of all, it is worth noting that a good number of the well-known *zījes* compiled in the Middle East in the medieval Islamic period deployed Ptolemy's values for the eccentricities of the superior planets: The *Mumtaḥan zīj* (Baghdad and Damascus, 829-833 AD), al-Battānī's *Ṣābī Zīj* (Raqqa and Antakya, d. 929 AD);[99] Ibn Yūnus's *Ḥākimī zīj* (d. 1007);[100] Bīrūnī's *al-Qānūn al-masʿūdī*;[101] al-Fahhād's *ʿAlāʾī Zīj* and some *zījes* that are based upon it, e.g., the Persian *Nāṣirī zīj* by Maḥmūd b. ʿUmar (Delhi, 1250) and the *Muẓaffarī zīj* by al-Fārisī (Yemen, *ca.* 1270);[102] the *Īlkhānī zīj*;[103] and Ibn al-Shāṭir's *Jadīd zīj*.[104]

Ibn al-Aʿlam is the first outstanding figure in the field of planetary astronomy in the Islamic period, and his now lost *ʿAḍudī zīj* exerted a great influence on the later medieval Middle Eastern astronomers. He was apparently the earliest medieval astronomer to occupy himself seriously with the derivation of the fundamen-

98. Bīrūnī 1954-1956, Vol. 3, pp. 1193, 1197.

99. Cf. Nallino [1899-1907] 1969, Vol. 2, pp. 110-123.

100. Ibn Yūnus, pp. 120-121; Caussin 1804, p. 219.

101. Bīrūnī 1954-1956, Vol. 3, pp. 1209-1210 (Saturn), 1225-1226 (Jupiter), 1242 (Mars).

102. See van Dalen 2004a, pp. 829, 848.

103. *Īlkhānī zīj*, C: pp. 99, 106, 115, P: ff. 34v, 36v, 38v; M1: ff. 61v, 65v, 67r. All the equation tables are displaced: Saturn: Min = 0;28°, Max = 13;32°; Jupiter: Min = 0;45°, Max = 11;15°; Mars: Min = 0;35°, Max = 23;25°.

104. Ibn al-Shāṭir, *Jadīd zīj*, K: ff. 59r, 61v, 64r, L1: ff. 76r, 79r, 82r, L2: ff. 59r, 61r, 63r, O: ff. 42v, 45v, 48v.

tal parameters of the Ptolemaic planetary models. The tables of his *zīj* are preserved in the *Ashrafī zīj* (written in Shiraz in the early 14th century),[105] which tells us that he measured new values for the eccentricities of Saturn (3;2), Jupiter (2;54),[106] and Mercury (3;35);[107] he, also, has an unprecedented value for the radius of the lunar epicycle.[108] The fact that no new tables for the equation of centre of Mars are associated with Ibn al-Aʿlam gives the strong impression that he probably did not measure a new value for its eccentricity. This situation is analogous to that of Jamāl al-Dīn. Contemporary to the Maragha Observatory, Khubilai Khan, the first emperor of the Mongolian Yuan dynasty of China (d. 1294 AD), founded an Islamic Astronomical Bureau in Beijing in 1271 AD, and appointed a certain Zhamaluding as its first director—quite probably an Iranian astronomer named Jamāl al-Dīn Muḥammad b. Ṭāhir b. Muḥammad al-Zaydī of Bukhārā. The observational activities at the Bureau produced a new set of values for the planetary parameters. Although the original work that was written on the basis of these parameter values seems to have been lost, some of them are preserved in

105. On the reconstruction of Ibn al-Aʿlam's parameters, cf. Kennedy 1977, Mercier 1989, and Dalen 2004a, esp. p. 22, note 7.

106. Ibn al-Aʿlam's tables of the equation of centre of these two superior planets are preserved in Kamālī's *Ashrafī zīj*. The table for Saturn's equation of centre is (F: f. 234v, G: f. 250r) and it is displaced with a minimum tabular value of 0;12° (for arguments 76°-81°) and a maximum value of 11;48° (for arguments 253°-258°). The table for Jupiter's equation of centre is (F: f. 235r, G: f. 250r) and it is also displaced with minimum 0;28° (for arguments 72°-78°) and maximum 11;32° (for arguments 246°-252°). Accordingly, the maximum equations of centre of Saturn and Jupiter respectively are derived as 5;48° and 5;32°. The modern values for the geocentric eccentricity of the two planets in Ibn al-Aʿlam's time are, respectively, 3;26 and 2;48 (see Mozaffari 2014c, p. 26). It should be noted that none of his values for the eccentricities of the two superior planets are more accurate than Ptolemy's. Ibn al-Aʿlam's value for the eccentricity of Saturn was used in the *zījes* of three Western Islamic astronomers (see Samsó and Millás 1998, p. 273).

107. Ibn al-Aʿlam's table of the equation of centre of Mercury is preserved in Kamālī's *Ashrafī Zīj* (F: f. 237r, G: f. 252v): the maximum equation of centre in this table is equal to 3;40° (for arguments 99°-101°). It should be noted that his value for the eccentricity of this planet is more exact than Ptolemy's three values 3;0, 2;45, 2;30, as found, respectively, in the *Almagest*, *Planetary Hypotheses*, and *Canobic Inscription* (*Almagest* IX.8,9: Toomer [1984] 1998, p. 459; Goldstein 1967, p. 19; Jones 2005, pp. 69, 86-87); the true value during the past two millennia has been about 3;50 (note that for the eccentricity of Mercury, we consider here half of the distance between the Earth and the centre of the hypocycle in Ptolemy's complicated model for this planet, on whose circumference the centre of its deferent revolves).

108. See Mozaffari 2014a, p. 105.

two later works: the one, *Huihuili*, a Chinese translation of a Persian *zīj* from the Bureau, prepared in Nanjing in 1382-1383; the other, the *Sanjufīnī zīj* written by a certain Sanjufīnī in Arabic in Tibet in 1366. Jamāl al-Dīn measured the new values for the eccentricities of Jupiter (2;40) and Saturn (3;19).[109]

As we have seen in 5.2.1, the geocentric eccentricity of Mars remained almost constant, about 6.0, over the past two millennia; therefore, even making extremely precise observations, no astronomer could find a noticeable change in the eccentricity of Mars from Ptolemy's value of 6;0 throughout the medieval period, and Muḥyī al-Dīn was no exception. This may be why neither Ibn al-Aʿlam nor Jamāl al-Dīn obtained a new value for it.

Nevertheless, Kushyār has a slightly greater value than Ptolemy for the eccentricity of Mars. It is not known whether he obtained it by carrying out a new observational programme (Table 6(a), no. 3). Moreover, three highly erroneous values for the eccentricity of the planet are reported in the Middle East, which are, interestingly, very close to each other; the first two go back to about 1250 AD, as will be explained in what follows. Among the *zījes* considered in the *Ashrafī zīj*, Kamālī informs us of the two (apparently lost) *zījes* named *Razāʾī zīj* and *Muntakhab zīj*, respectively, by a certain Abu 'l-Ḥasan and by a certain Muntakhab al-Dīn, both from Yazd (central Iran), which appear to have been written in the mid-13th century. Both works are now lost, but a *zīj* written in verse by Muntakhab al-Dīn, known as the *Manẓūm zīj* (*Versified zīj*), is extant. The *Muntakhab zīj* and *Razāʾī zīj* can be reconstructed to a large extent on the basis of the information that comes down to us through the *Ashrafī zīj*[110] and the anonymous *Sulṭānī zīj* written in Yazd in the 1290s (the latter work should not be confused with Wābkanawī's *Zīj al-Muḥaqqaq al-Sulṭānī*, nor with Ulugh Beg's *Sulṭānī zīj*), which is preserved in a unique manuscript in Iran (Library of Parliament, no. 184).[111] Some tables of the *Razāʾī zīj* are preserved in this *Sulṭānī zīj*: (a) the table

109. See Yabuuti 1987; 1997; van Dalen 2002a, esp. pp. 336-339; 2002b. On the accuracy of the values Jamāl al-Dīn measured for the eccentricities of Saturn and Jupiter, see Mozaffari 2014c, esp. p. 27. He found c1max = 6;19° for Saturn and 5;5° for Jupiter (see Sanjufīnī, ff. 47v and 48v; Yabuuti 1997, p. 33. Note that a very close value c1max = 5;6° (e ≈ 2.67) for Jupiter is also reported from the pre-Islamic Persian astronomy, the tradition of the *Shāh zīj*.

110. *Ashrafī zīj*, x.8 and x.9: F: f. 230v and ff. 231v-233r, 234r, 235v, G: f. 247v and ff. 248v-249r, 250v.

111. Despite the late E.S. Kennedy's conjecture (Kennedy 1956, no. 25 on p. 129), this work cannot be identical to the *Shāhī zīj*, since some materials from the latter work are explicitly quoted

of the longitude of the lunar node on f. 11r, (b) the procedure for the computation of the longitude of the superior planets in III.6 on f. 79r, (c) the planetary mean positions in longitude and in anomaly on f. 81v (the longitudes of the apogees of the Sun and Venus are equal), and (d) the tables of the "difference in equation" (*ikhtilāf-i taʿdīl*) for the superior planets on ff. 120v-121v. The latter corrective equation tables confirm Kamālī's statement that Ibn al-Aʿlam's values for the equations of centres of Jupiter and Saturn were used in both the *Muntakhab zīj* and *Razāʾī zīj*, and also indicate the new value that Abu 'l-Ḥasan used for the eccentricity of Mars: In the *Sulṭānī zīj*, all of the principal tables for the equation of centre of the superior planets are displaced, but based on Ptolemy's eccentricity values, and in steps of 0;5°, as follows: Saturn: Min = 0;28°, Max = 13;32° (ff. 16v-22r); Jupiter: Min = 0;45°, Max = 11;15° (ff. 30v-36r); and Mars: Min = 0;35°, Max = 23;25° (ff. 44v-50r). The equation tables from other *zījes* appear in the form of the auxiliary corrective tables, of the "difference in equation", in each of which the differences in entries between a principal equation table of this *zīj* and the corresponding table from another are tabulated. Consequently, the three corrective tables in the *Sulṭānī zīj* for the equation of centre of the superior planets referring to the *Razāʾī zīj* actually present the differences between Ptolemy's equation values and those originally tabulated in the *Razāʾī zīj*. The corrective table for Saturn is subtractive and displaced with Max = 1;44° and Min = 0;16° (f. 120v); thus, the maximum difference in Saturn's equation of centre between the *Razāʾī zīj* and *Almagest* is $\Delta c_{1\max}$ = -0;44°; therefore, according to the *Razāʾī zīj*, the maximum value of the equation of centre of Saturn is $c_{1\max}$ = 6;31 - 0;44 = 5;47°. The corrective table for Jupiter is symmetrical with $\Delta c_{1\max}$ = ±0;17° (f. 121r); therefore, $c_{1\max}$ = 5;15 + 0;17 = 5;32°. For Mars, the corrective table is additive and displaced with Min = 1;38° and Max = 2;28°; thus, $\Delta c_{1\max}$ = +0;25° (f. 121v); thus, $c_{1\max}$ = 11;25° + 0;25° = 11;50°. Therefore, the maximum values for the equation of centre of Jupiter and Saturn are equal to Ibn al-Aʿlam's (see above, note 106), in agreement with Kamālī's statement. However, the longitudes of the apogees show no obvious relation to Ibn al-Aʿlam's values. The source of the value 11;50° for the maximum equation of centre of Mars (corresponding to $e \approx 6;13$) is unknown. However, the table of the equation of centre of Mars from the *Razāʾī zīj* as preserved in the *Ashrafī zīj* is based on Ptolemy's eccentricity

and explained in it; e.g., the tables of the equation of time on ff. 7v and 15r, and Ḥusām al-Dīn al-Sālār's method for the construction of the planetary equation tables on f. 77r.

value (although displaced, with Min = 2;35° and Max = 25;25°).[112] A close value e = 6;15 is mentioned in *Ashrafī zīj* III.9.2,[113] where Kamālī lists the planetary eccentricities. Two centuries later, astronomers at the Samarqand observatory measured the other close value $e \approx$ 6;13,30.

Incidentally, the situation clarified above in the case of the eccentricity of Mars teaches us an important lesson. To measure unprecedented values for the various astronomical parameters new observations must often be carried out, but this does not always mean that a higher degree of precision will be attained in comparison with an already existent value; nor does it guarantee that using the new values will necessarily offer an effective advantage.

(B) Radius of the epicycle:

The radius r of the epicycle of a superior planet should be nearly equal to the ratio $Ra_{\circ}/a'$;[114] with $a_0 \approx 1$ and $a' \approx 1.52$ for Mars, the size of the epicycle of the planet should be equal to ~ 39;28, which is very close to Ptolemy's value of 39;30. The measurement of the size of the epicycle of a superior planet is entirely dependent on the orbital configuration of the Earth with respect to that of the planet at the moment of observation, and is also very sensitive to the input data. Three values measured for the radius of the epicycle of Mars in the medieval Islamic period are listed in Table 6(b). Ptolemy's value is more accurate than those obtained by his medieval followers.

(C) Longitude of the apogee:

The best-known values for the longitudes of the apogee of Mars are presented in Table 6(c) and plotted against time along with the graphs of the heliocentric and geocentric longitudes of the apogee of Mars in Figure 16. Except for Ptolemy's and Muḥyī al-Dīn's values, the others are the radixes of the *zījes*. As noted in the apparatus to the table, we see for example that Ibn Yūnus's values for the longi-

112. *Ashrafī zīj*, F: f. 235v, G: f. 250v.
113. *Ashrafī zīj*, F: f. 51r, G: f. 56r.
114. Neugebauer 1975, Vol. 1, p. 146.

tude of apogees of the planets appear to be dependent on those in the *Mumtaḥan zīj* with a difference of 1;3° between them, and that those applied in Kāshī's *Khāqānī zīj* are in reality updated from the *Īlkhānī zīj*. This illustrates the profound dependence of medieval *zījes* on each other.

A look at Figure 16 is sufficient to show that except for Ibn al-Aʿlam, the errors in the longitudes of apogee of Mars made in the classical Islamic period, before 1050 AD, with a *MAE* of 5.0°, are much larger than those made in the late period (*MAE* ≈ 1.5°). This is an example of the comments made above regarding the general situation of planetary astronomy in the early Islamic period.

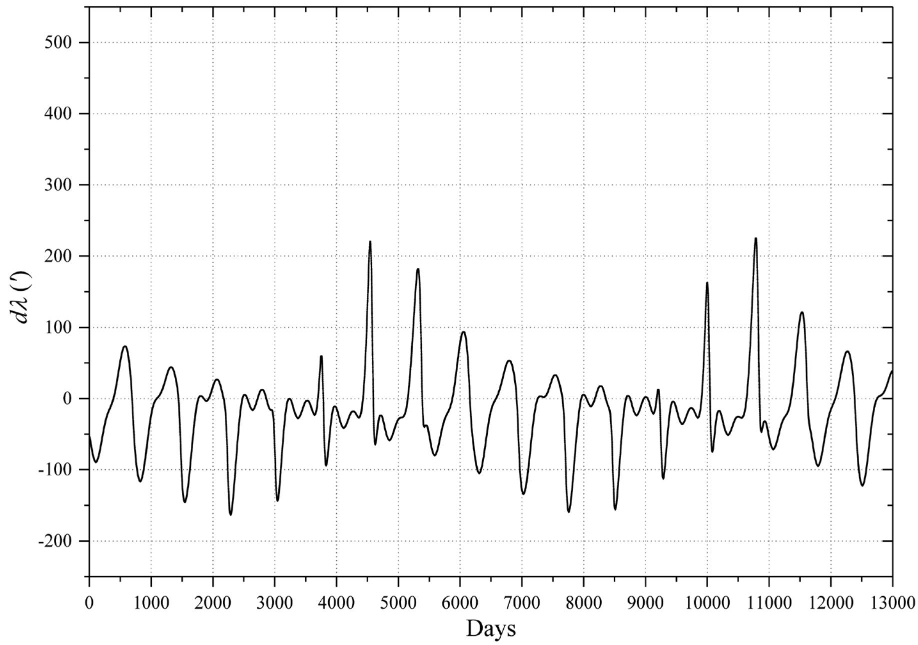
All the errors in the early period are negative. This may be because they are dependent upon the *Almagest*, as is the case of both al-Battānī's and Bīrūnī's values, which (we are certain) are based upon Ptolemy's value. The reason is that, as mentioned at the end of Section 5.2.1, the longitude of the apogee of Mars increases at a rate of 66.5″ per annum, which is substantially greater than the frequently-derived medieval values of 1°/66y or 1°/70y for the rate of precession/apogeal motion. Moreover, Ptolemy's value suffers from a negative error of ~ -1.23°. As a result, over a sufficiently long period, the errors in the values for the longitude of apogee of Mars that are dependent upon Ptolemy show a large negative shift. There is no explicit evidence that the *Mumtaḥan* value stems from Ptolemy, since the increase in the longitude of Regulus (which stands for the precession in the time interval between Ptolemy and the *Mumtaḥan* observations) in the Mumtaḥan star table is 10;30°, which is not equal to either the difference of about 9° in the longitude of the apogee of Mars, or the differences of about 11° and 11.5° in the longitudes of the apogees of the other planets between the *Mumtaḥan zīj* and the *Almagest*.[115] Nevertheless, it seems fair to assume that a large part (~ -3.5°) of the sizeable negative error found in the *Mumtaḥan* value may be due to its connection, in a way or another, to Ptolemy's value. The increase of 1;3° that Ibn Yūnus adopted to convert the *Mumtaḥan* values for the planetary apogees to his epoch is much less than the motion of the apogee of Mars in the time span of 174 years between them, which amounts to about 3.2°; hence, the error increases to about -7°.

Khāzinī has the most accurate value for the longitude of the apogee of Mars for his time; it is interesting that for Saturn, Jupiter, and Mercury he is dependent upon Ptolemy, but proposes independent values for Mars and Venus, both of

115. See Mozaffari 2017, p. 15.

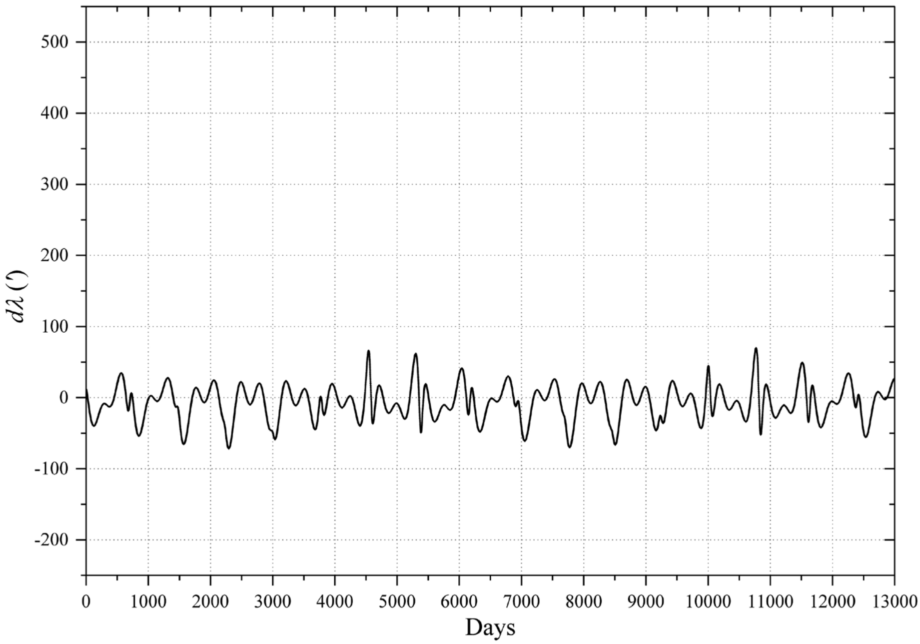
which are very precise (for Venus, his error is ~ -37′), which gives the impression that they might be the result of new observations. Muḥyī al-Dīn's value is second in order of accuracy, and Ibn al-Aʿlam's and Ibn al-Shāṭir are the next. Ibn al-Fahhād's and Ulugh Beg's values are nearly as precise. It is worth noting that, in the prologue of the *ʿAlāʾī zīj*, Ibn al-Fahhād praises Ibn al-Aʿlam's theory of Mars,[116] and so he probably adopted Ibn al-Aʿlam's value: al-Fahhād deploys the precessional rate of 1°/66^y in his work, and hence Ibn al-Aʿlam's value amounts to about 134;24° for 1172 AD, which is not far from al-Fahhād's radix; however, there is no relation between al-Fahhād's and Ibn al-Aʿlam's values for the longitudes of the apogees of the Sun and other planets. Jamāl al-Dīn's and al-Ṭūsī's values are also of the same degree of accuracy. Al-Kāshī's dependence on the latter makes the negative error greater, for the reason mentioned earlier.

5.3. Comparison with the Īlkhānī zīj



(a)

116. Al-Fahhād, *Zīj*, p. 4.



(b)

FIGURE 17. The errors in the longitude of Mars at midnight (0:0 LT) computed from the *Īlkhānī zīj* (a) and Muḥyī al-Dīn's parameter values derived in Maragha (b).

The errors $d\lambda$ in the longitude of Mars computed on the basis of the *Īlkhānī zīj* and Muḥyī al-Dīn's parameter values measured at the Maragha observatory for a long period of 13,000 days beginning on 1-1-1280 AD are plotted against time, respectively, in Figures 17(a) and 17(b), and the relevant statistical results are summarized in Table 7. Notably, Muḥyī al-Dīn's theory of Mars is significantly more precise than the one adopted in the official product of the observatory. The oscillation of the errors is an inevitable consequence of the mismatch between the complicated modern model and the simple Ptolemaic one; the geocentric longitude of the planet in the first is computed from the combination of the Keplerian motions of the planet and Earth through their elliptical orbits, and by taking the gravitational perturbations in the solar system into account; but, in the latter, only an equant motion and the two circular orbits (epicycle and deferent) are used to account for the motion of the planet with respect to the Earth. Accordingly, there is no benefit in tracing the sources of the errors we are confronted with by isolating the theoretical deviation existing between the models. The following considerations are presented just to give a general view of the impact of the errors in the fundamental parameters of Mars on the errors in its longitude, in the light of our lengthy discussion in the previous section.

As regards the Earth's motion, which is reflected in the solar theory, both Ibn Yūnus's modified solar theory adopted in the *Īlkhānī zīj* and Muḥyī al-Dīn's solar theory established at Maragha present almost the same degree of accuracy (with *MAE* ≈ 3′ and the amplitude of the errors within ±8′).[117]

As regards the mean motion in longitude and epoch mean longitude of the planet, the mean longitude of Mars at midnight on 1-1-1280 AD from the *Īlkhānī zīj* is equal to 61;23° and on the basis of Muḥyī al-Dīn's measurements 61;32. Their errors are, respectively, -11′ and -20′ (modern: **61;43**°), which is in principle responsible for the negative values of the mean errors μ. The mean daily motion in longitude of Mars in the *Īlkhānī zīj* is 0;31,26,39,35,29,27°; Muḥyī al-Dīn's value is 0;31,26,39,44,40,48° (see above, 4.2), and the modern value at the time: **0;31,26,39,19,40,34**°.[118] Consequently, the accuracy of the two values is acceptable, leading to an accumulated error of +1′ in the mean longitude of the planet, respectively, after 37.4 and 23.7 years.

As regards the structural elements, both theories use the same accurate value e = 6;0 for the eccentricity of Mars; Muḥyī al-Dīn's value for the longitude of the apogee ($d\lambda_A$ = -0.47°) is more precise (with an error of ~ 20%) than that adopted in the the *Īlkhānī zīj* ($d\lambda_A$ = -2.36°) (see Table 6(c) and Figure 16); however, this error has a small effect on the calculated ephemerides. In stark contrast to these two orbital elements, the value r = 40;18 for the radius of the epicycle of Mars in the *Īlkhānī zīj* (Table 6(b)) —which can be taken as one of the few remaining traces of the observational activities carried out at Maragha, other than Muḥyī al-Dīn's— is relatively large, and is mainly responsible for the increase in the amplitude of the errors $d\lambda$. For example, if we use Ptolemy's value of 39;30, it is clear (cf. Table 7) that both *MAE* and the amplitude of the errors are appreciably decreased (respectively, ~ 27% and ~ 39%).

The above comparison generally confirms the validity of Wābkanawī's conclusion regarding the comparative accuracy of the *Īlkhānī zīj* and the *Adwār al-anwār* in the particular case of the theories of Mars.

117. Mozaffari 2018, pp. 41, 39, 45.

118. Derived from the formula given in Simon et. al. 1994, p. 678. The mean motion in longitude of Mars changed from **31,26,39,16,56,19,58**° to **31,26,39,21,16,1,50**° in the past two millennia.

The errors in Muḥyī al-Dīn's early values for the orbital elements of Mars in his *Tāj al-azyāj* (see Tables 6(a)-6(c); Figures 15 and 16) were notably reduced in the new values he derived at Maragha. His value for the mean daily motion in longitude of Mars in his Damascus *zīj* is equal to 0;31,26,38,16, 2,26°; as a result, an accumulated error of -1′ in the mean longitude of the planet emerges after 9.3 years. A comparison of the ephemerides of Mars computed from his Damascus and Maragha *zījes* (Tables 3 and 7; Figures 12(c) and 17(b)) demonstrates that Muḥyī al-Dīn significantly improved his theory of Mars at the Maragha observatory.

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I	II	III	IV	V	VI	VII	VIII	IX
1	Night of Wednesday 1-11-633 Y 5 November 1264 JDN 2183043	Aldebaran, meridian transit	**0:28**	72; 2° −5′ **72; 7**	**23:59** (−1^{d})	0;33^{h} +4^{m}	+0;45° +2′ **+0;43**	53;19° −32′ **53;51**
2	Sunday 15-12-635 Y 19 December 1266 JDN 2183817	Sun, 3; 0° W	**16:29**	79;42 −4 **79;46**	**0: 0** (+1^{d})	7;30 −1	+3;37 −6 **+3;43**	94;41 −5 **94;46**
3	Night of Thursday 20-2-640 Y 26 February 1271 JDN 2185347	Regulus, meridian transit	**22:39** (−1^{d})	62;33 −2 **62;35**	**0:18**	1;34 −5	+3;34 −27 **+4; 1**	163;15 −51 **164; 6**
4	14;30 hours after true noon of Tuesday 10-10-639 Y 15 October 1270 JDN 2185213	Appulse to Regulus (see Section 3)						

TABLE 1: Muḥyī al-Dīn al-Maghribī's observations of Mars at the Maragha observatory.

	I	II	III	IV	V	VI	VII
Mars-1	4/5 November 1264 JDN 2183042/3	53;19° **53;51**	0;22°	53; 8° (+1^{d})	232; 3,44°	19;0$^{h\,(1)}$ after noon	6 Nov., 7: 0 **6 Nov., 15:52**
Mars-2	19/20 December 1266 JDN 2183817/8	94;41 **94;46**	0;22	94;52 (−1^{d})	274;57,15	1;33^{h} before noon	19 Dec., 10:27 **19 Dec., 11:58**
Mars-3	25/26 February 1271 JDN 2185346/7	163;15 **164; 6**	0;22	163; 4 (+1^{d})	342;59,48	1;15^{h} after noon	26 Feb., 13:15 **27 Feb., 3:52**

Table 2: Al-Maghribī's derivation of the oppositions of Mars to the mean Sun.

Notes:

1. Recomputed values: (1) 19;1.

2. The detailed data of the date and time of each observation are presented in Table 1. Since all of observations were made around midnight and the longitudes in Col. **IV** are given for noon either on the day preceding the observation or on that of observation, we indicate the dates in a double-day where the second day is the day at which midnight the observation was made: e.g., in Mars-1, the observation was made around midnight of 5 November 1264. The notions (−1^{d}) and (+1^{d}) in Col. **IV** are to display the day for which the noon longitudes are given; e.g., in Mars-3, the noon longitude is for 26 February 1271.

3. At the mean oppositions, the superior planets are at the perigee of their epicycles, namely, about the middle of their retrograde motion; thus, their longitudes decrease; e.g., as shown above, the longitude of Mars at noon on 5 November 1264 is 53;8°, but around midnight of the same day is equal to 53;19°, and so on.

	Mean (′)	σ (′)	*MAE* (′)	Min (′)	Max (′)
Ibn Yūnus	+29.5	105.0	77.3	−153.0	+489.5
Ibn al-Fahhād	+ 4.8	35.0	29.1	−100.9	+ 69.0
Tāj al-azyāj	−56.1	52.4	59.0	−228.0	+ 37.8

TABLE 3: The statistical results of the errors in the longitude of Mars in a time interval of 5,500 days since 1260.0.

	Date	Time (*LT*)	JDN	Days(1)	Ibn Yūnus	Ibn al-Fahhād	*Tāj al-azyāj*
1	1260–06–20	16:00	2181444	172	+488.9′	+ 15.9′*	−171.5′
2	1262–09–09	11:00	2182255	983	+371.4	− 61.9*	−187.9
3	1264–11–07	12:00	2183045	1773	+ 67.8	− 53.7*	−124.0
4	1266–12–19	09:00	2183817	2545	− 77.0	− 48.5*	−114.5
5	1269–01–22	13:00	2184582	3310	−152.8	− 76.3*	−157.0
6	1271–02–25	20:00	2185346	4074	−148.6	−100.2*	−205.8
7	1273–04–04	09:00	2186115	4843	− 7.2*	− 77.5	−218.7

TABLE 4: The date and time of the true oppositions of Mars occurred during the period of Muḥyī al-Dīn's observations in Maragha and the errors in the longitudes of Mars at midnight on the same days computed on the basis of the three *zījes*.

Notes:

– Asterisks (*) indicate the least errors.

1. The number of days counted from 1-1-1260 AD.

I	II	III	IV	V	VI	VII	VIII	IX	X
	Theoretical Longitudes			*Muḥyī al-Dīn's*		*Errors*			
Date	Ibn Yūnus	Ibn al-Fahhād	*Tāj al-azyāj*	*Observation*	**Modern**	*M*	IY	IF	*Tāj*
5–11–1264	54;56°	52;54°	51;46°	*53;19°*	**53;51**	***–32′***	*+ 97′* **+ 65**	*– 25′* **– 57**	*– 93′* **–125**
20–12–1266	93;29	93;58	92;51	*94;41*	**94;46**	***– 5***	*– 72* **– 77**	*– 43* **– 48**	*–110* **–115**
26– 2–1271	161;37	162;26	160;39	*163;15*	**164; 6**	***–51***	*– 98* **–149**	*– 49* **–100**	*–156* **–207**

Table 5: The errors in the longitude of Mars at the times of Muḥyī al-Dīn's trio of the observations of Mars (midnight on the three days given in Col. **I**) computed on the basis of the three *zījes* and measured by Muḥyī al-Dīn.

TABLE 6: The values for the orbital elements of Mars from the medieval Islamic period.

1	Ptolemy	6.0	**5.96**
2	Ibn al-A'lam	[6.0] (?)	
3	Kūshyār b. Labbān of Guilan	6.04	**5.99**
4a	Muḥyī al-Dīn	6.13	**6.0**
4b		6.0	**6.0**
5	Jamāl al-Dīn	[6.0] (?)	
6	Abu 'l-Ḥasan of Yazd	6.22	**5.99**
7	mentioned in the *Ashrafī Zīj*	6.25	
8	Ulugh Beg	6.23	**6.0**

(a) Eccentricity

Notes:

1. *Almagest*, X.7: Toomer [1984] 1998, p. 498.

2, 5, 6, 7. See Section 5.2.2.

3. The table of the equation of centre of Mars in Kūshyār b. Labbān's *Jāmi' Zīj* (cf. Brummelen 1998, p. 269) is displaced with Min = 0;30° for arguments 33°-38° and Max = 23;30° for arguments 204°-207°, from which $c_{1\max}$ = 11;30°.

4a. From Muḥyī al-Dīn's *Tāj al-azyāj*. The equation tables are represented in Dorce 2003, pp. 194, 199, 200, 209; see, also, Dorce 2002-2003, p. 206: for Jupiter $c_{2M\max} = c_{2M}(99°\text{-}102°) = 11;9°$; for Mars: $c_{1\max} = c_1(93°\text{-}96°) = 11;40°$, thus, $e = 6;7,47 \approx 6;8$, $c_{2M\max} = c_{2M}(130°\text{-}131°) = 41;40°$; and for Mercury: $c_{1\max} = c_1(94°\text{-}96°) = 3;24°$, $c_{2M\max} = c_{2M}(112°\text{-}113°) = 22;32°$.

8. Ulugh Beg's *Sulṭānī zīj* has $c_{1\max}$ = 11;50,48° for Mars (the table, P1: f. 141r, P2: f. 158r, is displaced with Min = 0;9,12° and Max = 23;50,48°). In his *Commentary on Zīj of Ulugh Beg* (N: pp. 320-324, P: pp. 187-189, PN: ff. 282v-286r), 'Alī b. Muḥamamd Qūshčī (*ca.* 1402-1474) explains the layout of the equation and mean motion tables in this *zīj*. The values he mentions for the parameter values agree with what can be extracted from the tables, with the exception of very slight differences in $c_{1\max}$ of Saturn and Jupiter (N: pp. 273-4, 371, P: pp. 158, 217, PN: ff. 241v, 329v). See, also, Mozaffari 2016c, p. 535.

1	Ptolemy	39;30
2	*Ilkhani zij*	40;18
3a	Muḥyī al-Dīn	39;53,16
3b		39;37,30
4	Ulugh Beg	39;43

(b) Radius of the epicycle

Notes:

1. *Almagest*, x.8: Toomer [1984] 1998, p. 501.

2. *Īlkhānī zīj* (C: p. 116, P: ff. 38v-39r, M1: ff. 70v-71v). In the first half of the table of the epicyclic equation of Mars for the adjusted anomaly at the mean distance, the region of arguments 0°-180°, the maximum value is $c_{2\mathrm{Mmax}}$ = 42;12°. The entries in the latter part of the table, arguments 180-360°, are increased by 360°. Al-Kāshī (IO: ff. 99r, 112r) notices that this value is different from Ptolemy, and that the entries of the table are produced by multiplying the entries of the *Almagest*'s table (with $c_{2\mathrm{Mmax}}$ = 41;9°; Toomer [1984] 1998, p. 551) by 42;12/41;9, which the spot checks show to be correct.

3a. From Muḥyī al-Dīn's *Tāj al-azyāj*. The equation tables are represented in Dorce 2003, pp. 194, 199, 200, 209; see, also, Dorce 2002-2003, p. 206: for Jupiter $c_{2\mathrm{Mmax}}$ = $c_{2\mathrm{M}}$(99°-102°) = 11;9°; for Mars: $c_{1\max}$ = c_1(93°-96°) = 11;40°, thus, e = 6;7,47 ≈ 6;8, $c_{2\mathrm{Mmax}}$ = $c_{2\mathrm{M}}$(130°-131°) = 41;40°; and for Mercury: $c_{1\max}$ = c_1(94°-96°) = 3;24°, $c_{2\mathrm{Mmax}}$ = $c_{2\mathrm{M}}$(112°-113°) = 22;32°.

4. Ulugh Beg (P1: f. 141v, P2: f. 158v): The tables of the equation of the epicyclic anomaly are displaced and use a *mixed* type of the displacements explained as follows. For all the planets, the first half of the table, in the region of the arguments 0°-180°, corresponds to the equation $c_{2\mathrm{A}}$ of the epicyclic anomaly at the greatest distance ($R + e$; for Mercury: $R + 3e$), when the centre of the epicycle is located at the apogee, but the latter half of the table, in the region of the arguments 180°-360°, to the equation $c_{2\Pi}$ of the epicyclic anomaly at the least distance ($R - e$, excepting Mercury), when the centre of the epicycle is located at the perigee. For Mars and the two inferior planets, the first half of the table gives $c_{2\mathrm{A}}$ while the entries in the latter part of the table are increased by k_2 = 360°, i.e., it gives 360° - $c_{2\Pi}$. For Mars, the first half of the table gives Max = 36;50,57° = $c_{2\mathrm{Amax}}$ (for argument 127°) and its latter part Min = 312;22,51°; hence, $c_{2\Pi\max}$ = 47;37,9° (for argument 101°). See, also, Mozaffari 2016c, p. 535-536.

No	Name	Original Source	Date	Value	Modern	Error	Origin
1	Ptolemy [1] (Pt) *ca*. 140 AD	*Almaegst*	12/13 Epiphi 2 Antoninus 27/28 May 139	115;30°	**116;44°**	−1.23°	
2	The Mumtaḥan team [2] (Mt) 829–833 AD	*Mumtaḥan zīj*	Murdād 198 Y August–September 829	124;33	**129;27**	−4.90	
3	al-Battānī [3] (Bt) d. 929 AD	*Ṣābi' zīj*	Beginning of 1191 Alexander 1 September 879	126;58	**130;22**	−3.40	Pt
4	Ibn al-A'lam [4] (IA) d. 985 AD	*'Aḍudī zīj*	365 H/345 Y 975–976 AD	131;25	**132; 8**	−0.72	
5	Ibn Yūnus [5] (IY) d. 1007 AD	*Ḥākimī zīj*	Beginning of 372 Y 16 March 1003	125;36	**132;38**	−7.03	Mt
6	Bīrūnī [6] (Br) d. 1048 AD	*al-Qānūn al-mas'ūdī*	Beginning of 400 Y 9 March 1031	128;29,29	**133;** 9,27	−4.67	Pt
7	Kūshyār [7] (Ku) d. 1029 AD	*Jāmi' zīj*	Beginning of 331 Y 26 March 962 JDN 2072513	**1:** 128;12 **2:** 135;33	**131;53**	−3.69 +3.66	Pt/Bt
8	al-Khāzinī [8] (Kh) *fl. ca*. 1115 AD	*Sanjarī zīj*	1 January 1120	135; 3	**134;48**	+0.25	
9	al-Fahhād [9] (F) *fl. ca*. 1172 AD	*'Alā'ī zīj*	Beginning of 541 Y 2 February 1172	134;44,43	**135;45**,23	−1.01	IA (?)

(Continued)

10	al-Ṭūsī [10] (T) *ca*. 1270 AD	*Īlkhānī zīj*	Beginning of 601 Y 18 January 1232	134;30, 8	**136;51,46**	−2.36	
11a	Muḥyī al-Dīn al-Maghribī (M) d. 1283 AD	*Tāj al-azyāj*	End of 630 H 5 October 1233 JDN 2171698	135;20	**136;53,41**	−1.56	
11b		*Adwār al-anwār*	20–2–640 Y 26 February 1271	137; 7, 6	**137;35, 4**	−0.47	
12	Jamāl al-Dīn al-Zaydī [11] (J) *fl. ca*. 1260 AD	——	660 H 1261–1262 AD	135; 4	**137;26**	−2.37	
13	Ibn al-Shāṭir [12] (ISh) d. *ca*. 1375 AD	*Jadīd zīj*	Beginning of 750 H 21 March 1349	138;10,10	**139; 1,30**	−0.86	
14	al-Kāshī [13] (K) d. 1429 AD	*Khāqānī zīj*	Beginning of 781 Y 4 December 1411	137; 4,26	**140;10,56**	−3.11	IK
15	Ulugh Beg [14] (UB) d. 1449 AD	*Sulṭānī zīj*	Beginning of 841 H 4 July 1437	141;56,48	**140;39,16**	+1.29	

(c) Longitude of the apogee

Notes:

1. *Almagest*, x.7: Toomer [1984] 1998, p. 498.

2. Yaḥyā, E: f. 86v on the date of the *Mumtaḥan* observations, which we have adopted as the epoch of the longitudes of the solar and planetary apogees, is given as *Murdād* 198 Y together with a summary of the apogee longitudes; for Mars: 128;50°, which is Ibn al-A'lam's value, as have been explicitly mentioned in the other surviving copy of the *Mumtaḥan zīj*, L: 88r. The reason is that the extant manuscripts of the *Mumtaḥan zīj* go back to a recension made in the 10th century, in which the original mean motion and equations tables of Mars have been replaced by Ibn al-A'lam's (See van Dalen 2004b). The original Mumtaḥan value can be known from Ḥabash's *zīj* (I: 111r; also, see Debarnot 1987, pp. 44–45), Ibn Yūnus's *zīj* (L: pp. 120–121; Caussin 1804, pp. 214–221; with a scribal error the longitude of the apogee of Mars, given as "***93***;33), as well as the comparative tables in the *Ashrafī zīj*. It should be in the Berlin copy of this work, there are also the two tables for the longitudes of the apogees of the Sun and five planets. The one gives the values up to the five sexagesimal fractional places, in which the fractions from the seconds to the fifths place are equal (…;…,24,2,43,53°) (f. 28r); the tabular values are by about 3;9° less than the apogee longitudes in the *Mumtaḥan*/Ḥabash's *zīj*, and thus they should be for the beginning of the Hijra era. Above it, Ibn al-A'lam's value is given in a different hand, presumably for the same date, as 125;36,24,29°, i.e., about 3;14° less than his value for the epoch of the apogee longitudes in the *Mumtaḥan zīj*. The other gives the apogees longitudes up to the ninth sexagesimal fractional place for the year 872 H (f. 17v), whose beginning was 1 August 1467, according to the astronomical Hijra calendar. In its beginning, we are explicitly told that the tabular values were updated from the *Mumtaḥan zīj*. The values are by about 9;44° more than those in Ḥabash's *zīj*. The apogee longitudes associated with the mean motion tables were updated for 878 H (B: 29r (Sun), 41r (Saturn), 45r, 46r (Jupiter), 49r (Mars), 53r (Venus), 57r (Mercury)), and the increment due to the apogeal motion with respect to the epoch of Ḥabash's *zīj* is 9;44° (there are some minor disagreements because of the confusing scribal errors in the alphanumerics with similar forms), which is in accordance with the second table for the apogee longitudes described above.

3. Al-Battānī, *Ṣābi' zīj*, chapter 45: E: f. 117v; Nallino [1899–1907] 1969, Vol. 3, p. 173; also, in the tables of planetary equations: E: ff. 208v, 211v, 214v, 217v, 220v; Nallino [1899–1907] 1969, Vol. 2, pp. 108, 114, 120, 126, 132. MS. E was written in the Western Arabic script from the late 11th or early 12th century, and the alphanumerics in it are in the Maghribī *abjad* sequence. The scribal errors in the *abjad* numerals can be found both in the end of chapter 45 and in the planetary equation tables in it; e.g., for Mars: **فكو يح** for قِكو نِح. About the

surviving materials of al-Battānī's *zīj* in Arabic, Castilian, and Latin as well as an assessment of the tables of this work, see van Dalen and Pedersen 2008 and the references mentioned therein. In the case of Saturn, Mars, and Mercury, the difference between al-Battānī's and Ptolemy's values amounts to 11;28°, which is very close to the apogeal/precessional motion of 11;25° which al-Battānī measured between Ptolemy's and his own times on the basis of a comparison between Ptolemy's and his own values for the longitudes of the three stars (β Sco, α Leo, and α CMa); see Mozaffari 2016a, pp. 304–305. Al-Battānī mentions, in chapter 51 of his *Ṣābi' zīj* (E: f. 127r; Nallino [1899–1907] 1969, Vol. 3, pp. 187–188), that he carried out the stellar observations in 1627 Nabonassar/1191 Alexander (879–880 AD), and in chapter 45 (E: f. 117v; Nallino [1899–1907] 1969, Vol. 3, p. 172), the longitudes of the apogees are given for 1191 Alexander, too. An explanation for the difference 0;3° between al-Battānī's values for the increments of the apogeal and precessional motions in the period from Ptolemy to him may be that he took a time about 3 years before the epoch of Ptolemy's star catalogue (the beginning of 885 Nabonassar/20 July 137 AD) as that of Ptolemy's values for the longitudes of the planetary apogees, i.e., about the mid-135s AD, which does not seem to be an unreasonable assumption with regard to the times of Ptolemy's derivations, as given earlier.

4. In his *Ashrafī zīj* (F: f. 232v, G: f. 249r), Kamālī gives Ibn al-A'lam's values for the longitudes of the solar and planetary apogees as updated for 13 Adhar 1614 Alexander/23 Rajab 702/13 Khurdād 672 (13 March 1303) (in MS. G, the Alexandrian date is wrongly given as 1***4*** Adhar 161***2***). They are ended with 19″, except for Mars, giving the impression that Ibn al-A'lam's original values were up to the minutes of arc, and this 19″ is due to Kamālī's precessional increment. The longitude of the apogee of Mars is 136;5,18°. The epoch of Ibn al-A'lam's *zīj* is unknown. As settled forth elsewhere (Mozaffari 2016–2017), it seems quite probable that (1) the second star table found in the preserved manuscripts of the *Mumtaḥan zīj* is a work by Ibn al-A'lam himself, in the sense that he updated the longitudes in the first, and in all likelihood original, star table in this *zīj* (for the year 198 Y/829–830 AD) for the year 380 Y (1011–1012 AD) by adding an increment of 2;36°, which is in accordance with his rate of precession of 1°/70^{y} and the interval of time of 182 between them. And (2) he attained this annual processional motion by a comparison between the value 135;6° he measured for the longitude of Regulus (α Leo) from his observation(s) carried out in 365 H (344–345 Y/975–976 AD) and the value 133;0° registered in the first *Mumtaḥan* star table. We convert the values for the longitudes of the apogees of Mars to the latter date, which is about 10 years before Ibn al-A'lam passed away, by subtracting the value 4;40,19° (≈ (672–345)/70) from them.

5. Ibn Yūnus, pp. 120–121; Caussin 1804, pp. 217–220. As established elsewhere (see Mozaffari 2017, pp. 14–16), Ibn Yūnus's values for the longitudes of the apogees of the five planets, except for Venus, appear to be more than Yaḥyā's values by 1;3°. Of course, this increment is less than the half of the expected values produced by either of the precessional rates $1°/66^{y}$ and $1°/70^{y}$, and so it is not known how Ibn Yūnus could have arrived at it, esepcially, considering that the difference in the longitude of the solar/Venus's apogee between Ibn Yūnus's and Yaḥyā's values (respectively, 86;10° and 82;39°; see Mozaffari 2013b, Part 1, p. 326) amounts to 3;31°.

6. Bīrūnī 1954–1956, Vol. 3, pp. 1193–1198. He simply converts Ptolemy's values for the longitudes of the planetary apogees to his epoch by an increment of about 13° calculated from his rate of precession of $1°/69^{y}$ (see Mozaffari 2017, pp. 13–14).

7. As he himself clarifies in the *Jāmi' zīj*, Kūshyār is entirely dependent upon al-Battānī's *Ṣābi' zīj* in the solar and planetary apogees, and thus in turn dependent upon the *Almagest* in the case of Saturn, Mars, and Mercury (see above, note 3). The apogee longitudes are given in the canons of his *zīj*, in I.4.4 (S1: f. 6v, S2: f. 238v) = I.19 (C1: f. 12r, B: —dropped) in a family of MSS, for the beginning of the Yazdigird era (= 16 June 632, JDN 1952063) (NB. In MSS. S1 and S2, each book is divided into *faṣl*s, "chapters", and each chapter is subdivided into *bāb*s, "sections", but in MSS. C and B, every book is continuously divided into sections) or, in I.25 in another category of MSS (L: f. 8r), for the beginning of 331 Y (= 26 March 962, JDN 2072513). Also, in II, the tables: S1: ff. 44r, 45v (for the years 1 Y, 249 Y = al-Battānī's epoch, and 331 Y), S2: f. 263r, L: f. 35v, C: f. 48v, B: p. 47 as well as in the headings of the mean motion tables. His values for 1 Y are less than al-Battānī's by 3;43° and those for 331 Y are more than al-Battānī's by 1;14°; both values are fairly consistent with his rate of precession of $\sim 1°/66^{y}$ (360° in 24000 years, as given in the canons) and the time intervals between either of these two dates and al-Battānī's epoch, 249 Y. Of course, especially in the case of the longitude of apogee of Mars, we are confronted with a very curious note Kūshyār appends to the list of the longitudes of apogees for 1 Y in the tables, as found in the three copies —B, S2, and C— of the five manuscripts consulted for the present study:

> According to what Theon mentions in *al-Qānūn*, the longitude of apogee of Mars is equal to 130;36°, which is equal to the longitude of *Qalb al-asad* [= Regulus], and this is closer to the truth.

This passage deserves some comments:

(1) In the *Handy Tables*, Ptolemy gives the longitudes of the solar and planetary apogees with respect to the longitude of Regulus; the apogees are sidereally fixed, and thus it would be practically simple to list the longitude of Regulus along with the constant distances c in longitude between it and the apogees. All values are dependent upon the *Almagest*, and hence for Mars $c = +353°$ (or $-7°$) which is the difference in longitude between Ptolemy's derivation of the longitude of apogee of the planet, 115;30° (no. **1**), and that of Regulus in the *Almagest* star catalogue, 122;30°. Nevertheless, nowhere in the edited texts of the *Handy Tables* and Theon's commentaries upon Ptolemy's writings, we can find any deviation from Ptolemy's value for c of Mars.

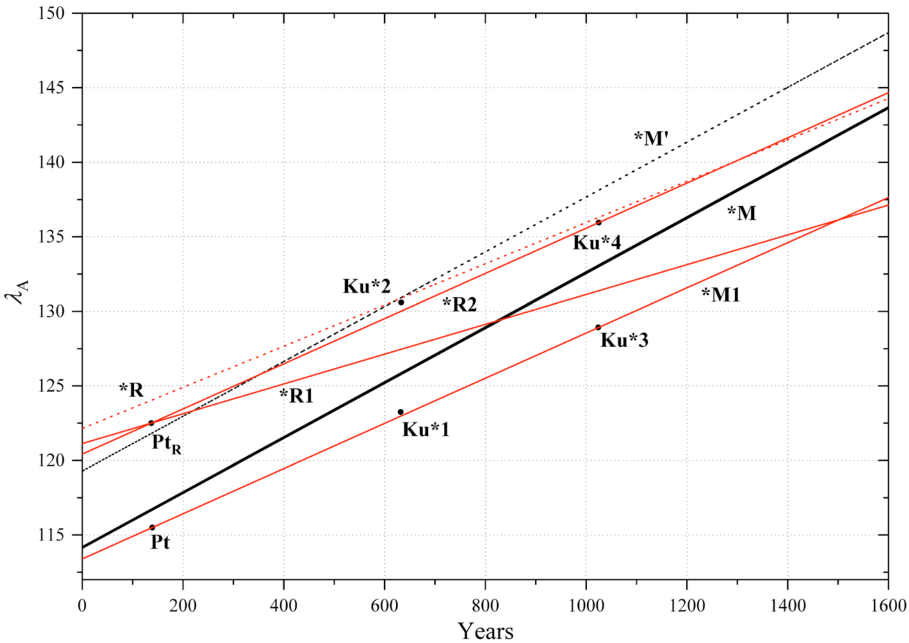

Figure 18. The graphs of the longitude of the geocentric apogee (***M**) and the heliocentric aphelion (***M′**) of Mars in comparison with the true longitude of Regulus (***R**), the longitude of Regulus as updated from the *Almagest* through the medieval period by the Ptolemaic precessional rate of 1°/100^y (***R1**) and the medieval rate of precession of 1°/66^y (***R2**), the longitude of apogee of Mars as updated from the *Almagest* by the apogeal motion of 1°/100^y (***M1**). $\mathbf{Pt_R}$ and **Pt** stand, respectively, for Ptolemy's values for the longitude of Regulus and of the apogee of Mars. The four values shown by **Ku*** are related to Kūshyār.

(2) The longitude of Regulus changed from 122.2° to 149.8° in the past two millennia; see Figure 18, where the dotted line, ***R**, displays the true longitude of Regulus. The

longitude of Regulus on the basis of Ptolemy's value 122;30° for 137 AD (indicated as $\mathbf{Pt_R}$ in Figure 18) and his value of 1°/100^y and the value 1°/66^y prevalently used in the medieval period for the rate of precession are depicted, respectively, as the continuous lines ***R1** and ***R2** in Figure 18. The first clearly becomes progressively less and less than the true values (***R**), obviously because the value 1°/100^y is less than the true rate of precession ~ 1°/71.2^y; but, in stark contrast to it, the second becomes progressively closer to the true values, because adopting the value 1°/66^y, which is greater than the true rate of precession, as the time passed, steadily compensated the striking error of ~ –1.5° in Ptolemy's value 122.5° for the longitude of Regulus; as discussed elsewhere, this situation explains why observing better values for the longitude of Regulus caused the precessional rate of 1°/66^y to be prevalently maintained in the medieval period.

(3) Among the planets, the apogee/aphelion of Mars (shown, respectively, as the thick continuous graph ***M** and the dotted graph ***M′** in Figure 18) is in reality located in the closest distance to Regulus. If the longitude of apogee of Mars was updated from Ptolemy's value (**Pt**) for 139 AD by the use of the rate of precession/apogeal motion of 1°/66^y (the graph ***M1** in Figure 18), as the early Islamic astronomers did, the negative error in it would become *progressively greater* in absolute value, due to the fact that this rate is in reality smaller than the motion of the apogee of Mars, ~ 66.5″/^{y} ≈ 1°/54.1^y (see above, 5.2.1). But, on the contrary, if the longitude of apogee of Mars was taken as equal to the longitude of Regulus and updated from the *Almagest* with the rate of precession of 1°/66^y (***R2**), the errors would be positive and *progressively decrease*.

(4) The longitude of Regulus in the above passage should have stemmed from Ptolemy's value, considering the prevalent, but wrong, idea among the Islamic astronomers that Ptolemy borrowed his star catalogue from Menelaus, composed 40 years before his era (137 AD), by adding 0;25° to the longitudes in it; hence, 122;30° – 0;25° + (632 – (137 – 40))/66 ≈ 130;11° which is not far from 130;36° given in the passage (perhaps, it was forgotten to subtract 0;25° from Ptolemy's value).

(5) The alternative value 130;36° given in this passage for the longitude of Mars seems in all probability to be for the epoch, i.e., 1 Y, (shown by **Ku*2** in Figure 18); the value 123;15° taken from al-Battānī's *zīj* is marked as **Ku*1** in Figure 18. As explained above (3), al-Battānī's value at the time suffers from a negative error of –2;33° (modern: **125;48°**), owing to its dependence upon the *Almagest*, but, on the contrary, the alternative value 130;36° shows a positive error of +4;48°. We convert these two values for Kūshyār's time and epoch, and compare the resultants with the modern values: We know that Kūshyār wrote the original of his *zīj* in 393 Y/1025 AD; so, for 393 Y, his formal, i.e., al-Battānī's, value would be about 129;9° (**Ku*3**) and the value he ascribed to Theon: 136;33° (**Ku*4**; NB Increment = 393/66 ≈ 5;57°). These

two values show the errors of approximately the same size, but with the opposite signs (respectively, –3;54° and +3;30°) in comparison with the modern value **133;3°** at the time. As shown in this table, for his epoch, 331 Y: from al-Battānī's value: λ_A = 126;58 + 1;14° = 128;12° and from the longitude of Regulus: λ_A = 130;36° + 3;43° + 1;14° = 135;33°. None of these values have any advantage over the others.

8. al-Khāzinī, *Zīj*, V: ff. 129r, 163v; L: ff. 102v, 125v; S: pp. 53–54. In a table in MS. V, the longitudinal differences between the apogee of the Sun and those of the five planets are given to the arc-minutes for the beginning of the Hijra era, which, added to the longitude of the solar apogee, are generally in agreement with the values given in the main table of the radixes of the Sun, Moon, and planets in this work. Kamālī (F: f. 232v, G: f. 249r) has added 10;18,48° to Khāzinī's values in order to update them for 23 Rajab 702 (13 March 1303); this increment is in accordance with the precessional motion of 1°/66^y and the period of about 681 Persian years elapsed from the beginning of the Hijra era to the date in question. Khāzinī has added 7;35° to the longitudes of the apogee of Saturn, Jupiter, and Mercury in the *Almagest*, which approximately agrees with his rate of precession of 1°/66^y and the interval of time of about 487 years, from the mid-130s AD to 622 AD, but for Mars and Venus, his values are, respectively, by 12° and 12;35° greater than Ptolemy's. We have added an increment of 7;33° to Khāzinī's values in order to convert them to 1 January 1120 AD, a date falling within the period of his fruitful career.

9. Ibn al-Fahhād, *Zīj*, p. 73; see, also, Mozaffari 2017, pp. 16–18.

10. *Īlkhānī zīj*, C: p. 111, P: —dropped, M1: f. 68r, M2: —dropped.

11. We know (Yabuuti 1997, pp. 22, 24) that Jamāl al-Dīn and his teams of Persian astronomers in China measured the longitude of the solar apogee as 89;21° in 660 H (1261/1262 AD). In Sanjufīnī's *zīj*, which is on the basis of their parameter values, the apogeal motion with a rate of 1°/60^y is clearly different from the precessional one with a rate of 1°/73^y, which can be derived from the values tabulated for them in the two separate columns in the table for the solar and planetary mean motions from 764 to 895 H (ff. 44v–46r). Sanjufīnī (f. 44v) gives the values 91;1,20° and 136;44°, respectively, for the longitudes of the apogees of the Sun and Mars for 24 Jumādā I 764 (10 March 1363, according to the astronomical Hijra calendar). Accordingly, it seems that he added an increment of 1;40,20° to Jamāl al-Dīn's value in order to update it for his time, which is compatible with the rate of apogeal motion of 1°/60^y and the period of about one century between them. If it is correct, then Jamāl al-Dīn's value for the longitude of the apogee of Mars for 1261/1262 AD would be equal to 135;4°. It should be noted that Jamāl al-Dīn precedes Ibn al-Shāṭir (see Mozaffari 2017) in putting a

clear distinction between the apogeal and precessional motions by one century.

12. Ibn al-Shāṭir, K: f. 52r, L1: f. 65v, L2: 50v, O: f. 31r, PR: f. 100r. A curious feature of Ibn al-Shāṭir's astronomy is that he correctly believed that the motion of the solar and planetary apogees (which he takes as equal to 1° in 60 Persian/Egyptian years) is not equal, but larger than, to the precession (which he takes as equal to $1°/70^{y}$); see Mozaffari 2017, esp. pp. 21–24.

13. al-Kāshī, *Zīj*, IO: f. 128v, P: p. 109. Note that al-Kāshī's values for the longitudes of the apogees exceed those in the *Īlkhānī zīj* by 2;37,17°, in the sense that they were updated from the *Īlkhānī zīj* with taking the precessional and apogee motion as equal to 1° in 70 Persian/Egyptian years; an interval of time of 180 years separates his epoch from that of the *Īlkhānī zīj*.

14. Ulugh Beg, P1: f. 140r, P2: f. 157r.

	Mean (′)	σ (′)	*MAE* (′)	Min (′)	Max (′)
Īlkhānī zīj (1)	−17.0	59.0	44.7	−163.3	+225.0
Īlkhānī zīj (2)	−17.2	37.3	32.5	− 94.9	+141.2
Muḥyī al-Dīn	− 7.1	24.4	19.7	− 71.7	+ 69.7

Table 7: The accuracy of the longitude of Mars as computed from the *Īlkhānī zīj* and Muḥyī al-Dīn's parameter values derived at the Maragha observatory.

Notes:

1. With radius of the epicycle = 40;18.
2. With radius of the epicycle = 39;30 (Ptolemaic).