Information Aggregation in a Financial Market with General Signal Structure[†]

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Abstract. We study a financial market with asymmetric, multidimensional trader signals that have general correlation structure. Each of a continuum of traders belongs to one of finitely many "information groups." There is a multidimensional aggregate signal for each group. Each trader observes an idiosyncratic signal about the fundamental, built from this group signal. Correlations across group signals are arbitrary. Several existing models serve as special cases, and new applications become possible. We establish existence and regularity of linear equilibrium, and demonstrate that the equilibrium price aggregates information perfectly as noise trade vanishes.

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[†]This paper combines (a) the results on existence and equilibrium structure in Lou et al. 2017, and (b) the information aggregation result in Parsa and Ray (2017), extending the analysis to multidimensional signals. Names are in random order, as proposed in Ray ⓡ Robson (2018). Lou and Li acknowledge the support of Hong Kong Research Grants Council under Grant 14204514. Lou also acknowledges the support of Hong Kong Scholars Program under Grant XJ2015049. Ray acknowledges research support under NSF Grant SES-1629370. We thank the editor Xavier Vives and three anonymous referees for helpful comments and suggestions. Lou would especially like to express his gratitude to his wife Shujuan for her continued support.

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1. Introduction

Consider an economy in which a single risky asset is traded, with unknown fundamental of common value to all of a continuum of individuals. Each trader belongs to one of a finite number of "information groups," which could vary in size. There is a multidimensional aggregate signal for every information group. Each individual receives an idiosyncratic signal built from her group signal plus iid noise. The setting is multivariate normal, with arbitrary correlation structure across fundamental and signals. In addition, each trader also observes the asset price and can make inferences about the fundamental using that price. As in the seminal contribution of Hellwig 1980, traders retain the incentive to use their private signals in the presence of noise trade. We will allow noise trade to converge to zero to obtain our information aggregation result in the limit.

We are therefore in a classical rational expectations equilibrium (REE) world, but with significantly added generality in information structure and dimensionality. Moreover, we allow traders to differ not only in their access to information, but also in their attitudes to risk. Indeed, we permit risk heterogeneity both within and across information groups. Of course, we can nest several existing REE models — among them Grossman (1976) and the finite-agent model of Hellwig (1980) — by properly selecting the mass and the risk-aversion coefficient of each trader type. But well beyond that, the multidimensional signal structure we work with can facilitate other investigations. Except for normality, we do not impose restrictions on aggregate signals and allow these to exhibit any degree of asymmetry, or heterogeneity in correlation structure. This generality is important. For instance, given different locations, risk attitudes or informational capacities, traders might have access to diverse sources (newsletters, advisory services etc.) for their private information, which leads to an asymmetric correlation structure not handled by the classical models. And multidimensionality acquires particular salience when traders share their private signals with their neighbors via a social network. Then the effective signals of traders are essentially many-dimensional, because two signals cannot be aggregated ex-ante without knowledge of the full equilibrium structure generated by the price system. Therefore information-sharing over a network cannot be handled by models with one-dimensional signals.

Apart from conceptual generality, new analytical issues arise when these considerations — asymmetry, multidimensionality, as well as arbitrary cross-signal correlation — are studied. Even the seemingly intuitive properties of equilibrium that are immediate in the Grossman-Hellwig setup must now be proved at a non-trivial level when the information structure is general. Among these properties is the *regularity* of any linear equilibrium: an increase in demand implies also an increase in price. More importantly, and owing to the generality of our signal structure, the existence arguments given in Hellwig (1980) cannot be applied to solve our model. We resort to a non-standard argument involving sequences of fixed points to establish the existence of a linear equilibrium price function.

An accompanying complication concerns the impact of signals on the equilibrium price. When those signals are independently generated — and positively related to fundamental value — they exert a positive influence on prices, as in Hellwig (1980). Because our model admits general correlation across signals, no parallel assertion is available here: the corresponding coefficients of signals in the price function are generally ambiguous in sign. They depend on the correlation pattern, the sizes of information groups, as well as the distribution of risk attitudes, though we can pin down the signs of the weights for some special cases. Nevertheless, it is in this general context that we are able to revisit a solution to the Grossman-Stiglitz paradox first investigated by Hellwig (1980).[1] If an imaginary "super-agent" were to observe every one of the signals, she would possess a best prediction of the fundamental, which is a linear function of the signal vector. Any price function which is the same linear function (up to an intercept term) would aggregate information perfectly. But an agent observing such a price function would entirely ignore her own signals. Indeed, she would *strictly* prefer *not* to use any of her information, whether or not it is freely available. Even the redundancy that tolerates some degree of mixing and allows information to seep in via indifference, is not to be had. But then: how can the market imitate the super-agent?

Certainly, with the existence of noisy movements in trades, prices lose their ability to perfectly aggregate information, contaminated as they must be by stochastic demand shocks.[2] Then traders

[1]See Grossman (1976, 1978) and Grossman and Stiglitz (1980) for more discussion on information aggregation.

[2]Variants of such REE models have been developed; see, for example, multi-asset settings (Admati 1985; Pálvölgyi and Venter 2015b; Chabakauri et al. 2017), equilibrium with a continuum of traders (Grossman and Stiglitz 1980; Ganguli and Yang 2009), signals about of the supply of tradable assets (Ganguli and Yang 2009; Manzano and

will use their own information at least to some degree, which therefore enters the price. Specifically, we show that the equilibrium price is positively correlated with the value of the fundamental, assuming, of course, that at least one of the observed signals is correlated with that value. But the more subtle question remains: as noise trade vanishes, must the price function converge to the *perfect information* aggregator? Our answer to this question is in the affirmative: as the variance of noise demand converges to zero, the equilibrium price aggregates information perfectly, fully capturing a linear relationship, including weights and correlation patterns, across the fundamental and aggregate signals. This is consistent with (and substantially generalizes) the observations in Grossman (1976) and Hellwig (1980).

Section 2 introduces the model. Section 3 characterizes linear equilibria. Section 4 states and discusses the information aggregation result. Section 5 discusses the weights of aggregate signals on prices. Section 6 discusses related literature. Section 7 concludes. All proofs are in the Appendix.

2. THE MODEL

There is a single risky asset and a single trading period. The risky asset is in fixed supply $X \in \mathbb{R}$ and has fundamental value θ, common to all agents. The value is not directly observed by market participants, but it generates signals, the structure of which will be described in detail below.

There is a unit measure of market participants or traders. Each trader has a CARA utility function and maximizes her conditional expected utility of her net profit W based on her information set $\mathcal{F}$:

$$\mathbb{E}\left[-\exp\{-\rho W\}|\mathcal{F}\right]. \tag{1}$$

In the setting at hand, $W = x(\theta - p)$, where x is the holdings of the asset and p its price.

The parameter ρ is the coefficient of absolute risk aversion and it will vary across traders, as will the sources of information. Specifically, a trader is described by her type r, which lies in some finite set R. Let $\tau(r)$ be the measure of individuals of type r. An individual type r has two components:

Vives 2011; Diamond and Verrecchia 1981; Verrecchia 1982); signal transmission and sharing in social networks (Han and Yang 2013; Ozsoylev and Walden 2011; Colla and Mele 2010; Walden 2019), etc.

$\{i(r), \rho(r)\}$, where $i(r)$ denotes her information group membership, and $\rho(r)$ denotes her risk-aversion. Each information group i has a positive mass. Members of that group access a distinct collection of newsletters, websites, and advice that effectively permit a multidimensional signal, $\boldsymbol{y}_i = (y_{i1}, ..., y_{im})' \in \mathbb{R}^m$, to be "directed" towards them. There is one such signal for every information group — the multiple dimensionality m of each group signal allows this structure to be quite general.[3] For instance, two information groups might have access to some common subset of signals. We refer to $\boldsymbol{y} = (\boldsymbol{y}_1', \ldots, \boldsymbol{y}_n')'$ as the *aggregate signal structure* of the economy.[4]

If trader j belongs to information group i, she observes a signal which communicates this aggregate signal $\boldsymbol{y}_i$ with idiosyncratic noise $\boldsymbol{\epsilon}_i(j) \in \mathbb{R}^m$:

$$\boldsymbol{z}_i(j) = \boldsymbol{y}_i + \boldsymbol{\epsilon}_i(j).$$[5]

This signal and the price will determine the trader's demand; see (2) below. In addition to such demands, there is noise demand u, to be interpreted as the stochastic demand of "noise traders" left unmodeled in this paper. Assumption 1 below will be maintained throughout.

Assumption 1. *All exogenous random variables are normal, with means normalized to zero. The variance-covariance matrix of $\boldsymbol{y}_i$ is positive definite for every i, and $\mathrm{Var}(\theta|\boldsymbol{y}) > 0$.[6] Idiosyncratic noise $\boldsymbol{\epsilon}_i(j)$ in each information group i could be degenerate, but if not, it is iid across individuals j,[7] independent of other random variables, with positive definite variance-covariance matrix. Noise demand u is independent of all other exogenous random variables, and has positive variance.*

[3] Given the flexible structure of variances, it is essentially without loss of generality to assume that all group signals have the same dimension m. We do so only for ease of exposition; all results hold for the more general case.

[4] The somewhat clumsy use of the double transpose ensures that all vector notation is for column vectors.

[5] This hierarchical structure of signals shares the same spirit as the one in Myatt and Wallace (2012), in which signals received by an individual traders contain both some "sender noise" regarding the fundamental, as well as receiver-specific noise around the sender's signal realization.

[6] The variance-covariance matrix V of a random vector $\boldsymbol{x}$ is always positive semi-definite because for any $\boldsymbol{d}$ with the same dimension as $\boldsymbol{x}$, $\boldsymbol{d}'V\boldsymbol{d} = \mathrm{Var}(\boldsymbol{d}'\boldsymbol{x}) \geq 0$. Therefore V is *not* positive definite if and only if there is nonzero $\boldsymbol{d}$ with $\boldsymbol{d}'\boldsymbol{x}$ degenerate. Thus, the positive definiteness of the variance-covariance matrix of $\boldsymbol{y}_i$ equivalently requires that any component of any group signal cannot be expressed as a linear combination of other components of this group signal.

[7] We adopt the usual convention for "iid" across a continuum of random variables without further comment. We allow for general correlation across the dimensional components of each idiosyncratic noise $\boldsymbol{\epsilon}_i(j)$.

Except for joint normality and the reasonable non-degeneracy requirement that group signals, taken together, cannot fully pin down the fundamental, we impose very little restriction on the aggregate signal structure $\boldsymbol{y}$ or on the precise relationship of its several components with the fundamental θ. Even the assumed positive-definiteness of $\boldsymbol{y}_i$ is essentially without any loss of generality: after all, it is always possible to remove the "redundant components" from the signal $\boldsymbol{y}_i$. We permit arbitrary correlation patterns across the private signals of traders, and in addition we allow each of these signals to be multidimensional. As a consequence, several existing models are nested within our formulation by the suitable choice of signals $\boldsymbol{y}$ and (if needed) by setting the dimension of signals to one and the variance of idiosyncratic signals in each information group to 0. As examples, we have:

(a) *Informed and uninformed.* A fraction of a continuum of traders receives the same signal: $y_i = \theta + \varepsilon$ (the noise ε is independent of θ), while the remainder receives no signal at all (Grossman and Stiglitz 1980).

(b) *Idiosyncratic information from a single source.* All traders have the same information source, but their signals are drawn independently: $y_i = \theta + \varepsilon_i$, where the ε_i's are mutually independent (Grossman 1976; Hellwig 1980). (We provide other specific connections to Hellwig 1980 below.) This case is sometimes referred to as the *common-values model*.

(c) *Multiple information sources with identical covariance.* Traders have different information sources, but these are correlated with a special pattern: for single-dimensional signals $\{y_i\}$ across n traders or trading groups, we have $\mathrm{Var}(y_i) = \mathrm{Var}(\theta)$ for any i and $\mathrm{Cov}(y_i, y_j) = \varsigma\,\mathrm{Var}(\theta)$ for any i, j, with $0 \leq \varsigma \leq 1$, where θ is the fundamental of the risky asset (Vives (2008), p. 381, but with notation changed to fit ours). The actual signal received may be contaminated by additional noise: $z_i = y_i + \varepsilon_i$, also accommodated in our setup. When $0 \leq \varsigma < 1$ and $\mathrm{Var}(\varepsilon_i) = 0$ for every i, this case is sometimes referred to as the *private-values model*, though we must add that the payoff still arises via a common fundamental θ.

(d) *Social connections.* Suppose that information is shared across a social network with node set $\mathcal{V} = \{1, \dots, n\}$ and arc set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Initially, each trader i receives a private signal $s_i = \theta + \varepsilon_i$, where the error terms ε_i are mutually independent with mean zero. These private signals are shared via social connections.

One approach to dealing with this setting is to presume that after neighboring signals are observed, each trader i forms some scalar statistic — say the average $\sum_{j\in\mathcal{N}_i} s_j/|\mathcal{N}_i| = \theta + (\sum_{j\in\mathcal{N}_i}\varepsilon_j)/|\mathcal{N}_i|$, where $\mathcal{N}_i = \{j|(j,i)\in\mathcal{E}\}$ denotes trader i's neighbor set — including herself — in the network (Ozsoylev and Walden 2011). Because the derived error terms $\sum_{j\in\mathcal{N}_i}\varepsilon_j/|\mathcal{N}_i|, i = 1,\ldots,n$ are correlated, this model cannot be nested under Hellwig (1980). But it is a special case of our formulation: set $y_i = \sum_{j\in\mathcal{N}_i} s_j/|\mathcal{N}_i|$.

This is a reasonable approach, but has its limits. Except for special situations, there is no reason why each trader should take an average of signals. Indeed, averaging isn't the problem: *any exogenous aggregation method* is suspect. The equilibrium price will affect the cross-weighting of signals in ways that cannot be pinned down a priori, except in very restrictive settings. A model of signal-sharing in networks must therefore, of necessity, need to handle the case of multidimensional signals at the individual level. How the individual aggregates those signals is part of the equilibrium structure. In the language of our model, we simply set $\boldsymbol{y}_i = \{s_i, j\in\mathcal{N}_i\}$.[8]

(e) *Informational hierarchies*. Another special case that is easily handled by our framework is one of informational hierarchies, generalizing models with the usual binary distinction between "informed" and "uninformed" traders. Think of a setting with "fundamental signals" $(s_1, s_2, \ldots, s_m)$. Traders have access to subsets of these signals. That induces a partial order: trader j is *more informed* than trader j' if trader j sees a larger subset of the signal space than trader j'. Once again, the multidimensional structure can be deployed to easily handle this case.

Notice how in cases (d) and (e) — and especially in (e) — it is imperative to impose the weaker condition that the variance-covariance matrix of $\boldsymbol{y}_i$ is positive definite for each i, rather than the stronger restriction that the variance-covariance matrix of $\boldsymbol{y}$ is positive definite. The latter condition will fail if one information group has access to a superset of signals compared to another.

[8]There are other REE models in which traders' signals are multidimensional. For instance, Goldstein and Yang (2017) study the implications of information disclosure in a multidimensional signal setting, where traders also observe the public signals released by the government, apart from the price and their own signals.

(f) *Two-stage models*. Our model does not fully exploit the special structures in (d) and (e) to provide results that are specific to these settings. At the same time, it does throw light on some basic questions of network formation or information acquisition. Consider a class of *two-stage* models, in which individuals choose to form costly links with other individuals in a network, or invest in acquiring costly information. Subsequently, they interact in the setting described in this paper.[9] One of our main results is Proposition 6, which show that with vanishing noise trade, prices efficiently aggregate all information in the signals. This has the implication that while *new* information not held by *anyone* is always welcome, the sharing or acquisition of *existing* information that is already held by others will have low priority as noise trades vanish. In particular, the better the job that a market does in aggregating information, the less incentive will there be to build social networks. To the extent that the destruction of such social networks could have implications for the decay of cultural connections and friendships, this is a depressing corollary of Proposition 6.

Particularly relevant is the finite-agent model studied in Hellwig (1980) that we extend and generalize. Apart from the dimensional generalization, there are also two additional distinct differences, the second of which is more important than the first: (i) Hellwig (1980) studies a finite-agent setting (along with a large economy which is the limit of a sequence of finite-agent economies), while our model has a continuum of agents, thereby making exact the competitive price-taking assumption; and (ii) Each agent i in Hellwig (1980) receives a private signal equal to the fundamental θ plus independent noise. Both (i) and (ii) can be represented as special cases of our model, in the following way. Think of each Hellwig agent i as a positive measure of atomless agents in our setting — information group i. Assume that the group signal y_i is unidimensional, with $y_i = \theta + \zeta_i$ for independent noises $\{\zeta_i\}$. Finally, set $\epsilon_i(j)$ equal to zero for all atomless individuals j in the group, so that everyone in i sees y_i exactly. This is, in essence, a representation of Hellwig's setting that fits perfectly with the perfect competition assumption.[10] The presence of cross-signal correlation

[9] Considerations of information-sharing are important in a variety of contexts. For instance, one might be interested in the trade-off between possible collusion via information-sharing and direct welfare losses that could arise if information-sharing is entirely prohibited (Boyarchenko, Lucca, and Veldkamp 2017).

[10] We say "in essence," because we still have a continuum of agents when the idiosyncratic variance is zero. But of course, this extreme case can mathematically mimic a finite number of price-taking agents.

and signal multidimensionality does mean that in our more general case, a different approach to information aggregation needs to be taken.

Details on Notation. For expositional ease in a notationally intensive model we use some non-standard notation. The operator Var will stand for variance (or sometimes a variance-covariance matrix, in which case we use boldface **Var**) and Cov will stand for covariance (or sometimes a vector of covariances, in which case we use boldface **Cov**). For instance, $\mathrm{Var}(p)$ will be the scalar variance of the price, whereas $\mathbf{Var}(z_i)$ will stand for the variance-covariance matrix of an individual's signal z_i in group i. For any vector $\boldsymbol{\mu} = (\boldsymbol{\mu}_1', \ldots, \boldsymbol{\mu}_n')'$ (with each component m-dimensional) and random variable x, $\mathrm{Cov}(\boldsymbol{\mu}, x)$ is shorthand for $\mathrm{Cov}(\sum_{k=1}^n \boldsymbol{\mu}_k' \boldsymbol{y}_k, x)$ and $\mathrm{Var}(\boldsymbol{\mu})$ stands for $\mathrm{Var}(\sum_{k=1}^n \boldsymbol{\mu}_k' \boldsymbol{y}_k)$, where $\boldsymbol{y} = (\boldsymbol{y}_1', \ldots, \boldsymbol{y}_n')'$ is the aggregate signal structure. These are *scalars* as $\sum_{k=1}^n \boldsymbol{\mu}_k' \boldsymbol{y}_k$ is a unidimensional random variable. We also use $\mathrm{Var}(\theta|i)$ to describe the variance of the fundamental θ conditional on a signal received by an individual in group i, and the the price. Neither the value of the signal nor the price will appear in the notation because variance updating is independent of the specific realizations; more on this below. For any random variable x and group signal $\boldsymbol{y}_i$, $\mathbf{Cov}(x, \boldsymbol{y}_i)$ is shorthand for the *vector* of covariances $(\mathrm{Cov}(x, y_{i1}), ..., \mathrm{Cov}(x, y_{im}))'$, and in the special case where $x = \sum_{k=1}^n \boldsymbol{\mu}_k' \boldsymbol{y}_k$ for some vector $\boldsymbol{\mu}$, we write $\mathbf{Cov}(\boldsymbol{\mu}, \boldsymbol{y}_i)$ for the vector $(\mathrm{Cov}(\boldsymbol{\mu}, y_{i1}), ..., \mathrm{Cov}(\boldsymbol{\mu}, y_{im}))'$. If $\boldsymbol{x}$ is a vector and γ is a scalar, then $\boldsymbol{x}/\gamma$ simply stands for a vector where all entries are divided by the scalar γ. For two matrices A and B, $A \preceq B$ (or $A \prec B$) means that the matrix $B - A$ is positive semi-definite (or positive definite). Finally, the 2-norm of a vector or matrix is denoted by $|\cdot|$.

3. Equilibrium

3.1. **Definition and Description.** A typical trader j of type r has information $\{\boldsymbol{z}_{i(r)}(j), p\}$.She maximizes her CARA payoff function in (1). When μ and v are respectively set equal to her conditional expectation $\mathbb{E}(\theta|\boldsymbol{z}_{i(r)}(j), p)$ and conditional variance $\mathrm{Var}(\theta|\boldsymbol{z}_{i(r)}(j), p)$ of the fundamental, it is well known from the CARA-Gaussian model that

$$\mathbb{E}\left[-\exp\{-\rho(r)W\}|\, \boldsymbol{z}_{i(r)}(j), p\right] = -\exp\left\{-\rho(r)\left[x(\mu - p) - \rho(r)\frac{x^2 v}{2}\right]\right\}.$$

It follows that maximizing (1) is equivalent to maximizing $x(\mu - p) - \rho\frac{x^2 v}{2}$. Consequently, the optimal demand for the risky asset by trader j of type r is given by

$$x_j^* = \frac{\mathbb{E}(\theta|\boldsymbol{z}_{i(r)}(j), p) - p}{\rho(r)\operatorname{Var}(\theta|\boldsymbol{z}_{i(r)}(j), p)}. \tag{2}$$

By Assumption 1, the variance of θ remains strictly positive after conditioning on any or all of the observables, so this object is well-defined. In this paper, we study linear equilibria, and so focus on the class of affine price functions given by

$$p = \sum_{k=1}^{n} \boldsymbol{\pi}_k' \boldsymbol{y}_k + \gamma u + c, \tag{3}$$

where $\boldsymbol{\pi} = (\boldsymbol{\pi}_1', ..., \boldsymbol{\pi}_n')'$ represents the weights on group-specific signal vectors (each component is m-dimensional so that $\boldsymbol{\pi}_k = (\pi_{k1}, ..., \pi_{km})' \in \mathbb{R}^m$ for each k), $\gamma \neq 0$ is the weight on noise trade, and c is an intercept term. Given this setting, we can describe the conditional expectation $\mathbb{E}(\theta|\boldsymbol{z}_{i(r)}(j), p)$ for a trader j of type r. The informational equivalent of the price p is the variable $p - c$, where the intercept term is netted out. Because idiosyncratic noise $\boldsymbol{\epsilon}_i(j)$ within each information group i is iid, the conditional expectation is therefore described by a system of weights $\{\boldsymbol{\alpha}_{i(r)}, \beta_{i(r)}\}$, with $\boldsymbol{\alpha}_{i(r)} \in \mathbb{R}^m$, $\beta_{i(r)} \in \mathbb{R}$, such that for any trader j of type r and signals $(\boldsymbol{z}_{i(r)}(j), p)$ received by her,

$$\mathbb{E}(\theta|\boldsymbol{z}_{i(r)}(j), p) = \boldsymbol{\alpha}_{i(r)}' \boldsymbol{z}_{i(r)}(j) + \beta_{i(r)}(p - c), \tag{4}$$

where this derivation invokes the projection theorem for normal random variables.

Now, the conditional *variance* $\operatorname{Var}(\theta|\boldsymbol{z}_{i(r)}(j), p)$ depends on trader j only via her information group identity $i(r)$, but not via the particular signal $\boldsymbol{z}_{i(r)}(j)$ she receives, nor her risk type, nor the particular realization of p. So we may write the conditional variance as $\operatorname{Var}(\theta|i)$, but with the understanding that $\operatorname{Var}(\theta|i)$ *does depend on the form of the price function*. We make this dependence explicit when needed. By (2) and (4), we see that aggregate demand for the risky asset by all individuals of type r depends on p and the group signal $\boldsymbol{y}_{i(r)}$, and is given by

$$\tau(r)\frac{\boldsymbol{\alpha}_{i(r)}' \boldsymbol{y}_{i(r)} + \beta_{i(r)}(p - c) - p}{\rho(r)\operatorname{Var}(\theta|i(r))}, \tag{5}$$

where recall that $\tau(r)$ is the population measure of type r.

We may now define an equilibrium price function. To this end, observe that while an agent's demand will depend on her risk-aversion $\rho(r)$, the coefficients $\boldsymbol{\alpha}_{i(r)}$ and $\beta_{i(r)}$ estimated from Bayes' Rule will be independent of her risk type. Writing these as $\boldsymbol{\alpha}_i$ and β_i, using the expression (5), aggregating across types, and adding on noise trade, the market-clearing condition becomes

$$\sum_{i=1}^{n} \frac{\boldsymbol{\alpha}_i' \boldsymbol{y}_i + (\beta_i - 1)p - \beta_i c}{\Delta_i \operatorname{Var}(\theta|i)} + u = X. \tag{6}$$

where we recall that X is the supply of the asset, and where we define, for each $i = 1, \ldots, n$,

$$\Delta_i = \frac{1}{\sum_{\{r|i(r)=i\}} \frac{\tau(r)}{\rho(r)}} > 0.$$

This implicitly defines an equilibrium price function. By matching terms from the price function (3) and the market-clearing condition (6), we see that

$$\gamma = \left[\sum_{i=1}^{n} \frac{1-\beta_i}{\Delta_i \operatorname{Var}(\theta|i)} \right]^{-1}, \tag{7}$$

$$c = - \left[\sum_{i=1}^{n} \frac{1}{\Delta_i \operatorname{Var}(\theta|i)} \right]^{-1} X, \text{ and} \tag{8}$$

$$\boldsymbol{\pi}_i = \frac{\gamma}{\Delta_i \operatorname{Var}(\theta|i)} \boldsymbol{\alpha}_i \text{ for } i = 1, \ldots, n, \tag{9}$$

where $\operatorname{Var}(\theta|i)$ will be given an explicit expression in (12) below.

We make six remarks on our price function. First, given a continuum of iid idiosyncratic signals within each information group, linearity guarantees that only aggregate group signals matter. So only $\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n\}$ enters the price function, and all idiosyncratic signal realizations disappear. Second, the matching equation (9) is unique when the variance-covariance matrix of the aggregate signal $\boldsymbol{y}$ is positive definite, but not otherwise. Because we do not impose this positive definiteness in our model, the matching equation (9) is only one of serval potential matchings. Third, the coefficient on each signal depends in an intimate way on the overall stochastic structure of signals, asset supply, information group sizes, and the distribution of risk attitudes as also the distribution of the fundamental. Fourth, the prior mean of θ is taken to be zero and so does not explicitly enter the price function, but of course the prior will cast its influence, so in general, $\sum_{k=1}^n \sum_{s=1}^m \pi_{ks} \neq 1$ (for an explicit computation in a special case, see (23)). Fifth, *even if* we were to reorder signals (without loss of generality) so that each is positively correlated with the fundamental, every weight π_{ij} cannot be guaranteed to be nonnegative, given the generality of correlation patterns across signals. Finally, $\gamma \neq 0$ is jointly implied by equilibrium and our definition of a price function as a non-trivial object, for if it were zero, the entire vector $(\boldsymbol{\pi}, \gamma)$ would be zero, by (9). That said, and in contrast to the special case of independent signals, a non-trivial argument is needed to show that γ must be *positive*.

3.2. **Existence and Regularity of Linear Equilibrium.** Consider trader j in information group i. By the independence of idiosyncratic noise (Assumption 1), $\mathbf{Var}(\boldsymbol{z}_i(j)) = \mathbf{Var}(\boldsymbol{y}_i) + \mathbf{Var}(\boldsymbol{\epsilon}_i(j))$. In addition, because all idiosyncratic noise within group i has the same distribution, $\mathbf{Var}(\boldsymbol{z}_i(j))$ depends only on the identity of the information group i, but not on agent j's risk-aversion. Therefore, we drop j and simply write $\mathbf{Var}(\boldsymbol{z}_i)$ instead of $\mathbf{Var}(\boldsymbol{z}_i(j))$. It follows from normality that for every i, the triple $(\theta, \boldsymbol{z}_i', p)'$ has mean zero and positive definite variance-covariance matrix[11]

$$\begin{pmatrix} \mathrm{Var}(\theta) & \mathbf{Cov}(\theta, \boldsymbol{y}_i)' & \mathrm{Cov}(\boldsymbol{\pi}, \theta) \\ \mathbf{Cov}(\theta, \boldsymbol{y}_i) & \mathbf{Var}(\boldsymbol{z}_i) & \mathbf{Cov}(\boldsymbol{\pi}, \boldsymbol{y}_i) \\ \mathrm{Cov}(\boldsymbol{\pi}, \theta) & \mathbf{Cov}(\boldsymbol{\pi}, \boldsymbol{y}_i)' & \mathrm{Var}(\boldsymbol{\pi}) + \gamma^2 \mathrm{Var}(u) \end{pmatrix} =: \begin{pmatrix} \mathrm{Var}(\theta) & \dots \\ \vdots & \Sigma_i \end{pmatrix},$$

where Σ_i is positive definite, and the inverse Σ_i^{-1} exists and equals[12]

$$\begin{pmatrix} \left[\mathbf{Var}(\boldsymbol{z}_i) - \frac{\mathbf{Cov}(\boldsymbol{\pi},\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{\pi},\boldsymbol{y}_i)'}{\mathrm{Var}(\boldsymbol{\pi})+\gamma^2\,\mathrm{Var}(u)}\right]^{-1} & -\frac{\mathbf{Var}^{-1}(\boldsymbol{z}_i)\,\mathbf{Cov}(\boldsymbol{\pi},\boldsymbol{y}_i)}{\mathrm{Var}(\boldsymbol{\pi})+\gamma^2\,\mathrm{Var}(u)-\mathbf{Cov}(\boldsymbol{\pi},\boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{z}_i)\,\mathbf{Cov}(\boldsymbol{\pi},\boldsymbol{y}_i)} \\ -\frac{\mathbf{Cov}(\boldsymbol{\pi},\boldsymbol{y}_i)'}{\mathrm{Var}(\boldsymbol{\pi})+\gamma^2\,\mathrm{Var}(u)}\left[\mathbf{Var}(\boldsymbol{z}_i) - \frac{\mathbf{Cov}(\boldsymbol{\pi},\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{\pi},\boldsymbol{y}_i)'}{\mathrm{Var}(p)}\right]^{-1} & \frac{1}{\mathrm{Var}(\boldsymbol{\pi})+\gamma^2\,\mathrm{Var}(u)-\mathbf{Cov}(\boldsymbol{\pi},\boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{z}_i)\,\mathbf{Cov}(\boldsymbol{\pi},\boldsymbol{y}_i)} \end{pmatrix}$$

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[11]If the variance-covariance matrix of $(\theta, \boldsymbol{z}_i', p)'$ is not positive definite for some i, then there exists a nonzero vector $(d_1, \mathbf{d}_2', d_3)$ such that $d_1\theta + \mathbf{d}_2'\boldsymbol{z}_i + d_3 p = 0$. Because $\gamma \neq 0$ and u is independent of other random variables, it must hold that $d_3 = 0$. Also observe that $d_1 \neq 0$ by the assumed positive definiteness of $\boldsymbol{y}_i$ and idiosyncratic noise. So $\theta = -\mathbf{d}_2'\boldsymbol{z}_i/d_1$. That is, θ can be expressed as a linear combination of $\boldsymbol{z}_i$. But then $\mathrm{Var}(\theta|\boldsymbol{y}_1, \dots, \boldsymbol{y}_n) \leq \mathrm{Var}(\theta|\boldsymbol{y}_i) \leq \mathrm{Var}(\theta|\boldsymbol{z}_i) = 0$, which contradicts Assumption 1.

[12]Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ be a nonsingular matrix with $\det(A_{11}) \neq 0$ and $\det(A_{22}) \neq 0$. Then

$$A^{-1} = \begin{pmatrix} [A_{11} - A_{12}A_{22}^{-1}A_{21}]^{-1} & -A_{11}^{-1}A_{12}[A_{22} - A_{21}A_{11}^{-1}A_{12}]^{-1} \\ -A_{22}^{-1}A_{21}[A_{11} - A_{12}A_{22}^{-1}A_{21}]^{-1} & [A_{22} - A_{21}A_{11}^{-1}A_{12}]^{-1} \end{pmatrix}.$$

So by the projection theorem for normal random variables, the conditional mean of θ equals

$$\begin{aligned}\mathbb{E}(\theta|\boldsymbol{z}_i(j),p) = \mathbb{E}(\theta|\boldsymbol{z}_i(j),p-c) &= \big(\mathbf{Cov}(\theta,\boldsymbol{y}_i)',\operatorname{Cov}(\boldsymbol{\pi},\theta)\big)\Sigma_i^{-1}(\boldsymbol{z}_i'(j),p-c)' \\ &\equiv \boldsymbol{\alpha}_i'\boldsymbol{z}_i(j)+\beta_i(p-c),\end{aligned}$$

where

$$\boldsymbol{\alpha}_i = \left[\mathbf{Var}(\boldsymbol{z}_i) - \frac{\mathbf{Cov}(\boldsymbol{\pi},\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{\pi},\boldsymbol{y}_i)'}{\operatorname{Var}(\boldsymbol{\pi})+\gamma^2\operatorname{Var}(u)}\right]^{-1}\left[\mathbf{Cov}(\theta,\boldsymbol{y}_i) - \frac{\operatorname{Cov}(\boldsymbol{\pi},\theta)}{\operatorname{Var}(\boldsymbol{\pi})+\gamma^2\operatorname{Var}(u)}\mathbf{Cov}(\boldsymbol{\pi},\boldsymbol{y}_i)\right], \quad (10)$$

$$\beta_i = \frac{\operatorname{Cov}(\boldsymbol{\pi},\theta) - \mathbf{Cov}(\theta,\boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{z}_i)\,\mathbf{Cov}(\boldsymbol{\pi},\boldsymbol{y}_i)}{\operatorname{Var}(\boldsymbol{\pi})+\gamma^2\operatorname{Var}(u) - \mathbf{Cov}(\boldsymbol{\pi},\boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{z}_i)\,\mathbf{Cov}(\boldsymbol{\pi},\boldsymbol{y}_i)}, \quad (11)$$

and the conditional variance of θ is given by

$$\begin{aligned}\operatorname{Var}(\theta|i) &= \operatorname{Var}(\theta) - \big(\mathbf{Cov}(\theta,\boldsymbol{y}_i)',\operatorname{Cov}(\boldsymbol{\pi},\theta)\big)\Sigma_i^{-1}\big(\mathbf{Cov}(\theta,\boldsymbol{y}_i),\operatorname{Cov}(\boldsymbol{\pi},\theta)\big)' \\ &= \operatorname{Var}(\theta) - [\boldsymbol{\alpha}_i'\,\mathbf{Cov}(\theta,\boldsymbol{y}_i)+\beta_i\operatorname{Cov}(\boldsymbol{\pi},\theta)], \quad (12)\end{aligned}$$

noticing, as mentioned before, that neither the signal nor price realization affects its value.

Now we can make further progress on the coefficients recorded in (7)–(9). First, substitute (10) into (9) to get

$$\boldsymbol{\pi}_i = \gamma\frac{\left[\mathbf{Var}(\boldsymbol{z}_i) - \frac{\mathbf{Cov}(\boldsymbol{\pi},\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{\pi},\boldsymbol{y}_i)'}{\operatorname{Var}(\boldsymbol{\pi})+\gamma^2\operatorname{Var}(u)}\right]^{-1}\left[\mathbf{Cov}(\theta,\boldsymbol{y}_i) - \frac{\operatorname{Cov}(\boldsymbol{\pi},\theta)}{\operatorname{Var}(\boldsymbol{\pi})+\gamma^2\operatorname{Var}(u)}\mathbf{Cov}(\boldsymbol{\pi},\boldsymbol{y}_i)\right]}{\Delta_i\operatorname{Var}(\theta|i)}. \quad (13)$$

Our goal is to find solutions for $\boldsymbol{\pi}$ and γ which can then be inserted into (10) and (11) to generate the coefficients $\{\boldsymbol{\alpha}_i,\beta_i\}$. To achieve this, it will be convenient to study the *ratio* of $\boldsymbol{\pi}$ to γ.[13] Recalling that $\gamma \neq 0$, define $\boldsymbol{Q}_i = \boldsymbol{\pi}_i/\gamma$ for $i = 1,\ldots,n$, and $\boldsymbol{Q} = (\boldsymbol{Q}_1',\ldots,\boldsymbol{Q}_n')'$. Using (13), we obtain the following equations involving only the variables $\{\boldsymbol{Q}_i\}$, but not γ:

$$\boldsymbol{Q}_i = \boldsymbol{f}_i(\boldsymbol{Q}) \equiv \frac{\left[\mathbf{Var}(\boldsymbol{z}_i) - \frac{\mathbf{Cov}(\boldsymbol{Q},\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{Q},\boldsymbol{y}_i)'}{\operatorname{Var}(\boldsymbol{Q})+\operatorname{Var}(u)}\right]^{-1}\left[\mathbf{Cov}(\theta,\boldsymbol{y}_i) - \frac{\operatorname{Cov}(\boldsymbol{Q},\theta)}{\operatorname{Var}(\boldsymbol{Q})+\operatorname{Var}(u)}\mathbf{Cov}(\boldsymbol{Q},\boldsymbol{y}_i)\right]}{\Delta_i\operatorname{Var}_{\boldsymbol{Q}}(\theta|i)} \quad (14)$$

[13] We borrow this technique from Hellwig (1980).

for $i = 1, \dots, n$. Notice how, in this fixed point mapping, we subscript the conditional variance by $\boldsymbol{Q}$, to emphasize that it does change with the price function, which is informationally equivalent to $\sum_{k=1}^{n} \boldsymbol{Q}_k' \boldsymbol{y}_k + u$ because $\gamma \neq 0$. Our existence theorem relies on a solution to (14).

PROPOSITION **1.** *The system of equations* (14) *has a solution.*

While postponing the formal details, we make some remarks on the existence argument. Because of the generality of our signal structure, we cannot guarantee that each $\boldsymbol{Q}_i$ is nonnegative (we return to this problem later). Consequently, the arguments in Hellwig (1980) cannot be applied here. A specific problem arises precisely in the Hellwig case with a finite number of traders, which corresponds in our model to the case of no idiosyncratic noise. When there is no idiosyncratic noise, we cannot guarantee the uniform boundedness of the mapping $f \equiv (f_1, \dots, f_n)$ with respect to all $\boldsymbol{Q} \in \mathbb{R}^{nm}$. The proof therefore proceeds differently, by first constructing a sequence of uniformly bounded mappings over the domain, obtaining a fixed point (using Brouwer's theorem) for each such mapping, and then taking the limit of the resulting sequence of fixed points to obtain a fixed point of the original mapping f. In contrast, when there is idiosyncratic noise, the uniform boundedness of the mapping f over all $\boldsymbol{Q} \in \mathbb{R}^{nm}$ can be established, permitting a direct application of Brouwer's fixed-point theorem to obtain a solution to the system (14).

Given the existence of a $\boldsymbol{Q}$ satisfying (14), and suppressing the subscript $\boldsymbol{Q}$ from the conditional variance, we can quickly solve for the coefficients of the accompanying price function. By virtue of (7) and (11), we have

$$\begin{aligned}\frac{1}{\gamma} &= \sum_{i=1}^{n} \frac{1}{\Delta_i \operatorname{Var}(\theta|i)} - \sum_{i=1}^{n} \frac{1}{\Delta_i \operatorname{Var}(\theta|i)} \frac{\operatorname{Cov}(\boldsymbol{\pi}, \theta) - \mathbf{Cov}(\theta, \boldsymbol{y}_i)' \mathbf{Var}^{-1}(\boldsymbol{z}_i) \mathbf{Cov}(\boldsymbol{\pi}, \boldsymbol{y}_i)}{\operatorname{Var}(\boldsymbol{\pi}) + \gamma^2 \operatorname{Var}(u) - \mathbf{Cov}(\boldsymbol{\pi}, \boldsymbol{y}_i)' \mathbf{Var}^{-1}(\boldsymbol{z}_i) \mathbf{Cov}(\boldsymbol{\pi}, \boldsymbol{y}_i)} \\ &= \sum_{i=1}^{n} \frac{1}{\Delta_i \operatorname{Var}(\theta|i)} - \frac{1}{\gamma} \sum_{i=1}^{n} \frac{1}{\Delta_i \operatorname{Var}(\theta|i)} \frac{\operatorname{Cov}(\boldsymbol{Q}, \theta) - \mathbf{Cov}(\theta, \boldsymbol{y}_i)' \mathbf{Var}^{-1}(\boldsymbol{z}_i) \mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)}{\operatorname{Var}(\boldsymbol{Q}) + \operatorname{Var}(u) - \mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)' \mathbf{Var}^{-1}(\boldsymbol{z}_i) \mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)}.\end{aligned}$$

Rearranging the terms in this equation, we see that

$$\gamma = \frac{1 + \sum_{i=1}^{n} \frac{\operatorname{Cov}(\boldsymbol{Q}, \theta) - \mathbf{Cov}(\theta, \boldsymbol{y}_i)' \mathbf{Var}^{-1}(\boldsymbol{z}_i) \mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)}{\Delta_i \operatorname{Var}(\theta|i)[\operatorname{Var}(\boldsymbol{Q}) + \operatorname{Var}(u) - \mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)' \mathbf{Var}^{-1}(\boldsymbol{z}_i) \mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)]}}{\sum_{i=1}^{n} \frac{1}{\Delta_i \operatorname{Var}(\theta|i)}}. \tag{15}$$

Observe that γ is finite for any $\boldsymbol{Q} \in \mathbb{R}^n$ because the denominator of the expression (15) is always strictly positive. Provisionally assume that γ is non-zero as well; then we can quickly establish the existence of linear equilibrium. With γ given by (15), we obtain a solution $\boldsymbol{\pi}$ to (13) using the relationship $\boldsymbol{\pi} = \gamma \boldsymbol{Q}$. Substituting the solution $(\boldsymbol{\pi}, \gamma)$ into (10) and (11) leads to $\boldsymbol{\alpha}_i$ and β_i. We then obtain $\mathrm{Var}(\theta|i)$ via (12). Finally, the value of c comes from (8). Taken together, we have a linear equilibrium.

So there is just one remaining step, which is to assure ourselves that the premise from which we started — $\gamma \neq 0$ — can be respected in this solution. In Hellwig (1980), there is no idiosyncratic noise and the aggregate signal for trader/group i takes the single-dimensional form $y_i = \theta + \varepsilon_i$. For this special case, Lemma 3.1 in Hellwig (1980) shows that the system of equations (14) has a strictly positive solution. This implies that γ in (15) is also positive.[14] This regularity result is also true — but far from immediate — in our setting. For instance, $\boldsymbol{Q}$ will not generally be positive, so the argument just made must be discarded.

PROPOSITION **2.** *For any solution $\boldsymbol{Q}$ to (14), we have $\gamma > 0$.*

Propositions 1 and 2 immediately yield:

PROPOSITION **3.** *There exists a linear equilibrium, and every such equilibrium is regular.*

Ozsoylev and Walden 2011, p. 2260, underline the non-triviality of Propositions 1 and 2: "We note that, in contrast to the analysis in [Hellwig 1980], the existence of a linear NREE [noisy rational expectations equilibrium] for a finite number of agents *is not guaranteed here*, because in our setup agents, who are each other's neighbors or who have common neighbors, receive signals with correlated error terms." However, our Propositions show that even though the error terms of private signals in a finite-agent economy are correlated with each other and the dimension of signals is arbitrary, there always exists a linear equilibrium and each linear equilibrium is regular.

[14] Refer to equation (8b) in Hellwig (1980).

3.3. **Uniqueness of Linear Equilibrium.** We now turn to a discussion of uniqueness of our equilibrium, within the class of all linear equilibria.[15] We therefore say that a linear equilibrium is *unique* if all linear equilibrium prices equal each other almost surely. Specifically, if p_1 and p_2 are two linear equilibrium price functions defined on $(\boldsymbol{y}', u)'$, then $p_1 = p_2$ almost surely.

Observe that this sort of "outcome-uniqueness" is different from the assertion that there is a single linear equilibrium price function. The latter cannot be obtained in situations in which the signal $\boldsymbol{y}$ fails to have a positive definite variance-covariance matrix, which is something we want to allow for; see earlier discussion. Specifically, consider any maximal linearly independent subset of $\{\boldsymbol{y}_{ij}, i = 1, \ldots, n, j = 1, \ldots, m\}$, denoted by $\bar{\boldsymbol{y}}$. Then $\mathbf{Var}(\bar{\boldsymbol{y}})$ is positive definite, and every y_{ij} not in $\bar{\boldsymbol{y}}$ can be expressed as a linear combination of $\bar{\boldsymbol{y}}$. So any linear equilibrium $p = \boldsymbol{\pi}'\boldsymbol{y} + \gamma u + c$ can be equivalently rewritten as an equilibrium of the form $p = \bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}} + \gamma u + c$. It is "outcome-uniqueness" that we are interested in, and that is quite generally satisfied in our model, provided that traders are sufficiently risk-averse.

PROPOSITION **4.** *There exists $\hat{\rho} < \infty$ such that if $\rho(\tau) \geq \hat{\rho}$ for every trader type τ, the resulting linear equilibrium is unique.*

It should be pointed out that Proposition 4 is different from uniqueness arguments made in the *absence* of noise trade. Given the information aggregation result in Proposition 6 that we prove below, we already have the asymptotic uniqueness of all equilibria in the limit as noise trade vanishes — at least for linear limit equilibria that survive small perturbations in noise trade.[16] This is because Proposition 6 shows that every limit point of every sequence of equilibrium price functions must serve as a perfect aggregator of information, and so must have the outcome-uniqueness property described above. (Additionally, a best estimator is fully pinned down when the set of signals is full-dimensional.) Nevertheless, Proposition 4 is of separate interest because it establishes uniqueness — albeit under some restrictions — *away* from the full-information limit.

[15] We do not address the more demanding question of uniqueness in the class of *all* potential equilibria. For instance, in the context of the simpler model of Grossman (1976), DeMarzo and Skiadas (1998) show that the linear equilibrium in is unique in the class of all possible equilibria, linear or not.

[16] This observation is directly related to uniqueness arguments in the literature (at least within the class of linear equilibria) in the case where there is no noise trade; see, e.g., Grossman (1976) and Nielsen (1996).

4. Information Aggregation

4.1. **Some Aggregation.** We begin with the question of whether the equilibrium price aggregates *some* information, even when there is noise. In what follows, we maintain the convention

$$\mathbf{Cov}(\theta, \boldsymbol{y}_i) \geq \mathbf{0}$$

for every i. This restriction is without any loss of generality, as a signal can always be reordered by a sign flip without changing anything of consequence. Next, say that the equilibrium price p *aggregates some information* about the fundamental θ if $\mathrm{Cov}(\theta, p) \neq 0$. We can now state:

Proposition **5.** *It is always true that* $\mathrm{Cov}(\theta, p) \geq 0$*. Strict inequality holds — i.e., the equilibrium price aggregates some information about* θ *— if and only if* $\mathbf{Cov}(\theta, \boldsymbol{y}_i) \neq \mathbf{0}$ *for some* i*.*

Proposition 5 is intuitive, and it is especially transparent in the Hellwig (1980) setting with the special one-dimensional signal and the special independent signal structure $y_i = \theta + \varepsilon_i$ because for this special structure, the solution to (14) is positive and the covariance $\mathrm{Cov}(\theta, y_i)$ is equal to the unconditional variance $\mathrm{Var}(\theta)$ for every signal y_i. Proposition 5, combined with the relationship $\mathrm{Var}(\theta|p) = \mathrm{Var}(\theta) - \frac{\mathrm{Cov}(\theta,p)^2}{\mathrm{Var}(p)}$, also tells us that the equilibrium price is informative, i.e., $\mathrm{Var}(\theta|p) < \mathrm{Var}(\theta)$, provided that at least one component in all the group signals is correlated with the fundamental.

4.2. **Full Aggregation With Vanishing Noise.** Next, suppose that a "super-agent" can see the entire set of aggregate signals $\boldsymbol{y} = (\boldsymbol{y}_1', \ldots, \boldsymbol{y}_n')'$, and is asked to infer the fundamental value θ. The solution to this problem is standard: choose a weighting vector $\boldsymbol{\pi}$ for the signals that satisfies the condition for perfect information aggregation: for every i,

$$\mathbf{Cov}(\theta, \boldsymbol{y}_i) = \mathbf{Cov}(\boldsymbol{\pi}, \boldsymbol{y}_i). \tag{16}$$

If the vector of signals $\boldsymbol{y}$ is linearly independent, then $\mathrm{Var}(\boldsymbol{y})$ is positive definite and the solution to (16) is unique. Let Σ stand for the variance-covariance matrix of signals $\boldsymbol{y}$, and $\mathbf{Cov}(\theta, \boldsymbol{y})$ for the vector of covariances

$$[\mathbf{Cov}(\theta, \boldsymbol{y}_1)', \ldots, \mathbf{Cov}(\theta, \boldsymbol{y}_n)']'.$$

Then the unique solution to (16) must be given by

$$\boldsymbol{\pi} = \Sigma^{-1}\,\mathbf{Cov}(\theta, \boldsymbol{y}). \tag{17}$$

When the vector of group signals is not linearly independent, as would be the case (for instance) when one group is unambiguously more informed than another, then in general there will not be a unique solution to the perfect aggregation condition. But it really does not matter, because all solutions do an equally good job. A typical solution can be described as follows.

Consider a maximal linearly independent subset of $\{\boldsymbol{y}_{ij}, i = 1, \ldots, n, j = 1, \ldots, m\}$, and without loss of generality, denote the vector of such a subset by $\bar{\boldsymbol{y}}$. That is, the variance-covariance matrix of $\bar{\boldsymbol{y}}$ is positive definite, and for every y_{ij} which is not in the maximal linearly independent subset, the variance-covariance matrix of $(\bar{\boldsymbol{y}}', y_{ij})'$ fails to have full rank. The signal $\bar{\boldsymbol{y}}$ is informationally equivalent to the aggregate signal $\boldsymbol{y}$. A solution to (16) could be described by an analogue of (17) applied to this linearly independent subset:

$$\bar{\boldsymbol{\pi}} = \bar{\Sigma}^{-1}\,\mathbf{Cov}(\theta, \bar{\boldsymbol{y}}), \tag{18}$$

where the associated notation using bars should be self-explanatory. Undoubtedly, if we fixed another maximally independent subset $\hat{\boldsymbol{y}}$, the corresponding weights $\hat{\boldsymbol{\pi}}$ would do just as well in aggregation, so the specific maximally independent subset does not matter.

We connect these remarks a bit more formally to a market context. Just for the discussion here, we suppose that there is no noise trade (i.e., $\mathrm{Var}(u) = 0$), and no idiosyncratic noise (i.e., $\mathbf{Var}(\boldsymbol{\epsilon}_i(j)) = \mathbf{0}$ for all i and j). Now consider the price function $\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}} + c$. We claim that for every i,

$$\mathbb{E}(\theta|\boldsymbol{y}_i, \bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}}) = \mathbb{E}(\theta|\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}}) \text{ and } \mathrm{Var}(\theta|\boldsymbol{y}_i, \bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}}) = \mathrm{Var}(\theta|\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}}), \tag{19}$$

which formally captures the idea that nothing of informational value can be added once the weights in (18) have been applied to predict the fundamental. We only show the first equality in (19), because the second will follow immediately from the first by invoking the Law of Total Variance: $\mathrm{Var}(\theta) = \mathrm{Var}(\mathbb{E}(\theta|\cdot)) + \mathbb{E}(\mathrm{Var}(\theta|\cdot))$. That equality is obvious when $\mathbf{Cov}(\theta, \bar{\boldsymbol{y}}) = \mathbf{0}$. When $\mathbf{Cov}(\theta, \bar{\boldsymbol{y}}) \neq \mathbf{0}$, we have $\bar{\boldsymbol{\pi}} \neq \mathbf{0}$ because $\bar{\Sigma}^{-1}$ is a positive definite matrix. By the projection theorem for normal random variables, we have

$$\mathbb{E}(\theta|\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}}) = \frac{\operatorname{Cov}(\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}},\theta)}{\operatorname{Var}(\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}})}\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}}.$$

Furthermore, it follows from (18) that

$$\mathbf{Cov}(\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}},\bar{\boldsymbol{y}}) = \mathbf{Cov}(\theta,\bar{\boldsymbol{y}}), \tag{20}$$

which is just (16) for the maximally independent set. Multiplying by $\bar{\boldsymbol{\pi}}$ on both sides, we obtain:

$$\operatorname{Var}(\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}}) = \operatorname{Cov}(\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}},\theta). \tag{21}$$

Now suppose that for every i, $\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}}$ cannot be expressed as a linear combination of $\boldsymbol{y}_i$. (For the opposite case, see this footnote.[17]) Then it follows from Lemma 3 in the Appendix, identifying $\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}}$ with $\boldsymbol{\mu}'\boldsymbol{y}$, that for any i, $\mathbf{Var}(\boldsymbol{y}_i) - \frac{\mathbf{Cov}(\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}},\boldsymbol{y}_i)\,\mathbf{Cov}(\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}},\boldsymbol{y}_i)'}{\operatorname{Var}(\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}})}$ is positive definite and $\operatorname{Var}(\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}}) - \mathbf{Cov}(\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}},\boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{y}_i)\,\mathbf{Cov}(\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}},\boldsymbol{y}_i) > 0$.

By setting $\gamma^2\operatorname{Var}(u) = 0$, $\boldsymbol{z}_i = \boldsymbol{y}_i$ and $p = \bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}} + c$, equations (20) and (21) imply that the coefficient $\boldsymbol{\alpha}_i$ in (10) equals zero, and β_i in (11) equals one. Therefore (19) holds for every i.[18]

In summary, under the perfect aggregation condition, all information in $\boldsymbol{y}$ is combined optimally in $\boldsymbol{\pi}$ to predict θ. Adding any signal to it in a way that matters is not only redundant, but reduces predictive ability for a market participant.

[17]If there is only one nonzero vector, say $\kappa_1 \neq \mathbf{0}$, such that $\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}} = \kappa_1'\boldsymbol{y}_1$, (20) implies that $\mathbf{Cov}(\kappa_1'\boldsymbol{y}_1,\boldsymbol{y}_1) = \mathbf{Cov}(\theta,\boldsymbol{y}_1)$ and consequently, $\kappa = \operatorname{Var}^{-1}(y_1)\,\mathbf{Cov}(\theta,\boldsymbol{y}_1)$. Hence

$$\mathbb{E}(\theta|\boldsymbol{y}_1,\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}}) = \mathbb{E}(\theta|\boldsymbol{y}_1) = \kappa'\boldsymbol{y}_1 = \mathbb{E}(\theta|\kappa_1'\boldsymbol{y}_1) = \mathbb{E}(\theta|\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}}),$$

where the first equality follows from the informational equivalence between $\{\boldsymbol{y}_1,\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}}\}$ and $\boldsymbol{y}_1$, and the second and third equalities follow from the projection theorem for normal random variables. Then (19) holds for $i = 1$. We next show that it also holds for $2 \leq i \leq n$. Consider some $2 \leq i \leq n$. If $\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}}$ cannot be expressed as a linear combination of $\boldsymbol{y}_i$, then we can show that (19) holds for this i by applying arguments similar to those in the main text. If $\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}} = \kappa_i'\boldsymbol{y}_i$ for some nonzero κ_i, then (19) also holds for this i by using arguments similar to those above in this footnote.

[18]Indeed, we can check that $\bar{p} = \bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}} - X/\sum_{i=1}^{n}\frac{1}{\Delta_i\operatorname{Var}(\theta|\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}})}$ is an equilibrium price in the sense of Grossman (1976). Optimal demands clear the market, given that $\sum_{i=1}^{n}\frac{\mathbb{E}(\theta|y_i,\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}})-\bar{p}}{\Delta_i\operatorname{Var}(\theta|y_i,\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}})} = X$, and given that the mean and variance of the noise demand equal zero.

The discussion is obviously related to that in Grossman (1976). When each of a finite number of traders obtains a conditionally independent signal ($y_i = \theta + \varepsilon_i$), Grossman showed that in the absence of noise trade, the equilibrium price perfectly aggregates all traders' private information. A trader who only observes the price can achieve the same expected utility as one acquires an additional private signal, and in fact *any* use of the private signal is payoff-reducing. But, of course, as is well known, Grossman's analysis only introduces the paradox of information aggregation, and does not solve it. Consult Grossman (1976), Grossman and Stiglitz (1980) and Hellwig (1980) for more discussion on the Grossman-Stiglitz paradox, and for Hellwig's solution to it, that we now proceed to extend and generalize.

With noise trade, it now makes sense to rely on private information, because the informativeness of the price is now clouded by stochastic shocks to demand. That reliance must fade as the noise trade approaches zero. The question we now ask is posed in an entirely general setting: does that reliance fade "slowly enough" so that private information seeps into the price system, allowing for *full* information aggregation as noise converges to zero? Our answer is in the affirmative:

PROPOSITION **6.** *Along any sequence of equilibria indexed by the variance of noise* $\mathrm{Var}_t(u)$ *converging to zero, any corresponding sequence of equilibrium price functions* $p = \boldsymbol{\pi}_t'\boldsymbol{y} + \gamma_t u + c_t$ *must have the following properties:*

(i) $\boldsymbol{\pi}_t$ *is bounded in* t *and every limit point* $\boldsymbol{\pi}$ *of* $\{\boldsymbol{\pi}_t\}$ *must satisfy perfect information aggregation:*

$$\mathbf{Cov}(\theta, \boldsymbol{y}_i) = \mathbf{Cov}(\boldsymbol{\pi}, \boldsymbol{y}_i), i = 1, \ldots, n, \tag{22}$$

and in particular, there is a unique limit (i.e., $\boldsymbol{\pi}_t \to \Sigma^{-1}\,\mathbf{Cov}(\theta, \boldsymbol{y})$*) when* $\mathrm{Var}(\boldsymbol{y})$ *is positive definite.*

(ii) $\gamma_t^2\,\mathrm{Var}_t(u) \to 0$ *and* $c_t \to \frac{\mathrm{Cov}(\boldsymbol{\pi},\theta) - \mathrm{Var}(\theta)}{\sum_{k=1}^n 1/\Delta_k} X$ *whenever* $\boldsymbol{\pi}_t \to \boldsymbol{\pi}$ *along some subsequence of* t*.*

The above proposition has two parts. The central assertion is Part (i). In general, an equilibrium price function will not aggregate information efficiently. Quite apart from the presence of noise trade, different signals are observed by groups that vary both in their sizes and in their within-group distribution of risk attitudes. Because the volume of group-specific trade also goes into determining the equilibrium price function, and because sizes and risk attitudes affect those volumes, the equilibrium price function will incorporate not just pure information but also group sizes and the full distribution of attitudes to risk. Finally, there are arbitrary correlations across possibly multidimensional signals. From this perspective, it is of interest that as the impact of noise trade vanishes, *all these additional effects* on the price function endogenously vanish, leaving only the efficient aggregation of information.

Could there be several limit points? In general, the answer is yes: it would depend on how much "slack" there is in the signal structure. But even then, as already discussed, each limit point would exhibit the perfect aggregation property, so in this sense, nothing of substance is lost. As a special case, if the set of group signals has full dimensionality, then we can assert that every sequence of equilibrium price functions must indeed converge to a well-defined, unique limit as noise trade vanishes. We remark that these considerations — many limit points all satisfying perfect aggregation, or just one — are orthogonal to the question of whether equilibria are unique for each parametric configuration: the results apply regardless.

Part (ii) states two ancillary observations. First, as the variance of the noise trade goes to zero, its overall impact on prices goes to zero as well. This is intuitive. Second, when the supply of the asset X is non-zero, the intercept term c of the price function does indeed retain the influence of group sizes and attitudes to risk, as captured by the Δ_i's. But confined as these influences are to the intercept term, they do not impede efficient information aggregation. The main point is that all group-level heterogeneity must completely vanish from the coefficients on $\boldsymbol{y}$, as already described in Part (i).

The above proposition significantly extends Proposition 4.3 in Hellwig (1980) to our fully general signal structure. Noise trade is bad for information aggregation, but that is precisely what allows the information to leak into the price in the first place, because traders see value in using their private signals in the presence of noise. As noise trade vanishes, the reliance on own signals vanishes as well, but the speed of that vanishing must be slow enough so that in the limit, full information aggregation is achieved.

5. A Remark on the Coefficients of the Equilibrium Price Function

In this section, we remark on the weights $\boldsymbol{\pi}$ that aggregate the signals in the equilibrium price function. We consider only the special case of one-dimensional signals for each information group. For simplicity we also assume that there is no idiosyncratic noise ($\mathrm{Var}(\epsilon_i(j)) = 0$). When aggregate signals take the special form of $y_i = \theta + \varepsilon_i$ (where $\{\varepsilon_i\}$ are iid), Lemma 3.1 in Hellwig (1980) shows that $Q_i > 0$ (equivalently, $\pi_i > 0$) for any i. Even with a general signal structure, when there is no noise trade, there is a linear correspondence between the weights $\boldsymbol{\pi}$ and the correlation $\mathbf{Cov}(\theta, \boldsymbol{y})$, as illustrated by (17). Indeed, when both these two cases apply, $\mathrm{Cov}(y_i, y_j) = \mathrm{Var}(\theta)$ for any $i \neq j$ and $\mathrm{Var}(y_i) = \mathrm{Var}(\theta) + \mathrm{Var}(\varepsilon_i)$. Consequently, we can explicitly obtain the weights $\boldsymbol{\pi}$ as follows:

$$\pi_i = \frac{\mathrm{Var}(\theta)/\,\mathrm{Var}(\varepsilon_i)}{1 + \sum_{k=1}^{n} \mathrm{Var}(\theta)/\,\mathrm{Var}(\varepsilon_k)}, \quad i = 1, \ldots, n. \tag{23}$$

This is a generalization of Theorem 1 in Grossman (1976).[19]

Beyond these cases, and once faced with the generality of the correlation pattern that we allow for, it is difficult to sign $\boldsymbol{\pi}$, even after imposing the convention that $\mathbf{Cov}(\theta, \boldsymbol{y}) \geq \mathbf{0}$. Consider the following three observations.

1: $\mathrm{Cov}(\theta, y_i) > 0$ for all i does not imply that $Q_i \neq 0$ for all i. As an example, suppose $n = 2$ and $\mathrm{Cov}(\theta, y_i) > 0$ for $i = 1, 2$, but that $\mathrm{Cov}(y_1, y_2) \neq 0$. Then (Q_1, Q_2) with $Q_1 = 0$ is a solution to (14) if and only if the following two equalities hold:

$$\mathrm{Cov}(\theta, y_1)[Q_2^2\,\mathrm{Var}(y_2) + \mathrm{Var}(u)] = \mathrm{Cov}(\theta, y_2) Q_2^2\,\mathrm{Cov}(y_1, y_2),$$
$$Q_2 = \frac{\mathrm{Cov}(\theta, y_2)}{\Delta_2[\mathrm{Var}(\theta)\,\mathrm{Var}(y_2) - \mathrm{Cov}(\theta, y_2)^2]},$$

which may well be true for some parameters.

2: $\mathrm{Cov}(\theta, y_i) = 0$ for some i does not imply that $Q_i = 0$. Again, let $n = 2$. Suppose that $\mathrm{Cov}(\theta, y_1) = 0$, $\mathrm{Cov}(\theta, y_2) > 0$ and $\mathrm{Cov}(y_1, y_2) > 0$. It is easy to show by contradiction that

[19] It also validates the observation that generally $\sum_k \pi_k$ is not equal to 1, given that some weight is always assigned to the prior on the fundamental.

$Q_1 \neq 0$. Otherwise, $Q_2 > 0$ by Proposition 5 and

$$Q_1 = -\frac{Q_2^2 \operatorname{Cov}(\theta, y_2) \operatorname{Cov}(y_1, y_2)}{\Delta_1 \operatorname{Var}(\theta | y_1, \sum_{k=1}^2 Q_k y_k + u)[\operatorname{Var}(y_1)(\operatorname{Var}(\boldsymbol{Q}) + \operatorname{Var}(u)) - \operatorname{Cov}(\boldsymbol{Q}, y_1)^2]},$$

which is a contradiction.

3: It is entirely possible for $Q_{i_0} < 0$ for some i_0, even under the assumption that $\operatorname{Cov}(\theta, y_i) \geq 0$ for all i. Consider the example in the preceding observation **2**. We can see that (Q_1, Q_2) satisfies the equation

$$Q_1 = -\frac{Q_2 \operatorname{Cov}(\theta, y_2)[Q_1 \operatorname{Var}(y_1) + Q_2 \operatorname{Cov}(y_1, y_2)]}{\Delta_1 \operatorname{Var}(\theta | y_1, \sum_{k=1}^2 Q_k y_k + u)[\operatorname{Var}(y_1)(\operatorname{Var}(\boldsymbol{Q}) + \operatorname{Var}(u)) - \operatorname{Cov}(\boldsymbol{Q}, y_1)^2]}.$$

Proposition 5 and the previous equation imply that $Q_1 \neq 0$ and $Q_2 \neq 0$. If $Q_2 < 0$, there is nothing to prove, and if $Q_2 > 0$, we can show by contradiction that $Q_1 < 0$.

Our last proposition signs Q_i (and therefore π_i) for some other special cases of possible interest.

PROPOSITION **7.** *Suppose that* $\operatorname{Cov}(\theta, y_k) > 0$ *for at least one* k*. Then:*

(i) *If* y_i *is uncorrelated with* y_j *for every pair* (i, j) *with* $j \neq i$*, then* $Q_i > 0$ *if* $\operatorname{Cov}(\theta, y_i) > 0$*, and* $Q_i = 0$ *if* $\operatorname{Cov}(\theta, y_i) = 0$.

(ii) *Suppose that* $\operatorname{Cov}(\theta, y_{i_0}) = 0$ *for some* i_0*, and that* $\operatorname{Cov}(y_{i_0}, y_k) > 0$ *for every* k*. Then there is at least one index* j *with* $Q_j < 0$.

(iii) *It is not possible that* $Q_i \leq 0$ *for all* i.

(iv) *There exists a threshold for the variance of noise trade,* v*, such that if* $\operatorname{Var}(u) \geq v$*, then* $Q_i > 0$ *for every* i *such that* $\operatorname{Cov}(\theta, y_i) > 0$.

The Proposition makes some progress in signing the coefficients in special cases, or by placing some overall restrictions, as in parts (ii) and (iii). But even the overall nihilism expressed in this section can be given a rich interpretation in applications. With many signals, some of which are not observed by an individual trader, there is scope for interesting inference in specific situations.

Suppose, for instance, that we are interested in the value of a pharmaceutical company, and there are just two information groups, so that $n = 2$. A representative trader in group 1 knows about the number of clinical trials are being performed by the pharmaceutical, but does not know the results of those trials. A trader in group 2 knows about the number of successes in these trials, but does not know the number of trials. Now, controlling for the number of successes, a larger number of trials is bad news, because it suggests a lower *percentage* of successes. Therefore the number of trials will enter negatively into the price function, when the number of successes exists as a separate signal for others. (In contrast, if the number of successes is entirely private, the number of trials could have entered positively, but that is another setting.)

Here is a concrete numerical example, though the reader is asked to forgive the replacement of positive integers by lognormal signals: $n = 2$, $\Delta_1 = \Delta_2 = 1$, $y_1 = \log(\text{trials})$, $y_2 = \log(\text{successes})$, $\text{Var}(y_1) = 12$, $\text{Var}(\theta) = \text{Var}(y_2) = 10$, $\text{Cov}(\theta, y_1) = 1$, $\text{Cov}(\theta, y_2) = 8$, $\text{Cov}(y_1, y_2) = 7$, and $\text{Var}(u) = 2$. The `fsolve` function in MATLAB yields $(Q_1, Q_2) = (-0.001, 0.2221)$ for the system in (14) and a solution $\gamma = 0.4858$ in (15). Therefore $(\pi_1, \pi_2) = \gamma(Q_1, Q_2) = (-0.0005, 0.1079)$, and the corresponding linear equilibrium price is $p = -0.0005y_1 + 0.10799y_2 + 0.4848u$.

Now consider what happens when a trader in group 1 sees a higher value of her signal (more trials), but observes the same market price. She can then infer that trader 2 *must* have received a really positive signal regarding the number of interim successes. (Note that we show in Proposition 5 that it is impossible that every sign is non-positive). She understands that trader 2 is therefore taking a larger position in the asset. But of course, she understands that trader 2 — who is making similar inferences — will also know that part of the reason for her good signals must be that the company is conducting more trials, so that trader 2 will be more cautious about the success *rate*. These two forces temper the reaction of each side of the market. Knowing the signs of the price coefficients is indispensable for making these arguments, and can help traders to understand well the positions of other traders.

6. Bibliographical Notes

Our paper contributes to the extensive literature on equilibrium in financial markets with rational expectations. The literature can be divided into two subareas — one set in the classical domain of competitive price-taking behavior, and another that approaches the problem via theories of imperfect competition and strategic interaction; see Vives (2008) for a comprehensive account. Our paper sits squarely in the former area. We discuss existence first, then information aggregation.

The competitive setting emanates from the pioneering contributions of Grossman (1976), Hellwig (1980) and Grossman and Stiglitz (1980). Pálvölgyi and Venter (2015a) show that the linear equilibrium in Grossman and Stiglitz (1980) is unique in the class of all continuous price functions.[20] We restrict ourselves to linear equilibria throughout. Barlevy and Veronesi (2000) extend Grossman and Stiglitz (1980) in a different direction, assuming that the fundamental is binomial and investors are risk-neutral, in place of the classical assumption of normality and exponential utility. García and Urošević (2013) analyze the effects of the level of aggregate supply of the risky asset on the acquisition, revelation, and aggregation of private information for a variant of Hellwig (1980) to allow for both the presence of informed and uninformed traders.

These contributions all presume that traders' private signals are expressible as the sum of the fundamental and idiosyncratic noise. As already mentioned, we depart from this structure. In this sense, our model is related to Breon-Drish (2015), which essentially analyzes the finite-agent model of Hellwig (1980), but with an extension to more general signal structures of the exponential family, which includes normal distributions.[21] For tractability, this paper restricts the main analysis to the case of two agents (one informed trader and one uninformed trader) and binomial distributions. While Breon-Drish (2015) also discusses equilibrium existence for the multi-agent model, their existence results all assume that some system of equations has a solution (refer to their Proposition 7 and Corollary 2 therein), without providing specific conditions that ensure the existence of

[20]The same result is generally not true for even larger spaces of functions: for idiosyncratic information structures the authors also prove that there exist multiple discontinuous equilibria.

[21]Bernhardt and Miao (2004) also consider a model — different from ours — with a general signal structure. Although this work characterizes necessary and sufficient conditions for linear equilibrium, it does not provide general results on equilibrium existence.

equilibrium. In contrast, we restrict ourselves to the normal case, but establish the existence and regularity of linear equilibrium.[22]

Ozsoylev (2006) considers a generalized REE model in which traders can observe their neighbors' actions (their demands for the risky asset) over a social network, and can glean information from those actions. Ozsoylev proposes a generalized concept of REE which accommodates this interaction, and analyzes equilibrium existence for some special networks (cycles, trees), owing to difficulties when dealing with more general network structures. In contrast, agents in our model observe not the actions of other agents, but their *signals*. So conceptually, we do not (and do not need to) extend the REE definition. However, because we establish our main results for any correlation structure, the implication is that the analysis here can handle arbitrary networks.

Turning now to information aggregation, our paper is related to a vast literature on the informational role of prices in competitive markets. In Grossman (1976), a price function that aggregates all information is an equilibrium, but there is no satisfactory explanation for how the information gets into the system. *Given* the price function, traders have no incentive to use their private information. One interpretation of Hellwig (1980) is that noise trade acts as a device to resolve the paradox in Grossman (1976),[23] one that we adopt and significantly extend in our setting.

We end by briefly mentioning the imperfect competition setting. Now prices naturally encode information, as they are typically set to clear the market "after" traders make their strategic choices. So there is no potential informational paradox here as in the competitive setting, but it is still of interest to learn when perfect aggregation is possible.[24] The standard approach is to pass to the limit as the number of traders becomes large.[25] A continuous auction consisting of a single insider,

[22]There is also a literature that studies the existence, multiplicity and uniqueness of equilibrium for multi-asset models; see, for instance, Pálvölgyi and Venter (2015b), Chabakauri et al. (2017) and Carpio and Guo (2017).

[23]Vives (2014) attempts a resolution of the Grossman-Stiglitz paradox by considering heterogeneous valuations for the risky asset, and shows that the incentives to acquire information could be preserved without resorting to noise trade — after all, the equilibrium price function caters to some average of all the valuations, leaving each trader with some incentive to use her own information. See also Rahi and Zigrand (2018), where this happens to a partial degree.

[24]It is still possible to write down models of imperfect competition where information is not "forced" into the price. See, e.g., Kyle (1989), where demand schedules as functions of the price are announced.

[25]On the strategic foundations of information aggregation and revelation when the number of traders/bidders (and objects) becomes large, see, e.g., Wilson (1977), Milgrom (1981), Pesendorfer and Swinkels (1997), Kremer (2002), Reny and Perry (2006) and Kovalenkov and Vives (2014). For a double auction in which the average of the correlation

noise traders and market makers is studied in Kyle (1985), in which all private information is incorporated into prices. Based on the single-period model of Kyle (1985), Lambert et al. (2018) study the trading behavior and the properties of prices in an general signal setting and obtain an information aggregation result. Our paper complements Lambert et al. (2018), in that they work with a similarly general signal structure but in the imperfectly competitive setting; in contrast, we work with a fully competitive model.

7. Concluding Summary

We revisit Hellwig (1980) by studying a financial market with correlated information received by traders. Our traders belong to finitely many "information groups," and there is an aggregate signal for each such group. Each trader observes an idiosyncratic signal — generated from that aggregate signal — about the fundamental, and acts on the basis of that signal and the market price of the traded security. Because signals are multidimensional and the information structure permits general correlations, several existing models serve as special cases. The existence and regularity of linear equilibrium are established, the former through a novel method involving sequences of fixed points. We show that the equilibrium price function serves well as an information aggregator of diverse and decentralized information in the market, as the variance of the noise demand converges to zero.

Appendix

In this appendix, we present all proofs. We begin with three useful preliminary steps.

Lemma **1.** *For each i, there is $\epsilon \geq 0$ such that for any price of the form (3),*

$$0 < \mathrm{Var}(\theta|\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n) \leq \mathrm{Var}(\theta|i) \leq \mathrm{Var}(\theta) - \epsilon,$$

with $\epsilon > 0$ when $\mathbf{Cov}(\theta, \boldsymbol{y}_i) \neq \mathbf{0}$.

of each bidder's value with other bidders' values is a constant, Rostek and Weretka (2012) show that the equilibrium price is privately revealing (one trader's private signal together with the price contains all information) if and only if correlations of values coincide across all bidders.

Proof. The first inequality is stated in Assumption 1. For the second inequality, recall that $\mathrm{Var}(\theta|i)$ is shorthand for $\mathrm{Var}(\theta|z(i),p)$, and now observe that a price function of the form (3), plus a private signal, has no more information than $\boldsymbol{y}$. The last inequality states that the unconditional variance of θ must weakly exceed any of the conditional variances. That inequality is necessarily strict, and *uniformly* so regardless of the price function, if any of the conditioning variables is informative about θ, as it will be when $\mathbf{Cov}(\theta,\boldsymbol{z}_i)\neq\mathbf{0}$, or equivalently, when $\mathbf{Cov}(\theta,\boldsymbol{y}_i)\neq\mathbf{0}$. $\square$

LEMMA **2.** *For any vector* $\boldsymbol{\mu}=(\boldsymbol{\mu}_1',...,\boldsymbol{\mu}_n')'\in\mathbb{R}^{nm}$, *define* $\Gamma_i(\boldsymbol{\mu})\equiv\mathbf{Cov}(\theta,\boldsymbol{y}_i)\,\mathrm{Var}(\boldsymbol{\mu})-\mathrm{Cov}(\boldsymbol{\mu},\theta)\,\mathbf{Cov}(\boldsymbol{\mu},\boldsymbol{y}_i)$. *Then* $\sum_{i=1}^n\boldsymbol{\mu}_i'\Gamma_i(\boldsymbol{\mu})=0$.

Proof. Combine $\sum_{i=1}^n\boldsymbol{\mu}_i'\,\mathbf{Cov}(\theta,\boldsymbol{y}_i)=\mathrm{Cov}(\boldsymbol{\mu},\theta)$ and $\sum_{i=1}^n\boldsymbol{\mu}_i'\,\mathbf{Cov}(\boldsymbol{\mu},\boldsymbol{y}_i)=\mathrm{Var}(\boldsymbol{\mu})$. $\square$

LEMMA **3.** *Let* $\boldsymbol{\mu}=(\boldsymbol{\mu}_1',...,\boldsymbol{\mu}_n')'\in\mathbb{R}^{nm}$ *with* $\boldsymbol{\mu}'\boldsymbol{y}\neq 0$. *Then the following hold:*

(i) *For any* i, *the matrix* $\mathbf{Var}(\boldsymbol{y}_i)-\frac{\mathbf{Cov}(\boldsymbol{\mu},\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{\mu},\boldsymbol{y}_i)'}{\mathrm{Var}(\boldsymbol{\mu})}$ *is positive semi-definite, and is additionally positive definite if* $\boldsymbol{\mu}'\boldsymbol{y}$ *cannot be expressed as a linear combination of* $\boldsymbol{y}_i$*;*

(ii) *For any* i, $\mathrm{Var}(\boldsymbol{\mu})-\mathbf{Cov}(\boldsymbol{\mu},\boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{z}_i)\,\mathbf{Cov}(\boldsymbol{\mu},\boldsymbol{y}_i)\geq 0$, *and the inequality is strict if* $\boldsymbol{\mu}'\boldsymbol{y}$ *cannot be expressed as a linear combination of* $\boldsymbol{y}_i$.[26]

Proof. (i) Observe that for any nonzero column vector $\boldsymbol{d}\in\mathbb{R}^m$,

$$\begin{aligned}\boldsymbol{d}'\left[\mathbf{Var}(\boldsymbol{y}_i)-\frac{\mathbf{Cov}(\boldsymbol{\mu},\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{\mu},\boldsymbol{y}_i)'}{\mathrm{Var}(\boldsymbol{\mu})}\right]\boldsymbol{d}&=\mathrm{Var}(\boldsymbol{d}'\boldsymbol{y}_i)-\frac{\mathbf{Cov}(\boldsymbol{\mu},\boldsymbol{d}'\boldsymbol{y}_i)^2}{\mathrm{Var}(\boldsymbol{\mu})}\\&\geq\mathrm{Var}(\boldsymbol{d}'\boldsymbol{y}_i)-\frac{\mathbf{Cov}(\boldsymbol{\mu},\boldsymbol{d}'\boldsymbol{y}_i)^2}{\mathrm{Var}(\boldsymbol{\mu})}\geq 0,\end{aligned}$$

where the second inequality follows from the Cauchy-Schwarz inequality. When $\boldsymbol{\mu}'\boldsymbol{y}$ cannot be expressed as a linear combination of $\boldsymbol{y}_i$, the second inequality is clearly strict: for any column vector $\boldsymbol{d}\in\mathbb{R}^m$, $\boldsymbol{\mu}'\boldsymbol{y}$ cannot be perfectly correlated with $\boldsymbol{d}'\boldsymbol{y}_i$.

[26]The two inequalities in Lemma 3 (though not in strict form as stated here) have been proved in Tripathi (1999).

(ii) Let $\boldsymbol{s} = (s_1, \dots, s_m)' = \mathbf{Var}^{-\frac{1}{2}}(\boldsymbol{y}_i)\boldsymbol{y}_i$. Simple computation shows that $\boldsymbol{s} \sim N(0, I_m)$. Observe that $\sum_{k=1}^n \boldsymbol{\mu}_k' \boldsymbol{y}_k$ can be linearly expressed as

$$\sum_{k=1}^n \boldsymbol{\mu}_k' \boldsymbol{y}_k = \sum_{i=1}^m \operatorname{Cov}\left(\sum_{k=1}^n \boldsymbol{\mu}_k' \boldsymbol{y}_k, s_i\right) s_i + s_{m+1},$$

where $s_{m+1} \equiv \sum_{k=1}^n \boldsymbol{\mu}_k' \boldsymbol{y}_k - \sum_{i=1}^m \operatorname{Cov}\left(\sum_k \boldsymbol{\mu}_k' \boldsymbol{y}_k, s_i\right) s_i$ is independent of $\{s_1, \dots, s_m\}$. Therefore, noting that $\mathbf{Var}^{-1}(\boldsymbol{z}_i) \preceq \mathbf{Var}^{-1}(\boldsymbol{y}_i)$,[27]

$$\begin{aligned}
\mathbf{Cov}(\boldsymbol{\mu}, \boldsymbol{y}_i)' \, \mathbf{Var}^{-1}(\boldsymbol{z}_i) \, \mathbf{Cov}(\boldsymbol{\mu}, \boldsymbol{y}_i) &\le \mathbf{Cov}(\boldsymbol{\mu}, \boldsymbol{y}_i)' \, \mathbf{Var}^{-1}(\boldsymbol{y}_i) \, \mathbf{Cov}(\boldsymbol{\mu}, \boldsymbol{y}_i) \\
&= \mathbf{Cov}(\boldsymbol{\mu}, \boldsymbol{s})' \, \mathbf{Cov}(\boldsymbol{\mu}, \boldsymbol{s}) = \sum_{i=1}^m \operatorname{Cov}\left(\sum_{k=1}^n \boldsymbol{\mu}_k' \boldsymbol{y}_k, s_i\right)^2 \\
&\le \sum_{i=1}^m \operatorname{Cov}\left(\sum_{k=1}^n \boldsymbol{\mu}_k' \boldsymbol{y}_k, s_i\right)^2 + \operatorname{Var}(s_{m+1}) = \operatorname{Var}(\boldsymbol{\mu}). \qquad \text{(A.1)}
\end{aligned}$$

When $\boldsymbol{\mu}' \boldsymbol{y}$ cannot be expressed as a linear combination of $\boldsymbol{y}_i$, it is easy to see that $s_{m+1} \neq 0$, and (A.1) holds with strict inequality. □

Proof of Proposition 1. For each $i = 1, \dots, n$ and for any $\delta > 0$, define

$$\boldsymbol{f}_{i\delta}(\boldsymbol{Q}) = \frac{\left[\mathbf{Var}(\boldsymbol{z}_i) + \delta I_m - \frac{\mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)'}{\operatorname{Var}(\boldsymbol{Q}) + \operatorname{Var}(u)}\right]^{-1} \left[\mathbf{Cov}(\theta, \boldsymbol{y}_i) - \frac{\operatorname{Cov}(\boldsymbol{Q}, \theta)}{\operatorname{Var}(\boldsymbol{Q}) + \operatorname{Var}(u)} \mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)\right]}{\Delta_i \operatorname{Var}_{\boldsymbol{Q}}(\theta | i)}, \qquad \text{(A.2)}$$

where I_m is the identity in $\mathbb{R}^m$. When $\delta = 0$, the system (A.2) coincides with (14). By Lemma 3 (i) and the fact that $\mathbf{Var}(\boldsymbol{y}_i) \preceq \mathbf{Var}(\boldsymbol{z}_i)$, we see that for any $\boldsymbol{Q} \in \mathbb{R}^{nm}$,

$$\delta I_m \preceq \mathbf{Var}(\boldsymbol{z}_i) + \delta I_m - \frac{\mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)'}{\operatorname{Var}(\boldsymbol{Q}) + \operatorname{Var}(u)}, \qquad \text{(A.3)}$$

along with

$$|\operatorname{Cov}(\boldsymbol{Q}, \theta)| \le \sqrt{\operatorname{Var}(\boldsymbol{Q}) \operatorname{Var}(\theta)} \text{ and } |\mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)| \le \sqrt{\operatorname{Var}(\boldsymbol{Q}) \sum_{j=1}^m \operatorname{Var}(y_{ij})}. \qquad \text{(A.4)}$$

---

[27] Because $\mathbf{Var}(\boldsymbol{z}_i)[\mathbf{Var}^{-1}(\boldsymbol{y}_i) - \mathbf{Var}^{-1}(\boldsymbol{z}_i)]\,\mathbf{Var}(\boldsymbol{z}_i) = \mathbf{Var}(\boldsymbol{\epsilon}_i(j)) + \mathbf{Var}(\boldsymbol{\epsilon}_i(j))\,\mathbf{Var}^{-1}(\boldsymbol{y}_i)\,\mathbf{Var}(\boldsymbol{\epsilon}_i(j))$ is positive semi-definite, $\mathbf{Var}^{-1}(\boldsymbol{z}_i) \preceq \mathbf{Var}^{-1}(\boldsymbol{y}_i)$, with "$\prec$" holding when there is idiosyncratic noise.

Invoking Lemma 1 and (A.2)–(A.4), we see that

$$
\begin{aligned}
|\boldsymbol{f}_{i\delta}(\boldsymbol{Q})| &\leq \frac{\left|\left[\mathbf{Var}(\boldsymbol{z}_i) + \delta I_m - \frac{\mathbf{Cov}(\boldsymbol{Q},\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{Q},\boldsymbol{y}_i)'}{\mathrm{Var}(\boldsymbol{Q})+\mathrm{Var}(u)}\right]^{-1}\right| \left[|\,\mathbf{Cov}(\theta,\boldsymbol{y}_i)| + \frac{\mathrm{Var}(\boldsymbol{Q})\sqrt{\mathrm{Var}(\theta)\sum_{j=1}^m \mathrm{Var}(y_{ij})}}{\mathrm{Var}(\boldsymbol{Q})+\mathrm{Var}(u)}\right]}{\Delta_i \,\mathrm{Var}(\theta|\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n)} \\
&\leq \frac{|\,\mathbf{Cov}(\theta,\boldsymbol{y}_i)| + \sqrt{\mathrm{Var}(\theta)\sum_{j=1}^m \mathrm{Var}(y_{ij})}}{\delta\Delta_i \,\mathrm{Var}(\theta|\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n)} \\
&\leq \max_{1\leq k\leq n} \frac{|\,\mathbf{Cov}(\theta,\boldsymbol{y}_k)| + \sqrt{\mathrm{Var}(\theta)\sum_{j=1}^m \mathrm{Var}(y_{kj})}}{\delta\Delta_k \,\mathrm{Var}(\theta|\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n)} =: B_\delta,
\end{aligned}
$$

where the second inequality uses the fact that $\left|\left[\mathbf{Var}(\boldsymbol{z}_i) + \delta I_m - \frac{\mathbf{Cov}(\boldsymbol{Q},\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{Q},\boldsymbol{y}_i)'}{\mathrm{Var}(\boldsymbol{Q})+\mathrm{Var}(u)}\right]^{-1}\right|$ is bounded above by the inverse of the smallest eigenvalue of the related matrix, which in turn is no greater than $1/\delta$, by (A.3). Noting that all components of the mapping $\boldsymbol{f}_\delta \equiv (\boldsymbol{f}_{1\delta},\ldots,\boldsymbol{f}_{n\delta})$ described in (A.2) are continuous in $\boldsymbol{Q}$, it follows from Brouwer's fixed point theorem that the restriction of that mapping to the subdomain $[-B_\delta, B_\delta]^{nm}$ has a fixed point $\boldsymbol{Q}_\delta$.

Next, as $\delta \to 0$, we claim that any sequence of fixed points $\{\boldsymbol{Q}_\delta\}_{\delta>0}$ is bounded. Evaluating (A.2) at each fixed point $\boldsymbol{Q}_\delta$, transposing terms, and adding over all components i, we have:

$$
\begin{aligned}
&\sum_{i=1}^n \boldsymbol{Q}'_{i\delta}\Big([\mathrm{Var}(\boldsymbol{Q}_\delta) + \mathrm{Var}(u)]\,(\mathbf{Var}(\boldsymbol{z}_i) + \delta I_m) - \mathbf{Cov}(\boldsymbol{Q}_\delta,\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{Q}_\delta,\boldsymbol{y}_i)'\Big)\boldsymbol{Q}_{i\delta}\Delta_i \,\mathrm{Var}_{\boldsymbol{Q}_\delta}(\theta|i) \\
&= \sum_{i=1}^n \boldsymbol{Q}'_{i\delta}\Big([\mathrm{Var}(\boldsymbol{Q}_\delta) + \mathrm{Var}(u)]\,\mathbf{Cov}(\theta,\boldsymbol{y}_i) - \mathrm{Cov}(\boldsymbol{Q}_\delta,\theta)\,\mathbf{Cov}(\boldsymbol{Q}_\delta,\boldsymbol{y}_i)\Big) \\
&= \sum_{i=1}^n \boldsymbol{Q}'_{i\delta}\,\mathbf{Cov}(\theta,\boldsymbol{y}_i)\,\mathrm{Var}(u), \qquad \text{(A.5)}
\end{aligned}
$$

where the second equality invokes Lemma 2 with $\boldsymbol{\mu}$ set equal to $\boldsymbol{Q}_\delta$. Applying the first relation in (A.3) to the left hand side of (A.5), we see that

$$\begin{aligned}\sum_{i=1}^n \boldsymbol{Q}'_{i\delta}\,\mathbf{Cov}(\theta,\boldsymbol{y}_i)\operatorname{Var}(u) &\geq \sum_{i=1}^n \boldsymbol{Q}'_{i\delta}\Big(\operatorname{Var}(u)[\mathbf{Var}(\boldsymbol{z}_i)+\delta I_m]+\delta\operatorname{Var}(\boldsymbol{Q})I_m\Big)\boldsymbol{Q}_{i\delta}\Delta_i\operatorname{Var}_{\boldsymbol{Q}_\delta}(\theta|i)\\ &\geq \sum_{i=1}^n \boldsymbol{Q}'_{i\delta}\big(\operatorname{Var}(u)\,\mathbf{Var}(\boldsymbol{z}_i)\big)\boldsymbol{Q}_{i\delta}\Delta_i\operatorname{Var}_{\boldsymbol{Q}_\delta}(\theta|i)\\ &\geq \operatorname{Var}(u)\operatorname{Var}(\theta|\boldsymbol{y})\sum_{i=1}^n \Delta_i\boldsymbol{Q}'_{i\delta}\,\mathbf{Var}(\boldsymbol{z}_i)\boldsymbol{Q}_{i\delta},\end{aligned}\tag{A.6}$$

where the last inequality uses Lemma 1. Because $\mathbf{Var}(\boldsymbol{z}_i)$ is positive definite, the right-hand side of (A.6) is quadratic in every component of $\boldsymbol{Q}_{i\delta}$, for every i, while the left hand side is linear. This implies that $\{\boldsymbol{Q}_\delta\}_{\delta>0}$ must be uniformly bounded in δ, establishing the claim.

Let $\boldsymbol{Q}$ be any limit point of $\{\boldsymbol{Q}_\delta\}_{\delta>0}$. Observing that the matrix $\mathbf{Var}(\boldsymbol{z}_i)-\frac{\mathbf{Cov}(\boldsymbol{Q},\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{Q},\boldsymbol{y}_i)'}{\operatorname{Var}(\boldsymbol{Q})+\operatorname{Var}(u)}$ is positive definite for any $\boldsymbol{Q}$ and passing to the limit as $\delta\to 0$ in (A.2) with $\boldsymbol{Q}$ set equal to $\boldsymbol{Q}_\delta$, we must conclude that $\boldsymbol{Q}$ solves the system (14), thus completing the proof. □

LEMMA **4.** *In any equilibrium,* $\operatorname{Cov}(\boldsymbol{Q},\theta)\geq 0$.

Proof. Follows from (A.5) and (A.6) by letting $\delta=0$ and replacing $\boldsymbol{Q}_{i\delta}$ with $\boldsymbol{Q}_i$. □

Proof of Proposition 2. For easy reference, we rewrite (15) here:

$$\gamma=\frac{1+\sum_{i=1}^n\frac{\operatorname{Cov}(\boldsymbol{Q},\theta)-\mathbf{Cov}(\theta,\boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{z}_i)\,\mathbf{Cov}(\boldsymbol{Q},\boldsymbol{y}_i)}{\Delta_i\operatorname{Var}(\theta|i)[\operatorname{Var}(\boldsymbol{Q})+\operatorname{Var}(u)-\mathbf{Cov}(\boldsymbol{Q},\boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{z}_i)\,\mathbf{Cov}(\boldsymbol{Q},\boldsymbol{y}_i)]}}{\sum_{i=1}^n\frac{1}{\Delta_i\operatorname{Var}(\theta|i)}}.\tag{A.7}$$

By Lemma 1, the denominator of (A.7) is positive. When $\mathbf{Cov}(\theta,\boldsymbol{y}_i)=\mathbf{0}$ for every i, $\operatorname{Cov}(\boldsymbol{Q},\theta)=0$, so that $\gamma>0$. We next suppose that $\mathbf{Cov}(\theta,\boldsymbol{y}_i)\neq\mathbf{0}$ for some i. Multiplying both sides of (14) by $\mathbf{Cov}(\theta,\boldsymbol{y}_i)'$ and then summing over all i, we obtain

$$\operatorname{Cov}(\boldsymbol{Q},\theta)=\sum_{i=1}^n\frac{\mathbf{Cov}(\theta,\boldsymbol{y}_i)'\Big[\mathbf{Var}(\boldsymbol{z}_i)-\frac{\mathbf{Cov}(\boldsymbol{Q},\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{Q},\boldsymbol{y}_i)'}{\operatorname{Var}(\boldsymbol{Q})+\operatorname{Var}(u)}\Big]^{-1}\Big[\mathbf{Cov}(\theta,\boldsymbol{y}_i)-\frac{\operatorname{Cov}(\boldsymbol{Q},\theta)}{\operatorname{Var}(\boldsymbol{Q})+\operatorname{Var}(u)}\mathbf{Cov}(\boldsymbol{Q},\boldsymbol{y}_i)\Big]}{\Delta_i\operatorname{Var}(\theta|i)},$$

which — given $\mathbf{Cov}(\theta, \boldsymbol{y}_i) \neq \mathbf{0}$ for some i and (A.3) — implies $\mathrm{Cov}(\boldsymbol{Q}, \theta) \neq 0$. Divide through by $\mathrm{Cov}(\boldsymbol{Q}, \theta)$; then

$$\sum_{i=1}^{n} \frac{\mathbf{Cov}(\theta, \boldsymbol{y}_i)' \Big[\mathbf{Var}(\boldsymbol{z}_i) - \frac{\mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)'}{\mathrm{Var}(\boldsymbol{Q}) + \mathrm{Var}(u)}\Big]^{-1} \Big[\frac{\mathbf{Cov}(\theta, \boldsymbol{y}_i)}{\mathrm{Cov}(\boldsymbol{Q}, \theta)} - \frac{\mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)}{\mathrm{Var}(\boldsymbol{Q}) + \mathrm{Var}(u)}\Big]}{\Delta_i \, \mathrm{Var}(\theta|i)} = 1.$$

Substituting this equality into the numerator of (A.7) to eliminate 1, we see that it suffices to show

$$\mathbf{Cov}(\theta, \boldsymbol{y}_i)' \Big[\mathbf{Var}(\boldsymbol{z}_i) - \frac{\mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)'}{\mathrm{Var}(\boldsymbol{Q}) + \mathrm{Var}(u)}\Big]^{-1} \Big[\frac{\mathbf{Cov}(\theta, \boldsymbol{y}_i)}{\mathrm{Cov}(\boldsymbol{Q}, \theta)} - \frac{\mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)}{\mathrm{Var}(\boldsymbol{Q}) + \mathrm{Var}(u)}\Big] \\ + \frac{\mathrm{Cov}(\boldsymbol{Q}, \theta) - \mathbf{Cov}(\theta, \boldsymbol{y}_i)' \, \mathbf{Var}^{-1}(\boldsymbol{z}_i) \, \mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)}{\mathrm{Var}(\boldsymbol{Q}) + \mathrm{Var}(u) - \mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)' \, \mathbf{Var}^{-1}(\boldsymbol{z}_i) \, \mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)} > 0 \tag{A.8}$$

for every i. By Lemma 4 together with $\mathrm{Cov}(\boldsymbol{Q}, \theta) \neq 0$, we have $\mathrm{Cov}(\boldsymbol{Q}, \theta) > 0$. Let $\boldsymbol{a}_i = \mathbf{Var}^{-\frac{1}{2}}(\boldsymbol{z}_i) \, \mathbf{Cov}(\theta, \boldsymbol{y}_i)$ and $\boldsymbol{b}_i = \mathbf{Var}^{-\frac{1}{2}}(\boldsymbol{z}_i) \, \mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)$. To establish (A.8), it is equivalent to show that

$$\boldsymbol{a}_i' \Big[\frac{\{\mathrm{Var}(\boldsymbol{Q}) + \mathrm{Var}(u)\} I_m - \boldsymbol{b}_i \boldsymbol{b}_i'}{\mathrm{Var}(\boldsymbol{Q}) + \mathrm{Var}(u) - \boldsymbol{b}_i' \boldsymbol{b}_i}\Big]^{-1} \Big[\boldsymbol{a}_i \, [\mathrm{Var}(\boldsymbol{Q}) + \mathrm{Var}(u)] - \boldsymbol{b}_i \, \mathrm{Cov}(\boldsymbol{Q}, \theta)\Big] + \mathrm{Cov}(\boldsymbol{Q}, \theta)^2 - \mathrm{Cov}(\boldsymbol{Q}, \theta) \boldsymbol{a}_i' \boldsymbol{b}_i > 0. \tag{A.9}$$

Observe that

$$\Big[\frac{\{\mathrm{Var}(\boldsymbol{Q}) + \mathrm{Var}(u)\} I_m - \boldsymbol{b}_i \boldsymbol{b}_i'}{\mathrm{Var}(\boldsymbol{Q}) + \mathrm{Var}(u) - \boldsymbol{b}_i' \boldsymbol{b}_i}\Big]^{-1} = \frac{\mathrm{Var}(\boldsymbol{Q}) + \mathrm{Var}(u) - \boldsymbol{b}_i' \boldsymbol{b}_i}{\mathrm{Var}(\boldsymbol{Q}) + \mathrm{Var}(u)} \Big[I_m - \frac{\boldsymbol{b}_i \boldsymbol{b}_i'}{\mathrm{Var}(\boldsymbol{Q}) + \mathrm{Var}(u)}\Big]^{-1} \\ = \Big[1 - \frac{\boldsymbol{b}_i' \boldsymbol{b}_i}{\mathrm{Var}(\boldsymbol{Q}) + \mathrm{Var}(u)}\Big] \Bigg[I_m + \frac{1}{1 - \frac{\boldsymbol{b}_i' \boldsymbol{b}_i}{\mathrm{Var}(\boldsymbol{Q}) + \mathrm{Var}(u)}} \frac{\boldsymbol{b}_i \boldsymbol{b}_i'}{\mathrm{Var}(\boldsymbol{Q}) + \mathrm{Var}(u)}\Bigg].$$

Substitute this equality into (A.9), which then becomes:

$$(\mathrm{Cov}(\boldsymbol{Q}, \theta) - \boldsymbol{a}_i' \boldsymbol{b}_i)^2 + \boldsymbol{a}_i' \boldsymbol{a}_i \{\mathrm{Var}(\boldsymbol{Q}) + \mathrm{Var}(u)\} \Big[1 - \frac{\boldsymbol{b}_i' \boldsymbol{b}_i}{\mathrm{Var}(\boldsymbol{Q}) + \mathrm{Var}(u)}\Big] > 0. \tag{A.10}$$

It follows from Lemma 3 (ii) that $\boldsymbol{b}_i' \boldsymbol{b}_i = \mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)' \, \mathbf{Var}^{-1}(\boldsymbol{z}_i) \, \mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i) \leq \mathrm{Var}(\boldsymbol{Q})$ and consequently, (A.10) is indeed true, completing the proof. □

Proof of Proposition 4. If $\mathbf{Cov}(\theta, \boldsymbol{y}_i) \neq \mathbf{0}$ for all i, $p = \frac{\mathrm{Var}(\theta)}{\sum_{k=1}^{n} \frac{1}{\Delta_k}}(u - X)$ is obviously the unique linear equilibrium. So assume $\mathbf{Cov}(\theta, \boldsymbol{y}_i) \neq \mathbf{0}$ for some i. We first focus on the case in which $\mathbf{Var}(\boldsymbol{y})$ is positive definite. In this case, the discussion in the main text before and after the statement of Proposition 1 makes it clear that linear equilibria are intimately connected to solutions

$\boldsymbol{Q}$ to (14). Specifically, every linear equilibrium price function can be mapped to a vector $\boldsymbol{Q}$ that solves (14), and conversely, every solution to (14) generates a particular linear equilibrium price function.

In what follows we fix all parameters of the model but consider a variety of risk attitudes. Specifically, fix some number $a > 0$. Let $\mathcal{P}(a)$ be the set of all linear equilibrium price functions under some configuration $\{\rho(r)\}$, with $\rho(r) \geq a$ for all types. We claim that the set

$$\mathcal{Q}(a) \equiv \left\{ \boldsymbol{Q} \neq 0 \middle| \boldsymbol{Q} = \frac{\boldsymbol{\pi}}{\gamma} \text{ for some linear equilibrium price } p = \boldsymbol{\pi}'\boldsymbol{y} + \gamma u + c \text{ in } \mathcal{P}(a) \right\}$$

is well-defined and bounded. It is well-defined because $\gamma > 0$ by Proposition 2. Let $\{\boldsymbol{Q}_k\}$ be some arbitrary sequence in $\mathcal{Q}(a)$. By taking a subsequence if needed, suppose that $\boldsymbol{Q}_k/|\boldsymbol{Q}_k| \to \boldsymbol{\mu}$ for some $\boldsymbol{\mu}$ with $|\boldsymbol{\mu}| = 1$. Consider a corresponding sequence of price functions in $\mathcal{P}(a)$, with accompanying values of $\{\rho^k(r)\}$ and therefore $\{\Delta_i^k\}$. By (6),

$$p^k = \frac{1}{\sum_{i=1}^n \frac{1-\beta_i^k}{\Delta_i^k \operatorname{Var}^k(\theta|i)}} \Big(\sum_{i=1}^n \frac{[\boldsymbol{\alpha}_i^k]'\boldsymbol{y}_i - \beta_i^k c^k}{\Delta_i^k \operatorname{Var}^k(\theta|i)} + u - X \Big), \tag{A.11}$$

where $\boldsymbol{\alpha}_i^k$ and β_i^k are such that $\mathbb{E}(\theta|\boldsymbol{z}_i(j), p^k) = [\boldsymbol{\alpha}_i^k]'\boldsymbol{z}_i(j) + \beta_i^k(p^k - c^k)$. (A.11) implies that

$$\boldsymbol{Q}_k'\boldsymbol{y} = \sum_{i=1}^n \frac{[\boldsymbol{\alpha}_i^k]'\boldsymbol{y}_i}{\Delta_i \operatorname{Var}^k(\theta|i)} \tag{A.12}$$

for all k. Because $p^k = \boldsymbol{\pi}_k'\boldsymbol{y} + \gamma_k u + c_k$ is informationally equivalent to $\boldsymbol{Q}_k'\boldsymbol{y}/|\boldsymbol{Q}_k| + u/|\boldsymbol{Q}_k|$, we have $\mathbb{E}(\theta|\boldsymbol{z}_i(j), p^k) \to \mathbb{E}(\theta|\boldsymbol{z}_i(j), \boldsymbol{\mu}'\boldsymbol{y})$ almost surely and $\operatorname{Var}^k(\theta|i) = \operatorname{Var}(\theta|\boldsymbol{z}_i(j), p^k) \to \operatorname{Var}(\theta|\boldsymbol{z}_i(j), \boldsymbol{\mu}'\boldsymbol{y})$. These objects have upper bounds that are uniform over $\boldsymbol{\mu}$, so it follows that $\boldsymbol{\alpha}_i^k$ is uniformly bounded in k (no matter which sequence $\{\boldsymbol{Q}_k\}$ we consider, while $\operatorname{Var}^k(\theta|i)$ is uniformly bounded below. Applying (A.12), that proves the uniform boundedness of any sequence $\{\boldsymbol{Q}_k\}$ in $\mathcal{Q}(a)$.

The map $\boldsymbol{f}_i$ in (14) is continuously differentiable, so for any $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in \mathbb{R}^{nm}$,

$$\boldsymbol{f}(\boldsymbol{Q}_1) - \boldsymbol{f}(\boldsymbol{Q}_2) = \Big(\int_0^1 D\boldsymbol{f}(\boldsymbol{Q}_2 + t(\boldsymbol{Q}_1 - \boldsymbol{Q}_2))dt \Big)(\boldsymbol{Q}_1 - \boldsymbol{Q}_2), \tag{A.13}$$

where $D\boldsymbol{f}$ denotes the Jacobian matrix of $\boldsymbol{f}$ and the integral of a matrix is to be understood componentwise. Because $\mathcal{Q}(a)$ is bounded, it follows from (A.13) that $|\boldsymbol{f}(\boldsymbol{Q}_1)-\boldsymbol{f}(\boldsymbol{Q}_2)| < |\boldsymbol{Q}_1-\boldsymbol{Q}_2|$ for any $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in \mathcal{Q}(a)$ when Δ_i's are sufficiently large, or equivalently when traders are sufficiently risk averse. That is, when Δ_i's are sufficiently large, $\boldsymbol{f}$ is a compressive mapping on the set $\mathcal{Q}(a)$, implying that the system of equations (14) has a unique fixed-point. So, given the equivalence analysis already conducted, there is a unique linear equilibrium.

We now extend the argument to $\boldsymbol{y}$ with $\mathbf{Var}(\boldsymbol{y})$ not of full rank. Select a maximal linearly independent subset of $\{\boldsymbol{y}_{ij}, i = 1, \ldots, n, j = 1, \ldots, m\}$, denoted by $\bar{\boldsymbol{y}}$. Then $\mathbf{Var}(\bar{\boldsymbol{y}})$ is positive definite, and every y_{ij} not in $\bar{\boldsymbol{y}}$ can be expressed as a linear combination of $\bar{\boldsymbol{y}}$. For any linear equilibrium $p = \boldsymbol{\pi}'\boldsymbol{y} + \gamma u + c$, we can rewrite $\boldsymbol{\pi}'\boldsymbol{y}$ as a linear combination $\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}}$ of $\bar{\boldsymbol{y}}$. So to show uniqueness, it suffices to prove that the vector $\bar{\boldsymbol{\pi}}$ with $p = \bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}} + \gamma u + c$ a linear equilibrium is unique. Inserting the price function $p = \bar{\boldsymbol{\pi}}'\bar{\boldsymbol{y}} + \gamma u + c$ into (6), we can get a system of equations involving $\bar{\boldsymbol{Q}} := \bar{\boldsymbol{\pi}}/\gamma$, which is similar to (14). The remainder of the proof consists in applying similar boundedness arguments, parallel to the positive definiteness case. □

Proof of Proposition 5. The sign of $\mathrm{Cov}(\theta, p)$ is the same as that of $\mathrm{Cov}(\boldsymbol{Q}, \theta)$ because $\gamma > 0$, as shown in Proposition 2, and because u is independent of θ. That $\mathrm{Cov}(\boldsymbol{Q}, \theta) \geq 0$ then follows from Lemma 4. We now show that equality holds if and only if $\mathbf{Cov}(\theta, \boldsymbol{y}_i) = \mathbf{0}$ for all i. Sufficiency is immediate by noting that $\mathrm{Cov}(\boldsymbol{Q}, \theta) = \sum_{i=1}^n \boldsymbol{Q}_i' \,\mathbf{Cov}(\theta, \boldsymbol{y}_i)$. For necessity, assume that $\mathrm{Cov}(\boldsymbol{Q}, \theta) = 0$. Using (14), we see that for each $i = 1, \ldots, n$,

$$\mathbf{Cov}(\theta, \boldsymbol{y}_i)'\boldsymbol{Q}_i = \frac{\mathbf{Cov}(\theta, \boldsymbol{y}_i)'\Big[\mathbf{Var}(\boldsymbol{z}_i) - \frac{\mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)'}{\mathrm{Var}(\boldsymbol{Q}) + \mathrm{Var}(u)}\Big]^{-1}\mathbf{Cov}(\theta, \boldsymbol{y}_i)}{\Delta_i \,\mathrm{Var}(\theta|i)}.$$

Summing over all i, we must conclude that

$$\sum_{i=1}^n \frac{\mathbf{Cov}(\theta, \boldsymbol{y}_i)'\Big[\mathbf{Var}(\boldsymbol{z}_i) - \frac{\mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{Q}, \boldsymbol{y}_i)'}{\mathrm{Var}(\boldsymbol{Q}) + \mathrm{Var}(u)}\Big]^{-1}\mathbf{Cov}(\theta, \boldsymbol{y}_i)}{\Delta_i \,\mathrm{Var}(\theta|i)} = \sum_{i=1}^n \mathbf{Cov}(\theta, \boldsymbol{y}_i)'\boldsymbol{Q}_i = \mathrm{Cov}(\boldsymbol{Q}, \theta) = 0,$$

implying that $\mathbf{Cov}(\theta, \boldsymbol{y}_i) = \mathbf{0}$ for every i. □

Proof of Proposition 6. In this proof, we consider a sequence of equilibria, indexed by t, along which the variance of noise trade vanishes parametrically. So in what follows, all equilibrium values will be indexed by t. In addition, we write $\mathrm{Var}_t(u)$ for the variance of noise trade, and $\mathrm{Var}_t(\theta|i)$ for the variance of θ conditional on group i's price and private signal, because this estimate will move as the equilibrium price function changes.

We first dispose of the case in which $\mathbf{Cov}(\theta, \boldsymbol{y}) = \mathbf{0}$. In this case, just as in the proof of Proposition 5, $\boldsymbol{Q_t} = \mathbf{0}$ and $\gamma_t = \frac{\mathrm{Var}(\theta)}{\sum_{k=1}^n \frac{1}{\Delta_k}}$, with $p_t = \gamma_t u + c_t$ for some sequence of intercepts $\{c_t\}$ that each solves (8). There is no dependence on the signals either in equilibrium or under the full-information aggregator, so Part (i) follows trivially. Moreover, in the case under consideration, $\gamma_t \mathrm{Var}_t(u) \to 0$ and $c_t = -[\sum_{k=1}^n \frac{1}{\Delta_k}]^{-1} \mathrm{Var}(\theta) X$ for all t, and Part (ii) follows as well.

So in what follows, assume $\mathbf{Cov}(\theta, \boldsymbol{y}_i) \neq \mathbf{0}$ for some i (and therefore $\mathbf{Cov}(\theta, \boldsymbol{y}_i) \geq \mathbf{0}$, by our convention on signals). Then by Proposition 5, $\boldsymbol{\pi}_t \neq \mathbf{0}$ and so $\boldsymbol{Q}_t \neq \mathbf{0}$. Define $\boldsymbol{\mu}_t \equiv \boldsymbol{\pi}_t / |\boldsymbol{\pi}_t|$ and $\iota_t \equiv \gamma_t / |\boldsymbol{\pi}_t|$. Note that $\iota_t > 0$ by Proposition 2. Then by (10) and (11), we have

$$\boldsymbol{\alpha}_{it} = \left[\mathbf{Var}(\boldsymbol{z}_i) - \frac{\mathbf{Cov}(\boldsymbol{\pi}_t, \boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{\pi}_t, \boldsymbol{y}_i)'}{\mathrm{Var}(\boldsymbol{\pi}_t) + \gamma_t^2 \mathrm{Var}_t(u)}\right]^{-1} \left[\mathbf{Cov}(\theta, \boldsymbol{y}_i) - \frac{\mathrm{Cov}(\boldsymbol{\pi}_t, \theta)\,\mathbf{Cov}(\boldsymbol{\pi}_t, \boldsymbol{y}_i)}{\mathrm{Var}(\boldsymbol{\pi}_t) + \gamma_t^2 \mathrm{Var}_t(u)}\right] \tag{A.14}$$

$$= \left[\mathbf{Var}(\boldsymbol{z}_i) - \frac{\mathbf{Cov}(\boldsymbol{\mu}_t, \boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{\mu}_t, \boldsymbol{y}_i)'}{\mathrm{Var}(\boldsymbol{\mu}_t) + \iota_t^2 \mathrm{Var}_t(u)}\right]^{-1} \left[\mathbf{Cov}(\theta, \boldsymbol{y}_i) - \frac{\mathrm{Cov}(\boldsymbol{\mu}_t, \theta)\,\mathbf{Cov}(\boldsymbol{\mu}_t, \boldsymbol{y}_i)}{\mathrm{Var}(\boldsymbol{\mu}_t) + \iota_t^2 \mathrm{Var}_t(u)}\right], \tag{A.15}$$

$$\beta_{it} = \frac{\mathrm{Cov}(\boldsymbol{\pi}_t, \theta) - \mathbf{Cov}(\theta, \boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{z}_i)\,\mathbf{Cov}(\boldsymbol{\pi}_t, \boldsymbol{y}_i)}{\mathrm{Var}(\boldsymbol{\pi}_t) + \gamma_t^2 \mathrm{Var}_t(u) - \mathbf{Cov}(\boldsymbol{\pi}_t, \boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{z}_i)\,\mathbf{Cov}(\boldsymbol{\pi}_t, \boldsymbol{y}_i)} \tag{A.16}$$

$$= \frac{1}{|\boldsymbol{\pi}_t|} \frac{\mathrm{Cov}(\boldsymbol{\mu}_t, \theta) - \mathbf{Cov}(\theta, \boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{z}_i)\,\mathbf{Cov}(\boldsymbol{\mu}_t, \boldsymbol{y}_i)}{\mathrm{Var}(\boldsymbol{\mu}_t) + \iota_t^2 \mathrm{Var}_t(u) - \mathbf{Cov}(\boldsymbol{\mu}_t, \boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{z}_i)\,\mathbf{Cov}(\boldsymbol{\mu}_t, \boldsymbol{y}_i)}. \tag{A.17}$$

Notice that $|\boldsymbol{\mu}_t| = 1$. Also, by Lemma 1, $0 < \mathrm{Var}(\theta|\boldsymbol{y}) \leq \mathrm{Var}_t(\theta|i) \leq \mathrm{Var}(\theta)$, and indeed, that $\mathrm{Var}(\theta|\boldsymbol{y})$ is just a constant independent of the realization of $\boldsymbol{y}$. So we can presume (using a subsequence if necessary, but retaining the original notation) that $\boldsymbol{\mu}_t \to \boldsymbol{\mu}$ for some $\boldsymbol{\mu}$ with $|\boldsymbol{\mu}| = 1$, and $\mathrm{Var}_t(\theta|i) \to v_i \in [\mathrm{Var}(\theta|\boldsymbol{y}), \mathrm{Var}(\theta)]$.

Claim 1: $\iota_t^2 \mathrm{Var}_t(u) \to 0$.

Suppose not. Then (using a subsequence of t if needed) we can presume that $\iota_t^2 \mathrm{Var}_t(u)$ is bounded away from zero. Because $\mathrm{Var}_t(u) \to 0$ and $\iota_t > 0$ by Proposition 2, it must be that $\iota_t \to \infty$. By

(9) and the definition of $\boldsymbol{\mu}$,

$$\boldsymbol{\mu}_{it} = \frac{\iota_t}{\Delta_i \operatorname{Var}_t(\theta|i)} \boldsymbol{\alpha}_{it} \tag{A.18}$$

for every i and t, and so, because $\{\operatorname{Var}_t(\theta|i)\}$ and $\{\boldsymbol{\mu}_{it}\}$ are bounded, we have $\boldsymbol{\alpha}_{it} \to \mathbf{0}$ for all i. Using $\Gamma_i(\boldsymbol{\mu}) = \mathbf{Cov}(\theta, \boldsymbol{y}_i) \operatorname{Var}(\boldsymbol{\mu}) - \operatorname{Cov}(\boldsymbol{\mu}, \theta) \mathbf{Cov}(\boldsymbol{\mu}, \boldsymbol{y}_i)$ in (A.15), we therefore see that

$$\boldsymbol{\alpha}_{it} = \Big(\big[\operatorname{Var}(\boldsymbol{\mu}_t) + \iota_t^2 \operatorname{Var}_t(u)\big] \mathbf{Var}(\boldsymbol{z}_i) - \mathbf{Cov}(\boldsymbol{\mu}_t, \boldsymbol{y}_i) \mathbf{Cov}(\boldsymbol{\mu}_t, \boldsymbol{y}_i)' \Big)^{-1} [\mathbf{Cov}(\theta, \boldsymbol{y}_i) \iota_t^2 \operatorname{Var}_t(u) + \Gamma_i(\boldsymbol{\mu}_t)] \to \mathbf{0} \tag{A.19}$$

for all i. Because there exists at least one nonzero vector of $\mathbf{Cov}(\theta, \boldsymbol{y}_i)$, say at $i = k$, it follows from (A.19) that $\iota_t^2 \operatorname{Var}_t(u)$ is bounded. (For if not, $\boldsymbol{\alpha}_{kt} \to \mathbf{Var}^{-1}(z_k) \mathbf{Cov}(\theta, \boldsymbol{y}_k) \neq \mathbf{0}$ along some subsequence of $\{t\}$, a contradiction.)

Thus, each limit point G of $\iota_t^2 \operatorname{Var}_t(u)$ must be finite, *and* it is strictly positive by our contradiction assumption. For any such limit point G, (A.19) tells us that $\mathbf{Cov}(\theta, \boldsymbol{y}_i) G + \Gamma_i(\boldsymbol{\mu}) = \mathbf{0}$ for all i. Summing over all indices i, $G \sum_i \boldsymbol{\mu}_i' \mathbf{Cov}(\theta, \boldsymbol{y}_i) + \sum_i \boldsymbol{\mu}_i' \Gamma_i(\boldsymbol{\mu}) = 0$. By Lemma 2, $\sum_i \boldsymbol{\mu}_i' \Gamma_i(\boldsymbol{\mu}) = 0$, and therefore $G \sum_i \boldsymbol{\mu}_i' \mathbf{Cov}(\theta, \boldsymbol{y}_i) = 0$, or (because $G > 0$) $\operatorname{Cov}(\boldsymbol{\mu}, \theta) = 0$.

Now, using (A.18) and the equality in (A.19), we have

$$\begin{aligned} &\Big[\{\operatorname{Var}(\boldsymbol{\mu}_t) + \iota_t^2 \operatorname{Var}_t(u)\} \mathbf{Var}(\boldsymbol{z}_i) - \mathbf{Cov}(\boldsymbol{\mu}_t, \boldsymbol{y}_i) \mathbf{Cov}(\boldsymbol{\mu}_t, \boldsymbol{y}_i)'\Big] \boldsymbol{\mu}_{it} \\ &= \frac{\iota_t}{\Delta_i \operatorname{Var}_t(\theta|i)} \Big[\{\operatorname{Var}(\boldsymbol{\mu}_t) + \iota_t^2 \operatorname{Var}_t(u)\} \mathbf{Cov}(\theta, \boldsymbol{y}_i) - \operatorname{Cov}(\boldsymbol{\mu}_t, \theta) \mathbf{Cov}(\boldsymbol{\mu}_t, \boldsymbol{y}_i)\Big]. \end{aligned} \tag{A.20}$$

Because $\{\boldsymbol{\mu}_{it}\}$ and $\iota_t^2 \operatorname{Var}_t(u)$ are bounded, so is the left hand side of (A.20). On the other hand, the right hand side of (A.20) diverges to ∞ with t for any i with $\mathbf{Cov}(\theta, \boldsymbol{y}_i) \neq \mathbf{0}$, because $\iota_t \to \infty$ and $\operatorname{Cov}(\boldsymbol{\mu}, \theta) = 0$, as just shown above. But that is a contradiction. Therefore, our claim is true and $\iota_t^2 \operatorname{Var}_t(u) \to 0$.

Claim 2: $\iota_t \to \infty$, and for each i, $\boldsymbol{\alpha}_{it} \to \mathbf{0}$, $\Gamma_i(\boldsymbol{\mu}) = \mathbf{0}$.

For each i and t, let $B_{it} := [\operatorname{Var}(\boldsymbol{\mu}_t) + \iota_t^2 \operatorname{Var}_t(u)] \mathbf{Var}(\boldsymbol{z}_i) - \mathbf{Cov}(\boldsymbol{\mu}_t, \boldsymbol{y}_i) \mathbf{Cov}(\boldsymbol{\mu}_t, \boldsymbol{y}_i)'$. Recall (9) for each i and index t, and multiply both sides by $\boldsymbol{\alpha}_{it}' B_{it}$ to obtain:

$$\boldsymbol{\alpha}_{it}' B_{it} \boldsymbol{\pi}_{it} = \frac{\gamma_t}{\Delta_i \operatorname{Var}_t(\theta|i)} \boldsymbol{\alpha}_{it}' B_{it} \boldsymbol{\alpha}_{it}.$$

Note that $\gamma_t > 0$ (by Proposition 2) and that B_{it} is positive semi-definite by Lemma 3 (i) and $\mathbf{Var}(\boldsymbol{y}_i) \preceq \mathbf{Var}(\boldsymbol{z}_i)$. It follows that for each i and t, $\boldsymbol{\alpha}'_{it} B_{it} \boldsymbol{\pi}_{it}$ — and so $\boldsymbol{\alpha}'_{it} B_{it} \boldsymbol{\mu}_{it}$ as well — are nonnegative for any t and all indices i.

Now multiply both sides of (A.19) by B_{it}, and then pass to the limit as $t \to \infty$, using Claim 1, to obtain for each i:

$$B_{it}\boldsymbol{\alpha}_{it} - \Gamma_i(\boldsymbol{\mu}_t) \to 0. \tag{A.21}$$

Multiplying both sides by $\boldsymbol{\mu}'_{it}$ and summing over i, we see that

$$\lim_{t\to\infty} \Big(\sum_i \boldsymbol{\mu}'_{it} B_{it} \boldsymbol{\alpha}_{it} - \sum_i \boldsymbol{\mu}'_{it} \Gamma_i(\boldsymbol{\mu}_t) \Big) = \lim_{t\to\infty} \sum_i \boldsymbol{\mu}'_{it} B_{it} \boldsymbol{\alpha}_{it} = 0,$$

where the latter equality follows from Lemma 2. Because we have just established that $\boldsymbol{\alpha}'_{it} B_{it} \boldsymbol{\mu}_{it} = \boldsymbol{\mu}'_{it} B_{it} \boldsymbol{\alpha}_{it} \geq 0$ for any t and every i, it follows that for all i,

$$\lim_{t\to\infty} \boldsymbol{\mu}'_{it} B_{it} \boldsymbol{\alpha}_{it} = 0. \tag{A.22}$$

Next, multiplying both sides of (A.18) by $\boldsymbol{\mu}'_{it} B_{it}$, we have

$$\boldsymbol{\mu}'_{it} B_{it} \boldsymbol{\mu}_{it} = \frac{\iota_t}{\Delta_i \operatorname{Var}_t(\theta|i)} \boldsymbol{\mu}'_{it} B_{it} \boldsymbol{\alpha}_{it}. \tag{A.23}$$

We now claim that $\iota_t \to \infty$. If not, then taking $t \to \infty$ (along some subsequence for which ι_t tends to a finite number) over both sides of (A.23) and using (A.22) along with $\operatorname{Var}_t(\theta|i) \to v_i \in (0, \infty)$, we have $\boldsymbol{\mu}'_i B_i \boldsymbol{\mu}_i = 0$, or

$$\operatorname{Var}(\boldsymbol{\mu}) \operatorname{Var}(\boldsymbol{\mu}'_i \boldsymbol{z}_i) - \mathbf{Cov}(\boldsymbol{\mu}, \boldsymbol{\mu}'_i \boldsymbol{y}_i)^2 = 0 \tag{A.24}$$

for all i, where $B_i = \operatorname{Var}(\boldsymbol{\mu}) \mathbf{Var}(\boldsymbol{z}_i) - \mathbf{Cov}(\boldsymbol{\mu}, \boldsymbol{y}_i) \mathbf{Cov}(\boldsymbol{\mu}, \boldsymbol{y}_i)'$. Because $\boldsymbol{\mu} \neq \mathbf{0}$, there exists k such that $\boldsymbol{\mu}_k \neq \mathbf{0}$. When there is idiosyncratic noise (i.e., $\mathbf{Var}(\boldsymbol{\epsilon}_i(j))$ is positive definite for all i and j), (A.24) cannot hold for k because

$$\mathbf{Cov}(\boldsymbol{\mu}, \boldsymbol{\mu}'_k \boldsymbol{y}_k)^2 \leq \operatorname{Var}(\boldsymbol{\mu}) \operatorname{Var}(\boldsymbol{\mu}'_k \boldsymbol{y}_k) < \operatorname{Var}(\boldsymbol{\mu})[\operatorname{Var}(\boldsymbol{\mu}'_k \boldsymbol{y}_k) + \boldsymbol{\mu}'_k \mathbf{Var}(\boldsymbol{\epsilon}_k(j)) \boldsymbol{\mu}_k] = \operatorname{Var}(\boldsymbol{\mu}) \operatorname{Var}(\boldsymbol{\mu}'_k \boldsymbol{z}_k).$$

But neither can (A.24) hold when there is no idiosyncratic noise (i.e., $\boldsymbol{\epsilon}_i(j) = \mathbf{0}$ for all i and j). For (A.24) would imply that for every i, $\sum_j \boldsymbol{\mu}'_j \boldsymbol{y}_j = \boldsymbol{\mu}'_i \boldsymbol{y}_i$ with probability one. That means that for every i, $\boldsymbol{\mu}'_i \boldsymbol{y}_i = 0$ with probability one. But this is impossible, given $\boldsymbol{\mu}_k \neq \mathbf{0}$ and the assumption of positive definiteness of $\mathbf{Var}(\boldsymbol{y}_k)$ in Assumption 1. Hence, $\iota_t \to \infty$. Reapplying (A.18) to all other indices i and recalling that $\{\boldsymbol{\mu}_{it}\}$ is bounded, we can further conclude that $\boldsymbol{\alpha}_{it} \to \mathbf{0}$ for *every* i and then, by (A.21), $\Gamma_i(\boldsymbol{\mu}) = 0$ for every i as well.

Claim 3: $\operatorname{Cov}(\boldsymbol{\mu}, \theta) > 0$.

Certainly, $\operatorname{Cov}(\boldsymbol{\mu}, \theta) \geq 0$, because

$$\operatorname{Cov}(\boldsymbol{\mu}_t, \theta) = \frac{1}{|\boldsymbol{\pi}_t|} \operatorname{Cov}(\boldsymbol{\pi}_t, \theta) = \frac{\gamma_t}{|\boldsymbol{\pi}_t|} \operatorname{Cov}(\boldsymbol{Q}_t, \theta) = \iota_t \operatorname{Cov}(\boldsymbol{Q}_t, \theta) > 0$$

for all sufficiently large t (because $\iota_t \to \infty$ by Claim 2 and $\operatorname{Cov}(\boldsymbol{Q}_t, \theta) > 0$ by Proposition 5). If, however, $\operatorname{Cov}(\boldsymbol{\mu}, \theta) = 0$, then multiplying both sides of (A.15) by the matrix $\mathbf{Var}(\boldsymbol{z}_i) - \frac{\mathbf{Cov}(\boldsymbol{\mu}_t, \boldsymbol{y}_i) \, \mathbf{Cov}(\boldsymbol{\mu}_t, \boldsymbol{y}_i)'}{\operatorname{Var}(\boldsymbol{\mu}_t) + \iota_t^2 \operatorname{Var}_t(u)}$ and subsequently passing to the limit, using Claim 1 and $\boldsymbol{\alpha}_{it} \to \mathbf{0}$ in Claim 2, we see that

$$\lim_{t\to\infty} \left[\mathbf{Var}(\boldsymbol{z}_i) - \frac{\mathbf{Cov}(\boldsymbol{\mu}_t, \boldsymbol{y}_i) \, \mathbf{Cov}(\boldsymbol{\mu}_t, \boldsymbol{y}_i)'}{\operatorname{Var}(\boldsymbol{\mu}_t) + \iota_t^2 \operatorname{Var}_t(u)} \right] \boldsymbol{\alpha}_{it} = \mathbf{Cov}(\theta, \boldsymbol{y}_i) = \mathbf{0}$$

for any i, which contradicts the assumption that $\mathbf{Cov}(\theta, \boldsymbol{y}_i) \neq \mathbf{0}$ for some i. So $\operatorname{Cov}(\boldsymbol{\mu}, \theta) > 0$.

Claim 4: $\lim_{t\to\infty} |\boldsymbol{\pi}_t| = \dfrac{\operatorname{Cov}(\boldsymbol{\mu}, \theta)}{\operatorname{Var}(\boldsymbol{\mu})}$ and $\lim_{t\to\infty} \beta_{it} = 1$ for all i.

By the Law of Total Variance, we know that for every t and information group i,

$$\operatorname{Var}(\theta) = \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{z}_i, p_t)) + \mathbb{E}(\operatorname{Var}(\theta|\boldsymbol{z}_i, p_t)) = \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{z}_i, p_t)) + \operatorname{Var}_t(\theta|i), \tag{A.25}$$

where we recall that $\operatorname{Var}(\theta|\boldsymbol{z}_i, p_t)$ is independent of the realizations of $\boldsymbol{z}_i$ and p. Combining (A.25) with (12), we must conclude that for every i and t,

$$\begin{aligned} \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{z}_i, p_t)) = \operatorname{Var}(\theta) - \operatorname{Var}_t(\theta|i) &= \boldsymbol{\alpha}_{it}' \, \mathbf{Cov}(\theta, \boldsymbol{y}_i) + \beta_{it} \operatorname{Cov}(\boldsymbol{\pi}_t, \theta) \\ &= \boldsymbol{\alpha}_{it}' \, \mathbf{Cov}(\theta, \boldsymbol{y}_i) + \beta_{it} |\boldsymbol{\pi}_t| \operatorname{Cov}(\boldsymbol{\mu}_t, \theta). \end{aligned} \tag{A.26}$$

Passing to the limit in (A.26) as $t \to \infty$, noting that $\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{z}_i, p_t))$ is bounded above by $\operatorname{Var}(\theta)$, using Claim 2 (every limit point of $\boldsymbol{\alpha}_{it}$ is zero), and using Claim 3, we must conclude that $\beta_{it}|\boldsymbol{\pi}_t|$ is bounded in t for every i. It's also the case that $\beta_{it}|\boldsymbol{\pi}_t|$ only has non-zero limit points. For if 0 were to be a limit point of $\beta_{it}|\boldsymbol{\pi}_t|$, it would then follow from (A.17) and $\operatorname{Cov}(\boldsymbol{\mu}, \theta) > 0$ that

$\mathbf{Cov}(\theta, \boldsymbol{y}_i) \neq \mathbf{0}$. But then the last inequality in Lemma 1 must hold strictly *and* uniformly in the price function, so that $\mathrm{Var}(\theta) > \mathrm{Var}(\theta|i)$ for any limit point $\mathrm{Var}_t(\theta|i)$. That would contradict (A.26), as the right hand must go to zero when $\beta_{it}|\boldsymbol{\pi}_t| \to 0$ ($\boldsymbol{\alpha_{it}} \to \mathbf{0}$ by Claim 2), while the middle term remains bounded away from 0 by virtue of $\mathrm{Var}(\theta) - \mathrm{Var}(\theta|i) > 0$. In summary, $\beta_{it}|\boldsymbol{\pi}_t|$ is bounded in t for every i and has non-zero limit points. Pick any such limit point ℓ_i for each i; in other words, extract a subsequence (but retain notation t) such that for every i,

$$\beta_{it}|\boldsymbol{\pi}_t| \to \ell_i \neq 0, \tag{A.27}$$

in addition to the convergence of $\boldsymbol{\mu}_t$ to $\boldsymbol{\mu}$ and $\boldsymbol{\alpha}_{it} \to \mathbf{0}$.

Now, there is a second way to write $\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{z}_i, p_t))$. Recall from (4) that for every i and t, $\mathbb{E}(\theta|\boldsymbol{z}_i, p_t) = \boldsymbol{\alpha}'_{it}\boldsymbol{z}_i + \beta_{it}(p_t - c_t)$, and so

$$\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{z}_i, p_t)) = \mathrm{Var}(\boldsymbol{\alpha}'_{it}\boldsymbol{z}_i + \beta_{it}p_t). \tag{A.28}$$

Passing to the limit with t in both (A.26) and (A.28), and using the fact that $\boldsymbol{\alpha}_{it} \to \mathbf{0}$ (Claim 2), we see that for each i,

$$\begin{aligned}\ell_i \,\mathrm{Cov}(\boldsymbol{\mu}, \theta) &= \lim_{t\to\infty} \mathrm{Var}(\boldsymbol{\alpha}'_{it}\boldsymbol{z}_i + \beta_{it}p_t) = \lim_{t\to\infty} \mathrm{Var}(\beta_{it}p_t) = \lim_{t\to\infty} \beta_{it}^2\Big[\mathrm{Var}\Big(\sum_k \boldsymbol{\pi}'_{kt}\boldsymbol{y}_k\Big) + \gamma_t^2\,\mathrm{Var}_t(u)\Big] \\ &= \lim_{t\to\infty} \beta_{it}^2|\boldsymbol{\pi}_t|^2\left[\mathrm{Var}\Big(\sum_k \boldsymbol{\mu}'_{kt}\boldsymbol{y}_k\Big) + \iota_t^2\,\mathrm{Var}_t(u)\right] = \ell_i^2\,\mathrm{Var}(\boldsymbol{\mu}),\end{aligned}$$

where the third equality uses the form of the price function and the independence of noise trade, and the fourth equality invokes Claim 1 ($\iota_t^2\,\mathrm{Var}_t(u) \to 0$) along with the fact that $\beta_{it}^2|\boldsymbol{\pi}_t|^2$ is bounded in t. Because $\ell_i \neq 0$, we can therefore conclude from these equalities that $\ell_i = \mathrm{Cov}(\boldsymbol{\mu}, \theta)/\,\mathrm{Var}(\boldsymbol{\mu})$ for every i, and using this information in (A.27), we have, for every i:

$$\lim_{t\to\infty} \beta_{it}|\boldsymbol{\pi}_t| = \frac{\mathrm{Cov}(\boldsymbol{\mu}, \theta)}{\mathrm{Var}(\boldsymbol{\mu})}. \tag{A.29}$$

To complete the proof of Claim 3, we argue first that $|\boldsymbol{\pi}_t|$ cannot have a zero limit point. For suppose that $|\boldsymbol{\pi}_t| \to 0$ along some subsequence, then — given (A.29) and the fact $\mathrm{Cov}(\boldsymbol{\mu}, \theta) > 0$

— $\beta_{it} \to \infty$ for all i along that same subsequence. That implies — using (7) — that γ_t is negative along a subsequence of t, which contradicts Proposition 5.

We can now combine this observation with Claim 2 ($\iota_t \to \infty$) to conclude that $\gamma_t = |\boldsymbol{\pi}_t|\iota_t \to \infty$ and so, by (7), $\sum_{i=1}^{n} \frac{1-\beta_{it}}{\Delta_i \operatorname{Var}_t(\theta|i)} \to 0$. At the same time, (A.29) informs us that $\lim_{t\to\infty} \beta_{it}$ is independent of i. Therefore, $\beta_{it} \to 1$ for all i. Applying this information to (A.29) again, the Claim is proved.

We now complete the proof of Proposition 6. Noting that we've already chosen a subsequence so that $\boldsymbol{\mu}_t$ converges to $\boldsymbol{\mu}$, and that $|\boldsymbol{\pi}_t|$ then converges by Claim 4, write $\boldsymbol{\pi} = \lim_t \boldsymbol{\pi}_t$. Recall that $\Gamma_i(\boldsymbol{\mu}) \equiv \mathbf{Cov}(\theta, \boldsymbol{y}_i) \operatorname{Var}(\boldsymbol{\mu}) - \mathbf{Cov}(\boldsymbol{\mu}, \theta)\, \mathbf{Cov}(\boldsymbol{\mu}, \boldsymbol{y}_i) = \mathbf{0}$ (by Claim 2), so that

$$\mathbf{Cov}(\theta, \boldsymbol{y}_i) = \frac{\operatorname{Cov}(\boldsymbol{\mu}, \theta)}{\operatorname{Var}(\boldsymbol{\mu})} \mathbf{Cov}(\boldsymbol{\mu}, \boldsymbol{y}_i) = |\boldsymbol{\pi}|\, \mathbf{Cov}(\boldsymbol{\mu}, \boldsymbol{y}_i) = \mathbf{Cov}(\boldsymbol{\pi}, \boldsymbol{y}_i) \tag{A.30}$$

for all i, where the second equality follows from Claim 4. This proves Part (i) of the Proposition. Next,

$$\gamma_t^2 \operatorname{Var}_t(u) = |\boldsymbol{\pi}_t|\iota_t^2 \operatorname{Var}_t(u) \to 0$$

by Claims 1 and 4. Finally, by Claims 2 and 4, $\boldsymbol{\alpha}_{it} \to \mathbf{0}$ and $\beta_{it} \to 1$ for all i, so invoking (12), we must conclude that $\operatorname{Var}_t(\theta|i) \to \operatorname{Var}(\theta) - \operatorname{Cov}(\boldsymbol{\pi}, \theta)$ whenever $\boldsymbol{\pi}_t \to \boldsymbol{\pi}$ along some subsequence of t. The asserted limit on $\{c_t\}$ then follows from (8), and the proof of Part (ii) is also complete. $\square$

Proof of Proposition 7. Part (i). If the condition there holds, equation (14) becomes

$$Q_i = \frac{\operatorname{Cov}(\theta, y_i)[\sum_{k=1}^{n} Q_k^2 \operatorname{Var}(y_k) + \operatorname{Var}(u)] - Q_i \operatorname{Var}(y_i) \operatorname{Cov}(\boldsymbol{Q}, \theta)}{\Delta_i \operatorname{Var}(\theta | y_i, \sum_{k=1}^{n} Q_k y_k + u)[\operatorname{Var}(y_i)(\operatorname{Var}(\boldsymbol{Q}) + \operatorname{Var}(u)) - \operatorname{Cov}^2(\boldsymbol{Q}, y_i)]}. \tag{A.31}$$

Recall that $\boldsymbol{Q} = \boldsymbol{\pi}/\gamma$, that $\gamma > 0$ by Proposition 2, and that $\operatorname{Cov}(p, \theta) = \operatorname{Cov}(\boldsymbol{\pi}, \theta) \geq 0$ by Proposition 5. It follows that $\operatorname{Cov}(\boldsymbol{Q}, \theta) \geq 0$. Now a simple inspection of (A.31) yields the desired conclusion.

(ii). Fix i_0 as in the statement of the Proposition. If $Q_{i_0} < 0$, there is nothing to prove, so suppose that $Q_{i_0} \geq 0$. By Proposition 5, we have $\operatorname{Cov}(\boldsymbol{Q}, \theta) > 0$ (because $\operatorname{Cov}(\theta, y_k) > 0$ for at least one k), and we also know that $\operatorname{Cov}(\theta, y_{i_0}) = 0$. Using all this information in (14), we can conclude that

$$\operatorname{Cov}(\boldsymbol{Q}, y_{i_0}) = \sum_{k=1}^{n} Q_k \operatorname{Cov}(y_k, y_{i_0}) \leq 0. \tag{A.32}$$

Now suppose, contrary to the assertion of the Proposition, that $Q_i \geq 0$ for all i. Then our assumption that $\operatorname{Cov}(y_k, y_{i_0}) > 0$ for all k, along with (A.32), implies that $\boldsymbol{Q} = \mathbf{0}$. Invoking (14) again, it is easy to see that this implies $\operatorname{Cov}(\theta, y_i) = 0$ for all i, a contradiction.

(iii). Because $\operatorname{Cov}(\theta, y_i) > 0$ for some i, Proposition 5 tells us that $\operatorname{Cov}(\boldsymbol{Q}, \theta) = \sum_{i=1}^{n} Q_i \operatorname{Cov}(\theta, y_i) > 0$. Our conclusion follows immediately from the convention that $\operatorname{Cov}(\theta, y_i) \geq 0$ for all i.

(iv). From (14) and the inequality $|\operatorname{Cov}(\boldsymbol{Q}, \theta) \operatorname{Cov}(\boldsymbol{Q}, y_i)| \leq \operatorname{Var}(\boldsymbol{Q})\sqrt{\operatorname{Var}(\theta)\operatorname{Var}(y_i)}$, we have

$$Q_i \geq \frac{\operatorname{Cov}(\theta, y_i)[\operatorname{Var}(\boldsymbol{Q}) + \operatorname{Var}(u)] - \operatorname{Var}(\boldsymbol{Q})\sqrt{\operatorname{Var}(\theta)\operatorname{Var}(y_i)}}{\Delta_i \operatorname{Var}(\theta | y_i, \sum_{k=1}^{n} Q_k y_k + u)[\operatorname{Var}(y_i)(\operatorname{Var}(\boldsymbol{Q}) + \operatorname{Var}(u)) - \operatorname{Cov}^2(\boldsymbol{Q}, y_i)]} \tag{A.33}$$

for every i. Moreover, similar to (A.6), we have

$$\sum_{i=1}^{n} Q_i \operatorname{Cov}(\theta, y_i) \geq \operatorname{Var}(\theta | y_1, \ldots, y_n) \sum_{i=1}^{n} Q_i^2 \Delta_i \operatorname{Var}(z_i),$$

so $\boldsymbol{Q}$ is bounded *uniformly over* $\operatorname{Var}(u)$. Use this information in (A.33) to complete the proof. □

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