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Structural Sparsity of Complex Networks: Bounded Expansion in Random Models and Real-World Graphs

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Abstract

This research establishes that many real-world networks exhibit *bounded expansion*[2], a strong notion of structural sparsity, and demonstrates that it can be leveraged to design efficient algorithms for network analysis. Specifically, we give a new linear-time fpt algorithm for motif counting and linear time algorithms to compute localized variants of several centrality measures.

To establish structural sparsity in real-world networks, we analyze several common network models regarding their structural sparsity. We show that, with high probability, (1) graphs sampled with a prescribed sparse degree sequence; (2) perturbed bounded-degree graphs; (3) stochastic block models with small probabilities; result in graphs of bounded expansion. In contrast, we show that the Kleinberg and the Barabási–Albert model have unbounded expansion. We support our findings with empirical measurements on a corpus of real-world networks.

Keywords: structural sparsity, bounded expansion, complex networks, random graphs, motif counting, centrality measures

1. Introduction

Complex networks vs. structural graph algorithms. Social networks (such as Facebook or physical disease propagation networks), biological networks (such as gene interactions or brain networks), and informatics networks (such as

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[2] Not to be confused with the notion of expansion related to expander graphs.

autonomous systems) are all examples of *complex networks*, which have been the attention of much study in recent years, given the surge of available network data. Viewed as graphs, complex networks seem to share several structural properties. Perhaps most famous is the *small-world* property, often referred to as "six degrees of separation": typical distances between vertex pairs are small compared to the size of the network. Another important property is that their degree distribution tends to be heavy-tailed, i.e. not exponentially bounded. In many cases, this degree distribution is close[2] to a *power-law*: the fraction of vertices of degree k is proportional to $k^{-\gamma}$, for some constant γ typically between 2 and 3. Networks furthermore often exhibit *high clustering* and admit a natural division into a *community structure*. Despite the clustering property, complex networks are *sparse*: the ratio of edges to vertices is usually small.

On the other hand, the field of structural graph algorithms has led to impressively efficient and precise algorithms including PTASs, subexponential fixed-parameter algorithms, and linear kernelizations for increasingly general families of graphs; see, e.g., [26–29, 33, 37, 48]. Many such results proved initially for planar graphs have since been extended to bounded-genus graphs, graphs of bounded local treewidth, and graphs excluding a fixed minor; yet such results are known to be impossible for general graphs. Can we apply these powerful algorithms to analyze complex networks?

We propose the following litmus test for whether a type of network sparsity is "useful": Does it enable efficient algorithms for a broad set of NP-hard problems? Unfortunately, the above-mentioned structural properties of complex networks seem too weak to enable better algorithms, while the discussed graph classes seem too restrictive to apply to complex networks. The goal of this paper is to bridge this gap, by identifying a more general graph class that simultaneously enables better algorithms and includes many complex networks.

Bounded expansion. In general, complex networks seem to exhibit an intermediate-scale structure composed of small dense parts—representing clusters or communities—that are sparsely interconnected. This hierarchical behavior has been established for many networks [59] and is consistent with the tree-like intermediate structure observed in [4].

How can this notion be captured formally? If we contract disjoint small-diameter subgraphs – representing potentially dense local substructures in the network – then the resulting graph, which represents the global connectivity of these substructures, should be sparse. This gives rise to the notion of an *r-shallow minor*, where r is the maximum diameter of the subgraphs that were contracted in the construction process. For formal definitions, see Section 2. We cannot expect the edge density of all r-shallow minors to be constant (then r would play no role), but we require it to grow as any function of r, and thus be independent of the size of the graph. A graph class for which this property

[2]Recent work [18, 25] has shown that this relationship is only approximate; instead these distributions typically are power-law with an exponential cut-off, making our results – which require polynomial tail-bounds – applicable. See Section 3 for discussion.

holds has *bounded expansion*, a concept introduced by Nešetřil and Ossona de Mendez [72].

Theoretical results. Since the definition of bounded expansion applies to graph classes instead of individual graphs, it is impossible to settle this question empirically. To ground our hypothesis in theory, we analyze several random graph models which were designed to mimic the properties of specific types of real world networks. Although not perfect, random graph models play a central role in network analysis, both to guide our understanding of complex networks and as a convenient source of synthetic data for algorithm testing and validation. In our case, random graph models allow us to determine whether (synthetic) complex networks have bounded expansion with high probability. We analyze several popular random graph models:

(i) the configuration model [70] and the Chung–Lu model [22, 23] with specified asymptotic degree sequences, which includes graphs with heavy-tailed degree distributions;

(ii) a variant of the configuration model which achieves high clustering [9];

(iii) a significant generalization of Erdős–Rényi graphs we call *perturbed bounded-degree graphs* (allowing the network to be built on top of a fixed or random base graph of bounded degree), which includes the stochastic block model with small probabilities;

(iv) the Kleinberg model [49, 50];

(v) and the Barabási–Albert model [6, 10].

We will show that the configuration model, the Chung–Lu model and perturbed bounded-degree graphs have bounded expansion, while the Kleinberg model and the Barabási–Albert model do not – actually they are not even *nowhere dense*, a strict generalization of bounded expansion.

Experimental results. We present an experimental study suggesting that important real-world networks have small *grad*, which is the density measure for single graphs that defines the expansion for graph classes. Interestingly, the algorithmic tools that become efficient when the grad of a graph is small can be directly applied without knowing its actual value.

We will make extensive use of *p-treedepth colorings*: for any integer p, a graph can be colored with $f(p)$ colors such that any set of at most $p-1$ colors induces a graph of treedepth at most $p-1$, where f only depends on the grad of the graph. Generally the running time of algorithms based on p-treedepth colorings depends heavily on the number of colors $f(p)$. In Section 5 we present experimental results obtained by computing and evaluating p-treedepth colorings with a simple algorithm in a variety of real-world networks.

Our results show that, in general, real networks exhibit even better structure (require fewer colors for a p-treedepth coloring) than randomly generated graphs with the same degree distribution via the configuration model. These results

support our hypothesis that "community structure," which is not captured by the degree distribution, further increases the algorithmic tractability.

Algorithmic results. With both theoretical and empirical results in hand, we exploit the aforementioned tools for graphs with small grad in Section 6 to solve typical problems for complex networks: First we develop a faster algorithm than the one presented in [74] to count subgraph homomorphisms based on p-treedepth colorings. Counting subgraphs is fundamental to *motif counting*, a widely used concept to analyze networks. Then we develop an algorithm which computes localized versions of several centrality measures in linear time on graphs of bounded expansion. Specifically, we present:

(i) A linear-time algorithm to count the number of times a fixed subgraph appears as an (induced) subgraph/homomorphism in graphs of bounded grad. We do this by improving the previously best known algorithm to count the appearances of a structure of size h on graphs of treedepth t from $O(2^{ht}ht \cdot n)$ to $O(t^h 6^h h^2 \cdot n)$, thus removing the exponential dependency on t, while keeping the algorithm simple and avoiding big hidden constants.

(ii) A linear-time algorithm to compute localized variants (i.e., computed in a constant-radius neighborhood around each vertex) of the *closeness centrality* and two other related measures. The constant in the running time depends on the radius and the grad of the graph.

For the second algorithm we provide experimental results which indicate that the local variants of these centrality measures can be used to estimate the top 10 percent of the most central nodes.

Previous results. There is substantial empirical work studying structural properties of complex networks, so we focus here on work closest to structural graph theory. In general, large real-world complex networks are not easily classified as either low- or high-dimensional. In particular, data-mining tools which implicitly assume low dimensionality (such as singular value decomposition) produce models and results incompatible with observed structure and dynamics, yet traditional high-dimensional tools like sampling often fail to achieve measure concentration due to the extreme sparsity of the networks. Adcock et al. [4] recently empirically established that, when compared with a suite of idealized graphs[3], realistic large social and informatics networks exhibit meaningful "tree-like" structure at an intermediate scale. Their work related the k-core structure – whose extremal statistic is the degeneracy – to the networks' Gromov hyperbolicity and tree decompositions. Unfortunately, they showed that straightforward applications of these measures are often insufficient to reveal meaningful structure because of noisy/random edges in the network which contradict the strict structural requirements. For example, several families of popular random graph models have been shown to have very large treewidth [38].

[3]Representing low-dimensional structures, common hierarchical models, constant-degree expanders, etc.

Gago and Schlatter [36] provided some simple preliminary observations about networks and bounded expansion, such as the *linear growth model* not having bounded expansion since it was known that it contains growing bi-cliques, and they conjecture that Barabási-Albert is somewhere dense a.a.s. Here we prove that it is at least somewhere dense with non-vanishing probability. For the *random intersection graph* model, which is used to model real world networks where connections depend on shared attributes, it was shown in [32] that it has bounded expansion exactly when it is degenerate.

2. Preliminaries

For a natural number n we use the notation $[n] := \{1, \dots, n\}$. For a graph G, we denote by $\Delta(G)$ its maximal degree and by $\omega(G)$ its clique number, e.g. the largest integer t such that the complete subgraph K_t is contained in G. We will make use of the following graph operations. For graphs G_1, G_2, the *complete join* $G_1 * G_2$ is the graph obtained by first taking the disjoint union of G_1, G_2 and then connecting every vertex of $V(G_1)$ to every vertex $V(G_2)$. For example, $G * \overline{K}_2$ is the graph obtained from G by adding two universal vertices. The *lexicographic product* $G_1 \bullet G_2$ is the graph with vertices $V(G_1) \times V(G_2)$ and edges

$$(u, x)(v, y) \in E(G_1 \bullet G_2) \iff uv \in E(G_1) \text{ or } (u = v \text{ and } xy \in E(G_2)).$$

We use the notation $H \simeq G$ to denote that the graphs H, G are isomorphic. In the following we sometimes employ the notation $\bar{k}^2 := \binom{k}{2}$, in particular when the expression appears as an exponent.

2.1. Bounded expansion classes

We will use $(\leqslant r)$*-subdivisions* to formalize the notion of shallow topological minors. A $(\leqslant r)$-subdivision of a graph H is any graph which can be obtained from H by replacing edges with disjoint paths of length at most $r + 1$.

Definition 1 (Shallow topological minor [72])**.** *A graph H is an r-*shallow topological minor *of G if a $(\leqslant 2r)$-subdivision of H is isomorphic to a subgraph G' of G. We call G' a* model of H in G*. For simplicity, we assume by default that $V(H) \subseteq V(G')$ such that the isomorphism between H and G' is the identity when restricted to $V(H)$. The vertices $V(H)$ are called* nails *and the vertices $V(G') \setminus V(H)$* subdivision vertices*. The set of all r-shallow topological minors of a graph G is denoted by $G \mathbin{\widetilde{\nabla}} r$.*

The theory of bounded expansion classes crucially relies on a 'parameterization' of shallow minors. The *grad* of a graph is one such parameterization:

Definition 2 (Topological grad)**.** *For a graph G and a half-integer $r \geqslant 0$, the* topological greatest reduced average density (topological grad) at depth r *is defined as*

$$\widetilde{\nabla}_r(G) = \max_{H \in G \mathbin{\widetilde{\nabla}} r} \frac{|E(H)|}{|H|}.$$

For a graph class $\mathcal{G}$, define $\widetilde{\nabla}_r(\mathcal{G}) = \sup_{G\in\mathcal{G}} \widetilde{\nabla}_r(G)$.

Given the notion of shallow topological minors and topological grad, we can now define what it means for a class to have bounded expansion.

Definition 3 (Bounded expansion)**.** *A graph class $\mathcal{G}$ has* bounded expansion *if and only if there exists a function f such that for all half-integers $r \geqslant 0$, it holds that $\widetilde{\nabla}_r(\mathcal{G}) < f(r)$.*

When introduced in [72], bounded expansion was originally defined using a characterization based on the notion of *shallow minors*: H is an r-shallow minor of G if H can be obtained from G by contracting disjoint r-balls and then taking a subgraph. Taking the maximum over the density of all r-shallow minors then defines the *grad* of graph $\nabla_r(G)$. An r-ball in a graph G is a subgraph $G' \subseteq G$ with the property that there exists $v \in V(G')$ such that for all $u \in V(G')$, $d_{G'}(u, v) \leqslant r$. Both characterizations are equivalent: for any graph G and integer r, we have the relation (cf. [74])

$$\widetilde{\nabla}_r(G) \leqslant \nabla_r(G) \leqslant 4(4\widetilde{\nabla}_r(G))^{(r+1)^2}$$

thus it follows that a graph class has bounded topological grad if and only if it has bounded grad, e.g. if either is bounded the class has bounded expansion. We will in the following exclusively use the topological grad and will therefore often drop the term 'topological'.

We note that graphs excluding a topological minor—in particular planar graphs and bounded-degree graphs—have bounded expansion. This generalizes to graphs excluding a minor (and thus to those of bounded treewidth). Finally, we point out that bounded expansion implies bounded degeneracy, where a graph G is d-degenerate if any subgraph of G contains a node of degree smaller than d. The converse does not hold.

The following alternative characterization of bounded expansion uses a special coloring number with nice algorithmic properties.

Definition 4 (p-centered coloring [72])**.** *Given a graph G, let $c\colon V(G) \to [r]$ be a vertex coloring of G with r colors. We say that the coloring c is* p-centered*, for $p \geqslant 2$, if every connected subgraph of G either receives at least p colors or contains some color exactly once. Define $\chi_p(G)$ to be the minimum number of colors needed for a $(p+1)$-centered coloring.*

While this definition looks rather cryptic, it is easy to see that every graph has a p-centered coloring for any p: simply assign a distinct color to each vertex of the graph. Note that p-centered colorings are proper colorings for $p \geqslant 2$ and in particular, χ_1 is precisely the chromatic number. Typically, the number of colors q is much larger than p and one is interested in minimizing q. Nešetřil and Ossona de Mendez [72] showed that for every graph G,

$$\widetilde{\nabla}_r(G) \leqslant \binom{\chi_{2r+2}(G)}{2r+2} \quad \text{and} \quad \chi_r(G) \leqslant P(\widetilde{\nabla}_{2^{p-2}+1}(G))$$

where P is some polynomial of degree around 2^{2^r}. Accordingly, for a graph class $\mathcal{G}$, the quantity $\chi_r(\mathcal{G})$ is bounded by a function of r if and only if the quantity $\widetilde{\nabla}_r(\mathcal{G})$ is, therefore bounded expansion can equivalently be defined in terms of p-centered colorings.

The following structural property, which follows directly from the equivalence between centered colorings and *treedepth*, make them an attractive tool for algorithm design.

Proposition 1 (p-treedepth colorings [73])**.** *Let $\mathcal{G}$ be a graph class of bounded expansion. There exists a function f such that for every $G \in \mathcal{G}$, $p \in \mathbf{N}$, the graph G can be colored with $f(p)$ colors so that any $i < p$ color classes induce a graph of treedepth $\leqslant i$ in G. This coloring can be computed in linear time.*

For those unfamiliar with this notion, the *treedepth* of a graph G is the lowest depth of a rooted forest whose closure contains G as a subgraph. A *treedepth decomposition* of G is simply a forest with vertex set $V(G)$ witnessing this fact. To put this width measure into perspective: a graph of treedepth at most t cannot contain a path of length 2^t *and* has pathwidth at most $t-1$. Importantly, a graph has treedepth t if and only if it admits a centered coloring with t colors. In one direction, imagine coloring the vertices of a graph according to their depth in the treedepth decomposition; then every connected subgraph receives a center (since it is connected, there is a single highest vertex in the decomposition, which thus receives a unique color). In the other direction, given a centered coloring with t colors, we can construct a treedepth decomposition of depth t by using the centers of its connected components as roots for the decomposition, removing these vertices from the graph and recursing on the resulting connected components. Thus the terms 'p-centered colorings' and 'p-treedepth colorings' are synonymous.

An example of a 5-centered coloring and treedepth-decompositions of a selected subset of colors can be found in Figure 1 on page 9. For more information about treedepth see e.g. [88].

Nešetřil and Ossona de Mendez show that graph classes of bounded expansion are precisely those for which there exists a function f such that every member G of the graph class satisfies $\chi_p(G) \leqslant f(p)$ (see Theorem 7.1 in [72]). In [73], the authors also showed how to obtain a p-centered coloring with at most $P(f(p))$ colors for each fixed p in linear time, where P is some polynomial of degree roughly 2^{2^p}. We will make use of this algorithm in Sections 5 and 6 and see that the actual number of colors is manageable.

When working with random graphs, we will make heavy use of the following alternative characterization of graphs of bounded expansion:

Proposition 2 (Nešetřil, Ossona de Mendez, Wood [74, 75])**.** *A class $\mathcal{G}$ of graphs has bounded expansion if and only if there exist real-valued functions $f_{thresh}, f_{\deg}, f_{\widetilde{\nabla}}, f_H \colon \mathbf{R}^+ \to \mathbf{R}$ such that the following two conditions hold:*

(i) For all $\varepsilon > 0$ and for all $G \in \mathcal{G}$ with $|G| > f_{thresh}(\varepsilon)$, it holds that

$$\frac{1}{|G|} \cdot |\{v \in V(G) \colon \deg(v) \geqslant f_{\deg}(\varepsilon)\}| \leqslant \varepsilon.$$

(ii) For all $r \in \mathbf{N}$ and for all $H \subseteq G \in \mathcal{G}$ with $\widetilde{\nabla}_r(H) > f_{\widetilde{\nabla}}(r)$, it follows that

$$|H| \geqslant f_H(r) \cdot |G|.$$

Intuitively, Proposition 2 characterizes classes of graphs with bounded expansion as those where:

(i) all sufficiently large members of the class have a small fraction of vertices of large degree;

(ii) all subgraphs of $G \in \mathcal{G}$ whose shallow topological minors are sufficiently dense must necessarily span a large fraction of the vertices of G.

Finding the first pair of functions $f_{\text{thresh}}, f_{\text{deg}}$ is usually straightforward, the real challenge lies in proving that functions $f_{\widetilde{\nabla}}, f_H$ exist. Consider the contra-positive of the second condition: in terms of random graph models, we want to show that subgraphs that span at most a $f_H(r)$-fraction of the vertices of G have their (topological) grad bounded by $f_{\widetilde{\nabla}}$ with high probability.

One of the useful properties of bounded expansion classes is that their expansion does not increase arbitrarily through the addition of a universal vertex.

Lemma 1. *For every graph G and half-integer $r \geqslant 0$ it holds that*

$$\widetilde{\nabla}_r(G) \leqslant \widetilde{\nabla}_r(G * K_1) < \widetilde{\nabla}_r(G) + 1$$

Proof. It is easy to see that if H is an r-shallow topological minor of G, then $H * K_1$ is a r-shallow topological minor of $G * K_1$. The density of $H * K_1$ is then given by

$$\frac{|E(H * K_1)|}{|V(H * K_1)|} = \frac{|E(H)| + |H|}{|H| + 1}$$

And therefore we obtain upper and lower bounds via

$$\frac{1}{1 + (1/|H|)} \left(\frac{|E(H)|}{|H|} + 1 \right) \leqslant \frac{|E(H * K_1)|}{|V(H * K_1)|} < \frac{|E(H)|}{|H|} + 1$$

Observing that $|H| \geqslant \widetilde{\nabla}_r(G)$ proves the claim. □

Therefore even a constant number of universal vertices will not influence the grad too much. In terms of graph classes this means that if $\mathcal{G}$ has bounded expansion, then the class $\{G * K_c\}_{G \in \mathcal{G}}$ for any constant c also has bounded expansion, albeit by a different function.

Similarly, the lexicographic product with a constant-sized graph does not change the expansion of a class arbitrarily.

Proposition 3 (Nešetřil and Ossona de Mendez [74])**.**
For every graph G, integer $p \geqslant 2$ and half-integer r it is true that

$$\widetilde{\nabla}_r(G \bullet K_p) \leqslant \max\{2r(p-1)+1, p^2\} \cdot \widetilde{\nabla}_r(G) + p - 1$$

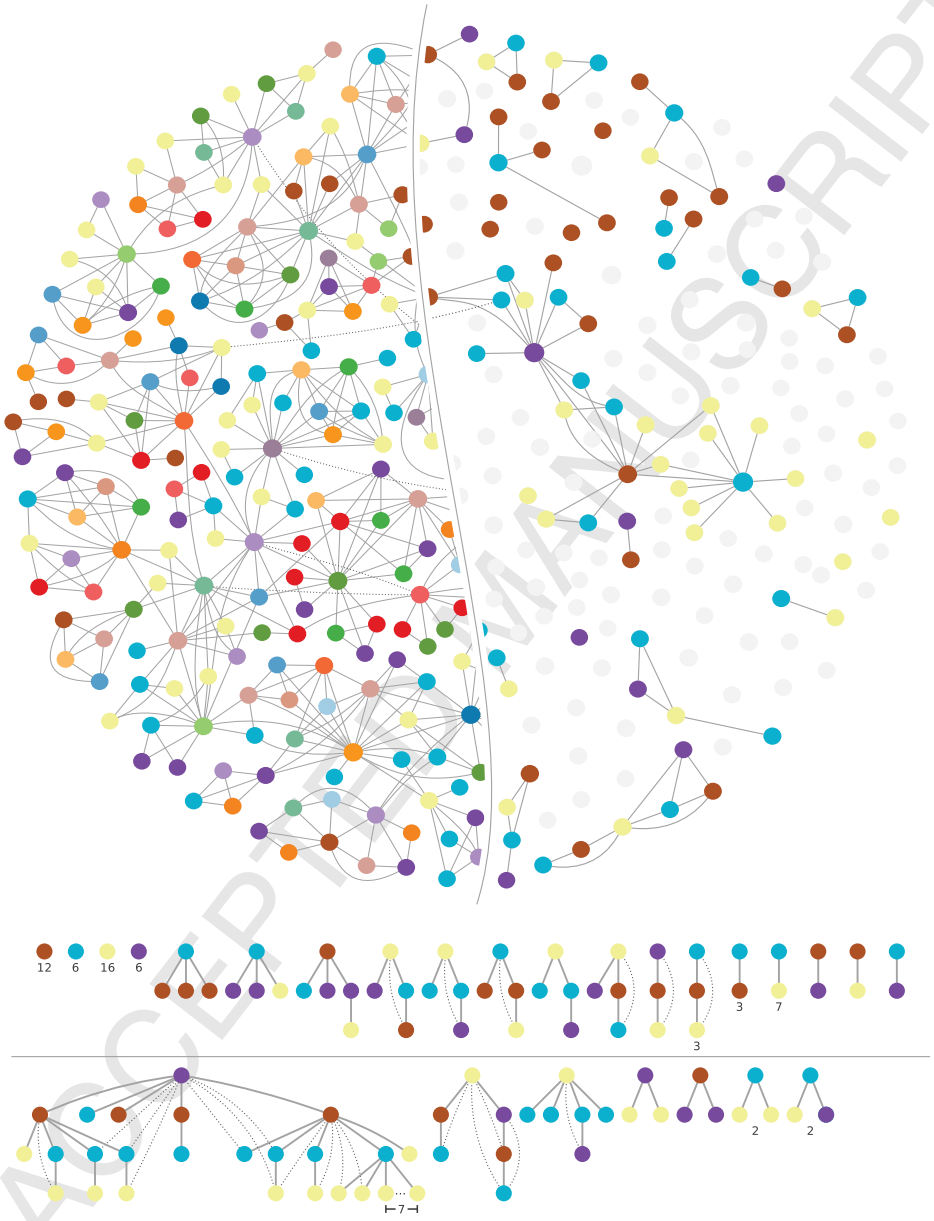


Figure 1: A 5-centered coloring (using 21 colors) of a real-world social network, Newman's Network Science [81] (giant component shown), which represents co-authorships between researchers in the field of network science as of 2006. The right half is restricted to a a subgraph formed by 4 color classes. Below, the corresponding representation by trees of depth $\leqslant 4$ (with multiplicities noted).

In conclusion, bounded expansion is a robust description of structural sparseness: it is closed under taking shallow minors and the above two 'algebraic' operations. However, bounded expansion classes are not the ultimate definition of sparseness. In order to obtain a dichotonomical view of structural sparseness, Nešetřil and Ossona de Mendez defined *nowhere dense* classes. Their definition mirrors that of bounded expansion classes, only that we now measure the clique number of shallow minors instead of their density.

Definition 5 (Nowhere dense)**.** *A graph class $\mathcal{G}$ is* nowhere dense *if and only if there exists a function f such that for all $r \geqslant 0$, we have $\omega(\mathcal{G} \mathbin{\widetilde{\nabla}} r) < f(r)$.*

See the book by Nešetřil and Ossona de Mendez for further definitions [74] and the proof that nowhere dense classes indeed provide a dichotomy of structural sparseness. We will need the following crucial result proved independently by Dvořák and Jiang.

Theorem 1 (Dvořák [30], Jiang [47])**.** *Let $\ell \in \mathbf{N}$ and $\varepsilon > 0$. There exists integers $n_{\ell,\varepsilon}$ and c_ε such that every graph G with $n > n_{\ell,\varepsilon}$ vertices and at least $n^{1+\varepsilon}$ edges contains a c_ε-subdivision of K_ℓ.*

2.2. Random graph models

We usually denote random variables by upper-case letters. Probabilities are denoted by $\mathrm{P}[\bullet]$, expectation, variance, and median by $\mathrm{E}[\bullet]$, $\mathrm{Var}[\bullet]$, $\mathrm{M}[\bullet]$, respectively. If we need to clarify which probability measures we employ, we use subscripts like $\mathrm{P}[\bullet]_M$. We will use the Iverson bracket notation $[\![\phi]\!]$ for boolean expressions ϕ where $[\![\phi]\!]$ is one if ϕ is true and zero otherwise. For a sequence of random variables $(X_n)_{n\in\mathbf{N}}$ and a random variable Y, recall that (X_n) *converges in distribution* to X if it holds that

$$\forall k \lim_{n\to\infty} \mathrm{P}[X_n \leqslant k] = \mathrm{P}[X \leqslant k].$$

We denote the convergence in distribution with $(X_n) \xrightarrow{d} X$.

A *random graph model* is a sequence of random variables $(G_n)_{n\in\mathbf{N}}$ over n-vertex graphs. For simplicity, we fix $V(G_n) = [n]$. The *parametrisation* of the model is a function $\rho\colon \mathbf{N} \to \mathbf{R}^t$ that creates a tuple of t parameters depending on n which in turn determine the probability distribution of each variable G_n. By $G(n, \rho(n))$ we denote the random variable G_n with the probability distribution prescribed by the model with parameters $\rho(n)$. In order to distinguish random models we will introduce superscripts like G^{CL} or G^{KL}.

For a random graph model $G(n, \rho(n))$ and an integer r the notation $G(n, \rho(n)) \mathbin{\widetilde{\nabla}} r$ denotes a random variable over sets of graphs with at most n vertices whose probability distribution is given by

$$\mathrm{P}\left[G(n, \rho(n)) \mathbin{\widetilde{\nabla}} r = A\right] = \sum_{G : A = G \mathbin{\widetilde{\nabla}} r} \mathrm{P}[G(n, \rho(n)) = G],$$

where A is a set of graphs. With this definition the quantity $\widetilde{\nabla}_r$ is well-defined for every integer r as a rational-valued random variable. As noted above, we

study the properties of random graphs in the limit and hence define the property of having bounded expansion as follows.

Definition 6. *A graph model $G(n, \rho(n))$ has* bounded expansion asymptotically almost surely (a.a.s.) *if there exists a function f such that for all $r \geqslant 0$*

$$\lim_{n\to\infty} \mathrm{P}\big[\, \widetilde{\nabla}_r\big(G(n,\rho(n))\big) < f(r) \,\big] = 1.$$

It has bounded expansion with high probability (w.h.p.) *if for every $c \geqslant 1$ there exists a function f such that, again for all $r \geqslant 0$,*

$$\mathrm{P}\big[\, \widetilde{\nabla}_r\big(G(n,\rho(n))\big) < f(r) \,\big] \geqslant 1 - O(n^{-c}).$$

The very same definition is possible to define when a random graph model is nowhere dense.

Definition 7. *A graph model $G(n, \rho(n))$ is* a.a.s. nowhere dense *if there exists a function f such that for all $r \geqslant 0$*

$$\lim_{n\to\infty} \mathrm{P}\big[\, \omega\big(G(n,\rho(n))\, \widetilde{\nabla}\, r\big) < f(r) \,\big] = 1.$$

It is nowhere dense w.h.p. *if for every $c \geqslant 1$ there exists a function f such that, again for all $r \geqslant 0$,*

$$\mathrm{P}\big[\, \omega\big(G(n,\rho(n))\, \widetilde{\nabla}\, r\big) < f(r) \,\big] \geqslant 1 - O(n^{-c}).$$

The following notions are needed to prove negative results about random models.

Definition 8. *A graph model $G(n, \rho(n))$ is* a.a.s. somewhere dense *if there exists $r \in \mathbf{N}$ such that for all functions f it holds that*

$$\lim_{n\to\infty} \mathrm{P}\big[\, \omega\big(G(n,\rho(n))\, \widetilde{\nabla}\, r\big) > f(r) \,\big] = 1.$$

It is not a.a.s. nowhere dense *if there exists $r \in \mathbf{N}$ such that for all functions f it holds that*

$$\lim_{n\to\infty} \mathrm{P}\big[\, \omega\big(G(n,\rho(n))\, \widetilde{\nabla}\, r\big) > f(r) \,\big] > 0.$$

Note that the above definition for random graphs is, in contrast to the definition of structural sparseness for graph classes, not a dichotomy: we can easily build an artificial random graph model where the probability that a dense shallow minor exists converges to some value bounded away from zero and one. This might now just seem like a technicality, but we will see two examples of random graph models developed to replicate certain aspects of complex networks that fall exactly in this category.

The following basic lemma will help to simplify the following proofs by enabling us to work with randomly chosen subgraphs.

Lemma 2. *Let $X_1, \ldots, X_n$ be binary random variables and let $S = \sum_{i=1}^{n} X_i$. Let further $I \in [n]$ be uniformly distributed. Then*

$$\mathrm{P}[S \geqslant 1] \leqslant n \cdot \mathrm{P}[X_I = 1].$$

Proof. By Markov's inequality we have that

$$\mathrm{P}[S \geqslant 1] \leqslant \mathrm{E}[S] = \sum_{i=1}^{n} \mathrm{P}[X_i = 1].$$

Observe that

$$\mathrm{P}[X_I = 1] = \sum_{i=1}^{n} \mathrm{P}[I = i] \cdot \mathrm{P}[X_i = 1] = \frac{1}{n} \sum_{i=1}^{n} \mathrm{P}[X_i = 1],$$

and hence

$$\mathrm{P}[S \geqslant 1] \leqslant n \cdot \mathrm{P}[X_I = 1]. \qquad \square$$

We apply this statement to subgraphs in a random graph and obtain the following corollary that lies closer to our application.

Corollary 1. *Let $G(n, \rho(n))$ be a random graph model with parametrisation ρ and Π be a graph property. Then*

$$\mathrm{P}[\exists X \subseteq V(G) \colon G[X] \in \Pi] \leqslant \sum_{k=1}^{n} \binom{n}{k} \mathrm{P}\big[G[Y_k] \in \Pi\big],$$

where Y_k is a k-vertex subset of $V(G)$ chosen uniformly at random.

2.3. Asymptotic degree distributions

Given a graph G, the *degree sequence* $D(G)$ of G is the sequence $(\deg(v))_{v \in G}$. We say that two graphs G_1 and G_2 have the same degree sequence, if $D(G_1) = \pi(D(G_2))$ for some permutation π. A sequence of n integers $(d_i)_{1 \leqslant i \leqslant n}$ is a degree sequence if it can be realized by a graph, i.e. there exists a graph G with $D(G) = (d_i)_{1 \leqslant i \leqslant n}$. Such sequences are also called *graphical*.

To define degree distributions, consider a random variable D_n describing the degree of a vertex chosen uniformly at random from an n-vertex graph G. The pmf f_n for D_n is then given by

$$f_n(d) = \frac{b_n(d)}{n} = \frac{1}{n} \sum_{v \in V(G)} [\![\deg(v) = d]\!],$$

where $b_n(d)$ denotes the number of degree-d vertices in G and $[\![\bullet]\!]$ is the Iverson bracket.

Definition 9 (Degree distribution)**.** *A n-vertex* degree distribution *is a random variable D with probability mass function f such that*

1. *$f(d) = 0$ for $d \leqslant 0$ and $d \geqslant n - 1$, and*
2. *$nf(d) \in \mathbf{N}_0$ for all $d \in \mathbf{N}$.*

Note that in the above definition we exclude vertices of degree zero, this greatly simplifies some of the following proofs. Note that the addition of any finite number of degree-zero vertices does not change the structural sparsity characterization of a graph class: under this operation, a bounded expansion class still has bounded expansion, a nowhere dense class remains nowhere dense, and a somewhere dense class is still somewhere dense. Thus, in our setting, this omission does not affect the generality of our results.

We say that a degree sequence $(d_i)_{1 \leqslant i \leqslant n}$ *matches* an n-vertex degree distribution D_n with pmf f_n if for every $1 \leqslant k \leqslant n-1$ it holds that

$$\sum_{i=0}^{n} [\![d_i = k]\!] = n f_n(k).$$

Consequently, a graph G *matches* a degree distribution D_n if its degree sequence does. Since we will consider sequences of random graphs we need to introduce a related notation of sequences of degree distributions.

Definition 10 (Degree distribution sequence, limit, sparse)**.** *A* degree distribution sequence *is an infinite sequence (D_n) of n-vertex degree distributions. A random variable D is the* limit *of $\mathcal{D}$ if $(D_n) \xrightarrow{d} D$. We say that $\mathcal{D}$ is* sparse *if* $\mathrm{E}[D] < \infty$ *and* $(\mathrm{E}[D_n])_{n \in \mathbf{N}} \longrightarrow \mathrm{E}[D]$.

To motivate the definition of *sparse* sequences, note that for a degree sequence D_n we have that

$$\mathrm{E}[D_n] = \sum_d d f_n(d) = \frac{1}{n} \sum_{d=1}^{n-1} d b_n(d),$$

thus for a graph G with degree distribution D_n it holds that $\mathrm{E}[D_n]$ is exactly its average degree $d_{avg}(G)$. The condition of a degree sequence being sparse will not quite suffice to prove structural sparseness: the situation parallels how graph class with bounded average degree can still harbour dense structures. Consider, for example, the degree distribution sequence of the class consisting of all one-subdivided cliques. While the sequence has constant mean, the graphs matching it are certainly not structurally sparse. This observation motivates the following stronger condition.

Definition 11 (Tail-bound)**.** *A degree distribution sequence (D_n) with limit D has the function h as its* tail-bound *if there exists a constant $\tau \geqslant 0$ such that for all $d \geqslant \tau$ and large enough n it holds that*

$$\mathrm{P}[\, D_n \geqslant d \,] = O\Big(\frac{1}{h(d)} \Big).$$

We observe that in Table 1, all listed functions have a tail-bound that is at least quadratic. This is self-evident for power-laws with $\gamma \geqslant 2$; for the other functions we simply note that their second moment exists and hence by Chebyshev's inequality satisfy

$$\mathrm{P}[\, D \geqslant d \,] \leqslant \frac{\mathrm{Var}[D]}{(d - \mathrm{E}[D])^2} = O\Big(\frac{1}{d^2} \Big).$$

for $d > \mathrm{E}[D]$.

We will need the following simple observation about the median $\mathrm{M}[D_n]$ of a degree sequence (D_n):

Observation 1. *Let (D_n) be a sparse degree distribution sequence with limit D. Then $\mathrm{M}[D_n] \longrightarrow \mathrm{M}[D]$.*

3. Graph Models with Bounded Expansion

Analytic methods are often selected based on their behavior when applied to random graphs that are believed to mimic characteristics of the particular networks under consideration. We will not digress into the arguments for and against this methodology, but simply note that it is a *de facto* part of standard practice at this point in time. Accordingly, we must be able to establish whether or not graphs generated by such models have bounded grad. For more information on random graph models, we refer the readers to the surveys in [70, 71, 80, 82]. In this section, we determine whether several such models have bounded expansion (as by Definition 6.)

We show that the popular configuration model [70, 71], including the version with households exhibiting high clustering [9], and the Chung–Lu model [22, 23], has bounded expansion w.h.p. in the typical parametric range found in its application to complex networks.

Prior work [75] has shown that the Erdős–Rényi model has bounded expansion a.a.s. Unfortunately, empirical analysis of real-world networks (including friendships/social networks, telephone/communication networks, and biological/neural networks) has shown that typical degree distributions are measurably different from the Poisson distribution exhibited by Erdős–Rényi graphs (see [82] and the references therein.)

We built upon the results of Nešetřil, Ossona de Mendez, and Wood in order to obtain two major results applicable to complex networks. First, the configuration and Chung–Lu model can be seen as an extension of Erdős–Rényi graphs that inherit some of the nice mathematical properties (edge probabilities are somewhat independent in the former and strict independent in the latter) while properly replicating the degree distribution of complex networks by design: Both models allow us to *prescribe* the desired distribution. Typical degree distributions found in real-world graphs are listed in Table 1. Since the prescribed distribution is the one parameter for both models, it is not surprising that the properties of these distributions ultimately determine their structural sparseness. In order to generate sparse graphs, it is necessary for the distributions to have a mean that is independent of n. To generate *structurally* sparse graphs, we show that the distribution in question must have a tail that shrinks at least as fast as a cubic polynomial. This particularly excludes power-law distributions with exponent less than three. However, recent findings have shown that *pure* power-law distributions seem to be rare and a precise statistical analysis often favors a power-law distribution with an exponential cut-off [18, 25]. Since the latter have

in particular tails that are dominated by any polynomial, we conclude that our findings are applicable to the majority of complex networks.

Name	Definition $f(d)$	Parameters
Power-law	$d^{-\gamma}$	$\gamma > 2$
Power-law w/ cutoff	$d^{-\gamma}e^{-\lambda d}$	$\gamma > 2$, $\lambda > 0$
Exponential	$e^{-\lambda d}$	$\lambda > 0$
Stretched exponential	$d^{\beta-1}e^{-\lambda d^{\beta}}$	$\lambda, \beta > 0$
Gaussian	$\exp(-\frac{(d-\mu)^2}{2\sigma^2})$	μ, σ
Log-normal	$d^{-1}\exp(-\frac{(\log d-\mu)^2}{2\sigma^2})$	μ, σ

Table 1: A selection of established functions used to model degree distributions of complex networks, listed without the necessary normalization factors. Here $f(d)$ is the fraction of nodes which have degree d. These functions were taken from an empirical analysis of degree distributions in real-world networks [25].

The other major result pertains to what we call the *perturbed bounded-degree model*, which allows the inclusion of an arbitrary (or random) bounded degree graph in addition to probabilistically generated edges (with non-identical probabilities, subject to a uniform bound). We show this model to have bounded expansion with high probability. This in particular strengthens the aforementioned result on the Erdős–Rényi model in terms of speed of convergence. Including a base graph and allowing non-uniform edge probabilities drastically increases the structural variability in the model's output. We will further argue that using graphs with many vertices of unbounded degree as the base graph will necessarily generate structurally dense graphs. Since this model includes as a special case of certain types of stochastic block models, it is relevant to the field of complex network.

3.1. The Chung–Lu and configuration model

Given a degree distribution sequence $\mathcal{D} = (D_n)_{n\in\mathbf{N}_0}$ with limit D and an integer n, we are faced with the task to sample graphs uniformly at random from the set

$$\{G \mid G \text{ has degree distribution } D_n\}.$$

Two methods to accomplish this task—with certain caveats—have been put forward: the *configuration model* as described by Bender and Canfield [11] and the model proposed by Chung and Lu [22, 23]. The latter is a special case of what has been discussed in the mathematical literature as inhomogeneous random graphs (see, e.g., the work by Bollobás, Janson, and Riordan on the phase transition of such graphs [13]).

To sample a graph according to the configuration model, we proceed as follows:

1. Build a degree sequence $(d_i)_{1\leqslant i\leqslant n}$ that matches D_n.

2. Construct a vertex set $V^C = \{v_i^1, \dots, v_i^{d_i}\}_{1 \leqslant i \leqslant n}$, i.e. create d_i copies (called *stubs*) for what will be the vertex v^i in the final graph.

3. Generate an auxiliary graph H with vertex set V^C and a random matching as its edge set.

4. Assemble the multi-graph G' with vertex set $\{v_i\}_{1 \leqslant i \leqslant n}$ and

$$|E(v_i, v_j)| = |E_H(\{v_i^1, \dots, v_i^{d_i}\}, \{v_j^1, \dots, v_j^{d_j}\})|,$$

that is, we connect the vertices v_i and v_j with as many edges as we find between their respective copy-classes in H.

5. Return the graph G derived from G' by removing all parallel edges and loops.

Graphs generated this way will henceforth be denoted by $G^{\mathrm{CF}}(D_n)$. The name *stubs* derives from the following picture of the process: we affix to every vertex a number of half-edges, the stubs, that matches its degree according to the generated degree sequence, and then obtain the multi-graph by randomly wiring the stubs to each other. This inspires two other methods of generating the configuration model: Instead of generating a matching between the stubs, we can instead choose a random permutation of them and match them up pair-by-pair according to that permutation. It is easy to see that this method is equivalent to the process described above since every matching of V^C has the same number of permutations that generate it. The second method works as follows. We prescribe an (arbitary) order $u_1, \dots, u_m$ on the set V^C and generate edges one by one as follows: pick the first yet unmatched vertex u_i as the first endpoint of the next edge and then choose the other endpoint u_j, $j > i$, uniformly at random among all yet unmatched vertices. To see that this is equivalent to drawing a matching uniformly at random, simply note that the probability for each fixed matching given the order $u_1, \dots, u_m$ is precisely $1/(m(m-2)(m-4)\dots) = 1/m!!$, or one over the number of matchings on m stubs. Both these alternative views of the process will be beneficial later on.

It is a priori not clear that either procedure generates graphs with the correct degree sequence. The intermediate multi-graph G' trivially exhibits the degree sequence $(d_i)_{1 \leqslant i \leqslant n}$ and thus has the degree distribution D_n. The last step, however, might skew the result by removing parallel edges and loops—we therefore need the probability that G' contains such offending edges to be reasonably low. The conditions under which this is the case have been proved by Molloy and Reed [70] and subsequently improved by Janson, whose result we present here using our own notation.

Theorem 2 (Janson [46])**.** *Let $(D_n)_{n \in \mathbf{N}_0}$ be a degree distribution sequence. Then we have that*

$$\liminf_{n \to \infty} \mathrm{P}[\, G^{CF}(D_n) \text{ is simple}] > 0 \iff \mathrm{E}[D_n^2] = O(\mathrm{E}[D_n]).$$

The original formulation of the theorem's condition is that

$$\sum_{d \geqslant 0} d^2 b_n(d) = O\big(\sum_{d \geqslant 0} d b_n(d)\big).$$

Note that in the case of sparse degree distribution sequences this condition is equivalent to saying that $\mathrm{Var}[D]$ is finite. Luckily, this is the case for all degree distributions listed in Table 1 with the exception of the power-law distribution where the variance is finite only for $\gamma > 3$.

The second method for sampling graphs with a prescribed degree distribution, the Chung–Lu model, forgoes the above problems by generating graphs whose *expected* degree distribution matches the given one. Given D_n, it constructs a random graph as follows[4]:

1. Build a degree sequence $(d_i)_{1 \leqslant i \leqslant n}$ that matches D_n. We will call d_i the *weight* of the vertex i.

2. Create a graph on n vertices $v_1, \dots v_n$ and connect each pair of vertices v_i, v_j, $i < j$, with probability $d_i d_j / m$ where $m = \sum_{k=0}^{n} d_k$.

As network models, both the configuration and the Chung–Lu model suffer from some shortcomings. While they, by design, generate graphs with the correct degree-distribution and small diameter, other statistics found in complex networks are not replicated. In particular, both models have a vanishing clustering-coefficient (see, e.g., Newman's survey [76]).

Since this statistic is critical in many real-world applications, methods to 'fix' these models have been put forward. A notable example is the *configuration model with household structure* as defined by Ball, Sirl, and Trapman [9]. For this variant, one samples a graph with a prescribed degree sequence and then replaces every vertex by a constant-sized 'household'-graph (for example a clique), distributing the edges incident to a household uniformly to the vertices that comprise it. The resulting graph has a provably constant clustering coefficient.

The main goal of this section will be the proof of the following Theorem 3. To this end, we will first show the behaviour of the Chung–Lu random graphs with respect to $\widetilde{\nabla}_0$ and ω. By a simple observation about the probability of short paths existings, we then relate the statistics $\widetilde{\nabla}_r$ and $\omega(\bullet \, \widetilde{\nabla} \, r)$ of different graph models to these base cases. As it turns out, the structural sparseness of the Chung–Lu and the configuration model is entirely determined by the *tail* of the prescribed degree distribution:

Theorem 3. *Let (D_n) be a sparse degree distribution sequence with tail $h(d)$. Both the configuration model $G^{CF}(D_n)$ and the Chung–Lu model $G^{CL}(D_n)$, with high probability,*

- *have bounded expansion for $h(d) = O(1/d^\gamma)$, $\gamma > 3$,*

[4]We excluded loops for simplicity here, including loops does not change our result.

- *are nowhere dense (with unbounded expansion) for* $h(d) = \Theta(1/d^3)$
- *and are somewhere dense for* $h(d) = \Omega(1/d^\gamma)$, $\gamma < 3$.

Since we can emulate household structures by simply taking the lexicographic product with some constant-size clique and then taking a subgraph, Theorem 3 immediately implies the same for those variants.

Corollary 2. *Let* (D_n) *be a sparse degree distribution sequence. Then the configuration model* $G^{CF}(D_n)$ *as well as the Chung–Lu model* $G^{CL}(D_n)$ *with households have bounded expansion w.h.p. if the sequence has a superquadratic tail-bound and are nowhere dense w.h.p. if it has a quadratic tail-bound.*

3.2. Tools for sparse degree distribution sequences

In both the Chung–Lu and the configuration model, the first phase consists of assigning weights to vertices according to a degree distribution D_n. This process chooses degrees without replacement, therefore we need to ensure that the important properties of D_n carry over even if a fraction of the vertices have been uncovered already, that is, we know their weight and hence cannot assume they are randomly distributed.

Since in the following proofs low weights are always preferable, we consider a 'worst-case' variable describing the degree of a vertex after at most n/c other vertices have been assigned the lowest available degrees. Luckily, sparse degree distributions are robust under truncation up to the median. Using this idea, the following lemma shows that we can, up to a point, assume that the vertex degrees are drawn independently according to a modified distribution.

Lemma 3. *Let* (D_n) *be a sparse degree distribution sequence with limit* D *and a tail-bound* $O(h(d)^{-1})$. *Let* $\mu_{1/2} = \mathrm{M}[D]$ *be the median of* D. *Then the sequence* $(\hat{D}_n)$ *defined via*

$$\mathrm{P}[\hat{D}_n = d\,] = \mathrm{P}\big[D_n = d \;\big|\; D_n \geqslant \mu_{1/2}\big]$$

is sparse and has tail-bound $O(h(d)^{-1})$.

Proof. Let $\hat{D}_n$ be defined as the conditioned random variable $(D_n \mid D_n > \mu_{1/2})$, where $\mu_{1/2}$ is the median of D_n. Then we have that

$$\begin{aligned}\mathrm{E}[\hat{D}_n] &= \sum_{d>0} d\,\mathrm{P}[D_n = d \mid D_n \geqslant \mu_{1/2}] = \sum_{d \geqslant \mu_{1/2}} d \cdot \frac{\mathrm{P}[D_n = d\,]}{\mathrm{P}[D_n \geqslant \mu_{1/2}]} \\ &= \frac{\mathrm{E}[D_n]}{\mathrm{P}[D_n \geqslant \mu_{1/2}]} \leqslant \frac{\mathrm{E}[D_n]}{2}.\end{aligned}$$

Hence $\mathrm{E}[\hat{D}_n]$ is finite and $(\hat{D}_n)$ is sparse.

Let τ be the threshold for which the tail-bound $O(h(d)^{-1})$ on (D_n) holds. To see that the same tail-bound works for $(\hat{D}_n)$, note that

$$\mathrm{P}[\hat{D} \geqslant d\,] = \mathrm{P}[D \geqslant d \mid D \geqslant \mu_{1/2}] = \frac{\mathrm{P}[D \geqslant \max\{d, \mu_{1/2}\}]}{\mathrm{P}[D \geqslant \mu_{1/2}]}.$$

Hence for $\hat{\tau} \geqslant \max\{\tau, \mu_{1/2}\}$ for all $d \geqslant \hat{\tau}$ the bound $\mathrm{P}[\hat{D} \geqslant d] = O(h(d)^{-1})$ holds, as claimed. □

In the remainder of this section, we work towards a proof of Theorem 3. Ultimately, our approach relates the density of shallow minors for graphs with degree distribution D to the density of subgraphs of a random graph with distribution ηD, where η is an appropriate scaling factor. This factor traces back to the value of the harmonic sum $\sum_{d=1}^{\Delta} \frac{1}{d^\gamma}$ which is a constant for $\gamma > 1$, roughly $\log \Delta$ for $\gamma = 1$ and approximately $\Delta^{1-\gamma}$ for $\gamma < 1$. We will make use of the following simple bounds for harmonic sums.

Lemma 4. *For all integers $0 < \delta \leqslant \Delta$ the bound $R_\gamma \leqslant \sum_{k=\delta}^{\Delta} \frac{1}{k^\gamma} \leqslant \frac{1}{\delta^\gamma} + R_\gamma$ holds where*

$$R_\gamma = \begin{cases} \frac{1}{\gamma-1}(\delta^{1-\gamma} - \Delta^{1-\gamma}) & \text{for } \gamma > 1, \\ \ln \Delta - \ln \delta & \text{for } \gamma = 1, \text{ and} \\ \frac{1}{1-\gamma}(\Delta^{1-\gamma} - \delta^{1-\gamma}) & \text{for } 0 < \gamma < 1. \end{cases}$$

Proof. The function $k^{-\gamma}$ is monotonically decreasing on $(0, \infty)$ and therefore it is true that

$$\int_\delta^\Delta \frac{1}{k^\gamma}\, dk \leqslant \sum_{k=\delta}^{\Delta} \frac{1}{k^\gamma} \leqslant \frac{1}{\delta^\gamma} + \int_\delta^\Delta \frac{1}{k^\gamma}\, dk.$$

For $\gamma > 1$, the indefinite integral $\int k^{-\gamma}\, dk$ equals $-\frac{1}{(\gamma-1)k^{\gamma-1}}$, for $\gamma = 1$ it is $\ln(k)$, and for $\gamma < 1$ it equals $\frac{1}{(1-\gamma)}k^{1-\gamma}$. Evaluating the integral on the interval $[\delta, \Delta]$ yields R_γ and the claim follows. □

We furthermore need the following bounds for harmonic sums that contain a logarithmic factor:

Lemma 5. *For $\gamma > 0$, $r \geqslant 1$ and integers $r^{2r} < \delta \leqslant \Delta$ the bound*

$$R'_\gamma \leqslant \sum_{k=\delta}^{\Delta} \frac{\ln^r k}{k^\gamma} \leqslant \frac{\ln^r \delta}{\delta^\gamma} + \zeta R'_\gamma$$

holds where ζ is a constant and

$$R'_\gamma = \begin{cases} \frac{1}{\gamma-1}(\delta^{1-\gamma} \ln^r \delta - \Delta^{1-\gamma} \ln^r \Delta) & \text{for } \gamma > 1, \\ \frac{1}{\zeta(r+1)}(\ln^{r+1} \Delta - \ln^{r+1} \delta) & \text{for } \gamma = 1, \text{ and} \\ \frac{1}{(1-\gamma)^{r+1}}(\Delta^{1-\gamma} \ln^r \Delta - \delta^{1-\gamma} \ln^r \delta) & \text{for } 0 < \gamma < 1. \end{cases}$$

Proof. Since $\ln^r(r^{2r}) \leqslant r^{2r\gamma}$ holds for all positive r, the function $\ln^r(k)k^{-\gamma}$ is monotonically decreasing on (r^{2r}, ∞) and therefore it is true that

$$\int_\delta^\Delta \frac{\ln^r k}{k^\gamma}\, dk \leqslant \sum_{k=\delta}^{\Delta} \frac{\ln^r k}{k^\gamma} \leqslant \frac{\ln^r \delta}{\delta^\gamma} + \int_\delta^\Delta \frac{\ln^r k}{k^\gamma}\, dk.$$

For $\gamma > 1$, the integral can be bounded by

$$\frac{\ln^r k}{(\gamma-1)k^{\gamma-1}} + \Theta(1) \leqslant \int \frac{\ln^r k}{k^\gamma}\,dk \leqslant \frac{\zeta \ln^r k}{(\gamma-1)k^{\gamma-1}} + \Theta(1),$$

for some constant ζ. For $0 < \gamma < 1$ we obtain the bounds

$$\Big(\frac{1}{1-\gamma}\Big)^{r+1} k^{1-\gamma} \ln^r k + \Theta(1) \leqslant \int \frac{\ln^r k}{k^\gamma}\,dk \leqslant \zeta\Big(\frac{1}{1-\gamma}\Big)^{r+1} k^{1-\gamma} \ln^r k + \Theta(1).$$

Finally, for $\gamma = 1$, the integral simply evaluates to

$$\int \frac{\ln^r k}{k}\,dk = \frac{\ln^{r+1} k}{r+1} + \Theta(1).$$

Applying the respective indefinite integral to the interval $[\delta, \Delta]$ yields the claimed bounds. □

We will use the following lemma to bound the probability that dense subgraphs appear in a Chung–Lu graph, assuming that the weights of the subgraph obey some bound.

Lemma 6. *Consider k vertices $\{v_i\}_{i\in[k]}$ with associated weights $\{d_i\}_{i\in[k]}$. Let G be a random graph on these vertices where each edge v_iv_j is independently present with probability $\leqslant \beta d_i d_j/n$. Then, for any positive constant ξ,*

$$\mathrm{P}\big[\,|E(G)| \geqslant \xi k\,\big] \leqslant \Big(\frac{e\beta d^2}{2n\xi k e^{d^2/2n}}\Big)^{\xi k}$$

if $d := \sum_i d_i$ satisfies $\beta d^2 \leqslant 2n\xi k$.

Proof. We associate a random variable X_{ij} with every edge v_iv_j. The expected number of edges is then

$$\mathrm{E}\big[\sum_{i<j} X_{ij}\,\big] = \sum_{i<j} \mathrm{E}[\,X_{ij}\,] \leqslant \sum_{i<j} \frac{\beta d_i d_j}{n} \leqslant \frac{\beta}{2n}\Big(\sum_i d_i\Big)^2 = \frac{\beta d^2}{2n}.$$

We apply the Chernoff-bound

$$\mathrm{P}\Big[\sum_{ij} X_{ij} \geqslant (1+\delta)\frac{\beta d^2}{2n}\Big] \leqslant \Big(\frac{e^\delta}{(1+\delta)^{1+\delta}}\Big)^{\beta d^2/2n}$$

choosing $\delta = \frac{2n\xi k}{\beta d^2} - 1$ and obtain

$$\mathrm{P}\Big[\sum_{ij} X_{ij} \geqslant \xi k\Big] \leqslant \frac{e^{\xi k - \beta d^2/2n\xi k}}{(\frac{2n\xi k}{\beta d^2})^{\xi k}} = \Big(\frac{e\beta d^2}{2n\xi k e^{d^2/2n}}\Big)^{\xi k},$$

as claimed. □

Let us convince ourselves that the above lemma will be applicable for *any* choice of k vertices for degree distributions with supercubic tails. To that end, we derive the following bound on the degree-sum of k vertices, which implies that Lemma 5 is applicable to such distributions with $\xi \geqslant 8$.

Lemma 7. *Let (D_n) be a sparse degree distribution sequence with tail-bound $\lambda/d^{\alpha+1}$, $\alpha \geqslant 2$. Let $d_1, \dots, d_k$ be the highest k degrees of D_n. Then, for large enough n,*

$$\Big(\sum_{i=1}^{k} d_i\Big)^2 \leqslant 16nk.$$

Proof. Let $\Delta = (\lambda n)^{\frac{1}{\alpha+1}}$ be the maximum realizable degree of D_n. To bound the sum-of-degrees for the top k vertices, let us determine a degree δ such that

$$\sum_{d=\delta}^{\Delta} \frac{n}{d^{\alpha+1}} \geqslant k.$$

Applying the bound from Lemma 4, we can solve for δ:

$$\sum_{d=\delta}^{\Delta} \frac{n}{d^{\alpha+1}} \geqslant \frac{n}{\alpha}\Big(\frac{1}{\delta^\alpha} - \frac{1}{\Delta^\alpha}\Big) \overset{!}{\geqslant} k \iff \delta \leqslant \big(\alpha k/n + 1/\Delta^\alpha\big)^{-\frac{1}{\alpha}}$$

Given the weight $\delta = (\alpha k/n + 1/\Delta^\alpha)^{-\frac{1}{\alpha}}$ we can now bound the degree-sum of the topmost k vertices by applying the other side of the bound given in Lemma 4:

$$\sum_{d=\delta}^{\Delta} \frac{n \cdot d}{d^{\alpha+1}} \leqslant \frac{n}{\alpha - 1}\Big(\frac{\delta + \alpha - 1}{\delta^\alpha} - \frac{1}{\Delta^{\alpha-1}}\Big) \leqslant \frac{2n}{\alpha-1} \cdot \frac{1}{\delta^{\alpha-1}},$$

were we used that $\delta > \alpha - 1$. Plugging in the above values for δ and Δ we obtain

$$\frac{2n}{\alpha-1}\big(\frac{\alpha k}{n} + (\lambda n)^{-\frac{\alpha}{\alpha+1}}\big)^{\frac{\alpha-1}{\alpha}}$$

as an upper bound for the degree-sum. Let $k := \tau n$ for some $0 < \tau \leqslant 1$. We claim the above expression can be upper-bounded by $\big(\frac{2}{1-\alpha}(2\alpha)^{\frac{\alpha-1}{\alpha}}\big)^2 nk = \big(\frac{2}{1-\alpha}(2\alpha)^{\frac{\alpha-1}{\alpha}}\big)^2 \tau n^2$:

$$\begin{aligned}
&\Big(\frac{2n}{\alpha-1}\big(\frac{\alpha\tau n}{n} + (\lambda n)^{-\frac{\alpha}{\alpha+1}}\big)^{\frac{\alpha-1}{\alpha}}\Big)^2 \overset{!}{\leqslant} \big(\frac{2}{\alpha-1}(2\alpha)^{\frac{\alpha-1}{\alpha}}\big)^2 \tau n^2 \\
\iff &(\alpha\tau)^{\frac{\alpha-1}{\alpha}}\big(1 + (\alpha\tau)^{-1}(\lambda n)^{-\frac{\alpha}{\alpha+1}}\big)^{\frac{\alpha-1}{\alpha}} \overset{!}{\leqslant} (2\alpha)^{\frac{\alpha-1}{\alpha}}\sqrt{\tau} \\
\impliedby \quad &(\alpha\tau)^{\frac{\alpha-1}{\alpha}} 2^{\frac{\alpha-1}{\alpha}} \overset{!}{\leqslant} (2\alpha)^{\frac{\alpha-1}{\alpha}}\sqrt{\tau} \\
\iff &\tau^{\frac{\alpha-2}{2\alpha}} \overset{!}{\leqslant} 1
\end{aligned}$$

where the backwards implication holds when n is large enough such that $(\alpha\tau)^{-1}(\lambda n)^{-\frac{\alpha}{\alpha+1}} \leqslant 1$. Note that since $\alpha \geqslant 2$, the exponent of τ is positive

and hence the last inequality holds. Finally, the function $\big(\frac{2}{1-\alpha}(2\alpha)^{\frac{\alpha-1}{\alpha}}\big)^2$ restricted to $[2,\infty)$ attains its maximum at $\alpha = 2$ with the value 16. Thus, $\big(\frac{2}{1-\alpha}(2\alpha)^{\frac{\alpha-1}{\alpha}}\big)^2 \tau n k \leqslant 16nk$, as claimed. □

We further need a bound on the distribution of the degree-sum for k vertices chosen at random. Note that we can apply the following lemma for drawing k vertices from the *same* distribution by applying Lemma 3 as long as $k \leqslant n/2$.

Lemma 8. *Let (D_n) be a sparse degree distribution with tail-bound $\lambda/d^{\alpha+1}$, $\alpha > 1$ which applies above the threshold τ. Then there exists a constant ζ such that, for large enough n, the probability distribution of the sum of k independent copies $D_n^{(i)}$, $1 \leqslant i \leqslant k$, is bounded by*

$$\mathrm{P}\Big[\sum_{i=1}^{k} D_n^{(i)} \geqslant d\Big] \leqslant \frac{(e\zeta k)^d}{d^d e^{\zeta k}}$$

for every $d \geqslant \zeta k$.

Proof. Let Δ be the largest possible degree that is realisable in D_n and let τ be the threshold at which the tail-bound $h(d)$ holds. Then the expected value of $\mathbb{D} := \sum_i D_n^{(i)}$ is given by

$$\begin{aligned}\mathrm{E}[\mathbb{D}] = k\,\mathrm{E}[D_n] = k\sum_{d=1}^{\Delta} d\,\mathrm{P}[D_n = d] &\leqslant k\Big(\sum_{d=1}^{\tau-1} d + \sum_{d=\tau}^{\Delta}\frac{\lambda}{d^\alpha}\Big)\\ &\leqslant k\Big(\tau^2 + \lambda\big(\frac{1}{\tau^\alpha} + \frac{1}{(\alpha-1)\tau^{\alpha-1}} - \frac{1}{(\alpha-1)\Delta^{\alpha-1}}\big)\Big) =: \zeta k,\end{aligned}$$

where we used the first bound of Lemma 4 for the second inequality. Applying the Chernoff-bound

$$\mathrm{P}\Big[\mathbb{D} \geqslant (1+\delta)\zeta k\Big] \leqslant \Big(\frac{e^\delta}{(1+\delta)^{1+\delta}}\Big)^{\zeta k}$$

with $\delta = \frac{d}{\zeta k} - 1$ results in the bound

$$\mathrm{P}\Big[\mathbb{D} \geqslant d\Big] \leqslant \frac{e^{d-\zeta k}}{(d/\zeta k)^d} = \frac{(e\zeta k)^d}{d^d e^{\zeta k}},$$

as claimed. □

Finally we will need the following bound on the *product* of k randomly chosen vertices. Again, we can apply the following lemma for drawing k vertices from the *same* distribution by applying Lemma 3 as long as $k \leqslant n/2$.

Lemma 9. *Let (D_n) be a sparse degree distribution with (upper) tail-bound $\lambda/d^{\alpha+1}$, $\alpha \geqslant 1$ which applies above the threshold τ. Then, for large enough n, the probability distribution of the product of r independent variables $D_n^{(i)}$, $1 \leqslant i \leqslant r$, is bounded by*

$$\mathrm{P}\Big[\prod_{i=1}^{r} D_n^{(i)} = d\Big] \leqslant \frac{\zeta'(\alpha+1)^r e^{(\alpha+1)\tau}}{(r-1)!}\,\frac{(\ln d)^{r-1}}{d^{(1+\alpha)}},$$

where ζ' is some constant.

Proof. Observe that

$$\mathrm{P}\Big[\prod_{i=1}^{r} D_n^{(i)} = d\Big] = \mathrm{P}\Big[\sum_{i=1}^{r} \ln D_n^{(i)} = \ln d\Big]$$

where, for each $1 \leqslant i \leqslant r$,

$$\mathrm{P}\big[\ln D_n^{(i)} = d\big] \leqslant \frac{\lambda'}{e^{(\alpha+1)(d-\tau)}},$$

for some normalising constant λ'. That is, the random variables $\ln D_n^{(i)}$ follow a (shifted) exponential distribution. Accordingly, the sum of r independent random variables $\ln D_n^{(i)}$ has an Erlang-distribution:

$$\mathrm{P}\Big[\sum_{i=1}^{r} \ln D_n^{(i)} = \hat{d}\Big] \leqslant \frac{\zeta'(\alpha+1)^r}{(r-1)!} \cdot \frac{(\hat{d}-\tau)^{r-1}}{e^{(\alpha+1)(\hat{d}-\tau)}} \leqslant \frac{\zeta'(\alpha+1)^r e^{(\alpha+1)\tau}}{(r-1)!} \cdot \frac{\hat{d}^{r-1}}{e^{(\alpha+1)\hat{d}}},$$

for some scaling constant ζ'. Therefore,

$$\mathrm{P}\Big[\prod_{i=1}^{r} \hat{D}_n^{(i)} = d\Big] = \mathrm{P}\Big[\sum_{i=1}^{r} \ln D^{(i)} = \ln d\Big] \leqslant \frac{\zeta'(\alpha+1)^r e^{(\alpha+1)\tau}}{(r-1)!} \frac{(\ln d)^{r-1}}{d^{(1+\alpha)}}.$$

□

We find that a similar lower-bound holds in cases where we have a lower-bound on the tail-distribution:

Lemma 10. *Let (D_n) be a sparse degree distribution with (lower) tail-bound $\lambda/d^{\alpha+1}$, $\alpha \geqslant 1$ which applies above the threshold τ. Then, for large enough n, the probability distribution of the product of r independent variables $D_n^{(i)}$, $1 \leqslant i \leqslant r$, is lower-bounded by*

$$\mathrm{P}\Big[\prod_{i=1}^{r} D_r^{(i)} = d\Big] \geqslant \frac{\zeta'(\alpha+1)^r e^{(\alpha+1)\tau}}{2^{r-1}(r-1)!} \cdot \frac{(\ln d)^{r-1}}{d^{(1+\alpha)}},$$

where ζ' is some constant and $\ln d \geqslant 2\tau$.

Proof. We proceed analogous to the proof of Lemma 9. Again, the random variables $\ln D_n^{(i)}$, $1 \leqslant i \leqslant r$, follow a (shifted) exponential distribution and the sum of all r variables follows an Erlang-distribution. We now obtain a lower bound using the lower bound on the tail of D_n:

$$\mathrm{P}\Big[\sum_{i=1}^{r} \ln D^{(i)} = \hat{d}\Big] \geqslant \frac{\zeta'(\alpha+1)^r}{(r-1)!} \cdot \frac{(\hat{d}-\tau)^{r-1}}{e^{(\alpha+1)(\hat{d}-\tau)}} \geqslant \frac{\zeta'(\alpha+1)^r e^{(\alpha+1)\tau}}{2^{r-1}(r-1)!} \cdot \frac{\hat{d}^{r-1}}{e^{(\alpha+1)\hat{d}}},$$

for some scaling constant ζ', where we used that $\tau \leqslant \hat{d}/2$. Therefore,

$$\mathrm{P}\Big[\prod_{i=1}^{r} D_n^{(i)} = d\Big] = \mathrm{P}\Big[\sum_{i=1}^{r} \ln D^{(i)} = \ln d\Big] \geqslant \frac{\zeta'(\alpha+1)^r e^{(\alpha+1)\tau}}{2^{r-1}(r-1)!} \cdot \frac{(\ln d)^{r-1}}{d^{(1+\alpha)}},$$

where we used that $\tau \leqslant \ln d/2$. □

3.3. Local density of Chung–Lu graphs

We can now show that Chung–Lu graphs generated with degree distributions that have a supercubic or cubic tail will not contain dense subgraphs. At this point supercubic and cubic tails seems to behave similarly, we will see in the next section why the bounds that hold for subgraphs in both cases will fail for the cubic case once we lift the result to shallow minors.

Lemma 11. *Let (D_n) be a sparse degree distribution sequence with limit D and tail-bound $\lambda/d^{\alpha+1}, \alpha \geqslant 2$. Let ζ be the constant from Lemma 8 associated with (D_n). Then for every $\xi \geqslant e^4\zeta^2 c$, every $c \geqslant 2e$ and every $n \geqslant 4\xi$ it holds that*

$$\mathrm{P}\big[\, \exists H \subseteq G^{CL}(D_n)) : |H| \leqslant n/c \text{ and } \nabla_0(H) \geqslant \xi \,\big] \leqslant \frac{1}{n^\xi}.$$

Proof. Using Lemma 2, we can bound the probability that a dense subgraph on k vertices exists by considering the probability that k randomly chosen vertices form a dense subgraph. Taking the union bound of all possible k (note that we need at least $2\xi + 1$ vertices for a subgraph of density ξ, we simplify this lower bound to 2ξ), the probability of the aforementioned event is at most

$$\sum_{k=2\xi}^{n/c} \binom{n}{k} \mathrm{P}\big[\, \|G^{\mathrm{CL}}(D_n)[X_k])\| \geqslant k\xi \,\big],$$

where X_k is a set of k vertices chosen uniformly at random. Let us write $\mathbb{D}$ to denote the degree-sum of k vertices chosen from D_n. By applying Lemma 6 to the random subgraph $G^{\mathrm{CL}}(D_n)[X_k]$, we can bound the above probability by

$$\begin{aligned}
&\sum_{k=2\xi}^{n/c} \binom{n}{k} \sum_{d=k}^{\Delta k} \left(\frac{ecd^2}{2n\xi k e^{d^2/2n}}\right)^{\xi k} \mathrm{P}[\mathbb{D} = d] \\
&\leqslant \sum_{k=2\xi}^{n/c} \left(\frac{en}{k}\right)^k \left(\frac{ec}{2n\xi k}\right)^{\xi k} \sum_{d=k}^{\Delta k} d^{2\xi k} \,\mathrm{P}[\mathbb{D} = d], \\
&\leqslant \sum_{k=2\xi}^{n/c} \left(\frac{e^2 c}{2\xi} \cdot \frac{k}{n}\right)^{\xi k} \frac{n^k}{k^{(2\xi+1)k}} \sum_{d=k}^{\Delta k} d^{2\xi k} \,\mathrm{P}[\mathbb{D} = d].
\end{aligned}$$

For $d \geqslant \zeta k$, we can apply the bound on $\mathrm{P}[\mathbb{D} \geqslant d]$ from Lemma 8: note that $\mathbb{D} \leqslant \sum_{i=1}^{k} D_n^{(i)}$ in the stochastic sense, i.e. $\mathrm{P}[\mathbb{D} \geqslant d] \leqslant \mathrm{P}[\sum_{i=1}^{k} D_n^{(i)} \geqslant d]$, since summing k independent copies of D_n will result in a larger degree-sum than drawing degrees from D_n without replacement. We split the inner sum at $d = \zeta k$ and obtain

$$\sum_{k=2\xi}^{n/c} \left(\frac{e^2 c}{2\xi} \cdot \frac{k}{n}\right)^{\xi k} \frac{n^k}{k^{(2\xi+1)k}} \left(\sum_{d=k}^{\zeta k - 1} d^{2\xi k} + \sum_{d=\zeta k}^{\Delta k} \frac{d^{2\xi k} (e\zeta k)^d}{d^d e^{\zeta k}} \right).$$

Let us first find an appropriate bound for the two innermost sums. The first one is easily bounded by

$$\sum_{d=k}^{\zeta k-1} d^{2\xi k} \leqslant (\zeta k)^{2\xi k+1} \leqslant k\zeta^{3\xi k} \cdot k^{2\xi k}$$

and we can find a comparable bound for the second part:

Claim.

$$\sum_{d=\zeta k}^{\Delta k} \frac{d^{2\xi k}(e\zeta k)^d}{d^d e^{\zeta k}} \leqslant 3\xi k(e\zeta)^{2\xi k} \cdot k^{2\xi k}$$

Proof. The sum in question has a tail that decreases supergeometrically. In order to identify the value for d from which on this property holds, we consider the ratio of two consecutive terms:

$$\begin{aligned}\frac{(d-1)^{2\xi k}(e\zeta k)^{d-1}}{(d-1)^{d-1}e^{\zeta k}} \cdot \frac{d^d e^{\zeta k}}{d^{2\xi k}(e\zeta k)^d} &= \frac{(d-1)^{2\xi k}}{(d-1)^{d-1}} \cdot \frac{d^d}{d^{2\xi k} e\zeta k} \\ \geqslant \Big(1-\frac{1}{d}\Big)^{d(2\xi k/d-1+1/d)} \cdot \frac{d}{e\zeta k} &\geqslant \Big(\frac{1}{2e}\Big)^{2\xi k/d} \cdot \frac{d}{e\zeta k},\end{aligned}$$

where we used that $(1-1/x)^x \geqslant 1/2e$ for $x \geqslant 2$ and that $d \geqslant k \geqslant 2$. For $d = 2\xi k$, the above expression is simply $\xi/e^2\zeta$ and for $\xi \geqslant e^4\zeta^2 c > 2e^2\zeta$ it is thus at least two. It is easy to verify that this holds also true for all $d \geqslant 2\xi k$, and we can bound the tail of the sum by twice its first summand:

$$\sum_{d=2\xi k}^{\Delta k} \frac{d^{2\xi k}(e\zeta k)^d}{d^d e^{\zeta k}} \leqslant 2\frac{(2\xi k)^{2\xi k}(e\zeta k)^{2\xi k}}{(2\xi k)^{2\xi k} e^{\zeta k}} \leqslant 2\Big(\frac{(e\zeta)^{2\xi}}{e^\zeta}\Big)^k k^{2\xi k} \leqslant 2(e\zeta)^{2\xi k} \cdot k^{2\xi k}.$$

To find a bound for the sum in the range $\zeta k \leqslant d < 2\xi k$, let us express d as $d = \tau k$ with $\zeta \leqslant \tau < 2\xi$. Then a single term of the sum has the form

$$\frac{(\tau k)^{2\xi k}(e\zeta k)^{\tau k}}{(\tau k)^{\tau k} e^{\zeta k}} = \Big(\frac{\tau^{2\xi} k^{2\xi-\tau}(e\zeta k)^\tau}{\tau^\tau e^\zeta}\Big)^k = \Big(\frac{\tau^{2\xi}(e\zeta)^\tau}{\tau^\tau e^\zeta}\Big)^k k^{2\xi k},$$

which again attains its maximum at $2\xi k$. Therefore the first part of the sum is bounded by

$$\sum_{d=\zeta k}^{2\xi k} \frac{d^{2\xi k}(e\zeta k)^d}{d^d e^{\zeta k}} \leqslant (2\xi-\zeta)k \cdot \Big(\frac{(e\zeta)^{2\xi}}{e^\zeta}\Big)^k \cdot k^{2\xi k} \leqslant 2\xi k(e\zeta)^{2\xi k} \cdot k^{2\xi k}$$

and we arrive at the claimed bound by adding the two bounds and bounding $(2\xi k + 2)$ by $3\xi k$ assuming $\xi k \geqslant 2$. □

We combine the bound on the two inner sums to bound the probability that a dense subgraph of $k \leqslant n/c$ vertices exists by

$$\sum_{k=2\xi}^{n/c} \Big(\frac{e^2 c}{2\xi} \cdot \frac{k}{n}\Big)^{\xi k} \frac{n^k}{k^{(2\xi+1)k}} \cdot 4\xi k (e\zeta)^{2\xi k} k^{2\xi k}$$
$$\leqslant \sum_{k=2\xi}^{n/c} 4\xi k \Big(\frac{e^4\zeta^2 c}{2\xi}\Big)^{\xi k} \cdot \Big(\frac{k}{n}\Big)^{\xi k} \frac{n^k}{k^k}.$$

For $\xi \geqslant e^4\zeta^2 c$, the term $4\xi k\big(\frac{e^4\zeta^2 c}{2\xi}\big)^{\xi k}$ is smaller than one, therefore we are left to bound the sum

$$\sum_{k=2\xi}^{n/c} \Big(\frac{k}{n}\Big)^{\xi k} \frac{n^k}{k^k}.$$

We prove that this sum is supergeometric by considering the ratio of two consecutive terms:

$$\Big(\frac{k-1}{n}\Big)^{\xi(k-1)} \frac{n^{k-1}}{(k-1)^{k-1}} \cdot \Big(\frac{n}{k}\Big)^{\xi k} \frac{k^k}{n^k} = \Big(1-\frac{1}{k}\Big)^{(\xi-1)(k-1)} \Big(\frac{n}{k}\Big)^{\xi-1} \geqslant \Big(\frac{n}{ek}\Big)^{\xi-1}.$$

Assuming that $c \geqslant 2e$, this expression is at least two for all term and it follows that we can bound the whole sum by twice its first term:

$$\sum_{k=2\xi}^{n/c} \Big(\frac{k}{n}\Big)^{\xi k} \Big(\frac{n}{k}\Big)^{k} \leqslant 2\Big(\frac{2\xi}{n}\Big)^{2\xi^2} \Big(\frac{n}{2\xi}\Big)^{2\xi} = 2\Big(\frac{2\xi}{n}\Big)^{2(\xi^2-\xi)}$$

which is bounded by $n^{-\xi}$ for $n \geqslant 4\xi$ and $\xi \geqslant 2.5$, proving the lemma. □

We prove a similar bound for the clique-number of Chung–Lu random graphs with degree distributions that have superquadratic tail-bounds:

Lemma 12. *Let (D_n) be a sparse degree distribution sequence with limit D and tail-bound $\lambda/d^{\alpha+1}, \alpha > 1$. Then for large enough n and $\xi \geqslant (9\alpha+9)/(8\alpha-8)$ it holds that*

$$\mathrm{P}\Big[\,\omega(G^{CL}(D_n)) \geqslant 4\sqrt{\xi(\alpha+1)/(\alpha-1)}\,\Big] \leqslant \frac{1}{n^\xi}.$$

Proof. Again using Lemma 2, we can bound the probability that a complete subgraph on $k \geqslant \xi$ vertices exists by considering the probability that k randomly chosen vertices form a clique. Recall that we use the notation $\bar{k}^2 := \binom{k}{2}$ for brevity. Taking the union bound of all possible k, the probability of the aforementioned event is at most

$$\sum_{k=\xi}^{n} \binom{n}{k} \mathrm{P}\big[\, G^{\mathrm{CL}}(D_n)[X_k]) \simeq K_k \,\big],$$

where X_k is a set of k vertices chosen uniformly at random. Given the weights $d_1, \dots, d_k$, of k vertices, the probability that they form a complete subgraph is

$$\frac{\prod_{i=1}^{k} d_i^{k-1}}{(\mu n)^{\bar{k}^2}} = \Big(\frac{\prod_{i=1}^{k} d_i}{(\mu n)^{k/2}}\Big)^{k-1}.$$

The probability in the statement of the lemma crucially depends on the value of the degree-product $\mathbb{D}^k := \prod_{i=1}^{k} D_n^{(i)}$. We condition the probability of a dense subgraph on X_k by the value of $\mathbb{D}^k$ and take the union bound over all possible values and then apply Lemma 9:

$$\begin{aligned}
&\sum_{k=\xi}^{n} \binom{n}{k} \sum_{d=1}^{\Delta^k} \Big(\frac{d}{(\mu n)^{k/2}}\Big)^{k-1} \mathrm{P}\big[\mathbb{D}^k = d\big] \\
&\leqslant \sum_{k=\xi}^{n} \Big(\frac{en}{k}\Big)^k \sum_{d=1}^{\Delta^k} \Big(\frac{d}{(\mu n)^{k/2}}\Big)^{k-1} \frac{\zeta'(\alpha+1)^k e^{(\alpha+1)\tau}}{(k-1)!} \cdot \frac{\ln^{k-1} d}{d^{(1+\alpha)}} \\
&\leqslant \sum_{k=\xi}^{n} \frac{\zeta''\,(2e\alpha)^k}{k^k (k-1)!} \frac{n^k}{(\mu n)^{\bar{k}^2}} \sum_{d=1}^{\Delta^k} d^{k-1-(\alpha+1)} \ln^{k-1} d,
\end{aligned}$$

where $\zeta'' = \zeta' e^{(\alpha+1)\tau}$. By bounding the inner sum by the number of terms times its largest term (which turns out to be the last one), we arrive at

$$\begin{aligned}
&\sum_{k=\xi}^{n} \frac{\zeta''\,(2e\alpha)^k}{k^k (k-1)!} \frac{n^k}{(\mu n)^{\bar{k}^2}} \Delta^{k(k-1-(\alpha+1))+k} (k \ln \Delta)^{k-1} \\
&\leqslant \sum_{k=\xi}^{n} \frac{\zeta''\,(2e\alpha)^k}{k!\,\mu^{\bar{k}^2}} \cdot \frac{n^k \Delta^{2\bar{k}^2 - \alpha k} \ln^{k-1} \Delta}{n^{\bar{k}^2}}.
\end{aligned}$$

Note that the first factor is bounded by a constant, we therefore focus on the second one: if it can be bounded by $n^{-\xi-1}$, the whole sum is bounded by $n^{-\xi}$ (where we invest one factor of $1/n$ to negate the number of terms $\leqslant n$), as claimed. Since Δ is bounded by $\lambda n^{1/(\alpha+1)}$ for some constant λ, this will be the case when

$$\begin{aligned}
&\bar{k}^2 - k - \frac{2}{\alpha+1}\bar{k}^2 + \frac{\alpha}{\alpha+1}k - (k-1) \overset{!}{\geqslant} \xi + 1 \\
\iff \qquad &(1 - \frac{2}{\alpha+1})\bar{k}^2 - (2 - \frac{\alpha}{\alpha+1})k \overset{!}{\geqslant} \xi, \qquad (1)
\end{aligned}$$

which certainly holds true for large enough k if $\alpha > 1$. To compute the threshold for k at which the above inquality holds true, consider quadratic inequalities of the form $A\bar{k}^2 - Bk \geqslant C$. The solutions we are interested in satisfy

$$k \geqslant \frac{A+2B}{2A}\Big(\sqrt{1 + \frac{8AC}{(A+2B)^2}} + 1\Big).$$

Assuming that $8AC \geqslant (A+2B)^2$, we can relax this condition to the more manageable form

$$k \geqslant \frac{A+2B}{A}\Big(\sqrt{\frac{16AC}{(A+2B)^2}}\Big) = \frac{A+2B}{A}\frac{4\sqrt{AC}}{A+2B} = 4\sqrt{C/A}.$$

With $A = (1-2/(\alpha+1))$, $B = (2-\alpha/(\alpha+1))$, and $C = \xi$ we therefore have that, assuming $\xi \geqslant (A+2B)^2/8A = (9\alpha+9)/(8\alpha-8)$, every k larger than $4\sqrt{C/A} = 4\sqrt{\xi(\alpha+1)/(\alpha-1)}$ satisfies inequality (1) and the claim follows. □

Note that the same proof still works if we scale the degree distribution by a small polynomial factor, a fact we will need later:

Corollary 3. *Let (D_n) be a sparse degree distribution with tail-bound $\lambda/d^{\alpha+1}$, $\alpha > 1$ and let $S = O(n^\beta)$ for $\beta < \frac{\alpha-1}{4(\alpha+1)}$. Then for large enough n and $\xi \geqslant \frac{(5\alpha+7)^2}{16(\alpha^2-1)}$ it holds that*

$$\mathrm{P}\Big[\,\omega(G^{CL}(SD_n))) \geqslant 4\sqrt{2\xi(\alpha+1)/(\alpha-1)}\,\Big] \leqslant \frac{1}{n^\xi}.$$

Proof. We proceed as in the proof of Lemma 12. The additional factor of $S \leqslant \zeta n^\beta$ in the weights of the vertices adds a factor of $S^{k(k-1)} \leqslant (\zeta n)^{2\beta\bar{k}^2}$ to the sum and we ultimately need to show that

$$\frac{n^k \Delta^{2\bar{k}^2-\alpha k}\ln^{k-1}\Delta}{n^{\bar{k}^2}}(\zeta n)^{2\beta\bar{k}^2} \overset{!}{\leqslant} \frac{1}{n^{\xi+1}}.$$

Note that we cannot take care of the factor $\zeta^{2\beta\bar{k}^2}$ by the lower-order factors (like $1/k!$) in the proof of Lemma 12, therefore we include it in the following calculation; let $\varepsilon = \log\zeta/\log n$. The above inequality is then equivalent to

$$\frac{n^k(\lambda n)^{(2\bar{k}^2-\alpha k)/(\alpha+1)}\ln^{k-1}\lambda n}{(\alpha+1)^{k-1}n^{\bar{k}^2}} n^{(1+\varepsilon)2\beta\bar{k}^2} \overset{!}{\leqslant} \frac{1}{n^{\xi+1}}.$$

We can ignore the lower-order factors ($\ln^{k-1}\lambda$, $(\alpha+1)^{k-1}$, etc.) in the following. To simplify the proof, we show that already

$$\frac{n^k \cdot n^{(2\bar{k}^2-\alpha k)/(\alpha+1)}n^{k-1}}{n^{\bar{k}^2}} n^{(1+\varepsilon)2\beta\bar{k}^2} \overset{!}{\leqslant} \frac{1}{n^{\xi+1}}$$

holds, where we replaced $\log n$ by n to avoid non-elementary functions like the product logarithm. The above then is equivalent to showing that

$$\begin{aligned} &\bar{k}^2 - k - \frac{2}{\alpha+1}\bar{k}^2 + \frac{\alpha}{\alpha+1}k - k + 1 - (1+\varepsilon)2\beta\bar{k}^2 \overset{!}{\geqslant} \xi + 1 \\ \iff \quad &\Big(1 - \frac{2}{\alpha+1} - (1+\varepsilon)2\beta\Big)\bar{k}^2 - (2 - \frac{\alpha}{\alpha+1})k \overset{!}{\geqslant} \xi, \end{aligned}$$

which is true if $(1+\varepsilon)\beta < \frac{\alpha-1}{2(\alpha+1)}$ and for large enough k. The following calculations become easier if we assume that $(1+\varepsilon)2\beta \leqslant \frac{\alpha-1}{2(\alpha+1)}$, then the above inequality becomes

$$\frac{1}{2}\Big(1-\frac{2}{\alpha+1}\Big)\bar{k}^2-(2-\frac{\alpha}{\alpha+1})k \overset{!}{\geqslant} \xi. \tag{2}$$

Using the formula from Lemma 12 with $A=\frac{1}{2}-\frac{1}{\alpha+1}=\frac{\alpha-1}{2(\alpha+1)}$, $B=2-\frac{\alpha}{\alpha+1}=\frac{\alpha+2}{\alpha+1}$, and $C=\xi$, we obtain that for $\xi \geqslant \frac{(5\alpha+7)^2}{16(\alpha^2-1)}$ and $k \geqslant 4\sqrt{2\xi(\alpha+1)/(\alpha-1)}$ Inequality 2 holds. The earlier condition $(1+\varepsilon)2\beta \leqslant \frac{\alpha-1}{2(\alpha+1)}$ holds for large enough n when $\beta < \frac{\alpha-1}{4(\alpha+1)}$, as claimed. □

3.4. From subgraphs to shallow minors

The next important puzzle piece is expressed in the following lemma: the probability that a short path exists in $G^{\mathrm{CL}}(D_n)$ crucially depends on the weight of its endpoints and the shape of the degree distribution's tail. At this point the difference between a cubic and a supercubic tail becomes visible:

Lemma 13. *Let (D_n) be a sparse degree distribution sequence with tail-bound $\lambda/d^{\alpha+1}$, $\alpha>1$ and maximum realizable degree Δ. Let s,t be vertices with weights d_s, d_t respectively. The probability that there exists an s-t–path of length r in $G^{CL}(D_n)$ is, for large enough n, at most*

$$f_\alpha(r)\frac{d_s d_t}{\mu n} g_\alpha(\Delta, r)$$

for some function f_α independent of n and $g_\alpha(\Delta,r)=1$ for $\alpha>2$, $g_2(\Delta,r)=\ln^r\Delta$, and $g_\alpha(\Delta,r)=\Delta^{r(2-\alpha)}\ln^{r-1}\Delta$ for $\alpha<2$.

Proof. The statement obviously holds for $r=1$, we will therefore assume $r\geqslant 2$ in the following. The probability that $r-1$ fixed vertices $v_1,\dots,v_{r-1}$ with weights $d_1,\dots,d_{r-1}$ form a path from s to t is given by

$$\frac{d_s d_t}{(\mu n)^r}\prod_{i=1}^{r-1} d_i^2 = \frac{d_s d_t}{(\mu n)^r}\Big(\prod_{i=1}^{r-1} d_i\Big)^2$$

where $\mu = E[D_n]$ is a constant for $\alpha>1$. We take a similar approach to the proof of Lemma 12 and condition on the degree-product of the $r-1$ vertices. With this approach, the probability of a path from s to t of length r is at most

$$\binom{n}{r-1}(r-1)!\frac{d_s d_t}{(\mu n)^r}\sum_{d=1}^{\Delta^r} d^2\,\mathrm{P}[\mathbb{D}=d]$$

where $\mathbb{D}$ is the product of r independent copies of D_n. We apply Lemma 9 to the above expression, letting $\zeta'' := \zeta'(\alpha+1)e^{(\alpha+1)\tau}$, and arrive at the upper bound

$$\begin{aligned}&\binom{n}{r-1}(r-1)!\frac{d_s d_t}{(\mu n)^r}\sum_{d=1}^{\Delta^r} d^2\frac{\zeta''}{(r-1)!}\cdot\frac{\ln^{r-1} d}{d^{1+\alpha}}\\ \leqslant\ &\zeta''\Big(\frac{en}{r-1}\Big)^{r-1}\frac{d_s d_t}{(\mu n)^r}\sum_{d=1}^{\Delta^r}\frac{\ln^{r-1} d}{d^{\alpha-1}}\ \leqslant \zeta''\Big(\frac{e}{(r-1)\mu}\Big)^{r-1}\frac{d_s d_t}{\mu n}\sum_{d=1}^{\Delta^r}\frac{\ln^{r-1} d}{d^{\alpha-1}}.\end{aligned}$$

We apply the upper bounds derived in Lemma 5 to the sum (were we split at r^{2r} instead of $(r-1)^{2(r-1)}$ for simplicity) and obtain

$$\begin{aligned}\sum_{d=1}^{\Delta^r}\frac{\ln^{r-1} d}{d^{\alpha-1}} &= \sum_{d=1}^{r^{2r}-1}\frac{\ln^{r-1} d}{d^{\alpha-1}} + \sum_{d=r^{2r}}^{\Delta^r}\frac{\ln^{r-1} d}{d^{\alpha-1}}\\ &\leqslant r^{2r}(2r)^{r-1}\ln^{r-1} r + \frac{(2r)^{r-1}}{r^{2r(\alpha-1)}}\ln^{r-1} r + R_\alpha\\ &\leqslant 2^r r^{3r}\ln^{r-1} r + R_\alpha,\end{aligned}$$

where R_α is bounded by

$$R_\alpha \leqslant \begin{cases}\frac{\zeta 2r}{\alpha-2}r^{2r(2-\alpha)}\ln^{r-1} r & \text{for } \alpha > 2,\\ \ln^r \Delta & \text{for } \alpha = 2, \text{ and}\\ \frac{\zeta r}{(2-\alpha)^r}\Delta^{r(2-\alpha)}\ln^{r-1}\Delta & \text{for } 1 < \alpha < 2.\end{cases}$$

Here ζ is the constant introduced in Lemma 5. By simple gathering of terms and small simplifications we can now bound the probability that a path of length r between vertices s, t exists by

$$\zeta''\Big(\frac{e}{(r-1)\mu}\Big)^{r-1}\frac{d_s d_t}{\mu n}(2^r r^{3r} + \frac{\zeta 2r}{\alpha-2})\ln^{r-1} r \leqslant f_\alpha(r)\frac{d_s d_t}{\mu n}$$

when $\alpha > 2$, bound it by

$$\zeta''\Big(\frac{e}{(r-1)\mu}\Big)^{r-1}\frac{d_s d_t}{\mu n}\cdot 2\ln^r\Delta \leqslant f_2(r)\cdot\frac{d_s d_t}{\mu n}\ln^r\Delta$$

when $\alpha = 2$ (and $\Delta \geqslant e^{2r^3} r$), and bound it by

$$\zeta''\Big(\frac{e}{(r-1)\mu}\Big)^{r-1}\frac{d_s d_t}{\mu n}\cdot\frac{2\zeta r\ln^{r-1}\Delta}{(2-\alpha)^r}\Delta^{r(2-\alpha)} \leqslant f_\alpha(r)\cdot\frac{d_s d_t}{\mu n}\Delta^{r(2-\alpha)}\ln^{r-1}\Delta$$

when $1 < \alpha < 2$ (and $\Delta \geqslant 4r^{3/(2-\alpha)}$) for some function $f_\alpha(r)$ that is independent of n. □

Note that Lemma 13 can be applied even if already up to $n/2$ weights have been uncovered by applying Lemma 3.

We now pair Lemma 11 and Proposition 2 to bound the probability that a dense shallor minor appears. We first formulate a property (in a sense a weak form of coupling) of random graph models that implies bounded expansion. For simplicity, we define $\mathcal{E}^X_{\xi,r}$ as the event that a random graph contains an r-shallow topological minor with nails X of density at least ξ.

Lemma 14. *Let $G^R(n)$ be a random graph model with the following property: for every r there exists a sparse degree distribution (D_n) with tail-bound $h(d) = \lambda/d^{\alpha+1}, \alpha > 1$ such that for every $\xi \geqslant 2e^6\zeta^2$ (where ζ is the constant from Lemma 5) it holds that*

$$\mathrm{P}[\mathcal{E}^X_{\xi,r}] \leqslant \mathrm{P}[\,\widetilde{\nabla}_0(G^{CL}(D_n)[X]) \geqslant \xi\,],$$

where X is a random set of at most $n/2e(r\xi+1)$ vertices. Then $G^R(n)$ has bounded expansion with high probability.

Proof. Note that if the event $\mathcal{E}^X_{\xi,r}$ occurs, it already occurs in a subgraph of size $|X| + r\xi|X|$. Therefore the maximal size of X that needs to be considered in order to apply Proposition 2 is

$$|X| + r\xi|X| \leqslant \frac{n}{2e} \iff |X| \leqslant \frac{n}{2e(r\xi+1)} \leqslant \frac{n}{2e}.$$

Exchanging the probability $\mathrm{P}[\mathcal{E}^X_{\xi,r}]$ by $\mathrm{P}[\widetilde{\nabla}_0(G^{\mathrm{CL}}(n)[X]) \geqslant \xi]$ in the proof of Lemma 11 immediately shows that

$$\sum_{k=2\xi}^{n/2e(r\xi+1)} \binom{n}{k} \mathrm{P}[\mathcal{E}^X_{\xi,r}] \leqslant \sum_{k=2\xi}^{n/2e} \binom{n}{k} \mathrm{P}[\mathcal{E}^X_{\xi,r}] \leqslant \frac{1}{n^\xi}$$

for suitably large n. By Lemma 2, therefore the probability that any set of at most $n/2e$ vertices form the nails of a dense r-shallow minor is at most $n^{-\xi}$. Accordingly, setting $f_{\widetilde{\nabla}} = \xi$ and $f_H = 1/2e$, the second condition of Proposition 2 holds with probability at least $1-n^{-\xi}$. It is left to show that functions $f_{\mathrm{thresh}}, f_{\mathrm{deg}}$ exist.

Claim. *Let (f_n) be the probability mass functions and D the limit of (D_n). Then every graph matching (D_n), for n sufficiently large, satisfies Condition 1 of Proposition 2.*

Recall that Condition 1 states that there exist functions $f_{\mathrm{thresh}}, f_{\mathrm{deg}}$ such that for all ε we either have $|G| \leqslant f_{\mathrm{thresh}}(\varepsilon)$ or it holds that

$$|\{v \in V(G)\colon \deg(v) \geqslant f_{\mathrm{deg}}(\varepsilon)\}| \leqslant \varepsilon \cdot |G|.$$

This translates to (D_n) as follows: for every $\varepsilon > 0$ there exists an integer $0 \leqslant d \leqslant n-1$ such that

$$n\sum_{k=d}^{n-1} f_n(k) \leqslant \varepsilon n \iff \sum_{k=d}^{n-1} f_n(k) \leqslant \varepsilon.$$

We apply Markov's inequality to find that

$$\sum_{k=d}^{n-1} f_n(k) = \mathrm{P}[D_n \geqslant d] \leqslant \frac{\mathrm{E}[D_n]}{d}.$$

Since $\mathrm{E}[D_n] \longrightarrow \mathrm{E}[D]$ and $\mathrm{E}[D]$ is finite, the right hand side can be made small enough by choosing $f_{\deg}(\varepsilon) = d = \mathrm{E}[D_n]/\varepsilon$ and n large enough. This proves the existence of appropriate functions f_{thresh} and $f_{\deg}$ and the claim.

Hence, we conclude that Proposition 2 is applicable to $G^{\mathrm{R}}(n)$ with probability at least $(1 - n^{-\xi})$ and the claim follows. □

Combining Lemma 14 with Lemma 11 gives us a proof for the first claim of Theorem 3 and combining Lemma 14 with Corollary 3 gives us a proof for the positive part of the second claim.

Having shown that supercubic tails produce structurally sparse graphs in the Chung–Lu model, we proceed to the next range of distributions.

3.5. The quadratic regime

As before, we will work towards an argument which translates probabilities for events on shallow topological minors to events on subgraphs. Starting at the bottom, we begin by proving an analogue of Lemma 11 for degree distributions that are scaled by a factor of $\log^{\Theta(1)}(n)$.

Let $\mathcal{K}_r^X$ denote the event that the vertices of X form the nails of an $(\leqslant r)$-subdivision of a complete graph. With this notation, the following is derived easily:

Corollary 4. *Let $G^R(n)$ be a random graph model with the following property: for every r there exists a sparse degree distribution (D_n) with tail-bound $h(d) = \Theta(d^{3+o(1)})$ and $\xi \geqslant 7$ such that*

$$\mathrm{P}[\mathcal{K}_r^X] \leqslant \mathrm{P}[G^{CL}(\log^{\Theta(1)}(n)D_n)[X] \simeq K_{|X|}],$$

where X is a random set of at most ξ vertices. Then $G^R(n)$ is nowhere dense with high probability.

Proof. By assumption, the probability that a clique of size $k \geqslant \xi$ appears as an $(\leqslant r)$-subdivision in $G^{\mathrm{R}}(n)$ is bounded by the probability that a clique of size k appears in the scaled model $G^{\mathrm{CL}}(\log^{\Theta(1)}(n)D_n)$. We apply Corollary 3 with $\alpha = 2 + o(1)$ to bound the probability of the latter event by

$$\mathrm{P}\left[\omega\big(G^{\mathrm{CL}}(\log^{\Theta(1)}(n)D_n)\big) \geqslant 4\sqrt{6\xi}\right] \leqslant \frac{1}{n^{\xi}}.$$

For this application of Corollary 3 we need that ξ is at least

$$\frac{(5\alpha+7)^2}{16(\alpha^2-1)} = \frac{(17+o(1))^2}{16(3+o(1))} = \frac{289+o(1)}{48+o(1)} \leqslant 7,$$

where the last inequality holds for large enough n. We conclude that $G^{\mathrm{R}}(n)$ is nowhere dense with high probability. □

The above corollary will later provide the positive statement, namely, that large shallow clique-minors are vanishingly improbable. However, Theorem 3 also states that cubic degree distributions do *not* result in graphs with bounded expansion. The following lemma provides us with the necessary negative statement.

Lemma 15. *Let (D_n) be a sparse degree distribution sequence with lower tail-bound λ/d^3. Then*

$$\widetilde{\nabla}_1(G^{CL}(D_n)) = \Omega(\log^2 n)$$

with high probability.

Proof. Let us write λ/d^3 for the tail-bound and let it hold for degree larger than the threshold τ. Hence the maximum realizable degree is $\Delta = (\lambda n)^{1/3}$. Let V_h contain all vertices of weight at least $\Delta/2$, applying Lemma 4 we find that

$$|V_h| = \sum_{d=\Delta/2}^{\Delta} \frac{\lambda n}{d^3} \geqslant \frac{\lambda n}{2}\big((\Delta/2)^{-2} - \Delta^{-2}\big) \geqslant \frac{3\lambda n}{2\Delta^2}$$

and

$$|V_h| = \sum_{d=\Delta/2}^{\Delta} \frac{\lambda n}{d^3} \leqslant \frac{2^3}{\Delta^3} + \frac{\lambda n}{2}\big((\Delta/2)^{-2} - \Delta^{-2}\big) \leqslant \frac{3\lambda n}{\Delta^2},$$

which holds for large enough n.

Let us further write V_δ for the set of all vertices of weight exactly δ. Then the expected number of V_δ-neighbors of a vertex $x \in V_h$ is

$$\mathrm{E}[|N(x) \cap V_\delta|]_{x \in V_h} \geqslant |V_\delta| \cdot \frac{\delta\Delta}{2\mu n} \geqslant \frac{\lambda n}{\delta^3} \cdot \frac{\delta\Delta}{2\mu n} = \frac{\lambda}{2\mu} \cdot \frac{\Delta}{\delta^2}.$$

The expected number of V_h-neighbors of a vertex $y \in V_\delta$, on the other hand, is

$$\mathrm{E}[|N(y) \cap V_h|]_{y \in V_\delta} \leqslant |V_h| \cdot \frac{\delta\Delta}{\mu n} \leqslant \frac{3\lambda n}{\Delta^2} \cdot \frac{\delta\Delta}{\mu n} = \frac{3\lambda}{\mu} \cdot \frac{\delta}{\Delta},$$

which for $\delta \leqslant \Delta/\log n$ is at most $\frac{3\lambda}{\mu} \cdot \frac{1}{\log n}$. We apply the Chernoff-bound

$$\mathrm{P}[|N(y) \cap V_h| \geqslant (1+\sigma)\eta] \leqslant \Big(\frac{e^\sigma}{(1+\sigma)^{1+\sigma}}\Big)^\eta$$

with $\eta = \frac{3\lambda}{\mu} \cdot \frac{1}{\log n}$ and $(1+\sigma)\eta = 1$. The latter implies that $1 + \sigma = 1/\eta$ and $\sigma = 1/\eta - 1$. Thus for $\eta < 1$ we have that $\sigma > 0$ and the bound applies. In that case, we have that

$$\mathrm{P}[|N(y) \cap V_h| \geqslant 1] \leqslant \eta e^{\eta\delta} = \eta e^{1-\eta} \leqslant \eta e = \frac{3\lambda}{\mu} \cdot \frac{e}{\log n}.$$

By assuming $\log n$ to be large enough, we can make the right hand side arbitrarily small. Thus of all the vertices in $N(x) \cap V_\delta$ for any $\delta \leqslant \Delta/\log n$ and $x \in V_h$,

we expect that at most, say, half of them have neighbors other than x in V_h. Since these events are independent, this will occur with high probability for every $x \in V_\delta$ for large enough n. For $x \in V_h$ let therefore $S_x \subseteq N(x)$ contain all neighbors of x that a) have weight at most $\Delta/\log n$ and b) are not connected to any other vertex in V_h. By the above arguments, we have that

$$\begin{aligned}\sum_{u\in S_x} \delta_u &\geqslant \sum_{\delta=\tau}^{\Delta/\log n} \delta \cdot |N(x) \cap V_\delta| \geqslant \sum_{\delta=\tau}^{\Delta/\log n} \delta \cdot \frac{\lambda}{4\mu} \cdot \frac{\Delta}{\delta^2} \\ &\geqslant \frac{\lambda}{4\mu}\Delta \sum_{\delta=\tau}^{\Delta/\log n} \frac{1}{\delta} \geqslant \frac{\lambda}{4\mu}\Delta(\ln(\Delta/\log n) - \ln\tau) \\ &\geqslant \frac{\lambda \ln 2}{4\mu}\Delta(\log\Delta - \log\log n - \log\tau) \geqslant \frac{\lambda}{16\mu}\Delta\log\Delta, \qquad (\star)\end{aligned}$$

where the last inequality holds when n is large enough such that $\frac{1}{2}\log\Delta \geqslant \operatorname{loglog} n - \log\tau$.

Now consider two sets S_x, S_z for distinct $x, z \in V_h$. By the above, we may assume that they both have a total weight of at least $\frac{\lambda}{16\mu}\Delta\log\Delta$. Therefore the expected number of edges between S_x and S_z is at least

$$\sum_{u\in S_x}\sum_{v\in S_z} \frac{\delta_u\delta_v}{\mu n} \geqslant \Big(\frac{\lambda}{16\mu}\Delta\log\Delta\Big)^2 \frac{1}{\mu n}.$$

Consider the graph H on vertices V_h obtained by contracting each set S_x into the respective vertex $x \in V_h$. In expectation we then have

$$\|H\| \geqslant \binom{|V_h|}{2}\Big(\frac{\lambda}{16\mu}\Delta\log\Delta\Big)^2 \frac{1}{\mu n}$$

and thus we expect the density of H to be tightly concentrated around

$$\begin{aligned}\frac{\|H\|}{|H|} &\geqslant \frac{1}{2}(|V_h|-1)\Big(\frac{\lambda}{16\mu}\Delta\log\Delta\Big)^2 \frac{1}{\mu n} \\ &\geqslant \frac{1}{4}\frac{|V_h|}{\mu n}\Big(\frac{\lambda}{16\mu}\Delta\log\Delta\Big)^2 \geqslant \frac{1}{4}\frac{3\lambda n}{2\mu n\Delta^2}\Big(\frac{\lambda}{48\mu}\Delta\log\lambda n\Big)^2 \\ &\geqslant \frac{\lambda^3}{6144\mu^3}\log^2\lambda n = \Omega(\log^2 n).\end{aligned}$$

The graph H is obtain by contracting sets of radius 1 and thus is a 1-shallow minor of G. This proves the claimed lower bound on $\widetilde{\nabla}_1(G)$. □

The construction for the configuration model is the same but relies on slightly different arguments. Central here is the argument that the probability that a *stub* x will be paired with another stub y after t pairs (none of which contain x or y) have already been drawn is given by $\frac{1}{\mu n - 2t - 1}$, i.e. the stub y is chosen uniformly at random among all still available stubs. The probability $p_{A,B}$ that

two sets of stubs A, B (for example stubs belonging to two different vertices) will be connected by a pair is therefore bounded by

$$1-\Big(1-\frac{|B|}{\mu n-2t-1}\Big)^{|A|/2}\leqslant p_{A,B}\leqslant 1-\Big(1-\frac{|B|}{\mu n-2t-|A|-1}\Big)^{|A|}\qquad(\star\star)$$

if t pairs have already been drawn. We prove the following lemma to obtain a more manageable bound:

Lemma 16. *Let t pairs have already been drawn in the random process to construct an instance of $G^{CF}(D_n)$ and let A, B be two sets of stubs that have not yet been paired with $|A|\leqslant|B|$. Then the probability $p_{A,B}$ that there exist stubs $a\in A, b\in B$ such that a will be paired with b in the remainder of the process is bounded by*

$$\frac{1}{4}\frac{|A||B|}{\mu n-2t}\leqslant p_{A,B}\leqslant 2\frac{|A||B|}{\mu n-2t},$$

assuming that $|A||B|\leqslant\mu n-2t$ and $\mu n-2t\geqslant 8$.

Proof. We will use in the following that $(1-x)^t\leqslant 1-\frac{xt}{2}$ for all $x\leqslant\frac{1}{t-1}$. Applying this inequality to the second term of the lower bound in $(\star\star)$, we find that

$$\Big(1-\frac{|B|}{\mu n-2t-1}\Big)^{|A|/2}\leqslant 1-\frac{|A|}{4}\frac{|B|}{\mu n-2t-1}\leqslant 1-\frac{1}{4}\frac{|A||B|}{\mu n-2t}$$

which holds if

$$\frac{|B|}{\mu n-2t-1}\leqslant\frac{1}{|A|-1}\iff|B|(|A|-1)\leqslant\mu n-2t-1.$$

The latter is implied by the lemma's simpler condition that $|A||B|\leqslant\mu n-2t$.

For the other direction, we use the well-known Bernoulli inequality $(1-x)^t\geqslant 1-xt$ which holds for $x\leqslant 1$. Applying it to the second term of the upper bound in $(\star\star)$ gives us

$$\Big(1-\frac{|B|}{\mu n-2t-|A|-1}\Big)^{|A|}\geqslant 1-\frac{|A||B|}{\mu n-2t-|A|-1}\geqslant 1-2\frac{|A||B|}{\mu n-2t},$$

where the second inequality holds when

$$1-\frac{|A|+1}{\mu n-2t}\geqslant\frac{1}{2}\iff|A|\leqslant\frac{1}{2}(\mu n-2t)-1.$$

For $\mu n-2t\geqslant 8$ (actually, $\mu n-2t\geqslant 4+2\sqrt{3}$) the latter inequality is implied by the lemma's conditions that $|A||B|\leqslant\mu n-2t$ and $|A|\leqslant|B|$. □

In particular, up to constant scaling the above probability looks very similar to the probability that two vertices with weights $|A|$,$|B|$ are connected in the Chung–Lu model and this remains true even if already a constant fraction of the stubs have been paired.

Lemma 17. *Let (D_n) be a sparse degree distribution sequence with lower tail-bound λ/d^3. Then*

$$\widetilde{\nabla}_1(G^{CF}(D_n)) = \Omega(\log^2 n)$$

with high probability.

Proof. We proceed as in the proof of Lemma 15. As there, let Δ denote the maximum realizable degree and let V_h contain all vertices of degree at least $\Delta/2$; the bounds

$$\frac{3\lambda n}{2\Delta^2} \leqslant |V_h| \leqslant \frac{3\lambda n}{\Delta^2}$$

hold as before for large enough n. We adapt the construction of the graph $G^{\mathrm{CF}}(D_n)$ by first pairing all stubs from V_h (potentially with other stubs from V_h). After these first $\leqslant 2|V_h|$ pairings, the partially constructed graph now contains all edges that have at least one endpoint in V_h, thus $N(V_h)$ is known.

At this stage, the number m of remaining stubs is bounded by

$$m \geqslant \mu n - \frac{3\lambda n}{\Delta^2} \cdot \Delta = \mu n - 3(\lambda n)^{2/3} \geqslant \frac{\mu n}{2},$$

where the inequality holds for $n \geqslant 216\lambda^2/\mu^3$, and

$$m \leqslant \mu n - \frac{3\lambda n}{2\Delta^2} \cdot \frac{\Delta}{2} < \mu n.$$

Accordingly, the expected number of V_δ-neighbors of a vertex $x \in V_h$

$$\mathrm{E}[|N(x) \cap V_\delta|]_{x \in V_h} \geqslant |V_\delta| \cdot \frac{\delta\Delta}{m} \geqslant \frac{\lambda n}{\delta^3} \cdot \frac{\delta\Delta}{\mu n} = \frac{\lambda}{\mu} \cdot \frac{\Delta}{\delta^2}.$$

The expected number of V_h-neighbors of a vertex $y \in V_\delta$, on the other hand, is

$$\mathrm{E}[|N(y) \cap V_h|]_{y \in V_\delta} \leqslant |V_h| \cdot \frac{\delta\Delta}{m} \leqslant \frac{6\lambda n}{\Delta^2} \cdot \frac{\delta\Delta}{\mu n} = \frac{6\lambda}{\mu} \cdot \frac{\delta}{\Delta},$$

which for $\delta \leqslant \Delta/\log n$ is at most $\frac{6\lambda}{\mu} \cdot \frac{1}{\log n}$. We skip the concentration-argument laid out in the proof of Lemma 15 since the calculations are exactly the same up to the change of the factor 3 to the factor 6. As in that proof, we can assume in the following that for $x \in V_h$ at most a constant fraction (say, half) of the vertices in $N(x) \cap V_\delta$ have neighbors other than x in V_h. Let us choose exactly $\frac{\lambda\Delta}{4\mu\delta^2}$ vertices from $N(x) \cap V_\delta$ whose only neighbor in V_h is x and collect all such vertices, for all $\delta \leqslant \Delta/\log n$, in a set S_x.

Note that at this stage we have only paired stubs belonging to vertices of V_h to other stubs. Hence for $u \in S_x$ we know that only one stub of u has been paired so far and $\delta_u - 1$ stubs are still left. Therefore the total number of remaining stubs that belong to vertices in S_x is now given by

$$\sum_{u \in S_x} (\delta_u - 1) \geqslant \sum_{u \in S_x} \delta_u - \sum_{\delta=1}^{\Delta/\log n} \frac{\lambda\Delta}{4\mu\delta^2}.$$

We can use the bound ($\star$) proved in Lemma 15 for the first sum (it differs by a factor of two because of the slight difference in the bounds proved above and we inherit the constraint that $\frac{1}{2}\log\Delta \geqslant \operatorname{loglog} n - \log\tau$), let us therefore focus on the second sum which bounds the size of S_x:

$$\sum_{\delta=1}^{\Delta/\log n} \frac{\lambda\Delta}{4\mu\delta^2} = \frac{\lambda\Delta}{4\mu}\sum_{\delta=1}^{\Delta/\log n}\frac{1}{\delta^2} \leqslant \frac{\lambda\Delta}{4\mu}\Big(\frac{1}{\tau} + \sum_{\delta=\tau}^{\Delta/\log n}\frac{1}{\delta^2}\Big)$$
$$\leqslant \frac{\lambda\Delta}{4\mu}\Big(\frac{1}{\tau}+\frac{1}{\tau^2}+\frac{1}{\tau}-\frac{\log n}{\Delta}\Big) \leqslant \frac{3\lambda\Delta}{4\mu},$$

which holds when $\log n \leqslant \Delta = (\lambda n)^{1/3}$. We therefore will have at least

$$\frac{\lambda}{8\mu}\Delta\log\Delta - \frac{3\lambda}{4\mu}\Delta \geqslant \frac{\lambda}{16\mu}\Delta\log\Delta$$

remaining stubs belonging to vertices in S_x (where the above inequality holds for $\log\Delta \geqslant 12$). We now proceed to construct the graph $G^{\mathrm{CF}}(D_n)$ by pairing all stubs in each S_x for all $x \in V_h$; notice that the total number of stubs paired this way is sublinear in μn. Accordingly, the probabilities involved will look very similar to those in the Chung–Lu model.

Let $\mathcal{S}_u$ contain all stubs belonging to vertices in S_u and let $\mathcal{S} = \bigcup_{u\in V_h}\mathcal{S}_u$. If we continue the pairing process by pairing all stubs in $\mathcal{S}$, we can use the following crude lower bound on the expected number of stub-pairs with *both* endpoints in $\mathcal{S}$: note that while we are pairing the first $|\mathcal{S}|/4$ stubs, the number of still available stubs in $\mathcal{S}$ is at least $|\mathcal{S}|/2$. Hence, the probability that any of the first $|\mathcal{S}|/4$ stubs is paired with another stub from $\mathcal{S}$ is at least $|\mathcal{S}|/2m$, accordingly we expect at least $|\mathcal{S}|^2/4m$ stub-pairs that have both endpoints in $\mathcal{S}$. Concerning the number of *edges* created through these pairings, note that the probability of a self-loop or parallel edge is on the order of $1/|V_h|$ and we can therefore expect a total of at least $|\mathcal{S}|^2/8m$ edges for large enough n (using the very crude bound that at most half the created edges are loops or parallel).

Let us finally assemble the minor H by contracting every set S_x onto x for all $x \in V_h$. As argued in Lemma 15, the density of H is, with high probability, at least

$$\frac{\|H\|}{|H|} \geqslant \frac{|\mathcal{S}|^2}{8m}\cdot\frac{1}{|V_h|} \geqslant \frac{(|V_h|\cdot\frac{\lambda}{16\mu}\Delta\log\Delta)^2}{|V_h|8\mu n} = \frac{|V_h|\cdot\lambda^2\Delta^2\log^2\Delta}{2048\mu^3 n} \geqslant \frac{\lambda^3\log^2\Delta}{1024\mu^3},$$

which is the claimed bound of $\Omega(\log^2\Delta) = \Omega(\log^2 n)$ on $\widetilde{\nabla}_1(G^{\mathrm{CF}}(D_n))$. □

Having characterised the cubic regime, we proceed to the final range; degree distributions with a subcubic tail.

3.6. The subcubic regime

We now prove that a degree distribution with a tail lower-bounded by $d^{3-\varepsilon}$ for any $\varepsilon > 0$ will with high probability result in the presence of shallow dense

clique minors of arbitrary size, making the model somewhere dense in this regime. Because we can leverage the powerful Theorem 1, the proof is quite straightforward.

Lemma 18. *Let (D_n) be a sparse degree distribution sequence with a tail lower-bounded by $\frac{\lambda}{d^{3-\varepsilon}}$ for some $\varepsilon > 0$. Then $G^{CL}(D_n)$ is somewhere dense with high probability.*

Proof. We proceed analogous to the proof of Lemma 15. Let us write λ/d^γ for the tail-bound with $\gamma = 3 - \varepsilon$, then the maximum realizable degree is $\Delta = (\lambda n)^{1/\gamma}$. Let again V_h contain all vertices of weight at least $\Delta/2$. The bounds

$$\frac{3\lambda n}{2\Delta^{\gamma-1}} \leqslant |V_h| \leqslant \frac{3\lambda n}{\Delta^{\gamma-1}}$$

established in the proof of Lemma 15 still apply for large enough n. Accordingly, we can with high probability find sets $\{S_x\}_{x\in V_h}$ where a) every $y \in S_x$ satisfies $N(y) \cap V_h = \{x\}$ and b) $|S_x \cap V_\delta| \geqslant \frac{\lambda}{4\mu}\frac{\Delta}{\delta^{\gamma-1}}$. Importantly, the total weight of these sets S_x differs from the previous case: because $\gamma - 2 < 1$, the application of Lemma 4 results in a different bound. Concretely:

$$\begin{aligned}\sum_{u\in S_x} \delta_u &\geqslant \sum_{\delta=\tau}^{\Delta/\log n} \delta \cdot |N(x) \cap V_\delta| \geqslant \frac{\lambda}{4\mu}\Delta \sum_{\delta=\tau}^{\Delta/\log n} \frac{1}{\delta^{\gamma-2}} \\ &\geqslant \frac{\lambda}{4\mu}\frac{\Delta}{3-\gamma}((\Delta/\log n)^{3-\gamma} - \tau^{3-\gamma}) \qquad (\star\star\star) \\ &\geqslant \frac{\lambda}{8\mu(3-\gamma)}\frac{\Delta^{4-\gamma}}{\log^{3-\gamma} n}\end{aligned}$$

holds with high probability, where $(\star\star\star)$ holds for $(\Delta/\log n)^{3-\gamma} \geqslant 2\tau^{3-\gamma}$ and therefore holds for large enough n.

Now consider two sets S_x, S_z for distinct $x, z \in V_h$. By the above, we may assume that they both have a total weight of at least $\frac{\lambda}{8\mu(3-\gamma)}\frac{\Delta^{4-\gamma}}{\log^{3-\gamma} n}$. Therefore the expected number of edges between S_x and S_z is at least

$$\sum_{u\in S_x}\sum_{v\in S_z} \frac{\delta_u\delta_v}{\mu n} \geqslant \Big(\frac{\lambda}{8\mu(3-\gamma)}\frac{\Delta^{4-\gamma}}{\log^{3-\gamma} n}\Big)^2 \frac{1}{\mu n}.$$

The graph H on vertices V_h obtained by contracting each set S_x into the respective vertex $x \in V_h$ has therefore, in expectation,

$$\binom{|V_h|}{2}\Big(\frac{\lambda}{8\mu(3-\gamma)}\frac{\Delta^{4-\gamma}}{\log^{3-\gamma} n}\Big)^2 \frac{1}{\mu n}$$

edges, thus H has (with high probability) a density of at least

$$\begin{aligned}\frac{\|H\|}{|H|} &\geqslant \frac{1}{2}(|V_h|-1)\Big(\frac{\lambda}{8\mu(3-\gamma)}\frac{\Delta^{4-\gamma}}{\log^{3-\gamma} n}\Big)^2\frac{1}{\mu n}\\ &\geqslant \frac{1}{4}\frac{|V_h|}{\mu n}\Big(\frac{\lambda}{8\mu(3-\gamma)}\frac{\Delta^{4-\gamma}}{\log^{3-\gamma} n}\Big)^2 \geqslant \frac{1}{6}\frac{\lambda n}{\mu n}\frac{1}{\Delta^{2-\varepsilon}}\Big(\frac{\lambda}{8\mu\varepsilon}\frac{\Delta^{1+\varepsilon}}{\log^{\varepsilon} n}\Big)^2\\ &\geqslant \frac{\lambda^3}{384\varepsilon^2\mu^3}\frac{\Delta^{3\varepsilon}}{\log^{2\varepsilon} n} = \Omega\Big(\frac{n^{3\varepsilon/(3-\varepsilon)}}{\log^{2\varepsilon} n}\Big) = \Omega\Big(n^{\frac{\varepsilon}{2(1-\varepsilon/3)}}\Big).\end{aligned}$$

The graph H is obtain by contracting sets of radius 1 and thus is a 1-shallow minor of G. Because it has (strict) superlinear density $\Omega(n^{\varepsilon'})$ for $\varepsilon' = \frac{\varepsilon}{2(1-\varepsilon/3)}$, Theorem 1 applies, meaning that for every $\ell \in \mathbf{N}$ there exists n large enough such that the constructed graph H (and hence G) contains K_ℓ as a shallow minor with high probability. In other words, $G^{\mathrm{CL}}(D_n)$ is somewhere-dense with high probability. □

3.7. The proof of Theorem 3

We begin with the proof for the Chung–Lu model since the application of Lemma 14 is straightforward. Afterwards, we will show how they can be adapted to extend the proof to the configuration model.

Lemma 19. *Let (D_n) be a sparse degree-distribution whose tail is upper-bounded by h for degrees above τ. Let s, t be vertices in $G^{CL}(D_n)$. Then for every $r \in \mathbf{N}$ it holds that*

$$\mathrm{P}\left[\exists P_{st} \subseteq G^{CL}(D_n), |P_{st}| = r \mid d_s, d_t\right] = \frac{d_s d_t}{n} O(\mathrm{E}[D_n^2]^{r-1})$$

and this bound still holds if up to $n/2$ weights have been uncovered.

Proof. Consider the probability that a path P_{st} with endpoints s, t is realized in $G := G^{\mathrm{CL}}(D_n)$ whose vertices have the weights $d_s, d_1, \ldots, d_{r-1}, d_t$:

$$\mathrm{P}\big[\, P_{st} \subseteq G \mid d_s, d_t \,\big] = \binom{n}{r-1}\frac{d_s d_t \prod_i d_i^2}{\mu^r n^r}\,\mathrm{P}[\mathbb{D}^r = (d_1, \ldots, d_{r-1})],$$

where $\mathbb{D}^r$ is a random $(r-1)$-tuple drawn from D_n without replacement (since we condition on d_s, d_t we only draw the inner $r-1$ weights of P_{st}). Since, by assumption, only a constant fraction of the vertex weights been uncovered, instead of the weights without replacement, we can use $\hat{D}_n$ from Lemma 3 to sample the weights d_i independently. Recall that $(\hat{D}_n)$ has the same tail-bound h as (D_n) and note that

$$\mathrm{E}[D_n^2] = \sum_{d=1}^{\tau-1}\mathrm{P}[D_n = d]\cdot d^2 + \sum_{d=\tau}^{\Delta}\frac{d^2}{h(d)} = \Theta(\mathrm{E}[\hat{D}_n^2]).$$

Let now $\hat{D}_{1,n}, \dots, \hat{D}_{r-1,n}$ be independent copies of $\hat{D}_n$ used to sample the weights $d_1, \dots, d_{r-1}$. Taking the union-bound over all possible weights, we have that

$$\begin{aligned}\prod_i d_i^2 \cdot \mathrm{P}\big[\bigwedge_i \hat{D}_{i,n} = d_i\big] &\leqslant \sum_{d_1,\dots,d_{r-1}} \prod_i d_i^2 \cdot \mathrm{P}[\hat{D}_{i,n} = d_i] \\ &= \prod_{d_1,\dots,d_{r-1}} \sum_i d_i^2 \cdot \mathrm{P}[\hat{D}_{i,n} = d_i] = \prod_{d_1,\dots,d_{r-1}} \mathrm{E}[\hat{D}_{i,n}^2] \\ &= \Theta(\mathrm{E}[D_n^2]^{r-1}).\end{aligned}$$

We arrive at the upper bound

$$\mathrm{P}[P_{st} \subseteq G \mid d_s, d_t, F] \leqslant \binom{n}{r-1} \frac{d_s d_t}{\mu^r n^r} \Theta(\mathrm{E}[D_n^2]^{r-1}) = \frac{d_s d_t}{n} O(\mathrm{E}[D_n^2]^{r-1}),$$

where we used that μ is a constant. □

We finally have all the ingredients for the main proof.

Proof of Theorem 3 for $G^{CL}(D_n)$. First consider a sparse degree distribution sequence (D_n) with tail-bound $h(d) = \Omega(d^{3+\varepsilon})$ for some $\varepsilon > 0$. By Lemma 19, the probability of an s-t-path of length $\leqslant r$ existing in $G := G^{\mathrm{CL}}(D_n)$ is

$$\begin{aligned}\mathrm{P}[\exists P_{st} \subseteq G] &\leqslant \sum_{r'=1}^{r} \frac{d_s d_t}{n} O(\mathrm{E}[D_n^2]^{r'-1}) = \frac{d_s d_t}{n} O(\mathrm{E}[D_n^2]^{r'-1}) \\ &= \frac{1}{n}\big(d_s O(\sqrt{\mathrm{E}[D_n^2]^{r-1}}) \cdot d_t O(\sqrt{\mathrm{E}[D_n^2]^{r-1}})\big)\end{aligned}$$

and we can interpret the right-hand side as the probability that an edge exists between s, t in a Chung–Lu graph with scaled distribution $O(\sqrt{\mathrm{E}[D_n^2]^{r-1}})D_n$. Because $h(d)$ is supercubic, $\mathrm{E}[D_n^2]$ is a constant and so is the scaling factor $c_r := O(\sqrt{\mathrm{E}[D_n^2]^{r-1}})$. Accordingly,

$$\mathrm{P}[\exists P_{st} \subseteq G] \leqslant \mathrm{P}[st \in G^{\mathrm{CL}}(c_r D_n)]$$

and this relation is still true if conditioned by the knowledge of up to $n/2$ vertex-weights.

Let $\mathcal{E}^X_{\xi,r}$ denote the event that an $(\leqslant r)$-subdivision of density at least ξ with nails X exists. Then for any random set X of at most $n/2$ vertices we have that

$$\mathrm{P}[\mathcal{E}^X_{\xi,r}]_{G^{\mathrm{CL}}(D_n)} \leqslant \mathrm{P}\big[\nabla_0(G^{\mathrm{CL}}(c_r D_n)[X])) \geqslant \xi\big] = \mathrm{P}[\mathcal{E}^X_{\xi,r}]_{G^{\mathrm{CL}}(c_r D_n)}.$$

Thus by Lemma 14 and the fact that $c_r D_n$ is sparse and has the same tail-bound as D_n, the model $G^{\mathrm{CL}}(D_n)$ has bounded expansion with high probability.

Next, assume (D_n) has a tail $h(d) = \Theta(d^3)$ and hence $\mathrm{E}[D_n^2] = \Theta(\log n)$. Applying Lemma 19, the probability of an s-t-path of length r existing in $G := G^{\mathrm{CL}}(D_n)$ is therefore

$$\begin{aligned}\mathrm{P}[\exists P_{st} \subseteq G] &\leqslant \mathrm{P}[st \in G^{\mathrm{CL}}(\Theta\big(\sqrt{\mathrm{E}[D_n^2]^{r-1}}\big) D_n)] \\ &\leqslant \mathrm{P}[st \in G^{\mathrm{CL}}(\Theta(\log^{\Theta(1)}(n) D_n))]\end{aligned}$$

and this relation is still true if conditioned by the knowledge of up to $n/2$ vertex-weights. Let again $\mathcal{K}_r^X$ denote the event that an $(\leqslant r)$-subdivision of a complete subgraph with nails X exists in G. Since the graph G is sparse with high probability, we focus on the case $r \geqslant 1$. Now for any random set X of at most $\sqrt{n/2r}$ vertices we have that

$$\begin{aligned}\mathrm{P}[\mathcal{K}_r^X]_{G^{\mathrm{CL}}(D_n)} &\leqslant \mathrm{P}\big[G^{\mathrm{CL}}(\log^{\Theta(1)}(n)D_n)[X]) \simeq K_{|X|}\big] \\ &= \mathrm{P}[\mathcal{K}_r^X]_{G^{\mathrm{CL}}(\log^{\Theta(1)}(n)D_n)}.\end{aligned}$$

and thus by Corollary 4 it follows that $G^{\mathrm{CL}}(D_n)$ is nowhere dense with high probability. By Lemma 15, we further have that already the measure $\widetilde{\nabla}_1(G)$ grows at a rate of at least $\Omega(\log^2 n)$, hence $G^{\mathrm{CL}}(D_n)$ has unbounded expansion.

Finally, assume (D_n) has a tail-bound $h(d) = O(d^{3-\varepsilon})$ for some $\varepsilon > 0$. By Lemma 18 we already have that $G^{\mathrm{CL}}(D_n)$ is somewhere dense with high probability. □

This proof can be extended to the configuration model, the main difficulty here is that edges are not sampled independently of each other. We first prove a variant of Lemma 19. The bound proved here crucially depends on the number of *unmatched* stubs: recall that, instead of matching up stubs by choosing a random matching, we can match them up pair-by-pair (cf. beginning of Section 3.1). From this perspective we can stop the process at any point and express the probabilities at this stage in terms of the remaining number of stubs.

Lemma 20. *Let (D_n) be a sparse degree-distribution whose tail is bounded by h for degrees above τ. Let s, t be vertices in $G^{CF}(D_n)$. Then for every $r \in \mathbf{N}$ it holds that*

$$\mathrm{P}[\exists P_{st} \subseteq G^{CF}(D_n), |P_{st}| = r \mid d_s, d_t] = \frac{d_s d_t}{m} O(\mathrm{E}[D_n^2]^{r-1}).$$

where m is the number of unmatched stubs.

Proof. Let $G := G^{\mathrm{CF}}(D_n)$. By $M(n) := (n-1)!!$ we denote the number of matchings on n vertices, where !! denotes the double factorial:

$$n!! := \begin{cases} n \cdot (n-2) \cdot \ldots \cdot 5 \cdot 3 \cdot 1 & \text{for } n > 0 \text{ odd,} \\ n \cdot (n-2) \cdot \ldots \cdot 6 \cdot 4 \cdot 2 & \text{for } n > 0 \text{ even, and} \\ 1 & n \in \{0, -1\}. \end{cases}$$

We will need the following bound for $k < n$:

$$\frac{M(n-k)}{M(n)} \leqslant \Big(\frac{(2e)^k (n-k)^{n-k}}{n^n}\Big)^{1/2} \leqslant \Big(\frac{2e}{n}\Big)^{k/2}.$$

The number of available stubs decreases with each edge added to the graph and hence the probability of an edge crucially depends on the number m of *remaining* stubs.

Fix a path P_{st} of length r and let $d_1, \dots, d_{r-1}$ denote the weights of its internal vertices. The probability of this path existing in G, conditioned on the weights of its endpoints, is bounded by

$$\begin{aligned}\mathrm{P}[P_{st} \subseteq G \mid d_s, d_t] &\leqslant d_s d_t \frac{M(m-2r)}{M(m)} \sum_{d_1,\dots,d_{r-1}} \prod_{i=1}^{r-1} d_i^2 \,\mathrm{P}[\hat{D}_n = d_i] \\ &\leqslant \frac{d_s d_t}{m^r} O(\mathrm{E}[D_n^2]^{r-1}).\end{aligned}$$

Therefore the probability that *some* s-t-path of length r exists is

$$\mathrm{P}[\exists P_{st} \subseteq G \mid d_s, d_t] \leqslant \frac{d_s d_t}{m} O(\mathrm{E}[D_n^2]^{r-1}),$$

as claimed. □

Proof of Theorem 3 for $G^{CF}(D_n)$. By Lemma 20, the probability of an s-t-path of length r existing in $G := G^{\mathrm{CF}}(D_n)$ is

$$\mathrm{P}[\exists P_{st} \subseteq G^{\mathrm{CF}}(D_n), |P_{st}| = r \mid d_s, d_t] = \frac{d_s d_t}{m} O(\mathrm{E}[D_n^2]^{r-1}).$$

Note that this probability looks almost identical to the one given by Lemma 19, provided that the number of remaining stubs m is $\Theta(n)$. Since we want to estimate the probability of the event $\mathcal{E}^X_{r,\xi}$, only up to $r\xi|X|$ edges need to be considered at once; meaning that at least

$$m - 2r\xi|X| \geqslant 2\mu n - 2r\xi n/4e(r\xi+1) \geqslant (\mu-1)2n$$

stubs remain (where $\mu = \mathrm{E}[D_n]$). As in the proof for the Chung–Lu model, we have that

$$\mathrm{P}[\mathcal{E}^X_{r,\xi}]_{G^{\mathrm{CF}}(D_n)} \leqslant \mathrm{P}[\mathcal{E}^X_{r,\xi}]_{G^{\mathrm{CL}}(\Theta(D_n))}$$

and we conclude that $G^{\mathrm{CF}}(D_n)$ has bounded expansion for distributions with tail-bound $\Omega(d^{3+\varepsilon})$. Similarly, the event $\mathcal{K}^X_r$ for any set of vertices $|X| \leqslant \sqrt{n/2r}$ concerns at most $n/2$ edges and hence the number of stubs left is $m = \Theta(n)$. Thus

$$\mathrm{P}[\mathcal{K}^X_r]_{G^{\mathrm{CF}}(D_n)} \leqslant \mathrm{P}[K^X_r]_{G^{\mathrm{CL}}(\Theta(D_n))}$$

and therefore $G^{\mathrm{CF}}(D_n)$ is nowhere dense for distributions with a tail that is in $\Theta(d^3)$. The lower bounds provided by Lemma 15 and Lemma 18 can be easily adapted in a similar way to apply to the configuration model. □

3.8. Perturbed bounded-degree graphs

An interesting application of the well-understood Erdős-Rényi random graphs is to generate a *perturbation* of some n-vertex base graph $G^\star$. We will use the notation $G = G^\star + G(n, \mu/n)$ to denote the graph obtained from $G^\star$ by adding every possible edge not already contained in $G^\star$ independently with

probability μ/n. We also allow the graph $G^\star$ to be random; in that case, $G^\star + G(n, \mu/n)$ denotes the random graph process of drawing a graph of size n according to $G^\star$ and then adding the perturbation edges as above.

This procedure is more flexible than many existing generalizations of the Erdős–Rényi model, like (sparse) *inhomogeneous random graphs*[13] or (sparse) *generalized random graphs*[17, 79]. Similar (e.g. [55]) and more general (e.g. [7]) models have been defined before, yet there seems to be no consensus on a name or notation.

In particular, uniform perturbation can be seen as the baseline for more complicated models, like the small-world model by Kleinberg (described below), models used in percolation theory (e.g. [12]) and the hybrid model by Chung and Lu [24] (described above). The central question is: what graph classes are still structurally sparse after the addition of few random edges?

We will call the graph $G^\star$ drawn in the first step the *base graph*. In the Erdős–Rényi model, the base graph $G^\star$ would be the edgeless graph and the edge probabilities p_n constant functions for all n.

Theorem 4. *Let $\mathcal{G}$ be a class of bounded-degree graphs and μ a constant. Let $G^{\mathcal{G}}$ be a random graph model which draws graphs from $\mathcal{G}$ with an arbitrary probability distribution. Then the composite model $G^{\mathcal{G}} + G(n, \mu/n)$ has bounded expansion with high probability.*

Note that this theorem in particular applies to $G(n, \mu/n)$ itself. The result carries over to the *stochastic block model*, if the parameters involved are small enough. This model was first studied in mathematical sociology by Holland, Laskey, and Leinhardt in 1983 [43] and extended by Wang and Wong to directed graphs [97]. We will supplement the above result by demonstrating that there exist very sparse graph classes of unbounded degree for which such a perturbation results in dense clique minors.

The following technical lemma subsumes Theorem 4. For a fixed graph G, let $D_{r,G}$ be a random variable which describes the size of the r-th neighbourhood $|N^r(x)|$ for a uniformly chosen random vertex $x \in G$. The distribution of $D_{r,G}$, given by

$$\Pr[D_{r,G} = d] = \frac{|\{x \in G : |N^r(x)| = d\}|}{|G|},$$

is an important factor in whether graph classes maintain bounded expansion under perturbation:

Lemma 21. *Let $\mathcal{G}$ be a class of graphs with the following properties:*

- *$\mathcal{G}$ has bounded expansion, and*
- *for $G \in \mathcal{G}$ and every $r \in \mathbf{N}$ the distribution of $\mathcal{N}^r$ has a tail-bound h with $h(d) = \Omega(d^{3+\varepsilon})$ for some $\varepsilon > 0$.*

Let $G^{\mathcal{G}}$ be a random graph model which draws graphs from $\mathcal{G}$ with an arbitrary probability distribution. Then $G^{\mathcal{G}}(n) + G(n, \mu/n)$ has bounded expansion with high probability.

Proof. Let $G_\nabla \in \mathcal{G}$, $\widetilde{G} = G(n, \mu/n)$ and let $G = G_\nabla + \widetilde{G}$. Assume H is an r-shallow topological minor of G and consider an embedding ϕ_V, ϕ_E of H witnessing this fact. Since $\widetilde{\nabla}_r(G_\nabla)$ is a constant, most of H's density must depend on random edges, i.e.

$$|\{e \in H \mid \phi_E(e) \cap E(\widetilde{G}) = \varnothing\}| \leqslant \widetilde{\nabla}_r(G_\nabla)|H|.$$

Therefore it suffices to bound the density of topological minors whose embedding use at least one edge of $\widetilde{G}$ for each edge of the minor. Consider a path P of length r in G that uses at least one edge of $\widetilde{G}$: each component of $P \setminus E(\widetilde{G})$ is contained in a subgraph $G_\nabla[N^r(v)]$ for some vertex v. Let $N_1, N_2, \ldots, N_p$ be these subgraphs of the path P: then we can bound the probability that P exists by considering the probability that there exist at least one edge between N_i and N_{i+1} in $\widetilde{G}$, for $1 \leqslant i \leqslant p-1$.

Since the probability that two r-neighbourhoods $N^r(u)$, $N^r(v)$ in G_∇ are connected by an edge in $\widetilde{G}$ is at most

$$\frac{\mu|N^r(u)||N^r(v)|}{n}$$

we can stochastically bound the occurrence of r-paths in G by the occurrence of edges in $G^{\mathrm{CL}}(D_{r,G})$. Hence we have that

$$\widetilde{\nabla}_r(G) - \widetilde{\nabla}_r(G_\nabla) \leqslant \widetilde{\nabla}_r(G^{\mathrm{CL}}(D_{r,G}))$$

in the stochastic sense. Since the latter has bounded expansion with high probability by Theorem 3, we conclude that $\mathcal{G} + G(n, \mu/n)$ does as well. □

The above result poses the question: are there structurally sparse classes which do *not* stay sparse under perturbation? The answer is yes: consider the class of graphs consisting of $\Theta(\sqrt{n})$ copies of $S_{\sqrt{n}}$. The probability that two such stars will be connected by a randomly added edge is lower-bounded by some constant, hence the minor obtained by contracting the former stars has density $\Theta(n)$ (while only having $\sqrt{n}$ vertices). Hence, the perturbed class is actually somewhere dense with high probability.

This example can be easily generalised: the presence of n^α vertices X to which we can assign at least n^β of their respective r-neighbours (not assigning any neighbour to more than on vertex of X), for some constant r, will yield an r-shallow minor whose density is concentrated around $n^{3\alpha+\beta-1}$. Hence for all α, β that satisfy $3\alpha + \beta > 2$, the perturbed class is somewhere dense with high probability.

4. Graph Models without Bounded Expansion

In this section we consider the Kleinberg [49, 50] and Barabási-Albert [6, 10] Models, which, respectively, were designed to replicate "small-world" properties and heavy-tailed (power-law) degree distributions observed in complex networks.

We show that both these models (with typical parameters) do not have bounded expansion, and in fact are somewhere dense w.h.p./non-vanishing probability, respectively. This is done by showing the existence of two/one-subdivisions of cliques respectively in the generated graphs with a certain probability.

4.1. The Kleinberg Model

Many social networks exhibit a property that is commonly referred to as the "small-world phenomenon." This property asserts that any two people in a network are likely to be connected by a short chain of acquaintances. This was first observed by Stanley Milgram in a study published in 1967 [68]. Milgram's study suggested that individuals in a social network who only knew the locations of their immediate acquaintances are collectively able to construct short chains between two points in the network. More recently, Kleinberg proposed a family of network models to explain the success of decentralized algorithms in finding short paths in social networks [50].

Kleinberg's model starts with a $n \times n$ grid as the base graph and allows edges to be directed. For a universal constant $p \geqslant 1$, a node u has a directed edge to every other node within lattice distance p. These are the *local neighbors* of u. For universal constants $q \geqslant 0$ and $r \geqslant 0$, node u has q *long range neighbors* chosen independently at random. The i^{th} directed outarc from u has endpoint v with probability $d(u,v)^{-r} / \sum_x d(u,x)^{-r}$.

When $r = 0$, the long-range contacts are uniformly distributed throughout the grid, and one can show that there exist paths between every pair of nodes of length bounded by a polynomial in $\log n$, exponentially smaller than the number of nodes. Kleinberg shows that in this case, the expected delivery time of *every* decentralized algorithm (one that uses only local information) is $\Omega(n^{2/3})$. When $p = q = 1$ and $r = 2$, then short chains continue to exist between the nodes of the network, but here is a decentralized algorithm to transmit a message that takes $O(\log^2 n)$ time in expectation between any two randomly chosen points.

What Kleinberg's model shows is that if the long-range contacts are formed independently of the geometry of the grid, then short chains exist between every pair of nodes, but nodes working with local knowledge are unable to find them. If the long-range contacts are formed by taking into account the grid structure in a specific way, then short chains exist and nodes working with local knowledge are able to discover them. We show that for those parameters where greedy routing is efficient, not only does the model not have bounded expansion, it is in fact, *somewhere dense* w.h.p.

Theorem 5. *The Kleinberg model with parameters $p = q = 1$ and $r = 2$ is somewhere dense w.h.p.*

Proof. Let Γ_n be an $n \times n$ grid. For $p = q = 1$ and $r = 2$, the probability that a node u has v as its long-range contact is proportional to $d_{\Gamma_n}(u,v)^{-1}$ and the normalizing factor in this case is $O(1/\log n)$. This can be easily seen by summing up $1/d_{\Gamma_n}(u,x)^2$ for all x and noticing that in the grid, there are $4d$

neighbors that are at a distance of d from u.

$$\sum_x \frac{1}{d_{\Gamma_n}(u,x)^2} = \sum_{d=1}^{n} \frac{4d}{d^2} \sim 4 \log n.$$

To show that the model is somewhere dense, we show that 2-subdivisions of cliques of a certain size $g(n)$ occur with high probability. Later we will see that $g(n) = \Omega(\log \log n)$. To this end, let $\Gamma'_{c \cdot g(n)}$ denote some fixed $c \cdot g(n) \times c \cdot g(n)$ subgrid of Γ_n, where c is some constant that we will fix later. Choose V' and E' to be, respectively, a set of $g(n)$ nodes and a set of $g(n)^2$ edges from the subgrid $\Gamma'_{c \cdot g(n)}$ with the following properties: (i) the endpoints of the edges in E' are different from the nodes in V'; (ii) no two edges in E' share an endpoint.

Given any pair of vertices $u, v \in V'$ and an edge $e \in E'$ with endpoints a, b, the probability that a has u as its long-range neighbor is $\Omega((d_{\Gamma_n}(a,u) \cdot \log n)^{-1})$. Similarly, the probability that b has v as its long-range neighbor is $\Omega((d_{\Gamma_n}(b,v) \cdot \log n)^{-1})$ The probability of both these events happening is

$$\frac{1}{d_{\Gamma_n}(a,u)^2 d_{\Gamma_n}(b,v)^2} \cdot \frac{1}{\log^2 n} \geqslant \frac{1}{c^4 g(n)^4 \log^2 n}, \tag{3}$$

where we upper-bounded distances $d_{\Gamma_n}(x,y)$ by $c \cdot g(n)$. Thus the probability that there exists a 2-subdivided $g(n)$-clique in $\Gamma'_{c \cdot g(n)}$ is at least:

$$\left(\frac{1}{c^4 g(n)^4 \log^2 n} \right)^{g(n)^2} =: f(n,c). \tag{4}$$

The probability that there does *not* exist a 2-subdivided $g(n)$-clique in $\Gamma'_{c \cdot g(n)}$ is at most $1 - f(n,c)$. Hence the probability that there does not exist a 2-subdivided $g(n)$-clique in *any* $c \cdot g(n) \times c \cdot g(n)$ subgrid is at most

$$(1 - f(n,c))^{\frac{n}{c^2 g(n)^2}} \leqslant \exp\left(- \frac{n}{c^2 \cdot g(n)^2 \cdot \left(c^4 \cdot g(n)^4 \cdot \log^2 n\right)^{g(n)^2}} \right) := e^{\frac{n}{h(n)}}.$$

This follows from the inequality $(1 - x/p)^p \leqslant e^{-x}$.

Choose $g(n) = \log \log n$ and $c = 3$ (actually any $c \geqslant 3$ works). Then it is easy to show that $h(n) < \sqrt{n}$. Thus the probability of a 2-subdivided $(\log \log n)$-clique *not* existing is at most $e^{-\sqrt{n}}$ (which goes to zero as $n \to \infty$), and we conclude that the graph model is somewhere dense. □

4.2. The Barabási-Albert model

The Barabási-Albert model uses a preferential attachment paradigm to produce graphs with a degree distribution that mimics the heavy-tailed distribution observed in many real-world networks [10]. This model uses a random graph process that works as follows: Start with a small number n_0 of nodes and at every time step, add a new node and link it to $q \leqslant n_0$ nodes already present in

the "system." To model preferential attachment, we assume that the probability with which a new node u is connected to node v already present in the system is proportional to the degree of v, so that $\mathrm{P}[u \to v] = \deg(v)/\sum_x \deg(x)$, where the sum in the denominator is over all vertices x that are already in the system. After t time steps, the model leads to a random network with $t + n_0$ vertices and qt edges.

Barabási and Albert suggested that such a network evolves into one in which the fraction $P(d)$ of nodes of degree d is proportional to $d^{-\gamma}$. They observed experimentally that $\gamma = 2.9 \pm 0.1$ and suggested that γ is actually 3. This model was rigorously analyzed by Bollobás, Riordan, Spencer and Tusnády in [15] who showed that it is indeed the case that the fraction of vertices of degree d fall off as d^{-3} as $d \to \infty$. In [14], Bollobás and Riordan showed that the diameter of the graphs generated by this model is asymptotically $\log n / \log \log n$.

We first provide a formal restatement of the Barabási-Albert Model. Note that this is slightly different from the formalization of Bollobás et al. in [15]. We start with a "seed" graph G_0 with n_0 nodes $u_1, \ldots, u_{n_0}$ with degrees $d_1, \ldots, d_{n_0}$. The number of edges in the seed graph is denoted by m_0. At each time step $t = 1, 2, \ldots$, we create a graph G_t by adding a new node v_t and linking it to q nodes in G_{t-1}. These q nodes are picked independently and with a probability that is proportional to their degrees in G_{t-1}. That is, we choose $u \in V(G_{t-1})$ to link to with probability

$$\mathrm{P}\big[\{v_t, u\} \in E(G_t)\big] = \frac{\deg_{G_{t-1}}(u)}{2 \cdot |E(G_{t-1})|} = \frac{\deg_{G_{t-1}}(u)}{2(m_0 + q(t-1))}.$$

Note that in this model, all edges between v_t and nodes of G_{t-1} are assumed to be added simultaneously (so that the increasing degrees of nodes which receive edges from v_t do not influence the probabilities for this time step.) In this restated version, n_0, $d_1, \ldots, d_{n_0}$, and q are the parameters of the model.

Lemma 22. *Given any fixed r, a graph G_n generated by the preferential attachment model with parameters n_0, $d_1, \ldots, d_{n_0}$ and $q \geqslant 2$ has a 1-subdivided K_r as a subgraph with probability at least $(4(\frac{m_0}{q} + r + r^2))^{-r^2}$, provided $n \geqslant r + r^2$.*

Proof. Choose any $r \in \mathbf{N}$. We will show that there exists a 1-subdivided K_r in the graph with probability that depends only on m_0, q, and r.

Consider the graph after the first $r + r^2$ time steps. Let $v_1, \ldots, v_r$ be the nodes that were added in the first r time steps and fix two nodes v_i, v_j from among these. The probability that a new node v_k (for any $r + 1 \leqslant k \leqslant r^2$) is connected to these two fixed nodes is at least

$$\left(\frac{q}{2(m_0 + q(r + r^2))}\right)^2 =: f(m_0, r, q),$$

where the denominator is the sum of the vertex degrees after $r + r^2$ time steps. Now if v_{r+1} is linked to v_1, v_2 and v_{r+2} is linked to v_1, v_3 and so on such that the nodes added after time step r connect the first r nodes in a pairwise fashion,

we would have a 1-subdivided K_r in the graph G_{r+r^2}. The probability of this happening is at least $f(m_0, r, q)^{r^2}$. Thus, for every r, the probability of 1-subdivided K_r existing is non-zero if the graph is large enough. □

It immediately follows that the Barabási-Albert model (and similar preferential attachment models) is not a.a.s. nowhere dense and in particular does not have bounded expansion a.a.s. We note that this result is more of theoretical interest, since the probabilities involved might be small enough to be irrelevant in practice. As such it would be worthwhile to investigate whether the Barabási-Albert model is somewhere dense a.a.s.

5. Experimental Evaluation

Although it has been established that real-world networks are sparse, and tend to have low degeneracy (relative to the size of the network), it is natural to ask whether there is empirical evidence that they satisfy the stronger conditions of bounded expansion. Unfortunately, since bounded expansion itself is a property of graph classes and not single graphs, it is impossible to determine whether individual instances have bounded expansion or not. One natural proxy would be to evaluate the *grad* of these graphs, calculating the maximum density of an r-shallow minor for each $r \in \mathbf{N}$ (obviously stopping when r is the diameter of the network), but it is not known how to find such minors in reasonable time.

In order to get around these difficulties, we calculate upper bounds on χ_{p-1} (the p-centered coloring number), a good proxy, since it and the grad are both related to each other by factors independent of the graph size. This is further justified by the fact that p-centered colorings are directly applicable to algorithm design, where the complexity of such algorithms depends heavily on the number of required colors (as will be shown in Section 6). Since it is very time-consuming to obtain a good p-centered coloring for large p (this is analogous to determining a reasonable bound for the maximum density of an r-shallow minor for large r), we evaluate this property for small values only. This is also roughly the range of p which is relevant to the algorithms presented later in this paper when applied to practical settings.

To obtain upper bounds on χ_{p-1}, we implemented the transitive-fraternal augmentation procedure of Nešetřil and Ossona de Mendez [73]. Our theoretical results predict that graphs generated with the configuration model for typical degree distributions of complex networks will likely have bounded expansion. We thus compared the results for χ_3 of this procedure between real networks and networks generated using the configuration model for the same degree distributions. We chose χ_3 since it is relatively easy to compute for large networks but still heavily influenced by one-subdivisions of cliques. The results of this experiment can be found in Figure 2.

For almost all networks the bound for the real-world network is either smaller or comparable to the values for the synthetic graphs. The one exception is "power", which is not surprising since this network has a relatively complex (grid-like) structure over low degree nodes.

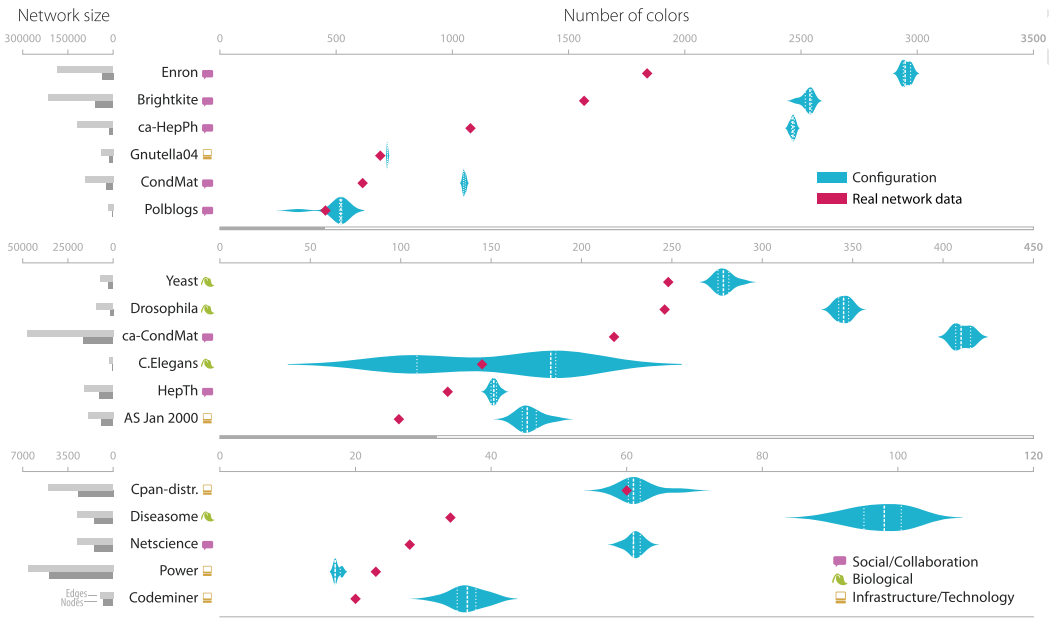


Figure 2: Comparison of 4-centered coloring numbers on real-world networks (red diamonds) compared to synthetic graphs (blue violins) with the same degree distribution. Each violin represents 10 random instances generated with the configuration model, with median and quartiles marked with dashed and dotted lines. Networks are partitioned into three groups by size (indicated on the left) to enable rescaling axes. See Table 2 for data sources.

We furthermore extended the algorithm based on transitive-fraternal augmentations with simple heuristic improvements (e.g. giving high degree nodes their own private color, merging color classes where possible), ran it to find p-centered colorings on a small corpus of well-known complex networks and verified the results. The results of the best colorings we were able to achieve with this relatively simple method can be found in Table 2.

The results show that some networks clearly have a moderately growing grad; in particular the larger networks *Netscience*, both *Cpan*-networks and *Diseasome*. Other networks, like *Twittercrawl*, have such quickly growing p-centered coloring numbers that we did not invest the time to determine the value for larger p. Since graphs of bounded crossing number (as infrastructure networks tend to be) and bounded degree have bounded expansion, we are not surprised at the small number of colors needed by *Power* and *Hex*. Finally some networks, like *CondMat* and *Hep-th*, start out reasonably well for small p but show a sudden jump at $p = 3$. At present we do not know whether this is an artifact of the procedure we use to obtain the coloring or whether the networks have indeed dense minors from a certain depth on. As shown in Section 4, some complex network models predict such an occurrence already for depth at most two. The growth behavior for small p as depicted in Table 2 might therefore serve as a

			p					
Network	Vertices	Edges	2	3	4	5	6	∞
Karate [99]	35	78	6	7	9	9	10	8
Dolphins [62]	62	159	7	11	17	18	19	24
Lesmiserables [52]	77	254	10	15	16	16	16	16
Polbooks [54, 77]	105	441	8	16	22	29	31	30
Word-adjacencies [81]	112	425	8	18	27	35	41	48
Football [39]	115	613	9	22	33	49	62	69
Airlines [2]	235	1297	11	28	39	47	55	64
Sp-data-school [93]	238	5539	23	100	138	157	168	171
C.Elegans [98]	306	2148	8	36	74	83	118	153
Hex-grid	331	930	3	9	20	21	25	69
Codeminer [2]	724	1017	5	10	15	17	23	51
Cpan-authors [1]	839	2212	9	24	34	43	47	224
Diseasome [40]	1419	2738	12	17	22	25	30	30
Polblogs [3]	1491	16715	30	118	286	354	392	603
Netscience [81]	1589	2742	20	20	28	28	28	20
Drosophila [94]	1781	8911	12	65	137	188	263	395
Yeast [19]	2284	6646	12	38	178	254	431	408
Cpan-distr. [1]	2719	5016	5	14	32	42	56	224
Twittercrawl [2]	3656	154824	89	561	1206	1285	1341	–
Power [98]	4941	6594	6	12	20	21	34	95
AS Jan 2000 [57]	6474	13895	12	29	70	102	151	357
Hep-th [78]	7610	15751	24	25	104	328	360	558
Gnutella04 [57, 90]	10876	39994	8	43	626	–	–	–
ca-HepPh [57]	12008	118489	239	296	1002	–	–	–
CondMat [78]	16264	47594	18	47	255	1839	–	1310
ca-CondMat [57]	23133	93497	26	89	665	–	–	–
Enron [51, 60]	36692	183831	27	214	1428	–	–	–
Brightkite [21]	58228	214078	39	193	1421	–	–	–

Table 2: Number of colors in p-centered colorings computed on real world networks and upper bounds of their respective treedepth. These networks were mostly taken from the datasets found in [2, 58, 77].

property to distinguish types of networks, meriting future research.

In the last column, we provide upper bounds for the treedepth of these graphs. Notice that the number of colors needed for a p-centered coloring will be fewer than the treedepth of the the graph for any $p \leqslant n$: Given a treedepth decomposition of a graph of depth t we can color every node by its depth in the treedepth decomposition. Since the treedepth is a hereditary property, the graph induced by any subset of nodes will have treedepth at most t. Notice that the simple coloring algorithm we used sometimes colors the graph with more than t colors. This is a good indication that a better coloring algorithm or heuristic exists.

We also note that the subgraph isomorphism algorithm presented in Section 6 should be directly applicable to some of these graphs, given their comparatively low treedepth.

Finally, we argue that for practical purposes the definition of p-treedepth colorings is too strict. Remember that per definition, in a p-centered coloring every graphs induced on $i < p$ colors has treedepth $\leqslant i$. Relaxing this latter condition, we arrive at the following variation of Proposition 1:

Observation 2. *Let $\mathcal{G}$ be a graph class of bounded expansion. There exists functions f and g such that for every $G \in \mathcal{G}$, $p \in \mathbf{N}$, the graph G can be colored with $f(p)$ colors so that any $i < p$ color classes induce a graph of treedepth $\leqslant g(i)$ in G.*

Obviously this follows from Proposition 1, taking g as the identity. However, algorithms based on p-treedepth colorings would run faster if g allows for a larger margin, provided we can decrease the number of colors f. This is owed to the fact that for large number of colors, iterating through all ($< p$)-sized subsets will be the deciding factor in the running time.

6. Algorithms

In this section, we present efficient algorithms for several important problems arising in the study of complex networks which exploit bounded expansion. Note that in both cases, only the algorithm's running time relies on the grad being small, not its correctness—in that sense, these algorithms are oblivious to whether the input graph is sparse or not (in contrast to, for example, algorithms that rely on planarity). The problems that we discuss revolve around the themes of *subgraph counting* and *centrality estimation*.

Computing the frequency of small fixed pattern graphs inside a network is the key algorithmic challenge in using *network motifs* and *graphlet degree distributions* to analyze network data (both of which are described in more detail in Section 6.1). We present a parameterized algorithm for counting the number of subgraphs with at most h vertices with a running time of $6^h \cdot t^h \cdot n$, where t is the treedepth of the input graph. In a graph class of bounded expansion, we use this algorithm in conjunction with p-centered colorings.

Another topic of interest in complex networks is estimating the relative importance of a vertex in the network (for example, how influential a person is inside a social network, which roads are busiest in a road-network, or which location is most attractive for business). The typical approach is to define/select an appropriate *centrality measure* (see [53] for a survey of common measures). We focus on the *closeness centrality*, which was introduced by Sabidussi [91], and related extensions. These measures are related in the sense that computing them requires knowledge of all the pairwise distances between the vertices of the network, which even in sparse networks takes time $O(n^2)$ to compute [16]. We introduce localized variants of the closeness-based centrality measures and design a linear-time algorithm to compute them in bounded expansion classes and provide experimental data that suggests that these measures are able to recover the topmost central vertices quite well.

6.1. Counting graphlets and subgraphs

In the following we highlight three domain-specific applications of computing the frequency of small fixed pattern graphs inside a network. In particular, the concept of *network motifs* and *graphlets* has proven very useful in the area of computational biology.

A *network motif* is a subgraph (not necessarily induced and possibly labeled) that appears with a significantly higher frequency in a real-world network than one would expect by pure chance. Introduced in [69] under the hypothesis that such frequently occurring structures have a functional significance, motifs have been identified in a plethora of different domains—including protein-protein-interaction networks [5], brain networks [92] and electronic circuits [44]. We point the interested reader to the surveys of Kaiser, Ribeiro and Silva [89] and Masoudi-Nejad, Schreiber, and Kashani [64] for a more extensive overview.

Graphlets are a related concept, though their application is in an entirely different scope. While motifs are used to identify and explain local structure in networks, graphlets are used to 'fingerprint' them. Pržulj [87] introduced the *graphlet degree distribution* as a way of measuring network similarity. To compute it, one enumerates all connected graphs up to a fixed size (five in the original paper) and computes for each vertex of the target graph how often it appears in a subgraph isomorphic to one of those patterns. Since some graphlets exhibit higher symmetry than others, the computation takes into account all possible automorphisms. The degree distribution then describes for each graphlet G_i, how many vertices of the target graph are contained in $0, 1, 2, \ldots$ subgraphs isomorphic to G_i—more precisely, in how many orbits of the respective automorphism groups it appears in. Note that if the set of graphlets only contains the single-edge graph this computation yields exactly the classical degree distribution.

The application of this distribution is two-fold: On the one hand, it can be used to measure similarity of multiple networks, in particular, networks related to biological data [42]. On the other hand, the local structure around a vertex can reveal domain-specific functions. This is the case for protein-protein interaction networks, where local structure correlates with biological activity [67]. This fact has been applied to identify cancer genes [66] and construct phylogenetic trees [56]. Graphlets have further been used in analysis of workplace dynamics [95], photo cropping [20] and DoS attack detection [86].

A third application of subgraph counting was given by Ugander et al. [96]: their empirical analysis and subsequent modeling of social networks revealed that there is an inherent bias towards the occurrence of certain subgraphs. Thus frequencies of small subgraphs are an important indicator for the social domain, similar to the role of graphlet frequencies in biological networks.

In Theorem 18.9 from [74] it was shown that for a graph class of bounded expansion counting the number of satisfying assignments of a fixed boolean query is possible in linear time on a labeled graph. This implies that (labeled) graphlet and motif counting are linear time on graph classes of bounded expansion. The result is achieved by using the algorithm presented in Lemma 17.3 in [74] to count

the number of satisfying assignments of a fixed boolean query parameterized by treedepth. For a graphlet with h nodes, this algorithm runs in time $O(2^{ht} \cdot ht \cdot n)$, where t is the treedepth of the host graph. We provide an algorithm with a running time of $O(6^h \cdot t^h \cdot h^2 \cdot n)$, which implies a faster algorithm on graph classes of bounded expansion, as explained below.

The tool of choice for applying a counting algorithm designed for bounded-treedepth graphs to a class of bounded expansion are p-treedepth colorings: to compute the frequency of a given graphlet H of size h, we first compute a $(h+1)$-treedepth coloring of the input graph in linear time as per Proposition 1[5].

We then enumerate all choices of $i \leqslant h$ colors and count the frequency of H in the graph induced by those colors: since this induced subgraph has treedepth at most i, we are able to apply the counting algorithm for bounded-treedepth graphs (thus imagine replacing t by h in the above running times, establishing the asymptotic improvement given by our approach in this setting). Afterwards it is a matter of simple inclusion-exclusion over the frequencies found for each collection of $\leqslant h$ colors to recover the frequency of H in the whole graph.

Central to the dynamic programming we will use to count isomorphisms is the following notion of a *k-pattern* which is very similar to the well-known notion of *boundaried graphs*; the main difference being that k-patterns describe specific decompositions of our input graph H.

Definition 12. *A* k-pattern *of the graph H is a triple $M = (W, X, \pi)$ where $X \subseteq W \subseteq V(H)$, $|X| \leqslant k$, such that $W \setminus X$ has no edge into $V(H) \setminus W$, and $\pi \colon X \to [k]$ is an injective function. We will call the set X the* boundary *of M. For a given k-pattern M we denote the underlying graph by $H[M] = H[W]$, the vertex set by $V(M) = W$, the boundary by $\mathrm{bd}(M) = X$ and the mapping by π^M.*

We denote by $\mathcal{P}_k(H)$ the set of all k-patterns of H. Note that every k-pattern (W, X, π) is also a $(k+1)$-pattern. In the following we denote by $|H| = |H|$.

Lemma 23. *Let H be a graph. Then $|\mathcal{P}_k(H)| \leqslant 3^{|H|} \cdot k^{|H|}$.*

Proof. The vertices of H can be partitioned in $3^{|H|}$ possible ways into boundary vertices, pattern vertices and remainder. The number of ways an injective mapping for a boundary of size $b \leqslant |H|$ into $[k]$ can be chosen is bounded by $k^{|H|}$. In total the size of $\mathcal{P}_k(H)$ is always less than $3^{|H|} \cdot k^{|H|}$. □

We will use k-patterns during dynamic programming via the basic join and forget operations defined below. Respectively, these correspond to gluing two patterns together and to demoting a boundary-vertex to a simple vertex.

Definition 13 (k-pattern join)**.** *Let H be a graph and $M_1 = (W_1, X_1, \pi_1)$, $M_2 = (W_2, X_2, \pi_2)$ k-patterns of H. Then the two patterns are* compatible *if $W_1 \cap W_2 = X_1 = X_2$ and for all $v \in X_1$ it holds that $\pi_1(v) = \pi_2(v)$. Their* join *is defined as the k-pattern $M_1 \oplus M_2 = (W_1 \cup W_2, X_1, \pi_1)$.*

[5]Since p-treedepth coloring is also a p'-treedepth coloring for $p' \leqslant p$, we only need to compute one such coloring.

Definition 14 (k-pattern forget)**.** *Let H be a graph, let $M = (W, X, \pi)$ be a k-pattern of H and $i \in [k]$. Then the* forget operation *is the k-pattern*

$$M \ominus i = \begin{cases} (W, X \setminus \pi^{-1}(i), \pi|_{X \setminus \pi^{-1}(i)}) & \text{if } \pi^{-1}(i) \neq \varnothing \text{ and } N_H(\pi^{-1}(i)) \subseteq W \\ \bot & \text{if } \pi^{-1}(i) \neq \varnothing \text{ and } N_H(\pi^{-1}(i)) \not\subseteq W \\ (W, X, \pi) & \text{otherwise} \end{cases}$$

Structurally, the k-pattern's boundaries will represent vertices from the path of the root vertex to the currently considered vertex in the treedepth decomposition, while the remaining vertices of the pattern represent vertices somewhere below it. The following two notations help expressing these properties.

Definition 15 (Subtree and root path)**.** *Let T be a treedepth decomposition of G rooted at $r \in G$ and let $v \in V(G)$ be a vertex. Then the* subtree of v *is the subtree T_v of T rooted at v. The* root path *of v is the unique path P_v from the root r to v in T. We let $P_v[i]$ denote the i^{th} vertex of the path (starting at the root), so that $P_v[1] = r$ and $P_v[\,|P_v|\,] = v$.*

We can now state the main lemma. The proof contains the description of the dynamic programming which works bottom-up on the vertices of the given treedepth decomposition (i.e. starting at the leaves and working towards the root of the decomposition).

Lemma 24. *Let H be a fixed graph on h vertices. Given a graph G on n vertices and a treedepth decomposition T of height t, one can compute the number of isomorphisms from H to induced subgraphs of G in time $O(6^h \cdot t^h \cdot h^2 \cdot n)$ and space $O(3^h \cdot t^h \cdot ht \cdot \log n)$.*

Proof. We provide the following induction that easily lends itself to dynamic programming over T. Denote by $M_H = (V(H), \varnothing, \varepsilon)$ the trivial t-pattern of H, here $\varepsilon\colon \varnothing \to \varnothing$ denotes the null function. Consider a set of vertices $v_1, v_2, \ldots, v_\ell \in G$ with a common parent v in T with respective subtrees T_{v_i} and root paths P_{v_i} for $1 \leqslant i \leqslant \ell$. Note that the root paths $P_{v_1}, \ldots, P_{v_\ell}$ all have the same length k and share the path P_v as a common prefix.

Let M_1 be a fixed t-pattern of H. We define the mapping $\psi_v^{M_1}\colon \mathrm{bd}(M_1) \to P_v$ via $\psi_v^{M_1}(u) = P_v[\pi^{M_1}(u)]$ which takes the pattern's boundary and maps it to the vertices of the root-path.

For patterns M_1 that satisfy that for all $u \in \mathrm{bd}(M_1)$, $\pi^{M_1}(u) \leqslant l$, we denote by $f[v_1, \ldots, v_\ell][M_1]$ the number of isomorphisms $\phi_1\colon V(M_1) \to V(G)$ such that

(i) $\phi_1|_{\mathrm{bd}(M_1)} = \psi_v^{M_1}$

(ii) $\phi_1(V(M_1) \setminus \mathrm{bd}(M_1)) \subseteq G[V(T_{v_1} \cup \cdots \cup T_{v_\ell})]$.

In other words we charge subgraphs to patterns whose boundaries lie on the shared root-path P_v, such that the labeling of the boundary coincides with the numbering induced by P_v while the rest of the pattern is contained entirely in the subtree below v.

Let r be the root of the treedepth decomposition. By the above definition, $f[r][M_H]$ counts exactly the number of isomorphisms of H into subgraphs of G.

We will show now how we can compute $f[r][M_H]$ recursively. For a leaf $v \in T$ and a t-pattern $M_1 = (W_1, X_1, \pi_1) \in \mathcal{P}_k(H)$ we compute $f[v][M_1]$ as follows: Defined the value $p_v^{M_1}$ to be 1 if the function $\psi\colon W_1 \to P_v$ defined as $\psi(w) = P_v[\pi_1[w]]$ is an isomorphism from $H[W_1]$ to $G[\psi(W_1)]$ and 0 otherwise. In particular, $p_v^{M_1}$ will be zero if $W_1 \neq X_1$ or $|W_1| > |P_v|$. Then for the leaf v, we compute

$$f[v][M_1] = \sum_{M_2 \ominus |P_v| = M_1} p_v^{M_2}$$

where $M_2 \in \mathcal{P}_t(H)$.

The following recursive definitions show how $f[\cdot][M_1]$ can be computed for all inner vertices of T.

$$f[v][M_1] = \sum_{M_2 \ominus |P_v| = M_1} f[v_1, \ldots, v_\ell][M_2] \qquad \text{(forget)}$$

$$f[v_1, \ldots, v_{j-1}, v_j][M_1] = \sum_{M_2 \oplus M_3 = M_1} f[v_1, \ldots, v_{j-1}][M_2] \cdot f[v_j][M_3] \qquad \text{(join)}$$

where $M_2, M_3 \in \mathcal{P}_t(H)$.

We need to prove that the table f correctly reflects the number of isomorphisms to subgraphs satisfying properties i and ii.

Consider the *join*-case first: fix a pattern $M_1 \in \mathcal{P}_t(H)$. By induction, the entries $f[v_1, \ldots, v_{j-1}][\cdot]$ and $f[v_j][\cdot]$ correspond to the number of isomorphisms to subgraphs that satisfy properties i and ii with the node tuples $v_1, \ldots, v_{j-1}$ and v_j, respectively. We need to show that $f[v_1, \ldots, v_j][M_1]$ as computed above counts the number of isomorphisms from $H[M_1]$ to subgraphs of G such that $\phi_1|_{\mathrm{bd}(M_1)} = \psi_v^{M_1}$ and $\phi_1(V(M_1) \setminus \mathrm{bd}(M_1)) \subseteq G[V(T_{v_1} \cup \cdots \cup T_{v_j})]$.

Consider the set Φ_1 of all isomorphisms from $H[M_1]$ to subgraphs of G satisfying properties i and ii for the vertex tuple $v_1, \ldots, v_j$. For any vertex subset $R \subseteq V(M_1) \setminus \mathrm{bd}(M_1)$, define the slice $\Phi_1(R) \subseteq \Phi_1$ as those isomorphisms ϕ that satisfy $\phi^{-1}(\phi(V(H)) \cap T_{v_j}) = R$. Let $L = (V(M_1) \setminus \mathrm{bd}(M_1)) \setminus R$ and define the patterns $M_L = (L \cup \mathrm{bd}(M_1), \mathrm{bd}(M_1), \pi^{M_1})$ and $M_R = (R \cup \mathrm{bd}(M_1), \mathrm{bd}(M_1), \pi^{M_1})$. Then by induction $|\Phi_1(R)| = f[v_1, \ldots, v_{j-1}][M_L] \cdot f[v_j][M_R]$. Since $M_1 = M_L \oplus M_R$ and clearly $M_L, M_R \in \mathcal{P}_t(H)$, the sum computes exactly $\sum_{R \subseteq V(M_1) \setminus \mathrm{bd}(M_1)} |\phi_1(R)| = |\phi_1|$.

Next, consider the *forget*-case. Again, fix $M_1 \in \mathcal{P}_t(H)$ and let u be the parent of v in T. Let Φ_1 be the set of those isomorphisms from $H[M_1]$ to subgraphs of G for which $\phi_1|_{\mathrm{bd}(M_1)} = \psi_u^{M_1}$ and $\phi_1(V(M_1) \setminus \mathrm{bd}(M_1)) \subseteq G[V(T_v)]$. We partition Φ_1 into $\Phi_1 = \Phi_{1,v} \cup \Phi_{1,\bar{v}}$ where $\Phi_{1,v}$ contains those isomorphisms ϕ for which $\phi^{-1}(v) \neq \varnothing$ and $\Phi_{1,\bar{v}}$ the rest. Since $|\Phi_{1,\bar{v}}| = f[v_1, \ldots, v_\ell][M_1]$ we focus on $\Phi_{1,v}$ in the following. For $w \in V(M_1) \setminus \mathrm{bd}(M_1)$, define $\Phi_{1,v}(w)$ as the set of those isomorphisms ϕ for which $\phi(w) = v$. Clearly, $\{\Phi_{1,v}(w) \mid w \in V(M_1) \setminus \mathrm{bd}(M_1)\}$ is a partition of $\Phi_{1,v}$. Define the pattern $M_w = (V(M_1), \mathrm{bd}(M_1) \cup \{w\}, \pi_w^{M_1})$ where $\pi_w^{M_1}$ is π^{M_1} augmented with the value $\pi_w^{M_1}(v) = |P_v|$. Note that by construction

$M_1 = M_w \ominus |P_v|$. By induction, $|\Phi_{1,v}(w)| = f[v_1, \ldots, v_\ell][M_w]$ and therefore

$$|\Phi_1| = |\Phi_{1,\bar{v}}| + \sum_{w \in V(M_1) \setminus \mathrm{bd}(M_1)} |\Phi_{1,v}(w)| = \sum_{M_2 \ominus |P_v|} f[v_1, \ldots, v_\ell][M_2]$$

It remains to prove the claimed running time. Initialization of f for a leaf takes time $O(|\mathcal{P}_t(H)|h^2)$ since we need to test whether the function ψ defined above is an isomorphism for each pattern $M_1 \in \mathcal{P}_t(H)$.

For the other vertices, a forget operation can be achieved in time $O(|\mathcal{P}_t(H)|)$ per vertex by enumerating all t-patterns, performing the forget operation and looking up the count of the resulting pattern in the previous table.

A join operation needs time $O(|\mathcal{P}_t(H)| \cdot h \cdot 2^h)$ per vertex, since for a given pattern M_1 those patterns M_2, M_3 with $M_1 = M_2 \oplus M_3$ are uniquely determined by partitions of the set $V(M_1) \setminus \mathrm{bd}(M_1)$.

In total the running time of the whole algorithm is $O(|\mathcal{P}_t(H)| \cdot 2^h \cdot h^2 \cdot n)$. Note that we only have to keep at most $O(t)$ tables in memory, each of which contains the occurrence of up to $|P_t(H)|$ patterns stored in numbers up to n^h. Thus in total the space complexity is $O(|\mathcal{P}_t(H)| \cdot t \cdot \log(n^h)) = O(|\mathcal{P}_t(H)| \cdot ht \cdot \log n)$. □

To count the occurrences of H as an induced subgraph instead of the number of subgraph isomorphisms, one can simply determine the number of automorphisms of H in time $2^{O(\sqrt{h \log h})}$ [8, 65] and divide the total count by this value (since this preprocessing time is dominated by our running time we will not mention it in the following). Counting isomorphisms to non-induced subgraphs can be done in the same time and space by changing the initialization on the leaves, such that it checks for an subgraph instead of an induced subgraph. Dividing again by the number of automorphisms gives the number of subgraphs. By allowing the mapping of the patterns to map several nodes to the same value, we can use them to represent homomorphisms. Testing the leaves accordingly the same algorithm can be used to count the number of homomorphisms from H to subgraphs of G. By keeping all tables in memory, thus sacrificing the logarithmic space complexity, and using backtracking we can also label every node with the number of times it appears as a certain vertex of H.

From these observations and Lemma 23 we arrive at the following theorem:

Theorem 6. *Given a graph H on h vertices, a graph G on n vertices and a treedepth decomposition of G of height t, one can compute the number of isomorphisms from H to subgraphs of G, homomorphisms from H to subgraphs of G, or (induced) subgraphs of G isomorphic to H in time $O(6^h \cdot t^h \cdot h^2 \cdot n)$ and space $O(3^h \cdot t^h \cdot ht \cdot \log n)$.*

Note that for graphs of unbounded treedepth the running time of the algorithm degenerates to $O(6^h \cdot h^2 \cdot n^{h+1})$, which is comparable to the running time of $2^{O(\sqrt{h \log h})} \cdot n^h$ of the trivial counting algorithm.

Theorem 7. *Given a graph H and a a graph G belonging to a class of bounded expansion, there exists an algorithm to count the appearances of H as a subgraph*

of G in time

$$O\left(\binom{f(h)}{h} \cdot 6^h \cdot h^{h+2} \cdot n\right)$$

where f is a function depending only on the graph class.

This immediately extends to nowhere dense classes, which have p-treedepth-colorings with at most n^ε colors (for sufficiently large graphs) for any $\varepsilon > 0$. Choosing the graphs large enough and setting $\varepsilon' = \varepsilon/h$, we can bound the term $\binom{f(h)}{h}$ by $n^{\varepsilon' \cdot h} = n^\varepsilon$.

Theorem 8. *Let $\mathcal{G}$ be a nowhere-dense class and let H be a graph. For every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbf{N}$, such that for any graph $G \in \mathcal{G}, |G| > N_\varepsilon$ there exists an algorithm to count the appearances of H as a subgraph of G in time*

$$O\left(6^h \cdot h^{h+2} n^{1+\varepsilon}\right).$$

Finally, we would like to point out that this counting algorithm is trivially parallelizable.

6.2. Localized Centrality

Centrality is a notion used to ascribe the relative importance of a vertex in the network. A centrality measure is a real-valued function that assigns each vertex of the network some value with the understanding that higher values correspond to more central vertices. Depending on the application, "central" vertices need not be those with high degree (for example, a cut-vertex may have high centrality as it is the only way for information to flow between two large subgraphs). There have been a wide variety of centrality scores introduced in the literature, including degree centrality, closeness centrality, eigenvector centrality, betweenness centrality and others. For a comprehensive introduction to centrality measures in social networks see, for instance, [34, 35]. There are several recent articles devoted to the topic of centrality measures in general [53, 85]. In this section, we consider localized variants of measures similar to the *closeness centrality* introduced by Sabidussi [91] (see Table 3). The global versions of these measures require one to compute the distance between all vertex pairs in the network, a sub-routine where the fastest known algorithm (due to Brandes [16]) is $O(n(n+m))$ which in the context of sparse networks reduces to quadratic time.

In these localized variants, we compute the measure of a vertex with respect to its r^{th} neighborhood rather than with respect to the whole graph. We give linear time algorithms for computing these measures on graphs of bounded expansion for every constant r. As the value of r increases, the value of the measure computed for a vertex approaches its unlocalized variant at the expense of an increase in running time.[6] The measures in question and their localized variants are listed in Table 3.

[6] For general graphs, we can compute these localized variants in time $O(n(n+m))$, by performing a breadth-first search from every vertex, for instance. We do not know whether a better running time is possible.

Measure	Definition	Localized
Closeness [91]	$c_C(v) = (\sum_{u \in V(G)} d(v,u))^{-1}$	$c^r_C(v) = (\sum_{u \in N^r(v)} d(v,u))^{-1}$
Harmonic [83]	$c_H(v) = \sum_{u \in V(G)} d(v,u)^{-1}$	$c^r_H(v) = \sum_{u \in N^r(v)} d(v,u)^{-1}$
Lin's index [61]	$c_L(v) = \frac{\lvert\{u \mid d(v,u) < \infty\}\rvert^2}{\sum_{u \in V(G): d(v,u) < \infty} d(v,u)}$	$c^r_L(v) = \frac{\lvert N^r[v]\rvert^2}{\sum_{u \in N^r[v]} d(v,u)}$

Table 3: Distance-based centrality measures with localized variants that can be computed in linear time on graphs of bounded expansion.

One natural question is the utility of localized variants (and their accuracy in reflecting the global measure). We remark that Marsden demonstrated that for some networks, calculating the measure for a vertex v inside its closed neighborhood $G[N[v]]$ can be used as a viable substitute for the full measure [63]. In the context of computer networks, Pantazopoulos et al. [84] consider local variants which lend themselves to distributed computing and found a close correlation to the full measures on a sample of networks. We can show experimentally that our variants reliably capture the top ten percent in (arbitrarily) selected networks of our real-world corpus. To that end, we compare the top 10 percent as identified by our localized variants to those 10 percent identified by the respective full centrality measure. Specifically, we use the Jaccard index [45]—defined as $|A \cap B|/|A \cup B|$ for two sets A, B—to measure similarity[7].

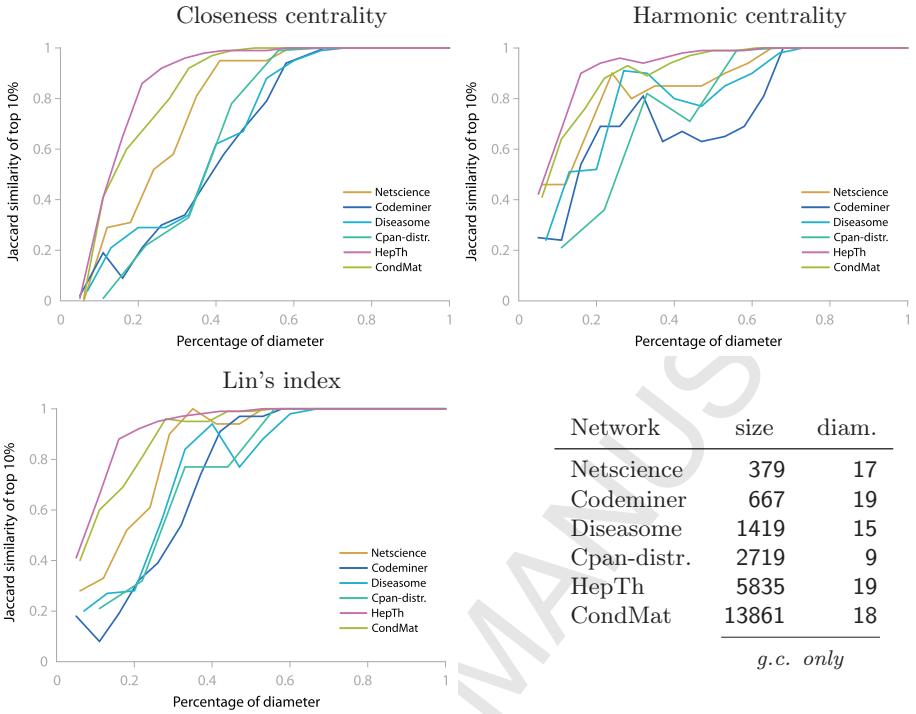
The results in Figure 3 suggest that already a value of r equal to about half the diameter yields very good results across all three measures. Note that we do not compare rankings; but rather only the difference between the sets of the identified top vertices—our experiments showed that rank ordering is not preserved reliably. Furthermore, there seems to be a slight positive tendency towards the localized version being better in larger networks, though it is hard to draw any conclusions on a small experiment like that.

While the localized Lin's index and the localized harmonic closeness work as depicted in Table 3, the localized closeness needs a small normalization tweak in order to yield good results: this is achieved by treating the $(r+1)$-st neighborhood of very vertex as if it would contain all remaining vertices and adding this value accordingly (this obviously does not change the running time of the algorithm).

We will rely on the following proposition to compute the localized centrality measures.

Proposition 4 (Truncated distances [73])**.** *Let G be a graph of bounded expansion. For every r one can compute in linear time a directed graph $\vec{G}_r$ with in-degree bounded by $f(r)$ – for some function f – on the same vertex set as*

[7]Since we compare sets of equal size, the measures precision and recall—and accordingly the F_1-score—are all the same. The Jaccard index is better suited for this situation.



Network	size	diam.
Netscience	379	17
Codeminer	667	19
Diseasome	1419	15
Cpan-distr.	2719	9
HepTh	5835	19
CondMat	13861	18

g.c. only

Figure 3: Quality of localized centrality measures in terms of similarity of the top ten percent vertices against the full centrality measure. Measurements were taken only in the giant component of the networks displayed in the table.

G and an arc labeling $\omega\colon \vec{E}(\vec{G}_r) \to \mathbf{N}$ such that for every pair $u, v \in G$ with $d_G(u,v) \leqslant r$ one of the following holds:

(i) $uv \in \vec{G}_r$ *and* $\omega(uv) = d_G(u,v)$;

(ii) $vu \in \vec{G}_r$ *and* $\omega(vu) = d_G(u,v)$;

(iii) there exists $w \in N^-_{\vec{G}_r}(u) \cap N^-_{\vec{G}_r}(v)$ *such that* $\omega(wu) + \omega(wv) = d_G(u,v)$.

Let G be a graph from a class of bounded expansion and let $\vec{G}_r$ be the directed graph with in-degree bounded by $f(r)$, for some function f, that is obtained from G as by Proposition 4. In the following, we let $N^-_r(v)$ denote the in-neighborhood of the vertex v in the directed graph $\vec{G}_r$. We assume that the vertices of G are ordered so that every vertex set has a unique representation as a tuple and, by slight abuse of notation, we use both representations interchangeably. For $v \in V(G)$ and $A = (a_1, a_2, \ldots, a_p) \subseteq N^-_r(v)$, define the *distance vector* from v to A as $\mathrm{dist}(v, A) := (\omega(a_1 v), \omega(a_2 v), \ldots, \omega(a_p v))$, where ω denotes the arc-labeling from Proposition 4. Since $a_i \in N^-_r(v)$, $\omega(a_i, v) = d_G(a_i, v)$.

Definition 16. *Let $v \in \vec{G}_r$, $\varnothing \neq X \subseteq N_r^-(v)$, $\alpha : V(G) \to \mathbf{R}$ a vertex weighting and let $\bar{d} \in [r]^{|X|}$ be a distance vector. We define*

$$N(v, X, \bar{d}) := \{v \neq u \in V(G) \mid N_r^-(v) \cap N_r^-(u) = X \text{ and } dist(u, X) = \bar{d}\}$$

as those vertices whose in-neighborhood in $\vec{G}_r$ overlap with the in-neighborhood of v in exactly X and whose distance-vector to X is exactly $\bar{d}$. Then the query-function c_α *is defined as*

$$c_\alpha(v, X, \bar{d}) := \sum_{u \in N(v,X,\bar{d})} \alpha(u).$$

Lemma 25. *Given $\vec{G}_r$, one can compute a data structure in time $O(n)$ such that queries $c_\alpha(v, X, \bar{d})$ as in Definition 16 can be answered in constant time.*

Proof. We define an auxiliary dictionary R indexed by vertex sets $X \subseteq N^- r(v)$, for some vertex v. At each entry $v \in \vec{G}_r$, we will store another dictionary indexed by distance vectors which in turn stores a simple counter. We initialize R as follows: for every $v \in \vec{G}_r, X \subseteq N_r^-(v)$ and every distance vector $\bar{d} \in [r]^{|X|}$, set $R[X][\bar{d}] = 0$. Note that in total, R contains only $O(n)$ entries since all in-neighborhoods in $\vec{G}_r$ have constant size. We can implement R as a hash-map to achieve the desired (expected) constant-time for insertion and look-up, though this would yield a randomized algorithm. A possible way to implement R on a RAM deterministically is the following: We store the key $X = \{x_1, x_2, \ldots, x_p\}$ at address $x_1 + n \cdot x_2 + \ldots + n^p \cdot x_p$. This uses addresses up to size n^c, for some constant c, but since we only insert $O(n)$ keys the setup takes only linear time. Our later queries to R will be restricted to keys that are guaranteed to be contained in the dictionary, thus we will never visit a register that has not been initialized.

For every $v \in \vec{G}_r$ and $X \subseteq N_r^-(v)$, increment the counter $R[X][\text{dist}(v, X)]$ by $\alpha(v)$. We now claim that queries of the form $c_\alpha(v, X, \bar{d})$ can be computed using inclusion-exclusion as follows:

$$c_\alpha(v, X, \bar{d}) = \sum_{X \subseteq Y \subseteq N_r^-(v)} (-1)^{|Y \setminus X|} \sum_{\bar{d}' : \bar{d}'|_X = \bar{d}} R[Y][\bar{d}'].$$

The computation of the sum clearly takes constant time[8]. We now prove that it indeed computes the quantity $c_\alpha(v, X, \bar{d})$.

Consider a vertex $u \in \vec{G}_r$, such that $N_r^-(u) \cap N_r^-(v) = X$ and $\text{dist}(u, X) = \bar{d}$. We argue that $\alpha(u)$ is counted once by the above sum: $\alpha(u)$ is not counted by any $R[Y][\cdot]$ with $Y \supsetneq X$, therefore only the term where $X = Y$ counts $\alpha(u)$ and does so exactly once. It remains to be shown that the weight of vertices that do not conform with Definition 16 are either not counted by the sum or cancel out.

[8] We tacitly assume that the weights α only assign numbers polynomially bounded by the size of the graph.

Consider a vertex $w \in \vec{G}_r$ with such that $\mathrm{dist}(w, X) \neq \bar{d}$. The weight of such a vertex is not counted by the above sum, since $\alpha(w)$ is only counted in entries of R that do not occur as summands.

Finally, consider a vertex $w' \in \vec{G}_r$ with $N_r^-(w') \cap N_r^-(v) = Z$ where $X \subsetneq Z \subseteq N_r^-(v)$ and such that $\mathrm{dist}(w', X) = \bar{d}$. The weight of this vertex is counted in each term of

$$\sum_{X \subseteq Y \subseteq Z} (-1)^{|Y \setminus X|} R[Y][\mathrm{dist}(w', Z)|_Y]$$

since

$$\sum_{X \subseteq Y \subseteq Z} (-1)^{|Y \setminus X|} = \sum_{0 \leqslant k \leqslant |Z \setminus X|} (-1)^k \binom{|Z \setminus X|}{k} = 0$$

we know that the signs cancel out and thus $\alpha(w')$ does not contribute to $c_\alpha(v, X, \bar{d})$. Hence the above sum computes exactly the query $c_\alpha(v, X, \bar{d})$. □

Theorem 9. *Let $\mathcal{G}$ be a graph class of bounded expansion, $G \in \mathcal{G}$ a graph and $r \in \mathbf{N}$ an integer. Then one can compute the quantities $\alpha_d(v) = \sum_{w \in N^d(v)} \alpha(w)$ for all $v \in G, d \leqslant r$ in linear time.*

Proof. By Theorem 4 we can compute $\vec{G}_r$ in linear time, thus we can employ Lemma 25 to answer queries as defined in Definition 16 in constant time. To compute the quantity $\alpha_d(v)$ for all $0 < d \leqslant r$ and $v \in V(G)$, we proceed as follows. Initialize an array C by setting $C[v][d] = 0$ for every $v \in G, 0 < d \leqslant r$.

Now for every $v \in V(G)$, every $X \subseteq N_r^-(v)$ and every distance vector $\bar{d} \in [r]^{|X|}$, update C via

$$C[v][\min(\bar{d} + \mathrm{dist}(v, X))] \leftarrow C[v][\min(\bar{d} + \mathrm{dist}(v, X))] + c_\alpha(v, X, \bar{d})$$

and then apply the correction

$$C[v][\min(\mathrm{dist}(v, X) + \mathrm{dist}(v, X))] \leftarrow C[v][\min(\mathrm{dist}(v, X) + \mathrm{dist}(v, X))] - 1$$

in both cases with the convention that we dismiss entries where $\min(\bar{d} + \mathrm{dist}(v, X)) > r$. The second case corrects the query $c_\alpha(v, N^-(v), \mathrm{dist}(v, N^-(v)))$ counting the vertex v itself.

At this point, $C[v][d]$ contains the sum of weights of vertices u for which $\min(\mathrm{dist}(v, X) + \mathrm{dist}(u, X)) = d$ where $X = N_r^-(v) \cap N_r^-(u) \neq \varnothing$. This follows directly from the definition of c_α.

By Theorem 4, every pair of vertices of distance $< r$ in G either is connected by an arc or they share a common in-neighbor in $\vec{G}_r$. Accordingly, we update the values of C as follows: for every $uv \in \vec{E}(\vec{G}_r)$

- if $N_r^-(u) \cap N_r^-(v) = \varnothing$, the weights of the vertices u and v were not counted in $C[v][\cdot], C[u][\cdot]$ respectively, thus we update C via

$$C[v][\omega(uv)] \leftarrow C[v][\omega(uv)] + \alpha(u)$$
$$C[u][\omega(uv)] \leftarrow C[u][\omega(uv)] + \alpha(v)$$

- if $X = N_r^-(u) \cap N_r^-(v) \neq \varnothing$, the weights of the vertices u and v were counted in $C[v][d']$ and $C[u][d']$ for $d' = \min(\mathrm{dist}(u, X) + \mathrm{dist}(v, X))$, respectively. Since d' might be larger than $\omega(uv)$ (but cannot be smaller), we update C via

$$\begin{aligned} C[v][d'] &\leftarrow C[v][d'] - \alpha(u) \\ C[u][d'] &\leftarrow C[u][d'] - \alpha(v) \\ C[v][\omega(uv)] &\leftarrow C[v][\omega(uv)] + \alpha(u) \\ C[u][\omega(uv)] &\leftarrow C[u][\omega(uv)] + \alpha(v) \end{aligned}$$

  where we again ignore the update of $C[\cdot][d']$ if $d' > r$.

Note that this procedure is problematic if both uv and vu are present in the graph, since then the this correction would (wrongly) be applied twice. The simple solution is that in the case of both arcs being present we only apply the above update for that arc where the start vertex is smaller than the end vertex, i.e. to uv if $u < v$ and vu otherwise.
At this point, $C[v][d]$ contains the sum of weights of vertices u for which either

- the value $d = \min(\mathrm{dist}(v, X) + \mathrm{dist}(u, X))$ where $X = N_r^-(v) \cap N_r^-(u) \neq \varnothing$ and $uv \notin \vec{E}(\vec{G}_r)$,
- or $d = \omega(uv)$ and $uv \in \vec{E}(\vec{G}_r)$.

Thus by Theorem 4 we have that $C[v][d] = \alpha_d(v)$ for $d < r$ and $v \in G$. Since all of the above operations take time linear in $|G|$, the claim follows. □

If we take $\alpha(\cdot) = 1$, the above algorithm counts exactly the sizes of the d^{th} neighborhoods of each vertex, for $d < r$. Thus it can be used to compute the r-centric centrality measures presented in Table 3.

Corollary 5. *Let $\mathcal{G}$ be a graph class of bounded expansion, $G \in \mathcal{G}$ a graph and $r \in \mathbf{N}$ an integer. Then the r-centric closeness, harmonic centrality and Lin's index can be computed for all vertices of G in total time $O(|G|)$.*

7. Conclusion and Open Problems

We propose unifying structural graph algorithms with complex network analysis by searching for observable structural properties that satisfy the litmus test of enabling efficient algorithms for network analysis. We presented theoretical and empirical results that support our hypothesis that complex networks are structurally sparse in a well-defined and robust sense. Efficient algorithms are known for networks of bounded expansion [31, 41, 74], and we show that for key network analysis problems these algorithms can be even further improved. On the theoretical side, we show that several random graph models of complex networks exhibit bounded expansion with high probability, although not all do—suggesting an interesting dichotomy of networks. On the experimental side, we confirm

these mathematical results, and show that many real-world complex networks additionally appear to exhibit bounded expansion as measured using specialized colorings. This new approach enables fast algorithms to analyze features including communities, centrality, and motifs while more broadly providing a rigorous framework for a deeper understanding of real-world networks and related models.

There are a plethora of random graph models specifically designed to mimic properties of complex networks. Which of these models exhibit structural sparsity (and which ones do not)? There is also room for debate about how to establish that a model will generate graphs with certain properties *in practice*. Asymptotic behavior is only a proxy, although we took care to provide details on the speed of convergence in our proofs where possible. As exemplified by the relatively weak result about the Barabási-Albert model, the practical implications are sometimes difficult to judge.

On the algorithmic side, there are several key challenges remaining. Does there exist a better algorithm/heuristic to obtain p-treedepth colorings, in particular taking into consideration the special structure of complex networks? Does a good coloring algorithm exist that provides a trade-off between the number of colors and the treedepth of subgraphs induced by few color classes? Can we compute or approximate lower bounds for either χ_p or $\widetilde{\nabla}_r$ with reasonable margins of error? Both would likely improve our current empirical understanding of the grad of networks. We would also like to investigate whether the grad for small depths is a reliable measure to differentiate networks; both our empirical and theoretical results seem to indicate so.

Finally, algorithms exploiting low grad should be tested extensively via computational experiments, to ascertain the feasibility of applying these techniques to real-world networks.

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