Introduction to Renormalization Group and Ward Identities in Critical Phenomena and in Fermi and Bose Liquids

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Abstract. We review some of the main applications of the renormalization-group technique in condensed matter physics. The first relevant example is the description of critical phenomena. Here perturbation theory is affected by singularities, which are a consequence of the long-ranged character of the dominant fluctuations when approaching criticality. The use of renormalization group allows to sum up these singularities into a power-law behavior of the physical quantities, which is experimentally observed near a continuous phase transition. The second example is provided by the description of the physical properties of interacting Fermi and Bose systems. Here perturbation theory is affected by infrared divergences within stable liquid phases, due to the presence of massless excitations, in reduced dimensionality. However, the condition of stability of the system implies exact cancellations among the singular contributions, controlled by additional Ward identities, which must be considered besides the standard Ward identities related to the conservation of the total particle and spin density. The combined use of renormalization group and these new Ward identities allows for the closure of the renormalization-group equations, leading to the description of the asymptotic behavior of the system at low energies.

I INTRODUCTION

The main tool for the investigation of the properties of interacting systems is an expansion in powers of coupling constants about some non-interacting reference system, which is exactly soluble. Unfortunately, too many interesting results lie beyond any finite power expansion. This becomes dramatically evident if the physical problem is such that the individual terms of the expansion become very

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large. Nevertheless, perturbation expansions do not lose their usefulness, and the renormalization-group (RG) approach has been devised to deal with cases where the individual contributions of the naive perturbative expansion are formally divergent [1–3].

In these notes we discuss some applications of RG in condensed-matter physics. The first, most well-known, is the RG approach to second-order (continuous) phase transitions and the related critical phenomena [4–8]. In this case the divergences which affect perturbation theory are a manifestation of approaching criticality, where the most relevant fluctuations in the system become long-ranged [9–11].

The prototype model of an interacting system at a temperature T is the $\hat{\varphi}^4$ model, described by the effective Hamiltonian in d spatial dimensions

$$\frac{1}{T}\mathcal{H}_{\text{eff}}[\hat{\varphi}] = \int [(\nabla\hat{\varphi})^2 + t\hat{\varphi}^2 + u\hat{\varphi}^4] d^d\mathbf{r},$$

which is made dimensionless by the inclusion of the factor $1/T$ in the definition (we measure the temperature in energetic units). The part which is quadratic in the classical field $\hat{\varphi}$ is the free (exactly soluble) part, and the interaction term is $u\hat{\varphi}^4$. In the theory of critical phenomena the thermal average of the field $\hat{\varphi}$ specifies the order parameter φ [12] which characterizes the ordered (less symmetric) phase, and the parameter t provides a measure of the distance from criticality, e.g., $t = T - T_c$, where T_c is the critical (phase-transition) temperature. The dimensionless coupling constant which appears in the perturbative expansion is $u/|t|^{\epsilon/2}$, where $\epsilon = 4-d$. It is then evident that the dimensionless coupling constant diverges when the phase transition is approached ($t \to 0$), for $d < 4$ ($\epsilon > 0$). The RG successfully deals with this singular perturbation theory, and allows for a resummation of the singularities into the experimentally observed power-law behavior of the physical quantities, like the order parameter $\varphi \simeq |t|^{\beta}$, the susceptibility $\chi \simeq |t|^{-\gamma}$, which measures the response of the order parameter to an external field linearly coupled (i.e., thermodynamically conjugated) to it, and the specific heat $C \simeq |t|^{-\alpha}$.

A different, and more recent, application of RG deals with cases when perturbation theory is singular within stable phases (far from phase transitions) as for example in:

1) Interacting Fermi systems for $d = 1$, the paradigm of which is described by the so-called Luttinger model [13]. Here the lowest-order correction to the bare coupling constant is logarithmically divergent, despite the fact that the system is not at criticality. This divergence leads to a power-law behavior of the wave-function renormalization factor, which vanishes when approaching the Fermi level, leading to the suppression of the low-laying single-particle excitations.

2) Interacting Bose systems in the broken-symmetry phase, i.e., in the presence of the Bose-Einstein condensate. The lowest-order correction to the single-particle propagator has a singular contribution in $d < 3$. In this case, as we shall see, the presence of the condensate changes the dimensional analysis and infrared singularities affect perturbation theory, due to the Goldstone mode associated with the broken continuous symmetry.

Both systems are, however, stable liquid phases, in which physical responses like χ, K (compressibility), C, must be finite. The stability must then imply the *exact cancellation* of singularities at any order in perturbation theory. These cancellations are controlled by *additional* symmetries, and hence new Ward identities, which must be considered besides the standard Ward identities related to the conservation of the total particle and spin density.

In general, the use of Ward identities simplifies the structure of the underlying RG. The main advantages of their use can be summarized as follows:
i) Ward identities provide constraints on the renormalization parameters. As a textbook example let us recall the quantum theory of electrodynamics. Denoting by Z_2, Z_3, and Z_1^{-1} the renormalization parameters of the electron propagator, the photon propagator, and the electron-photon vertex, respectively, the charge renormalization is given by $e_R^2 = e^2 Z_2^{-2} Z_3^{-1} Z_1^2$. The $U(1)$ electromagnetic gauge transformations read

$$A_\nu \to A_\nu + \partial_\nu \alpha \ , \qquad \psi \to \psi \mathrm{e}^{ie\alpha} \simeq (1 + ie\alpha),$$

where A_ν is the photon field, ψ is the fermion field, and $\partial_\nu \alpha$ is an infinitesimal gauge field. However, the renormalized fermion field is $\psi_R = \sqrt{Z_2}\psi$, and hence the changes under a gauge transformation are related by $\delta\psi_R = \sqrt{Z_2}\delta\psi$. Using $\delta\psi = i\psi e\alpha$ and $\delta\psi_R = i\psi_R e_R \alpha_R$, one finds $e_R\alpha_R = e\alpha$, and, since $\alpha_R = \sqrt{Z_3}\alpha$, gauge invariance implies $Z_1 = Z_2$, thus reducing the required renormalization parameters.
ii) Ward identities simplify the identification of the proper running variables (effective coupling constants) and the connection with physical quantities. For example, in interacting disordered electron systems the scattering amplitudes are dynamically dressed by disorder. The required renormalization parameters (of the related field-theoretical description based on the non-linear σ-model) are identified in terms of physical quantities of the Fermi-liquid theory via gauge invariance applied to the specific dynamical skeleton structure of the response functions. The resulting scale-dependent Landau parameters in χ, K, C describe various universality classes of the metal-insulator transition besides the Anderson localization [14]. In particular, the possibility arises for a non-Fermi-liquid metallic phase in $d = 2$, as opposed to the insulating behavior of the non-interacting system [15].
iii) Additional Ward identities related to specific symmetries control the exact cancellation of singularities in the response functions within stable phases. For example, the low-energy properties of interacting fermions in $d = 1$ are highly constrained by additional conservation laws. In the presence of forward scattering, charge and spin are conserved separately at each Fermi point $+k_F$ and $-k_F$. As a consequence, we have new relations between different skeleton structures that allow for the closure of the hierarchical equations, leading to the solution of the model [16].

The combination of the three procedures indicated above will be used to describe the asymptotic (low-energy) behavior of:
1) The Luttinger liquid in $d = 1$ [16–18], and the crossover to Fermi liquid in $d > 1$ [19,20];

2) The non-Fermi-liquid phase in $d > 1$ induced by singular scattering [21–23].
3) The Bose liquid in the presence of condensate [24].

The program of these notes is the following: In Sec. II the Landau theory of second-order phase transition and critical phenomena is briefly reviewed. Its range of validity is expressed by means of the Ginzburg criterion and, by simple dimensional analysis, it is shown that going beyond the Gaussian approximation within a perturbative expansion leads to the appearance of infrared singularities in $d < 4$.

The first attempt to reach the fluctuation domain is described by introducing the crucial concepts of universality and scaling. According to the block-variable idea proposed by Kadanoff [11,25] (see also Ref. [7]), the difficult problem of dealing with a huge number of degrees of freedom strongly correlated within a distance of the order of the correlation length ξ, which diverges at criticality, is reduced to the determination of the critical indices of few relevant physical quantities. The gradual elimination of the irrelevant degrees of freedom, while describing the approach to criticality and the emergence of the scaling behavior, entails universality, i.e., independence of the microscopic (irrelevant) details.

Moreover, the scaling property allows to relate different critical exponents in such a way that only two of them are independent. The corresponding relations are known as scaling laws.

The theoretical framework of the idea of universality is presented in Sec. III, where the RG approach is sketched. After listing the general properties of a RG transformation, we present three different implementations of the RG idea: the Kadanoff block-variable transformation in real space; the Wilson RG transformation in momentum space; the field-theoretic RG approach. The first two implementations are the direct mathematical formulation of the hypothesis that the elimination of the short-distance (large-momentum) degrees of freedom can be translated into a proper rescaling of all the physical quantities. Scaling is obtained asymptotically, after an infinite number of iterations of the RG transformations is performed and a fixed point of the transformation is reached. Within the field-theoretic approach, the RG transformations act multiplicatively on the renormalized relevant quantities while changing the effective coupling, whose value is considered as irrelevant, and the evolution under renormalization is such that the physics is left unchanged. Therefore, the field-theoretic RG implements the other universality concept that one can change irrelevant variables (namely the coupling constant, which is the first irrelevant variable), provided the relevant fields are suitably rescaled.

After the three implementations of RG are briefly described, we use the Wilson RG approach to study critical phenomena near four dimensions. We show that below four dimensions there is the non-trivial Wilson-Fisher fixed point, which moves away from the Gaussian fixed point (stable for $d > 4$) by increasing the parameter $\epsilon = 4 - d$. All the corresponding anomalous (with respect to the Landau values) critical exponents can be computed in power-series of the small parameter ϵ and we give the first-order corrections.

In Sec. IV the low-energy behavior of some interacting fermion systems is investigated. They can be classified in terms of a small number of universality classes, which can be understood in terms of the Wilson RG, that selects few types of relevant scattering processes at small energies (near the Fermi surface).

We present a detailed analysis of interacting Fermi systems with strong forward scattering. In particular, by means of the RG ideas, we first recall that, when specific conditions on the initial coupling constants hold, the g-ology model, which is usually introduced to describe the various dominant interactions in these systems, reduces to the Luttinger model. In this model only forward scattering is present. As a consequence, in addition to the total charge (spin), also the charge (spin) associated with the left- and right-moving particles is conserved separately at each Fermi point. Additional Ward identities then follow, which allow to obtain the exact asymptotic behavior of the Luttinger liquid [16].

Once the Luttinger-liquid solution is obtained from the Luttinger model, we discuss its extension to $d > 1$. This is done in Sec. V, where we describe a first attempt to extend the non-Fermi-liquid behavior to $d > 1$ by adding a small interchain hopping $t_\perp$ within a system of parallel chains [26–29]. The RG analysis shows that $t_\perp$ is a relevant perturbation which drives the system away from the Luttinger-liquid fixed point, towards strong coupling. Thus the strongly anisotropic multichain approach is not appropriate to determine at which dimension the Luttinger liquid turns into a Fermi liquid. To this purpose we present a generalization of the Ward-identity method previously discussed, and control the evolution of the system via Ward identities to any order in perturbation theory. We illustrate the dimensional crossover from Luttinger liquid to Fermi liquid as $d > 1$ in systems with short-range forward interaction forces [19,20]. Then, we show how the same results can be obtained, both for $d = 1$ and $d = 2$, within the framework of the bosonization technique [30–36].

Finally, Within the framework of the perturbative RG, we analyze a model of fermions interacting through a singular potential [21–23], which leads to non-Fermi-liquid behavior in $d > 1$.

In Sec. VI we discuss interacting Bose systems in the presence of the Bose-Einstein condensate. We show that the presence of the condensate changes the dimensional analysis of the problem, and infrared singularities affect perturbation theory in $d < 3$, due to the soft Goldstone mode associated with the broken continuous symmetry. As anticipated above, one is able to control the exact cancellation of the singularities in all the physical quantities via Ward identities, and describe the asymptotic (low-energy) properties of the model [24]. We show that, although the resulting low-laying modes are always sound excitations, characteristic of the superfluid, the linear (sound-like) spectrum is realized in a completely different way in $d < 3$, where the RG fixed point is nontrivial, and in $d \geq 3$, where the Bogoliubov mean-field theory is correct.

In these notes we do not intend to be comprehensive, and we only refer to papers directly considered while lecturing.

II CRITICAL PHENOMENA

A Landau Theory

In second-order phase transitions the symmetry of the high-temperature phase is usually higher than that of the low-temperature phase. Although there are noticeable cases in which the opposite is true, we assume for the sake of definiteness that the low-temperature phase is the ordered (less symmetric) one. To characterize the symmetry change between the two phases, Landau introduced the concept of order parameter [12], which we indicate with φ throughout the paper. φ is defined as the thermal average of a suitable field $\hat{\varphi}$ which is chosen in such a way that:
1) $\varphi \neq 0$ below the critical temperature T_c and $\varphi = 0$ above it;
2) φ is a continuous function of the temperature T.

For example, in a ferromagnet φ is the thermal average of the local magnetic moment in the sample.

Usually, the order parameter is linearly coupled to an external field h (although there are cases in which the field coupled to the order parameter cannot be realized experimentally). Assuming that φ is constant throughout the system, the free energy $F(t = T - T_c, h)$ is such that

$$\frac{1}{V}\frac{\partial F}{\partial h} = -\varphi, \tag{1}$$

where V is the volume of the system. It is customary to introduce the thermodynamic potential $\Gamma(t, \varphi)$, which is defined via the Legendre transformation $\Gamma(t, \varphi) = F(t, h(t, \varphi)) + Vh(t, \varphi)\varphi$, where h is expressed as a function of t and φ by inverting Eq. (1). Then Γ is such that

$$\frac{1}{V}\frac{\partial \Gamma}{\partial \varphi} = h.$$

Near the critical temperature, the properties 1) and 2) ensure that φ is small and hence Landau assumed that $\Gamma(t, \varphi)$ could be expanded in power series of φ, taking $\Gamma = Vg_L$, with

$$g_L(t, \varphi) = g_n(t) + a(t)\varphi^2 + b(t)\varphi^4 + \ldots, \tag{2}$$

where g_n is the thermodynamic potential per unit volume in the normal (symmetric) phase and we are neglecting spatial fluctuations, assuming that φ is uniform throughout the system. The absence of odd powers of φ is related to the assumption that the underlying symmetry is such that φ and $-\varphi$ correspond to the same phase (in the absence of the external field h). Terms with powers higher than φ^4 are neglected within the Landau theory, since they are small for $t \to 0$, and the phenomenology of the phase transition is qualitatively accounted for by the minimal model, Eq. (2). We point out that the expansion of g_L in powers of the order

parameter and the absence of thermal fluctuations are the main assumptions of the Landau theory.

The order parameter can be generalized to be a vector with n components, such that $\varphi^2 = \sum_{i=1}^{n} \varphi_i^2$. In the case of a ferromagnet, $n = 1$ corresponds to the easy-axis ferromagnet, with up or down ordering along a preferential axis. The cases $n = 2$ and $n = 3$ correspond to the easy-plane and isotropic ferromagnet, respectively.

The t dependence of φ is obtained by imposing that the system is in thermal equilibrium, or, equivalently, by minimizing g_L with respect to φ (in the absence of an external field coupled to φ):

$$\left.\frac{\partial g_L}{\partial \varphi}\right|_{\varphi=\varphi_0} = 2a(t)\varphi_0 + 4b(t)\varphi_0^3 = 0 \quad \Rightarrow \quad \varphi_0^2 = \begin{cases} 0 & t > 0, \\ -\frac{1}{2}\frac{a(t)}{b(t)} & t < 0. \end{cases}$$

The stability of the above solution requires $a(t)/t > 0$ and $b(0) > 0$. Assuming a and b to be regular analytic functions of t, to the lowest order in $t \ll T_c$ we can write $a(t) \simeq At$, with $A > 0$, and $b(t) \simeq b(0) \equiv b > 0$. Taking into account the t dependence of $a(t)$ we get the critical exponent β of the Landau theory, which controls how the order parameter $\varphi_0 \sim |t|^{\beta}$ vanishes as the critical temperature is approached,

$$\varphi_0 = \sqrt{-\frac{a(t)}{2b}} \sim |t|^{1/2} \quad \Rightarrow \quad \beta = \frac{1}{2}. \tag{3}$$

In the presence of a uniform external field h coupled to φ the equilibrium condition $\partial g_L/\partial\varphi|_{\varphi=\bar{\varphi}} = h$ yields

$$2a(t)\bar{\varphi} + 4b(t)\bar{\varphi}^3 - h = 0. \tag{4}$$

The susceptibility $\chi = [\partial\bar{\varphi}/\partial h]_{h=0}$ is obtained from Eq. (4) by taking the derivative with respect to h and letting $h = 0$ at the end. The critical exponents γ, γ' are defined through the power-law behavior of the susceptibility near the critical temperature, $\chi \sim t^{-\gamma'}$, for $t > 0$, and $\chi \sim (-t)^{-\gamma}$, for $t < 0$. The direct calculation within the Landau theory gives

$$2a\chi + 12b\varphi_0^2\chi = 1 \quad \Rightarrow \quad \chi = \frac{1}{2a + 12b\varphi_0^2} = \begin{cases} \frac{1}{2a} \sim t^{-1} & t > 0, \\ -\frac{1}{4a} \sim (-t)^{-1} & t < 0, \end{cases}$$

where Eq. (3) has been used, and we see that $\gamma = \gamma' = 1$. The positivity of χ ensures in both cases thermodynamic stability.

The critical exponent δ is defined through the power-law behavior $\bar{\varphi}(t = 0) \sim h^{1/\delta}$ on the critical isotherm. Within the Landau theory $a(t = 0) = 0$ on the critical isotherm, and hence, from Eq. (4),

$$\bar{\varphi} \sim h^{1/3} \quad \Rightarrow \quad \delta = 3.$$

Another physical quantity which exhibits a power-law behavior near criticality is the specific heat C. The Landau theory gives

$$C(T) = -T\frac{\partial^2 g_L}{\partial T^2} = -T\frac{\partial^2}{\partial T^2}[a\varphi_0^2 + b\varphi_0^4] + C_n(T),$$

where $C_n(T)$ is the specific heat in the normal phase. C is discontinuous at T_c, and namely

$$\lim_{T\to T_c^+} C(T) = C_n, \quad \lim_{T\to T_c^-} C(T) = C_n + \frac{A^2 T_c}{2b}, \tag{5}$$

where $C_n \equiv C_n(T_c)$, and $C_n(T)$ is assumed continuous at $T = T_c$. This implies that the exponents α, α' defined by

$$C - C_n \sim \begin{cases} t^{-\alpha'} & t > 0, \\ (-t)^{-\alpha} & t < 0, \end{cases}$$

are both vanishing within the Landau theory. We point out that a vanishing exponent may indicate a discontinuity (as in the Landau theory), or a logarithmic singularity. Experimentally, the exponents α, α' are found either positive (in the case of a divergent specific heat), or negative (in the case of a finite specific heat, with a cusp at the phase transition).

B Fluctuations and the Ginzburg Criterion

The spatial variation of the order parameter, either due to fluctuations, or to a local external field, cannot be neglected, i.e., $\varphi = \varphi(\mathbf{r})$, although in general the uniform configuration is energetically favored in the absence of a local external field. This corresponds to the fact that an additional term of the form $c(\nabla\varphi)^2$, with $c > 0$, appears in g_L, Eq. (2), such that $\Gamma = \int g_L d^d\mathbf{r}$. Higher-order derivatives of $\varphi(\mathbf{r})$ are usually neglected, assuming slowly varying fluctuations.

In the presence of a non-homogeneous external field $h(\mathbf{r})$ the equilibrium condition is written as a functional derivative and reads $\delta\Gamma/\delta\varphi(\mathbf{r}) = h(\mathbf{r})$, i.e.,

$$2a\varphi(\mathbf{r}) + 4b\varphi^3(\mathbf{r}) - 2c\nabla^2\varphi(\mathbf{r}) = h(\mathbf{r}).$$

For an infinitesimal field variation $\delta h(\mathbf{r})$, $\varphi(\mathbf{r}) = \varphi_0 + \delta\varphi(\mathbf{r})$, and the above equation becomes

$$2a\delta\varphi(\mathbf{r}) + 12b\varphi_0^2\delta\varphi(\mathbf{r}) - 2c\nabla^2\delta\varphi(\mathbf{r}) = \delta h(\mathbf{r}), \tag{6}$$

to first order in $\delta\varphi(\mathbf{r})$ and $\delta h(\mathbf{r})$. A relation between $\delta\varphi$ and δh at zero field is expressed in terms of the two-point order-parameter correlation function $f_\varphi(\mathbf{r}, \mathbf{r}') = \langle\delta\hat{\varphi}(\mathbf{r})\delta\hat{\varphi}(\mathbf{r}')\rangle$, via the linear-response theory

$$\delta\varphi(\mathbf{r}) = \frac{1}{T}\int d^d\mathbf{r}' f_\varphi(\mathbf{r}, \mathbf{r}')\delta h(\mathbf{r}') + O(\delta h^2).$$

According to Eq. (6)

$$\frac{1}{T}\int d^d\mathbf{r}'[2a + 12b\varphi_0^2 - 2c\nabla_{\mathbf{r}}^2]f_\varphi(\mathbf{r},\mathbf{r}')\delta h(\mathbf{r}') = \int d^d\mathbf{r}'\delta(\mathbf{r}-\mathbf{r}')\delta h(\mathbf{r}'),$$

or in Fourier space[1]

$$2(a + 6b\varphi_0^2 + c|\mathbf{q}|^2)\tilde{f}_\varphi(\mathbf{q}) = T.$$

The Fourier-transformed correlation function $\tilde{f}_\varphi(\mathbf{q})$ is the thermal average of $|\delta\hat{\varphi}(\mathbf{q})|^2$, where $\delta\hat{\varphi}(\mathbf{q}) = \int \mathrm{e}^{-i\mathbf{q}\cdot\mathbf{r}}\delta\hat{\varphi}(\mathbf{r})d^d\mathbf{r}$. From the above equation we get the correlation function in the so-called Gaussian approximation

$$\tilde{f}_\varphi(\mathbf{q}) = \langle|\delta\hat{\varphi}(\mathbf{q})|^2\rangle = \frac{T/2c}{\frac{a+6b\varphi_0^2}{c} + |\mathbf{q}|^2} \equiv \frac{T/2c}{\xi^{-2} + |\mathbf{q}|^2}, \tag{7}$$

where ξ is the correlation length, i.e., the characteristic length scale for the decay of spatial correlations of the order parameter. In this approximation, $\xi^2 = 2c\chi$. Since $a \sim t$ and $\varphi_0 \sim |t|^{1/2}$, the correlation length diverges approaching the critical temperature as

$$\xi \sim |t|^{-1/2},$$

and hence within the Landau theory $\nu = \nu' = 1/2$, where ν and ν' are defined by

$$\xi \sim \begin{cases} t^{-\nu'} & t > 0, \\ (-t)^{-\nu} & t < 0, \end{cases}$$

for $t \to 0$. Evidently $2\nu = \gamma$.

The correlation function behaves like

$$f_\varphi(|\mathbf{r}|) \sim \frac{\mathrm{e}^{-|\mathbf{r}|/\xi}}{|\mathbf{r}|^{d-2}} \underset{t\to 0}{\longrightarrow} \frac{1}{|\mathbf{r}|^{d-2}},$$

in real space, or

$$\tilde{f}_\varphi(\mathbf{q}) \underset{t\to 0}{\longrightarrow} \frac{1}{|\mathbf{q}|^2},$$

in momentum space. The extrapolation at T_c of the structure factor measured in neutron scattering experiments on ferromagnets, leads to $\tilde{f}_\varphi(\mathbf{q}) \sim 1/|\mathbf{q}|^{2-\eta}$ with a small ($\sim 10^{-2}$) correcting critical index η. In the Gaussian approximation $\eta = 0$.

We conclude this section by deriving the criterion of applicability of the Landau theory. The results presented in Sec. II A neglected thermal fluctuations and

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[1] Here we are assuming translational and rotational invariance at $h = 0$, in such a way that $f_\varphi(\mathbf{r},\mathbf{r}') = f_\varphi(|\mathbf{r}-\mathbf{r}'|)$.

therefore they are plausible results only if the fluctuations of the order parameter within a volume of linear size ξ are much smaller than the Landau value, i.e.,

$$\frac{1}{\xi^d}\int_{|\mathbf{r}|<\xi} d^d\mathbf{r} f_\varphi(|\mathbf{r}|) \sim \frac{1}{\xi^{d-2}} \ll \varphi_0^2 \sim \frac{1}{\xi^2},$$

or

$$\xi^{d-4} \gg \text{const} \quad \Rightarrow \quad t^{d-4} \ll \text{const}. \tag{8}$$

This is known as the Ginzburg criterion. Eq. (8) is always satisfied in $d > 4$ if t is sufficiently small, i.e., for T close enough to T_c. On the other hand, when $d < 4$ the critical point $t = 0$ cannot be reached within the Landau theory. Here the dominant contribution comes from the fluctuations and the corresponding region is called the fluctuation region, or critical region. We point out that, for the Landau theory to be valid for $d < 4$, the temperature must be outside the fluctuation region, but not too far from the critical point, since the Landau expansion in Eq. (2) is justified only if $|t| \ll T_c$.

The relevance of fluctuations for $d < 4$ is witnessed by the fact that the measured physical quantities, although following a power-law behavior, have critical indices different from those found within the Landau theory, which are reported in Tab. 1. In particular the measured indices, as well as the indices obtained either by the exact solution, or by numerical simulation of various models, depend on the space dimensionality d, as well as on the number of components of the order parameter n, as it is shown in Tab. 2, where the indices for the Ising model in $d = 2$ are obtained from the exact solution [37], and the indices for $d = 3$, reported from Ref. [38] as an example, were obtained as the average of the best available numerical estimates (see also Ref. [39]). On the other hand, the experimental critical indices display a large degree of universality, in so far as they are the same for completely different physical systems, provided d, n, and any other underlying symmetry of the system are the same[2]. They do not depend on the range of the interaction forces (provided it is finite), on the crystal structure, on the specific values of the coupling constants, as far as these do not change the symmetry of the problem. For instance, as it has been verified by numerical calculations [39], the model for ferromagnetic ordering described by the classical Heisenberg Hamiltonian $H = -\sum_{ij}(J_x S_{ix}S_{jx} + J_y S_{iy}S_{jy} + J_z S_{iz}S_{jz})$, where $S_{i\alpha}$ is the α component of the spin vector at the site i of a given lattice, and i, j are neighboring sites, has the same exponents as the Ising model ($n = 1$) for all $J_x > J_y, J_z \geq 0$, i.e., for an easy-axis ferromagnet, the same exponents as the xy model ($n = 2$) for all $J_x = J_y > J_z \geq 0$. The isotropic (Heisenberg) case corresponds $J_x = J_y = J_z > 0$, and the critical indices do not depend on the strength of the coupling constant.

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[2] For instance, the exponents measured in liquids at their critical point are $\beta = 0.321 \div 0.328$, $\gamma = 1.23 \div 1.28$, $\nu = 0.61 \div 0.65$ [38], which compare well with the indices of the $d = 3$, $n = 1$ Ising universality class, reported in Tab. 2. See, e.g., Ref. [8] for a detailed list of the values of the critical indices in various physical systems.

TABLE 1. In the first column we summarize the critical (power-law) behavior of the various thermodynamic quantities. The second column shows the values of the critical exponents according to the Landau theory.

Critical Laws as $t, h \to 0$	Landau Critical Exponents
$C(h=0) \sim \lvert t\rvert^{-(\alpha,\alpha')}$,	$\alpha = \alpha' = 0$
$\varphi(h=0) \sim (-t)^{\beta}, \quad \beta > 0$	$\beta = 1/2$
$\chi(h=0) \sim \lvert t\rvert^{-(\gamma,\gamma')}, \quad \gamma, \gamma' > 0$	$\gamma = \gamma' = 1$
$\varphi(t=0) \sim h^{1/\delta}, \quad \delta > 1$	$\delta = 3$
$f_{\varphi}(\lvert\mathbf{r}\rvert)(t=0, h=0) \sim \lvert\mathbf{r}\rvert^{-(d-2+\eta)}, \quad \eta \geq 0$	$\eta = 0$
$\xi(h=0) \sim \lvert t\rvert^{-(\nu,\nu')}, \quad \nu, \nu' > 0$	$\nu = \nu' = 1/2$

With respect to universality, the Landau theory is too universal, as it yields critical indices which do not depend on d and n, and their values, different from those measured experimentally, are determined only by the expansion (2) of the thermodynamic potential in terms of the order parameter.

C Dimensional Analysis

Tab. 1 shows the values of the critical exponents calculated in the mean-field Landau theory. The Ginzburg criterion ensures that above the critical dimension $d = 4$ they are exact. What can we say when $d < 4$? To give an answer to this question, we have to find a new scheme which improves the idea that the thermodynamic potential is expanded in series of the order parameter, as fluctuations must be taken into account. However, the idea that the underlying symmetry plays a relevant role in selecting the starting model, which originally guided Landau in writing Eq. (2), is sound. Moreover, the phenomenology suggests that the power-law behavior of the physical quantities, and the self-similarity properties entailing universality, which are related to the divergence of the correlation length at criticality, must survive in the new reduction scheme, leading to the effective model for a system near criticality.

The functional-integral formulation of specific models (e.g., the Ising model, see Ref. [40]), as well as the physical idea that the microscopic details are irrelevant, and that there must exist a coarse-graining procedure which allows to eliminate the microscopic (fastly thermalizing) degrees of freedom for a given configuration of the slowly fluctuating field $\hat{\varphi}$ [7], suggest that, near the critical point, the partition function Z is written as a functional integral over $\hat{\varphi}$, $Z = \int D\hat{\varphi} \mathrm{e}^{-H_{\mathrm{eff}}[\hat{\varphi}]/T}$. The

TABLE 2. Dependence of the critical indices on the space dimensionality d, and on the number of components of the order parameter n.

Index	$d=2, \quad n=1$ (Ising)	$d=3, \quad n=1$ (Ising)	$d=3, \quad n=2$ (xy)	$d=3, \quad n=3$ (Heisenberg)
α, α'	0	0.110	−0.007	−0.115
β	1/8	0.325	0.346	0.365
γ, γ'	7/4	1.240	1.316	1.387
δ	15	4.82	4.80	4.80
η	1/4	0.03	0.03	0.03
ν, ν'	1	0.630	0.669	0.705

simplest form of a coarse-grained Hamiltonian is

$$\frac{1}{T}H_{\text{eff}}[\hat{\varphi}] = \int [c(\nabla\hat{\varphi})^2 + a\hat{\varphi}^2 + b\hat{\varphi}^4 - h\hat{\varphi}]d^d\mathbf{r}, \tag{9}$$

where $\hat{\varphi}(\mathbf{r})$ is a continuous scalar field, which can be generalized to a n-component multiplet $(\hat{\varphi}_1, ..., \hat{\varphi}_n)$, with $\hat{\varphi}^2 = \sum_{i=1}^n \hat{\varphi}_i^2$.

The order parameter φ is the thermal average of $\hat{\varphi}$. The absence of terms with powers of $\hat{\varphi}$ larger than 4, and of spatial derivatives higher than the second is assumed here, and finds a justification only within the dimensional analysis and the RG approach. Within the Landau theory the model Eq. (9) is the minimal model which produces the phase transition and the critical exponents. The Landau theory corresponds to compute Z by evaluating H_{eff} at the functional extremal, i.e., for $\hat{\varphi} = \bar{\varphi}$. The Gaussian approximation includes the effect of the fluctuation in an approximate manner, by expanding the integrand in Eq. (9) up to quadratic terms in the fluctuation $\delta\hat{\varphi} = \hat{\varphi} - \bar{\varphi}$, giving rise to the expression (7) for the order-parameter correlation function.

The question we want to address in this section is: Is it possible to improve the Landau theory beyond the Gaussian approximation?

Let x_Q^0 be the physical dimension of the quantity Q in units of the inverse length. Since H_{eff}/T is a dimensionless quantity we derive from Eq. (9)

$$x_\varphi^0 = \frac{d-2}{2}, \quad x_a^0 = 2, \quad x_b^0 = 4 - d, \tag{10}$$

where $x_c^0 = 0$ has been assumed. Using the above relations one can express the effective Hamiltonian H_{eff}/T in terms of dimensionless quantities only[3]. Let us define

$$\phi = a^{\frac{2-d}{4}}\hat{\varphi}, \quad \mathbf{x} = a^{\frac{1}{2}}\mathbf{r}, \quad b_0 = a^{\frac{d-4}{2}}b,$$

and let us assume, for the sake of simplicity, that $t > 0$ and $h = 0$. Then

$$\frac{1}{T}H_{\text{eff}} = \frac{1}{T}H_0 + \frac{1}{T}H_{\text{int}} = \int d^d\mathbf{x}[(\nabla\phi)^2 + \phi^2] + b_0\int d^d\mathbf{x}\phi^4,$$

where H_{int} takes into account the interaction between the fluctuations. The value $c = 1$ has been chosen without loss of generality, by a redefinition of $\hat{\varphi}$. If $b_0 = 0$ the Gaussian approximation is exact. In general $b_0 \neq 0$. The case $b_0 \ll 1$ is a good starting point for perturbation theory. Nevertheless, we recall that $b_0 = a^{\frac{d-4}{2}}b$ and $a \sim t = T - T_c$. Therefore, when $t \to 0$ and $d < 4$, $b_0 \to \infty$ and perturbation theory becomes inappropriate. On the other hand, when $d > 4$, $b_0 \to 0$ for $t \to 0$ and the Landau mean-field theory becomes more and more accurate when approaching the critical point.

We observe that the dimensional analysis is very powerful. It predicts not only the correct critical dimension $d = 4$, but also the Landau critical indices of Tab. 1. Indeed, by definition, $f_\varphi(|\mathbf{r}|)$ has the same dimension of φ^2: $x^0_{f_\varphi} = d - 2$. At the critical point $f_\varphi(|\mathbf{r}|) \sim |\mathbf{r}|^{-(d-2+\eta)}$, and hence $\eta = 0$. Analogously, since $\xi \sim 1/\sqrt{a} \sim |t|^{-1/2}$, we deduce $\nu = 1/2$. It is not hard to see that the dimensional analysis cannot predict values for the critical exponents different from the ones assigned by the Landau theory. At a first sight the critical index β seems to be not reproducible by simple dimensional analysis. Indeed, $x^0_\varphi = (d-2)/2$ and $a \sim t$ so that one could say $\varphi \sim |t|^{(d-2)/4}$ and $\beta \neq 1/2$ when $d \neq 4$. The problem can be solved by noting that the value of the order parameter within the mean-field theory is $\varphi \sim (a/b)^{1/2}$. Therefore, one should write

$$\varphi \sim \frac{|t|^{(d-2)/4}}{\sqrt{b_0}} \sim \frac{|t|^{(d-2)/4}}{\sqrt{b}t^{(d-4)/4}} \sim t^{1/2},$$

which yields the correct result. We shall see later that, above the critical dimension $d = 4$, b is a "dangerous" parameter and we have to pay attention in calculating physical relations near criticality.

As a matter of fact, the critical exponents within the Landau theory are too universal, and do not depend on the dimensionality d and on the number of components n of the order parameter. As we anticipated above, this is a consequence of the fact that the Landau theory incorrectly assumes that the thermodynamic potential can be expanded in powers of the order parameter[4], and neglects the

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[3] The following procedure is not necessary, but it allows to immediately identify the proper dimensionless coupling constant, without computing Feynman diagrams.

[4] The expansion in powers of the field $\hat{\varphi}$ in Eq. (9) can be justified a posteriori via scaling for $3 < d < 4$. By no means it is correct in the thermodynamic potentials.

effect of strong fluctuations, which for $d < 4$ become more and more relevant when the critical point is approached. Once these are taken into account in a proper way, one may expect that the bare physical dimensions x^0 are corrected, and that these corrections crucially depend on d and n.

D Universality Principle

The first breakthrough in understanding the physics near the critical point came from the universality principle [11,25]: provided the proper choice of the relevant variables is made, say the field h conjugate to the order parameter and the deviation t from the critical temperature, we can safely change the other variables, as long as they do not assume a value which changes the symmetry of the problem, leaving the physics unchanged. Said in other words, systems which differ only in irrelevant variables share the same critical behavior.

Let $\{\zeta_i\}$ be the parameters specifying the details of the interaction. Then, the critical behavior of two systems differing only in the values of the ζ's is the same (similarity property). The universality principle establishes a one-to-one correspondence between the quantities of two similar systems. In formulæ

$$\begin{aligned} F(t,h;\{\zeta_i\}) &= F(t',h';\{\zeta_i'\}), \\ \varphi(t,h;\{\zeta_i\}) &= \mathcal{A}(\{\zeta_i\},\{\zeta_i'\})\varphi(t',h';\{\zeta_i'\}), \\ f_\varphi(t,h;\{\zeta_i\}) &= \mathcal{B}(\{\zeta_i\},\{\zeta_i'\})f_\varphi(t',h';\{\zeta_i'\}), \end{aligned} \tag{11}$$

where $F/T = -\ln Z$ is the dimensionless free energy, t' and h' are the rescaled values of the relevant quantities t and h, and $\{\zeta_i\}$, $\{\zeta_i'\}$ are the irrelevant parameters entering in the Hamiltonian of the two systems. $\mathcal{A}$ and $\mathcal{B}$ are two functions of the irrelevant variables only. We remark that the relations (11) are written for the *most singular part* of the corresponding functions.

F is the generating functional of connected correlation functions

$$f_\varphi^{(n)}(\mathbf{r}_1,\ldots,\mathbf{r}_n) = -\frac{1}{T}\frac{\delta^n F}{\delta h(\mathbf{r}_1)\ldots\delta h(\mathbf{r}_n)}.$$

In particular $\varphi(\mathbf{r}) = f_\varphi^{(1)}(\mathbf{r})$, and $f_\varphi(\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2) = f_\varphi^{(2)}(\mathbf{r}_1,\mathbf{r}_2)$.

We can also express the universality principle by eliminating the "field" variables in terms of the "density" variables. This is done by Legendre-transforming the dimensionless free energy F/T and yields the thermodynamic potential

$$\Gamma(t,\varphi;\{\zeta_i\}) = \frac{1}{T}F(t,h(t,\varphi;\{\zeta_i\});\{\zeta_i\}) + \int d^d\mathbf{r}\; h(t,\varphi;\{\zeta_i\})\varphi,$$

which generates the n-point vertex functions

$$\Gamma^{(n)}(\mathbf{r}_1,\ldots,\mathbf{r}_n) = \frac{\delta^n\Gamma}{\delta\varphi(\mathbf{r}_1)\ldots\delta\varphi(\mathbf{r}_n)}.$$

In particular $\Gamma^{(1)}(\mathbf{r}) = h(\mathbf{r})$. The Fourier transform of $\Gamma^{(2)}(\mathbf{r}_1, \mathbf{r}_2)$ is $\tilde{\Gamma}^{(2)}(\mathbf{q}) = [\tilde{f}_\varphi^{(2)}(\mathbf{q})]^{-1}$, where we have factorized a δ function in momentum space, which implements momentum conservation when $\Gamma^{(2)}$ depends only on $\mathbf{r}_1 - \mathbf{r}_2$. The universality principle now reads

$$\begin{aligned} \Gamma(t, \varphi; \{\zeta_i\}) &= \Gamma(t', \varphi'; \{\zeta_i'\}), \\ h(t, \varphi; \{\zeta_i\}) &= \bar{\mathcal{A}}(\{\zeta_i\}, \{\zeta_i'\}) h(t', \varphi'; \{\zeta_i'\}), \\ \Gamma^{(2)}(t, \varphi; \{\zeta_i\}) &= \bar{\mathcal{B}}(\{\zeta_i\}, \{\zeta_i'\}) \Gamma^{(2)}(t', \varphi'; \{\zeta_i'\}). \end{aligned} \tag{12}$$

Eqs. (11) and (12) are the formal content of the idea of irrelevant and relevant variables.

E Scaling Hypothesis

One step further in building a theoretical framework for critical phenomena was the phenomenological scaling theory [9–11,25,41].

On the basis that the correlation length ξ diverges for $T \to T_c$, Kadanoff assumed the length scale as one of the irrelevant details: whatever is the length scale we choose, near criticality the system appears the same. Two obvious relevant fields are t and h, which drive the system out of, or toward criticality.

Let us measure all the physical quantities in terms of the inverse lattice spacing Λ:

$$\mathbf{r}_\Lambda = \Lambda \mathbf{r}, \quad t_\Lambda = \frac{t}{\Lambda^{x_t^0}}, \quad \varphi_\Lambda = \frac{\varphi}{\Lambda^{x_\varphi^0}}, \quad h_\Lambda = \frac{h}{\Lambda^{x_h^0}}, \quad \zeta_{i,\Lambda} = \frac{\zeta_i}{\Lambda^{x_{\zeta_i}^0}},$$

where $x_t^0 = x_a^0$, since $a = At$ and we can take $x_A^0 = 0$ without loss of generality. The scaling hypothesis states that if $\mathbf{r}_\Lambda \to \mathbf{r}_\Lambda / s$, where s is the scaling parameter, it is possible to rescale t_Λ, h_Λ, φ_Λ and all the irrelevant parameters $\zeta_{i,\Lambda}$ in such a way that, sufficiently near criticality, the *most singular parts* of all the physical relations do not change, i.e., the physical content, as far as the critical behavior is concerned, remains unchanged. In other words, there must be scaling dimensions x_t, x_φ, x_h, x_{ζ_i} in the transformations

$$t_\Lambda \to s^{x_t} t_\Lambda, \quad \varphi_\Lambda \to s^{x_\varphi} \varphi_\Lambda, \quad h_\Lambda \to s^{x_h} h_\Lambda, \quad \zeta_{i,\Lambda} \to s^{x_{\zeta_i}} \zeta_{i,\Lambda},$$

such that the s factor disappears everywhere when $\mathbf{r}_\Lambda \to \mathbf{r}_\Lambda / s$. It is worth to note that the scaling dimensions x are *a priori* different from the bare dimensions x^0.

We can easy identify the parameter x_t. Near criticality $\xi_\Lambda \sim |t_\Lambda|^{-\nu}$ and the rescaling implies $\xi_\Lambda \to \xi_{\Lambda/s} = \xi_\Lambda / s$ and $t_\Lambda \to t_{\Lambda/s} = t_\Lambda s^{x_t}$. Therefore

$$\frac{1}{s} = \frac{\xi_{\Lambda/s}}{\xi_\Lambda} = \left(\frac{t_{\Lambda/s}}{t_\Lambda} \right)^{-\nu} = \frac{1}{s^{x_t \nu}} \quad \Rightarrow \quad x_t = \frac{1}{\nu}.$$

Analogously, we can identify the parameter x_φ. At criticality $f_{\varphi_\Lambda} \equiv \langle \hat{\varphi}_\Lambda \hat{\varphi}_\Lambda \rangle \sim |\mathbf{r}_\Lambda|^{-(d-2+\eta)}$. The requirement of invariance under a scaling transformation yields

$$x_\varphi = \frac{d-2+\eta}{2}. \tag{13}$$

Moreover, since for any couple of conjugate variables (like φ and h) the sum of the two scaling dimensions is equal to the space dimension d we have $x_\varphi + x_h = d$, i.e.,

$$x_h = \frac{d+2-\eta}{2}. \tag{14}$$

While all the irrelevant parameters of the theory should asymptotically disappear, the two phenomenological parameters x_t and x_h (or x_φ), related to the relevant variables, are identified in terms of the physical indices ν and η.

The difference between the scaling dimension x and the bare dimension x^0 is called the anomalous dimension. For instance, $x_\varphi - x_\varphi^0 = \eta/2$.

F Scaling Laws

The scaling hypothesis is a very powerful tool to generate: *i*) general homogeneous expressions for the physical quantities, and *ii*) exact relations between the critical indices. Below, we show that α, β, γ, δ, η, and ν are related by 4 scaling laws: only two critical exponents are independent[5], which correspond to the two parameters x_t and x_h introduced in the phenomenological theory.

Let us denote with a bar the thermodynamic potentials expressed in terms of dimensionless quantities. The scaling hypothesis on the dimensionless free energy per unit volume $\bar{F} = F/VT$ reads

$$\bar{F}(t_\Lambda, h_\Lambda; \{\zeta_{i,\Lambda}\}) = \frac{1}{s^d} \bar{F}(t_\Lambda s^{x_t}, h_\Lambda s^{x_h}; \{\zeta_{i,\Lambda} s^{x_{\zeta_i}}\}), \tag{15}$$

which is a generalized homogeneity condition. A possible solution for this functional equation is[6]

$$\bar{F}(t_\Lambda, h_\Lambda; \{\zeta_{i,\Lambda}\}) = |t_\Lambda|^{d/x_t} f\left(\frac{h_\Lambda}{|t_\Lambda|^{x_h/x_t}}; \left\{ \frac{\zeta_{i,\Lambda}}{|t_\Lambda|^{x_{\zeta_i}/x_t}} \right\} \right), \quad t_\Lambda < 0. \tag{16}$$

The crossover index $\phi_i = x_{\zeta_i}/x_t = \nu x_{\zeta_i}$ distinguishes between relevant and irrelevant variables:

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[5] The generalized homogeneity relation (15) does not distinguish between the two limits from above and from below the critical temperature, therefore $\gamma = \gamma'$, $\alpha = \alpha'$, and $\nu = \nu'$.

[6] Depending on the sign of t_Λ, we have in principle two different solutions for a given choice of the variable to put in front of f. In the following we work with $t_\Lambda < 0$ without loss of generality since the critical exponents do not depend on from where we approach the critical point.

i) If $\phi_i < 0$, ζ_i disappears as $t \to 0$ and is thus irrelevant. The corresponding variable has no critical fluctuations.

ii) If $\phi_i > 0$, ζ_i is relevant. When $|t| \gg \zeta_i^{1/\phi_i}$ the system behaves as if approaching criticality with $\zeta_i = 0$. However, by further reducing t, the crossover region $\zeta_i \sim |t|^{\phi_i}$ is reached. The scaling dependence on t is violated and the behavior of the system is controlled by the non vanishing relevant variables when $|t| \ll \zeta_i^{1/\phi_i}$.

iii) If $\phi_i = 0$ the corresponding variable is called marginal. Here we may have an apparent violation of universality with critical indices depending continuously on a parameter [42].

Assuming t and h as the only relevant variables, all the ζ's disappear from the equations when $s \to \infty$, or $t_\Lambda \to 0$. The thermodynamic potential, the order parameter, and the correlation function become homogeneous functions of the variables t, h (and $\mathbf{r}$, or $\mathbf{q}$) with the appropriate degree of homogeneity. This implication has been tested in several systems [43].

We now proceed to derive the four scaling relations. If

$$f(h_\Lambda = 0; \{\zeta_{i,\Lambda}/|t_\Lambda|^{x_{\zeta_i}/x_t}\} \to 0) \neq 0, \pm\infty$$

when $t_\Lambda \to 0$, the most singular part of the specific heat is given by

$$C \sim \left(\frac{\partial^2 \bar{F}}{\partial t_\Lambda^2}\right)_{h=0} \sim |t_\Lambda|^{\nu d - 2}. \tag{17}$$

Comparing this result with $C \sim |t_\Lambda|^{-\alpha}$ we get the first scaling law

$$\boxed{\nu d = 2 - \alpha.} \tag{18}$$

The power-law behavior of the order parameter in zero external field can be obtained from Eq. (16) together with the relation $\varphi_\Lambda \sim -(\partial \bar{F}/\partial h_\Lambda)_{h_\Lambda = 0}$:

$$\varphi_\Lambda \sim |t_\Lambda|^{\nu(d - x_h)},$$

apart from a finite multiplicative constant. By definition of the index β, $\varphi_\Lambda \sim |t_\Lambda|^\beta$. Therefore

$$\boxed{\nu(2 - \eta) = 2 - \alpha - 2\beta,} \tag{19}$$

where Eqs. (13), (14) and (18) have been used.

Analogously, we obtain a scaling law involving the critical exponent γ. Indeed, from Eq. (16) we have

$$\chi_\Lambda \sim \left(\frac{\partial \varphi_\Lambda}{\partial h_\Lambda}\right)_{h_\Lambda = 0} \sim -\left(\frac{\partial^2 \bar{F}}{\partial h_\Lambda^2}\right)_{h_\Lambda = 0} \sim |t_\Lambda|^{\nu(d - 2x_h)},$$

and using Eq. (14) together with the previous two scaling laws and the definition $\chi_\Lambda \sim |t_\Lambda|^{-\gamma}$, one obtains

$$\boxed{\alpha + 2\beta + \gamma = 2.} \tag{20}$$

The last scaling law can be derived as follows. Let us come back to Eq. (15) and write the solution in the equivalent form

$$\bar{F}(t_\Lambda, h_\Lambda; \{\zeta_{i,\Lambda}\}) = h_\Lambda^{d/x_h} \tilde{f}\left(\frac{t_\Lambda}{h_\Lambda^{x_t/x_h}}; \left\{\frac{\zeta_{i,\Lambda}}{h_\Lambda^{x_{\zeta_i}/x_h}}\right\}\right), \quad h_\Lambda > 0.$$

Assuming $\tilde{f}(t_\Lambda = 0; \{\zeta_{i,\Lambda}/h_\Lambda^{x_{\zeta_i}/x_h}\} \to 0) \neq 0, \pm\infty$ when $h_\Lambda \to 0$, we get the power-law behavior of the order parameter on the critical isotherm

$$\varphi_\Lambda \sim -\left(\frac{\partial \bar{F}}{\partial h_\Lambda}\right)_{t_\Lambda = 0} \sim h_\Lambda^{d/x_h - 1}. \tag{21}$$

The exponent of h_Λ in Eq. (21) is exactly $1/\delta$ and the fourth scaling law reads

$$\boxed{\beta + \gamma = \beta\delta,} \tag{22}$$

where Eq. (14) and the previous scaling laws have been used.

Taking the mean-field values of the critical exponents from Tab. 1, the last three scaling laws, Eqs. (19), (20) and (22), are satisfied. What about Eq. (18)? We observe that, among the four scaling laws, Eq. (18) is the only one in which the space dimensionality d appears explicitly. Since ν and α do not depend on d at mean-field level, Eq. (18) is violated for the mean-field exponents if $d \neq 4$. Nevertheless, $\nu = 1/2$ and $\alpha = 0$ are the right values when $d > 4$. How to solve this paradox? In $d > 4$ there is a dangerous irrelevant parameter among the ζ's, i.e., the interaction strength b. Let us consider Eq. (17) once more; from the explicit solution of the Ginzburg-Landau model we have $C \sim 1/b$. According to the scaling relation

$$C \sim \left(\frac{\partial^2 \bar{F}}{\partial t_\Lambda^2}\right)_{h_\Lambda = 0} \sim |t_\Lambda|^{\nu d - 2} f\left(0; \frac{b_\Lambda}{t_\Lambda^{x_b/x_t}}, \ldots\right).$$

However, the explicit expression (5) for the specific heat in the Landau theory has the irrelevant variable b in the denominator, and therefore $f\left(0; b_\Lambda/t_\Lambda^{x_b/x_t}, \ldots\right)$ does not behave as a constant for $t_\Lambda \to 0$, as it is required to derive the scaling relation (18). Taking into account the extra power of t_Λ coming from f via the dependence on b, we get $C \sim |t_\Lambda|^{\nu(d + x_b) - 2}$. Moreover, the vanishing of the anomalous dimensions in $d > 4$ implies $x_b = x_b^0 = 4 - d$ and hence Eq. (18) becomes $4\nu = 2 - \alpha$. We conclude that the correct scaling law is

$$\boxed{2 - \alpha = \begin{cases} d\nu & d \leq 4, \\ 4\nu & d \geq 4. \end{cases}}$$

We can now express all the indices in terms of ν and η:

$$\alpha = 2 - \nu \min(d, 4),$$
$$\beta = \frac{\nu}{2} \left[\min(d, 4) - 2 - \eta\right],$$
$$\gamma = \nu(2 - \eta),$$
$$\delta = \frac{\min(d, 4) + 2 - \eta}{\min(d, 4) - 2 + \eta},$$

where $\eta = 0$ and $\nu = 1/2$ for $d \geq 4$.

For $d = 3$, the scaling properties are well satisfied by the experimental values of the critical indices, and for $d = 2$ by the exact solution of models, e.g., the Ising model (see Tab. 2). We have therefore to provide a microscopic foundation to this phenomenological theory.

III RENORMALIZATION GROUP

A Introduction

As discussed in Sec. II, the phenomenological theory of critical phenomena finds a sound basis in the logical sequence of universality, scaling, relevant and irrelevant variables [10,11,25,41].

The difficult problem of a large number of degrees of freedom strongly correlated within the coherence distance ξ which diverges at criticality, is thereby in principle reduced to the determination of the homogeneous form of the relevant response functions, and the evaluation of few critical indices. Once the mechanism which controls the approach to the scaling behavior of the relevant quantities is understood, universality follows immediately.

In order to give a basic microscopic foundation to universality and to the scaling theory, an exact transformation is needed to reproduce the scaling transformation asymptotically near the critical point.

Alongside with the scaling theory, the field-theoretic approach was introduced into this problem simultaneously in the USSR and in Italy. In 1969 it was noticed [44] that the field-theoretic RG equations [1,2], generalize the universality relations in the sense that they relate one model system to another by varying the coupling and suitably rescaling the other variables and the correlation functions. Under the assumption that the coupling disappears from the equations near to the critical point, the homogeneous form of the static order-parameter correlation function is obtained and a microscopic definition of the critical indices is given [44–46].

Migdal [47] and Poljakov [48] started from a detailed analysis of the diagrammatic structure of the correlation functions rather than from global conditions such as the

RG equations, and, using Ward identities, tried to build an *ad hoc* renormalization procedure[7].

The scaling picture had then been given a theoretical basis and the RG appeared as the right tool to investigate the scaling behavior of the correlation functions. Nevertheless, the mechanism by which the memory of the original coupling disappears from the RG equations via the fixed point of the transformation was not yet made explicit since no way to make reliable calculations of the critical indices was found.

Wilson [3] gave a great contribution to the understanding of the physics underlying the RG procedure and proposed a simple mathematical realization of the other very physical idea of Kadanoff's universality of grouping together degrees of freedom associated with larger and larger cells. He proposed a procedure of elimination of the short-wavelength fluctuations, with momenta between the upper momentum cutoff Λ and Λ/s, and a suitable rescaling of the resulting variables. The mechanism of disappearance of the original coupling constants was then clarified and the calculation of the critical indices was performed via the numerical evaluation of the resulting recurrence equations. As an alternative scheme to evaluate the critical indices, the ϵ expansion ($\epsilon = 4 - d$) of the transformation around four dimensions was also proposed by Wilson and Fisher [50] (see Section III D).

After Wilson's great advances, it was not difficult to develop the analogous mechanism for the field-theoretic RG also, by realizing the crucial role played by the dimension of the coupling constant in a field-theory model for a correct description of critical phenomena [51]. For a $\hat{\varphi}^4$ model, this was actually noticed in the classical limit of an interacting Bose gas, studied via the Matsubara technique for the correlation functions. The theory was then completed [52,53] within the field-theoretic RG approach (including the Callan-Symanzik equation) by the introduction of the thermodynamic potential as the generating functional of the relevant correlation functions, which permitted the discussion of the thermodynamic scaling.

Few theories have been supported by so many successes, nevertheless a certain knowledge of the fundamental symmetries inherent to the problem, in particular of the order parameter, has to be assumed in order to make the proper choice of the basic variables on which to operate the transformation (i.e., the partial trace over the degrees of freedom associated with the short-range behavior in the Wilson scheme, and the renormalization of the relevant fields in the field-theoretic RG).

One must therefore translate a given problem in terms of an effective Hamiltonian expressed through physically significant variables. This effective model should account for a qualitative thermodynamic description of the system at mean-field level. This is then assumed as the zeroth approximation suitable for the application of the RG transformation. A bad choice of the starting point and of the corresponding group transformation may lead to unphysical results [54].

Most of the so-called complex problems in statistical mechanics are those for

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[7] This approach was later developed into the skeleton expansion method for a practical calculation of the critical indices [49].

which it is difficult to identify the relevant physical quantities, and in particular, a suitable parameter to write an effective action and the corresponding mean-field theory. In this section, as a matter of an introduction, we deal with the simplest case of critical phenomena, where the starting mean-field theory is well characterized. For instance, in the case of an ordinary magnetic critical point, the spontaneous magnetization (the order parameter φ), the deviation from the critical temperature t, and the coupling constant b, are considered as the basic variables. Three coefficients are therefore present in the RG equations for the thermodynamic potential. One of these, the coupling constant, will assume an asymptotic value by iterating the RG transformations towards the fixed point. The corresponding scale-invariant theory is fully determined by the remaining two coefficients evaluated at the fixed-point value of the coupling constant, thus yielding two independent critical indices.

Of course different aspects of critical phenomena and more involved scaling theories manifest themselves when further symmetries derived from the phenomenological analysis are taken into account.

In the remaining part of this section we closely follow Refs. [7,55,56].

In Sec. III B we summarize the general requirements that a "good" group transformation has to satisfy in order to describe the physics of critical phenomena. Then, in Section III C we describe the Kadanoff-Wilson transformation. In Section III D we apply the Kadanoff-Wilson transformation to study the critical phenomena near $d = 4$. Finally, in Section III E we review the field-theoretic RG approach.

B General Properties of a Group Transformation

Let $\mathcal{H}$ be a preassigned space of Hamiltonians H specified by a given set $\{\mu_j\}$ of parameters and R_s be a RG transformation depending on a parameter s and acting on $H \in \mathcal{H}$. Then the following properties must be satisfied [57]:

a) The transformed Hamiltonian H' belongs to the preassigned space

$$R_s(H) = H' \in \mathcal{H},$$

or, denoting with $\{\mu'_j\}$ the parameters of H', $\mu'_i = f_{s,i}(\{\mu_j\})$. In principle the preassigned space must be larger than the subspace of the initial Hamiltonian, since new coupling constants are generated by the RG procedure. However, most of the coupling constants (the irrelevant ones) are subsequently washed away by the RG flow, so that transient variables are only important in determining the approach (and the corrections) to scaling.

b) Composition law. Since we are going to interpret the parameter s as a change in the length scale (scaling factor), the subsequent application of two RG transformations with parameters s and s' must infact coincide with a single transformation with parameter ss', i.e.,

$$R_s(R_{s'}(H)) = R_{ss'}(H). \tag{23}$$

c) Invariance of the thermodynamic potentials

$$F[R_s(H)] = F[H], \quad \Gamma[R_s(H)] = \Gamma[H].$$

To ensure an invariant critical behavior it is sufficient that this equality holds for the most singular part of the thermodynamic potentials.

d) Smoothness (or analyticity) postulate. Let $H + \tau\delta H$ be a Hamiltonian "close" to H for $\tau \ll 1$, i.e., if $\{\mu_{\tau,j}\}$ are the parameters specifying $H + \tau\delta H$, then $\delta\mu_j \equiv \mu_{\tau,j} - \mu_j \sim O(\tau)$. We assume that

$$R_s(H + \tau\delta H) = R_s(H) + \tau L_s(H)\delta H, \tag{24}$$

where L_s is a linear operator which depends on H. This amounts to say that spurious (unphysical) singularities are not introduced by the transformation. In terms of the parameters of the Hamiltonian, Eq. (24) is equivalent to

$$\delta\mu_i' = \sum_j \frac{\partial f_{s,i}}{\partial \mu_j}\delta\mu_j.$$

e) The RG transformation must be constructed to extract the dominant large-distance behavior of the system near criticality. Associated with a given RG transformation there must be a set of fixed points $H^* \in \mathcal{H}$ defined by $R_s(H^*) = H^*$. Two trivial examples of fixed points are found when the system is completely ordered ($T = 0$) or completely disordered ($T = \infty$): in both cases a RG transformation should leave the system unchanged. These are the low-temperature and high-temperature fixed points respectively. Any system out of criticality is attracted by one of them, depending on whether $T < T_c$ or $T > T_c$ respectively. At criticality $\xi = \infty$, and it remains infinite under any finite number of RG transformations. The corresponding fixed point is a critical fixed point. When the transient behavior disappears, as we shall see, scaling is achieved. In the following we assume that there is at least one non-trivial fixed-point Hamiltonian H^* for an interacting system, separating the two non-critical regions. Then, if $\{\mu_j^*\}$ are the parameters specifying H^*, they are such that

$$\mu_i^* = f_{s,i}(\{\mu_j^*\}).$$

From the composition law, Eq. (23), and the smoothness postulate, Eq. (24), we have

$$R_s(R_{s'}(H + \tau\delta H)) = R_{ss'}(H + \tau\delta H) = R_{ss'}(H) + \tau L_{ss'}(H)\delta H,$$

or

$$\begin{aligned} R_s(R_{s'}(H + \tau\delta H)) &= R_s(R_{s'}(H) + \tau L_{s'}(H)\delta H) \\ &= R_{ss'}(H) + \tau L_s(R_{s'}(H))L_{s'}(H)\delta H, \end{aligned}$$

and hence, in general $L_{ss'}(H) = L_s(R_{s'}(H))L_{s'}(H)$, and only at a fixed point the composition law for the linearized transformation simplifies as

$$L_{ss'}(H^*) = L_s(H^*)L_{s'}(H^*). \tag{25}$$

Let us assume that $L_s(H^*)$ is diagonalizable and indicate with $\{\lambda_l^*\}$ its eigenvalues. Eq. (25) implies

$$\lambda_l^* = s^{x_l}, \quad l = 1, 2, \ldots,$$

and x_l are the scaling indices (or scaling dimensions), when s is identified with the change of the length scale. They are an intrinsic property of the fixed-point Hamiltonian H^*. The Hamiltonians which flow to the same fixed point under the RG transformations belong to the domain of attraction of H^*, and lead therefore to the same scaling indices. Universality amounts then to say that all the Hamiltonians within the same domain of attraction share the same critical behavior.

We denote by h_l^* the eigenvectors of $L_s(H^*)$ which correspond to the eigenvalues λ_l^*. We can use the eigenvectors to expand any perturbation of the fixed-point Hamiltonian

$$\tau\delta H = \tau \sum_l \bar{\mu}_l h_l^* \quad \Rightarrow \quad (\tau\delta H)' = L_s(H^*)\tau\delta H = \tau \sum_l s^{x_l} \bar{\mu}_l h_l^*. \tag{26}$$

From Eq. (26) we read

$$\bar{\mu}_l' = s^{x_l}\bar{\mu}_l.$$

The $\{\bar{\mu}_l\}$ are combination of the original $\{\delta\mu_j\}$ and are named linear scaling fields [58]. If $x_l > 0$ and the corresponding $\bar{\mu}_l \neq 0$ the RG transformations drive the system away from H^*, eventually moving it toward another fixed point. The eigenvectors with positive scaling dimensions define the directions of escape from the critical surface and are relevant fields. On the other hand, the eigenvectors with negative scaling dimensions define the tangent plane to the critical surface at H^*, and correspond to the irrelevant variables. They determine the transient behavior of the system, when approaching criticality.

C Kadanoff-Wilson Transformation

In Sec. III B we have listed the general properties of a RG transformation. Here, we explicitly build up two examples of a RG transformation. The first one is conceptually simpler and follows the block-variable idea put forward phenomenologically by Kadanoff [11]. It deals with transformations in real (configuration) space, which correspond to grouping variables belonging to a cell of size s [55,57,59,60]. The second is due to Wilson [3,5], and deals with transformations in momentum space. Although the main ingredient is again the elimination of degrees of freedom at short distance, it is of much simpler implementation.

1 *Transformation in Configuration Space*

Let us consider a statistical system defined by a given probability distribution

$$P[\{\hat{\varphi}_i\}] = \frac{1}{Z} \mathrm{e}^{-H[\{\hat{\varphi}_i\}]},$$

where $\hat{\varphi}_i$ is a continuous random variable defined on the site i of a d-dimensional lattice (as it is customary, the temperature has been absorbed in the parameters of the dimensionless Hamiltonian H), the order parameter being the thermal average of $\hat{\varphi}$ over the distribution P, $\varphi = \langle \hat{\varphi} \rangle_P$.

If Λ^{-1} is the original lattice spacing, we construct a new lattice with lattice spacing $s\Lambda^{-1}$ as follows. If j labels the position of the center of the cell with s^d points i, the block variables are defined as

$$\hat{\varphi}_j^{(s)} = \frac{1}{s^d} \sum_{i \in \mathcal{C}_j} \hat{\varphi}_i,$$

where $\mathcal{C}_j$ is the cell centered at j. Before we can compare the new probability distribution P_s, associated with the block variables, with P, we have to measure all the physical quantities in units of the new lattice spacing $s\Lambda^{-1}$. In order to obtain a non-trivial asymptotic probability distribution[8] we also need to rescale the field variable while measuring the distances in the new units: $j \to j/s \;\Rightarrow\; \hat{\varphi}_j^{(s)} \to \hat{\varphi}_j^{(s)} s^{x_\varphi} \equiv \hat{\varphi}_{j/s}$. The exponent x_φ coincides with the scaling index, as we show in the following. The probability distribution for the block variables is

$$P_s[\{\hat{\varphi}_{l/s}\}] = \int \left[\prod_i d\hat{\varphi}_i \right] P[\{\hat{\varphi}_i\}] \prod_j \delta \left(\hat{\varphi}_{j/s} - \frac{s^{x_\varphi}}{s^d} \sum_{i \in \mathcal{C}_j} \hat{\varphi}_i \right). \tag{27}$$

We define a new Hamiltonian H_s such that $P_s = \mathrm{e}^{-H_s - F_s^{(0)}}$, and the transformation associated to the probability distribution corresponds to a transformation R_s acting on $H \in \mathcal{H}$, $H_s = R_s(H)$, which is defined modulo a constant term that we can fix putting $H_s[\{\hat{\varphi}_{j/s} = 0\}] = 0$, and associating the part not containing the field $\hat{\varphi}$ to $F_s^{(0)}$.

2 *Scaling*

We have seen that for a given Hamiltonian $H \in \mathcal{H}$, the RG transformation R_s generates a new Hamiltonian $H_s \in \mathcal{H}$ with the same large-distance properties. Let us make this statement more precise in the case of the two-point correlation

[8] In Sec. III C 3 we shall see that also in the trivial case of non-correlated, identically distributed, random variables one has to properly rescale the field to avoid a totally flat or extremely peaked asymptotic probability distribution.

function, showing how the correlation function for H is related to the one for H_s. We have

$$\begin{aligned}\langle\hat{\varphi}_{j/s}\hat{\varphi}_{j'/s}\rangle_{P_s} &= \int\left[\prod_{l/s} d\hat{\varphi}_{l/s}\right] P_s[\{\hat{\varphi}\}]\hat{\varphi}_{j/s}\hat{\varphi}_{j'/s} \\ &= \int\left[\prod_{l/s} d\hat{\varphi}_{l/s}\right]\int\left[\prod_{i} d\hat{\varphi}_{i}\right] P[\{\hat{\varphi}\}]\delta\left(\hat{\varphi}_{l/s} - s^{-d+x_\varphi}\sum_{i\in\mathcal{C}_l}\hat{\varphi}_i\right)\hat{\varphi}_{j/s}\hat{\varphi}_{j'/s} \\ &= s^{-2d+2x_\varphi}\sum_{i\in\mathcal{C}_j}\sum_{i'\in\mathcal{C}_{j'}}\langle\hat{\varphi}_i\hat{\varphi}_{i'}\rangle_P \sim s^{2x_\varphi}\langle\hat{\varphi}_j\hat{\varphi}_{j'}\rangle_P.\end{aligned}$$

In the last equality we made explicit the fact that all the sites i within a given cell j can be labelled with j provided $|j-j'| \gg s\Lambda^{-1}$. Thus

$$f_\varphi[|\mathbf{r}|/s, R_s(H)] = s^{2x_\varphi} f_\varphi[|\mathbf{r}|, H], \quad \tilde{f}_\varphi[|\mathbf{q}|s, R_s(H)] = s^{2x_\varphi - d}\tilde{f}_\varphi[|\mathbf{q}|, H].$$

In particular, taking the limit $s \to \infty$, and assuming $R_s(H) \to H^*$, we get

$$f_\varphi[|\mathbf{r}|, H] = s^{-2x_\varphi} f_\varphi[|\mathbf{r}|/s, R_s(H)] \sim s^{-2x_\varphi} f_\varphi[|\mathbf{r}|/s, H^*], \quad \text{for } s \to \infty.$$

Since s must disappear in the right-hand side of the equation, $f_\varphi[|\mathbf{r}|/s, H^*] \sim f_\varphi[1, H^*](s/|\mathbf{r}|)^{2x_\varphi}$. Provided $f_\varphi[1, H^*]$ is finite, we obtain the asymptotic scaling behavior

$$f_\varphi(|\mathbf{r}|) \sim \frac{1}{|\mathbf{r}|^{2x_\varphi}} \equiv \frac{1}{|\mathbf{r}|^{(d-2+\eta)/2}} \quad \Rightarrow \quad x_\varphi = \frac{d-2+\eta}{2}.$$

The Hamiltonian H^* corresponds to the asymptotic distribution $P_\infty \sim \mathrm{e}^{-H^*}$. In this case the Hamiltonian H describes a critical system. The susceptibility χ is proportional to $\tilde{f}_\varphi(\mathbf{q}=0)$. Since $\tilde{f}_\varphi(\mathbf{q}) \sim |\mathbf{q}|^{2x_\varphi - d}$, with $d - 2x_\varphi = \gamma/\nu$, χ is infinite, and the system is critical, only if $x_\varphi < d/2$ ($\gamma > 0$), i.e., $\eta < 2$. If $x_\varphi = d/2$ ($\eta = 2$) the susceptibility is finite and the system is not critical. In this case, when $s \to \infty$ and the block variable becomes extensive, its variance is proportional to s^d and diverges, unless the variable itself is normalized as

$$\hat{\varphi}_j^{(s)} = \frac{1}{s^{d/2}}\sum_{i\in\mathcal{C}_j}\hat{\varphi}_i.$$

3 Probabilistic Considerations

In general, let $\{\hat{\varphi}_i\}$ be a set of random independent variables, identically distributed with zero average and finite variance [7,59]

$$\langle\hat{\varphi}_i\rangle = 0, \quad \langle\hat{\varphi}_i\hat{\varphi}_{i'}\rangle = \delta_{ii'}\sigma^2.$$

Then, construct a block of N variables $\bar{\varphi}_N = \sum_{i=1}^N \hat{\varphi}_i$ (with $N \sim s^d$ in the previous example), whose variance is

$$\langle \bar{\varphi}_N^2 \rangle = N\sigma^2, \tag{28}$$

and the fluctuations satisfy the standard square-root law $\sqrt{\langle \bar{\varphi}_N^2 \rangle}/N \sim 1/\sqrt{N}$. Since the variance of $\bar{\varphi}_N$ increases linearly with N, the variance of the normalized variable $\bar{\varphi}/\sqrt{N}$ is finite, i.e., it has a well defined asymptotic probability distribution and if we take $N = s^d$ we read the non-critical index $x_\varphi = d/2$, as anticipated above. This result should not be a surprise. Indeed, the distribution for the sum $\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \varphi_i$ of independent variables tends to a Gaussian distribution when $N \to \infty$ according to the central limit theorem.

If the variables are not independent, Eq. (28) becomes

$$\langle \bar{\varphi}_N^2 \rangle = \sum_{ii'} \langle \varphi_i \varphi_{i'} \rangle \equiv \sum_{ii'} f_\varphi(i, i') \sim N \sum_{l=0}^{N} f_\varphi(l) \sim N\chi \quad \text{for } N \to \infty,$$

where in the last step we made use of translational invariance and of the expression of the susceptibility as the Fourier transform of the correlation function at $|\mathbf{q}| = 0$. If the system is not critical, the finite susceptibility χ replaces the variance σ^2 and the square-root law for the fluctuations is still valid. This means that, although there are correlations in the system, these are short-ranged, so that one can always find independent subsystems (block variables), and the central limit theorem follows. In this case, the only scaling dimension giving a well-defined asymptotic probability distribution is $x_\varphi = d/2$ as in the case of independent variables. On the other hand, if the system is critical the correlations are long-ranged and

$$\langle \bar{\varphi}_N^2 \rangle \sim s^d \int_0^s r^{d-1-2x_\varphi} dr = s^{2d-2x_\varphi},$$

with $2d - 2x_\varphi > d$, i.e., $x_\varphi < d/2$. The square-root law is no longer valid and the variables are strongly correlated. The correct normalization of the block variables still guarantees the existence of a well-defined asymptotic probability distribution, but we do not expect it to have in general a simple Gaussian form. The choice of the proper normalization of the block variables is related to the choice of the correct asymptotic behavior of the correlation function of the order parameter, which is the procedure used by Wilson to select the proper fixed point [50].

4 Transformation in Momentum Space

Although intuitively simple, the idea of constructing block variables encounters several technical difficulties when it is applied in practice. First of all, the integral in Eq. (27) is involved, and cannot be worked out analytically in all the relevant cases. Secondly, *ad hoc* decimation procedures which have been proposed to circumvent the problem of directly calculating the integral in Eq. (27), sometimes are very successful, but sometimes lead to contradictory results, and in the worst cases, to a change of class (or to spurious classes) of universality under RG transformation.

The above problems are partially avoided by a technically simpler realization of the physical idea that short-distance degrees of freedom should be eliminated, which relies on the momentum-space formulation proposed by Wilson. We first introduce the Fourier transform for the random variables $\hat{\varphi}_i$:

$$\hat{\varphi}_{\mathbf{k}} = \frac{1}{\sqrt{N}} \sum_i \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}_i} \hat{\varphi}_i, \qquad \hat{\varphi}_i = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}_i} \hat{\varphi}_{\mathbf{k}}, \tag{29}$$

where $\mathbf{r}_i$ is the vector position of the site i and N is the number of lattice sites ($N \to \infty$ in the thermodynamic limit). Then we realize that the short-distance details of a given configuration $\{\hat{\varphi}_i\}$ are related to the large-momentum components of the Fourier transform $\hat{\varphi}_{\mathbf{k}}$. The elimination of the irrelevant degrees of freedom in real space, for $0 < |\mathbf{r}| < s\Lambda^{-1}$, can be reformulated as the elimination of the large-momentum components of $\hat{\varphi}_{\mathbf{k}}$, for $\Lambda/s < |\mathbf{q}| < \Lambda$. The RG-transformed probability distribution is

$$P_s[\{\hat{\varphi}_{\mathbf{q}}\}] = \left(\int \left[\prod_{\Lambda/s<|\mathbf{k}|<\Lambda} d\hat{\varphi}_{\mathbf{k}} \right] P[\{\hat{\varphi}_{\mathbf{k}}\}] \right)_{\mathbf{q}\to\mathbf{q}/s,\ \hat{\varphi}_{\mathbf{q}}\to s^{d/2-x_\varphi}\hat{\varphi}_{s\mathbf{q}}} \tag{30}$$

where the inverse lattice spacing Λ acts as an ultraviolet cutoff for momentum integrals, and we have taken into account Eq. (29) to find the proper rescaling for the Fourier mode $\hat{\varphi}_{\mathbf{k}}$. As before, x_φ is a parameter of the Wilson transformation, introduced to obtain a non-trivial fixed-point distribution. Like Eq. (27), Eq. (30) induces a transformation R_s on the space $\mathcal{H}$. Even if the two transformations are technically different, both of them lead to a transformation for the correlation function which, asymptotically near the fixed point, reduces to a scaling transformation.

D Critical Phenomena near $d = 4$: ϵ Expansion

In this section we sketch a systematic procedure to take intro account the interaction among the fluctuations, and apply the concepts discussed so far to study the critical phenomena near $d = 4$, following Ref. [7]. We use the Wilson approach in momentum space, with $P[\{\hat{\varphi}\}] \sim \mathrm{e}^{-H[\hat{\varphi}]}$, applied to the simplest form of a coarse-grained Landau-Wilson dimensionless Hamiltonian, Eq. (9), which we rewrite as

$$H[\hat{\varphi}] = \int d^d\mathbf{r} [t_0 \hat{\varphi}^2 + u\hat{\varphi}^4 + c(\nabla\hat{\varphi})^2 - h\hat{\varphi}], \tag{31}$$

where, as it is customary, t_0 and u are now used instead of a and b.

As already noted, the thermodynamic potential of a system described by the Hamiltonian in Eq. (31) coincides with the Landau form, Eq. (2), if the partition function is evaluated in the saddle-point approximation. In momentum space, $H[\hat{\varphi}]$ can be written as

$$H[\hat{\varphi}] = \sum_{|\mathbf{k}|<\Lambda} (t_0 + c|\mathbf{k}|^2)\hat{\varphi}_{\mathbf{k}}\hat{\varphi}_{-\mathbf{k}} + \frac{u}{L^d} \sum_{|\mathbf{k}_i|<\Lambda} \hat{\varphi}_{\mathbf{k}_1}\hat{\varphi}_{\mathbf{k}_2}\hat{\varphi}_{\mathbf{k}_3}\hat{\varphi}_{-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3} - \frac{1}{\sqrt{L^d}} h\hat{\varphi}_0, \quad (32)$$

where L is the linear size of the system and $\hat{\varphi}_0$ is the zero mode of the field variable $\hat{\varphi}$.

The idea which supports the simple form (31) is that near criticality the microscopic details are not relevant, which means that only the most slowly varying contribution of the spatial fluctuations $\sim (\nabla\hat{\varphi})^2$, should be considered. As far as the absence of powers of the field $\hat{\varphi}$ larger than the fourth is concerned, the idea is that, even if these interactions are generated in the intermediate steps of the coarse-graining procedure, they are washed out by the RG flow when approaching criticality, since the corresponding coupling constants have negative bare dimensions near $d = 4$, and are thus irrelevant in the RG sense. This is also true for terms containing powers of the spatial derivatives higher than the second. In principle, one can add to the above Hamiltonian (31) terms of the form $u_{n,m} \int d^d\mathbf{r} (\nabla^m\hat{\varphi})^n$ with integers n and m. For the specific case under consideration

$$u_{1,0} = -h, \quad u_{2,0} = t_0, \quad u_{3,0} = 0, \quad u_{4,0} = u, \quad u_{2,1} = c. \quad (33)$$

According to our convention, c is dimensionless; hence, the bare dimension of the parameter $u_{n,m}$ in unit of the inverse length is

$$x^0_{n,m} = d - nm - \frac{n}{2}(d-2). \quad (34)$$

It is a simple matter to verify that for the parameters in Eq. (33) we get the values in Eq. (10). The interaction parameter $u_{n,0}$ has positive bare dimension when $d < 2n/(n-2)$. Thus all $u_{n,0}$ with $n > 4$ are irrelevant for $3 < d \leq 4$. Moreover, at fixed d, $x^0_{n,m} < x^0_{n,0}$, i.e., terms with gradients are more irrelevant than terms without gradients for a given n. This analysis can be corrected by the presence of the anomalous index η, which, however, is experimentally small ($\sim 10^{-2}$).

The $\hat{\varphi}^4$ model, Eq. (32), is then a good starting point as long as we work with $3 < d \leq 4$.

Following the Wilson procedure, the renormalized Hamiltonian is given by

$$\mathrm{e}^{-H_s - F_s^{(0)}} = \left(\int \left[\prod_{\Lambda/s<|\mathbf{k}|<\Lambda} d\hat{\varphi}_{\mathbf{k}} \right] \mathrm{e}^{-H[\hat{\varphi}]} \right)_{\mathbf{q}\to\mathbf{q}/s, \hat{\varphi}_{\mathbf{q}}\to s^{1-\eta/2}\hat{\varphi}_{s\mathbf{q}}} \quad (35)$$

where Eq. (30) and the relation $d/2 - x_\varphi = 1 - \eta/2$ have been used.

1 Gaussian Fixed Points

In the simplest case $u = 0$ the elimination of the large-$\mathbf{k}$ degrees of freedom can be performed exactly, and the integral in Eq. (35) gives

$$F_s^{(0)} = -\frac{1}{2} \sum_{\Lambda/s<|\mathbf{k}|<\Lambda} \ln \frac{\pi}{t_0 + c|\mathbf{k}|^2}, \tag{36}$$

$$H_s = \sum_{|\mathbf{q}|<\Lambda} (s^{2-\eta} t_0 + s^{-\eta} c|\mathbf{q}|^2) \hat{\varphi}_{\mathbf{q}} \hat{\varphi}_{-\mathbf{q}},$$

where $h = 0$ has been considered for simplicity. If the summation over $\mathbf{k}$ would extend from zero to Λ, Eq. (36) would give the free energy of the Gaussian model, with the complete description of the system. Nevertheless, we consider here the RG approach to show how the formal apparatus works in this simple, exactly soluble, case. The renormalized parameters are

$$t_s = s^{2-\eta} t_0, \qquad c_s = s^{-\eta} c, \qquad u_s = u \equiv 0,$$

so that three fixed points are possible:

A) $\eta = 0$ and $t_0 = 0$, for any $c > 0$. The constraints $h = t_0 = 0$ define the critical surface in $d > 4$ (see below). Since $x_\varphi = (d-2)/2 < d/2$ the block variables are strongly correlated (via the gradient term). This fixed point is a critical (Gaussian) fixed point with $\eta = 0$, $\nu = 1/2$, and all the critical indices are those derived in the context of the Landau theory, with no anomalous dimensions.

B-C) $\eta = 2$ and $c^* = 0$ for any t_0. In this case, since $x_\varphi = d/2$, the central limit theorem holds and the fluctuations of the block variables obey the square-root law. In particular, for $t_0 > 0$ ($T > T_c$) each block variable has an independent Gaussian distribution and we get the high-temperature fixed point. For $t_0 < 0$ ($T < T_c$) we need a small $u > 0$ to stabilize the system (otherwise large values of the block variables are energetically favored). The asymptotic probability distribution is still normal and the associated fixed point is the low-temperature one.

2 Wilson-Fisher Fixed Point

The Gaussian fixed point A is stable for $d > 4$ and describes the correct critical behavior, whereas it is unstable for $u \neq 0$ in $d < 4$. This can be easily seen by computing how the u parameter renormalizes to lowest order in perturbation theory, and linearizing the transformation around the fixed point. Let

$$H^* = c \int d^d\mathbf{r} (\nabla\hat{\varphi})^2, \qquad \delta H = \int d^d\mathbf{r} [t_0 \hat{\varphi}^2 + u\hat{\varphi}^4]$$

be the fixed-point Hamiltonian and the perturbation respectively. It is convenient to write the field variable $\hat{\varphi}$ as

$$\hat{\varphi}(\mathbf{r}) = \hat{\varphi}_>(\mathbf{r}) + \hat{\varphi}_<(\mathbf{r})$$

where

$$\hat{\varphi}_>(\mathbf{r}) = \frac{1}{\sqrt{L^d}} \sum_{\Lambda/s<|\mathbf{k}|<\Lambda} \hat{\varphi}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}, \qquad \hat{\varphi}_<(\mathbf{r}) = \frac{1}{\sqrt{L^d}} \sum_{|\mathbf{k}|<\Lambda/s} \hat{\varphi}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}},$$

are the fast and slow modes, respectively. Then,

$$\int \left[\prod_{\Lambda/s<|\mathbf{k}|<\Lambda} d\hat{\varphi}_{\mathbf{k}}\right] \mathrm{e}^{-H[\hat{\varphi}]} =$$
$$\mathrm{e}^{-H^*[\hat{\varphi}_<]} \int \left[\prod_{\Lambda/s<|\mathbf{k}|<\Lambda} d\hat{\varphi}_{\mathbf{k}}\right] \mathrm{e}^{-H^*[\hat{\varphi}_>]} \left[1 - \delta H + \frac{1}{2}(\delta H)^2 + ...\right].$$

When re-exponentiated, the expansion in powers of δH gives rise to a cumulant expansion for the renormalized Hamiltonian

$$H_s[\hat{\varphi}] =$$
$$\left(H^*[\hat{\varphi}_<] + \langle \delta H\rangle_{\text{fast}} - \frac{1}{2}[\langle(\delta H)^2\rangle_{\text{fast}} - \langle \delta H\rangle^2_{\text{fast}}] + \dots\right)_{\mathbf{r}\to s\mathbf{r}, \hat{\varphi}_<(\mathbf{r})\to s^{-x_\varphi}\hat{\varphi}(\mathbf{r}/s)} \tag{37}$$

where

$$\langle A\rangle_{\text{fast}} = \frac{\int \left[\prod_{\Lambda/s<|\mathbf{k}|<\Lambda} d\hat{\varphi}_{\mathbf{k}}\right] \mathrm{e}^{-H^*[\hat{\varphi}_>]} A}{\int \left[\prod_{\Lambda/s<|\mathbf{k}|<\Lambda} d\hat{\varphi}_{\mathbf{k}}\right] \mathrm{e}^{-H^*[\hat{\varphi}_>]}}$$

defines the average of a functional $A[\hat{\varphi}]$ with respect to the fast modes $\hat{\varphi}_>$.

To first order in δH, we read from Eq. (37) the linearized RG transformation. The proper normalization for the block variables is still $\eta = 0$, since at this order there are no corrections to c, but now t_s receives a contribution coming from the interaction between the fluctuations and u_s has a s^d factor coming from the integration over the configuration variable and $s^{-4x^0_\varphi} = s^{4-2d}$ coming from the rescaling of the fields. The RG equations become

$$t_s = s^2\left[t_0 + uA(n,d)\left(1 - \frac{1}{s^{d-2}}\right)\right] \equiv s^2\tilde{t}, \qquad u_s = s^\epsilon u, \tag{38}$$

with $\epsilon = 4-d$. $A(n,d)$ is a constant, which comes out from the explicit evaluation of the cumulant $\langle \delta H\rangle_{\text{fast}}$ and depends on c, on the number n of components of the field variable $\hat{\varphi}$ ($n = 1$ in the Ising case), and on the spatial dimensionality d. Eq. (38) shows that, for $d < 4$, u grows under the action of the RG transformation and the Gaussian fixed point is unstable with respect to a new direction introduced by the interaction (besides the t direction). On the other hand, if $d > 4$, u is irrelevant and the Gaussian fixed point describes the critical behavior of the originally interacting system. This result can be interpreted as the RG version of the Ginzburg criterion.

Considering ϵ as a small positive parameter, we can ask if the slow running away of u_s can be compensated by the non-linear terms, giving rise to a new fixed point with $t^*, u^* \sim \epsilon$. Indeed, it can be shown that the contribution to the renormalization of the coupling u coming from the quadratic terms in δH in Eq. (37), starting at $d = 4$ and considering terms up to $O(\epsilon^2)$, yields

$$u_s = s^\epsilon u[1 - g(n, d=4)u\ln s],$$

where $g(n,d)$ is a positive constant for $d = 4$ ($\epsilon = 0$), which comes out from the evaluation of the cumulants $\langle \delta H^2 \rangle_{\text{fast}} - \langle \delta H \rangle_{\text{fast}}$. It is worth to note that the only true perturbative parameter here is ϵ: as $\epsilon \to 0$ the Wilson-Fisher fixed point continuously moves toward the Gaussian fixed point, ending on it when $\epsilon = 0$. Therefore, to get a meaningful expansion from the cumulant expansion in Eq. (37), we have to group those terms of the same order in ϵ. This is what is called the ϵ expansion (and it can be shown that it is equivalent to a loop expansion).

The RG flow equation in differential form, up to $O(\epsilon^2)$, reads

$$\frac{du_s}{d\ln s} = \epsilon u_s - g(n, d = 4) u_s^2. \tag{39}$$

From Eq. (39) we see that, besides the unstable Gaussian fixed point $u^* = 0$, we have a new fixed point $u^* = \epsilon/g > 0$. Then, from the first equation in (38), we find $t^* = -A(n, d = 4)u^*$, i.e., $t^*, u^* \sim \epsilon$.

The linearization of Eq. (39) with respect to this new fixed point yields

$$\frac{d\Delta u_s}{d\ln s} = \epsilon \Delta u_s - 2u^* g \Delta u_s = -\epsilon \Delta u_s, \tag{40}$$

with $u_s = u^* + \Delta u_s$. Hence, $\Delta u_s = s^{-\epsilon} \Delta u$, and the non-trivial Wilson-Fisher fixed point is stable for $d < 4$. Universality is recovered since, along the critical line $\tilde{t} = 0$, the initial coupling constant is driven to the fixed point u^* by the RG flow, and the deviation Δu_s from it is irrelevant[9]. This is the subtle way in which the original coupling constant disappears for $d < 4$.

As far as $t_s = t^* + \Delta t_s$ is concerned, the linearization of the RG equation [evaluated up to $O(\epsilon^2)$, which gives the critical indices to $O(\epsilon)$], around the Wilson-Fisher fixed point yields

$$\frac{d\Delta t_s}{d\ln s} = \left[2 - \epsilon B(n, d = 4)\right] \Delta t_s + C(c, n, \epsilon) \Delta u_s, \tag{41}$$

with $B(n, d = 4) = (n + 2)/(n + 8)$, as obtained through the explicit evaluation of the integrals coming from the cumulant expansion. The coefficient C does not need to be evaluated to obtain the critical indices at this order. Indeed, Eqs. (40)-(41) constitute a system of linear differential equations which can be solved by diagonalizing the associated 2×2 matrix of the coefficients in the right-hand sides. However, since the 2×2 matrix has a zero off-diagonal element, its eigenvalues can be directly read off the diagonal elements, and give the scale dimensions at the Wilson-Fisher fixed point

$$x_1 = 2 - \frac{n+2}{n+8}\epsilon, \qquad x_2 = -\epsilon.$$

[9] u is relevant with respect to the Gaussian fixed point. Its presence leads to the non-Gaussian fixed point, for which Δu is irrelevant. A further condition to enforce criticality is necessary if we want the Gaussian fixed point to maintain its effectiveness at $d < 4$. The additional condition enforces a critical behavior related to the so-called tricritical point [58].

The eigenvector which corresponds to x_1, is a linear combination of t and u which defines the critical line for $d < 4$. Whereas for $d > 4$ the Wilson-Fisher is unphysical ($u^* < 0$), for $d < 4$ we have only one relevant perturbation at $h = 0$, since $x_2 < 0$. It is therefore the non-trivial, critical fixed point at small ϵ. x_2 controls the approach to the fixed point of the first irrelevant variable Δu. Now the critical indices depend both on n and d (through ϵ). The critical index ν coincides with the reciprocal of the scale dimension of the relevant perturbation

$$\nu \equiv \frac{1}{x_1} \simeq \frac{1}{2} + \frac{\epsilon}{4}\frac{n+2}{n+8},$$

to $O(\epsilon)$. For instance, when $n = 1$ (Ising model) $\nu = 1/2 + \epsilon/12$, which is different from the Landau value and in better agreement with numerical and experimental results. Since at order ϵ the index η is still zero (it is of order ϵ^2), two indices are known, and the other indices can be calculated to order ϵ by means of the scaling laws.

To conclude this section, we point out that the solution of the RG equations corresponds to the exponentiation of the logarithmic singularities which affect perturbation theory (i.e., the cumulant expansion) near $d = 4$. Indeed, the explicit evaluation of a cumulant gives a contribution $1 - \frac{1}{2}B\epsilon \ln s$ for Δt_s, and we can approximate this expression for small ϵ as $\exp(-\frac{1}{2}B\epsilon \ln s) = s^{-B\epsilon/2}$, which gives the correction to $1/\nu$ obtained by solving Eq. (41).

E Field-Theoretic Approach

A different technical support to the evaluation of the critical indices and of the approach to the scaling behavior of a critical system is provided by the *field-theoretic* RG approach [6,40].

The idea is to start with a field theory described by the coarse-grained Hamiltonian (31). We are taking advantage of the previous discussion of the dimension $x^0_{n,m}$ of the coefficient of the generalized Landau Hamiltonian, and we limit ourselves to consider the Hamiltonian (31) with $c = 1$ and a momentum cutoff Λ for $3 < d \leq 4$. For $d \leq 3$ further terms should be considered in the Hamiltonian, down to $d = 2$ where all the $u_{n,0}$ terms become relevant. With this precautions taken, the model we are considering is the $\hat{\varphi}^4$ field-theoretical model with bare inverse propagator

$$\Gamma^{(2)}_0 = |\mathbf{k}|^2 + t_0,$$

and a momentum cutoff Λ. The perturbative structure of the theory in the interaction u takes advantage of the standard procedure based on the Feynman diagrams. These permit to represent in a graphic fashion the perturbative contributions to various physical quantities, once the general rules for calculating the analytical expressions corresponding to the different diagrams have been established. Continuous lines represent bare propagators $[\Gamma^{(2)}_0]^{-1}$, whereas empty squares represent

the bare interaction vertex u, and momentum conservation is understood in each vertex. The closed continuous lines must be interpreted as integrals over the corresponding momenta, which will therefore explicitly depend on the momentum cutoff Λ. The great advantage of using Feynman diagrams is that they allow for immediate resummations of the perturbative series which establish connections between the various dressed quantities, for instance, the relation which connects the dressed inverse propagator $\Gamma^{(2)}$ to the bare inverse propagator $\Gamma_0^{(2)}$ (Dyson equation, see Fig. 1), the bare vertex $\Gamma_0^{(4)}$, and the dressed vertex $\Gamma^{(4)}$ (full square).

There is no simple representation of the dressed vertex $\Gamma^{(4)}$ in terms of the bare vertex $\Gamma_0^{(4)}$, and higher-order vertex functions should be introduced in a hierarchy of linked equations (see Fig. 2). However, we shall not pursue this procedure, which is not relevant to the purpose of identifying the building blocks of the field-theoretical RG near criticality, as analyzed in Refs. [47,48].

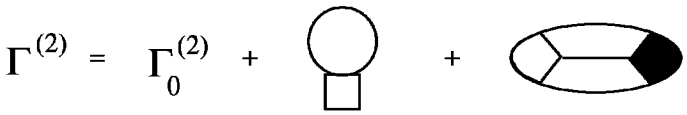

FIGURE 1. Dyson equation for the inverse propagator $\Gamma^{(2)}$.

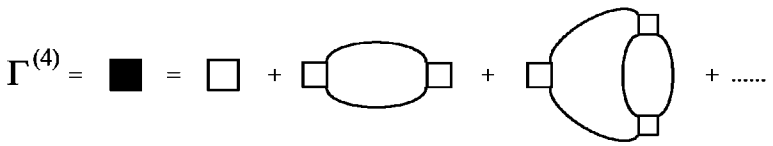

FIGURE 2. Diagrams contributing to the dressed vertex $\Gamma^{(4)}$.

The first evident effect of introducing a non-zero coupling constant u is a shift of the transition temperature with respect to $t_0 = 0$. The deviation from the critical temperature will be denoted by $t = t_0 - t_{0,c}(u, \Lambda)$. Hereafter we choose to express $\Gamma^{(2)}$ in terms of t, which vanishes at the true critical temperature. Perturbation theory in terms of the bare coupling constant u is ill-defined in $d < 4$ due to the infrared divergences which arise when the critical point is approached. We need a renormalized theory, where the renormalized coupling constant remains finite in the limit $t \to 0$, i.e., when the square of the inverse coherence length ξ^{-2}, which plays the role of the mass of the field $\hat{\varphi}$, vanishes. Therefore we are interested in a renormalized theory in which the cutoff Λ is much larger than the mass of the propagator. As in a traditional renormalization procedure, if we can control the ultraviolet divergences generated by the momentum integrations, we can take the limit $\Lambda \to \infty$ and make the cutoff disappear from the RG equations. Once the theory is well defined in the limit $\Lambda \to \infty$, it is easy to investigate the infrared scaling behavior since the parameters of the RG are independent of Λ. The resulting scaling theory will also be valid at finite Λ as long as $\Lambda \gg \xi^{-1}$. It can be shown that whenever the interaction vertices in the diagrammatic language are such that

$x^0_{n,m} \geq 0$, the theory is ultraviolet renormalizable. For our derivation of the infrared renormalized theory we shall be guided by physical considerations on the invariance properties of the diagrammatic equations shown in Figs. (1) and (2), although the forthcoming analysis can be confirmed by a direct calculation of the corresponding Feynman diagrams.

In constructing the RG transformation, the relevant variables which enter in the description of the system are t and h (or its conjugate variable $\hat{\varphi}$). As we have seen previously, another quantity which is needed to describe the approach to the fixed point is the first irrelevant coupling, related to u. We are therefore led to find a transformation of these variables via the introduction of the standard wave function $\hat{\varphi}$, mass t, and vertex u renormalizations, multiplicatively implemented by the parameters Z_φ, Z_t, Z_u, and the corresponding correlation functions. Let us examine the separate invariance property under each renormalization:
Under a *wave-function renormalization*

$$\hat{\varphi} \to Z_\varphi^{-1/2}\hat{\varphi}, \quad u \to Z_\varphi^2 u, \quad \gamma_\varphi \to Z_\varphi \gamma_\varphi;$$

Under a *mass renormalization*

$$t \to Z_t^{-1} t, \quad \gamma_t \to Z_t \gamma_t;$$

Under a *vertex renormalization*

$$u \to Z_u^{-1} u, \quad \gamma_u \to Z_u \gamma_u;$$

with

$$\gamma_\varphi = \frac{\partial \Gamma^{(2)}}{\partial |\mathbf{k}|^2}, \quad \gamma_t = \frac{\partial \Gamma^{(2)}}{\partial t}, \quad \gamma_u = \frac{\Gamma^{(4)}}{u},$$

where the diagrammatic expansion for γ_t is shown in Fig. 3.

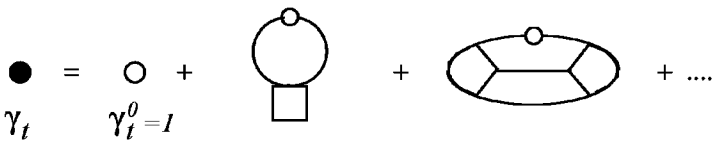

FIGURE 3. Diagrammatic expansion for γ_t (full dot). Each empty dot represents the mass insertion due to the derivative of $\Gamma^{(2)}$ (see Fig. 1) with respect to t.

When we make use of the three transformations simultaneously, the new variables read

$$\hat{\varphi}' = Z_\varphi^{-1/2}\hat{\varphi}, \quad t' = Z_\varphi Z_t^{-1} t, \quad u' = Z_\varphi^2 Z_u^{-1} u.$$

If we define the invariant functional in terms of the new variables $\hat{\varphi}', t', u'$ as

$$\bar{\Gamma}(\hat{\varphi}', t', u') = \Gamma(Z_\varphi^{1/2}\hat{\varphi}', Z_\varphi^{-1} Z_t t', Z_\varphi^{-2} Z_u u'),$$

with $c = 1$, it follows that

$$\bar{\Gamma}^{(n)}(\hat{\varphi}', t', u') = Z_\varphi^{n/2}\Gamma^{(n)}(Z_\varphi^{1/2}\hat{\varphi}', Z_\varphi^{-1}Z_t t', Z_\varphi^{-2}Z_u u').$$

$\bar{\Gamma}$ and $\bar{\Gamma}^{(n)}$ are given by the same expansion as Γ and $\Gamma^{(n)}$ provided the correspondence between old and new variables is made. Similarly we define

$$\bar{\gamma}_i(\hat{\varphi}', t', u') = Z_i\gamma_i(Z_\varphi^{1/2}\hat{\varphi}', Z_\varphi^{-1}Z_t t', Z_\varphi^{-2}Z_u u'),$$

where $i = \varphi, t, u$.

Above we have assumed that each variable is transformed independently from the others. However we can still use the arbitrariness of the multiplicative factors in the correlation functions to obtain a one-parameter transformation. This can be achieved by imposing suitable normalization conditions on the $\bar{\gamma}$'s at a given normalization point (hereafter n.p.) specified by an auxiliary variable λ with the dimensions of (length)$^{-1}$. The normalization condition permits to express the multiplicative factors Z_i in terms of λ and only one of the variable $\hat{\varphi}, t, u$ or $\hat{\varphi}', t', u'$. We can therefore change $\hat{\varphi}', t', u'$ by varying λ at fixed $\hat{\varphi}, t, u$, or vice versa. Since we expect $\hat{\varphi}'$ and t' to be the relevant variables that drive the system towards the critical point, we can parametrize the transformation in terms of u' only. This procedure implements the physical idea that, when the first irrelevant coupling u is changed, a proper multiplicative rescaling of the relevant variables yields a physical system which shares the same critical properties with the initial system. Then we make the following choice:

$$\bar{\gamma}_\varphi|_{\text{n.p.}} = 1, \quad \bar{\gamma}_t|_{\text{n.p.}} = 1, \quad \bar{\gamma}_u|_{\text{n.p.}} = 1, \tag{42}$$

where the normalization point is chosen as n.p.$= (|\mathbf{k}|^2 = 0, \hat{\varphi}' = 0, t' = \lambda^2; \forall u')$. In this way the transformation is multiplicative and linear in the variables $\hat{\varphi}'$ and t'. Different choices of the normalization point would give different parametrizations of the transformation, without changing the final expression for the critical exponents. Other parametrizations can be found in Ref. [53]. The asymptotic equivalence among the different versions of the field-theoretic approach and the Wilson approach is discussed in Refs. [61,62].

We note that the Z's determined via the Eqs. (42) will depend on the cutoff Λ, but since from now on we deal with renormalized functions only, we omit the dependence on Λ. It is therefore useful to introduce the following dimensionless variables:

$$t_\lambda = \frac{t'}{\lambda^{x_t^0}}, \quad \varphi_\lambda = \frac{\hat{\varphi}'}{\lambda^{x_\varphi^0}}, \quad u_\lambda = \frac{u'}{\lambda^{x_u^0}},$$

where the bare dimensions are $x_t^0 = 2$, $x_\varphi^0 = 1 - \epsilon/2$, and $x_u^0 = \epsilon$. We also define the parameters of the transformation leading from a n.p. λ to a n.p. λ' as

$$Z_{\varphi,\lambda'/\lambda} = \frac{Z_{\varphi,\lambda'}}{Z_{\varphi,\lambda}}, \quad Z_{t,\lambda'/\lambda} = \frac{Z_{t,\lambda'}}{Z_{t,\lambda}}, \quad Z_{u,\lambda'/\lambda} = \frac{Z_{u,\lambda'}}{Z_{u,\lambda}}.$$

The variables change now according to the following multiplicative rules:

$$
\begin{aligned}
t_{\lambda'} &= \frac{t_\lambda Z_{\varphi,\lambda'/\lambda} Z^{-1}_{t,\lambda'/\lambda}}{(\lambda'/\lambda)^2}, \\
\varphi_{\lambda'} &= \frac{\varphi_\lambda Z^{-1/2}_{\varphi,\lambda'/\lambda}}{(\lambda'/\lambda)^{1-\epsilon/2}}, \\
u_{\lambda'} &= \frac{u_\lambda Z^2_{\varphi,\lambda'/\lambda} Z^{-1}_{u,\lambda'/\lambda}}{(\lambda'/\lambda)^\epsilon}.
\end{aligned} \tag{43}
$$

In this way the correlation functions are multiplicatively related by

$$
\begin{aligned}
\bar{\gamma}_\varphi\left(\frac{|\mathbf{k}|^2}{\lambda^2}, t_\lambda, \varphi_\lambda, u_\lambda\right) &= Z^{-1}_{\varphi,\lambda'/\lambda}\bar{\gamma}_\varphi\left(\frac{|\mathbf{k}|^2}{\lambda'^2}, t_{\lambda'}, \varphi_{\lambda'}, u_{\lambda'}\right) \\
\bar{\gamma}_t\left(\frac{|\mathbf{k}|^2}{\lambda^2}, t_\lambda, \varphi_\lambda, u_\lambda\right) &= Z^{-1}_{t,\lambda'/\lambda}\bar{\gamma}_t\left(\frac{|\mathbf{k}|^2}{\lambda'^2}, t_{\lambda'}, \varphi_{\lambda'}, u_{\lambda'}\right) \\
\bar{\gamma}_u\left(\frac{|\mathbf{k}_i|}{\lambda}, t_\lambda, \varphi_\lambda, u_\lambda\right) &= Z^{-1}_{u,\lambda'/\lambda}\bar{\gamma}_u\left(\frac{|\mathbf{k}_i|}{\lambda'}, t_{\lambda'}, \varphi_{\lambda'}, u_{\lambda'}\right)
\end{aligned} \tag{44}
$$

and the thermodynamic functional is invariant

$$
\bar{\Gamma}(t_\lambda, \varphi_\lambda, u_\lambda) = \bar{\Gamma}(t_{\lambda'}, \varphi_{\lambda'}, u_{\lambda'}). \tag{45}
$$

The normalization (42) for $t_{\lambda'} = 1, |\mathbf{k}|^2/\lambda'^2 = 0, \varphi_{\lambda'} = 0$, implicitly determines now the Z's in terms of the correlation functions

$$
\begin{aligned}
Z^{-1}_{\varphi,\lambda'/\lambda} &= \bar{\gamma}_\varphi\left(0, \frac{\lambda'^2}{\lambda^2} Z^{-1}_{\varphi,\lambda'/\lambda} Z_{t,\lambda'/\lambda}, 0, u_\lambda\right), \\
Z^{-1}_{t,\lambda'/\lambda} &= \bar{\gamma}_t\left(0, \frac{\lambda'^2}{\lambda^2} Z^{-1}_{\varphi,\lambda'/\lambda} Z_{t,\lambda'/\lambda}, 0, u_\lambda\right), \\
Z^{-1}_{u,\lambda'/\lambda} &= \bar{\gamma}_u\left(0, \frac{\lambda'^2}{\lambda^2} Z^{-1}_{\varphi,\lambda'/\lambda} Z_{t,\lambda'/\lambda}, 0, u_\lambda\right),
\end{aligned}
$$

once the invertibility of the correlation functions with respect to their arguments is assumed. Indeed, the above equations can be inverted, as it is shown term by term in perturbation theory, yielding

$$
Z_{\varphi,\lambda'/\lambda} = Z_\varphi\left(\frac{\lambda'}{\lambda}, u_\lambda\right), \quad Z_{t,\lambda'/\lambda} = Z_t\left(\frac{\lambda'}{\lambda}, u_\lambda\right), \quad Z_{u,\lambda'/\lambda} = Z_u\left(\frac{\lambda'}{\lambda}, u_\lambda\right),
$$

where it is evident that u_λ appears as a parameter. Eq. (45) reproduce the scaling relations asymptotically near to the critical point. Iteration of the transformation is achieved by changing λ into $\lambda' = \lambda/s$. The group equations (44) and (45), which

are valid for any $\mathbf{k}, \varphi, t$, can be casted in differential form by taking the derivatives with respect to λ' and then setting $\lambda' = \lambda$.

If we denote by $v_{i,\lambda}$ the variables appearing in (43), we have the following differential equation:

$$\lambda \frac{d}{d\lambda} v_{i,\lambda} = \psi_i, \tag{46}$$

where

$$\psi_t = -t_\lambda \beta_t(u_\lambda), \quad \psi_\varphi = -\varphi_\lambda \beta_\varphi(u_\lambda), \quad \psi_u = -u_\lambda \beta_u(u_\lambda), \tag{47}$$

and

$$\beta_t(u_\lambda) = x_t^0 - \left.\frac{\partial Z_\varphi}{\partial \lambda'/\lambda}\right|_{\lambda'=\lambda} + \left.\frac{\partial Z_t}{\partial \lambda'/\lambda}\right|_{\lambda'=\lambda}, \tag{48}$$

$$\beta_\varphi(u_\lambda) = x_\varphi^0 + \frac{1}{2}\left.\frac{\partial Z_\varphi}{\partial \lambda'/\lambda}\right|_{\lambda'=\lambda}, \tag{49}$$

$$\beta_u(u_\lambda) = \epsilon - 2\left.\frac{\partial Z_\varphi}{\partial \lambda'/\lambda}\right|_{\lambda'=\lambda} + \left.\frac{\partial Z_u}{\partial \lambda'/\lambda}\right|_{\lambda'=\lambda}. \tag{50}$$

On the critical surface $t = 0, \varphi = 0$, and from Eqs. (46), (47), and (50) the fixed point of the transformation is given by

$$u^* \beta_u(u^*) = 0,$$

which yields two solutions, the trivial one $u^* = 0$, which corresponds to the Gaussian fixed point, and $u^* = u^*(\epsilon)$ which is defined implicitly through Eq. (50)

$$\beta_u(u^*) = 0 \quad \Rightarrow \quad \epsilon = 2\left.\frac{\partial Z_\varphi}{\partial \lambda'/\lambda}\right|_{\lambda'=\lambda;u=u^*} - \left.\frac{\partial Z_u}{\partial \lambda'/\lambda}\right|_{\lambda'=\lambda;u=u^*}. \tag{51}$$

If we linearize Eqs. (46) around this fixed point, denoting $\beta_i(u^*) = \beta_i^*$, we see immediately that

$$\beta_t^* = x_t = \frac{1}{\nu}, \quad \beta_\varphi^* = x_\varphi = 1 - \frac{\epsilon}{2} + \frac{\eta}{2},$$

where $\eta = \partial Z_\varphi / \partial(\lambda'/\lambda)|_{\lambda'=\lambda;u=u^*}$ is the anomalous dimension of the field. The critical index for u determines the rate of approach to the fixed point and is given by

$$x_u = -\left.\frac{d\psi_{u,\lambda}}{du}\right|_{u=u^*}. \tag{52}$$

When we linearize the transformations (44) at the fixed point, Eqs. (44) and (45) imply that the correlation functions and the functional $\bar{\Gamma}$ per unit volume are homogeneous functions with degrees of homogeneity

$$
\begin{aligned}
&d - 2\beta_\varphi^* - 2 \quad \text{for} \quad \bar{\gamma}_\varphi, \\
&d - 2\beta_\varphi^* - \beta_t^* \quad \text{for} \quad \bar{\gamma}_t, \\
&d - 4\beta_\varphi^* \quad \text{for} \quad \bar{\gamma}_u, \\
&d \quad \text{for} \quad \bar{\Gamma} \quad \text{(per unit volume)}.
\end{aligned}
$$

From these, the scaling laws follow naturally.

The effective values of the exponents can be calculated within perturbation theory. At first order in ϵ only the diagram for Z_u shown in Fig. 4 enters into Eqs. (50) and (51). In fact, given the definition of γ_u, this diagram is of first order in u. The first diagram for Z_φ (see Fig. 5) is of second order in u and does not enter in determining the fixed point $u^* \sim O(\epsilon)$. For the sake of simplicity we discuss the Ising case $n = 1$. The explicit evaluation of the diagram in Fig. 4 gives then a correction to the bare coupling constant u_0, so that, within perturbation theory, the dressed coupling constant is

$$
u = u_0 - 36 u_0^2 \int_{\Lambda/s<|\mathbf{q}|<\Lambda} \frac{d^4\mathbf{q}}{(2\pi)^4} \frac{1}{|\mathbf{q}|^2} = u_0 \left(1 - \frac{9u_0}{2\pi^2} \log s\right),
$$

where the prefactor 36 comes from the multiplicity of the diagram, and the integral is evaluated for $d = 4$ ($\epsilon = 0$), to obtain the leading order in ϵ, since at the fixed point $u_0 \to u^* \sim \epsilon$. By definition, the factor in front of u_0 in the right-hand side, with $s = \lambda/\lambda'$ and $u_0 \to u$, is Z_u^{-1} (at the lowest order in perturbation theory). Then, at $O(\epsilon)$,

$$
Z_u \simeq 1 + \frac{9u}{2\pi^2} \log s,
$$

and from Eqs. (50) and (51) $\epsilon = 9u^*/2\pi^2$, i.e., $u^* = 2\pi^2\epsilon/9$, which coincides with the solution of Eq. (39), with the explicit expression $g(n = 1, d = 4) = 9/2\pi^2$. Then Eq. (52) gives $x_u = -\epsilon$, with $u - u^*$ as the first irrelevant variable.

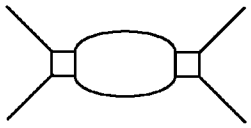

FIGURE 4. Lowest-order diagram contributing to γ_u.

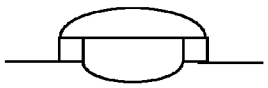

FIGURE 5. Lowest-order diagram contributing to Z_φ.

The anomalous part of x_t, equal to $x_t - 2$, is of first order in ϵ and comes from the γ_t correction shown in Fig. 6, whereas the contribution coming from Z_φ (see Fig. 5) is of higher order in ϵ. Calculating the diagram in Fig. 6 we find

$$
t = t_0 - 12 t_0 u_0 \int_{\Lambda/s<|\mathbf{q}|<\Lambda} \frac{d^4\mathbf{q}}{(2\pi)^4} \frac{1}{|\mathbf{q}|^2} = t_0 \left(1 - \frac{3u_0}{2\pi^2} \log s\right),
$$

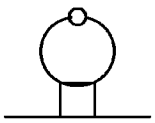

FIGURE 6. Lowest-order diagram contributing to γ_t.

where the prefactor 12 comes from the multiplicity of the diagram, and the integral is evaluated for $d = 4$ ($\epsilon = 0$), since we are interested in the leading contribution in ϵ. By definition, the factor in front of t_0 in the right-hand side, with $s = \lambda/\lambda'$ and $u_0 \to u$, is Z_t^{-1} (at the lowest order in perturbation theory). Then, at $O(\epsilon)$,

$$Z_t \simeq 1 + \frac{3u}{2\pi^2} \log s,$$

so that, from Eq. (48) at $u = u^*$, we find $x_t = x_t^0 - 3u^*/2\pi^2 = x_t^0 - \epsilon/3$, which coincides with the exponent x_1 found in Sec. III D 2, for $n = 1$.

The first contribution to η is $O(\epsilon^2)$. Let us indicate with $\Sigma(\mathbf{q})$ the diagram for Z_φ shown in Fig. 5, with incoming momentum $\mathbf{q}$. Then

$$\begin{aligned} \Sigma(\mathbf{q}) - \Sigma(0) &= -96 u_0^2 \int_{\Lambda^{-1}<|\mathbf{r}|<s\Lambda^{-1}} d^4\mathbf{r} [G(\mathbf{r})]^3 \left(\mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}} - 1\right) \\ &\simeq 12 u_0^2 |\mathbf{q}|^2 \int_{\Lambda^{-1}<|\mathbf{r}|<s\Lambda^{-1}} d^4\mathbf{r} [G(\mathbf{r})]^3 |\mathbf{r}|^2 = \frac{3u_0^2}{8\pi^4} |\mathbf{q}|^2 \log s, \end{aligned} \tag{53}$$

where the prefactor 96 comes from the multiplicity of the diagram, the integral is evaluated for $d = 4$ ($\epsilon = 0$), since we are interested in the leading contribution $O(\epsilon)$, the term $\sim |\mathbf{q}|^2$ has been considered for $|\mathbf{q}| \ll \Lambda$, and

$$G(\mathbf{r}) = \int \frac{d^4\mathbf{p}}{(2\pi)^4} \frac{\mathrm{e}^{i\mathbf{p}\cdot\mathbf{r}}}{|\mathbf{p}|^2} = \frac{1}{4\pi^2 |\mathbf{r}|^2}$$

is the Fourier transform of the propagator $[\Gamma_0^{(2)}(\mathbf{k})]^{-1}$ for $t_0 = 0$ and $d = 4$. By definition, the factor in front of $|\mathbf{q}|^2$ in Eq. (53), with $s = \lambda/\lambda'$ and $u_0 \to u$, is $Z_\varphi^{-1} - 1$ (at the lowest order in perturbation theory). Then, at the fixed point $u = u^*$, we obtain

$$Z_\varphi \simeq 1 - \frac{\epsilon^2}{54} \log s,$$

and, from Eq. (49) at $u = u^*$, we find $\eta = \epsilon^2/54$.

The explicit calculation of the Feynman diagrams thus yields exponents which coincide at $O(\epsilon)$ with those obtained within the ϵ expansion discussed in Section III D 2.

In general, the fixed-point condition (51) implies the scaling index $\epsilon - 2\eta$ for the four-point vertex $\bar{\gamma}_u \sim Z_u^{-1}$ and $-\eta$ for $\bar{\gamma}_\varphi \sim Z_\varphi^{-1}$. In the renormalized coupling their asymptotic behaviors compensate each other for the anomalous part, and what is left compensates for the bare dimension ϵ of u. The renormalized coupling

has in this way a well-defined asymptotic limit. The index η controlled the suitable normalization of the block variables in the probabilistic approach, to get a well-defined asymptotic probability distribution and the fixed point with the proper behavior of the order-parameter correlation function in the Wilson approach.

IV LUTTINGER LIQUID AND FERMI LIQUID

The low-energy behavior of most interacting electron systems can be classified in terms of a small number of *universality* classes. This classification can be understood in terms of Wilson's RG that selects few types of relevant scattering processes near the Fermi surface [13,20,63,64] and allows to determine the entire low-energy behavior in terms of a small set of parameters. For pure systems, without symmetry breaking, two types of metallic phases are well known: the Luttinger-liquid phase in $d = 1$ [13], and the Fermi-liquid phase in $d > 1$ [65]. Concerning systems with symmetry breaking, two superfluid behaviors are well understood: the one due to the Bose-Einstein condensation (e.g., ^{4}He, local pairs of fermions, etc.), and the one related with the coherent superposition of pairs in momentum space (with a fixed phase of the pair wave function) with strong spatial overlap within the characteristic spatial extent ξ_0 of the pair, as described by the BCS state.

The ratio of the pair size ξ_0 over the mean inter-particle distance k_F^{-1} controls the crossover between these two regimes. The limit $\xi_0/k_F^{-1} \gg 1$ concerns ordinary superconductors described by the BCS state ($\xi_0 \simeq 10^4$Å, $T_c \simeq 10 - 20$K). The limit $\xi_0/k_F^{-1} \simeq 1$ corresponds the Bose-Einstein condensation of tightly bound pairs which do not overlap with one another (small bipolarons). The high-temperature superconducting cuprates (HTS) are in the intermediate regime $\xi_0 k_F \simeq 10$, in which k_F is considerably reduced with respect to ordinary superconductors (e.g., $YBa_2Cu_3O_7$ has $3 \cdot 10^{21}$ charge carriers per cm^3) and $\xi_0 \simeq 10 - 30$Å. The great difficulty in explaining high-temperature superconductivity is that, while ordinary superconductivity is the result of electron-phonon interaction with a characteristic energy scale $\omega_D \ll E_F$, for HTS the energy scale for superconductivity is comparable with the characteristic electronic energy (i.e., E_F is reduced and T_c increased). For the ordinary superconductors we can therefore integrate out high-energy processes falling in a Landau Fermi-liquid theory characterized by normal-state quasiparticles plus a small residual attractive interaction. The mean-field analysis of such an interaction leads to the superconducting instability. Such procedure is no more applicable to HTS, due to the absence of the separation between the energy scales. Moreover, the normal state of HTS is not a Fermi liquid [66]. The HTS are strongly anisotropic and almost two-dimensional, as it is witnessed by the much smaller value of the electrical resistivity in the direction perpendicular to the copper-oxygen planes $\rho_\perp$ with respect to the resistivity within the planes $\rho_\parallel$ ($\rho_\perp/\rho_\parallel \simeq 10^{-2} - 10^{-5}$). In contrast with the Fermi-liquid behavior, $\rho_\parallel$ is linear in T in a large range of temperatures [67], when the density of charge carriers, which can be varied by chemical doping, is such that the superconducting critical

temperature T_c is the highest possible within each family of HTS (optimal doping). Moreover, below the optimal doping, in the so-called underdoped regime, where the charge-carrier density is small, the anomalous behavior of the normal state is much stronger and a pseudogap opens at a doping-dependent temperature T^*. Another evidence for a non-Fermi-liquid behavior comes from the linearity in the quasiparticle inverse scattering time, $\tau_{scatt}^{-1} \propto (T, \omega)$, as measured, e.g., in angle-resolved photoemission spectroscopy [68] by the slow narrowing of the single-particle spectral density as the energy of the quasiparticle peak approaches the Fermi level, in contrast with the usual Fermi-liquid behavior $\tau_{scatt}^{-1} \propto (T^2, \omega^2)$, which can be deduced by simple phase-space arguments for low-energy scattering processes near the Fermi surface [65].

A clarification of the properties of the normal phase from which high-temperature superconductivity emerges, apart from being an interesting problem by itself, would give also hints on the basic mechanism for the formation of superconductivity. A relevant question which arises when looking at the properties of the normal phase of HTS is whether these are characteristic of a new quantum metallic state or are simply the result of an anomalous scattering mechanism. Experimentally we observe a continuous gradual change from the underdoped regime, where the metal is anomalous, to the regime where the doping is larger than the optimal one (overdoped regime), where the metal seems to be more Fermi-liquid-like.

A perturbative analysis at the lowest orders is not enough. Suggestions for the anomalous behavior of the normal phase are the breakdown of the Fermi liquid, as an extension of the Luttinger-liquid behavior for interacting electron systems in $d = 2$ [66,69,70] (which is the dimensionality proper of the HTS, due to their strong anisotropy), or the onset of the marginal Fermi liquid [71], in which quasiparticles are not well defined, as their inverse lifetime is of the same order as their energy. A third possibility is that the system is close to an instability, and that the critical fluctuations couple to the charge carriers and give rise to a singular effective interaction which spoils the Fermi-liquid behavior [72]. Critical fluctuations can be due to a charge instability (phase separation, or incommensurate charge-density wave [72]), to a magnetic instability [73], or to a combination of the two, with a stripe-like modulation of the charge and spin density [74]. This last case can result from the tendency of an antiferromagnetic background to expel charges at low doping [75], or from an instability of the Fermi liquid towards the formation of a charge modulation guiding a spin modulation [72], which is reached upon reducing the density of charge carriers, starting from the markedly metallic overdoped region.

The proposal that the Luttinger-liquid behavior can be extended to a two-dimensional system stems from the observation that all HTS are insulators in the absence of charge carriers introduced by chemical doping, and the ground-state is antiferromagnetically ordered. Doping an antiferromagnet results in a frustrated state where the spin of the charge carriers introduced by doping couple with the antiferromagnetically ordered spins localized on the copper ions, disturbing the ordered state. This frustrated state, which in $d = 1$ is characterized by resonating spin singlets without magnetic long-range order (resonating-valence-bond state)

with no fermionic low-lying excitations, and decoupled charge and spin collective excitations, could then extend to $d = 2$ [69]. The underlying hypothesis is thus that the doped system never behaves as a Fermi liquid, and, in the absence of a symmetry breaking, rather reaches a new fixed point, characterized by a non-Fermi-liquid behavior. Does this new fixed point exist for $d > 1$? Can we extend the Luttinger-liquid behavior to $d = 2$, as appropriate to the case of the conducting copper-oxygen planes of HTS? What are the conditions? Before answering these questions, let us review the main characteristics of a normal Fermi liquid and a Luttinger liquid. Then we discuss the dimensional crossover from a Luttinger to a Fermi liquid, as soon as $d > 1$ [19,20], unless singular long-range forces are present which may extend the non-Fermi-liquid behavior to a higher dimension [21–23].

Many reviews are available on the topic of fermion systems in $d = 1$, starting from the one by Sólyom, which summarizes the understanding of the problem at the end of 1970s [13]. More recent accounts of the problem can be found in Refs. [29,76], whereas the case of interacting fermions in the presence of forward scattering is dealt with in Ref. [20], and a pedagogical approach to the RG for Fermi systems is found in Ref. [64]. In these notes we closely follow Refs. [18,20].

A Normal Fermi Liquid

The normal Fermi-liquid theory, which generically applies to liquid ^{3}He and ordinary metals in $d = 3$, relies on the existence of well-defined fermionic low-lying excitations (quasiparticles). In terms of the RG approach, the corresponding fixed-point Hamiltonian of quasiparticles in the presence of residual Hartree-like interactions can be obtained by integrating out states far from the Fermi surface via a Wilson-like approach, and all the momentum-transferring scattering processes in $d > 1$ become ineffective. The effective Hamiltonian, representative of a metallic state (no Cooper instability will be considered here) can be written as

$$H = H_0 + \frac{1}{V} \sum_{\mathbf{k},\mathbf{k}',\sigma,\sigma'} f^{\sigma\sigma'}_{\mathbf{k}\mathbf{k}'} \delta n_{\mathbf{k}\sigma} \delta n_{\mathbf{k}'\sigma'},$$

where H_0 is the term which describes the metal as a collection of free quasiparticles, and the term $f^{\sigma\sigma'}_{\mathbf{k}\mathbf{k}'}$ is the (weak) effective interaction which couples the changes in the occupation numbers of the quasiparticle states $\delta n_{\mathbf{k}\sigma}$. The presence of the interaction term leads to the following effects:

i) The various response functions are affected by the interactions via a small set of parameters (the Landau parameters). The equilibrium properties, modified by the presence of the Landau parameters, are similar to those of a Fermi gas, i.e., the specific-heat coefficient $\gamma = C_V/T$, the spin susceptibility χ, and the compressibility $K = \partial n/\partial \mu$ are temperature-independent, as long as the temperature is much smaller than the Fermi temperature.

ii) A finite wave-function normalization Z appears as the residual at the simple pole of the single-particle Green's function

$$G(\mathbf{k},\omega) = \frac{Z}{\omega - v_F k_r + i0^+ \mathrm{sgn}\omega}, \tag{54}$$

where $k_r = |\mathbf{k}| - k_F$. This implies that there still exists a Fermi surface, where the quasiparticle occupation number in momentum space is discontinuous, $n_{\mathbf{k}_F - 0} - n_{\mathbf{k}_F + 0} = Z$ (see Fig. 7). In the Fermi gas $Z = 1$.
iii) An inverse lifetime of quasiparticles appears, associated with subleading corrections, which vanishes as the square of the deviation of the momentum from the Fermi momentum or the square of the temperature T

$$\tau^{-1} \simeq E_F^{-1} \mathrm{max}[(v_F k_r)^2, T^2].$$

As a consequence, for instance, angle-resolved photoemission experiments should show a single-particle spectral-density peak narrowing as ω^2 when $|\mathbf{k}| \to k_F$ and, at finite temperature, the resistivity due to electron-electron scattering is $\rho(T) \propto T^2$.

The Fermi-liquid theory breaks down in a variety of ways even in the absence of symmetry breaking, as for instance in Kondo systems [77], or in electronic systems near the metal-insulator transition in the presence of strong-disorder [78]. In the following, we discuss how the breakdown of the Fermi liquid occurs in clean electronic systems in $d = 1$.

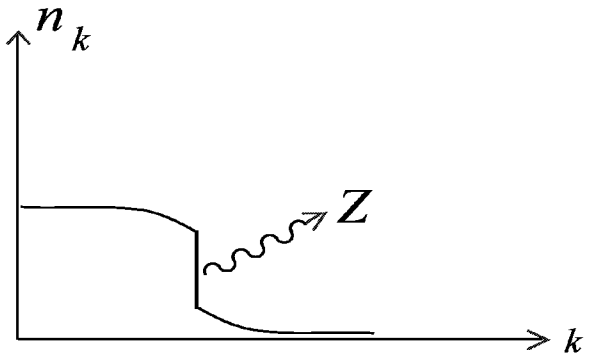

FIGURE 7. Quasiparticle occupation number in momentum space in a Fermi liquid. The discontinuity marks the Fermi surface.

B Luttinger Liquids in $d = 1$

The breakdown of the Fermi-liquid theory in one-dimensional interacting systems is witnessed by the presence of logarithmic divergences in the second-order perturbative contribution to the single-particle self-energy and to the four-point vertex function. The problem of treating these divergences has been first solved by a perturbative RG approach [13]. Assuming a scaling ansatz for the vertex functions, one approaches the Fermi surface by rescaling the fields and the coupling constants.

Depending on the values of the bare coupling constants, the renormalized coupling constants flow either to strong-coupling, out of the controlled regime, or to the exactly soluble Luttinger model [33,79]. In the latter case the system is a Luttinger liquid [13,30–33]. The Luttinger liquid is a normal (non-symmetry-broken) metallic phase characterized by:
i) A finite renormalization in correlation functions: C_v/T, χ, K, and the weight of the Drude peak in the optical conductivity are finite. These quantities are all characterized by changes in the values of the parameters, but not in the form, with respect to a Fermi liquid;
ii) A vanishing wave-function renormalization Z, which suppresses the discontinuity of the occupation number in momentum space, leading to an algebraic singularity with non-universal exponent $n_{\mathbf{k}} - n_{\mathbf{k}_F} \sim -\mathrm{sgn}(k_r)|k_r|^{\eta}$;
iii) A single-particle density of states which vanishes as ω^{η} at low energy;
iv) The presence of collective charge and spin excitations, which implies the so-called charge and spin separation, due to the vanishing of the single-particle spectral density at the Fermi surface, and the consequent suppression of low-lying quasiparicle excitations.

Therefore, the collective properties for $|\mathbf{q}|, \omega \to 0$ are trivial as from the random-phase approximation (RPA) or the Landau-Boltzmann kinetic equation, whereas the single-particle properties are anomalous.

In the next sections we explicitly discuss the Luttinger liquid in $d = 1$.

C The g-ology Model

The problem of treating the divergences of an interacting fermion system in $d = 1$ has been first solved by a weak-coupling RG method applied to an effective low-energy theory known as the "g-ology" model [13]. It is a continuum model with a linear dispersion relation and two-particle interactions, that incorporates the generic low-energy dynamics of a one-dimensional Fermi system. It can be written as $H = H_0 + H_I$, where H_0 describes a linear band limited by a momentum cutoff Λ

$$H_0 = \sum_{|\mathbf{k}|<\Lambda} \sum_{\alpha\sigma} \alpha v_F \mathbf{k}\; a^{\dagger}_{\alpha,\sigma}(\mathbf{k}) a_{\alpha,\sigma}(\mathbf{k}),$$

where $a^{\dagger}_{\alpha,\sigma}(\mathbf{k})$ and $a_{\alpha,\sigma}(\mathbf{k})$ create and annihilate fermions with spin σ close to $+k_F$ ($\alpha = +$) and $-k_F$ ($\alpha = -$). We have a boldface notation for the momentum $\mathbf{k}$, measured relative to $\pm k_F$, to distinguish it from the bi-vectors $k = (\mathbf{k}, \omega)$. v_F is the Fermi velocity. The ultraviolet cutoff of the theory acts also as momentum-transfer cutoff in the interaction. The interaction Hamiltonian H_I is given by

$$H_I = H_1 + H_2 + H_3 + H_4, \tag{55}$$

where

$$H_1 = \frac{1}{V}\sum_{\mathbf{q}}\sum_{\sigma\sigma'} g_1^{\sigma\sigma'}\rho_\sigma^+(\mathbf{q})\rho_{\sigma'}^-(-\mathbf{q}),$$
$$H_2 = \frac{1}{V}\sum_{\mathbf{q}}\sum_{\sigma\sigma'} g_2^{\sigma\sigma'}\rho_{+,\sigma}(\mathbf{q})\rho_{-,\sigma'}(-\mathbf{q}),$$
$$H_3 = \frac{1}{2V}\sum_{\mathbf{q}}\sum_{\sigma\sigma'} g_3^{\sigma\sigma'}[\rho_\sigma^+(\mathbf{q})\rho_{\sigma'}^+(-\mathbf{q}) + \rho_\sigma^-(\mathbf{q})\rho_{\sigma'}^-(-\mathbf{q})],$$
$$H_4 = \frac{1}{2V}\sum_{\mathbf{q}}\sum_{\sigma\sigma'} g_4^{\sigma\sigma'}[\rho_{+,\sigma}(\mathbf{q})\rho_{+,\sigma'}(-\mathbf{q}) + \rho_{-,\sigma}(\mathbf{q})\rho_{-,\sigma'}(-\mathbf{q})].$$

V is the volume of the system, which for $d = 1$ is equal to the length of the chain, and

$$\rho_\sigma^+(\mathbf{q}) = \sum_{\mathbf{k}} a_{+,\sigma}^\dagger(\mathbf{k}-\mathbf{q})a_{-,\sigma}(\mathbf{k}),$$
$$\rho_\sigma^-(\mathbf{q}) = \sum_{\mathbf{k}} a_{-,\sigma}^\dagger(\mathbf{k}-\mathbf{q})a_{+,\sigma}(\mathbf{k}),$$
$$\rho_{\alpha,\sigma}(\mathbf{q}) = \sum_{\mathbf{k}} a_{\alpha,\sigma}^\dagger(\mathbf{k}-\mathbf{q})a_{\alpha,\sigma}(\mathbf{k}) \qquad (\text{with } \alpha = \pm),$$

where $\rho_{\pm,\sigma}$ are density operators for right- $(+)$ and left- $(-)$ moving particles.

The interaction Hamiltonian H_I describes various scattering processes shown in Fig. 8: the small-momentum-transfer processes (H_2 and H_4) are expressed in terms of $\rho_{\alpha,\sigma}$, the back scattering (H_1) and the Umklapp scattering (H_3) in terms of $\rho_\sigma^\pm$. Since the coupling constants may be spin dependent, one writes

$$g_i^{\sigma\sigma'} = g_{i\parallel}\delta_{\sigma\sigma'} + g_{i\perp}\delta_{\sigma,-\sigma'},$$

where $g_{i\parallel}$ and $g_{i\perp}$ refer to parallel and opposite spins respectively. Owing to momentum conservation, Umklapp processes in H_3 can be relevant only if $4k_F$ is equal to a reciprocal-lattice vector $2\pi/a$ (where a is the lattice spacing), as in the Hubbard model at half filling.

The g-ology Hamiltonian is the general outcome of a renormalization procedure, applied to an interacting one-dimensional model, such as the one-dimensional Hubbard model, by integrating out the momentum states far from the Fermi surface ($|\mathbf{k}| > \Lambda$). The model parameters g_i are related to specific forms of the four-point vertex of the resulting effective low-energy action [64]. In the case of Coulomb interactions and for repulsive Hubbard-like systems all the g's are positive. A nonlinear dispersion of the quasiparticle band and the momentum dependence of the coupling constants scale to zero under partial RG iteration, with Eq. (55) as a final result.

The properties of the g-ology model can be extracted from the quantum analog of the correlation and the vertex functions introduced in Sec. II D, i.e, the Green's functions defined as ground-state expectation values of the time-ordered operator products,

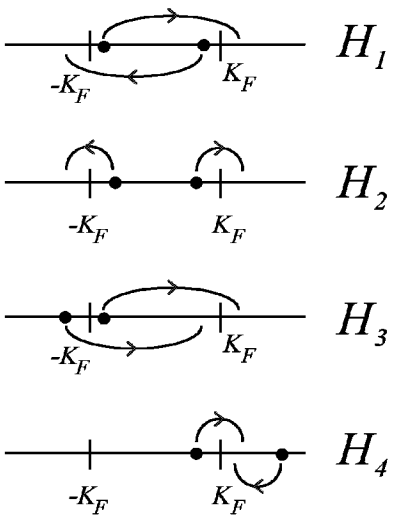

FIGURE 8. Interaction terms of the g-ology model; $+k_F$ and $-k_F$ indicate the right and left Fermi points.

$$G^{(2n,l)}_{\alpha_1,\sigma_1;\ldots;\alpha'_1,\sigma'_1;\ldots;\alpha''_1,\sigma''_1;\ldots}(\mathbf{k}_1,t_1;\ldots;\mathbf{k}'_1,t'_1;\ldots;\mathbf{k}''_1,t''_1;\ldots) =$$
$$= (-i)^{n+l}\langle\tau a_{\alpha_1,\sigma_1}(\mathbf{k}_1,t_1)\ldots a_{\alpha_n,\sigma_n}(\mathbf{k}_n,t_n)a^\dagger_{\alpha'_n,\sigma'_n}(\mathbf{k}'_n,t'_n)\ldots a^\dagger_{\alpha'_1,\sigma'_1}(\mathbf{k}'_1,t'_1)$$
$$\times\rho_{\alpha''_1,\sigma''_1}(\mathbf{k}''_1,t''_1)\ldots\rho_{\alpha''_l,\sigma''_l}(\mathbf{k}''_l,t''_l)\rangle,$$

where $a_{\alpha,\sigma}$,$a^\dagger_{\alpha,\sigma}$, and $\rho_{\alpha,\sigma}$, are the operators in the Heisenberg representation; τ is the time ordering. The Green's function in the frequency representation will be denoted by $G^{(2n,l)}(k_1\ldots;k'_1\ldots;k''_1\ldots)$ where $k_i = (\mathbf{k}_i,\omega_i)$ and $\Gamma^{(2n,l)}$ will denote the corresponding vertex functions, i.e., the one-particle irreducible contributions. The Fourier transform of single-particle Green's function $G(\mathbf{k},t) = -i\langle\tau a_{\alpha,\sigma}(\mathbf{k},t)a^\dagger_{\alpha,\sigma}(\mathbf{k},0)\rangle$ with respect to the time variable, in the noninteracting case leads to the expression (54) with $Z = 1$. The function $G^{(0,2)}$ describes the charge- and spin-density correlations with small $\mathbf{q}$ (i.e., $|\mathbf{q}| \ll k_F$).

In $d = 1$ only the existence of the two Fermi points is important, not their distance in momentum space. Thus the inverse cutoff Λ^{-1} is the only length scale in the g-ology model; Λ substitutes the natural cutoff given by the Brillouin-zone boundary or by the non-linear terms in the momentum dependence of band dispersion of the underlying microscopic system. Since the inverse cutoff Λ^{-1} is the only length scale, all dimensionless quantities depend only via the ratio p/Λ on the momenta p and Λ. Hence, the infrared limit $p \to 0$ is directly related to the ultraviolet limit $\Lambda \to \infty$.

In general, an exact solution giving the expression of the correlation functions in the g-ology model is not possible. Perturbation theory with respect to the coupling constants g_i diverges at low energies.

Simple dimensional analysis yields the canonical dimensions in inverse length:

$[H] = 1$, $[a(k)] = -1/2$, and therefore $[g_i] = 0$, i.e., the coupling constants are dimensionless. The bare propagator $G^0(k)$ has dimension -1. Each energy-momentum integration in a Feynman diagram introduces a power 2. Hence power counting predicts that primitive divergences may occur in $\Gamma^{(2)}$, $\Gamma^{(4)}$, $\Gamma^{(0,2)}$, and $\Gamma^{(2,1)}$, but not in higher-order vertex functions.

A first understanding and successful treatment of these divergences is described in Ref. [13]. Taking a scaling ansatz for the vertex functions, the low-energy limit is approached by rescaling fields and coupling constants

$$\Gamma^{(2n)}(\ldots, g', \Lambda') = Z^n(g, \Lambda'/\Lambda)\Gamma^{(2n)}(\ldots, g, \Lambda),$$

where $g' = g'(g, \Lambda'/\Lambda)$ is the running coupling with flow determined by the behavior of the β function, $\beta(g) = [\partial g'/\partial(\Lambda'/\Lambda)]_{\Lambda'=\Lambda}$. Depending on the values and on the signs of the bare coupling constant, or on whether $4k_F$ is equal to, or different from, a reciprocal lattice vector, the renormalized coupling constant may flow to strong coupling, or reach a fixed point. In the first case (e.g., $g_3 \neq 0$, $g_1 < 0$), the low-energy Hamiltonian undergoes a dramatic change, with dynamical generation of a gap in the charge and spin spectrum, while in the latter case (as for g_i derived from repulsive Hubbard-like models with $g_3 = 0$) one reaches a fixed-point Hamiltonian given by the Luttinger model [30,79], where g_2 and g_4 only are present. Therefore the low-energy behavior of a generic one-dimensional fermion system interacting via repulsive interaction is well described in terms of the Luttinger model Hamiltonian $H = H_0 + H_2 + H_4$. As already stated, the Luttinger model is exactly soluble either by employing Ward identities [16,18,80], or by the bosonization procedure [30–33]. Many features of the Luttinger liquids are already clear within perturbative RG calculations [13], as, for instance, the anomalous scaling behavior of the single-particle propagator

$$G(s\omega, s\mathbf{k}) = s^{\eta-1}G(\omega, \mathbf{k}), \tag{56}$$

where $\eta > 0$ is a non-universal constant[10] which determines the power-law behavior of the momentum distribution function, to be contrasted with the case of the Fermi liquid, where $\eta = 0$. Indeed, as usual, the RG approach leads to a power-law resummation of the logarithmic singularities which affect the self-energy correction at second order in perturbation theory. At this order, the critical index

$$\eta = \frac{g_{2\parallel}^2 + g_{2\perp}^2}{8\pi^2 v_F^2} + 0(g^3), \tag{57}$$

is the anomalous dimension entering the scaling form (56) for the Green's function G. The exponent η also characterizes the momentum distribution function

[10] The anomalous dimension depends on the value of the coupling constants of the model since the β function vanishes identically, owing to the peculiar left and right conservation (see below), which implies a cancellation of singularities at any order in perturbation theory [17,18].

$n_{\mathbf{k}} = -(1/2\pi^2)\sum_\sigma \int d\omega\, \mathrm{Im}G_{\alpha\sigma}(\mathbf{k},\omega) \propto |\mathbf{k}|^\eta$, and the single-particle spectral function $D(\omega) = -(1/2\pi^2)\sum_\sigma \int d\mathbf{k}\, \mathrm{Im}G_{\alpha\sigma}(\mathbf{k},\omega) \propto \omega^\eta$.

We explore in the next section the exact solution of the Luttinger model by exploiting the conservation laws and the related Ward identities, as reviewed in Refs. [18,20].

D Analysis of the Luttinger Model via Ward Identities

1 Global Charge and Spin Conservation: Ward Identities

In the g-ology model the *global* charge and spin conservation laws give rise to the Ward identities which yield relations between different correlation functions, and some information on their asymptotic behavior for small momentum and energy transfer.

Let us define the left ($\alpha = -$) and the right ($\alpha = +$) charge or spin density operators

$$\rho_\alpha^{c,s}(\mathbf{q}) = \rho_{\alpha,\uparrow}(\mathbf{q}) \pm \rho_{\alpha,\downarrow}(\mathbf{q}),$$

which satisfy the commutation relations [18]

$$[\rho_\alpha^a(\mathbf{q}), \rho_{\alpha'}^a(\mathbf{q}')] = \frac{V}{2\pi}\delta_{\alpha\alpha'}\delta_{\mathbf{q},-\mathbf{q}'}\ \alpha\mathbf{q}, \tag{58}$$

where $a = c, s$ (observe that $|\mathbf{q}|V/2\pi$ gives the number of points in the segment $|\mathbf{q}|$). This non commutativity of the theory is a consequence of the restriction of the sums in momentum space up to a cutoff Λ, within the g-ology model.

The Ward identities come from the continuity equation which is obtained by calculating the commutator of the g-ology Hamiltonian with the total charge (or spin) density operator at small $\mathbf{q}$, $\rho^a(\mathbf{q}) = \rho_+^a(\mathbf{q}) + \rho_-^a(\mathbf{q})$, i.e.,

$$i\partial_t\rho^a(\mathbf{q},t) = [\rho^a(\mathbf{q},t), H] = \mathbf{q}v^a j^a(\mathbf{q},t), \tag{59}$$

where j^a is the current operator given by $j^a(\mathbf{q}) = \rho_+^a(\mathbf{q}) - \rho_-^a(\mathbf{q})$, v^a is a coupling-dependent velocity

$$v^a = v_F + \frac{1}{\pi}(g_4^a - g_2^a), \tag{60}$$

and $g_i^{c,s} = \frac{1}{2}(g_{i\|} \pm g_{i\perp})$. The physical current operator, $J^a = v^a j^a$ depends only on the forward scattering because $\rho^a(\mathbf{q})$ commutes with H_1 and H_3. Eq. (59) is the continuity equation corresponding to the conservation of the total charge (or spin). Our aim is to derive from Eq. (59) the Ward identities for the vertex functions and the correlation functions, to reduce the number of independent relevant structures to be considered. We derive, first, the Ward identity for the correlation functions, $J_{\mu\nu}^a(\mathbf{q},t) = -\frac{i}{V}\langle \tau j_\mu^a(\mathbf{q},t) j_\nu^a(-\mathbf{q},0)\rangle$, where the *bi-currents* j_μ^a ($\mu = 0, 1$), are defined

by $j_0^a(\mathbf{q}) = \rho^a(\mathbf{q})$ and $j_1^a(\mathbf{q}) = j^a(\mathbf{q})$. J_{00}^a is the density-density correlation function and J_{11}^a is the current-current correlation function. Applying $i\partial_t$ to $J_{0\nu}^a$ and using the continuity equation, one obtains

$$i\partial_t J_{0\nu}^a(\mathbf{q}, t-t') = -\frac{i}{V}\langle \tau \mathbf{q} v^a j_1^a(\mathbf{q}, t) j_\nu^a(-\mathbf{q}, 0)\rangle + \frac{1}{V}\delta(t-t')\langle [j_0^a, j_\nu^a]\rangle.$$

After Fourier-transforming, we obtain the following Ward identity

$$\omega J_{0\nu}^a - v^a \mathbf{q} J_{1\nu}^a = \frac{2}{\pi}\mathbf{q}\delta_{\nu,1}, \tag{61}$$

where we have used the commutators (58). This Ward identity puts constrains on the limiting behavior of charge ($a = c$) and spin ($a = s$) correlation functions as $q = (\omega, \mathbf{q}) \to (0, \mathbf{0})$. As in Fermi liquids, this limit is not unique, but depends on the ratio $r = |\mathbf{q}|/\omega$. If we define $J_{\mu\nu,0}^a = J_{\mu\nu}^a(q \to 0, r \to 0)$, and $J_{\mu\nu,\infty}^a = J_{\mu\nu}^a(q \to 0, r \to \infty)$, then

$$J_{00,0}^a = J_{01,0}^a = J_{10,\infty}^a = 0$$
$$J_{11,\infty}^a = -2/\pi v^a.$$

The limit $|\mathbf{q}| \to 0$ at finite ω of the density-density correlation function must always vanish to conserve the total number of particles. The Ward identity (61) yields no information on $J_{00,\infty}^a$ and $J_{11,0}^a$. These limits are related to important physical quantities in the charge sector: the first one to the compressibility ($K = \partial n/\partial \mu = -J_{00,\infty}^c$) and the second one to the conductivity via the Einstein relation. The total charge (and spin) conservation cannot fully determine the physical behavior of the system.

We now proceed with the vertex functions. The Ward identity for the charge and spin vertices can be derived from the equation of motion of the Green's function

$$F_{\alpha,\sigma;\mu}^a(\mathbf{p}, t'; \mathbf{q}, t) = -\langle \tau j_\mu^a(\mathbf{q}, t) a_{\alpha,\sigma}(\mathbf{p} - \mathbf{q}/2, t') a_{\alpha,\sigma}^\dagger(\mathbf{p} + \mathbf{q}/2, t')\rangle.$$

Differentiating $F_{\alpha,\sigma;0}^a$ with respect to t and using the continuity equation (59), one obtains

$$i\partial_t F_{\alpha,\sigma;0}^a(\mathbf{p}, t'; \mathbf{q}, t) = -\langle \tau \mathbf{q} v^a j^a a a^\dagger\rangle - \delta(t-t')\langle [\rho^a, a a^\dagger]\rangle,$$

where, for simplicity, we have omitted the dependence on the variables. After transforming the previous relation to frequency representation, we have

$$\omega F_{\alpha,\sigma;0}^a(p, q) - v^a \mathbf{q} F_{\alpha,\sigma;1}^a(p, q) = \epsilon_\sigma^a [G_{\alpha,\sigma}(p - q/2) - G_{\alpha,\sigma}(p + q/2)],$$

where $\epsilon_\sigma^a = -1$ if $a = s$ and $\sigma = \downarrow$, and $\epsilon_\sigma^a = +1$ otherwise. If we introduce the vertex function $\Gamma_{\alpha,\sigma;\mu}^a(p, q) = F_{\alpha,\sigma;\mu}^a(p, q)|_{\rm tr} = F_{\alpha,\sigma;\mu}^a/[G_{\alpha,\sigma}(p - q/2) G_{\alpha,\sigma}(p + q/2)]$, obtained by truncating the external fermion lines in a diagrammatic representation, we have the following Ward identity

$$\omega \Gamma_{\alpha,\sigma;0}^a(p, q) - v^a \mathbf{q} \Gamma_{\alpha,\sigma;1}^a(p, q) = \epsilon_\sigma^a [G_{\alpha,\sigma}^{-1}(p + q/2) - G_{\alpha,\sigma}^{-1}(p - q/2)].$$

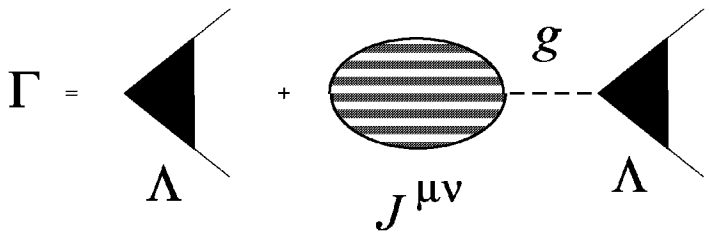

FIGURE 9. The Dyson equation for Γ^a.

The vertex functions $\Gamma^a_{\alpha,\sigma;\mu}$ are reducible with respect to cutting single interaction lines in a diagrammatic representation. The above equations are valid both for left- and right-moving particles. The skeleton structure of the g-ology model is obtained in terms of the *irreducible* charge and spin vertices, $\Lambda^a_{\alpha,\sigma;\mu} = (\Gamma^a_{\alpha,\sigma;\mu})^{irr}$, with respect to cutting an interaction line. The vertex $\Lambda^a_{\alpha,\sigma;\mu}$ is related to $\Gamma^a_{\alpha,\sigma;\mu}$ by the Dyson equation

$$\Gamma^a_{\alpha,\sigma;\mu}(p,q) = \Lambda^a_{\alpha,\sigma;\mu} + J^a_{\mu\nu}(q) g^a_{\nu\nu} \Lambda^a_{\alpha,\sigma;\nu}(p,q),$$

which is illustrated diagrammatically in Fig. 9. The coupling constants are given by $g^a_{00} = (g^a_4 + g^a_2)/2$ and $g^a_{11} = (g^a_4 - g^a_2)/2$. Using the Ward identity for $J^a_{\mu\nu}$ and $\Gamma^a_{\alpha,\sigma;\mu}$ yields

$$\begin{aligned}
&\omega \Gamma^a_{\alpha,\sigma;0}(p,q) - v^a \mathbf{q} \Gamma^a_{\alpha,\sigma;1}(p,q) \\
&\qquad = \omega \Lambda^a_{\alpha,\sigma;0}(p,q) - v^a \mathbf{q} \Lambda^a_{\alpha,\sigma;1}(p,q) + \frac{2}{\pi} \mathbf{q} g^a_{11} \Lambda^a_{\alpha,\sigma;1}(p,q) \\
&\qquad = \omega \Lambda^a_{\alpha,\sigma;0}(p,q) - v_F \mathbf{q} \Lambda^a_{\alpha,\sigma;1}(p,q) \\
&\qquad = \epsilon^a_\sigma [G^{-1}_{\alpha,\sigma}(p - q/2) - G^{-1}_{\alpha,\sigma}(p + q/2)],
\end{aligned} \tag{62}$$

where we have used the expression (60) for v^a. Note that the form of this latter identity is independent of the coupling, i.e., it involves v_F instead of v^a. Total charge and spin conservation also holds in Fermi liquids, and so do the Ward identities following from these properties. The Ward identity (62) relates the density and the current vertex functions to the single-particle Green's function. Its validity is generic to all systems conserving the total number of particles and, by itself, it is not enough to solve any model, and in particular the Luttinger model in $d = 1$.

2 Ward Identities from Left and Right Conservation Laws

In addition to the usual total charge and spin conservation, the discrete structure of the Fermi surface in $d = 1$ allows for more stringent conservation laws when large momentum scattering processes are absent ($g_1 = g_3 = 0$), i.e., in the Luttinger model: *charge (or spin) near the left and right Fermi points is conserved separately.* This separate conservation guarantees, even in the presence of singularities in perturbation theory, the finiteness of charge- (or spin-) density response

in normal one-dimensional metals, and the velocities associated to the corresponding currents provide a complete parametrization of the low-energy physics of the model.

In the Luttinger model, let us consider the charge (or spin) difference operator $\tilde{\rho}^a(\mathbf{q}) = \rho^a_+(\mathbf{q}) - \rho^a_-(\mathbf{q})$. These, hereafter referred to as axial densities, are now conserved quantities and obey the continuity equation

$$i\partial_t \tilde{\rho}^a(\mathbf{q},t) = [\tilde{\rho}^a, H] = \mathbf{q}\tilde{v}^a \tilde{j}^a(\mathbf{q},t),$$

with the axial current operator given by $\tilde{j}^a(\mathbf{q}) = \rho^a_+(\mathbf{q}) + \rho^a_-(\mathbf{q})$, where $\tilde{v}^a = v_F + (g^a_4 + g^a_2)/\pi$. Note that

$$\tilde{\rho}^a(\mathbf{q}) = j^a(\mathbf{q}), \quad \tilde{j}^a(\mathbf{q}) = \rho^a(\mathbf{q}).$$

In complete analogy with the preceding section, we can define the axial correlation functions $\tilde{J}^a_{\mu\nu}$ and the vertices $\tilde{\Lambda}^a_{\alpha,\sigma;\mu}$. The continuity equation for the axial bi-current $\tilde{j}^a_\mu = (\tilde{\rho}^a, \tilde{j}^a)$ implies the Ward identities analogue to Eqs. (61) and (62), with v^a substituted by $\tilde{v}^a$ and ϵ^a_σ by $\tilde{\epsilon}^a_{\alpha,\sigma} = \alpha\epsilon^a_\sigma$.

Combining the Ward identities from global and axial conservation laws one completely determines the correlation functions $J^a_{\mu\nu}$ and $\tilde{J}^a_{\mu\nu}$ in terms of v^a and $\tilde{v}^a$. This yields an expression of the vertices as functions of the propagator G.

The results concerning physical quantities will be discussed in the next section where we calculate the density-density correlation function. Before ending this section, we want to discuss an important consequence of separate charge (and spin) left and right conservation. In this case the density vertex $\Lambda^a_{\alpha,\sigma;0}$ and the current vertex $\Lambda^a_{\alpha,\sigma;1}$ are simply related

$$\Lambda^a_{\alpha,\sigma;1} = v_F \alpha \Lambda^a_{\alpha,\sigma;0}.$$

Therefore the Ward identity, Eq. (62), reads

$$(\omega - \alpha v_F \mathbf{q})\Lambda^a_{\alpha,\sigma;0} = \epsilon^a_\sigma \left[G^{-1}_{\alpha,\sigma}(p+q/2) - G^{-1}_{\alpha,\sigma}(p-q/2)\right], \tag{63}$$

which now involves the density vertex only. The density vertex can now be eliminated in favor of the single-particle Green's function.

E The Luttinger Liquid as the Solution of the Luttinger Model

1 Density-Density Correlation Function

We discuss first the left and right charge-density correlation function. It is defined by

$$J^c_{\alpha\beta;00} \equiv C_{\alpha\beta}(\mathbf{q},t) = -\frac{i}{V}\langle \tau \rho^c_\alpha(\mathbf{q},t)\rho^c_\beta(-\mathbf{q},t)\rangle.$$

We can proceed either by using the Ward identities for C which follow from right and left conservation laws, or from the Ward identity (63) for the vertex Λ, by considering its effect on the polarization bubble. We follow here this second procedure. C can be expressed in terms of the polarization function $\Pi^c_{\alpha\alpha}(\mathbf{q},\omega)$ by the resummation of the geometric series analytically given by the Dyson equation

$$C_{\alpha\beta}(q) = \Pi^c_{\alpha\beta}(q)\delta_{\alpha\beta} + \Pi^c_{\alpha\alpha}(q)\sum_\gamma g^c_{\alpha\gamma}C_{\gamma\beta}(q), \tag{64}$$

where $g^c_{++} = g^c_{--} = g^c_4$ and $g^c_{+-} = g^c_{-+} = g^c_2$, and we have used the condition $\Pi^c_{\alpha,-\alpha} = 0$. The charge polarization bubble can be written in terms of the irreducible charge-density vertex Λ^c and of the exact propagator G, as shown in Fig. 10,

$$\Pi^c_{\alpha\alpha}(q) = -i\sum_\sigma \int \frac{d^2k}{(2\pi)^2}\Lambda^c_{\alpha,\sigma;0}(k,k+q;q)G_{\alpha,\sigma}(k)G_{\alpha,\sigma}(k+q).$$

Π^c contains in principle singular contributions due to vertex and self-energy corrections. However, the Ward identity (63) for Λ^c implies that

$$\Pi^c_{\alpha\alpha}(q) = \frac{-i}{\omega - \alpha v_F \mathbf{q}}\sum_\sigma \int \frac{d^2k}{(2\pi)^2}\left[G_{\alpha,\sigma}(k) - G_{\alpha,\sigma}(k+q)\right].$$

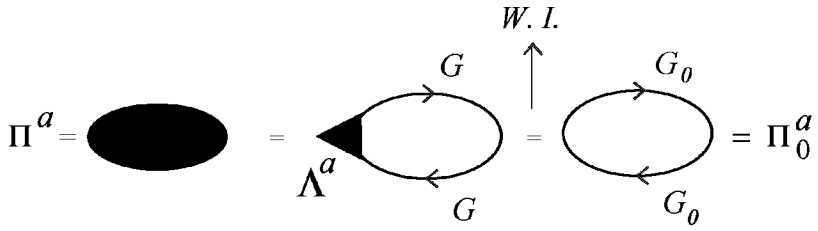

FIGURE 10. The Dyson equation for Π^a; Λ^a represents the bare density vertex ($a = c$ in the case under discussion).

Integrating over frequencies (remembering that all allowed momenta are restricted by a cutoff, $|\mathbf{k}|, |\mathbf{k}+\mathbf{q}| < \Lambda$) one obtains, for $\mathbf{q} > 0$,

$$\Pi^c_{\alpha\alpha}(q) = \frac{1}{\omega - \alpha v_F \mathbf{q}}\sum_\sigma \int_{-\Lambda}^{\Lambda-q}\frac{d\mathbf{k}}{2\pi}\left[n_{\alpha,\sigma}(\mathbf{k}) - n_{\alpha,\sigma}(\mathbf{k}+\mathbf{q})\right],$$

where $n_{\alpha,\sigma}$ is the momentum distribution function of the interacting system. Shifting the integration variable in the second term by $-\mathbf{q}$, one obtains the same expression as for the noninteracting system

$$\Pi^c_{\alpha\alpha}(q) = \frac{\alpha}{\pi}\frac{\mathbf{q}}{\omega - \alpha v_F \mathbf{q}}. \tag{65}$$

All vertex and self-energy corrections have cancelled each other, due to the peculiar conservation laws, and the density-density response is given by the simple RPA series, with the appropriate coupling constants ($g^c_{++} = g^c_{--} = g^c_4$, $g^c_{+-} = g^c_{-+} = g^c_2$), and with the bare bubble

$$C(q) = \sum_{\alpha\beta} C_{\alpha\beta}(q) = \frac{2v^c}{\pi}\frac{|\mathbf{q}|^2}{\omega^2 - (u^c|\mathbf{q}|)^2},$$

which correctly vanishes as $|\mathbf{q}| \to 0$, at finite ω.

The density-density response has poles at $\omega = \pm u^c \mathbf{q}$, where $u^c = (v^c \tilde{v}^c)^{1/2}$. A similar result holds also for the correlation function of the spin density with $u^s = (v^s \tilde{v}^s)^{1/2}$. The Luttinger-liquid collective modes are therefore an undamped charge-density mode (zero sound) and an undamped spin-density mode, with velocity u^c and u^s, respectively. The compressibility $K = \partial n/\partial \mu = -C(|\mathbf{q}| \to 0, \omega = 0)$ is now fully determined

$$K = \frac{2}{\pi \tilde{v}^c}.$$

In the same way, the spin susceptibility is derived from the spin-spin correlation function, and reads

$$\chi = \frac{2}{\pi \tilde{v}^s}.$$

By using the generalized Einstein relation one can derive the electrical conductivity $\sigma(\omega)$ for the Luttinger liquid. The Drude weight is determined by v^c,

$$\mathrm{Re}\sigma(\omega) = 2v^c \delta(\omega).$$

Let us finally note that the continuity equations for ρ^a and $\tilde{\rho}^a$ can be combined into a harmonic-oscillator equation for $\rho^a(\mathbf{q}, t)$, i.e.,

$$\partial_t^2 \rho^a(\mathbf{q}, t) + v^a \tilde{v}^a |\mathbf{q}|^2 \rho^a(\mathbf{q}, t) = 0,$$

describing undamped harmonic oscillations with frequency $u^a|\mathbf{q}|$. The low-temperature specific heat of the Luttinger liquid is therefore given by

$$C_V = \frac{\pi}{6}\frac{u^c + u^s}{u^c u^s} T,$$

and is linear in temperature, as in the Fermi liquid, but with different coefficients, coming from the collective modes rather than from the fermion quasiparticles.

2 The Single-Particle Green's Function

The most dramatic change of the Luttinger-liquid solution with respect to the Fermi liquid is in the single-particle Green's function. It can be calculated in the bosonization approach [31,33]. We use here the Ward identity (63), derived in Sec. IV D via the separate charge and spin conservation at each Fermi point. The Ward identities obtained from the continuity equation associated to these conservation laws give a complete system of equations for any correlation function, and in particular we have discussed the explicit calculation of the density-density correlation function. The same results are implied by another important property of the Luttinger model, the loop cancellation (see, e.g., Ref. [20]). For small momentum transfer Feynman diagrams involving fermion loops with more than two insertions cancel each other. This cancellation has been first exploited for the Luttinger model in Ref. [16]. As a consequence of loop cancellation, the effective interaction (including polarization) and response functions for small $\mathbf{q}$ are given by the RPA resummation. Indeed, in general, the bubbles shown in Fig. 11 must be dressed by the self-energy and vertex corrections, however, as we have shown for the density-density correlation function, only the bare bubble remains. Indeed, these corrections involve fermion loops with more than two insertions, and must therefore cancel each other for small $\mathbf{q}$ and cutoff Λ, at least at leading order in these small parameters.

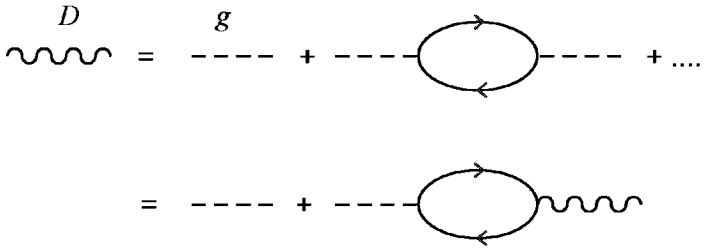

FIGURE 11. The effective interaction D.

The effective interaction D for small q is obtained by resummation of the diagrams illustrated in Fig. 11:

$$D^{\sigma\sigma'}_{\alpha\alpha'}(q) = g^{\sigma\sigma'}_{\alpha\alpha'} + \sum_{\alpha'',\sigma''} g^{\sigma\sigma''}_{\alpha\alpha''}\Pi^{c}_{\alpha''\alpha''}(q) D^{\sigma''\sigma'}_{\alpha''\alpha'}(q),$$

with $g^{\sigma\sigma'}_{\alpha\alpha} = g^{\sigma\sigma'}_{4}$, $g^{\sigma\sigma'}_{\alpha,-\alpha} = g^{\sigma\sigma'}_{2}$, and $\Pi^{c}_{\alpha\alpha}$ given by Eq. (65).

The effective interaction (for $\alpha = \alpha'$) $D(q)$ contains all the information of the charge and spin collective modes via their velocities u_a, and can be cast in the form

$$D(q) = (\omega - \alpha v_F \mathbf{q})\pi \sum_{a=c,s} \left[\frac{(2-\eta^a)(u^a - v_F)}{\omega - u^a \mathbf{q}} + \frac{\eta^a (u^a + v_F)}{\omega + u^a \mathbf{q}} \right], \tag{66}$$

where $\eta^a = (K^a - 2 + 1/K^a)/4$, and $K^a = (v^a/\tilde{v}^a)^{1/2}$.

Once D is known, since the density vertex can be eliminated via the Ward identity (63), the Dyson equation for the single-particle propagator (Fig. 12) becomes a closed equation for G [16]

$$(\omega - \alpha v_F \mathbf{p}) G(p) = 1 + i \int \frac{d\mathbf{q} d\omega'}{4\pi^2} \frac{D(p-q)G(q)}{\omega - \omega' - \alpha v_F(\mathbf{p}-\mathbf{q})}.$$

The solution of the above integral equation can be obtained by transforming it to real space and time in the form

$$G(\mathbf{r},t) = \mathrm{e}^{L(\mathbf{r},t)-L(0,0)} G_0(\mathbf{r},t),$$

where $L(\mathbf{r},t)$ is the Fourier transform of $iD(q)[\omega - \alpha v_F \mathbf{q} + i0^+ \mathrm{sgn}(\omega)]^{-2}$, and $G_0(\mathbf{r},t) = (1/2\pi)[|\mathbf{r}| - v_F t + i0^+ \mathrm{sgn}(t)]^{-1}$. Using the expression (66) for D, $L(\mathbf{r},t)$ behaves logarithmically in $|\mathbf{r}|$ and t. The logarithmic expression appears at the exponent in the expression for G, giving rise to a power-law behavior with anomalous exponents depending on g_2 and g_4 via v^a and $\tilde{v}^a$. For large $|\mathbf{r}|$ and/or large t, one thus finds [13]

$$G(\mathbf{r},t) = \frac{1}{2\pi\Lambda^\eta} \prod_{a=c,s} \frac{1}{[|\mathbf{r}| - u^a t + i\mathrm{sgn}(t)/\Lambda]^{1/2+\eta^a/2}} \frac{1}{[|\mathbf{r}| + u^a t - i\mathrm{sgn}(t)/\Lambda]^{1/2+\eta^a/2}},$$

where Λ is the momentum-transfer cutoff, and $\eta = \eta^c + \eta^s$.

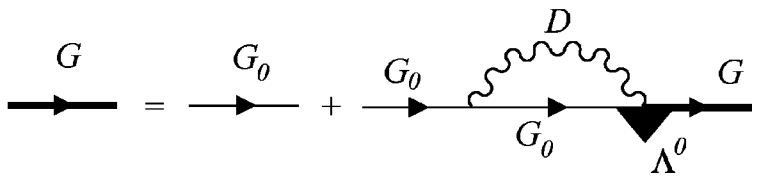

FIGURE 12. Dyson equation relating the dressed and bare fermion propagators G, and G_0, the effective interaction D, and the irreducible vertex Λ^0.

In conclusion, in $d = 1$, all the scattering processes with small momentum transfer are marginal and the corresponding Luttinger model has been solved exactly via the Ward identities which, in addition to the total charge and spin conservation, specify the separate left and right conservation at each Fermi point. Due to this additional Ward identities, no sign of the infrared divergences is left in the response functions, giving rise to finite compressibility, spin susceptibility, Drude peak in the conductivity, and a linear-in-T specific heat.

The main anomalies appear in the single-particle propagator, with a power-law behavior and a vanishing wave-function renormalization Z when approaching the Fermi points. The single-particle propagation is realized in a complex way, with

velocities related to charge and spin collective modes and with vanishing low-energy spectral weight. The wave-function renormalization is the only true renormalization left, after the Ward identities have been applied.

We have not dealt with the properties of the charge and spin collective modes, and of the singlet and superconducting fluctuations, for momenta close to $2k_F$. For the sake of completeness we recall [13] that the corresponding correlation functions are characterized by a power-law long-distance (and long-time-interval) decay. The exponents of the power-law behavior can be expressed through K^c and K^s. These power-law behviors translate into a power-law dependence on T at finite temperature. Therefore, the Luttinger liquids are "critical" systems, although a symmetry breaking is forbidden by the one-dimensional character of the fluctuations. Predominantly repulsive ($g_2 > 0$) or attractive ($g_2 < 0$) Luttinger liquids have thus a tendency towards a spin- (charge-) density-wave, or triplet- (singlet-) superconducting instability, respectively.

V NON-FERMI-LIQUID BEHAVIOR IN HIGHER DIMENSIONS AND SINGULAR INTERACTIONS

The attempt to describe the anomalous properties of the normal phase of HTS has motivated a search for metallic non-Fermi-liquid phases in $d > 1$, and in particular in $d = 2$ [69,71]. In this section we address the issue of non-Fermi-liquid metals in $d > 1$, emphasizing in particular the role of Ward identities. The problem of finding metallic non-Fermi-liquid fixed points can be addressed in various ways: i) by considering a system of chains coupled by an interchain-hopping $t_\perp$ much smaller than the intrachain hopping [26–28]; ii) by formally taking an isotropic system in dimension $1 \leq d < 2$, d being an arbitrary real number, and trying to extend the solution for $d = 1$ to $d > 1$ [19,20]; iii) by considering singular interactions and long-range forces [21–23].

A Multichain Model

In view of explaining the anomalous normal state of the HTS, an important question is the possible extension of the Luttinger liquid to higher dimensions. As a first attempt to study the evolution of the Luttinger liquid with increasing dimensionality a system of Luttinger chains coupled by a $t_\perp$ interchain hopping was considered [27]. The expectation was that by first solving the problem of strong correlation in reduced dimensionality, and then introducing a weak interchain hopping as a perturbation, a Luttinger-liquid-like behavior would persist down to temperatures (or to energies, at $T = 0$) of the order of a renormalized $t_\perp^R < t_\perp$.

The transverse hopping in the y direction, treated as a perturbation with respect to the Luttinger-liquid solution, can be written in a mixed real-momentum-space

representation as

$$H_\perp = -t_\perp \sum_l \sum_{k_x,\sigma} (c^\dagger_{l,k_x\sigma} c_{l+1,k_x\sigma} + c^\dagger_{l,k_x\sigma} c_{l-1,k_x\sigma}),$$

where $c^\dagger_{l,k_x\sigma}(c_{l,k_x\sigma})$ creates (annihilates) an electron with spin orientation σ on the chain l with momentum k_x along the chain. The linear response of the momentum distribution function $n_{\mathbf{k}\sigma}$ to a transverse-hopping perturbation is

$$\delta n_{\mathbf{k}\sigma} = -i \int_0^\infty \langle [n_{\mathbf{k}\sigma}(t), H_\perp] \rangle dt = -4t_\perp \cos(k_y) \int_0^\infty \mathrm{Im}[G(k_x,t)G(k_x,-t)]dt,$$

where G is the exact propagator of a one-dimensional Luttinger liquid with momentum distribution $n_{\mathbf{k}\sigma} = n_{k_F} - A\mathrm{sgn}(|k_x| - k_F)||k_x| - k_F|^\eta$. Therefore

$$\begin{aligned} &\delta n(k_x, k_y) \sim t_\perp \cos(k_y)||k_x| - k_F|^{2\eta-1}, \quad \eta < 1, \\ &\text{i.e.,} \;\; t^R_\perp \sim t_\perp ||k_x| - k_F|^{\eta-1}. \end{aligned} \tag{67}$$

As shown in Eq. (67), $t_\perp$ is a relevant perturbation if $\eta < 1$ (in the repulsive Hubbard model $\eta < 1/8$), in the sense that it drives the system away from the Luttinger-liquid fixed point, independently of the value of the interaction which leads to the Luttinger-liquid solution in reduced dimensionality. Thinking in terms of standard scaling concepts, the results (67), are in agreement with dimensional analysis [26,81], and with an explicit two-loop RG study [28] for Hubbard repulsion $U < U_c \simeq 12t_\parallel$[11].

The introduction of $t_\perp$ on Luttinger chains is not a good procedure for studying possible non-Fermi-liquid systems in $d > 1$, since i) the regime $t_\perp \ll t_\parallel$ is too anisotropic for most realistic systems; ii) the system flows to strong coupling out of the Luttinger-liquid line of fixed points; iii) as soon as $t_\perp \neq 0$ the number of effective coupling constants increases and the system goes out of control.

The alternative approach would be to extend the exact procedure from $d = 1$ to $d > 1$ via bosonization in $d = 2$ [36,82] or via Ward identities [18–20], as we now discuss.

B Tomographic Luttinger Model

We want to determine at which dimension the Luttinger liquid turns into a Fermi liquid, i.e., the dimensional crossover from Luttinger liquid to Fermi liquid. To this purpose, we can generalize to $d > 1$ the method proposed by Dzyaloshinskii and Larkin [16], by showing how the additional Ward identities can still be used asymptotically near the Fermi surface, provided $1 \leq d < 2$.

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[11] In this section $t_\parallel$ is the hopping term along the chains. In Ref. [28] it is found that $t_\perp$ scales to zero for $U > U_c$. However, this parameter range falls outside the region of validity of the calculations.

In going from $d = 1$ to $d = 2$ many more new coupling constants than in the g-ology model appear, although the low-energy properties of the metallic phase are still determined by the excitation close to the Fermi surface. For small momentum-transfer cutoff Λ, constraints imposed by momentum conservation in the narrow shell around the Fermi surface introduce strong simplifications and lead to a generalization of the g-ology model to $d > 1$. Consider the model given by $H = H_0 + H_I$, where

$$H_0 = \sum_{k\sigma} v_F k_r a_\sigma^\dagger(\mathbf{k}) a_\sigma(\mathbf{k})$$

and

$$H_I = \frac{1}{V} \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}} \sum_{\sigma\sigma'} g_{\mathbf{k}\mathbf{k}'}^{\sigma\sigma'}(\mathbf{q}) a_\sigma^\dagger(\mathbf{k}+\mathbf{q}) a_\sigma(\mathbf{k}) a_{\sigma'}^\dagger(\mathbf{k}'-\mathbf{q}) a_{\sigma'}(\mathbf{k}'),$$

with transferred momentum $\mathbf{q} < \Lambda \ll k_F$. Here k_r is the distance of $\mathbf{k}$ from the Fermi surface, $k_r = |\mathbf{k}| - k_F$. The model represents an effective Hamiltonian for the excitations near the Fermi surface and we assume that $g_{\mathbf{k}\mathbf{k}'}^{\sigma\sigma'}$ is a slowly varying function of $\mathbf{k}$ and $\mathbf{k}'$ on the scale set up by Λ. As $\mathbf{k}$ and $\mathbf{k}'$ are both near the Fermi surface, the scattering processes are severely restricted. We consider, for example, a spherical Fermi surface in $d = 2$. For incoming and outgoing particles strictly on the Fermi surface, there are three distinct scattering processes, which can be parametrized by a single angle each

forward (F) scattering: $\mathbf{k_1} = \mathbf{k_1'}$ and $\mathbf{k_2'} = \mathbf{k_2}$,

exchange (E) scattering: $\mathbf{k_1'} = \mathbf{k_2}$ and $\mathbf{k_2'} = \mathbf{k_1}$,

Cooper (C) scattering: $\mathbf{k_1} + \mathbf{k_2} = \mathbf{0}$.

They are depicted in Figure 13. Forward and exchange scattering can be parametrized in terms of the angle between the momenta of the incoming particles $\theta = \angle(\mathbf{k_1}, \mathbf{k_2})$, while Copper scattering is parametrized by the angle defined by the momentum transfer $\phi = \angle(\mathbf{k_1}, \mathbf{k_1'})$. The asymptotic behavior is therefore described by three functions $g_F(\theta), g_E(\theta)$ and $g_C(\phi)$ as a generalization of the g-ology model in $d = 1$, where only two angles (0 and π) exist, with the following correspondence

$$\begin{aligned} g_F(0) &= g_4, \;\; g_F(\pi) = g_2, \\ g_E(0) &= g_4, \;\; g_F(\pi) = g_1, \\ g_C(0) &= g_2, \;\; g_C(\pi) = g_1. \end{aligned}$$

Since we want to discuss a normal metallic state, we do not consider the Cooper channel here. In a Fermi liquid, g_F plays a special role, giving rise to Landau's quasiparticle interactions [65]. For parallel spins, exchange and forward scattering are asymptotically indistinguishable, while for opposite spins exchange scattering generalizes the spin backscattering process in $d = 1$, which renormalizes to zero for repulsive interactions. For our purpose, we would like to ignore both g_C and g_E. In $d = 2$, for a system with dominant forward scattering the angle between particles

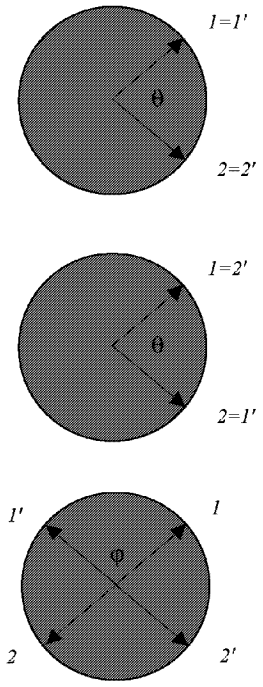

FIGURE 13. Scattering processes on the Fermi surface in $d = 2$: forward, exchange, and Cooper scattering, from top to bottom. The numbers $1, 1', 2, 2'$ label the momenta, according to the convention adopted in the text.

involved in low-lying scattering processes is asymptotically conserved. This is the generalization to each point of the Fermi surface of left and right charge and spin conservation, determining the behavior of the Luttinger liquid in $d = 1$, and allows to describe the low-energy properties of the system in terms of a tomographic Luttinger model, i.e., a collection of one-dimensional Luttinger models, each labelled by the angle θ.

C Dimensional Crossover

The extension of the theory to non-integer dimension $1 \leq d < 2$ is obtained, as usual, by analytic continuation of the Feynman diagrams, defined for arbitrary d, to the complex d plane. It is sufficient to continue momentum integrals of functions $f(\mathbf{k})$ which depend on $\mathbf{k}$ only via the modulus $|\mathbf{k}|$ and the angle θ, between $\mathbf{k}$ and another fixed momentum. In these cases one can use

$$\int d^d\mathbf{k} f(|\mathbf{k}|, \mathbf{p}\cdot\mathbf{k}) = S_{d-1} \int d|\mathbf{k}|\, |\mathbf{k}|^{d-1} \int_0^\pi d\theta\, (\sin\theta)^{d-2} f(|\mathbf{k}|, |\mathbf{p}||\mathbf{k}|\cos\theta), \qquad (68)$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface of the d-dimensional unit sphere. In the limit $d \to 1$ one has $S_{d-1} \simeq d-1$, and thus $S_{d-1}(\sin\theta)^{d-2} \to \delta(\theta) + \delta(\theta - \pi)$. The steps leading in $d = 1$ to the additional Ward identities of Sec. IV D, and in particular to Eqs. (62),(63) are no longer strictly valid when $d > 1$. However, the geometric restriction at small $\mathbf{q}$ and the condition $d < 2$ make both equations valid asymptotically for small exchanged momentum, since the typical integrals in Eq. (68) are peaked at $\theta = 0, \pi$. In particular, since almost all relevant $\mathbf{k}$ vectors are still parallel or antiparallel, asymptotically near the Fermi surface we can still write

$$\mathbf{\Lambda}(p;q) \simeq v_F \hat{\mathbf{p}} \Lambda^0(p;q), \tag{69}$$

where $\mathbf{\Lambda}$ and Λ^0 are the current and density vertex, respectively, and $\hat{\mathbf{p}} = \mathbf{p}/p_F$. Asymptotically we have, therefore, the same Ward identity for the density vertex, which controls the infinite resummation of possibly relevant terms, leading to the Luttinger liquid in $d = 1$,

$$\Lambda^0 = \frac{G^{-1}(p+q/2) - G^{-1}(p-q/2)}{\omega - v_F \hat{\mathbf{p}} \cdot \mathbf{q}}. \tag{70}$$

Eq. (70) is valid apart from a correction in the denominator, which vanishes as $d \to 1$, and is small for $|\mathbf{q}| < \Lambda \ll k_F$.

By inserting (70) in the Dyson equation, one again obtains the Green's function in a form similar to that found for $d = 1$,

$$G_{\hat{\mathbf{p}}}(\mathbf{r},t) = \mathrm{e}^{L(\mathbf{r},t) - L(0,0)} G_0(\mathbf{r},t), \tag{71}$$

where $L(\mathbf{r},t)$ is the Fourier transform of $iD(q)[\omega - v_F q_r + i0^+\mathrm{sgn}(\omega)]^{-2}$, and $q_r = \hat{\mathbf{p}}\cdot\mathbf{q}$ is the radial component of $\mathbf{q}$. The expression for $L(\mathbf{r},t)$ involves now an *angular* average $\bar{D}$ of the effective dynamical interaction $D(q)$,

$$L(r,t) = \frac{1}{(2\pi)^2} \int d\omega \int dq_r \mathrm{e}^{i(q_r|\mathbf{r}| - \omega t)} \frac{i\bar{D}_\Lambda(q_r,\omega)}{[\omega - v_F q_r + i0^+\mathrm{sgn}(\omega)]^2}, \tag{72}$$

where

$$\bar{D}_\Lambda(q_r,\omega) = \frac{S_{d-1}}{(2\pi)^{d-1}} \int_0^{\sqrt{\Lambda^2 - q_r^2}} dq_t q_t^{d-2} D\left(\frac{q_r}{\omega}, \frac{q_t}{\omega}\right). \tag{73}$$

In Eq. (73), we have integrated over the $d-1$ components of the tangent momentum. The low-energy behavior of G is now described in terms of a tomographic Luttinger model [70]. In the limit $d \to 1$ one has $\bar{D} = D$ and one recovers the exact expression for the propagator of the Luttinger model in $d = 1$. In the case we considered (regular coupling, $1 < d < 2$) the effective interaction scales to zero at low-energy since Eq. (73) implies the scaling relation

$$\bar{D}_{s\Lambda}(sq_r, s\omega) = s^{d-1} \bar{D}_\Lambda(q_r,\omega),$$

i.e., $\bar{D}(q_r,\omega) \simeq \omega^{d-1}\bar{D}(q_r/\omega)$. This result clearly illustrates the marginality of small-$\mathbf{q}$ scattering processes in $d = 1$, and their irrelevance in higher dimensions where the effective interaction has to be averaged over the angles. The physical origin of this irrelevance is therefore related to the reduction of the phase space for the scattering of the particles by the collective modes at $d > 1$, where the Landau Fermi-liquid theory is recovered.

Fourier transforming $G(\mathbf{r}, 0)$ yields the momentum distribution $n_{\mathbf{k}}$ near the Fermi surface. A signature of the validity of the Fermi-liquid theory is given by a finite discontinuity $\Delta n_{\mathbf{k}}$ at the Fermi surface, which vanishes only if $d \to 1$. The quasiparticle inverse lifetime is proportional to k_r^d $(d < 2)$, and is therefore anomalous when compared to the standard k_r^2 term present for $d > 2$. However, it is not sufficient to destroy the Fermi-liquid quasiparticle features as soon as $d > 1$. These results, which rely on Ward identities, are reproduced for $d = 2$ by various authors [35,36], by extending the bosonization approach to $d = 2$, along the line proposed by Haldane [34].

The above results show that the mechanism for non-Fermi-liquid behavior in $d = 1$ does not extend to $d > 1$ (specifically to $d = 2$). The small-$\mathbf{q}$ scattering processes which generalize those leading to the Luttinger-liquid behavior in $d = 1$, can still be controlled by asymptotic Ward identities, and do not destroy the Fermi liquid, as soon as $d > 1$. A non-Fermi-liquid behavior in $d = 2$ must therefore have a different and more subtle origin than in $d = 1$. For example, strongly singular interaction can produce such an effect, as we discuss in Sec. V E.

D Bosonization

For completeness, in this section we give a brief description of the bosonization procedure to treat systems whose low-energy physics is dominated by forward scattering. It consists in expressing the fermion creation and annihilation operators in terms of boson density-fluctuation operators. This technique has been originally invoked to analyze one-dimensional systems [30–33]. A generalization to higher dimension has been pioneered by Haldane [34] and elaborated in detail for $d = 2$ in Refs. [35,36], and later in Refs. [82,83]. The basic conclusion is that the results obtained by bosonization procedure are equivalent to those obtained by the asymptotic Ward identities.

Let us first assume spinless fermion for simplicity. The main idea behind the bosonization approach in $d = 1$ is that the particle-hole excitation created by the operator $a^\dagger_{\mathbf{k}+\mathbf{q}}a_{\mathbf{k}}$ has energy $\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}} = v_F\mathbf{q}$, independent of $\mathbf{k}$. This gas of bosons accounts for the free-particle part of the Luttinger Hamiltonian for the right- $(\alpha = +)$ and left- $(\alpha = -)$ moving particles as

$$H_0 = \sum_{\mathbf{q}\alpha} \alpha v_F \mathbf{q} b^\dagger_\alpha(\mathbf{q}) b_\alpha(\mathbf{q}),$$

where $b^\dagger_\alpha(\mathbf{q})$ and $b_\alpha(\mathbf{q})$ satisfy the bosonic commutation relation

$$[b_\alpha(\mathbf{q}), b^\dagger_{\alpha'}(\mathbf{q}')] = \delta_{\alpha,\alpha'}\delta_{\mathbf{q},\mathbf{q}'}.$$

The formal correspondence between the density operators $\rho_\alpha(\mathbf{q})$, whose commutators are given as in $d = 1$ by Eq. (58), and the boson operators is established via the relation

$$\rho_+(\mathbf{q}) = \sqrt{\frac{V|\mathbf{q}|}{2\pi}} b_+(\mathbf{q}), \quad \mathbf{q} > 0,$$
$$\rho_+(\mathbf{q}) = \sqrt{\frac{V|\mathbf{q}|}{2\pi}} b^\dagger_+(-\mathbf{q}), \quad \mathbf{q} < 0, \tag{74}$$

and similarly

$$\rho_-(\mathbf{q}) = \sqrt{\frac{V|\mathbf{q}|}{2\pi}} b^\dagger_-(-\mathbf{q}), \quad \mathbf{q} > 0,$$
$$\rho_-(\mathbf{q}) = \sqrt{\frac{V|\mathbf{q}|}{2\pi}} b_-(\mathbf{q}), \quad \mathbf{q} < 0. \tag{75}$$

The term H_4 of the Luttinger model is also simply expressed in terms of $b^\dagger b$. The term H_2 has instead the bilinear anomalous form

$$H_2 = \frac{g_2}{2\pi} \sum_{\mathbf{q}} |\mathbf{q}| \left[b^\dagger_+(\mathbf{q}) b^\dagger_-(-\mathbf{q}) + b_-(-\mathbf{q}) b_+(\mathbf{q}) \right].$$

The full Hamiltonian can be diagonalized by a Bogoliubov transformation, leading to the collective charge modes. In presence of spin variables we would have a correspondence between ρ^a_α and the boson operators b^a_α, and the charge ($a = c$) and spin ($a = s$) sectors would decouple with charge and spin modes propagating with velocities v^a, leading to the charge and spin separation.

We now briefly recall the extension of the bosonization approach to $d > 1$. The main idea of bosonization in dimension higher than one is a decomposition of momentum space in disjoint sectors K_α, $\alpha = 1, \ldots, M$, which consists in a partition of the Fermi surface into patches, the area of each patch being Λ^{d-1}_α. The condition $\Lambda_\alpha \ll k_F$ ensures that the Fermi surface is almost flat and the velocity is constant within each patch. The condition for constant velocity in the patch box is the same as the velocity conservation required by the Ward identities at each point of the Fermi surface. As in $d = 1$, the basic ingredient in the bosonization procedure is the density-fluctuation operator, that now is defined for each patch α as

$$\rho_\alpha(\mathbf{q}) = \sum_{\mathbf{k} \in \text{patch } \alpha} a^\dagger_{\mathbf{k}+\mathbf{q}} a_{\mathbf{k}}.$$

For $|\mathbf{q}| \ll \Lambda_\alpha$, the above operators obey commutation relations [34,35] which generalize Eq. (58),

$$[\rho_\alpha(\mathbf{q}), \rho_{\alpha'}(\mathbf{q}')] \simeq \frac{V}{(2\pi)^d} \Lambda^{d-1}_\alpha \delta_{\alpha\alpha'} \delta_{\mathbf{q},-\mathbf{q}'} (\mathbf{n}_\alpha \cdot \mathbf{q}),$$

where $\Omega_\alpha = V\Lambda_\alpha^{d-1}/(2\pi)^d$ is the number of states in the patch box divided by the size Λ in the radial direction, and $\mathbf{n}_\alpha$ is a unit vector normal to the patch. The density fluctuation operator $\rho_\alpha(\mathbf{q})$ can be related to boson operators $b_\alpha(\mathbf{q})$ and $b_\alpha^\dagger(\mathbf{q})$ by the following generalization of Eqs. (74) and (75),

$$\rho_\alpha(\mathbf{q}) = (\Omega_\alpha|\mathbf{n}_\alpha \cdot \mathbf{q}|)^{1/2}[\Theta(\mathbf{n}_\alpha \cdot \mathbf{q})b_\alpha(\mathbf{q}) + \Theta(-\mathbf{n}_\alpha \cdot \mathbf{q})b_\alpha^\dagger(-\mathbf{q})],$$

Θ being the characteristic function.

The main point of the bosonization procedure is that also in $d > 1$, if we consider forward scattering with momentum transfer restricted by $|\mathbf{q}| \ll \Lambda_\alpha$, the interaction Hamiltonian can still be written in a quadratic form in terms of the density operators. Owing to the quadratic structure of the bosonized representation of the Hamiltonian, the dynamics of the operators ρ_α can be calculated exactly. The result in $d = 2$ is the same as that obtained by the Ward identities presented before. In particular, using the bosonic representation of fermion operators, the single-particle propagator G_α can be expressed in terms of the expectation value of the boson (density) operator for particles belonging to a small patch α of the Fermi surface. G_α coincides with the expression (71) derived via Ward identity, when calculated for $d = 2$. The only difference is that, instead of the radial variable, a vector is now present in Eq. (72). The bosonization approach, indeed, introduces an overcomplete set of fermion fields. The Ward-identity approach avoids instead the patch construction with artificial overcompletness of states and interpatch scattering processes. Moreover the limitations of the bosonization approach in $d = 2$ are not so transparent as in the Ward-identity approach in $d = 2$. Indeed, in $d = 2$, the generic integrals (68) in momentum space are no longer peaked at $\theta = 0, \pi$. The relation between the current vertex and the density vertex and its consequences are therefore much less evident, when extended to $d = 2$. The dimensional crossover from Luttinger to Fermi liquid as $d > 1$ derived via Ward identities is instead a firm result.

E Singular Interactions and Long-Range Forces

In this section we consider singular scattering as a possible source of non-Fermi-liquid behavior in $d > 1$. Singular scattering is quite plausible to occur in strongly correlated fermion systems [72]. In ordinary metals the kinetic-energy term is strong and ensures a homogeneous phase. However, in the presence of strong correlation due, e.g., to local repulsion, the kinetic-energy term is strongly suppressed. In this way the homogenizing contribution of the kinetic term may become insufficient against the effect of forces of different nature (magnetic, nearest-neighbour repulsion, etc.) which give rise to phase separation [84]. On the other hand, long-range Coulomb interactions, by forbidding charge unbalance on a macroscopic scale, prevent the thermodynamic phase separation [85]. However, Coulomb interactions being strongly effective at small $\mathbf{q}$, may lead to an instability at finite wave vectors $\mathbf{q} = \mathbf{q}_c$, with strong dynamic charge fluctuations. Informations about such

an instability are provided by the quasiparticle scattering amplitude $\Gamma(\omega, \mathbf{q})$. Near phase separation in the absence of long-range Coulomb forces, Γ is strongly singular at zero momentum. When long-range forces are present, Γ is anisotropic, and is still strongly singular, although at a finite momentum $\mathbf{q} = \mathbf{q}_c$, which signals an incommensurate charge-density-wave instability [72]. A similar instability could be present in the magnetic sector, giving rise to strong magnetic fluctuations [73]. In both cases (short- and long-range forces) the singular interactions may lead to a breakdown of the Fermi-liquid picture in the normal phase. Anomalous transport properties of the metal arise in a region determined by a crossover temperature, which depends in general on the density of charge carriers.

The previous analysis for the effective interaction D, can be pursued in the presence of static long-range forces with spin independent singular coupling $g(\mathbf{q}) = g_0^2/|\mathbf{q}|^\alpha$, which corresponds to a $|\mathbf{r}|^{-(d-\alpha)}$ behavior in real space [22]. One could expect that long-range forces change the scaling dimension of $\bar{D}$ from $d-1$ to $d-1-\alpha$. However, summing up the RPA series for the effective (screened) dynamical interaction, the leading contribution in the asymptotic regime $|\mathbf{q}|/\omega \to 0$, which is relevant to the present problem, is found to be

$$D(\omega, \mathbf{q}) = \frac{g_0^2 \omega^2}{|\mathbf{q}|^\alpha (\omega^2 - C^2 |\mathbf{q}|^{2-\alpha})}, \tag{76}$$

where C is a constant which depends on the dimension d. From Eq. (76) one sees that the long-range forces change the dispersion of the collective mode from a sound mode $\omega_q \propto |\mathbf{q}|$ to a *plasma* mode $\omega_q \propto |\mathbf{q}|^{1-\alpha/2}$, changing the the dimension of ω from 1 to $1-\alpha/2$ and reducing the contribution from the poles to the integrand for L in Eq. (72). The scaling dimension for $L(\mathbf{r}, t)$ is now $d-1-\alpha/2$. The outcome is that the breakdown of the Fermi liquid occurs only if the long-range interaction is strongly singular, $\alpha \geq 2d-2$ [21,22]. From this considerations it comes out that in $d=2$ the Coulomb interaction $g(\mathbf{q}) \propto 1/|\mathbf{q}|$ is not singular enough to produce a non-Fermi-liquid behavior.

F Renormalization-Group Approach in the Presence of Singular Forward Scattering

The result discussed in Sec. V E is based on the assumption that the RPA resummation leading to the dynamically dressed effective interaction D is still valid in the presence of a singular interaction. This amounts to say that (singular contributions associated with) self-energy and vertex corrections to the polarization bubble must cancel. This cancellation can be controlled by means of the Ward identities, associated with the specific conservation laws of a system with singular forward scattering, within a perturbative RG approach.

To develop the RG approach in the case of long-range interaction [23], we exploit

the equivalence with a problem of fermions interacting with a *scalar* boson field ϕ

$$H_I = ig_0 \int \frac{d^d\mathbf{k}d^d\mathbf{q}d\omega d\epsilon}{(2\pi)^{2d+2}} \bar{\psi}(\mathbf{k}+\mathbf{q}/2;\epsilon+\omega/2)\phi(\mathbf{q};\omega)\psi(\mathbf{k}-\mathbf{q}/2;\epsilon-\omega/2),$$

where the RPA propagator of the field ϕ is given by $D^{LR} = D/g_0^2$, with D given by Eq. (76), in the region $\omega \gg v_F|\mathbf{q}|$. Here $C = g_0\sqrt{v_F S_d k_F^{d-1}/d(2\pi)^d}$, and $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface of the d-dimensional unit sphere. If $\omega \ll v_F|\mathbf{q}|$ the effective interaction assumes the usual short-ranged expression due to the screening by the particle-hole fluctuations. Here we restrict to consider the regime $\omega \gg v_F|\mathbf{q}|$, in the case $\alpha < 2$ where the collective plasma mode is propagating and gapless.

Within a functional-integral formulation, by eliminating the fermions, one obtains an effective action for the field ϕ

$$\mathcal{S}_{\text{eff}}[\phi] = \frac{1}{2}\int \frac{d^d\mathbf{q}d\omega}{(2\pi)^{d+1}} \phi^*[D^{LR}]^{-1}\phi + \sum_{n\geq 3}\int \Gamma_n \phi^n,$$

where the first term in the right-hand side is the RPA quadratic part, and the interacting part is symbolically represented by the second term, where the n-th contribution contains $n-1$ integrals over momenta and frequencies.

The dimensional analysis yields the scaling dimensions $[q] = 1$, $[\omega] = 1-\alpha/2$, $[\phi] = -(2d+2+\alpha)/4$, and $[\Gamma_n] = d+1+[(3n-2)\alpha - 2n(d+1)]/4$. Γ_n seems therefore to be relevant for $d < (3n-2)\alpha/(2n-4) - 1$. However, performing a gauge transformation $\phi \to \mathbf{A} = -\mathbf{q}\phi/\omega$ from a scalar to a vector field, one is able to make the momentum dependence explicit and show that, for $n \geq 3$, Γ_n vanishes at least as $\Gamma_n \sim (|\mathbf{q}|/\omega)^n$ in the dynamical regime $|\mathbf{q}|/\omega \to 0$, which is the dominant regime in the present problem [23]. The expression for the ϕ-field propagator is therefore not changed by corrections beyond RPA. This amounts to say that the field ϕ does not acquire an anomalous dimension with respect to its Gaussian form, and the wave-function renormalization Z_ϕ remains finite.

We are thus left with the problem of renormalizing the fermion Green's function G and the fermion-boson vertex. We use conservation laws to establish relations among the various singularities of the theory, thus reducing the number of independent renormalization parameters. The first conservation law to be exploited is the conservation of the total number of particles which is generically valid. The corresponding Ward identity, which is the analog of Eq. (62) for $a = c$, then reads

$$\begin{aligned}&\mathbf{q}\cdot\Lambda(\mathbf{k},\mathbf{q};\epsilon,\omega) - \omega\Lambda^0(\mathbf{k},\mathbf{q};\epsilon,\omega) = \\ &G^{-1}(\mathbf{k}-\mathbf{q}/2,\epsilon-\omega/2) - G^{-1}(\mathbf{k}+\mathbf{q}/2,\epsilon+\omega/2),\end{aligned} \tag{77}$$

which is valid both for the bare and for the fully renormalized vertices and Green's functions, and allows to relate the renormalization parameters of the vertices with those of the propagator.

Since frequency and momentum are inequivalent variables in the present problem, we introduce two different renormalization parameters in the fermion propagator

$G^{-1}(\mathbf{k},\epsilon) \equiv \epsilon - \xi_{\mathbf{k}} - \Sigma(\mathbf{k},\epsilon) = Z^{-1}\epsilon - (ZZ_{v_F})^{-1}\xi_{\mathbf{k}}$, where $\xi_{\mathbf{k}} \equiv (|\mathbf{k}|^2 - k_F^2)/2m \simeq v_F(|\mathbf{k}| - k_F)$ is the free-particle dispersion and Σ is the fermion self-energy. The renormalization parameters are

$$Z^{-1} = 1 - \frac{\partial\Sigma}{\partial\epsilon},$$
$$(ZZ_{v_F})^{-1} = 1 - \frac{1}{v_F}\frac{\partial\Sigma}{\partial|\mathbf{k}|},$$

and have to be related with the singular parts of the current and density vertices in the left-hand side of Eq. (77). In particular, one has to extract the coefficient of $\mathbf{q}$ in the right-hand side of (77) and compare it with the static limit of the current vertex in the left-hand side. In that way, defining the renormalization of the current vertex Λ in the static limit, $Z_{stat}^{-1}\mathbf{v}_F \equiv \Lambda(\mathbf{k}, \mathbf{q} \to \mathbf{0}; \epsilon, \omega = 0)$, one finds

$$Z_{stat} = ZZ_{v_F}.$$

To extract the divergences of the frequency part of the propagator, one defines the dynamical limit of the density vertex $Z_{dyn}^{-1} = \Lambda^0(\mathbf{k}, \mathbf{q} \to \mathbf{0}; \epsilon, \omega \to 0)$, and similarly finds

$$Z_{dyn} = Z. \tag{78}$$

As we show below, the renormalization which is relevant for the long-range case is Eq. (78).

The second conservation law to be considered is the conservation of the number of particles at each point of the Fermi surface, which is obviously approximate in $d > 1$ and holds if forward scattering is dominant at low energies as in the tomographic Luttinger model. The associated Ward identity, as in Eq. (69), is

$$\Lambda(\mathbf{k}, \mathbf{q}; \epsilon, \omega) = \mathbf{v}_F\Lambda^0(\mathbf{k}, \mathbf{q}; \epsilon, \omega).$$

Assuming the above approximate equation and combining it with Eq. (77) we can once again write down the equation that expresses the density vertex as a function of the single-particle propagator only. Finally, as it was shown in Sec. IV, by inserting this into the expression for the polarization bubble, the latter does not get dressed, since the vertex corrections exactly cancel the self-energy corrections, i.e., the validity of the RPA expression for the boson propagator is under control.

The renormalization parameters Z and Z_{v_F} are calculated through a perturbative expansion of the fermion self-energy Σ, and are defined in terms of Σ at a proper normalization point λ in momentum space. It can be shown that the one-loop self-energy contribution due to the long-range singular interaction is such that in the dynamical limit

$$\frac{1}{v_F}\left|\frac{\partial\Sigma}{\partial|\mathbf{k}|}\right| \simeq \left|\frac{\partial\Sigma}{\partial\epsilon}\right|,$$

i.e., the derivative of Σ with respect to $\mathbf{k}$ and ϵ have the same singular behavior in the infrared [23]. Therefore, the renormalization of the momentum part of fermion propagator is equal to the renormalization of the frequency part, $ZZ_{v_F} = Z$, i.e., Z_{v_F} is finite and the Fermi velocity does not acquire singular corrections, as in the one-dimensional Luttinger model.

When considering the general expression for the fermion self-energy

$$\Sigma = ig \int \Lambda^0 D^{LR} G,$$

one realizes that the most singular contribution to this integral comes from the pole of the boson propagator $\omega^2 \simeq |\mathbf{q}|^{2-\alpha}$, i.e., $\omega \gg |\mathbf{q}|$. We introduce, therefore the renormalization parameter in the relevant kinematic regime

$$Z_{dyn}^{-1} = \Lambda^0(\mathbf{k}, \mathbf{q} = 0; \epsilon, \omega \to 0).$$

The renormalization of the coupling constant g is now

$$g_0 = \frac{g Z_{dyn}}{Z},$$

which, using Eq. (78), becomes $g_0 = g$, i.e., the divergences in the vertex and self-energy diagrams cancel out thanks to the Ward identity coming from the conservation of the total number of particles. In conclusion, due to Ward identities, the Fermi velocity, the boson field, and the coupling constant do not get renormalized,

At one-loop level, one determines the effective coupling constant which appears in the skeleton structure of the perturbative expansion. The effective coupling is specified by observing that at the dominant pole $\omega = C|\mathbf{q}|^{1-\alpha/2}$, the energy of the internal fermion propagator is $\epsilon + C|\mathbf{q}|^{1-\alpha/2}$. But, since at small momenta and for $\alpha < 2$,

$$\xi_{\mathbf{k}+\mathbf{q}} - \xi_{\mathbf{k}} \ll C|\mathbf{q}|^{1-\alpha/2},$$

one can neglect the contribution from $\xi_{\mathbf{k}+\mathbf{q}}$ when computing the contribution to Σ. As a consequence, the effective coupling coupling constant coincides with g^2 apart from a multiplicative factor coming from the integration in the diagram. Defining $u_0 \propto g_0^2$, the one-loop calculation of the self-energy [23] leads to the one-loop wave-function renormalization parameter

$$Z^{-1}(\epsilon) = 1 + u_0 \frac{2-\alpha}{2+\alpha-2d} |\epsilon|^{-\frac{2+\alpha-2d}{2-\alpha}}. \tag{79}$$

Since this quantity must be dimensionless, we can compute the scaling dimension of u_0, thus obtaining, for the dimensionless coupling constant, the expression

$$u = \frac{u_0}{\lambda^{x_u}}, \tag{80}$$

with $x_u = \frac{2+\alpha-2d}{2-\alpha} \times [\omega] = 1 - d + \alpha/2$. Thus u does not acquire any anomalous dimension. Eq. (80) shows that a line of non-universal fixed points exist only for

$x_u = 0$, i.e., for $d = d_c \equiv 1 + \alpha/2$, while for $x_u > 0$ or $x_u < 0$ the effective coupling constant either scales to strong coupling or to the free fixed point $u^* = 0$. In the strong-coupling and non-universal fixed-point cases one obtains a non-Fermi-liquid behavior in agreement with the result discussed in Sec. V E.

From the one-loop result, Eq. (79), we can obtain a differential equation for Z, $\dot{Z} = -(1 - \alpha/2)uZ$, which can be integrated, taking Eq. (80) into account. We find therefore

$$Z \sim \exp\left[-(2-\alpha)u_0|\epsilon|^{-(2+\alpha-2d)/(2-\alpha)}/(2+\alpha-2d)\right],$$

i.e., Z vanishes as a stretched exponential, for $d < d_c$, and

$$Z \sim |\epsilon|^{u_0},$$

i.e., Z vanishes with a non-universal exponent for $d = d_c$. We point out that, due to the absence of an anomalous dimension for the effective coupling constant, the solution of the RG equation for Z corresponds to the exponentiation of the one-loop result, Eq. (79) [23].

VI INTERACTING BOSONS AT ZERO TEMPERATURE

In this section we discuss the application of RG to interacting Bose systems at zero temperature, which represents a further example of a stable liquid phase of the matter, with singular perturbation theory. As anticipated in Sec. I, and applied in the context of one-dimensional fermion systems in Secs. IV and V, the use of Ward identities provides a powerful tool to control the divergences, and show that they cancel exactly in all physical quantities.

The development of the theory of interacting Bose systems is motivated by the fact that, due to the small atomic mass, and to the weakness of the interactions, helium remains liquid below the degeneracy temperature, where quantum effect become relevant. At $T = T_\lambda \simeq 2.17$K the Bose liquid ^{4}He undergoes a transition to a superfluid phase. The phenomenon of superfluidity is a consequence of the form of the spectrum of the low-lying excitation, according to the Landau criterion, which is based on quasiparticle energy and momentum conservation. The transition itself is instead attributed to the Bose-Einstein condensation, as suggested by the estimate of the transition temperature for a Bose gas, $T_{BE} = 6\hbar^2 n^{2/3}/m$, which is of the correct order of magnitude $\sim T_\lambda$ when the density n and the mass m are taken as appropriate to ^{4}He. The transition is then characterized by the appearance of a non-zero expectation value of the Bose field $\langle\psi\rangle = \langle\psi^\dagger\rangle = \sqrt{n_0} > 0$. One of the major problem in condensed matter physics during the '50s and the '60s was therefore the description of the elementary excitation of an interacting Bose system, in the presence of the condensate.

The first solution to this problem was achieved by Bogoliubov [86] in 1947, within a generalized Hartree-Fock approximation. The condensate density n_0 is assumed to coincide with the particle density n, and the interaction term, containing four boson fields, is dealt with in mean field, by replacing two of them with their mean value $\sqrt{n_0}$ in all possible ways, generating an anomalous self-energy, proportional to $\langle\psi\psi\rangle = \langle\psi^\dagger\psi^\dagger\rangle = n_0$. This approximation leads to the correct expression for the sound velocity c_0 in terms of the compressibility of the system, and to a linear (sound-like) spectrum at low momenta and energy, as required by the Landau superfluidity condition. The first attempt to improve the Bogoliubov theory, including the effect of the depletion of the condensate within the so-called pairing approximation, led to a spurious gap in the excitation spectrum [87], together with the non-vanishing anomalous self-energy. A series of exact results proved that the gap in the excitation spectrum must vanish (Hugenoltz-Pines theorem) [88], and that the anomalous self-energy is identically zero [89].

All the attempts to improve the Bogoliubov approximation encountered the problem of facing infrared divergences due to the presence of the Bose-Einstein condensate and of the Goldstone mode associated with the global broken symmetry [90,91]. The problem of a singular perturbation theory was first recognized by Gavoret and Nozières [91], posing the question of the validity of perturbation theory. The infrared divergences appear in the intermediate steps of the calculations, while they must cancel out in all physical quantities [90], since the Bose superfluid is a stable phase of the matter. A way to take care of the infrared divergences (in $d > 1$, which is the necessary condition for the existence of the condensate at $T = 0$) is the RG approach, together with the exploitation of the underlying local gauge symmetry and of the related Ward identity. This allows to show that singularities do cancel exactly in all physical quantities, that the excitation spectrum is linear, in agreement with the Hugenoltz-Pines theorem, and that the anomalous self-energy vanishes. This approach to obtain the solution of the problem at zero temperature, which reduces to the Bogoliubov result for $d > 3$, was employed in Ref. [24], and we discuss it in the following. Ward identities for a Bose system were previously partially exploited in Refs. [91,92]. The Wilson-like RG approach for bosons in $d = 3$ was also discussed in Ref. [93]. Here we follow Ref. [24].

We start from the real-time zero-temperature action of a Bose system in d spatial dimensions

$$S = \int dt \int d^d\mathbf{x}\Big\{\psi^*(x)(i\partial_t + \mu)\psi(x) - |(\nabla - i\mathbf{A})\psi(x)|^2 - \frac{v}{2}|\psi(x)|^4 + \psi(x)\eta^*(x) + \psi^*(x)\eta(x)\Big\}\,, \tag{81}$$

where $\psi(x)$ is a boson field, $x \equiv (t, \mathbf{x})$, $\eta(x)$ and $(\mu(x), \mathbf{A}(x)) = A(x)$ are external sources to obtain, by functional derivatives, the order-parameter, density, and current correlation functions. At the end of the calculation μ recovers the constant value of the chemical potential while η and $\mathbf{A}$ are set equal to zero. We have taken units such that the mass of the boson is $m = 1/2$. Finally the interaction potential is taken to be short-ranged in real space (i.e., constant in momentum space). The

generating functional associated with the action (81) is $F = \log \int D(\psi, \psi^*) \exp(iS)$. In the case of spontaneously broken symmetry, it is convenient to introduce the notation $\psi(x) = \psi_l(x) + i\psi_t(x)$ and $\psi^*(x) = \psi_l(x) - i\psi_t(x)$, to distinguish between the longitudinal and transverse components along the broken-symmetry direction (the order parameter $\psi_{l0} = \langle \psi_l(x) \rangle_{\eta=0}$ is assumed to be real). By differentiating the generating functional F one obtains the connected correlation functions as the wave function of the condensate, $\psi_{i0} = \langle \psi_i(x) \rangle = -i\delta F/\delta \eta_i(x)$ and the single-particle Green's function $G_{ij} = -\delta^2 F/\delta\eta_i \delta\eta_j$ with $i, j = l, t$ and we defined η_l, η_t through $\eta = \eta_l + i\eta_t$, $\eta^* = \eta_l - i\eta_t$. Introducing the Legendre transform of F with respect to η_i, $\Gamma = i \int dx \eta_l \psi_{l0} - F$, one obtains the vertex functions associated with the one-particle irreducible diagrams, where the functional derivatives with respect to A_ν generate the composite density (current) vertices $\Gamma_{i_1 \dots i_n; \nu_1 \dots \nu_m} = (-i)^{n+m} \delta^{(n+m)} \Gamma / \delta\psi_{i_1 0} \dots \psi_{i_n 0} \delta A_{\nu_1} \dots \delta A_{\nu_m}$. In the broken-symmetry phase one keeps the value of the condensate fixed ψ_{l0} and introduces the fluctuation fields $\tilde{\psi}_i$ with vanishing averages in the absence of external sources, such that

$$\psi_l(x) = \psi_{l0} + \tilde{\psi}_l,$$
$$\psi_t(x) = \tilde{\psi}_t.$$

Within mean field, taking into account the mean-field condition $\psi_{l0} \equiv \sqrt{n_0} = \sqrt{\mu/v}$, the quadratic part of the action (81) leads to the single-particle Green's functions $\mathcal{G}_{tt} \simeq c_0^2(\omega^2 - c_0^2 \mathbf{k}^2)^{-1}$, $\mathcal{G}_{lt} \simeq \omega(\omega^2 - c_0^2 \mathbf{k}^2)^{-1}$, $\mathcal{G}_{ll} \simeq \mathbf{k}^2(\omega^2 - c_0^2 \mathbf{k}^2)^{-1}$, where $c_0 = \sqrt{2\mu}$ is the mean-field sound velocity, and the expressions are approximated at small momenta and energies. At this point one could set up the usual perturbation expansion in which the quadratic action is treated as free and the rest as a perturbation. The resulting perturbation theory is affected by infrared divergences, due to the presence of the sound Goldstone mode, already at one-loop level for $d \leq 3$. A RG treatment is then required. Since perturbation theory generates (divergent) corrections to the mean-field quadratic part of the action (81), we generalize the inverse Green-function matrix to include all its possible renormalizations

$$\mathcal{G}^{-1} = \begin{pmatrix} v_{ll} + z_{ll}\mathbf{k}^2 - u_{ll}\omega^2 & v_{lt} + iw_{lt}\omega \\ v_{lt} - iw_{lt}\omega & v_{tt} + z_{tt}\mathbf{k}^2 - u_{tt}\omega^2 \end{pmatrix}, \tag{82}$$

where the bare (mean-field) values are $v_{ll}^0 = 2\mu \neq 0$, $z_{ll}^0 = z_{tt}^0 = w_{lt}^0 = 1$, and $v_{tt}^0 = u_{ll}^0 = u_{tt}^0 = v_{lt}^0 = 0$.

Apart from the quadratic terms, one also introduces the running coupling constants, for the cubic (three-leg) and the quartic (four-leg) interaction vertices, (v_{ttt}, v_{ltt}), etc.) and (v_{tttt}, v_{ltttt}), etc.) respectively. We point out that the cubic terms only appear in the presence of the condensate.

The RG treatment requires a preliminary power counting for the running coupling constants. The mean-field sound velocity c_0 is assumed dimensionless from the point of view of scaling (see also below) and may be reabsorbed by redefining

momenta, frequencies, and Bose fields, suitably multiplied by powers of c_0. Therefore, we take units such that $[k] = [\omega] = 1$, as suggested by the bare single-particle Green's functions. The most singular bare propagator is $\mathcal{G}_{tt}$, with $[\mathcal{G}_{tt}] = -2$, followed by $\mathcal{G}_{lt}$, with $[\mathcal{G}_{lt}] = -1$, and by $\mathcal{G}_{ll}$, with $[\mathcal{G}_{ll}] = 0$. Thus we have that the bare dimensions of the fields are

$$[\psi_l(x)] = (d+1)/2,$$
$$[\psi_t(x)] = (d-1)/2.$$

The bare dimensions of the longitudinal and transverse fields differ by one since in the free action a quadratic term in the longitudinal field appears (a finite mass v_{ll}^0 is present), whereas the transverse field only appears through the square of its gradient ($v_{tt}^0 = 0$), i.e., the mass of the transverse mode vanishes, since this is the Goldstone mode.

z_{ll} and u_{ll} are irrelevant. For $d \leq 3$ there are nine running coupling constants controlling the infrared behavior. They have bare dimensions $[v_{ll}] = [w_{lt}] = [u_{tt}] = [z_{tt}] = 0$ (marginal), $[v_{lt}] = 1$, $[v_{tt}] = 2$, $[v_{ttt}] = 1+\epsilon/2$ (strongly relevant), $[v_{ltt}] = \epsilon/2$ and $[v_{tttt}] = \epsilon$, where $\epsilon = 3 - d$. For instance, $[v_{tttt}] - d - 1 + 4[\psi_t(x)] = 0$ gives the previous result.

Although v_{lt}, v_{tt}, and v_{ttt} would be strongly relevant, they vanish identically for vanishing external sources, as we show below. The continuity equation will, in the present case, allow for the elimination of two coupling constants.

As already mentioned, in presence of a stable phase, we expect the singular perturbation theory to result in finite response functions. On the other hand, the cancellation of infrared divergences implies a connection between the various running coupling constants. We shall identify three running variables in terms of physical quantities, which indeed remain finite. We are finally left with only one running coupling, whose closed equation can be solved as for the case of the Luttinger model in $d = 1$. We now proceed to sketch how all this is achieved by analyzing the Ward identities, which result from the local gauge invariance of the functional Γ, i.e.,

$$\Gamma[A_\nu(x) + \partial_\nu \alpha(x), R_{ij}[\alpha(x)]\psi_j(x)] = \Gamma[A_\nu(x), \psi_i(x)], \tag{83}$$

where $R_{ij}[\alpha(x)] \simeq I + i\alpha(x)\tau_2$ is a rotation matrix by an angle α in the (ψ_l, ψ_t) space, I is the identity matrix, τ_2 is the Pauli matrix, and we wrote the explicit expression for R_{ij} at small α, which is needed to derive the Ward identities. Eq. (83) follows from the invariance of the action (81) under the gauge transformation $\psi(x) \to \mathrm{e}^{i\alpha(x)}\psi(x), \eta(x) \to \mathrm{e}^{-i\alpha(x)}\eta(x)$ and $A_\nu(x) \to A_\nu(x) + \partial_\nu \alpha(x)$, where $\alpha(x)$ is a real function. By taking the derivatives of the functional (83) with respect to α, ψ_i and A_ν one obtains an infinite set of Ward identities. The following Ward identities are relevant for $k_\nu \to 0$

$$\Gamma_{lt}(0)\psi_{l0} + \Gamma_t(0) = 0,$$
$$\Gamma_{tt}(0)\psi_{l0} - \Gamma_l(0) = 0. \tag{84}$$

Since $\psi_{l0} \neq 0$ in the superfluid phase, as the external sources $\Gamma_t(0) = \eta_t$ and $\Gamma_l(0) = \eta_l$ vanish, $\Gamma_{lt}(0)$ and $\Gamma_{tt}(0)$, which are directly connected to v_{lt} and v_{tt} must vanish. By inspection of Eq. (82) one realizes that this implies that no gap appears in the single-particle spectrum, in agreement with the Hugenoltz-Pines theorem [88].

Other three relations are established via Ward identities, specifying the continuity equation, which in the presence of the condensate connects 3- and 4-point vertices to the 2-point vertices. These Ward identities, for vanishing external momenta and frequencies, and taking into account the vanishing of $\Gamma_{lt}(0), \Gamma_{tt}(0)$, read

$$\Gamma_{ltt}(0,0)\psi_{l0} - \Gamma_{ll}(0) = 0, \tag{85}$$

$$\Gamma_{ttt}(0,0)\psi_{l0} = 0, \tag{86}$$

$$\Gamma_{tttt}(0,0,0)\psi_{l0} - 3\Gamma_{ltt}(0,0) = 0. \tag{87}$$

The first two equations imply $v_{ltt}\psi_{l0} = v_{ll}$ and $v_{ttt} = 0$, while the last one yields $v_{tttt}\psi_{l0} = 3v_{ltt}$.

We are left with four marginal running coupling constants, namely, $v_{ll} = -\partial^2\Gamma/\partial\psi_{l0}^2, w_{lt}, u_{tt}$ and z_{tt}, whose infrared behavior can be determined exactly, owing to the identification of the renormalization parameters with physical quantities. In particular, w_{lt}, u_{tt} and z_{tt}, which are related to the frequency and wave-vector dependence of Γ_{lt} and Γ_{tt}, can be identified in terms of physical quantities, by exploiting the full Ward identities in terms of frequency and wave vector, generalizations of Eqs. (85), (86), and (87), to include now the derivatives with respect to μ also. As the scaling parameter $s \to 0$, one obtains $z_{tt} \to 2n_s/n_0$, $v_{ll}/w_{lt} \to -2n_0/(dn_0/d\mu)_\eta$, $v_{ll}z_{tt}/(v_{ll}u_{tt} + w_{lt}^2) \to c^2 = 2n_s/(dn/d\mu)_\eta$, where n_s is the superfluid density, c is the sound velocity, and $(dn_0/d\mu)_\eta$ is the condensate compressibility. By the very stability of the boson system, all these quantities are free from infrared divergences in the limit of vanishing sources.

The RG equations at one-loop level reproduce this situation with the following asymptotic behavior for v_{ll} as $s \to 0$: $v_{ll} \sim s^\epsilon \to 0$, for $\epsilon > 0$ (i.e., $d < 3$), $v_{ll} \sim -1/\log(s) \to 0$, for $\epsilon = 0$ (i.e., $d = 3$), whereas v_{ll} is finite for $\epsilon < 0$ (i.e., $d > 3$), where the Bogoliubov result $v_{ll} = 2\mu$ is recovered.

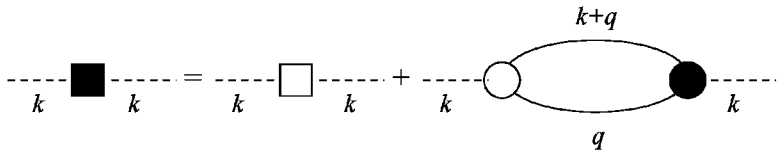

FIGURE 14. Equation for the most singular part of Γ_{ll} (filled square). The empty square represents the bare interaction v_{ll}^0, whereas the empty and black circles represents v_{ltt}^0 and Γ_{ltt}, respectively. The external dashed lines, which represent longitudinal fluctuations, are amputated, and are only drawn to indicate the corresponding ingoing and outgoing momenta. The solid lines represent the propagator of transverse fluctuations, $\mathcal{G}_{tt}$.

The behavior $v_{ll} \sim s^\epsilon$, or $\Gamma_{ll} \sim k^\epsilon$, is again fixed exactly by the Ward identities

which allow to close the equation for Γ_{ll}. The true asymptotic behavior of $\Gamma_{ll}(s)$ follows by expressing the most singular part of Γ_{ll} in terms of the exact Γ_{ltt} and $\mathcal{G}_{tt}$ (see Fig. 14)

$$\Gamma_{ll}(k) = v_{ll}^0 - \frac{v_{ltt}^0}{2} \sum_q \mathcal{G}_{tt}(q)\mathcal{G}_{tt}(q+k)\Gamma_{ltt}(q,-q-k). \tag{88}$$

The Ward identity (85) allows one to express Γ_{ltt} in terms of Γ_{ll} thus closing Eq. (88) and providing the exact asymptotic behavior $\Gamma_{ll} \simeq k^{\epsilon}$, since the loop of integration over momentum and frequency, with two transverse Green's functions, gives a kernel going as $k^{-\epsilon}$.

We point out that, although the spectrum is always linear at small momenta at energy, leading to sound-like excitations and superfluidity, the sound mode is realized in completely different ways for $d > 3$ and $d \leq 3$. Indeed, from Eq. (82), once v_{lt} and v_{tt} have been set to zero, and the Hugenoltz-Pines result is recovered, the scale dependent quantity $c^2(s) = v_{ll}z_{tt}/(v_{ll}u_{tt} + w_{lt}^2)$ must be interpreted as the sound velocity for $s \to 0$. In the Bogoliubov case $v_{ll} \equiv 2n_0 v$, $z_{tt} \equiv 1$, $u_{tt} \equiv 0$, and $w_{lt} \equiv 1$, so that the sound velocity $c = \sqrt{v_{ll}}$. In $d \leq 3$, instead, as $s \to 0$, $w_{lt} \sim v_{ll} \to 0$, and the scale-dependent function $c^2(s) \to c^2(0) \equiv z_{tt}(0)/u_{tt}(0) = 2n_s/(dn/d\mu)_\eta$.

In conclusion, Ward identities have been used to reduce the number of running coupling constants to the single v_{ll}, for which a non-linear equation has been solved [24], thus providing the exact infrared behavior of the system, as for the $d = 1$ fermion case. The final results turn out to be quite different from the Bogoliubov result in $d \leq 3$, despite the coincidence of the sound-like spectrum.

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