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The boundary of the deformation space of the fundamental group of some hyperbolic 3–manifolds fibering over the circle

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Abstract By using Thurston’s bending construction we obtain a sequence of faithful discrete representations ρ_n of the fundamental group of a closed hyperbolic 3–manifold fibering over the circle into the isometry group $Iso \mathbf{H}^4$ of the hyperbolic space \mathbf{H}^4 . The algebraic limit of ρ_n contains a finitely generated subgroup F whose 3–dimensional quotient $\Omega(F)/F$ has infinitely generated fundamental group, where $\Omega(F)$ is the discontinuity domain of F acting on the sphere at infinity $S_\infty^3 = \partial\mathbf{H}^4$. Moreover F is isomorphic to the fundamental group of a closed surface and contains infinitely many conjugacy classes of maximal parabolic subgroups.

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1 Introduction and statement of results

By a Kleinian (discontinuous) group G we mean a subgroup of the group $\text{Conf}(\mathbf{S}^n) \cong SO_+(1, n + 1)$ of conformal transformations of $\bar{R}^n = S^n = R^n \cup \{\infty\}$ which acts discontinuously on a non-empty set $\Omega(G) \subset S^n$ called its domain of discontinuity. It may be connected or not; we will say that G is a function group if there is a connected component $\Omega_G \subset \Omega(G)$ that is invariant under the action of the whole group: $G\Omega_G = \Omega_G$. The quotient spaces $M_G = \Omega_G/G$ and $M(G) = \Omega(G)/G$ are n –manifolds in the case in which G is torsion-free. The complement $\Lambda(G) = (S^n \setminus \Omega(G)) \subset \partial\mathbf{H}^{n+1}$ is called the limit set of G .

A finitely generated Kleinian group G is called geometrically finite if for some $\varepsilon > 0$ there exists an ε –neighbourhood of H_G/G in \mathbf{H}^{n+1}/G which is of finite hyperbolic volume. Here $H_G \subset \mathbf{H}^{n+1}$ is the convex hull of $\Lambda(G)$.

Let us consider for $n = 3$ a hyperbolic 3-manifold $M = H^3/\Gamma$ ($\Gamma \subset PSL_2\mathbf{C}$) fibering over the circle S^1 with fiber a closed surface σ . The notation is $M = \sigma \tilde{\times} S^1$. A representation $\rho: \pi_1(M) \rightarrow \text{Conf}(\mathbf{S}^3)$ is called admissible if the following conditions are satisfied.

- (1) $\rho: \Gamma \rightarrow \text{Conf}(\mathbf{S}^3)$ is faithful and $\rho(\Gamma) = \Gamma_0$ is Kleinian.
- (2) ρ preserves the type of each element, ie $\rho(\gamma)$ is loxodromic for all $\gamma \in \Gamma$.
- (3) ρ is induced by a homeomorphism $f_\rho: \Omega(\Gamma) \rightarrow \Omega(\Gamma_0)$, namely $f_\rho \gamma f_\rho^{-1} = \rho(\gamma)$, $\gamma \in \Gamma$.

The set of all admissible representations modulo conjugation in $\text{Conf}(\mathbf{S}^3)$ is called the deformation space $\text{Def}(\Gamma)$ of the group Γ .

The set $\text{Def}(\Gamma)$ inherits the topology of convergence on generators of Γ on compact subsets in \mathbf{S}^3 because $\text{Def}(\Gamma) \subset (\text{Conf}(\mathbf{S}^3))^k / \sim$, $k \in \mathbf{N}$ (\sim is conjugation in $\text{Conf}(\mathbf{S}^3)$). As $\text{Def}(\Gamma)$ is a bounded domain [13] two questions have arisen. The first is to describe the cases when $\text{Def}(\Gamma)$ is non-trivial and the second is to study the boundary $\partial \text{Def}(\Gamma)$, as was done for the classical Teichmüller space [2], [10]. The answer to the first question is still unknown even in the case when M is Haken. We will consider the case when M contains many totally geodesic surfaces. Each of them produces a curve in $\text{Def}(\Gamma)$ by Thurston's "bending" construction [19]. Our main interest is in groups which appear on the boundary $\partial \text{Def}(\Gamma)$. These are higher dimensional analogs of B -groups which arise as the limits of sequences of quasifuchsian groups in classical Teichmüller space.

One of the most fundamental questions is to describe the topological type of the orbifold $M(\Gamma) = \Omega(\Gamma)/\Gamma$ (a manifold in the case when Γ is torsion-free), in particular, when Γ is a function group it is important to know when the fundamental group $\pi_1(M_G = \Omega_\Gamma/\Gamma)$ turns out to be finitely generated, or even more generally when it has finite homotopy type.

In dimension 2 the famous theorem of Ahlfors [1] says that a finitely generated non-elementary Kleinian group $G \subset \text{Conf}(\mathbf{R}^2)$ has a factor-space $\Omega(G)/G$ consisting of a finite number of Riemann surfaces S_1, \dots, S_n each having a finite hyperbolic area.

We discovered in [7] that the weakest topological version of Ahlfors' theorem does not hold starting already with dimension 3. Namely we constructed a finitely generated function group $F \subset \text{Conf}(\mathbf{S}^3)$ such that the group $\pi_1(\Omega_F/F)$ is not finitely generated. Afterwards it was pointed out in [15] that this group is in fact not finitely presented.

It has also been shown that there exists a finitely generated Kleinian group with infinitely many conjugacy classes of parabolics [6].

In [14] we constructed a finitely generated group F_1 such that $\pi_1(\Omega_{F_1}/F_1)$ is not finitely generated and having infinitely many non-conjugate elliptic elements; moreover F_1 appears as an infinitely presented subgroup of a geometrically finite Kleinian group in \mathbf{H}^4 without parabolic elements. On the other hand, it was shown in [4] that a finitely generated but infinitely presented group can also appear as a subgroup of a cocompact group in $SO(1,4)$.

Theorem 1 *Let $\Gamma = \pi_1(M)$ be the fundamental group of a hyperbolic 3-manifold M fibering over the circle with fiber a closed surface σ . Suppose that Γ is commensurable with the reflection group R determined by the faces of a right-angular polyhedron $D \subset \mathbf{H}^3$. Then there exists a finite-index subgroup $L \subset \Gamma$ and a path $\beta_t: [0, 1[\rightarrow \text{Def}(\Gamma)$ such that β_t converges to a faithful representation $\beta_1 \in \partial\text{Def}(\Gamma)$ (as $t \rightarrow 1$) and the following hold:*

- (1) $\beta_1(F_L)$ contains infinitely many conjugacy classes of maximal parabolic subgroups,
- (2) $\pi_1(\Omega_{\beta_1(F_L)})/\beta_1(F_L)$ is infinitely generated,

where $F_L = L \cap \pi_1\sigma$ is isomorphic to the fundamental group of a closed hyperbolic surface which finitely covers σ and $\beta_1(F_L)$ acts discontinuously on an invariant component $\Omega_{\beta_1(F_L)} \subset \mathbf{S}^3$.

Remark Groups satisfying all the conditions of Theorem 1 do exist. An example of Thurston, of the reflection group in the faces of the right-angular dodecahedron, which is commensurable with a group of a closed surface bundle, is given in [18].

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2 Outline of the proof

Before giving a formal proof of the Theorem let us describe it informally.

Our construction is inspired essentially by papers [6], [8] and [14]. In the first two a free Kleinian group of finite rank satisfying the conclusion (2) was produced, whereas now we give an example of a closed surface group with this property. Our present construction is essentially easier than that of [14]. Also, we produce a curve in the deformation space whose limit point is the group in question.

Step 1 We start with an uniform lattice $\Gamma \subset PSL_2\mathbf{C}$ commensurable with the reflection group R whose limit set is the Euclidean 2-sphere ∂B_1 – the boundary of the ball $B_1 \subset \mathbf{S}^3$. There exists a Fuchsian subgroup $H_2 \subset \Gamma$ leaving invariant a vertical plane π whose intersection with B_1 is a round circle, its limit set $\Lambda(H_2)$ (see figure 1). The group H_2 also leaves invariant a geodesic plane $w_2 \subset B_1$. Consider the action of the group Γ in the outside ball $B_1^* = \mathbf{S}^3 \setminus B_1$. For some finite-index subgroup Γ_1 of Γ we construct a new group G_1 obtained by Maskit’s Combination theorem from Γ_1 and $\tau_\pi \Gamma_1 \tau_\pi$ combined along the common subgroup $H_2 = \text{Stab } w_2$, where τ_π is the reflection in π . The new group G_1 is still isomorphic to some subgroup $G^* \subset R$ of finite index essentially because the same construction can be done inside B_1 by reflecting the picture along the geodesic plane w_2 . Thus G_1 belongs to the deformation space $\text{Def}(G_1^*)$. One can obtain a fundamental domain $R(G_1) \subset B_1^*$ of G_1 which is situated in a small neighbourhood of the spheres ∂B_1 and $\tau_\pi(\partial B_1)$.

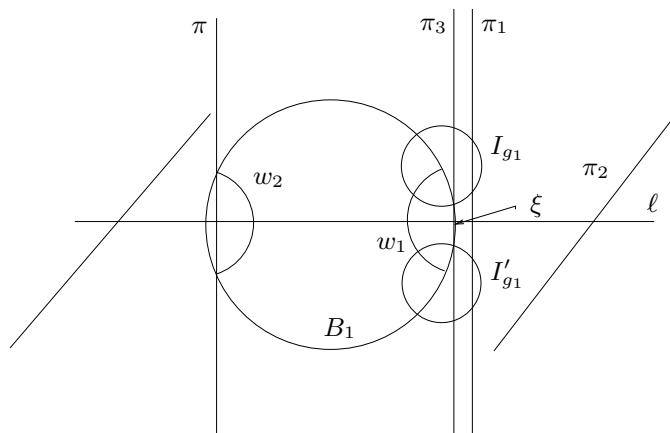


Figure 1

Step 2 There is another geodesic plane $w_1 \subset B_1$ disjoint from w_2 whose stabilizer in Γ_1 is H_1 (see figure 2). Denote by B_2 the ball $\tau_\pi(B_1)$. Take a sphere $\Sigma \subset B_1^*$ passing through the circle $w_3 \cap B_2$ – the limit set of the group $\tau_\pi H_1 \tau_\pi$ – and tangent to the isometric spheres of some element $g_1 \in \Gamma_1$, where H_1 is a subgroup of Γ_1 stabilizing w_1 . We now construct a family of Euclidean spheres Σ_t ($0 \leq t \leq 1$, $\Sigma_1 = \Sigma$) and corresponding groups \mathcal{G}_t obtained as before from G_1 and $\tau_{\Sigma_t} G_1 \tau_{\Sigma_t}$ by using the combination method along common closed surface subgroups. We prove then that there is a path $\beta_t: t \in [0, 1[\mapsto \beta \in \text{Def}(L')$ such that $\beta_0 = L'$, $\beta_t = \mathcal{G}_t$ where L' is some finite-index subgroup of R . One can equally say that β_t is obtained by using Thurston’s bending deformation. The main point is now to prove that the limit

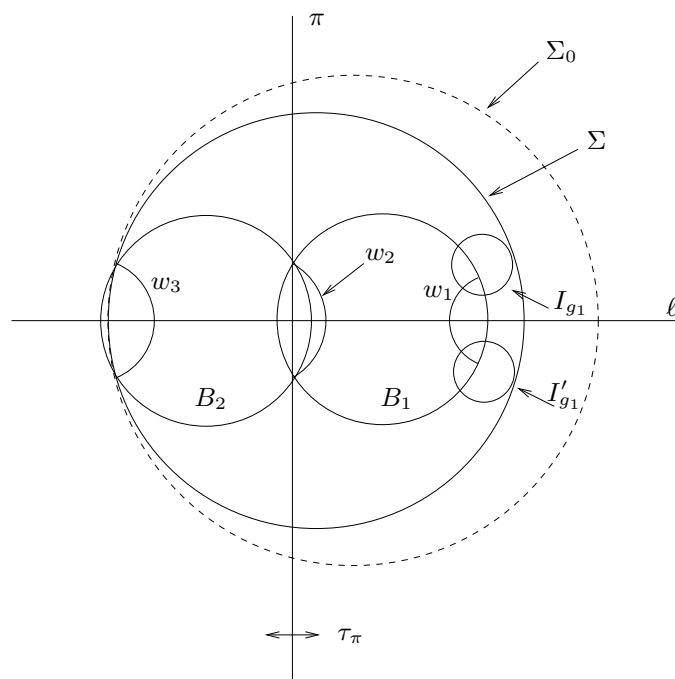


Figure 2

group $\mathcal{G}_1 = \lim_{t \rightarrow 1} \beta_t(L')$ is discontinuous and has a fundamental domain obtained from the part of $R(G_1)$ by doubling along the sphere Σ . The group \mathcal{G}_1 is also isomorphic to L' and so contains a fundamental group \mathcal{N} of a closed surface bundle over the circle which is isomorphic to the group $L = \Gamma \cap L'$. Let \mathcal{F} be the fundamental group of the fiber given by $\beta_1(F_L = F \cap L)$. Since two isometric spheres of the element $g_1 \in \Gamma_1$ are tangent to Σ , we get a new accidental parabolic element $g = g_1 \cdot g_2$, $g_2 = \tau_\Sigma g_1 \tau_\Sigma$ in the group \mathcal{G}_1 . By a choice of g_1 made from the very beginning we assure that $g \in \mathcal{F}$, so we have a pseudo-Anosov action of some element $t \in \mathcal{N} \setminus \mathcal{F}$ such that the orbit $t^n \cdot g \cdot t^{-n}$ ($n \in \mathbf{Z}$) gives us infinitely many conjugacy classes of maximal parabolic subgroups of \mathcal{F} . Now Scott's compact core theorem implies that $\pi_1(\Omega_{\mathcal{F}})/\mathcal{F}$ is not finitely generated. *End of outline*

3 Preliminaries

We will consider the Poincaré model of hyperbolic space \mathbf{H}^3 in the unit ball B_1 equipped with the hyperbolic metric ρ . By a right-angled polyhedron $D \subset \mathbf{H}^3$ we mean a polyhedron all of whose dihedral angles are $\pi/2$.

Consider the tessellation of \mathbf{H}^3 by images of D under the reflection group R from Theorem 1. Denote by $W \subset \mathbf{H}^3$ the collection of geodesic planes w such that there exists $r \in R$, for which $r(w) \cap \partial D$ is a face of D .

It is easy to see that if σ_1 and σ_2 are two faces of D with $\sigma_1 \cap \sigma_2 = \emptyset$, then also the geodesic planes $\tilde{\sigma}_1 \supset \sigma_1$ and $\tilde{\sigma}_2 \supset \sigma_2$ have no point in common. One can easily show that the distance between σ_1 and σ_2 , as well as that of $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$, is realized by a common perpendicular ℓ for which $\ell \cap \text{int}D \neq \emptyset$.

Let $\Gamma_0 = R \cap \Gamma$ which is a subgroup of a finite index in both groups R and Γ . By passing to a subgroup of a finite index and preserving notation, we may assume that Γ_0 is a normal subgroup in R , $|R : \Gamma_0| < \infty$. For a plane $w \in W$ we write $H_w = \text{Stab}(w, \Gamma_0) = \{g \in \Gamma_0, gw = w\}$. It is not hard to see that H_w is a Fuchsian group of the first kind commensurable with the reflection group determined by the edges of some face of the polyhedron $r(D_1)$, $r \in R$.

Let us now fix two disjoint planes w_1 and w_2 from W containing opposite faces of D and let ℓ be their common perpendicular; up to conjugation in $\text{Isom } \mathbf{H}^3$ we can assume that ℓ is a Euclidean diameter of B_1 . Denote $B_1^* = \mathbf{S}^3 \setminus \text{cl}(B_1)$ as well (where $\text{cl}(\cdot)$ is the closure of a set). We have the following:

Lemma 1 *For every horosphere π_3 in B_1^* centered at the point $\xi \in \ell \cap \partial B_1$ (see figure 1) there exists $\varepsilon_0 > 0$ such that for every ε -close sphere $\pi_1 \subset B_1^*$ to π_3 ($\varepsilon < \varepsilon_0$) orthogonal to the plane π_2 there exists a geodesic plane w and an element $g_1 \in [H_w, H_w]$ (commutator subgroup) such that:*

$$I_{g_1} \cap \pi_1 \neq \emptyset \quad \text{and} \quad g_1(I_{g_1} \cap \pi_1) = I'_{g_1} \cap \pi_1, \tag{1}$$

where $I_{g_1}, I'_{g_1} = I_{g_1^{-1}}$ are isometric spheres of g_1 .

Proof Up to further conjugation in $\text{Isom } B_1$ preserving ℓ we may assume that π_3 is the vertical plane tangent to ∂B_1 at $\xi \in \ell \cap \partial B_1$. Take $w = w_1$ and let $g_1 \in [H_{w_1}, H_{w_1}]$ be any primitive element corresponding to a simple dividing loop on the surface w_1/H_{w_1} .

Suppose first that $I_{g_1} \cap \pi_3 = \emptyset$. In this case we proceed as follows. Put $\chi = \tau_{w_1} \circ \tau_{w_2} \in R$, where τ_{w_i} denotes the reflection in plane w_i ($i = 1, 2$). Then χ is a hyperbolic element whose invariant axis is ℓ . Consider the sequence of planes $\chi^n(w_1)$. We claim that, for some n , $\chi^n(I_{g_1}) \cap \pi_3 \neq \emptyset$. In fact this follows directly from the fact that the fixed point ξ of the hyperbolic element χ is a conical limit point of Γ_0 , and so the approximating sequence $\chi^n(I_{g_1})$ should intersect a fixed horosphere (or equivalently by sending ξ to the infinity and passing to the half-space model one can see that χ becomes now a dilation $z \mapsto \lambda z$ ($\lambda > 0$) which implies that the translations of the image of I_{g_1} by

powers of the dilation will intersect a fixed horosphere at infinity). Since Γ_0 is normal in R it now follows that $\chi^n g_1 \chi^{-n} \in [H_{\chi^n(w_1)}, H_{\chi^n(w_1)}] \subset \Gamma_0$ and $\chi^n(I_{g_1}) = I_{\chi^n g_1 \chi^{-n}}$. The latter is true since χ preserves each Euclidean plane passing through $B_1 \cap \ell$ and, hence $(\chi^n g_1 \chi^{-n})|_{\chi^n(I_{g_1})}$ is an Euclidean isometry. So up to replacing w_1 by $\chi^n(w_1)$ and g_1 by $\chi^n g_1 \chi^{-n}$ if needed, we may assume that $I_{g_1} \cap \pi_3 \neq \emptyset$. The same conclusion is then obviously true for a plane $\pi_1 \subset B_1^*$ sufficiently close to π_3 .

For $\ell_1 = I_{g_1} \cap \pi_1$ we now claim that $g_1(\ell_1) = \ell_2 = I'_{g_1} \cap \pi_1$. Indeed, $g_1 = \tau_{\pi_2} \cdot \tau_{I_{g_1}}$ where π_2 is orthogonal to π_1 and contains ℓ (figure 1). Evidently

$$g_1(\ell_1) = \tau_{\pi_2}(I_{g_1} \cap \pi_1) = \tau_{\pi_2}(I_{g_1}) \cap \pi_1 = I'_{g_1} \cap \pi_1 \tag{2}$$

since $\tau_{\pi_2}(\pi_1) = \pi_1$. The lemma is proved. □

So we can suppose that $w_1 \in W$ is chosen satisfying all the conclusions of Lemma 1. Let $w_2 \in W$ be a geodesic plane disjoint from w_1 and let ℓ be their common perpendicular passing through the origin of B_1 . Now consider the Euclidean plane π orthogonal to ℓ (figure 2) such that

$$\pi \cap \partial B_1 = \pi \cap w_2 .$$

It is not hard to see that $\text{Stab}(\pi, \Gamma) = \text{Stab}(w_2, \Gamma) = H_{w_2}$. Reflecting our picture in the plane π we get

$$B_2 = \tau_\pi(B_1) , \quad w_3 = \tau_\pi(w_2) \quad \text{and} \\ H_{w_3} = \tau_\pi H_{w_1} \tau_\pi .$$

By Lemma 1 we can now find a Euclidean sphere Σ centered on ℓ which goes through the circle $w_3 \cap \partial B_2$ and is tangent to I_{g_1} (figure 2). Moreover, by Lemma 1, Σ is tangent also to I'_{g_1} .

Denote $\Sigma' = \tau_\pi^{-1}(\Sigma)$.

Lemma 2 *There exists a subgroup $\Gamma_1 \subset \Gamma_0$ of finite index such that the following conditions hold:*

- (a) *The boundary of the isometric fundamental domain $\mathcal{P}(\Gamma_1) \subset B_1^*$ lies in a regular ε -neighbourhood of ∂B_1^* ($B_1^* = \mathbf{S}^3 \setminus cl(B_1)$, $\varepsilon > 0$).*
- (b) $\Sigma \cap I_\gamma = \emptyset$, $\gamma \in \Gamma_1 \setminus \{g_1, g_1^{-1}\}$.
- (c) *For subgroups $H_1 = \Gamma_1 \cap H_{w_1}, H_2 = \Gamma_1 \cap H_{w_2}$ there exists another fundamental domain $R(\Gamma_1) \subset B_1^*$ of Γ_1 such that*

$$R(\Gamma_1) \cap (\pi \cup \Sigma') = \mathcal{P}(H) \cap (\pi \cup \Sigma'),$$

where $\mathcal{P}(H)$ is an isometric fundamental domain for the group $H = \langle H_1, H_2 \rangle$.

- (d) $g_1 \in \Gamma_1 \cap [H_1, H_1]$.

Proof This Lemma can be obtained by repeating the arguments of [14, Main Lemma]. We just sketch these considerations. First, we choose a subgroup $\tilde{\Gamma} \subset \Gamma_0$ of a finite index satisfying conditions (a) and (b) such that $g_1 \in \tilde{\Gamma}$ by using the property of separability of infinite cyclic subgroups in Γ_0 [9].

To obtain (c) we will find Γ_1 by using Scott's *LERF*-property of the group Γ_0 with respect to its geometrically finite subgroups (see [16], [17]). To this end we proceed as follows: the group H is geometrically finite as a result of Klein–Maskit free combination from H_1 and H_2 , which are both geometrically finite subgroups of Γ_0 . The *LERF* property now says that for the element g_1 there exists a subgroup of Γ_0 of finite index which contains H and does not contain g_1 . Call this subgroup Γ_1 . Evidently, $g_1 \in [H_1, H_1] \subset \Gamma_1$ by construction. For the complete proof, see [14, Main Lemma]. \square

Let us introduce the following notation: $\Omega_1^- = B_1^* \setminus \bigcup_{\gamma \in \Gamma_1} \gamma(\pi^-)$ where π^- is the component of $\mathbf{S}^3 \setminus \pi$ for which $w_3 \in \pi^-$. Let $\Gamma_1' = \text{Stab}(\Omega_1^-, \Gamma_1)$.

The complete proof of the following assertion can be also found in [14, Lemma 3].

Lemma 3 *The group $G_1 = \langle \Gamma_1', \tau_\pi \Gamma_1' \tau_\pi \rangle$ is discontinuous and*

- (1) $G_1 \cong \Gamma_1' *_{H_2} (\tau_\pi \Gamma_1' \tau_\pi)$.
- (2) G_1 is isomorphic to a subgroup $G_1^* \subset R$ of finite index.

Sketch of proof (1) This follows from the fact that the plane π is strongly invariant under H_2 in Γ_1' by [14, Lemma 3.c], which means $H_2\pi = \pi$ and $\gamma\pi \cap \pi = \emptyset$, $\gamma \in \Gamma_1' \setminus H_2$. One can now get assertion (1) from Maskit's First Combination theorem [11].

(2) Consider the reflection τ_{w_2} in the geodesic plane $w_2 \subset B_1$. We claim that the group $G_1^* = \langle \Gamma_1', \tau_{w_2} \Gamma_1' \tau_{w_2} \rangle$ is isomorphic to G_1 . Indeed, w_2 is also strongly invariant under H_2 in Γ_1' and we again observe that $G_1^* = \Gamma_1' *_{H_2} (\tau_{w_2} \Gamma_1' \tau_{w_2}) \cong G_1$ because $\tau_{w_2}|_{w_2} = \tau_\pi|_\pi = id$.

Now $\tau_{w_2} \in R$. Therefore, $G_1^* \subset R$ and G_1^* has a compact fundamental domain $R(G_1^*) = R(\Gamma_1') \cap \tau_{w_2}(R(\Gamma_1'))$. The covering $\mathbf{H}^3 / (G_1^* \cap \Gamma_0) \rightarrow \mathbf{H}^3 / G_1^*$ is finite since $|R : \Gamma_0| < \infty$ and, hence, the manifold $M(G_1^* \cap \Gamma_0) = \mathbf{H}^3 / (G_1^* \cap \Gamma_0)$ is compact. Thus, the covering $M(G_1^* \cap \Gamma_0) \rightarrow M(\Gamma_0)$ is finite as well and so $|\Gamma_0 : G_1^* \cap \Gamma_0| < \infty$. \square

Corollary 4 *There exists a path $\alpha_t: [0, 1] \rightarrow \text{Def}(G_1^*)$ such that $\alpha_0 = G_1^*$ and $\alpha_1 = G_1$.*

Proof By choosing a continuous family of spheres μ_t for which $\mu_t \cap \pi = w_2 \cap \pi = \Lambda(H_2)$, $\mu_0 \supset w_2$, $\mu_1 = \pi$, $t \in [0, 1]$, we construct the family of groups $G_t = \langle \Gamma'_1, \tau_{\mu_t} \Gamma'_1 \tau_{\mu_t} \rangle$ by the arguments of Lemma 3. Consider now the action of Γ'_1 in B_1^* where $p_1: B_1^* \rightarrow B_1^*/\Gamma_1$ is the covering map. The surfaces $p_1(\mu_t)$ are all embedded and parallel due to condition (b). If now Ω_{G_t} is the component of G_1 containing ∞ then the manifold $M_{G_t} = \Omega_{G_t}/G_t$ is homeomorphic to the double of the manifold $M_1^- = \Omega_1^-/\Gamma'_1$ along the boundary $p_1(\pi)$. Thus, for all $t \in [0, 1]$, M_{G_t} are all homeomorphic and there exists a continuous family of homeomorphisms $f_t: \Omega(G_1^*) \rightarrow \Omega(G_t)$ such that $G_t = f_t G_1^* f_t^{-1}$, $G_1 = f_1 G_1^* f_1^{-1}$. \square

By construction the domain $R(G_1) = R(\Gamma'_1) \cap \tau_\pi(R(\Gamma'_1))$ is fundamental for the action of G_1 in Ω_{G_1} .

Claim 5 $R(G_1) \cap \Sigma = (\mathcal{P}(H_3) \cup I_{g_1} \cup I'_{g_1}) \cap \Sigma$.

Proof Recall that $\pi^+(\pi^-)$ means the right (left) component of $\mathbf{S}^3 \setminus \pi$ ($I_{g_1} \in \pi^+$). Then $\pi^+ \cap \Sigma \cap R(\Gamma'_1) = \mathcal{P}(H_1) \cap \Sigma = (I_{g_1} \cup I'_{g_1}) \cap \Sigma$ by (b) and (c) of Lemma 2.

Also, $\tau_\pi(\pi^- \cap \Sigma \cap \tau_\pi(R(\Gamma'_1))) = \pi^+ \cap \tau_\pi(\Sigma) \cap R(\Gamma'_1) \subset \mathcal{P}(H_1) \cap \Sigma'$, so $\pi^- \cap \Sigma \cap R(G_1) = \tau_\pi(\mathcal{P}(H_1)) \cap \Sigma = \mathcal{P}(H_3) \cap \Sigma$. \square

Let us consider now the family of spheres Σ_t centered on the y -axis (figure 2) such that $\Sigma_t \cap w_3 = \Sigma \cap w_3$, $\sigma_1 = \Sigma$, $\sigma_0 = \Sigma_0$, $t \in [0, 1]$, where $\Sigma_t \cap \text{ext}(B_1) \cap \text{ext}(B_2) \subset \text{ext}(\Sigma) \cap \text{ext}(B_1) \cap \text{ext}(B_2)$ (recall $\text{ext}(\cdot)$ is the exterior of a set in $\overline{\mathbf{R}}^3$), $\Sigma_t \cap I_{g_1} = \emptyset$ ($t > 0$). Denote by τ_{Σ_t} the corresponding reflections. As before take the domain $\Omega^* = \Omega_{G_1} \setminus G_1(\Sigma_0^-)$ and the group $G'_1 = \text{Stab}(\Omega^*, G_1)$, where $\Sigma_0^- = \text{ext}(\Sigma_0)$ is the unbounded component of $\overline{\mathbf{R}}^3 \setminus \Sigma_0$.

Denote $\mathcal{G}_t = \langle G'_1, \tau_{\Sigma_t} G'_1 \tau_{\Sigma_t} \rangle$. Evidently, $\mathcal{G}_1 = \lim_{t \rightarrow 1} \mathcal{G}_t$.

Lemma 6 *The groups \mathcal{G}_t are discontinuous, $t \in [0, 1]$.*

Proof First, let us prove the lemma for $t \neq 1$. By Claim 5 we have now that $R(G_1) \cap \Sigma_t = \mathcal{P}(H_3) \cap \Sigma_t$. Moreover we claim also that

$$g\Sigma_t \cap \Sigma_t = \emptyset, \quad g \in G_1 \setminus H_3, \quad H_3 \Sigma_t = \Sigma_t, \tag{3}$$

where $H_3 = \tau_\pi H_1 \tau_\pi$.

To prove (3) we only need to show that $g(\Sigma_t \cap \Lambda(H_3)) \cap (\Sigma_t \cap \Lambda(H_3)) = \emptyset$, but this can be shown from the fact that each point of $\Lambda(H_3)$ is a point of approximation (see [14, Claim 1]).

All conditions of Maskit’s First Combination theorem are now satisfied for the groups G'_1 and $\tau_{\Sigma_t} G'_1 \tau_{\Sigma_t}$ ($t \neq 1$) [11] and we obtain also

$$\mathcal{G}_t \cong G'_1 *_{H_3} (\tau_{\Sigma_t} G'_1 \tau_{\Sigma_t}) \tag{4}$$

where the \mathcal{G}_t are all discontinuous, $t \in [0, 1)$.

Let us now consider the group \mathcal{G}_1 and the domain $R(\mathcal{G}_1) = R(G_1) \cap \tau_{\Sigma}(R(G_1))$. Our goal now is to show that $R(\mathcal{G}_1)$ is a fundamental domain for the action of \mathcal{G}_1 in $\Omega_{\mathcal{G}_1}$ ($\infty \in \Omega_{\mathcal{G}_1}$). If now $\langle g_1, \gamma_1, \dots, \gamma_\ell \rangle$ is a set of generators of G'_1 then $S = \langle g_1, \gamma_1, \dots, \gamma_\ell, g_2, \gamma'_1, \dots, \gamma'_\ell \rangle$ are generators of \mathcal{G}_1 , where $\gamma'_i = \tau_{\Sigma} \cdot \gamma_i \cdot \tau_{\Sigma}$ and $g_2 = \tau_{\Sigma} \cdot g_1 \cdot \tau_{\Sigma}$. Observe that the element g_1 is included in S because some of its isometric spheres belong to the boundary $\partial R(G'_1)$

We want to apply the Poincaré Polyhedron theorem [12]. Indeed, an arbitrary cycle of edges in $\partial R(\mathcal{G}_1)$ consists either of edges situated in $\partial(R(G_1)) \cap \text{int}(\Sigma)$, and $\partial(\tau_{\Sigma}(R(G_1))) \cap \text{ext}(\Sigma)$, or is an edge cycle $\ell_1 = I_{g_1} \cap I_{g_2}$, $\ell_2 = I'_{g_1} \cap I'_{g_2}$, where I_{g_k}, I'_{g_k} are the isometric spheres of g_k and g_k^{-1} ($k = 1, 2$). The sum of angles in any cycle of the first type is 2π because $R(G_1)$ is a fundamental domain [12].

We now claim that the element $g = g_2^{-1} \cdot g_1$ is parabolic with a fixed point $d = I_{g_1} \cap I_{g_2}$. Indeed, $g_2^{-1} \cdot g_1 = (\tau_{\Sigma} \cdot \tau_{I_{g_1}})^2$ because $g_1 = \tau_{\pi_2} \cdot \tau_{I_{g_1}}$ and π_2 is orthogonal to Σ (figure 2). Now it is easy to check that $g(d) = d$, $gI_{g_1} \subset \text{int}(I_{g_2})$ and $g(\text{int}(I_{g_1})) = \text{ext}(I_{g_1})$, therefore the elements g and $g' = g_1 \cdot g \cdot g_1^{-1}$ are parabolics.

All conditions of the Maskit–Poincaré theorem are valid at the edges ℓ_i also and, hence, \mathcal{G}_1 is discontinuous. Lemma 6 is proved. □

Lemma 7 *The group \mathcal{G}_0 is isomorphic to a subgroup $L' \subset R$ of a finite index.*

Proof We repeat our construction of \mathcal{G}_0 by modelling it in \mathbf{H}^3 so as to get the required isomorphism.

Recall that we started from the group $\Gamma'_1 \subset \text{Isom}(\mathbf{H}^3)$ and showed that $G_1 = \langle \Gamma'_1, \tau_{\pi} \Gamma'_1 \tau_{\pi} \rangle \cong G_1^* = \langle \Gamma'_1, \tau_{w_2} \Gamma'_1 \tau_{w_2} \rangle$ (see Lemma 4). Next we constructed \mathcal{G}_0 by using reflection in $\sigma_0 = \Sigma_0$ such that $\sigma_0 \cap w_3 = \Lambda(H_3)$, $\sigma_0 \cap B_1 = \emptyset$, $w_3 = \tau_{\pi}(w_1)$.

Let $\eta = \tau_{w_2}(w_1) \subset \mathbf{H}^3$, $\eta \in W$. Again let us take the subgroup G_1^{**} of G_1^* which is $G_1^{**} = \text{Stab}(\mathbf{H}^3 \setminus G_1^*(\eta^-), G_1^*)$, where η^- is a subspace $\mathbf{H}^3 \setminus \eta$ not containing w_2 .

By construction the fundamental domain $R(G_1^*) = R(\Gamma'_1) \cap \tau_{w_2}(R(\Gamma'_1))$ of the group G_1^* satisfies $R(G_1^*) \cap \eta = \mathcal{P}(H'_3 = \text{Stab}(\eta, G_1^*))$. Again by Maskit's First Combination theorem we have a group L' :

$$L' = G_1^{**} *_{{H'_3}} (\tau_\eta G_1^{**} \tau_\eta) \tag{5}$$

We constructed an isomorphism $\varphi_1: G_1^* \rightarrow G_1$ in Lemma 4 such that $\tau_\pi \cdot \varphi_1 \cdot \tau_{w_2} = \varphi_1$, therefore $\varphi_1(H'_3) = H_3$ and $\varphi_1(G_1^{**}) = G_1'$. It follows now from (4) and (5) that the map $\varphi_1|_{G_1^{**}}$ can be extended to an isomorphism $\varphi: L' \rightarrow \mathcal{G}_0$.

Index $|R : L'|$ is finite because L' has a compact fundamental domain. The Lemma is proved. \square

Recall that we identify $[\rho] \in \text{Def}(L')$ with $\rho(L')$.

Lemma 8 *There exists a path $\beta_t: [0, 1] \rightarrow \text{cl}(\text{Def}(L'))$ such that $\beta_0 = L'$, $\beta_1 = \mathcal{G}_1 \in \partial \text{Def}(L')$, $\beta_t([0, 1]) \subset \text{Def}(L')$.*

Proof We have constructed a path $\alpha_t: [0, 1] \rightarrow \text{Def}(G_1^*)$ in Corollary 4 such that $\alpha_0 = G_1^*$, $\alpha_1 = G_1$ and α_t is a family of admissible representations. Let further $\alpha_t|_{G_1^{**}} = \alpha'_t$. Obviously, the representations α'_t are also admissible and $\alpha'_t(G_1^{**}) = G_1'$. We can easily extend our family α'_t to a family of admissible representations $\theta_t: L' \rightarrow \text{Def}(L')$ by the formula $\theta_t = \tau_{\mu_t} \alpha'_t \tau_{\mu_t}$, where μ_t are the spheres constructed in Corollary 4.

Observe that $\mu_1 = \pi$ and now take a new continuous family of spheres ν_t for which $\nu_t \cap w_3 = \Lambda(H_s) = w_3 \cap B_2$ and $\nu_1 = \tilde{w}_3$, $\nu_2 = \Sigma_0$ where \tilde{w}_3 is the sphere containing w_3 ($t \in [0, 1]$).

Again we have a path $\theta'_t(L') = \langle G'_1, \tau_{\nu_t} G'_1 \tau_{\nu_t} \rangle$. Composing the path θ_t with θ'_t and with the path corresponding to spheres Σ_t connecting Σ_0 with Σ_1 we get required path β_t . The Lemma is proved. \square

4 Proof of Theorem 1

(1) Denote by $F = \pi_1 \sigma$ a fixed fiber group of our initial manifold M , and let also $F_0 = \Gamma_0 \cap F$.

By Jørgensen's theorem [5] the limit $\beta_1 = \lim_{t \rightarrow 1} \beta_t$ is an isomorphism $\beta_1: L' \rightarrow \mathcal{G}_1$. Let us consider the subgroup $L = L' \cap \Gamma_0$, $|\Gamma_0 : L| < \infty$. Put also $F_L = L \cap F_0$ for its normal subgroup. We have also the curve $\beta_t(L) \subset \text{Def}(L)$. Let $\mathcal{N} = \beta_1(L)$, $\mathcal{F} = \beta_1(F_L)$. Let us show that $g = g_2^{-1} \cdot g_1 \in \mathcal{F}$. To this

end let us recall that the element g_1 was chosen from the very beginning being in $[H_{w_1}, H_{w_1}]$ (Lemma 1). Recalling also that $\beta_1^{-1}(g_1) = g_1$ and denoting $\beta_1^{-1}(g_2) = g'_2$, by construction we get $g'_2 = \tau_\eta \cdot g_1 \cdot \tau_\eta$, $\eta = \tau_{w_2}(w_1)$, $g_1 \in [H_{w_1}, H_{w_1}] \subset [F_0, F_0]$ (see Lemma 1). The group Γ_0 was chosen to be normal in the reflection group R , and since $[\Gamma_0, \Gamma_0] \subset F$, it is straightforward to see that

$$r[F_0, F_0]r^{-1} \subset F_0, \quad r \in R.$$

Hence, $g'_2 \in F_0$, and for the element $g' = (g'_2)^{-1} \cdot g_1$ we immediately obtain $g' \in F_L = F_0 \cap L'$. It follows that $\beta_1(g') = g = g_2^{-1} \cdot g_1 \in F_0 \cap \mathcal{G}_1 = \mathcal{F}$ as was promised.

We have that \mathcal{N} is isomorphic to the semi-direct product of \mathcal{F} and the infinite cyclic group \mathbf{Z} , so taking the element $t \in \mathcal{N} \setminus \mathcal{F}$ projecting to the generator of \mathcal{N}/\mathcal{F} , we observe that the elements

$$g_n = t^n g t^{-n} \in \mathcal{F}, \quad g \in \mathcal{F}, \quad n \in \mathbf{Z} \tag{6}$$

are all parabolics. Since \mathcal{N} contains no abelian subgroups of rank bigger than 1 and $t^n \notin \mathcal{F}$ ($n \in \mathbf{Z}$) one can easily see that the elements (6) are also non-conjugate in \mathcal{F} . We have proved (1) of the Theorem.

(2) By the construction, the fundamental polyhedron $R(\mathcal{G}_1)$ of the group \mathcal{G}_1 contains only one conjugacy class of parabolic elements g of rank 1. There is a strongly invariant cusp neighborhood $B_g \cong [0, 1] \times R^1 \times [0, \infty)$ which comes from the construction of $R(\mathcal{G}_1)$. So each parabolic g_n of type (6) gives rise to submanifold

$$B_{g_n} / \langle g_n \rangle \cong T_n \times [0, \infty), \quad T_n \cong S^1 \times S^1 \tag{7}$$

in the manifold $M(\mathcal{F}) = \Omega_N / \mathcal{F}$. Therefore $M(\mathcal{F})$ contains infinitely many parabolic ends (7) bounded by tori T_n . They all are non-parallel in $M(\mathcal{F})$ and therefore by Scott's "core" theorem the group $\pi_1(M(\mathcal{F}))$ is not finitely generated [16]. □

Remark By using the argument of [14] one can prove:

Theorem 2 *There is a (non-faithful) representation $\beta_{1+\varepsilon}$ which is ε -close to β_1 for some small $\varepsilon > 0$ such that the group $\beta_{1+\varepsilon}(F_L)$ is infinitely generated, has infinitely many non-conjugate elliptic elements. Moreover, $\beta_{1+\varepsilon}(F_L)$ is a normal infinitely presented subgroup of a geometrically finite group $\beta_{1+\varepsilon}(L)$ without parabolics.*

To prove the theorem one can continue to deform the group for $1 < t \leq 1 + \varepsilon$ (these representations will no longer be faithful) in order to get an elliptic element g_t whose isometric spheres form an angle $\theta(t)$ instead of being tangent. To do this in our Lemma 2, instead of the sphere Σ tangent to the isometric spheres of g_1 , one needs to consider a nearby sphere $\Sigma_{1+\varepsilon}$ forming angle $\theta(\varepsilon)$ with them. If $\theta(\varepsilon) = \frac{\pi}{2n}$ and $n > 0$ is large enough the group $\beta_{1+\varepsilon}(F_L)$ is Kleinian, has infinitely many non-conjugate elliptic elements of the order n (obtained as above as an orbit of $g_{1+\varepsilon}$ by a pseudo-Anosov automorphism of the $\beta_{1+\varepsilon}(F_L)$). The construction gives us that $\beta_{1+\varepsilon}(F_L)$ is a normal and finitely generated but infinitely presented subgroup of the geometrically finite group $\beta_{1+\varepsilon}(L)$ without parabolic elements. In particular $\beta_{1+\varepsilon}(L)$ is a Gromov hyperbolic group (see [14, Lemmas 5–7]).

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