Induced trees in graphs of large chromatic number

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Abstract. Gyárfás and Sumner independently conjectured that for every tree T and integer k there is an integer $f(k,T)$ such that every graph G with $\chi(G) > f(k,T)$ contains either K_k or an induced copy of T. We prove a 'topological' version of the conjecture: for every tree T and integer k there is $g(k,T)$ such that every graph G with $\chi(G) > g(k,T)$ contains either K_k or an induced copy of a *subdivision* of T.

§1. Introduction

What can we say about the induced subgraphs of a graph G with large chromatic number? Of course, one way for a graph to have large chromatic number is if it contains a large complete subgraph. However, if we consider graphs with large chromatic number and small clique number then we can ask what other subgraphs must occur. We can avoid any graph H that contains a cycle since, as Erdős and Hajnal ([3], [1], [2]) showed, there are graphs with arbitrarily high girth and chromatic number; but what can we say about trees? Gyárfás [5] and Sumner [17] independently made the following beautiful and difficult conjecture.

Conjecture A. *For every integer k and tree T there is an integer $f(k,T)$ such that every graph G with*

$$\mathrm{cl}(G) \le k$$

and

$$\chi(G) \ge f(k,T)$$

contains an induced copy of T.

Let us rephrase this, using the notation of Gyárfás [6]. We call a class $\mathcal{G}$ of graphs χ-*bounded* if there is a function f such that $\chi(G) \le f(\mathrm{cl}(G))$ for every $G \in \mathcal{G}$; we call f a χ-*binding* function. For instance, the class of perfect graphs is χ-bounded with $f(k) = k$ as a χ-binding function.

For a graph H, we write $\mathrm{Forb}(H)$ for the class of graphs that do not contain H as an induced subgraph. For a family of graphs $\mathcal{H}$, we write $\mathrm{Forb}(\mathcal{H})$ for the class of graphs that contain no member of $\mathcal{H}$ as an induced subgraph. As we have remarked, $\mathrm{Forb}(H)$ is not χ-bounded when H contains a cycle. The conjecture of Gyárfás and Sumner asserts that $\mathrm{Forb}(T)$ is χ-bounded for every tree T. In fact, an easy argument shows that the conjecture is equivalent to the assertion that $\mathrm{Forb}(H)$ is χ-bounded iff H is a forest.

If we do not demand that T be induced, then the problem becomes much easier. Indeed, both Gyárfás, Szemerédi and Tuza [8] and Sumner [17] showed that if

$\chi(G) = |T| = t$ and T is coloured with $1, \ldots, t$, then any proper t-colouring of G contains a subgraph isomorphic to T and has the same labels.

It is also known that graphs with low clique number contain large induced trees. Erdős, Saks and Sós [4] proved that, for $k \geq 3$ and $n \geq 4$, every connected graph G of order n such that $\mathrm{cl}(G) \leq k$ contains an induced tree of order at least

$$\frac{2\log n}{(k-2)\log\log n} - 3.$$

However, little can be said about the structure of such a tree.

Recently, attention has been given to the on-line version of the conjecture. Gyárfás and Lehel [7] proved that $\mathrm{Forb}(P_5)$ is on-line χ-bounded and noted that $\mathrm{Forb}(P_6)$ is not on-line χ-bounded; Kierstead, Penrice and Trotter [13] gave a better binding function and further results. Finally, Kierstead, Penrice and Trotter [14] proved the difficult result that, for any tree T, $\mathrm{Forb}(T)$ is on-line χ-bounded iff T has radius at most two. For a survey of these results, see Kierstead [9].

Returning to the conjecture, it follows easily from Ramsey's Theorem that $\mathrm{Forb}(K_{1,n})$ is χ-bounded for every n. (Indeed, suppose $\chi(G) > R(n,k)$ and $\mathrm{cl}(G) \leq k$. Then G contains a vertex x of degree at least $R(n,k)$, and so $\Gamma(x)$ must contain an independent set S of size at least n, since $\mathrm{cl}(G) \leq k$; then $\{x\} \cup S$ induces $K_{1,n}$.) Gyárfás [6] showed that $\mathrm{Forb}(P_n)$ is χ-bounded for every path P_n, and Hajnal and Rödl (see [12], [15], [16]) proved that $\mathrm{Forb}(T, K_{n,n})$ is χ-bounded for every tree T and integer n. Significant progress was made by Gyárfás, Szemerédi and Tuza [8], who proved a special case of the conjecture for trees of radius two: for every tree T of radius two there is a constant $c(T)$ such that every triangle-free graph G such that $\chi(G) \geq c(T)$ contains an induced copy of T. Kierstead and Penrice [12] succeeded in generalizing this argument to prove the following.

Theorem B. $\mathrm{Forb}(T)$ *is χ-bounded for every tree T of radius two.*

Very little else is known, however (though some special cases of the conjecture have been proved by Kierstead and Penrice [11] and Kierstead [10]; these results are both special cases of Corollary 2 below), and it seems that the full conjecture is rather difficult. Even partial results are therefore of interest. For instance, Sauer

[16] notes that the conjecture is not even known to hold for subdivisions of stars; this will follow as a special case of Theorem 1 below.

For a graph H, let us write $\mathrm{Forb}^*(H)$ for the class of graphs that contain no subdivision of H as an induced subgraph. (For instance, $\mathrm{Forb}^*(C_3)$ is the class of forests.) The main result of this paper is the following.

Theorem 1. $\mathrm{Forb}^*(T)$ *is χ-bounded for every tree T.*

Equivalently, for every tree T and positive integer k, every graph with sufficiently large chromatic number contains either K_k or an induced copy of a subdivision of T.

This can be seen as a 'topological' version of the Gyárfás-Sumner conjecture, and allows us to demand trees with more structure than was previously possible (stars, paths, trees of radius two and a few other trees).

Consider now a tree T that is a subdivision of a star (equivalently, T contains at most one vertex of degree greater than two): whenever we subdivide T, we get a tree that contains T as an induced subgraph. (In fact, subdivisions of stars are the only connected graphs with this property.) We therefore get the following result as an immediate corollary of Theorem 1.

Corollary 2. *Let T be a subdivision of a star. Then* $\mathrm{Forb}(T)$ *is χ-bounded.*

This answers the question mentioned by Sauer (and solves Problem 2.13 from [6]).

In order to prove Theorem 1, we will in fact prove a rather stronger result, which gives a bound on the extent to which our induced copy of T is subdivided.

Theorem 3. *For every tree T there is an integer $t(T)$ such that the following assertion holds. For every integer k there is an integer $c(k, T)$ such that every graph G with*

$$\chi(G) \geq c(k, T)$$

either contains K_k or contains a subdivision of T as an induced subgraph, with each edge of T subdivided at most $t(T)$ times.

Now for a fixed tree T, there are only finitely many subdivisions of T such that each edge of T is subdivided at most $t(T)$ times. Theorem 3 can therefore be reformulated as follows.

Corollary 4. *For every tree T there is a finite family $T_1, \ldots, T_s$ of subdivisions of T such that*

$$\bigcap_{i=1}^{s} \mathrm{Forb}(T_i)$$

is χ-bounded.

In §2 we give a proof of the main result. After a technical lemma, the proof is divided into two sections, depending on whether or not the 'local' chromatic number of our graph G is large.

In §3 we make some remarks and suggest possible further applications of our method.

We use standard notation. For a graph G and vertices $v, w \in V(G)$, we write $d_G(v, w)$ for the distance between v and w, i.e. the length of a shortest path between them ($d_G(v, w) = \infty$ if there is no such path). For $v \in V(G)$ and a positive integer d, we define

$$B_G(v, d) = \{x \in V(G) : d_G(v, x) \leq d\}$$

and

$$S_G(v, d) = \{x \in V(G) : d_G(v, x) = d\}.$$

If there is no ambiguity we write $d(v, w)$ for $d_G(v, w)$, etc.

For positive integers a and b, let T_a^b be the rooted tree of radius b in which the root has degree a, every endvertex is distance b from the root and every vertex that is not the root or an endvertex has degree $a + 1$. Thus $T_m^1 \cong K_{1,m}$.

§2. The main result

In this section we prove Theorem 3. Our proof proceeds in several stages. We begin with a technical lemma about subdivisions, which will be used several times in the proof.

Lemma 5. *For every triple of integers m, d, k there is an integer $M(m, d, k)$ such that the following is true. Let G be a graph with $\mathrm{cl}(G) \leq k$. Let $x_1, \ldots, x_M$ and $y_1, \ldots, y_M$ be vertices in G such that, for $i = 1, \ldots, M$,*

$$\Gamma(y_i) = \{x_i\}$$

and

$$d_G(x_1, x_i) \leq d.$$

Then G contains an induced subdivision of the star $K_{1,m}$ with endvertices y_1 and $m-1$ vertices from $y_2, \ldots, y_M$ such that each edge of $K_{1,m}$ is subdivided at most d times.

Proof. We proceed by induction on d. If $d = 1$, then x_1 is joined to $x_2, \ldots, x_M$. If $M \geq R(k, m-1) + 1$ then since $\mathrm{cl}(G) \leq k$, there must be an independent set of size $m-1$ among $x_2, \ldots, x_M$, say $\{x_2, \ldots, x_m\}$. Then $G[\{x_1, \ldots, x_m\}]$ is a star $K_{1,m-1}$ with centre x_1, and $G[\{x_1, \ldots, x_m, y_1, \ldots, y_m\}]$ is a subdivision of $K_{1,m}$ with centre x_1 and each edge subdivided once.

Now suppose $d > 1$ and the lemma is true for smaller values of d (and any m, k). Let $x_1, \ldots, x_M, y_1, \ldots, y_M$ be as in the statement of the lemma. If we have $d_G(x_1, x_i) \leq d-1$ for at least $M(m, d-1, k)$ values of i (from $2, \ldots, M$), then by the inductive hypothesis we can find the required subdivision of $K_{1,m}$. Thus we may assume that we have at least

$$M_0 = M - M(m, d-1, k) \tag{1}$$

vertices x_i with $d_G(x_1, x_i) = d$, say $x_2, \ldots, x_{M_0+1}$. For $i = 2, \ldots, M_0 + 1$, let P_i be a path of length d from x_1 to x_i. Let

$$S = \bigcup_{i=2}^{M_0+1} V(P_i)$$

and, for $i = 0, \dots, d$, let

$$S_i = \{x \in S : d_G(x_1, x) = i\}.$$

Thus $S_0 = \{x_1\}$ and $S_d = \{x_2, \dots, x_{M_0+1}\}$. Now consider $G[S]$. If any $x \in S_{d-1}$ has $R(k, m-1)$ neighbours in S_d then we are done: since $\mathrm{cl}(G) \le k$, there must be an independent set of size $m-1$ in $\Gamma(x) \cap S_d$, say $\{x_2, \dots, x_m\}$. Let P be a path of length $d-1$ from x_1 to x. Then

$$V(P) \cup \{x_2, \dots, x_m\} \cup \{y_1, \dots, y_m\}$$

induces a subdivision of $K_{1,m}$ with centre x, endvertices $y_1, \dots, y_m$, and each edge subdivided at most d times.

Otherwise, we have

$$|\Gamma(x) \cap S_d| < R(k, m-1) \qquad (2)$$

for every $x \in S_{d-1}$. Let $Z = \{z_2, \dots, z_{M_1}\} \subset S_{d-1}$ be a minimal set such that every $x \in S_d$ has a neighbour in Z. It is clear from (2) that

$$|Z| \ge |S_d|/R(k, m-1). \qquad (3)$$

Furthermore, for every $z_i \in Z$ we can find $x_{z_i} \in S_d$ such that

$$\Gamma(x_{z_i}) \cap Z = \{z_i\},$$

or else we could replace Z by $Z \setminus \{z_i\}$. Renumbering if necessary, we may assume that $x_{z_i} = x_i$ for each i.

We now find a large independent set among $x_2, \dots, x_{M_1}$. Indeed, if

$$|Z| \ge R(k+1, M(m, d-1, k)), \qquad (4)$$

then $x_2, \dots, x_{M_1}$ contains an independent set of size $r = M(m, d-1, k) - 1$, say $x_2, \dots, x_r$. Consider the subgraph H of G induced by

$$\{x_1, \dots, x_r\} \cup \{z_2, \dots, z_r\} \cup \{y_1\} \cup \bigcup_{i=1}^{d-2} S_i.$$

We have $d_H(x_1, z_i) = d-1$ and $\Gamma_H(x_i) = \{z_i\}$, for $i = 2, \ldots, r$, and $\Gamma_H(y_1) = \{x_1\}$. By the inductive hypothesis, H contains an induced subdivision of $K_{1,m}$ with endvertices y_1 and $m-1$ vertices from $\{x_2, \ldots, x_s\}$, say $x_2, \ldots, x_m$, where each edge of $K_{1,m}$ is subdivided at most $d-1$ times. Adding $y_2, \ldots, y_m$, we get an induced subdivision of $K_{1,m}$ with endvertices $y_1, \ldots, y_m$ and each edge subdivided at most d times.

From (1), (3) and (4), we deduce that

$$M(m, d, k) \leq R(k+1, M(m, d-1, k)) \cdot R(k, m-1) + M(m, d-1, k).$$

□

We now turn to proving the main theorem. The proof is split into two lemmas: in the first we consider graphs that have chromatic number much larger than their 'local' chromatic number; in the second we consider graphs with large 'local' chromatic number. The main result will then follow by an easy argument.

Let us define a little notation. For any integer r and graph G, we define the *r-local chromatic number of G* to be

$$\chi^{(r)}(G) = \max_{v \in V(G)} \chi(G[B(v, d)]).$$

Clearly $\chi^{(0)}(G) = 1$, and $\chi^{(r)}(G) = \chi(G)$ whenever $r \geq \operatorname{diam}(G)$.

We prove a lemma about graphs G for which $\chi^{(r)}(G)$ is much smaller than $\chi(G)$, for suitable r. In essence, the lemma states that for any tree T, every graph with small clique number, small local chromatic number and sufficiently large chromatic number contains a subdivision of T.

Lemma 6. *For every tree T and integer k there exists a function $g\colon \mathbb{N} \to \mathbb{N}$ and integers d, t such that for every integer c, every graph G satisfying*

$$\chi^{(d)}(G) \leq c$$
$$\operatorname{cl}(G) \leq k$$

and

$$\chi(G) \geq g(c)$$

contains an induced subdivision T^* of T, in which every edge is subdivided at most t times.

Proof. It is enough to prove the theorem for trees of form T_a^b, since every tree T is contained in T_s^s for s sufficiently large. Note that T_a^b can be decomposed into a copies of T_a^{b-1}, say $T_1, \dots, T_a$, and an additional vertex x, where x is joined to the root of T_i, for $i = 1, \dots, a$.

The idea of the proof is simple: arguing by induction on $b = \mathrm{rad}(T_a^b)$, we take induced copies of a tree containing T_a^{b-1} and join a of them together to get T_a^b. However, there are a couple of technical difficulties: a vertex in one copy of T_a^{b-1} may be adjacent to vertices in another copy; and a vertex adjacent to one vertex in a given copy of T_a^{b-1} may be adjacent to other vertices in that copy as well. Thus we demand that our copies of T_a^{b-1} are not too close together, and that each copy is 'spread out' in G, in the following sense.

We say that an induced subgraph T^* of G is a (T_a^b, t)-*structure in* G if the following two conditions are satisfied.

1. T^* is a subdivision of T_a^b such that each edge is subdivided at most t times.
2. Let x^* be the centre of T^*, and let $T_1^*, \dots, T_a^*$ be the induced subdivisions of T_a^{b-1} corresponding to $T_1, \dots, T_a$ in the decomposition of T_a^b given above. Then, for $1 \leq i \leq a$,
$$d_G(x, T_i^*) \geq 3 \tag{5}$$
and, for $1 \leq i < j \leq a$,
$$d_G(T_i^*, T_j^*) \geq 3. \tag{6}$$

Now fix k. We prove by induction on b that for every pair of integers a and b there exists a function $g_{a,b}\colon \mathbb{N} \to \mathbb{N}$ and integers d, t such that, for every integer c, whenever G is a graph such that

$$\chi^{(d)}(G) \leq c$$

and

$$\mathrm{cl}(G) \le k,$$

and $X \subset V(G)$ satisfies

$$\chi(G[X]) \ge g_{a,b}(c),$$

then we can find an induced (T_a^b, t)-structure with all its endvertices in X. Note that this is stronger than demanding an induced (T_a^b, t)-structure in $G[X]$, since we may have $d_{G[X]}(x,y) < d_G(x,y)$ for vertices $x, y \in X$.

For $b = 0$ the assertion is trivial. Suppose that $b > 0$ and that the assertion is true for smaller values of b. By the inductive hypothesis, we may pick constants d, t and a function $g : \mathbb{N} \to \mathbb{N}$ such that, for every integer c, whenever G is a graph such that

$$\chi^{(d)}(G) \le c$$

and

$$\mathrm{cl}(G) \le k$$

and $X \subset V(G)$ satisfies

$$\chi(G[X]) \ge g(c)$$

then we can find a (T_{2a}^{b-1}, t)-structure in G, with all its endvertices in X. Increasing t if necessary, we may assume $t \ge d$. Let us note that any (T_{2a}^{b-1}, t)-structure has radius at most

$$D = (t+1)(b-1). \tag{7}$$

We show that, for every integer c, if

$$\chi^{(3D+10)}(G) \le c \tag{8}$$

and

$$\mathrm{cl}(G) \le k,$$

and $X \subset V(G)$ satisfies

$$\chi(G[X]) \ge M(c + g(c)) \tag{9}$$

for sufficiently large M (depending on T and k), then G contains a $(T_a^b, 14D+42)$-structure.

Suppose that G and X satisfy (8) and (9). Let $T_1, \ldots, T_p$ be a maximal set of (T_{2a}^{b-1}, t)-structures in G with centres, say, $x_1, \ldots x_p$, such that each T_i has all endvertices in X and, for $i \neq j$,

$$d(x_i, x_j) \geq 2D + 10. \tag{10}$$

Note that $V(T_i) \subset B(x_i, D)$ for each i, so for $i \neq j$ we have

$$d(T_i, T_j) \geq 10.$$

Now consider

$$W = \bigcup_{i=1}^{p} B(x_i, 3D + 10). \tag{11}$$

There are no (T_{2a}^{b-1}, t)-structures in G with endvertices in $X \setminus W$, or else $T_1, \ldots, T_p$ would not be maximal. (If T_{p+1} were another such (T_{2a}^{b-1}, t)-structure, with centre x_{p+1}, say, then since all endvertices of T_{p+1} are contained in $X \setminus W$ and the radius of T_{p+1} is at most D, we would have $d(x_{p+1}, X \setminus W) \leq D$. Therefore $d(x_i, x_{p+1}) \geq d(x_i, X \setminus W) - d(x_{p+1}, X \setminus W) \geq 2D + 10$, and so we could take $T_1, \ldots, T_{p+1}$ instead of $T_1, \ldots, T_p$.) Thus, by the inductive hypothesis, we must have

$$\chi(G[X \setminus W]) < g(c)$$

and so

$$\begin{aligned} \chi(G[W]) &\geq \chi(G[X]) - \chi(G[X \setminus W]) \\ &\geq (M-1)(c + g(c)). \end{aligned} \tag{12}$$

We now try to join some of the (T_{2a}^{b-1}, t)-structures $T_1, \ldots, T_p$ together to get a $(T_a^b, 14D + 42)$-structure. We begin by showing that some x_i is not too far from M other vertices amongst $x_1, \ldots, x_p$. Let μ be the following colouring of W: for each $x \in W$ let

$$j(x) = \min_{i=1,\ldots,p} \{d(x, x_i)\}$$

and define

$$\mu(x) = \min\{i : d(x, x_i) = j(x)\}.$$

(Note that it follows from (11) that $j(x)$ and $\mu(x)$ do not depend on whether we take the distance in G or the distance in $G[W]$.) Let the μ-colour classes be $C_1, \ldots, C_p$. It is easily checked that $G[C_i]$ is connected for each i: indeed, if $x \in C_i$ and P is a path of length $d(x, x_i)$ from x_i to x then $V(P) \subset C_i$. Now from (8) and (11) we have

$$\chi(G[C_i]) \leq c$$

for $i = 1, \ldots, p$, since $C_i \subset B(x_i, 3D + 10)$. Let $\lambda_i : C_i \to [c]$ be a colouring of $G[C_i]$, for $i = 1, \ldots, p$, and for $x \in W$ define

$$\lambda(x) = \lambda_{\mu(x)}(x).$$

Thus adjacent vertices in W have the same λ-colour only if they are in different μ-colour classes.

Now consider the graph H with vertices $1, \ldots, p$ and an edge between i and j iff

$$e(C_i, C_j) > 0. \tag{13}$$

If $\chi(H) < M$, then let ν be a colouring of H with $M - 1$ colours. We get a proper colouring of X by colouring each $x \in X$ with the ordered pair

$$\langle \lambda(x), \nu(\mu(x)) \rangle.$$

Thus $\chi(G[W]) \leq c(M - 1)$, which contradicts (12). Therefore, $\chi(H) \geq M$ and so some C_i satisfies (13) for at least $M - 1$ values of j.

Let us suppose

$$e(C_1, C_j) > 0 \tag{14}$$

for $j = 2, \ldots, M$.

The idea now is to take the (T_{2a}^{b-1}, t)-structures $T_2, \ldots, T_M$ in $C_2, \ldots, C_M$ and connect them together through C_1. We know that $G[C_i]$ is connected for each i; it follows from (14) that $G[C_1 \cup C_i]$ is connected for $i = 2, \ldots, M$. It also follows from (10) and the definition of the C_i that there are no edges between C_1 and $B(x_i, D + 3)$. Let P_i be a shortest path in $G[C_1 \cup C_i]$ from x_1 to $B(x_i, D + 2)$; by (11) we have

$$|P_i| \leq 5D + 19. \tag{15}$$

Suppose

$$P_i = x_1 \dots w_i v_i,$$

where $d(w_i, x_i) = D+3$, and $d(v_i, x_i) = D+2$. Let

$$S = \bigcup_{i=2}^{M} V(P_i). \tag{16}$$

Now it is clear from (10) and the definition of the C_i that $\Gamma(v_i) \cap S = \{w_i\}$ for $i = 2, \dots, M$, and $d(v_i, v_j) \geq 6$ for $i \neq j$. It follows from (15) and (16) that $d_{G[S]}(x_1, v_j) \leq 5D + 19$. Applying Lemma 5 to $G[S]$ (with an extra pendant vertex attached to x_1), we see that if

$$M > M(a+2, 5D+19, k)$$

then $G[S]$ contains an induced subdivision of the star $K_{1,a+2}$ with endvertices from $w, v_2, \dots, v_M$, and thus an induced subdivision U of $K_{1,a+1}$ with endvertices from $v_2, \dots, v_M$, where each edge is subdivided at most $5D+19$ times. Let the centre of U be v; we may assume that U has endvertices $v_2, \dots, v_{a+2}$. Then $d(v, v_i) < 3$ for at most one i (since $d(v_i, v_j) \geq 6$ for $i \neq j$), so we may assume

$$d(v, v_i) \geq 3 \tag{17}$$

for $i = 2, \dots, a+1$.

U will form the centre of our induced subdivision of T_a^b. Our aim now is to join U to $T_2, \dots, T_{a+1}$. Recall that, by definition, $d(v_i, x_i) = D+2$ and $d(w_i, x_i) = D+3$. Let Q_i be a shortest path of the form $w_i v_i \cdots y_i t_i$, where every vertex after v_i belongs to $B(x_i, D+1)$ and $t_i \in V(T_i)$.

Now, since T_i is a (T_{2a}^{b-1}, t)-structure, it has subtrees $T_1^*, \dots, T_{2a}^*$, where T_j^* is joined to x_i by a path $R_j = x_i \cdots x_l^*$ of length at least three. Let $U_1, \dots, U_{2a}$ be the components of $T_i \setminus \{x_i\}$, where $V(T_j^*) \subset V(U_j)$, for each j. We construct an induced subdivision of T_a^{b-1} with its root joined by a path to y_i. If y_i has neighbours in at most one of the U_j, say in U_s, then delete U_s and join y_i to x_i by a shortest path P in $G[\{x_i, y_i\} \cup V(U_s)]$; our subdivision is induced by $V(P)$ and any $b-1$ sets from $\{V(U_j) : j \neq s\}$. Otherwise, y_i has neighbours in more than one U_i. In this case, it follows from (6) that y_i can have neighbours in at most one T_i^*; we may suppose y_i has no neighbours in $T_1^*, \ldots, T_{2a-1}^*$. If y has neighbours in at most $a-1$ of the U_i, say among $U_1, \ldots, U_{a-1}$, then join y_i to x_i by a shortest path P in $G[\{x_i, y_i\} \cup \bigcup_{i=1}^{a-1} U_i]$; our subdivision is induced by $V(P) \cup \bigcup_{j=a}^{2a-1} V(U_j)$. Otherwise, we may assume that y_i has neighbours in $U_1, \ldots, U_a$. In this case, join y_i to T_j^* by a shortest path S_j in $\{y_i\} \cup U_j$, and take $\bigcup_{j=1}^{a}(V(S_j) \cup V(T_j^*))$. It is easily checked that for each i we obtain an induced subdivision of T_a^{b-1}, joined to U by a path; adding U, we obtain an induced subdivision of T_a^b. Furthermore, it follows from (7) and (15) that this induced subdivision is a $(T_a^b, 14D + 42)$-structure. □

We have now dealt with graphs that have low 'local' chromatic number. How will a more general proof of Theorem 3 proceed? Well, our aim is to argue by induction on $|T|$. Suppose we have proved the theorem for smaller trees: let

$$N = \max\{c(k, S) : S \text{ is a tree and } |S| < |T|\},$$

where $c(k, S)$ is the minimum c satisfying Theorem 3. Let g, k, d be as in Lemma 6, and let G be a graph with large chromatic number.

How can we find an induced subdivision of T? If we have some $X \subset V(G)$ such that $\chi(G[X]) > g(\chi^{(d)}(G[X]))$ then we are done immediately by Lemma 6. We are also done, by the inductive hypothesis, if $\chi(G[\Gamma(x)]) > c(k-1, T)$.

What other structures guarantee an induced copy of T? Let us call a subset $X \subset V(G)$ *well-covered in* G if for each $x \in X$ there exists $x' \in V(G) \setminus X$ such that $\Gamma(x') \cap X = \{x\}$. If we can find a well-covered subset X of $V(G)$ that induces a graph with chromatic number at least N, then $G[X]$ contains an induced copy of $T \setminus \{t\}$, where t is an endvertex of T. However, since X is well-covered, we can add a vertex from $V(G) \setminus X$ to get an induced copy of T.

What do we do if none of these structures can be found in G? The next lemma says that, provided that a ball around some vertex has high enough chromatic number, then we can build a tree from that vertex.

Lemma 7. *Let T be a tree, let $g\colon \mathbb{N} \to \mathbb{N}$ be an unbounded increasing function and let N, L, d be constants. Let G be a graph such that*

$$\chi(G[B(x,1)]) < N \tag{18}$$

for every $x \in G$, such that no well-covered subset $X \subset V(G)$ satisfies

$$\chi(G[X]) > L \tag{19}$$

and such that whenever H is an induced subgraph of G, we have

$$\chi(H) < g(\chi^{(d)}(H)). \tag{20}$$

Then there exist constants $C(T,N,L,d)$ and $t(T,d)$ such that if, for some $x \in V(G)$,

$$\chi(G[B_G(x,d)]) > C,$$

then there exists an induced subdivision of T, or T with a pendant vertex, that contains x, and in which each edge is subdivided at most t times.

Proof. We prove this for trees of form T_a^b by induction on $b = \mathrm{rad}(T)$, with the additional condition that x corresponds to the root of T_a^b or else corresponds to a pendant vertex added to the root of T_a^b.

For $b = 0$ the result is trivial. Suppose $b > 0$, and we have proved the lemma for smaller values of b. As in Lemma 6, we remark that T can be decomposed into a copies of T_a^{b-1} with their centres joined to a central vertex y. The idea of the proof is to take copies of T_a^{b-1} rooted in $S_G(x,i)$, for some $i < d$, and join them together in $B_G(x, i-1)$.

Let $C_0 = C(T_a^{b-1}, N, L, d+1)$, and let C be a large constant. Suppose

$$\chi(G[B_G(x,d)]) > C,$$

so for some $d_0 \le d$,

$$\chi(G[S_G(x,d_0)]) \ge C/2, \tag{21}$$

since

$$\chi(G[B_G(x,d)]) \le \max_{i=1,\dots,d}(\chi(G[S_G(x,i)]) + \chi(G[S_G(x,i-1)])).$$

Let $X = S_G(x, d_0)$, and let $T_1 \subset S_G(x, d_0 - 1)$ be minimal such that

$$|\Gamma(x) \cap T_1| > 0$$

for all $x \in X$. By minimality of T_1, for each $s \in T_1$ we can find $x_s \in X$ such that

$$\Gamma(x_s) \cap T_1 = \{s\}.$$

Define $U_1 = \{x_s : s \in T_1\}$.

We define sets $T_1, \dots, T_p$ and $U_1, \dots, U_p$ as follows. Given the sets $T_1, \dots, T_j$ and $U_1, \dots, U_j$, if $\bigcup_{i=1}^{j} U_i = X$ then set $p = j$ and stop. Otherwise, let $X_j = X \setminus \bigcup_{i=1}^{j} U_i$, and let $T_{j+1} \subset T_j$ be minimal such that $|\Gamma(x) \cap T_{j+1}| > 0$ for all $x \in X_j$. As before, for each $y \in T_{j+1}$ we can find $x_y \in X_j$ such that $\Gamma(x_y) \cap T_{j+1} = \{y\}$. Define

$$U_{j+1} = \{x_y : y \in T_{j+1}\}.$$

Clearly, for each j, U_j is well-covered in G by T_j, so by (19) we have

$$\chi(G[U_j]) \le L. \tag{22}$$

Furthermore, since $T_j \supset T_{j+1} \supset \cdots$ it follows that every vertex $x \in U_j$ has at most one neighbour in T_i for $i \ge j$. We know from (21) that

$$\chi(G[\bigcup_{j=1}^{p} U_j]) = \chi(G[X]) \ge C/2,$$

and from (22) that, for each l,

$$\begin{aligned}\chi(G[\bigcup_{j=1}^{l+1} U_j]) &\le \chi(G[\bigcup_{i=1}^{l} U_j]) + \chi(G[U_{l+1}]) \\ &\le \chi(G[\bigcup_{j=1}^{l} U_j]) + L.\end{aligned}$$

Let s be minimal such that

$$\chi(G[\bigcup_{j=1}^{s} U_j]) \geq g(C_0) \tag{23}$$

(s is well-defined provided C is sufficiently large) and let

$$Y_1 = \bigcup_{j=1}^{s} U_j.$$

Our aim now is to find (a subdivision of) an induced copy of T_a^{b-1} with one vertex (its root, or a pendant vertex attached to its root) in T_1 and the remainder of its vertices in Y_1. Now from (22) and (23) it follows that

$$g(C_0) \leq \chi(G[Y_1]) \leq g(C_0) + L. \tag{24}$$

By (20), we have $\chi(G[Y_1]) < g(\chi^{(d)}(G[Y_1]))$, and so, for some $y \in Y_1$,

$$\chi(B_{G[Y_1]}(y, d)) > C_0.$$

Pick $z_1 \in \Gamma(y) \cap T_1$, and consider $H = G[\{z_1\} \cup X]$. Since $B_H(z_1, d+1) \supset B_{G[Y_1]}(y, d)$ we have $\chi(B_H(z_1, d+1)) > C_0$. Thus by our inductive hypothesis, we can find an induced subdivision of T_a^{b-1} in H such that z_1 corresponds to its root, or an induced subdivision of T_a^{b-1} with a pendant vertex corresponding to z_1 added to its root, where each edge has been subdivided at most $t(T_a^{b-1})$ times. Call this H_1. Note that H_1 has at most

$$h = (|T_a^{b-1}| + 1)(t(T_a^{b-1}) + 1)$$

vertices.

We want now to define further induced trees $H_2, H_3, \ldots$, with roots $z_2, z_3, \ldots$, such that there is no edge in G between $H_i \setminus z_i$ and $H_j \setminus z_j$ for $i \neq j$. Thus we will have to avoid the vertices adjacent to $H_1 \setminus z_1$. With this in mind, we define

$$\begin{aligned} S_1 &= \left(V(H_1) \cup \bigcup_{x \in V(H_1)} \Gamma(x) \right) \cap X \\ S_2 &= \{x \in T_{s+1} : |\Gamma(x) \cap V(H_1) \setminus \{z_1\}| > 0\} \\ S_3 &= \bigcup_{x \in S_2} \Gamma(x) \cap X. \end{aligned} \tag{25}$$

Clearly, by (18),

$$\begin{aligned}\chi(G[S_1]) &\leq \sum_{x\in V(H_1)} \chi(G[(\Gamma(x)\cap X)\cup\{x\}])\\ &\leq \sum_{x\in V(H_1)} \chi(G[B(x,1)])\\ &\leq hN.\end{aligned}$$

Now $|S_2| \leq h$, since each $x \in V(H_1)\setminus\{z_1\}$ belongs to U_l for some $l \leq s$, and so since (as remarked above) $T_l \supset T_{s+1}$ we have

$$|\Gamma(x)\cap T_{s+1}| \leq |\Gamma(x)\cap T_l| = 1.$$

Thus $|S_2| \leq |H_1| = h$, and so by (18) we have

$$\begin{aligned}\chi(G[S_3]) &\leq \sum_{x\in S_2} \chi(G[\Gamma(x)\cap X])\\ &\leq hN.\end{aligned}$$

Now define

$$X' = X \setminus (Y_1 \cup S_1 \cup S_3)$$

and let $T_1' \subset T_{s+1}\setminus S_2$ be a minimal cover for X'. We have by (24) that

$$\begin{aligned}\chi(G[X']) &\geq \frac{C}{2} - \chi(G[Y_1]) - \chi(G[S_1]) - \chi(G[S_3])\\ &\geq \frac{C}{2} - g(C_0) - L - 2hN.\end{aligned}$$

Provided $C \geq 2M(a,d+1,k)(g(C_0)+L+2hN)$, we can repeat the process $M(a,d+1,k)$ times, to get induced subdivisions $H_1,\ldots,H_{M(a,d+1,k)}$ of T^{b-1}_{a+1} rooted at $z_i \in S(x,i-1)$, or with root joined to z_i by a path of length at most $t(T^{b-1}_{a+1})$, and all remaining vertices in $S(x,i)$. By the definition of (25), the only possible edge between H_i and H_j, for $i \neq j$, is z_iz_j. Applying Lemma 5 to the graph formed by joining x to $z_1,\ldots,z_{M(a,d+1,k)}$ by shortest paths (and adding a pendant vertex to the z_i), we get the required induced subdivision of T^b_a, or T^b_a with a pendant vertex, with each edge subdivided at most

$$2t(T^{b-1}_{a+1}) + 2d$$

times. □

Theorem 3 now follows easily by a double induction on k and T. Indeed, for $k = 1$ the result is immediate. Suppose $k > 1$ and we know the result for smaller values of k (for all T), and for k and smaller trees. As remarked above, if

$$\chi(G[B_G(x, 1)]) > c(k - 1, T)$$

for any $x \in V(G)$, or we have a well-covered subset $X \subset V(G)$ with

$$\chi(G[X]) > c(k, T \setminus \{x\}),$$

where x is an endvertex of T, then we are done. By Lemma 6, there are constants t and d, and a function g, such that if

$$\chi(G[X]) \geq g(\chi^{(d)}(G[X])) \tag{26}$$

for any $X \subset V(G)$, then G contains an induced subdivision of T with each edge subdivided at most t times. If (26) is not satisfied for any $X \subset V(G)$, and $\chi(G) > g(C(T, N, L, d))$, then

$$\chi(G[B_G(x, d)]) > C(T, N, L, d)$$

for some $x \in V(G)$. But then Lemma 7 gives us the required induced subdivision of T. □

§3. Remarks

We can actually strengthen Theorem 3 slightly, in that we can take t to be dependent only on the radius of T. In other words, for every integer r there is an integer $t(r)$ such that, for any tree T of radius r and any integer k, every graph with sufficiently large chromatic number contains a copy of K_k or else an induced copy of T in which each edge is subdivided at most t times. This follows with fairly easy modifications to the proofs of Lemmas 5, 6 and 7.

The bounds for t that follow from the proof above are rather large, and grow exponentially in $|T|$. It would be interesting to give a smaller bound.

We believe that the methods we have developed here should have further application. Motivated by the Strong Perfect Graph Conjecture, Gyárfás [6] has made a number of conjectures about χ-bounded families of graphs. For any integer m, let

$$\mathcal{H}_m = \{C_{2m+1}, C_{2m+3}, \ldots\}$$

and

$$\mathcal{C}_m = \{C_m, C_{m+1}, \ldots\}.$$

Gyárfás conjectured that Forb($\mathcal{H}_2$) is χ-bounded and that Forb($\mathcal{C}_m$) is χ-bounded for $m \geq 4$. (Forb($\mathcal{C}_4$) is the family of triangulated graphs, which is known to be perfect.) He also made the stronger conjecture that Forb($\mathcal{H}_m$) is χ-bounded for $m \geq 2$. We have so far been able to prove only that Forb($\mathcal{H}_2 \cup \mathcal{C}_m$) is χ-bounded for every integer m, but hope that our methods can be exploited for further questions of this type.

All the conjectures mentioned above ask whether, for some family $\mathcal{F}$ of graphs, Forb($\mathcal{F}$) is χ-bounded; clearly, many similar questions can be asked. In particular, the case when $\mathcal{F}$ consists of the subdivisions of a single graph H, so that Forb($\mathcal{F}$) = Forb$^*(H)$, seems interesting: Theorem 1 deals with the case when H is a tree, and one of the conjectures of Gyárfás concerns Forb($\mathcal{C}_m$) = Forb$^*(C_m)$. We make the following stronger conjecture.

Conjecture 8. Forb$^*(H)$ *is* χ*-bounded for every graph* H.

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