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Sub-Hopf algebras of the Steenrod algebra and the Singer transfer

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The Singer transfer provides an interesting connection between modular representation theory and the cohomology of the Steenrod algebra. We discuss a version of "Quillen stratification" theorem for the Singer transfer and its consequences.

55S10, 55U15, 18G35, 55U35, 55T15, 55P42, 55Q10, 55Q45; 46E25, 20C20

Dedicated to Professor Huỳnh Mùi on the occasion of his 60th birthday

1 Introduction

Let $\mathcal{A}$ denote the mod 2 Steenrod algebra (see Steenrod and Epstein [28]). The problem of computing its cohomology $H^{*,*}(\mathcal{A})$ is of great importance in algebraic topology, for this bigraded commutative algebra is the E^2 term of the Adams spectral sequence (see Adams [1]) converging to the stable homotopy groups of spheres. But despite intensive investigation for nearly half a century, the structure of this cohomology algebra remains elusive. In fact, only recently was a complete description of generators and relations in cohomological dimension 4 given, by Lin and Mahowald [12, 11]. In higher degrees, several infinite dimensional subalgebras of $H^*(\mathcal{A})$ have been constructed and studied. The first such subalgebra [1], called the Adams subalgebra, is generated by the elements $h_i \in H^{1,2^i}(\mathcal{A})$ for $i \geq 0$. Mahowald and Tangora [14] constructed the so-called *wedge* subalgebra which consists of some basic generators, propagated by the Adams periodicity operator P^1 and by multiplication with a certain element $g \in H^{4,20}(\mathcal{A})$[1]. The wedge subalgebra was subsequently expanded by another kind of periodicity operator M, discovered by Margolis, Priddy and Tangora [16]. On the other hand, perhaps the most important result to date on the structure of $H^*(\mathcal{A})$ is a beautiful theorem of Palmieri [24] which gives a version of the famous Quillen stratification theorem in group cohomology for the cohomology of the Steenrod algebra. Loosely

[1] This element is also denoted as g_1 in literature.

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speaking, Palmieri's theorem says that modulo nilpotent elements, the cohomology of the Steenrod algebra is completely determined by the cohomology of its elementary sub-Hopf algebras. The underlying ideas in both type of results mentioned above are the same: One can obtain information about $H^{*,*}(\mathcal{A})$ by studying its restriction to various suitably chosen sub-Hopf algebras of $\mathcal{A}$.

In this paper, we take up this idea to investigate the Singer transfer [27]. To describe this map, we first need some notation. We work over the field $\mathbb{F}_2$ and let V_n denote the elementary abelian 2–group of rank n. It is well-known that the mod 2 homology $H_*(BV_n)$ is a divided power algebra on n generators. Furthermore, there is an action of the group algebra $\mathcal{A}[GL(n,\mathbb{F}_2)]$, where the Steenrod algebra acts by dualizing the canonical action in cohomology, and the general linear group $GL(n) := GL(n,\mathbb{F}_2)$ acts by matrix substitution. Let $\overline{\mathcal{A}}$ be the augmentation ideal of $\mathcal{A}$. Let $P_{\mathcal{A}}H_*(BV_n)$ be the subring of $H_*(BV_n)$ consisting of all elements that are $\overline{\mathcal{A}}$–annihilated. In [27], Singer constructed a map from this subring to the cohomology of the Steenrod algebra

$$\mathtt{Tr}_n^{\mathcal{A}}\colon\ P_{\mathcal{A}}H_*(BV_n) \longrightarrow H^{n,n+*}(\mathcal{A}),$$

in such a way that the *total* transfer $\mathtt{Tr}^{\mathcal{A}} = \bigoplus_n \mathtt{Tr}_n^{\mathcal{A}}$ is a bigraded algebra homomorphism with respect to the product by concatenation in the domain and the usual Yoneda product for the Ext group. Moreover, there is a factorization through the coinvariant ring $[P_{\mathcal{A}}H_*(BV_n)]_{GL(n)}$,

$$\begin{tikzcd} P_{\mathcal{A}}H_*(BV_n) \arrow[rr, "Tr_n^{\mathcal{A}}"] \arrow[dr, "q"'] & & H^{n,n+*}(\mathcal{A}), \\ & {[P_{\mathcal{A}}H_*(BV_n)]_{GL(n)}} \arrow[ur, dotted, "\varphi_n^{\mathcal{A}}"'] & \end{tikzcd}$$

and $\varphi^{\mathcal{A}} = \bigoplus_{n\geq 0}\varphi_n^{\mathcal{A}}$ is again a homomorphism of a bigraded algebra. The map $\mathtt{Tr}^{\mathcal{A}}$ can be thought of as the E_2 page of the stable transfer $B(\mathbb{Z}/2)^n_+ \to S^0$ (see Mitchell [21]) hence the name "transfer". Singer computed this map in small ranks, and found that $\varphi_n^{\mathcal{A}}$ is an isomorphism for $n \leq 2$. Later, Boardman [4], with additional calculations by Kameko [10], showed that $\varphi_3^{\mathcal{A}}$ is also an isomorphism. In fact, Singer has conjectured that $\varphi^{\mathcal{A}}$ is always a monomorphism. Our main interest in this paper is the image of $\varphi^{\mathcal{A}}$. It appears from the calculations above that the image of the transfer $\mathtt{Tr}^{\mathcal{A}}$ is a large, interesting and accessible subalgebra of $H^{*,*}(\mathcal{A})$. In particular, this image contains the Adams subalgebra on generators h_i. Our calculation strongly suggests that $\varphi^{\mathcal{A}}$ also detects many elements in the wedge subalgebra. In fact, in some sense, elements in the wedge subalgebra have a better chance to be in the image of the transfer than others.

The study of the Singer transfer is intimately related to the problem of finding a minimal set of generators for the cohomology ring $H^*(BV_n)$ as a module over the Steenrod algebra[2]. The $\mathcal{A}$–indecomposables in $H^*(BV_n)$ were completely calculated by Peterson [26] for $n \leq 2$, and by Kameko [10] for $n = 3$, and in "generic degrees" for all n by Nam [23]. Here we prefer to work with the dual $P_{\mathcal{A}}H_*(BV_n)$ because of its ring structure, and also because we are interested in the algebra structure by considering this subring for all n. We should mention that Smith and Meyer [18] have recently found a surprising connection between the subring $P_{\mathcal{A}}H_*(BV_n)$ and a certain type of Poincaré duality quotient of the polynomial algebra, a subject of great interest in modular invariant theory.

An important ingredient in Kameko's calculation [10] of the $\mathcal{A}$–generators for $H^*(BV_3)$ is the existence of an operator

$$Sq^0\colon\ P_{\mathcal{A}}H_d(BV_n) \to P_{\mathcal{A}}H_{2d+n}(BV_n),$$

for all $d, n \geq 0$. To explain the notation, recall that there are Steenrod operations $\widetilde{Sq}^i$ acting on the cohomology of any cocommutative Hopf algebra (see May [17] or Liulevicius [13]) such that the operation $\widetilde{Sq}^0$ is not necessarily the identity. It turns out that Kameko's operation commutes with $\widetilde{Sq}^0$ via the Singer transfer (see Boardman [4]). This key property is used by Bruner, Hà and Hưng [5] to show that the family of generators $g_i \in H^{4,*}(\mathcal{A})$ is not in the image of the transfer. As a result, 4 is the first degree where $\varphi_4^{\mathcal{A}}$ is not an epimorphism. In another direction, Carlisle and Wood [6] showed that as a vector space, the dimension of $P_{\mathcal{A}}H_d(BV_n)$ is uniformly bounded; that is, it has an upper bound which depends only on n. It follows that some sufficiently large iteration of the endomorphism Sq^0 must become an isomorphism. In fact, Hưng [7] showed that the number of iterations needed is precisely $(n-2)$. This beautiful observation allowed him to obtain many further information on the image of the Singer transfer. Among other results, Hưng showed that for each $n \geq 5$, $\varphi_n^{\mathcal{A}}$ is not an isomorphism in infinitely many degrees (the same conclusion for $\varphi_4^{\mathcal{A}}$ can be deduced from the main result of [5].) Moreover, using some computer calculations by Bruner (using MAGMA) and by Shpectorov (using GAP), Hưng also made a comprehensive analysis of the image of the transfer in rank 4 and gave a conjectural list of elements in $H^{4,*}(\mathcal{A})$ that should, or should not, be in the image of φ_4 provided Singer's conjecture is true in rank 4.

Despite these successful calculations, it seems that one has arrived at the computation limit on both sides of the Singer transfer. What we really need now are some global

[2]This problem is called "the hit problem" in the literature (see Wood [31]).

results on the structure of the graded algebra $\bigoplus_n P_{\mathcal{A}}H_*(BV_n)$. This paper is the first step in our investigation of the multiplicative structure of the graded algebra $P_{\mathcal{A}}H_*(BV_*)$ using suitably chosen sub-Hopf algebras of $\mathcal{A}$. We shall show that the tranfer can be constructed not only for the Steenrod algebra, but also any of its sub-Hopf algebras. We then propose a conjecture which is the analog of the Quillen stratification theorem for $P_{\mathcal{A}}H_*(BV_*)$. We also make some calculations for the transfer with respect to an important class of sub-Hopf algebras of $\mathcal{A}$. One of the main results in this paper is the following application to the study of the original Singer transfer that, in our opinion, demonstrates the potential power of our approach.

Theorem 1.1 (i) *The element $g \in H^{4,24}(\mathcal{A})$ is not in the image of the Singer transfer.*
(ii) *The elements $d_0 \in H^{4,18}(\mathcal{A})$ and $e_0 \in H^{4,21}(\mathcal{A})$ are in the image of the Singer transfer.*

We refer to Mahowald–Tangora [14] and Zachariou [33] for detailed information about the generators that appear in this theorem. The fact that g, and in fact the whole family of generators $g_i = (\widetilde{Sq}^0)^i g$, are not in the image of the Singer transfer was already proven in [5]. We give a different proof, which is much less computational. The second part of our theorem is new and give an affirmative answer to a part of Hưng's conjecture [7, Conjecture 1.10]. It should be noted that our method seems applicable to many other generators in the wedge subalgebra, but the calculation seems much more daunting.

Organization of the paper

The first two sections are preliminaries. In Section 1, we recall basic facts about the Steenrod algebra and its sub-Hopf algebras. Detailed information about the action of the operation P^s_t on $H_*(BV_n)$ is also given. In Section 3, we present a convenient resolution, called the *Hopf bar resolution* by Anderson and Davis [3], to compute the cohomology of a Hopf algebra. This resolution is then used to construct a chain-level representation of the Singer transfer for any sub-Hopf algebra B of $\mathcal{A}$. The idea of using this particular resolution to study the Singer transfer is due to Boardman [4] (See also Minami [20].) The remaining three sections are related, but independent of each other and can be read separately. We present our stratification conjectures for the domain of the Singer transfer in Section 4. Section 5 is devoted to properties of the B–transfer for various sub-Hopf algebras B of $\mathcal{A}$. Section 6 contains the proof of Theorem 1.1, which is one of the main applications of our approach.

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2 Sub-Hopf algebras of the Steenrod algebra

In this section, we review some basic facts about the Milnor basis of the Steenrod algebra and the classification of sub-Hopf algebras of $\mathcal{A}$. There are several excellent references on these materials in literature; among them we highly recommend Margolis's book [15] and Palmieri's memoir [25]. The original sources are Milnor [19] and Anderson and Davis [3].

2.1 Milnor's generators

It is generally more convenient to describe sub-Hopf algebras of the Steenrod algebra in terms of their dual, as quotient algebras of the dual Steenrod algebra $\mathcal{A}^*$. We recall some necessary materials about the dual of the Steenrod algebra. According to Milnor [19], there is an algebra isomorphism

$$\mathcal{A}^* \cong \mathbb{F}_2[\xi_0, \xi_1, \ldots, \xi_n, \ldots],$$

where ξ_t is in degree $2^t - 1$, and ξ_0 is understood to be the unit 1. The coproduct Δ is given by

$$\Delta(\xi_n) = \sum_i \xi_{n-i}^{2^i} \otimes \xi_i. \tag{2–1}$$

For any $s \geq 0$ and $t > 0$, let $P_t^s \in \mathcal{A}$ denote the dual of $\xi_t^{2^s}$. These generators are very important for our purpose. We briefly review some of their fundamental properties. If $s < t$, then P_t^s is a differential, ie $(P_t^s)^2 = 0$. The effect of P_t^s on $H^*(\mathbb{R}P^\infty) \cong \mathbb{F}_2[x]$ is completely determined by the formula

$$P_t^s x^{2^k} = x^{2^{s+t}}$$

if $k = s$, and zero for all $k \neq s$. Let $b_k \in H_*(B\mathbb{Z}/2)$ denote the dual of x^k. We will be working extensively with the dual action which reads

$$b_k P^s_t = \binom{k - 2^s(2^t-1)}{2^s} b_{k-2^s(2^t-1)}, \tag{2–2}$$

where binomial coefficients are taken modulo 2. Write $2^s \in k$ if 2^s appears in the binary expansion of k and $2^s \notin k$ if the opposite happens. We will need the following simple but useful lemma.

Lemma 2.1 *With the notation as above,*

(i) $b_k P^s_t = 0$ *if and only if either* $k < 2^{s+t}$, *or* $k \geq 2^{s+t}$ *and* $2^s \in k$.

(ii) b_k *is in the image of* P^s_t *if and only if* $k \geq 2^{s+t}$ *and* $2^s \in k$.

Proof $\binom{k-2^s(2^t-1)}{2^s}$ is non-zero if and only if $k \geq 2^{s+t}$, and $2^s \in k - 2^s(2^t-1)$. The latter condition is clearly equivalent to $2^s \notin k$. □

2.2 Sub-Hopf algebras of the Steenrod algebra

We are mainly interested in two families of sub-Hopf algebra of $\mathcal{A}$: the elementary ones, which essentially play similar role as elementary abelian subgroups in group cohomology; and the normal ones, which serve as intermediate between the elementary sub-Hopf algebra and the whole Steenrod algebra.

Let A be a Hopf algebra. A sub-Hopf algebra $E \subset A$ is called *elementary* if it is bicommutative, and $e^2 = 0$ for any element e in the augmentation ideal $\bar{E}$ of E. This definition is due to Wilkerson [29, page 138]. We now specialize to the case $A = \mathcal{A}$. Each elementary sub-Hopf algebra E of $\mathcal{A}$ is isomorphic, as an algebra, to the exterior algebra on the operation P^s_ts that it contains. In particular,

$$H^*(E) \cong \mathbb{F}_2[h_{ts} | P^s_t \in E], \tag{2–3}$$

where h_{ts} is represented by $[\xi_t^{2^s}]$ in the cobar complex for E, so $|h_{ts}| = (1, 2^s(2^t-1))$. Among these elementary sub-Hopf algebras, the maximal ones have the form

$$E(m)^* = \mathcal{A}^* /(\xi_1, \ldots, \xi_{m-1}, \xi_m^{2^m}, \xi_{m+1}^{2^m}, \ldots), \tag{2–4}$$

for each $m \geq 1$. Equivalently, $E(m)$ is generated by those P^s_t for which $s < m \leq t$. We now discuss normal sub-Hopf algebras of $\mathcal{A}$. We say that a sub-Hopf algebra B is normal in $\mathcal{A}$ if the left and the right ideal generated by $\bar{B}$ are equal; that is,

$\bar{B}\mathcal{A} = \mathcal{A}\bar{B}$. If B is normal in $\mathcal{A}$, then one can define the quotient Hopf algebra $\mathcal{A}/\!/B$ as $\mathcal{A}\otimes_B \mathbb{F}_2 = \mathbb{F}_2 \otimes_B \mathcal{A}$. The short exact sequence of vector spaces $B \to \mathcal{A} \to \mathcal{A}/\!/B$ is called a *Hopf algebra extension*. Of course, this definition applies not just for the Steenrod algebra, but also to any cocommutative Hopf algebra over $\mathbb{F}_2$. The normal sub-Hopf algebras of $\mathcal{A}$ are completely classified (see Margolis [15, Theorem 15.6]). They correspond to non-decreasing sequences $n_1 \le n_2 \le \cdots \le \infty$ via the correspondence

$$(n_1, n_2, n_3, \dots) \to \mathcal{A}^* /(\xi_1^{2^{n_1}}, \xi_2^{2^{n_2}}, \xi_3^{2^{n_3}} \dots).$$

In particular, maximal elementary sub-Hopf algebras $E(m)$ are normal in $\mathcal{A}$. The union of $E(m)$s, denoted by D, turns out to be another normal sub-Hopf algebra of $\mathcal{A}$. In fact, as observed by Palmieri [24], there is a whole sequence of normal sub-Hopf algebras, starting at D

$$D = \bigcap_m D(m) \to \cdots \to D(m) \to D(m-1) \cdots \to D(1) \to D(0) = \mathcal{A}, \tag{2–5}$$

where $D(m)$ is defined in terms of its dual as follows,

$$D(m)^* = \mathcal{A}^* /(\xi_1^2, \xi_2^4, \dots, \xi_m^{2^m}). \tag{2–6}$$

In other words, $D(m)$ is generated by the operations P^s_ts where either $t > m$ or $s < t \le m$. The quotient $D(m-1)/\!/D(m)$ is the exterior algebra on generators P^{m+i}_m, $i \ge 0$. In particular, $D(m-1)/\!/D(m)$ is $(2^m(2^m-1)-1)$-connected.

3 Construction of the B–transfer

Let B be any sub-Hopf algebra of $\mathcal{A}$. Clearly B is also a graded connected and cocommutative, so that results from Section 3 can be applied to B. We will construct a chain level representation $P_B H_*(BV_n) \to (\bar{B}^*)^n$ for the analogue of the Singer transfer for any sub-Hopf algebra B of $\mathcal{A}$. We begin with a review of the so-called (reduced) Hopf bar resolution that we will use.

3.1 Hopf bar resolution

Let A be a graded cocommutative connected Hopf algebra over $\mathbb{F}_2$ and let M be an A–module. We denote by $\mu\colon A\otimes A \to A$ the product and $\Delta\colon A \to A\otimes A$ the coproduct maps. In this section, we present the (normalized) Hopf bar construction of A for M, introduced by Anderson and Davis [3], to calculate the cohomology $H^{*,*}(A, M)$ of A

with coefficient in M. This construction is functorial with respect to maps between Hopf algebras as well as maps between A–modules when A is fixed. This particular resolution is well-suited to our purposes rather than the usual bar resolution because, as we shall see later, there exists an explicit description of the representing map for the transfer.

Observe that if A is a cocommutative connected Hopf algebra, then the tensor product of any two A–modules is again an A–module via the coproduct Δ. Let $\bar{A}$ be the augmentation ideal of A. From the obvious short exact sequence of A–modules $0 \to \bar{A} \to A \to \mathbb{F}_2 \to 0$, tensoring (over $\mathbb{F}_2$) with $\bar{A}^{\otimes k} \otimes M$ (from now on, we will write $\bar{A}^k$ instead of $\bar{A}^{\otimes k}$ to avoid clustering) and splicing together the resulting short exact sequences, we obtain a chain complex $\mathcal{H}(M)$ which is visibly exact:

$$\cdots \to A \otimes \bar{A}^k \otimes M \to A \otimes \bar{A}^{(k-1)} \otimes M \to \cdots \to A \otimes M \to M. \tag{3–1}$$

We claim that $\mathcal{H}(M)$ is an A–free resolution of M. Indeed, it suffices to verify that $A \otimes \bar{A}^k \otimes M$ is a free A–module for each $k \geq 0$. This is not quite as obvious as it seems because by its construction, the A–module structure of $A \otimes \bar{A}^k \otimes M$ is via the iterated coproduct. However, this A–action can be modified by mean of a well-known trick that for any A–module N, the usual (ie A–module structure via coproduct) A–module $A \otimes N$ is isomorphic to the A–module $A \otimes tN$ where A acts only on the copy of A, and tN signifies the same $\mathbb{F}_2$–vector space N, but with trivial A–action (see Anderson and Davis [3, Proposition 2.1]).

Let $M^* = \mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_2)$ be the $\mathbb{F}_2$–linear dual of M. Taking $\mathrm{Hom}_A(\mathcal{H}(M), \mathbb{F}_2)$ and simplify, we obtain a cochain complex

$$0 \longrightarrow M^* \xrightarrow{\partial_0} \bar{A}^* \otimes M^* \xrightarrow{\partial_1} \cdots \longrightarrow (\bar{A}^*)^k \otimes M^* \xrightarrow{\partial_k} (\bar{A}^*)^{(k+1)} \otimes M^* \longrightarrow \cdots .$$

whose homology is $H^{*,*}(A; M)$. Since we have modified the A–action on $A \otimes \bar{A}^k \otimes M$, the differential ∂_0 becomes a kind of twisted coaction map,

$$\partial_0 \colon M^* \xrightarrow{\alpha_*} \bar{A}^* \otimes M^* \xrightarrow{\chi \otimes id} \bar{A}^* \otimes M^*, \tag{3–2}$$

where $\alpha \colon A \otimes M \to M$ is the map that defines the A–module structure on M. There is similar description for ∂_k, $k > 0$. Namely, if $m \in M^*$ with $\alpha_*(m) = \sum_\nu a_\nu \otimes m_\nu$, then

$$\partial_k(a_1|\cdots|a_k|m) = \sum_\nu \sum \chi(a'_1 \cdots a'_k a_\nu)|a''_1|\cdots|a''_k|m_\nu. \tag{3–3}$$

if $\mu^*(a_i) = \sum a'_i \otimes a''_i$ (See Anderson and Davis [3, pages 320–321].) In particular, if $M = \mathbb{F}_2$, then equation (3–3) becomes

$$\partial_k(a_1|\cdots|a_k) = \sum (\chi(a'_1 \cdots a'_k)|a''_1|\cdots|a''_k). \tag{3–4}$$

3.2 Construction of the B–transfer

We begin with the existence of the B–transfer.

Theorem 3.1 *Let B be a sub-Hopf algebra of $\mathcal{A}$. For each $n \geq 0$, there exists a map*

$$P_B H_*(BV_n) \xrightarrow{\mathtt{Tr}_n^B} H^{n,n+*}(B),$$

natural with respect to B, such that it factors through the coinvariant ring

$$\begin{tikzcd} P_B H_*(BV_n) \arrow[rr, "\mathtt{Tr}_n^B"] \arrow[dr, "q"'] & & H^{n,n+*}(B). \\ & {[P_B H_*(BV_n)]_{GL(n)}} \arrow[ur, dotted, "\varphi_n^B"'] & \end{tikzcd}$$

Furthermore, the total B–transfer $\varphi^B = \bigoplus_n \varphi_n^B$ is a homomorphism of bigraded algebras.

Proof A careful look at Singer's construction shows that his proof also works for any sub-Hopf algebra B of $\mathcal{A}$. We follow Boardman's approach because it provides an explicit description of a map $P_B H_*(BV_n) \to (\bar{B}^*)^n$ which represents $\mathtt{Tr}_n^B$.

To begin, we need to introduce several $\mathcal{A}$–modules related to $P = \mathbb{F}_2[x] = H^*(B\mathbb{Z}/2)$. Let $\hat{P}$ be obtained from P by formally adding a basis element x^{-1} in degree -1 and equip $\hat{P}$ with an $\mathcal{A}$–action such that it is the unique extension of the $\mathcal{A}$–action on P that also satisfies the Cartan formula. According to the referee, $\hat{P}$ was first introduced by Adams in [2].

Let $\hat{f}$ be the $\mathcal{A}$–epimorphism $\mathcal{A} \to \hat{P}$ such that $\hat{f}(1) = x^{-1}$. Denote by f its restriction to $\overline{\mathcal{A}}$. It is clear that f maps into P. Moreover, f is B–linear for any sub-Hopf algebra B of $\mathcal{A}$. On the other hand, the inclusion $B \to \mathcal{A}$ provides $\mathcal{A}$ with a B–module structure, so the dual $\mathcal{A}^* \to B^*$ is a map of right B–modules. It follows that the following composition is a map of right B–modules:

$$f_n^B \colon\ H_*(BV_n) \xrightarrow{f_*^{\otimes n}} (\overline{\mathcal{A}}^*)^n \to (\bar{B}^*)^n. \tag{3–5}$$

The image of this map of any B–annihilated element in $H_*(BV_n)$ is a cocycle in the Hopf resolution for B. Thus f_n^B induces a map

$$\mathtt{Tr}_n^B\colon\ P_B H_*(BV_n) \longrightarrow H^{n,n+*}(B),$$

which is our version of the Singer transfer for the sub-Hopf algebra B. It is clear that the construction just described is natural with respect to B. □

Singer's proof that $\mathtt{Tr}_n$ factors through $GL(n)$ coinvariants is very simple and elegant. Recall that $GL(n)$ is generated by the symmetric group Σ_n and an element denoted by τ of order 3 in $GL(2)$, considered as a subgroup of $GL(n)$ in the obvious way. In terms of the chain-level map that we have just constructed, the fact that $\mathtt{Tr}_n$ factors through Σ_n–coinvariants is precisely because the Steenrod algebra $\mathcal{A}$ is cocommutative. That $\mathtt{Tr}_2$ is invariant under τ seems much less obvious. In fact, we have to use a smaller resolution, which is the Lambda algebra. The author plans to write about this elsewhere.

Remark 3.2 According to Boardman [4], the $\mathcal{A}$–linear map f_* has an explicit description as follows. Let b_k be the generator of $H_k(B\mathbb{Z}/2)$ in degree k, dual to $x^k \in H^k(B\mathbb{Z}/2)$. Then the image of b_k under f_* is the coefficient of x^{k+1} in the expansion of

$$\prod_{i=0}^{\infty}(1\otimes 1 + x^{2^i(2^1-1)}\otimes \xi_1^{2^i} + x^{2^i(2^2-1)}\otimes \xi_2^{2^i} + x^{2^i(2^3-1)}\otimes \xi_3^{2^i} + \cdots). \tag{3–6}$$

Of course, for a fixed degree k, one needs only to consider finite products. The situation is even simpler for f_n^B when B is small because many elements of the form $\xi_t^{2^s}$ get killed when mapped down to $\bar{B}^*$.

For example, if $B = E(1)$, then the only nontrivial factor in the above infinite product is $i = 0$ and so the only nontrivial images are $f_*(b_{2^t-2}) = \xi_t$, $t \geq 1$.

Now let $\mathcal{E}$ be the collection of all elementary sub-Hopf algebras of $\mathcal{A}$. By the naturality of the transfer $\mathtt{Tr}^B$, we have the commutative diagram

$$\begin{tikzcd}
P_{\mathcal{A}}H_*(BV_n) \arrow[r, "i_D^{\mathcal{A}}"] \arrow[d, "\mathtt{Tr}_n^{\mathcal{A}}"'] & P_D H_*(BV_n) \arrow[r, "i_{\mathcal{E}}^D", "\cong"'] \arrow[d, "\mathtt{Tr}_n^D"'] & \varprojlim_{\mathcal{E}} P_E H_*(BV_n) \arrow[d, "\lim \mathtt{Tr}_n^E"] \\
H^{n,n+*}(\mathcal{A}) \arrow[r, "\mathrm{res}_D^{\mathcal{A}}"] & H^{n,n+*}(D) \arrow[r, "\mathrm{res}_{\mathcal{E}}^D"] & \varprojlim_{\mathcal{E}} H^{n,n+*}(E),
\end{tikzcd} \tag{3–7}$$

where the horizontal maps are, or are induced by, the obvious inclusions. Note that $i_D^{\mathcal{A}}$ is a monomorphism, and $i_{\mathcal{E}}^D$ is an isomorphism because D is generated by the elementary sub-Hopf algebras of $\mathcal{A}$. Of course, the induced maps φ_n^B after passing to $GL(n)$–coinvariant rings for various Bs in the diagram above between need not be mono nor epi.

For convenience, write $\mathtt{Tr}_n^{\mathcal{E}}$ for the inverse limit $\varprojlim_E P_E H_*(BV_n)$, and call it the $\mathcal{E}$–transfer. The factorization of the $\mathcal{E}$–transfer to the coinvariant ring is denoted by $\varphi^{\mathcal{E}}$.

3.3 Kameko's Sq^0

Perhaps the the most useful tool in the study of the hit problem and the Singer transfer is a certain operator defined by Kameko [10]. We discuss the behavior of this operator with respect to the B–transfer. According to Liulevicius [13] (see also May [17]), the cohomology of any cocommutative Hopf algebra A is equipped with an action of the Steenrod algebra where $\widetilde{Sq}^0$ may act nontrivially (ie it is not necessarily the identity). In fact, $\widetilde{Sq}^0\colon\ H^{n,q}(A) \to H^{n,2q}(A)$ is induced by the Frobenius $z \mapsto z^2$ in the cochain level.

Kameko's operation on $P_{\mathcal{A}}H_*(BV_n)$ behaves much like $\widetilde{Sq}^0$. By definition, it is a map $H_d(BV_n) \to H_{2d+n}(BV_n)$, given by the formula

$$Sq^0\colon\ b_{i_1}\cdots b_{i_n} \mapsto b_{(2i_1+1)}\cdots b_{(2i_n+1)}.$$

In fact, it is easy to check that under the inclusion f_*, this operation corresponds precisely with the Frobenius homomorphism in $(\mathcal{A}^*)^n$. The following easy lemma describes the relation between Kameko's Sq^0 and the operations P^s_t.

Lemma 3.3 $(Sq^0z)P^0_t = 0$ *and* $(Sq^0z)P^s_t = Sq^0(zP^{s-1}_t)$ *when* $s > 1$ *for any* $z \in H_*(BV_n)$.

Proof This is immediate from formula (2–2). □

It follows from this lemma that Kameko's operation also induces an automorphism on $P_BH_*(BV_n)$ where B is one of the sub-Hopf algebras $E(m), D(m)$ or D. Thus Kameko's Sq^0 commutes with the Liulevicius–May $\widetilde{Sq}^0$ via the B–transfer for any sub-Hopf algebras of the types above.

Remark 3.4 In [6], Carlisle and Wood proved a striking property that the dimension of the vector space $P_{\mathcal{A}}H_*(BV_n)$ is bounded, for each n fixed. It can be shown that the same is true when replacing $\mathcal{A}$ by any sub-Hopf algebra $D(m)$. This is essentially because the quotient algebra $\mathcal{A}//D(m)$ is finite. On the other hand, $P_DH_*(BV_n)$ is not uniformly bounded. Here is an example in rank 2. Choose a number $a > 1$, and for all $1 \le i \le a-1$, let $k_i = 2^a(2\times 2^i - 1) - 1$, and $\ell_i = 2^{a+i+1}(2\times 2^{a-i-1} - 1) - 1$. One can verify, using Corollary 5.3 below, that in degree $d = 2^{2a+1} - 2^a - 1$, all $(a-1)$ monomials $b_{k_i}b_{\ell_i}$ are D–annihilated. Since a is chosen arbitrarily, it follows that $P_DH_*(BV_2)$ is not uniformly bounded.

4 Two stratification conjectures

In this section, we discuss our conjectures on the domain of the Singer transfer, which are the analogues of Quillen stratification theorem about the cohomology of finite groups, and Palmieri's version for cohomology of the Steenrod algebra. These conjectures, if true, would provide some global information on the mysterious algebra $\bigoplus_n P_{\mathcal{A}} H_*(BV_n)$.

The lower horizontal maps in (3–7) were studied by Palmieri [24]. We shall first summarize his results.

4.1 Palmieri's stratification theorems

Just as in the classical case of group cohomology, there is an action of the Hopf algebra $\mathcal{A}$ on the cohomology of its sub-Hopf algebra $H^{*,*}(D)$ such that the image of the restriction map $\mathrm{res}^{\mathcal{A}}_{D}$ actually lands in the subring $[H^{*,*}(D)]^{\mathcal{A}}$ of elements that are invariant under this action. Since D is normal in $\mathcal{A}$ and since $D \subset \mathcal{A}$ acts trivially on the cohomology of itself, we can write $H^{*,*}(D)^{\mathcal{A}} = H^{*,*}(D)^{\mathcal{A}//D}$. The following is Palmieri's version of Quillen stratification for the Steenrod algebra.

Theorem 4.1 (Palmieri [24]) *The canonical maps*

$$H^{*,*}(\mathcal{A}) \xrightarrow{\mathrm{res}^{\mathcal{A}}_{D}} [H^{*,*}(D)]^{\mathcal{A}//D},$$

and

$$H^{*,*}(D) \xrightarrow{\mathrm{res}^{D}_{\mathcal{E}}} \varprojlim_{\mathcal{E}} H^{*,*}(E),$$

are F–isomorphisms. That is, their kernel and cokernel are nilpotent (in an algebraic sense).

Moreover, the limit $R = \varprojlim_{\mathcal{E}} H^{*,*}(E)$ is computable. To describe it, recall that the cohomology of each elementary sub-Hopf algebra E is a polynomial algebra on elements $h_{t,s}$ which corresponds to each generator P^s_t that it contains. The second theorem of Palmieri says:

Theorem 4.2 (Palmieri [24, Theorem 1.3]) *There is an isomorphism of $\mathcal{A}$–algebras*

$$R = \mathbb{F}_2[h_{t,s} | s < t]/(h_{t,s}h_{v,u} | u \geq t),$$

where $|h_{t,s}| = (1, 2^s(2^t-1))$. *The action of* $\mathcal{A}$ *is given by the Cartan formula and the following formula on the generators*

$$Sq^{2^k} h_{t,s} = \begin{cases} h_{t-1,s+1} & \text{if } k = s \text{ and } t-1 > s+1, \\ h_{t-1,s} & \text{if } k = s+t-1 \text{ and } t-1 > s, \\ 0 & \text{otherwise.} \end{cases}$$

It is important to note that the $\mathcal{A}$–action here is induced by the inclusion $E \subset \mathcal{A}$ for each $E \in \mathcal{E}$, not the kind of $\mathcal{A}$–action of Liulevicius–May on cohomology of cocommutative Hopf algebras that we mentioned earlier. For the latter action, it can be verified in the cochain level that $\widetilde{Sq}^0$ acts on R by sending $h_{t,s}$ to $h_{t,s+1}$ if $s+1 < t$, and sending $h_{t,t-1}$ to zero.

Combining the two F-isomorphisms in Theorem 4.1, we obtain an isomorphism mod nilpotents $H^{*,*}(\mathcal{A}) \to R^{\mathcal{A}}$. There are two major difficulties if we want to use this map to study $H^{*,*}(\mathcal{A})$. First of all, the invariant ring $R^{\mathcal{A}}$, which is another kind of "hit problem", seems to be very complicated because it has lots of zero divisors. Secondly, given an invariant element in $R^{\mathcal{A}}$, we do not know which power of it is lifted to $H^{*,*}(\mathcal{A})$. On the other hand, several families of elements in $R^{\mathcal{A}}$ are known and we will discuss the problem whether the Singer transfer or its B–analogues detects several such families in Sections 5 and 6.

4.2 Two conjectures

Base on Palmieri's results and the Singer conjecture, we make the following conjectures.

Conjecture 4.3 *The following canonical homomorphism of algebras,*

$$i_D^{\mathcal{A}}\colon \bigoplus_n P_{\mathcal{A}} H_*(BV_n)_{GL(n)} \longrightarrow \bigoplus_n [P_D H_*(BV_n)_{GL(n)}]^{\mathcal{A}//D}, \quad \text{and} \tag{4–1}$$

$$i_{\mathcal{E}}^D\colon \bigoplus_n [P_D H_*(BV_n)]_{GL(n)} \longrightarrow \bigoplus_n \varprojlim_{E\in\mathcal{E}} [P_E H_*(BV_n)]_{GL(n)}, \tag{4–2}$$

are both F*–isomorphisms.*

Here is our evidence for making the above conjecture. First of all, they are true for trivial reasons in degree $n = 1$. In fact, both maps are isomorphisms in this degree. Secondly, recall the Singer conjecture that $\varphi^{\mathcal{A}}$ is a monomorphism. If this conjecture is true, then according to Theorem 4.1, it is necessary that the kernel of the two homomorphisms above be nilpotent. Our final evidence is the fact that $\varphi_n^{\mathcal{A}}$ is an isomorphism for $n \le 3$. But in these degrees, only $h_0 \in H^{1,1}(\mathcal{A})$ has infinite height, so the kernel of $i_{\mathcal{E}}^{\mathcal{A}}$ is nilpotent for $n \le 3$.

We now describe a refinement, by adapting Palmieri's proof of his theorem, of the first part of the above conjecture. If M is an $\mathcal{A}$–module of finite type, then the family of normal sub-Hopf algebras $D(m)$ induces a filtration

$$P_D M \supseteq \cdots \supseteq P_{D(m)} M \supseteq P_{D(m-1)} M \supseteq \cdots \supseteq P_{\mathcal{A}} M,$$

which stabilizes in each degree. The first part of Conjecture 4.3 will be a consequence of the following.

Conjecture 4.4 *For each $m \ge 1$. The canonical map*

$$\bigoplus_n [P_{D(m-1)} H_*(BV_n)]_{GL(n)} \to \bigoplus_n ([P_{D(m)} H_*(BV_n)]_{GL(n)})^{D(m-1)//D(m)},$$

is an F–isomorphism.

Of course, $P_{D(m-1)} H_*(BV_n) = [P_{D(m)} H_*(BV_n)]^{D(m-1)//D(m)}$. The problem alluded to is the order of taking $GL(n)$–coinvariant and taking invariant under Steenrod operations.

For the second part of Conjecture 4.3, we have the following observation. Since the family of maximal elementary sub-Hopf algebra $E(m)$ is cofinal in the category $\mathcal{E}$, we can define an element in $\varprojlim_E [P_E H_*(BV_n)]_{GL(n)}$ as a family of compatible elements $z_m \in [P_{E(m)} H_*(BV_n)]_{GL(n)}$. Compatibility means that the restriction of z_m and $z_{m'}$ to $[P_{E(m)\cap E(m')} H_*(BV_n)]_{GL(n)}$ must be the same for any m and m'.

On the other hand, in each fixed degree d, any element of $H_d(BV_n)$ is $E(m)$–annihilated for m large. It follows that in a compatible family $\{z_m\}$, one has

$$z_m = z_{m+1} = \cdots \in [H_d(BV_n)]_{GL(n)},$$

for m sufficiently large. It is well-known that the algebra $[H_*(BV_n)]_{GL(n)}$, which is dual to the Dickson algebra of $GL(n)$–invariants of polynomial algebra on n generators, is $(2^{n-1} - 2)$–connected. It follows that any element in the algebra

$$\bigoplus_n [H_*(BV_n)]_{GL(n)}$$

is nilpotent. Indeed, if $z \in [H_d(BV_n)]_{GL(n)}$, then $z^k \in [H_{dk}(BV_{nk})]_{GL(nk)}$ is in degree dk, which is less that the connectivity $(2^{nk-1} - 2)$ for k large.

We have shown that if

$$\bigoplus_n \lim\nolimits^f_E [P_E H_*(BV_n)]_{GL(n)}$$

denotes the subalgebra of

$$\bigoplus_n \varprojlim_E \, [P_E H_*(BV_n)]_{GL(n)}$$

consisting of *finite* sequences $\{z_m\}$ (ie $z_m = 0$ for all but finitely many values of m). Then

Lemma 4.5 *The inclusion of algebras*

$$\bigoplus_n \varprojlim_E{}^f [P_E H_*(BV_n)]_{GL(n)} \to \bigoplus_n \varprojlim_E \, [P_E H_*(BV_n)]_{GL(n)},$$

is an F–isomorphism.

This result is relevant to the first part of the conjecture above as well, for there is a theorem of Hung and Nam [8, 9] which says that the canonical homomorphism

$$[P_{\mathcal{A}} H_*(BV_n)]_{GL(n)} \to P_{\mathcal{A}}([H_*(BV_n)]_{GL(n)}),$$

is trivial in positive degrees, as soon as $n \geq 3$. Thus the image of the canonical map

$$\bigoplus_n [P_{\mathcal{A}} H_*(BV_n)]_{GL(n)} \to \bigoplus_n \varprojlim_E \, [P_E H_*(B(BV_n))]_{GL(n)},$$

actually lands in the subalgebra of finite sequences in Lemma 4.5.

It would be interesting also to see whether the theorem of Hung and Nam mentioned above remain true when the Steenrod algebra is replaced by a sub-Hopf algebra, say the family $D(m)$, or even the sub-Hopf algebra D.

5 Study of the E–transfer

In this section, we investigate the subring $P_E H_*(BV_n)$ where E is an elementary sub-Hopf algebra of $\mathcal{A}$. We pay particular attention to the case when E is a maximal one.

5.1 $E(m)$–annihilated elements in $H_*(B\mathbb{Z}/2)$

The formula (3–6) serves as an efficient means to compute the image of b_k in $\mathcal{A}^*$ as well as E^*. Recall that we have $f_*(b_{2k+1}) = (f_* b_k)^2$, so in principle, we need only to compute the image of evenly graded elements.

Notation 5.1 For each $k \geq 0$, write $k+1 = 2^\kappa(2\rho - 1)$. Clearly, κ and ρ are uniquely determined by k. In fact, κ is the smallest non-negative integer such that $2^\kappa \notin k$. We reserve the letters κ and ρ for such a presentation.

Lemma 5.2 *b_k is $E(m)$–annihilated if and only if either (i) $\kappa \geq m$ or (ii) $\kappa < m$ and $\rho \leq 2^{m-1}$.*

Proof Recall that $E(m)$ is generated by the operations P^s_t in which $s < m \leq t$. Thus to be $E(m)$–annihilated, k must satisfies the condition that for any $s < m$, if $2^s \notin k$, then $k < 2^{s+t}$ for all $t \geq m$.

If $\kappa \geq m$, then $2^s \in k$ for any $s < m$, so b_k is clearly $E(m)$–annihilated. If $\kappa < m$, then $k = 2^\kappa(2\rho - 1) - 1 < 2^{\kappa+m}$ which implies $\rho \leq 2^{m-1}$. □

The following Corollary is immediate.

Corollary 5.3 *b_k is D–annihilated if and only if $\rho \leq 2^\kappa$.*

5.2 Some properties of the $E(m)$–transfer

Recall from the proof of Theorem 3.1 that the composition

$$f_n^{E(m)}\colon\ P_{E(m)}H_*(BV_n) \xrightarrow{f_*^{\otimes n}} (\overline{\mathcal{A}}^*)^n \to (\overline{E(m)}^*)^n,$$

is a chain level representation for $\mathtt{Tr}_n^{E(m)}$. Moreover,

$$E(m)^* = \mathcal{A}^*/(\xi_1, \dots, \xi_{m-1}, \xi_m^{2^m}, \xi_{m+1}^{2^m}, \dots).$$

Our first result says that the image of $f_n^{E(m)}$ has a rather strict form.

Lemma 5.4 *Under $f_1^{E(m)}\colon H_*(B\mathbb{Z}/2) \to \overline{E(m)}^*$, the image of b_k is nontrivial if and only if k can be written in the form*

$$k = 2^{s_1}(2^{t_1} - 1) + \cdots + 2^{s_\ell}(2^{t_\ell} - 1) - 1, \tag{5–1}$$

for some $0 \leq s_1 < \cdots < s_\ell \leq (m-1)$, and $m \leq t_1, \dots, t_\ell$. Moreover, the ℓ–tuple $(s_1, \dots s_\ell)$ is unique, up to a permutation.

Proof We make use of Boardman's formula (3–6). When projecting down to $\overline{E(m)}^*$, the infinite product in this formula is actually a finite product since $\xi_t^{2^s} = 0$ in $\overline{E(m)}_*$ for all $s \leq m$. So the image of b_k under $T_1^{E(m)}$ is the coefficient of x^{k+1} in the finite product

$$\prod_{i=0}^{m-1}(1 \otimes 1 + x^{2^i(2^m-1)} \otimes \xi_m^{2^i} + x^{2^i(2^{m+1}-1)} \otimes \xi_{m+1}^{2^i} + x^{2^i(2^{m+2}-1)} \otimes \xi_{m+2}^{2^i} + \cdots).$$

Thus b_k is mapped to the sum of $\prod_{i=0}^{\ell} \xi_{t_i}^{2^{s_i}}$ for each presentation of k in the form (5–1). For the uniqueness of the set $\{s_1, \ldots, s_\ell\}$. Observe that the equality

$$2^{s_1}(2^{t_1} - 1) + \cdots + 2^{s_i}(2^{t_i} - 1) = 2^{u_1}(2^{v_1} - 1) + \cdots + 2^{u_j}(2^{v_j} - 1),$$

implies that

$$2^{s_1} + \cdots + 2^{s_i} \equiv 2^{u_1} + \cdots + 2^{u_j} \pmod{2}^m,$$

But both sides in the above equation are at most $2^0 + \cdots + 2^{m-1} = 2^m - 1$, hence they must be actually equals. We then obtain two binary expansion of the same number, which implies that the two set of indices $\{s_1, \ldots, s_i\}$ and $\{u_1, \ldots, u_j\}$ are the same. □

Lemma 5.5 *The image of the $E(1)$–transfer is the polynomial subalgebra generated by $h_{1,0}$.*

Proof From the previous lemma, we see that $f_1^{E(1)}(b_m)$ is nontrivial iff $m = 2^t - 2$ for some $t \geq 1$. Note that if $t \geq 2$, then m is even. So a monomial consisting of only b_m of the form above with at least one index $m > 0$ cannot be a summand of a P_1^0–annihilated (hence $E(1)$–annihilated) element. □

For $m > 1$, even in the case of $E(2)$, we have not yet been able to calculate the whole image of the $E(m)$–transfer, however, we have the following result.

Proposition 5.6 *For each $m \geq 1$, the subalgebra $\mathbb{F}_2[h_{m,s} | 0 \leq s < m]$ of $H^{*,*}(E(m)) = \mathbb{F}_2[h_{t,s} | s < m \leq t]$ is in the image the $E(m)$–transfer.*

Proof Since the transfer is an algebra homomorphism, it suffices to show that for each $0 \leq s \leq (m-1)$,

- $b_{2^s(2^m-1)-1} \in P_{E(m)}H_*(\mathbb{Z}/2)$, and
- $f_1^{E(m)}$ sends $b_{2^s(2^t-1)-1}$ to $[\xi_t^{2^s}]$.

The first item is immediate from Lemma 5.2. In fact, $b_{2^s(2^m-1)-1}$ is $E(n)$–annihilated for all $n > m$. For the second item, it suffices to note that in a presentation of $2^s(2^t-1)-1$ in the form (5–1),

$$2^s(2^t-1)-1 = 2^{s_1}(2^{t_1}-1)+\cdots+2^{s_\ell}(2^{t_\ell}-1)-1,$$

it follows that $2^s = 2^{s_1}+\cdots+2^{s_\ell}$ by Lemma 5.4. This equation clearly has only one possible solution: $\ell = 1$ and $s_1 = s$. But then $t_1 = t$, so we are done. □

Taking limits over all $m \geq 1$, we obtain the following result.

Corollary 5.7 *For each $m \geq 1$, any element of the form*

$$h_{i_0,\dots i_{m-1}} = h_{m,0}^{i_0}\cdots h_{m,m-1}^{i_{m-1}},$$

where $i_{m-1} > 0$ in the algebra R of Theorem 4.2 is contained in the image of the total $\mathcal{E}$–transfer $\varphi^{\mathcal{E}}$.

Proof First of all, observe that the family of maximal sub-Hopf algebras $E(m)$ forms a cofinal system in $\mathcal{E}$. Thus to construct an element in the inverse limit

$$\varprojlim_E \bigoplus_n [P_E H_*(BV_n)]_{GL(n)},$$

it suffices to define it on the $E(m)$s.

Now for any element of the form in the Corollary, we can define its preimage as the compatible sequence which is zero for $E \neq E(m)$, and

$$b_{i_0,\dots i_{m-1}} = b_{2^0(2^m-1)-1}^{i_0}\cdots b_{2^{m-1}(2^m-1)-1}^{i_{m-1}}$$

when $E = E(m)$. The fact that this sequence is compatible is because the restriction of $b_{i_0,\dots i_{m-1}}$ to any intersection $E(m)\cap E(m')$ is trivial. Here we have used the condition that $i_{m-1} \geq 1$. □

Corollary 5.8 *The subalgebra of $H^{*,*}(D)$ generated by $h_{t,t-1}$, $t \geq 1$, is contained in the image of the D–transfer.*

Proof It is clear that $b_{2^{t-1}(2^t-1)-1}$ is D–annihilated. On the other hand, we have that $f_*(b_{2^{t-1}(2^t-1)-1}) = (f_* b_{2^t-2})^{2^{t-1}}$. Since $\xi_t^{2^s} = 0$ in D_* for all $s \geq t$, it follows easily that the image of $b_{2^{t-1}(2^t-1)-1}$ in D_* is $\xi_t^{2^t-1}$ which is a cycle representing $h_{t,t-1}$. □

Remark 5.9 The set of elements of the form $h_{i_0,\dots i_{m-1}}$ with $i_{m-1} > 0$ is the complete set of $\mathcal{A}$–invariant monomials in R (see Palmieri [24, page 433]). Palmieri's theorem then predicts that some power, say α_m, of it comes from $H^{*,*}(\mathcal{A})$. If our Conjecture 4.3 is true, then there would be another exponent β_m for which $b_{i_0,\dots i_{m-1}}^{\beta_m}$ comes from a GL–coinvariant element of $P_{\mathcal{A}}H_*V$. The two exponents are probably not equal in general. For example, if $m = 1$, then $h_{m,m-1}$ corresponds to the element called h_0 in $H^{1,1}(\mathcal{A})$ and $\alpha_1 = \beta_1 = 1$. But in the case $m = 2$, then $\alpha_2 = 4$ where $h_{2,1}^4$ corresponds to the element called g in $H^{4,24}(\mathcal{A})$. However, g is not an element of the $\mathcal{A}$–transfer (see Bruner, Hà and Hung [5]; also see the next section for a quick proof). We do not know what α_m and β_m are in general. But for $m = 2$, we conjecture that one can take $\beta_2 = 6$ (Equivalently, this means that the element denoted by $r \in H^{6,36}(\mathcal{A})$ is in the image of $\varphi_6^{\mathcal{A}}$.)

One may wonder whether Proposition 5.6 describes all elements in the image of the $E(m)$–transfer. This is not the case, as our next example show.

Proposition 5.10 *$h_{3,0}h_{2,1}$ is in the image of the $\mathcal{E}$–transfer.*

Proof We will construct an element in the inverse limit $\varprojlim_E [P_E H_{11}(BV_2)]_{GL(2)}$ whose image in R is $h_{3,0}h_{2,1}$. Let $b \in H_{11}(BV_2)$ be the sum $b = b_6b_5 + b_3b_8 + b_9b_2 + b_{10}b_1 + b_7b_4$. By direct inspection, b is $E(2)$–annihilated. We claim that b represents a nontrivial element in the coinvariant ring $[P_{E(2)}H_{11}(BV_2)]_{GL(2)}$ but represents a trivial class when replacing $E(2)$ by any other $E \subset E(2)$.

Indeed, one verifies that the only $E(2)$–annihilated elements in $H_{11}(BV_2)$ are b_0b_{11}, b and their obvious permutations. It follows that b represent a nontrivial element in $[P_{E(2)}H_{11}(BV_2)]_{GL(2)}$.

On the other hand, if $\sigma \in GL(2)$ denote the matrix $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$ and τ the standard permutation, then we have

$$b_{11}b_0 + \sigma(b_{11}b_0) = b + \tau b + b_0b_{11} \qquad b_9b_2 + \sigma(b_9b_2) = b + b_{11}b_0.$$

Thus if E is a sub-algebra of $E(2)$ such that b_9b_2 is E–annihilated, then b represents the trivial class in the corresponding coinvariant ring. This condition is true for the subalgebras $E(2) \cap E(m)$ for any other m. We can now define an element in $\varprojlim_E [P_E H_{11}(BV_2)]_{GL(2)}$ by taking its value to be $[b]$ when $E = E(2)$ and zero for other $E = E(m)$.

It remains to verify that the image of this element under the $\mathcal{E}$–transfer is $h_{3,0}h_{2,1}$. For this purpose, note that under the map $f_*\colon H_*(BZ/2) \to \overline{\mathcal{A}}_*$, we see that $f_*(b_3) = \xi_1^4$, $f_*(b_9) = \xi_1^{10} + \xi_1^4\xi_2^2$, $f_*(b_1) = \xi_1^2$, and $f_*(b_4) = \xi_1^5 + \xi_1^2\xi_2$, all are mapped to zero in $\overline{E(2)}^*$. It follows that the restriction of b to $(\overline{E(2)}^*)^2$ is b_6b_5 which represents $h_{3,0}h_{2,1}$.

We remark that this element does not come from $[P_DH_{11}(BV_2)]_{GL(2)}$. For the only nontrivial element in the latter is $\sum_{i=1}^{10} b_ib_{11-i}$ and its image in $[P_{E(2)}H_{11}(BV_2)]_{GL(2)}$ is $b + \tau b \equiv 0$. Our Conjecture 4.3 then predicts that $[b]$ is nilpotent. □

Another useful question is to find a necessary criteria so that an element in $H^{*,*}(E(m))$ is not in the image of the $E(m)$–transfer. For this, we have the following.

Proposition 5.11 *Any element in $H^{*,*}(E(m))$ which contains a monomial of the form $h = h_{t_1,s}\cdots h_{t_k,s}$ where not all $t_i = m$, is not in the image of the $E(m)$–transfer.*

Proof By way of contradiction, suppose that h is a nontrivial summand of an element which is in the image of the $E(m)$–transfer. This implies that there is an $E(m)$–annihilated element which contains the monomial

$$b = b_{2^s(2^{t_1}-1)-1}\cdots b_{2^s(2^{t_k}-1)-1},$$

as a nontrivial summand. Equivalently, the dual of b

$$x = x_1^{2^s(2^{t_1}-1)-1}\cdots x_k^{2^s(2^{t_k}-1)-1},$$

is $E(m)$–indecomposable. We will show that this last statement is not true. Indeed, observe that for each t, $2^s \notin 2^s(2^t-1)-1 = (2^{s+t}-1)-2^s$. Thus by Lemma 2.1, $x^{2^s(2^t-1)-1}$ is P^s_ℓ–annihilated for any ℓ. It follows that if there exist $t_i > n$, say t_1, then $P^s_{t_1-1} \in E(m)$ and

$$P^s_{t_1-1}(x_1^{2^{s+t_1-1}}x_2^{2^s(2^{t_2}-1)-1}\cdots x_k^{2^s(2^{t_k}-1)-1}) = x_1^{2^s(2^{t_1}-1)-1}\cdots x_k^{2^s(2^{t_k}-1)-1}.$$

The lemma is proved. □

Example 5.12 Here is an example of the usefulness of the last proposition. In [24], Palmieri asserted that there is an $\mathcal{A}$–invariant element of R of the form

$$z^{12,80} = h_{2,0}^8h_{3,1}^4 + h_{3,0}^8h_{2,1}^4 + h_{2,1}^{11}h_{3,1}.$$

This element contains the monomial $h_{2,1}^{11}h_{3,1}$ of the form described in Proposition 5.11, thus $z^{12,80}$ or any of its powers cannot be in the image of the $E(2)$–transfer.

6 Applications to the rank 4 transfer

The cohomology of the Steenrod algebra in cohomological degree 4 has been completely determined by Lin and Mahowald [12]. In particular, there are three generators, namely $d_0 \in H^{4,18}(\mathcal{A}), e_0 \in H^{4,21}(\mathcal{A})$, and $g \in H^{4,24}(\mathcal{A})$ whose restriction to the sub-Hopf algebra $E(2)$ are nontrivial. According to Zachariou [32, 33], their images in $H^{4,*}(E(2))$ are $h_{2,0}^2 h_{2,1}^2$, $h_{2,0}h_{2,1}^3$ and $h_{2,1}^4$ respectively. Furthermore, $i^{\mathcal{A}}_{E(2)}$ is a monomorphism in these degrees. In this section, we prove Theorem 1.1. That g, in fact the whole family of generators $g_i = (Sq^0)^i g$, are not in the image of the rank 4 transfer was first proved by Bruner, Hà and Hung [5]. We provide here a different proof which is less calculational. The elements d_0 and e_0 are in Hung's list of conjectural elements that must be in the image of the transfer, provided the Singer conjecture holds. Our results thus partially complete his list[3]. Recall first that under the map $f_4^{E(2)}$, the images of b_k for $k \leq 20$ are all trivial, except the following:

$$
\begin{array}{lllll}
b_2 \longmapsto \xi_2 & b_5 \longmapsto \xi_2^2 & b_6 \longmapsto \xi_3 & b_8 \longmapsto \xi_2^3 & b_{11} \longmapsto \xi_2^4 \\
b_{12} \longmapsto \xi_2^2\xi_3 & b_{13} \longmapsto \xi_3^2 & b_{14} \longmapsto \xi_4 & b_{16} \longmapsto \xi_2\xi_3^2 & b_{20} \longmapsto \xi_2\xi_4 + \xi_3^3
\end{array}
$$

The proof of Theorem 1.1 is divided into three parts, corresponding to the three generators in question.

6.1 g is not in the image of the transfer

We prove by contradiction. Suppose that there is an element $z \in P_{\mathcal{A}}H_{20}(BV_4)$ such that $\varphi_4^{\mathcal{A}}([z]) = g$. Since the restriction of g to $H^*(E(2))$ is $h_{2,1}^4$, it follows easily that z must contains the monomial $b_5b_5b_5b_5$ as a nontrivial summand. Dually, we have that the monomial $x_1^5x_2^5x_3^5x_4^5 = 5555$ is indecomposable in the $\mathcal{A}$–module $H^{20}(BV_4)$. We will show that this latter statement is not true. In [30], Wood proved Peterson's conjecture which says that a monomial in degree d with exactly r odd exponents such that $\alpha(d+r) > r$ is $\mathcal{A}$–decomposable; where $\alpha(m)$ counts the number of non-zero digits in the binary expansion of m. His proof makes use of an important observation, nowadays known as Wood's χ–trick, that for any two homogeneous polynomials u and v in the polynomial algebra $H^*(BV_n)$ and any Steenrod operation θ, $u(\theta v)$ is $\mathcal{A}$–decomposable iff $(\chi(\theta)u)v$ is, where χ is the canonical conjugation in $\mathcal{A}$.

[3] Recently, Nam [22] claimed to have verified most cases in Hung's list by using a completely different method.

Writing $u \equiv v$ whenever $u - v$ is $\mathcal{A}$–decomposable, we have

$$\begin{aligned}5555 = Sq^8(2222) \times 1111 \equiv& 2222 \times (\chi Sq^8)1111\\ \equiv& 2222 \times [(4422) + (8211)]\\ \equiv& (6,6,4,4) + (10,4,3,3) \equiv (10,4,3,3),\end{aligned}$$

where a monomial in brackets means that we take the sum of all possible permutations of that monomial. But the monomial $(10,4,3,3)$ is $\mathcal{A}$–decomposable because $\alpha(20+2) = 3 > 2$. We have a contradiction.

It can be easily seen that the proof above also works for any other generator $g_i = (\widetilde{Sq}^0)^i g$.

6.2 d_0 is in the image of the transfer

We will show that there exists an element $z \in P_{\mathcal{A}} H_{14} BV_4$ such that it contains an odd number of permutations of 2255. Assuming that such an element exists, then its image under the canonical maps

$$P_{\mathcal{A}} H_{14}(BV_4) \hookrightarrow P_{E(2)} H_{14}(BV_4) \to [P_{E(2)} H_{14}(BV_4)]_{GL(4)},$$

is the equivalent class of 2525 because in degree 14, there are only two possible type of monomials, namely 2255 and 2228 that maps nontrivially to $\overline{E(2)}^*$. The latter monomial obviously cannot be a nontrivial summand of any $\mathcal{A}$–annihilated element. It follows that the image of z under the $\mathcal{A}$–transfer is an element in $H^{4,14}(\mathcal{A})$ whose restriction to $H^{4,14}(E(2))$ is $h_{2,0}^2 h_{2,1}^2$. Thus z is a chain-level representation of d_0.

In fact, one can verify that the $\mathcal{A}$–annihilated element

$$z = x + (2,3)x + (1,3)x + (3155 + 5513 + 5135 + 5315 + 5333),$$

where

$$x = (2255+2165+1256+1166+4253+4163+3263+2435+1436+2336+4433),$$

is a representation of d_0. Indeed, to verify that this sum is indeed $\mathcal{A}$–annihilated, we need only consider the effects of Sq^1, Sq^2 and Sq^4 because of the unstable condition. By direct calculation, we have

$$\begin{aligned}Sq^1 x = Sq^1 4433 =& 4333 + 3433,\\ Sq^2 x =& 3153 + 1335 + 3333,\\ Sq^4 x =& 1333 + 3133.\end{aligned}$$

It follows that $x+(2,3)x+(1,3)x$ is Sq^1– and Sq^4–nil. Moreover,

$$Sq^2[x+(2,3)x+(1,3)x] = 3153+3513+3315+5133+3333.$$

The extra summand $(3155+5513+5135+5315+5333)$ is needed to kill off the effect of Sq^2.

6.3 e_0 is in the image of the transfer

There is only one type of monomial, namely 2555, in $H_{17}(BV_4)$ whose image in $(\overline{E(2)}^*)^4$ is nontrivial. As in the proof for d_0, it suffices to show that there exists an element of $P_{\mathcal{A}}H_{17}BV_4$ which contains an odd number of permutations of 2555.

We exhibit an explicit element of such form below.

$$\begin{aligned}
&2555+1655+18(53)+17(63)+14(75)+13(76)+14(93)+23(93)\\
&\quad+12(95)+11,10,5+1169+12(11,3)+4(355)+11,12,3+114,11+\\
&\quad+1187+2177+112,13+111,14+3356+3635+3563\\
&\quad+5336+5633+5363+6(335)+8333+7433+7253+7163\\
&\quad+2933+1,10,33+2735+2375+2357+1736+1376+1367.
\end{aligned}$$

References

[1] **J F Adams**, *On the structure and applications of the Steenrod algebra*, Comment. Math. Helv. 32 (1958) 180–214 MR0096219

[2] **J F Adams**, *Operations of the nth kind in K–theory, and what we don't know about RP^∞*, from: "New developments in topology (Proc. Sympos. Algebraic Topology, Oxford, 1972)", London Math. Soc. Lecture Notes 11, Cambridge Univ. Press, London (1974) 1–9 MR0339178

[3] **D W Anderson**, **D M Davis**, *A vanishing theorem in homological algebra*, Comment. Math. Helv. 48 (1973) 318–327 MR0334207

[4] **J M Boardman**, *Modular representations on the homology of powers of real projective space*, from: "Algebraic topology (Oaxtepec, 1991)", Contemp. Math. 146, Amer. Math. Soc., Providence, RI (1993) 49–70 MR1224907

[5] **R R Bruner**, **L M Hà**, **N H V Hưng**, *On the behavior of the algebraic transfer*, Trans. Amer. Math. Soc. 357 (2005) 473–487 MR2095619

[6] **D P Carlisle**, **R M W Wood**, *The boundedness conjecture for the action of the Steenrod algebra on polynomials*, from: "Adams Memorial Symposium on Algebraic Topology 2 (Manchester, 1990)", London Math. Soc. Lecture Note Ser. 176, Cambridge Univ. Press, Cambridge (1992) 203–216 MR1232207

[7] **N H V Hưng**, *The cohomology of the Steenrod algebra and representations of the general linear groups*, Trans. Amer. Math. Soc. 357 (2005) 4065–4089 MR2159700

[8] **N H V Hưng**, **T N Nam**, *The hit problem for the Dickson algebra*, Trans. Amer. Math. Soc. 353 (2001) 5029–5040 MR1852092

[9] **N H V Hưng**, **T N Nam**, *The hit problem for the modular invariants of linear groups*, J. Algebra 246 (2001) 367–384 MR1872626

[10] **M Kameko**, *Generators of the cohomology of BV_3*, J. Math. Kyoto Univ. 38 (1998) 587–593 MR1661173

[11] **W H Lin**, *Some differential in Adams spectral sequence for spheres*, Trans. Amer. Math. Soc. to appear (1999)

[12] **W-H Lin**, **M Mahowald**, *The Adams spectral sequence for Minami's theorem*, from: "Homotopy theory via algebraic geometry and group representations (Evanston, IL, 1997)", Contemp. Math. 220, Amer. Math. Soc., Providence, RI (1998) 143–177 MR1642893

[13] **A Liulevicius**, *The factorization of cyclic reduced powers by secondary cohomology operations*, Proc. Nat. Acad. Sci. U.S.A. 46 (1960) 978–981 MR0132543

[14] **M Mahowald**, **M Tangora**, *An infinite subalgebra of* $\mathrm{Ext}_A(\mathbb{Z}_2, \mathbb{Z}_2)$, Trans. Amer. Math. Soc. 132 (1968) 263–274 MR0222887

[15] **H R Margolis**, *Spectra and the Steenrod algebra*, North-Holland Mathematical Library 29, North-Holland Publishing Co., Amsterdam (1983) MR738973

[16] **H Margolis**, **S Priddy**, **M Tangora**, *Another systematic phenomenon in the cohomology of the Steenrod algebra*, Topology 10 (1970) 43–46 MR0300272

[17] **J P May**, *A general algebraic approach to Steenrod operations*, from: "The Steenrod Algebra and its Applications (Proc. Conf. to Celebrate N. E. Steenrod's Sixtieth Birthday, Battelle Memorial Inst., Columbus, Ohio, 1970)", Lecture Notes in Mathematics 168, Springer, Berlin (1970) 153–231 MR0281196

[18] **D M Meyer**, **L Smith**, *Poincaré duality algebras, Macaulay's dual systems, and Steenrod operations*, Cambridge Tracts in Mathematics 167, Cambridge University Press, Cambridge (2005) MR2177162

[19] **J Milnor**, *The Steenrod algebra and its dual*, Ann. of Math. (2) 67 (1958) 150–171 MR0099653

[20] **N Minami**, *The iterated transfer analogue of the new doomsday conjecture*, Trans. Amer. Math. Soc. 351 (1999) 2325–2351 MR1443884

[21] **S A Mitchell**, *Splitting $B(\mathbb{Z}/p)^n$ and BT^n via modular representation theory*, Math. Z. 189 (1985) 1–9 MR776532

[22] **T N Nam**, *Transfert algébrique et représentation modulaire du groupe linéare*, preprint (2003)

[23] **T N Nam**, *$\mathcal{A}$–générateurs génériques pour l'algèbre polynomiale*, Adv. Math. 186 (2004) 334–362 MR2073910

[24] **J H Palmieri**, *Quillen stratification for the Steenrod algebra*, Ann. of Math. (2) 149 (1999) 421–449 MR1689334

[25] **J H Palmieri**, *Stable homotopy over the Steenrod algebra*, Mem. Amer. Math. Soc. 151 (2001) xiv+172 MR1821838

[26] **F P Peterson**, *A-generators for certain polynomial algebras*, Math. Proc. Cambridge Philos. Soc. 105 (1989) 311–312 MR974987

[27] **W M Singer**, *The transfer in homological algebra*, Math. Z. 202 (1989) 493–523 MR1022818

[28] **N E Steenrod**, **D B A Epstein**, *Cohomology operations*, Annals of Mathematics Studies 50, Princeton University Press, Princeton, N.J. (1962) MR0145525

[29] **C Wilkerson**, *The cohomology algebras of finite-dimensional Hopf algebras*, Trans. Amer. Math. Soc. 264 (1981) 137–150 MR597872

[30] **R M W Wood**, *Steenrod squares of polynomials and the Peterson conjecture*, Math. Proc. Cambridge Philos. Soc. 105 (1989) 307–309 MR974986

[31] **R M W Wood**, *Problems in the Steenrod algebra*, Bull. London Math. Soc. 30 (1998) 449–517 MR1643834

[32] **A Zachariou**, *A subalgebra of* $\mathrm{Ext}^*_A(\mathbb{Z}_2, \mathbb{Z}_2)$, Bull. Amer. Math. Soc. 73 (1967) 647–648 MR0214060

[33] **A Zachariou**, *A polynomial subalgebra of the cohomology of the Steenrod algebra*, Publ. Res. Inst. Math. Sci. 9 (1973/74) 157–164 MR0341489

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