Total Termination of Term Rewriting

M.C.F. Ferreira and H. Zantema

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Utrecht University

Department of Computer Science

Padualaan 14, P.O. Box 80.089,
3508 TB Utrecht, The Netherlands,
Tel. : ... + 31 - 30 - 531454

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M. C. F. Ferreira and H. Zantema
Utrecht University, Department of Computer Science
P.O. box 80.089, 3508 TB Utrecht, The Netherlands
e-mail: maria@cs.ruu.nl hansz@cs.ruu.nl

Abstract

We investigate proving termination of term rewriting systems by interpretation of terms in a compositional way in a total well-founded order. This kind of termination is called *total termination*. On one hand it is more restrictive than simple termination, on the other it generalizes most of the usual techniques for proving termination. For total termination it turns out that below ϵ_0 the only orders of interest are built from the natural numbers by lexicographic product and the multiset construction. By examples we show that both constructions are essential. For a wide class of term rewriting systems we prove that total termination is a modular property. Most of our techniques are based on ordinal arithmetic.

1 Introduction

One of the main problems in the theory of term rewriting systems (TRS) is the detection of termination: for a fixed system of rewrite rules, detect whether there exist infinite rewrite chains or not. In general this problem is undecidable ([7, 2]). However, there are several methods for deciding termination that are successful for many special cases. Roughly these methods can be divided into two main types: *syntactical* methods and *semantical* methods. In a syntactical method terms are ordered by a careful analysis of the term structure. A well-known representative of this type is the *recursive path order* ([3]). All of these orderings are simplification orderings, i.e., a term is always greater than its proper subterms. An overview and comparison of simplification orderings is given in [14].

Here we focus on a semantical method: terms are interpreted compositionally in some well-founded ordered set. This is done in such a way that each rewrite chain will map to a descending chain, and hence will terminate. The general framework has been introduced in [15]. One problem is how to choose a suitable well-founded ordered set. The variation among well-founded ordered sets is so unwieldy that some restriction is reasonable. A natural way is the restriction to total orders: then the ordered sets correspond to ordinal numbers, having a very elegant structure that has been studied extensively in the past. This kind of termination of term rewriting systems is called *total termination.*

Total termination turns out to be a slightly stronger restriction than simple termination. However, most of the general techniques of proving termination like *polynomial interpretations* ([11, 1]), *elementary interpretations* ([12]), recursive path order with status and Knuth-Bendix order with status, all fit in the notion of total termination.

This paper is an investigation of total termination, in particular of which totally ordered sets are useful. One of the main conclusions is that apart from some minor exceptions only ordinals of the shape ω^α are of interest. The basic observation leading to this result is the following. The existence of a binary operation in a total well-founded order that is strictly monotonous in both coordinates implies that the order type is ω^α. Stated without ordinals this means that the order is isomorphic to the finite multisets over another order. Below the ordinal ϵ_0 this implies that all totally ordered sets of interest can be constructed from the natural numbers in finitely many steps using only lexicographic product and the multiset construction. We show that these constructions are essential by presenting examples of TRS's for which a termination proof can be given (by an interpretation) in ω^η, for any fixed $\eta \leq \omega$, but not in a totally ordered set of a smaller order type.

Another main topic of this paper is the modularity of total termination. Surprisingly the tree structure of mixed terms that is essential in other modularity questions ([13]) does not play a role here. The essential problem is how to lift an interpretation in an ordinal to an interpretation in a greater ordinal without affecting monotonicity and compatibility. We did not succeed in proving modularity of total termination in full generality. However, we found some interesting partial results. For example, if two systems are totally terminating and not both of them contain duplicating rules, then the direct sum is also totally terminating.

2 Monotone algebras

Let $\mathcal{F}$ be a set of operation symbols each having a fixed arity. We define a *well-founded monotone $\mathcal{F}$-algebra* $(A, >)$ to be an $\mathcal{F}$-algebra A for which the underlying set is provided with a well-founded order $>$ and each algebra operation is strictly monotone in all of its coordinates, more precisely: for each operation symbol $f \in \mathcal{F}$ and all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ for which $a_i > b_i$ for some i, and $a_j = b_j$ for all $j \neq i$, we have $f_A(a_1, \ldots, a_n) > f_A(b_1, \ldots, b_n)$.

Let $(A, >)$ be a well-founded monotone $\mathcal{F}$-algebra. Let $A^\mathcal{X} = \{\sigma : \mathcal{X} \to A\}$. We define $\phi_A : \mathcal{T}(\mathcal{F}, \mathcal{X}) \times A^\mathcal{X} \to A$ inductively by

$$\begin{aligned} \phi_A(x, \sigma) &= \sigma(x), \\ \phi_A(f(t_1, \ldots, t_n), \sigma) &= f_A(\phi_A(t_1, \sigma), \ldots, \phi_A(t_n, \sigma)) \end{aligned}$$

for $x \in \mathcal{X}, \sigma : \mathcal{X} \to A, f \in \mathcal{F}, t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. This function induces a partial order $>_A$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ as follows:

$$t >_A t' \iff (\forall \sigma \in A^\mathcal{X} : \phi_A(t, \sigma) > \phi_A(t', \sigma)).$$

Intuitively $t >_A t'$ means that for each interpretation of the variables in A the interpreted value of t is greater than that of t'.

We say that a non-empty well-founded monotone algebra $(A, >)$ *normalizes* a TRS if $l >_A r$ for every rule $l \to r$ of the TRS. This terminology is motivated by the following proposition.

Theorem 2.1 *A TRS is terminating if and only if it is normalized by a non-empty well-founded monotone algebra.*

For the proof we refer to [15]. The way of proving termination of a TRS is now as follows: choose a well-founded poset A, define for each operation symbol a corresponding operation that is strictly monotone in all of its coordinates, and for which $\phi_A(l, \sigma) >_A \phi_A(r, \sigma)$ for all rewrite rules $l \to r$ and all $\sigma : \mathcal{X} \to A$. Then according to the above proposition the TRS is terminating. A typical example is the system

$$f(f(x, y), z) \to f(x, f(y, z)).$$

Choose $(A, >) = (\mathbb{N}, >)$ where $\mathbb{N}$ is defined to be the set of strictly positive integers, and choose $f_A(x, y) = 2x + y$. Clearly f_A is strictly monotone in both coordinates, and

$$f_A(f_A(x, y), z) = 4x + 2y + z > 2x + 2y + z = f_A(x, f_A(y, z))$$

for all $x, y, z \in A$. Hence $f(f(x, y), z) >_A f(x, f(y, z))$, proving termination.

Definition 2.2 *A TRS is called* totally terminating *if it is normalized by a non-empty well-founded monotone algebra in which the underlying order is total.*

Every totally terminating TRS allows a simplification order (as defined in [4] and many other texts); in fact this follows from lemma 3.1 presented below. The converse does not hold, for example, termination of the system

$$\begin{array}{rcl} f(a) & \rightarrow & f(b) \\ g(b) & \rightarrow & g(a) \end{array}$$

is easily proved by a simplification order, but the system is not totally terminating since the interpretations of a and b have to be incomparable.

However, most of the existing methods of proving termination of TRS also prove total termination. By definition the methods of polynomial interpretations ([11, 1]) and elementary interpretations ([12]) are nothing else than our approach in which A is chosen to be the naturals and the operations have a particular shape. Hence a termination proof by these interpretations implies total termination. The same can be said for recursive path order and Knuth-Bendix order, both with status. Here we choose A to be the set of ground terms modulo some congruence. If there are no constants, one constant can be added to force the existence of ground terms. The congruence is generated by interchanging the arguments of the operations that have multiset status. The order on these congruence classes is defined by the RPO or KBO itself, where the precedence is extended to a total precedence. For both RPO and KBO it can be proved by induction on the size of the terms that any two terms, modulo this congruence, are comparable. As a consequence, the orders are total and prove total termination. For a finite TRS proved terminating by recursive path order with only multiset status, Hofbauer ([6]) proved that a proof of total termination can be given in the natural numbers with primitively recursive operations.

A main topic of this paper is the investigation of useful total orders for total termination. The main tool is the arithmetic of ordinals, i.e., of total well-founded orders modulo order-isomorphism. We say that a proof of total termination is in an ordinal α if the underlying order of the monotone algebra has order type α. Since in this algebra we allow all possible monotone functions this does not mean that the proof can be given in α in the proof-theoretical sense. For example, the term rewriting system describing the Ackermann function can be proven terminating by a monotone algebra of which the underlying order corresponds to the natural numbers, so in our notion its termination proof is in ω.

We first summarize notions and results about well-ordered sets and ordinals needed. For many of the proofs we refer to [9].

3 Well-ordered sets

A well-ordered set $(\mathcal{A}, >)$ is a set with a partial (or strict) order $>$ that is totally (linearly) ordered and well-founded, i. e. $<$ is irreflexive, transitive and linear (as usual we write $x > y$

meaning $y < x$),and there are no infinite decreasing chains $x_0 > x_1 > x_2 > \dots$
It can be shown that the last restriction is equivalent to every non-empty subset of $\mathcal{A}$ having a minimal element.

A simple but useful lemma is the following:

Lemma 3.1 *Let $\mathcal{A} = (A, >)$ be well-ordered and let $f : A \to A$ be any monotone*[1] *function. Then $f(x) \geq x$ for every $x \in A$.*

Proof Suppose there is $x \in A$ such that $x > f(x)$. Monotonicity of f leads to an infinite decreasing sequence

$$x > f(x) > f(f(x)) > f(f(f(x))) > \dots$$

contradicting well-foundedness. □

Two ordered sets are *similar* if they are *order-isomorphic*, i. e. there is a monotone bijection between them. Since monotonicity implies injectivity, we have:

Lemma 3.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be totally ordered sets. Then $f : \mathcal{A} \to \mathcal{B}$ is monotone and surjective $\iff$ f is an order-isomorphism between $\mathcal{A}$ and $\mathcal{B}$.*

As an aside, we remark that from the two previous results follows the unicity of the isomorphism between two well-ordered sets.

Another important notion is defined below:

Definition 3.3 *Given a totally ordered set $\mathcal{A}$ and X a proper subset of $\mathcal{A}$, we say that X is an initial segment of $\mathcal{A}$ if $\forall x \in X\ \forall y \in \mathcal{A}\ (y < x \Rightarrow y \in X)$.*

Lemma 3.4 *Each initial segment of a well-ordered set $\mathcal{A}$ is of the form $W(x) = \{y \mid y < x\}$, for some $x \in \mathcal{A}$.*

Proof Given I an initial segment of $\mathcal{A}$ take $x = min(\mathcal{A} - I)$, whose existence is guaranteed by the well-foundedness of $\mathcal{A}$.
We see that $I = W(x)$. First we show that $i \in I \Rightarrow i \in W(x)$. Suppose $x < i$ (equality is ruled out since $x \in (\mathcal{A} - I)$). By definition of initial segment it follows that $x \in I$, which is a contradiction. On the other hand, if $i < x$ then $i \notin (\mathcal{A} - I)$, by definition of min. Therefore $i \in I$. □

Theorem 3.5 *If $\mathcal{A}$ and $\mathcal{B}$ are well-ordered sets then either:*

- *$\mathcal{A}$ is similar to $\mathcal{B}$.*
- *$\mathcal{A}$ is similar to an initial segment of $\mathcal{B}$.*
- *$\mathcal{B}$ is similar to an initial segment of $\mathcal{A}$.*

Furthermore if A is order-isomorphic to an initial segment B_0 of B via order-isomorphism ϕ then both B_0 and ϕ are unique.

The proof of this theorem can be found in [9].

[1] By monotone we mean ***strictly increasing***.

4 Tools from Ordinal Theory

Let Ord denote the class of ordinal numbers. Roughly speaking, ordinals are types of well-ordered sets, that is, an ordinal number is an equivalence class under similarity of well-ordered sets. For finite well-ordered sets their ordinals coincide with their cardinality and are denoted by natural numbers. We define a relation $<$ in Ord by:
$\alpha < \beta \iff$ any set of type α is similar to an initial segment of a set of type β.
From theorem 3.5 it follows that $<$ totally orders Ord.

Treating ordinals as sets is quite convenient; therefore instead of taking ordinals to be equivalence classes we will identify them with canonical representatives of those classes. If we take a representative $(\mathcal{A}, >)$ of an ordinal α, it can be seen that $\mathcal{A}$ is similar to the set $\{\beta \mid \beta < \alpha\}$, that is $\alpha = \{\beta \mid \beta < \alpha\}$ and this implies that $\beta < \alpha \iff \beta \in \alpha$. So the sets well-ordered by the $\in$-relation are in fact canonical representatives of ordinals.[2]

We shall freely switch between equivalence classes and canonical representatives.

We list below some basic properties of Ord.

I. $<$ well-orders the class Ord, that is:

- $<$ is a total ordering in Ord.
- Every non-empty class $B \subseteq Ord$ has a minimal element (with respect to $<$) in B.
- For every $\alpha \in Ord$, $\{\xi \in Ord \mid \xi < \alpha\}$ is a set.

II. For every set of ordinals U there is an ordinal α such that $\alpha = sup(U) = \bigvee U$. If $U = \{f(\xi) \mid p(\xi)\}$ (for any predicate p) we sometimes use the notation $\bigvee_{p(\xi)} f(\xi)$.

III. $Lim \neq \emptyset$. λ is a limit ordinal if λ is not the successor of any ordinal and is not 0. Alternatively, $0 \neq \lambda$ is a limit ordinal iff $\xi < \lambda \Rightarrow \xi' < \lambda$, or still $\lambda = \bigvee_{\xi<\lambda} \xi$.

IV. $W(\alpha) = \{\xi \mid \xi < \alpha\}$ is well-ordered and has type α.

The ordinal 0 is defined to be the minimal element of Ord; it is the type of the empty set. For every ordinal ξ, its *successor* ξ' is defined by $\xi' = min\{\alpha \mid \xi < \alpha\}$. We use the notation $0' = 1$, $1' = 2$, and so forth. We will sometimes denote the successor ordinal by $\xi + 1$. Clearly $\xi < \xi'$ and there is no ordinal α such that $\xi < \alpha < \xi'$.

An ordinal ξ is defined to be a limit ordinal if $(\exists\alpha < \xi) \wedge (\forall\alpha < \xi\ \exists\eta < \xi : \ \alpha < \eta)$. The first condition states that a limit ordinal is non-empty, and the second condition says that it has no maximal element. An ordinal ξ is a limit ordinal if and only if $\alpha < \xi \Rightarrow \alpha' < \xi$, if and only if $\xi = \bigvee_{\alpha<\xi} \alpha$. The class of limit ordinals is denoted by Lim. The ordinal ω is defined to be the minimum of Lim; it is the type of the natural numbers.

Every ordinal is either 0, a *successor* ordinal or a *limit* ordinal. These three kinds often appear in inductive proofs and definitions.

Theorem 4.1 (Principle of Transfinite Induction) *If $\mathcal{A}$ is a class well-ordered by $>$ (see* **(I)** *above) and F is a propositional function such that $\forall x \in \mathcal{A} :\ (\forall y < x :\ F(y)) \Rightarrow F(x)$. Then $\forall x \in \mathcal{A} :\ F(x)$.*

[2]These canonical representatives are also known as Von Neumann's ordinals.

Proof Define $B = \langle x \in \mathcal{A} \mid \neg F(x)\rangle$[3]. Assume B is non-empty. Then according to the second point in **I**, B has a minimal element x and therefore, for $y < x$ we have $F(y)$. But then, by hypothesis, we also have $F(x)$ contradicting the fact that $x \in B$. □

We have the following useful lemma:

Lemma 4.2 *$\emptyset \neq U \subseteq Ord$ and $sup(U) \notin Lim \Rightarrow sup(U) \in U$.*

Proof

Take $\alpha = sup(U)$.

If $\alpha = 0$, then $\xi \leq \alpha \Rightarrow \xi = 0$ and since $U \neq \emptyset$, we have $0 \in U$.

If $\alpha = \alpha_0'$, assume that $\alpha \notin U$. Then, by definition of supremum and assumption, we have, for every $\xi \in U$, $\xi < \alpha \Rightarrow \xi \leq \alpha_0$. Again by definition of supremum, we conclude that $\alpha = sup(U) \leq \alpha_0 < \alpha$, which is a contradiction. □

The operations of addition, multiplication and exponentiation are inductively defined in Ord as follows:

	$\alpha+\beta$	$\alpha.\beta$	α^β
$\beta = 0$	α	0	1
$\beta = \beta_0'$	$(\alpha+\beta_0)'$	$\alpha.\beta_0 + \alpha$	$\alpha^{\beta_0}.\alpha$
$\beta \in Lim$	$\bigvee_{\xi<\beta}(\alpha+\xi)$	$\bigvee_{\xi<\beta}(\alpha.\xi)$	$\bigvee_{\xi<\beta}(\alpha^\xi)$

We will not go into details about these operations and we will limit ourselves to the presentation of some results that will be necessary later.

We remark that:

- $+$ and $.$ are both associative and non-commutative and $.$ left-distributes over $+$.
- $+$ is (strictly) monotone in the right argument and weakly monotone in the left argument. Consequently there is a left-cancellation law: if $\alpha+\beta = \alpha+\gamma$ then $\beta = \gamma$. Right-cancellation does not hold. For example $0+\omega = 1+\omega$ and $0 \neq 1$.
- for any $\delta > 0$, if $\alpha < \beta$ then $\delta.\alpha < \delta.\beta$ and $\alpha.\delta \leq \beta.\delta$. Again we have a left-cancellation law.
- for a fixed base greater than 1, exponentiation is strictly monotone in the exponent; consequently there is a cancellation law for the base.
- $0.\alpha = 0$, for any α. Also $\alpha.\beta = 0 \iff \alpha = 0$ *or* $\beta = 0$.
- for any $\alpha,\ \beta,\ \gamma$, $(\alpha^\beta)^\gamma = \alpha^{\beta.\gamma}$
- for any α, if $\beta \in Lim$, then $\alpha+\beta \in Lim$. Additionally if $\alpha \neq 0$ then $\alpha.\beta,\ \beta.\alpha \in Lim$.

The following lemma gives a different characterization of limit ordinals.

Lemma 4.3 $\lambda \in Lim \iff \lambda = \omega\beta,\ \beta \neq 0$

[3]The symbols $\langle\rangle$ denote a class.

Proof $\Longleftarrow$

We proceed by transfinite induction on β.

If $\beta = 1$ then $\lambda = \omega.1 = \omega \in Lim$.

If $\beta = \beta_0 + 1$ then $\lambda = \omega\beta_0 + \omega$. Now by induction hypothesis $\omega\beta_0 \in Lim$ and since ω is also a limit ordinal and Lim is closed for addition, we get $\lambda \in Lim$.

If $\beta \in Lim$, since $\omega \in Lim$ and Lim is closed for multiplication, the result follows.

$\Longrightarrow$

Suppose then that $\lambda \in Lim$. By lemma 4.11 we can write $\lambda = \omega\beta + \alpha$, with $\alpha < \omega$. If $0 < \alpha < \omega$ then λ has a direct predecessor and so cannot be a limit ordinal. Therefore $\alpha = 0$ and $\lambda = \omega\beta$. Since $\lambda \neq 0$ we have $\beta \neq 0$. □

Remark. If we include 0 in Lim and remove the restriction $\beta \neq 0$ above, the result is still valid.

Some ordinals are closed under the operations of addition and/or multiplication; they are crucial in this paper. We first define those ordinals and then give a characterization of them.

Definition 4.4 *An ordinal α is additive principal if $\alpha \neq 0$ and $(\xi, \eta < \alpha \Rightarrow \xi + \eta < \alpha)$. An ordinal α is multiplicative principal if $\alpha > 1$ and $(\xi, \eta < \alpha \Rightarrow \xi.\eta < \alpha)$.*

Lemma 4.5 *Let $\alpha \in Ord$. Then the following conditions are equivalent:*

- *α is additive (multiplicative) principal.*
- *$\alpha = \omega^\eta$ (respectively $\alpha = \omega^{\omega^\eta}$ or $\alpha = 2$), for some $\eta \geq 0$*
- *$\forall \beta < \alpha : \ \beta + \alpha = \alpha$ (respectively $\beta.\alpha = \alpha$).*

The next results are also standard results that we present without proof.

Lemma 4.6 *If $\alpha \leq \beta$ then there is a unique ordinal δ such that $\beta = \alpha + \delta$.*

The ordinal δ is usually written as $\beta - \alpha$ and we speak of subtraction of ordinals.

It is not difficult to see that subtraction is weakly anti-monotone, i. e., if $a < b \leq \alpha$ then $(\alpha - b) \leq (\alpha - a)$. As we see below, in certain conditions difference can be performed in an initial segment.

Lemma 4.7 *If $\tau \leq \alpha$ then $(\alpha + \delta) - \tau = (\alpha - \tau) + \delta$.*

Proof First we remark that the difference $\alpha - \tau$ is well-defined given that $\tau \leq \alpha$.

By definition of difference, we have $(\alpha + \delta) - \tau = \tau_1 \iff \tau + \tau_1 = \alpha + \delta$. Also $\alpha - \tau = \tau_2 \iff \tau + \tau_2 = \alpha$ and therefore $\tau + \tau_2 + \delta = \alpha + \delta$.

So we have $\tau + \tau_1 = \tau + \tau_2 + \delta$ and by the left cancellation law we conclude that $\tau_1 = \tau_2 + \delta$ as we wanted. □

Lemma 4.8 *Let $f : \alpha \to \alpha$ be a monotone function. Then for any ordinals a, b such that $a + b < \alpha$ we have $f(a + b) \geq f(a) + b$.*

Proof Fix $a \in \alpha$. Define $g(x) = f(a + x) - f(a)$, for any $x \in (\alpha - a)$. Since f is monotone, g is well-defined and is a function from $\alpha - a$ to $\alpha - f(a)$. Furthermore g is also monotone. From lemma 3.1 we conclude that $f(a + x) - f(a) = g(x) \geq x$, hence $f(a + x) \geq f(a) + x$. □

As we would expect, additive principal ordinals are closed under subtraction.

Lemma 4.9 *If $\alpha < \omega^\eta$, for some ordinal η, then $\omega^\eta - \alpha = \omega^\eta$.*

Proof By definition of subtraction, $\omega^\eta - \alpha = \gamma \iff \alpha + \gamma = \omega^\eta \Rightarrow \gamma \leq \omega^\eta$. Suppose that $\gamma < \omega^\eta$, then since ω^η is additive principal we would get $\alpha + \gamma < \omega^\eta$. Therefore $\gamma = \omega^\eta$. □

We conclude this section with some useful standard results.

Lemma 4.10 *Let $f : \alpha \to \beta$ be monotone. Then $\alpha \leq \beta$.*

Proof Suppose that $\beta < \alpha$. Then there is a unique δ such that $\alpha = \beta + \delta$ and since $\delta > 0$, $\beta \in \beta + \delta$. We remark that in particular f is also a monotone function from $\beta + \delta$ to $\beta + \delta$ and therefore, by lemma 3.1, we have that $x \leq f(x)$, for any $x \in \beta + \delta$. Consequently $\beta \leq f(\beta) < \beta$ (since $f(\beta) \in \beta$), giving a contradiction. □

Lemma 4.11

1. $\forall \lambda, \delta \in Ord\ \exists! \beta, \alpha \in Ord: \ \lambda = \delta\beta + \alpha, \ \alpha < \delta$
2. $\forall \beta \geq 1\ \forall \alpha \geq 2\ \exists! \eta: \ \alpha^\eta \leq \beta < \alpha^{\eta+1}$.
3. *If $\alpha < \beta.\gamma$ then $\exists! \beta_1, \gamma_1 : \beta_1 < \beta \ \wedge \ \gamma_1 < \gamma \ \wedge \ \alpha = \beta.\gamma_1 + \beta_1$.*

Proof For **1**, define $U = \{\xi \mid \delta\xi \leq \lambda\}$ and take $\beta = sup(U)$ (possible since U is a set).
We have:
$\delta.\beta = \delta. \bigvee_{\delta\xi \leq \lambda} \xi \leq \bigvee_{\delta\xi \leq \lambda} \delta.\xi = \bigvee_{\alpha \leq \lambda, \exists \xi : \alpha = \delta\xi} \alpha \leq \bigvee_{\alpha \leq \lambda} \alpha = \lambda$
Now suppose $\delta.(\beta + 1) \leq \lambda$. Then, by definition of U, we have $\beta + 1 \in U$ implying $\beta + 1 \leq \beta$, which is clearly a contradiction. So $\delta.(\beta + 1) > \lambda$. Using lemma 4.6 and monotonicity of addition, we see that there is an ordinal α such that $\lambda = \delta\beta + \alpha$ and $\alpha < \delta$.

We now prove **2**. Define $U = \{\xi \mid \alpha^\xi \leq \beta\}$ and take $\eta = sup(U)$ (possible since U is a set; notice also that U is non-empty since $0 \in U$). By lemma 4.2, $\eta \in U$ or $\eta \in Lim$. In the first case $\omega^\eta \leq \beta$. If $\eta \in Lim$ then $\omega^\eta = \bigvee_{\xi < \eta} \omega^\xi$, and since for each ξ, $\omega^\xi \leq \beta$, then also $\omega^\eta \leq \beta$.
Suppose now that $\alpha^{\eta+1} \leq \beta$. Then by definition of U, $\eta + 1 \in U$, and by definition of *sup*, we get $\eta + 1 \leq \eta$, which is not possible. Therefore $\beta < \alpha^{\eta+1}$ and since $\alpha^\eta \leq \beta$, we get the result.

Finally for **3**, fixing α and β, we get by **1** that there are unique ordinals γ_1, β_1 such that $\alpha = \beta.\gamma_1 + \beta_1$ and $\beta_1 < \beta$. Suppose it would be $\gamma \leq \gamma_1$. Then $\beta.\gamma \leq \beta.\gamma_1 \leq \alpha$, which is a contradiction. □

Theorem 4.12 (Cantor Normal Form) *For every ordinal $\alpha \neq 0$ there are uniquely determined ordinals $\eta_1 \geq \ldots \geq \eta_k$, with $k \geq 1$, such that $\alpha = \omega^{\eta_1} + \ldots + \omega^{\eta_k}$.*

Given an ordinal α we can determine its unique normal form $\overline{\alpha}$. This unique normal form is the expansion of α with base ω, that is $\overline{\alpha} = \omega^{\eta_1}.p_1 + \ldots + \omega^{\eta_k}.p_k$, with $\eta_1 > \eta_2 > \ldots > \eta_k$, $0 \leq p_i < \omega$, for $1 \leq i \leq k$ and $k \geq 1$. Using this normal form we can define natural addition, denoted by $\bigoplus$. Given ordinals x, y, $\overline{x} \bigoplus \overline{y}$ is performed by adding the expansions of both x and y as polynomials in ω (well-defined since ordinal addition is commutative for ordinals smaller than ω). Natural addition is commutative, associative and strictly increasing in each argument. Furthermore ordinals of the form ω^γ, for $\gamma \geq 0$, are principal ordinals for addition, and therefore closed for natural addition. Similarly we can define natural multiplication (for details see [9]).

5 Exponentiation Revisited

In this section we give a constructive description of ordinal exponentiation.

Definition 5.1 *Let $Exp(\alpha, \eta) = \{\sigma | \eta \rightarrow \alpha \mid \{y \in \eta \mid \sigma(y) \neq 0_\alpha\}$ is finite$\}$, for $\alpha, \eta \in Ord$. In $Exp(\alpha, \eta)$ we define the relation $>$ by:*
$\sigma > \sigma' \iff \exists x \in \eta :\ (\sigma(x) >_\alpha \sigma'(x))\ \wedge\ (\forall y \in \eta : y >_\eta x \Rightarrow \sigma(y) = \sigma'(y))$
for any $\sigma, \sigma' \in Exp(\alpha, \eta)$.

Lemma 5.2 *Let $(Exp(\alpha, \eta), >)$ be defined as above. Then $>$ is a well-order on $Exp(\alpha, \eta)$.*

Proof $>$ is indeed a strict order in $Exp(\alpha, \eta)$, since

- $\forall \sigma \in Exp(\alpha, \eta) :\ \sigma \not> \sigma$
- $>$ is transitive, for if $\sigma, \sigma', \sigma'' \in Exp(\alpha, \eta)$, $\sigma > \sigma'$ and $\sigma' > \sigma''$, then
  - $\exists x \in \eta :\ (\sigma(x) >_\alpha \sigma'(x))\ \wedge\ (\forall y \in \eta :\ y >_\eta x \Rightarrow \sigma(y) = \sigma'(y))$
  - $\exists x' \in \eta :\ (\sigma'(x') >_\alpha \sigma''(x'))\ \wedge\ (\forall y \in \eta :\ y >_\eta x' \Rightarrow \sigma'(y) = \sigma''(y))$

  Since η is totally ordered, we know that either $x \geq_\eta x'$ or $x' >_\eta x$. If $x \geq_\eta x'$ then $\sigma'(x) \geq_\alpha \sigma''(x)$ and so $\sigma(x) >_\alpha \sigma''(x)$. Also, if $y >_\eta x$ then $y >_\eta x'$ so $\sigma(y) = \sigma'(y) = \sigma''(y)$, thus we conclude that $\sigma > \sigma''$.
  If $x' >_\eta x$, then $\sigma(x') = \sigma'(x') >_\alpha \sigma''(x')$ and for $y \in \eta, y >_\eta x'$, $\sigma(y) = \sigma'(y) = \sigma''(y)$, so again $\sigma > \sigma''$.

It is also not difficult to see that $>$ is total.
Let then $\sigma, \sigma' \in Exp(\alpha, \eta)$ and $\sigma \neq \sigma'$. That means that there is $x \in \eta$ such that $\sigma(x) \neq \sigma'(x)$. Take $\overline{x} = max\{y \in \eta | \ \sigma(y) \neq \sigma'(y)\}$ (this set is finite by definition of $Exp(\alpha, \eta)$ and non-empty by hypothesis). Since α is totally ordered, we have $\sigma(\overline{x}) >_\alpha \sigma'(\overline{x})$ or $\sigma'(\overline{x}) >_\alpha \sigma(\overline{x})$. In both cases, if $y >_\eta \overline{x}$, by definition of $\overline{x}$, we have $\sigma(y) = \sigma'(y)$. Therefore $\sigma > \sigma'$ or $\sigma' > \sigma$, as we wanted.

Finally we see that $>$ is well-founded following an approach similar to the approach presented in [5] for multisets.
We extend η with a (new) least element $\perp$. Clearly $\eta_\perp = \eta \cup \{\perp\}$ is still well-ordered (and corresponds to the same ordinal η).
Suppose that $Exp(\alpha, \eta)$ is not well-founded. Then there exists an infinite descending chain of the form:

$$\sigma_0 > \sigma_1 > \sigma_2 > \ldots\ldots$$

For this chain we will build a tree with nodes labelled in $\eta_\perp \times \alpha$, in the following way: the root of the tree has no label and it has a child labelled $(x, \sigma_0(x))$, for each $x \in \eta$ such that $\sigma_0(x) \neq 0_\alpha$ (recall that those elements are in finite number).
Since $\sigma_0 > \sigma_1$, then there is $x_0 \in \eta$ such that $\sigma_0(x_0) >_\alpha \sigma_1(x_0)$ and for $y >_\eta x_0$, $\sigma_0(y) = \sigma_1(y)$. Now for $x <_\eta x_0$, we do the following:

- if $\sigma_0(x) >_\alpha \sigma_1(x) >_\alpha 0_\alpha$, then the node labelled $(x, \sigma_0(x))$ has a unique child labelled $(x, \sigma_1(x))$.
- if $\sigma_0(x) > 0_\alpha$ and $\sigma_1(x) = 0_\alpha$, then the node labelled $(x, \sigma_0(x))$ has a unique child labelled $(\perp, \sigma_0(x))$.
- if $\sigma_1(x) >_\alpha \sigma_0(x)$, then add to the node labelled $(x_0, \sigma_0(x_0))$ the child labelled $(x, \sigma_1(x))$. Additionally, if $\sigma_0(x) > 0$ then the node $(x, \sigma_0(x))$ has a unique child labelled $(\perp, \sigma_0(x))$.

For x_0, we have to consider two cases, namely:

- if $\sigma_1(x_0) > 0_\alpha$, then add to the children of $(x_0, \sigma_0(x_0))$ the node labelled $(x_0, \sigma_1(x_0))$.
- if $\sigma_1(x_0) = 0_\alpha$ and the node $(x_0, \sigma_0(x_0))$ has no children, then it will get a unique child labelled $(\perp, \sigma_0(x_0))$[4].

We repeat the process for $\sigma_1 > \sigma_2$ and so on.

We remark that

- a node has finitely many children.
- at the i-th iteration of the construction of the tree, the leaves not labelled $(\perp, .)$, describe the function σ_i.
- every element σ_i in the initial sequence contributes with at least one node to the tree.

By the last remark and since the sequence is infinite, then the tree is also infinite and, by König's Lemma, it has an infinite path. But that path (eliminating the root node) corresponds to an infinite descending chain in $\eta_\perp \times \alpha$ (the lexicographic product with weight on $\eta_\perp$) contradicting its well-foundedness. □

Theorem 5.3 *Let $\alpha, \eta \in Ord$ and $(Exp(\alpha, \eta), >)$ as defined in 5.1. Then $(Exp(\alpha, \eta), >)$ is order-isomorphic to ordinal exponentiation α^η.*

Proof We proceed by transfinite induction on η; for $\eta = 0$ the assertion is trivial.

If $\eta = \beta + 1$ then $\eta = \beta \cup \{\beta\}$. But $\alpha^{\beta+1} = \alpha^\beta.\alpha$ and corresponds to the lexicographic product with weight on α. By induction hypothesis, $(Exp(\alpha, \beta), >) \cong \alpha^\beta$.
We define $\Psi : \alpha^\beta.\alpha \to Exp(\alpha, \beta + 1)$ by

$$\begin{array}{rccl} \Psi : & \alpha^\beta.\alpha & \to & Exp(\alpha, \beta+1) \\ & (\sigma_b, a) & \longmapsto & \sigma, \quad \text{with } \sigma(x) = \begin{cases} \sigma_b(x) & if\ x \in \beta \\ a & if\ x = \beta \end{cases} \end{array}$$

[4] Actually we can always add this child, whether the set of children is empty or not.

In the definition above, we abuse the notation: σ_b stands actually for $\Phi_\beta(\sigma_b)$, where Φ_β is the unique order-isomorphism between α^β and $Exp(\alpha,\beta)$ postulated by the induction hypothesis. Note that Ψ is well-defined since the set $\{y \in \beta \mid \sigma_b(y) \neq 0_\alpha\}$ is finite and therefore $\{y \in \beta+1 \mid \sigma(y) \neq 0_\alpha\}$ is also finite.

We shall prove that Ψ is an order-isomorphism. For any $\sigma \in Exp(\alpha,\beta+1)$, define $\sigma_b = \sigma|_\beta$ (the restriction of σ to β) and $a = \sigma(\beta)$. Then it is trivial to see that $(\sigma_b, a) \in \alpha^\beta.\alpha$ and $\Psi(\sigma_b, a) = \sigma$. So we conclude that Ψ is surjective.
We show now that Ψ is monotone. Let then $(\sigma_b, a), (\sigma'_b, a') \in \alpha^\beta.\alpha$ such that $(\sigma_b, a) >_l (\sigma'_b, a')$ ($>_l$ is the lexicographic order with weight on the second coordinate). We have to see that $\sigma = \Psi(\sigma_b, a) > \sigma' = \Psi(\sigma'_b, a')$. If $(\sigma_b, a) >_l (\sigma'_b, a')$ then either $a >_\alpha a'$ or both $a = a'$ and $\sigma_b > \sigma'_b$.
If $a >_\alpha a'$ then $\sigma(\beta) >_\alpha \sigma'(\beta)$ and since in $\beta+1$ there is no element greater than β, $\sigma > \sigma'$ holds.
In the second case, we have that $\exists x \in \beta : \ (\sigma_b(x) >_\alpha \sigma'_b(x)) \ \wedge \ (\forall y \in \beta : \ y >_\beta x \Rightarrow \sigma_b(y) = \sigma'_b(y))$.
Then for the same $x \in \beta$, $\sigma(x) = \sigma_b(x) >_\alpha \sigma'_b(x) = \sigma'(x)$ and also if $y \in \beta$ and $y >_\beta x$ then $\sigma(y) = \sigma'(y)$. Since $\sigma(\beta) = a = a' = \sigma'(\beta)$, we have $\sigma > \sigma'$.
Given that $\alpha^\beta.\alpha$ and $Exp(\alpha,\beta+1)$ are both totally ordered, we can apply lemma 3.2 to conclude that Ψ is an order-isomorphism, and therefore $(Exp(\alpha,\beta+1), >) \cong \alpha^\beta.\alpha = \alpha^{\beta+1}$.

If $\eta \in Lim$ then $\eta = \bigvee_{\xi<\eta} \xi$ and $\alpha^\eta = \bigvee_{\xi<\eta} \alpha^\xi = \bigcup_{\xi<\eta} \alpha^\xi$. By induction hypothesis, for every $\xi < \eta$, we have $\alpha^\xi \cong (Exp(\alpha,\xi), >)$. Furthermore for $\xi < \gamma < \eta$ we have that $Exp(\alpha,\xi)$ is an initial segment of $Exp(\alpha,\gamma)$, and the unique order-isomorphism between $Exp(\alpha,\xi)$ and the correspondent initial segment of $Exp(\alpha,\gamma)$ is given by

$$\begin{array}{rccl} f_\xi : & Exp(\alpha,\xi) & \longrightarrow & Exp(\alpha,\gamma) \\ & \sigma & \longmapsto & \overline{\sigma}, \ where \ \ \overline{\sigma}(x) = \begin{cases} \sigma(x) & if\ x \in \xi \\ 0_\alpha & otherwise \end{cases} \end{array}$$

We will see that $Exp(\alpha,\eta)$ is order-isomorphic to $\bigcup_{\xi<\eta} \alpha^\xi$. For that we define

$$\begin{array}{rccl} \phi : & \bigcup_{\xi<\eta} \alpha^\xi & \longrightarrow & Exp(\alpha,\eta) \\ & \tau & \longmapsto & \sigma_\tau, \ with \ \ \sigma_\tau(x) = \begin{cases} \phi_\nu(\tau)(x) & if\ x \in \nu \\ 0_\alpha & otherwise \end{cases} \end{array}$$

where $\nu = \min\{\gamma < \eta \mid \tau \in \alpha^\gamma\}$ and ϕ_ν is the unique order-isomorphism between α^ν and $Exp(\alpha,\nu)$ (whose existence is guaranteed by induction hypothesis).

It is easy to see that ϕ is well-defined. We check that ϕ is surjective. For $\sigma \in Exp(\alpha,\eta)$ we define $\xi = \min\{\gamma < \eta \mid \forall x \geq \gamma : \ \sigma(x) = 0_\alpha\}$. Defining $\sigma|_\xi$ as the restriction of σ to ξ, it is trivial to verify that $\sigma|_\xi \in Exp(\alpha,\xi)$. Since $Exp(\alpha,\xi)$ and α^ξ are order-isomorphic, there is an element $\tau \in \alpha^\xi$ such that $\phi_\xi(\tau) = \sigma|_\xi$. Considering the definition of ϕ, we check now that $\nu_\tau = \xi$. By definition of ν_τ, it holds $\nu_\tau \leq \xi$. Suppose $\nu_\tau < \xi$ then, since by induction hypothesis α^{ν_τ} is order-isomorphic to $Exp(\alpha,\nu_\tau)$ which is an initial segment of $Exp(\alpha, \xi)$ (modulo order-isomorphism) and $\tau \in \alpha^{\nu_\tau}$, there must be an element $\varsigma \in Exp(\alpha, \nu_\tau)$ such that $f_{\nu_\tau}(\varsigma) = \sigma|_\xi$. But then, for all $x \geq \nu_\tau$, $f_{\nu_\tau}(\varsigma)(x) = 0_\alpha \Rightarrow \nu_\tau \geq \xi$, giving a contradiction. Consequently $\nu_\tau = \xi$ and therefore $\phi(\tau)$ is given by:

$$\sigma_\tau(x) = \begin{cases} \phi_{\nu_\tau}(\tau)(x) = \sigma|_\xi(x) & if\ x \in \xi \\ 0_\alpha & otherwise \end{cases}$$

That is, $\phi(\tau) = \sigma$.

Next we prove the monotonicity of ϕ. If $\tau > \gamma$ then either $\nu_\tau > \nu_\gamma$ or $\nu_\tau = \nu_\gamma$. In the last case $\phi(\tau) > \phi(\gamma)$ is a consequence of the definition of ϕ and the monotonicity of ϕ_{ν_τ} $(= \phi_{\nu_\gamma})$. In the case $\nu_\tau > \nu_\gamma$ we apply theorem 3.5 to α^{ν_γ}, α^{ν_τ} and $Exp(\alpha, \nu_\tau)$, to get $\phi_{\nu_\gamma}(\gamma) \leq \phi_{\nu_\tau}(\gamma) < \phi_{\nu_\tau}(\tau)$ in $Exp(\alpha, \nu_\tau)$ and where the last inequality is justified by monotonicity of ϕ_{ν_τ}. Consequently $\phi(\gamma) < \phi(\tau)$.

We now apply lemma 3.2 to conclude that ϕ is an order-isomorphism and therefore $\alpha^\eta = \bigcup_{\xi<\eta} \alpha^\xi \cong Exp(\alpha, \eta)$. □

We would like to point out that the proof of well-foundedness of $>$ in $Exp(\alpha, \eta)$ we give in 5.2 is redundant. Well-foundedness of $>$ is a direct consequence of the above result.

Note that the definition of $Exp(\omega, \eta)$ coincides with that of the set $M(\eta)$ of finite multisets over η, together with its multiset order as described in [5]. So the order type of $M(\eta)$ is ω^η. In the sequel we shall freely switch between $M(\eta)$ and ω^η. For example, considering multisets in $M(\eta)$ as functions from η to ω multiset union is pointwise addition. This corresponds exactly to natural addition of ordinals below ω^η.

6 Multisets and binary functions

We shall prove that the existence of an operation of arity greater than one in some ordinal implies that the ordinal has the form ω^η. As a consequence, for a TRS containing operation symbols of arity > 1 the only monotone algebras of interest are those whose underlying order is a multiset order. First we need two lemmas.

Lemma 6.1 *Let λ be an ordinal for which $\exists \alpha < \lambda : \lambda - \alpha \leq \alpha$. Then no function from $\lambda \times \cdots \times \lambda$ to λ exists which has more than one argument and is monotone in all arguments.*[5]

Proof Suppose such a function exists. Then by fixing all arguments but two we obtain a binary function f that is monotone in both arguments. Define $\varphi : \lambda \to \lambda$ by $\varphi(x) = f(x, \alpha) - \alpha$. We have to see that φ is well-defined. If we fix the first argument of f to 0_λ, the minimum of λ, we have, since $f(0_\lambda, x)$ is strictly monotone and by lemma 3.1, that $f(0_\lambda, \alpha) \geq \alpha$. So $f(x, \alpha) \geq \alpha$, for any x, hence φ is well-defined and is actually a function from λ to $\lambda - \alpha$. If $x > y$ then $\alpha + \varphi(x) = f(x, \alpha) > f(y, \alpha) = \alpha + \varphi(y)$. Due to the left cancellation law, we conclude that φ is (strictly) monotone. By lemma 4.10 we conclude that $\lambda \leq \lambda - \alpha$. Since $\alpha < \lambda$ we get $\alpha < \lambda - \alpha$, contradicting the hypothesis. □

Lemma 6.2 *Let $\lambda \neq 0$. Then $\lambda = \omega^\gamma$, for some γ, if and only if $\forall \alpha < \lambda : \lambda - \alpha > \alpha$.*

[5] $\times$ denotes cartesian product.

Proof We will prove that $\forall\alpha,\beta < \lambda : \alpha+\beta < \lambda$ if and only if $\forall\alpha < \lambda : \lambda - \alpha > \alpha$; then the result follows from lemma 4.5.
For the only-if part, let $\alpha < \lambda$. We always have $\lambda - \alpha \leq \lambda$. If $\lambda - \alpha < \lambda$, by hypothesis we get $\alpha + (\lambda - \alpha) < \lambda$, giving a contradiction. Therefore $\lambda - \alpha = \lambda$, so $\lambda - \alpha > \alpha$.

For the if part, take $\alpha, \beta < \lambda$. The hypothesis implies $\alpha < \lambda - \alpha$ and $\beta < \lambda - \beta$. If $\beta \leq \alpha$ then $\alpha + \beta \leq \alpha + \alpha < \alpha + (\lambda - \alpha) = \lambda$. If $\alpha < \beta$ then $\alpha + \beta \leq \beta + \beta < \beta + (\lambda - \beta) = \lambda$. Since either $\alpha < \beta$ or $\beta \leq \alpha$, we get the result. □

Theorem 6.3 *Let $\mathcal{A} = (A, >)$ be a well-ordered set such that $A \neq \emptyset$. Then there is a function from $A \times \cdots \times A$ to A with more than one argument, monotone in all arguments if and only if $\mathcal{A}$ is order-isomorphic to $M(\mathcal{B})$ for some well-ordered set $\mathcal{B}$.*

Proof Assume that $\mathcal{A} \cong M(\mathcal{B})$, for some well-order $\mathcal{B}$. It is easy to see that multiset union is strictly increasing in both operands and that if we extend the union to k arguments, the strict monotonicity in each argument is still respected. By the isomorphism we get a similar function in $\mathcal{A} \times \ldots \times \mathcal{A}$.

Suppose now $\mathcal{A} \neq \emptyset$ is well-ordered and such that we can define a function $f : \mathcal{A}^n \to \mathcal{A}$, with $n \geq 2$, strictly increasing in each coordinate. Let λ be the type of $\mathcal{A}$. Then by lemma 6.1 we know that, for any $\alpha < \lambda$, $\lambda - \alpha > \alpha$. So we can apply lemma 6.2 to conclude that $\lambda = \omega^\gamma$ for some γ. But ω^γ is precisely the type of $M(\mathcal{B})$, for some well-ordered set $\mathcal{B}$ of type γ, and since two well-ordered sets have the same type iff they are order-isomorphic, we conclude our result. □

Stated in different words, the previous result says that if we have a TRS R containing at least a function symbol of arity $n \geq 2$ and totally terminating in an algebra $\mathcal{A}$, then $\mathcal{A}$ has type ω^γ, for some $\gamma \geq 0$.

7 Extension to higher ordinals and modularity

In this section we look at modularity of total termination. If two TRS's are totally terminating, what can be said about their disjoint union? From [10] follows that the disjoint union is simply terminating, but is it also totally terminating? This is not clear if the proofs of total termination are given in distinct ordinals. That arises the question whether a total termination proof in some ordinal can be lifted to a similar proof in another ordinal.

Definition 7.1 *For a TRS R we define $U(R)$ to be the class of ordinals in which a proof of total termination of R can be given. The minimum of $U(R)$ is denoted by u_R.*

By definition $U(R)$ is non-empty for every totally terminating TRS R. For example, if R consists of one rule involving two different constants then $U(R)$ is the class of all ordinals > 1. Note that the disjoint union $R_1 \oplus R_2$ of two TRS's R_1 and R_2 is totally terminating if and only if $U(R_1 \oplus R_2) = U(R_1) \cap U(R_2) \neq \emptyset$.

The next lemmas state some basic properties of $U(R)$.

Lemma 7.2 *Let $\alpha \in U(R)$ and let β be an arbitrary non-zero ordinal. Let either all function symbols in R have arity ≤ 1 or $\beta = \omega^\gamma$ for some ordinal γ. Then $\beta.\alpha \in U(R)$.*

Proof Remember that $\beta.\alpha$ is the lexicographic product with weight on α. Its elements will be denoted by pairs (b,a), with $a \in \alpha$ and $b \in \beta$. Since $\alpha \in U(R)$, we have an interpretation f_α of every function symbol f of R in α, strictly monotone in each argument, such that for every rule $l \to r$ in R and every substitution $\tau : X \to \alpha$, it holds $\phi_\alpha(l,\tau) >_\alpha \phi_\alpha(r,\tau)$. For every function symbol f we introduce an interpretation f_β in β: for constants c we choose $c_\beta = 0$ and for unary f we choose f_β to be the identity on β. If there are symbols of arity > 1 we assumed β to be the finite multisets over γ, in this case we define f_β to be the multiset union of all of its arguments. For every f define

$$f_{\beta.\alpha}((b_1,a_1),\ldots,(b_n,a_n)) = (f_\beta(b_1,\ldots,b_n), f_\alpha(a_1,\ldots,a_n)).$$

Monotonicity of $f_{\beta.\alpha}$ is in all arguments is easily verified. We still have to check that $\phi_{\beta.\alpha}(l,\tau) >_{\beta.\alpha} \phi_{\beta.\alpha}(r,\tau)$ for every rule $l \to r$ in R and every $\tau : X \to \beta.\alpha$. For this we need a lemma, which is easily proven by induction on terms.

Lemma 7.3 *Let t be any term, let $\tau : X \to \beta.\alpha$, and let π_j be the projection on the j^{th} coordinate for $j = 1,2$. Then $\phi_{\beta.\alpha}(t,\tau) = (\phi_\beta(t,\pi_1 \circ \tau), \phi_\alpha(t,\pi_2 \circ \tau))$.*

Since $\phi_\alpha(l,\sigma) >_\alpha \phi_\alpha(r,\sigma)$ for any $\sigma : X \to \alpha$, we conclude
$\phi_{\beta.\alpha}(l,\tau) = (\phi_\beta(l,\pi_1 \circ \tau), \phi_\alpha(l,\pi_2 \circ \tau)) >_{\beta.\alpha} (\phi_\beta(r,\pi_1 \circ \tau), \phi_\alpha(r,\pi_2 \circ \tau)) = \phi_{\beta.\alpha}(r,\tau)$.
This concludes the proof of lemma 7.2. We now prove lemma 7.3.

Proof By induction on t.
If $t = x \in X$ then $\phi_{\beta.\alpha}(x,\tau) = (\pi_1 \circ \tau(x), \pi_2 \circ \tau(x)) = (\phi_\beta(x,\pi_1 \circ \tau), \phi_\alpha(x,\pi_2 \circ \tau))$.
If $t = c$ the result is also trivial.
If $t = f(t_1,\ldots,t_n)$ then

$$\begin{aligned}
&\phi_{\beta.\alpha}(f(t_1,\ldots,t_n),\tau) \stackrel{def}{=} f_{\beta.\alpha}(\phi_{\beta.\alpha}(t_1,\tau),\ldots,\phi_{\beta.\alpha}(t_n,\tau))\\
&\stackrel{IH}{=} f_{\beta.\alpha}((\phi_\beta(t_1,\pi_1 \circ \tau), \phi_\alpha(t_1,\pi_2 \circ \tau)),\ldots,(\phi_\beta(t_n,\pi_1 \circ \tau), \phi_\alpha(t_n,\pi_2 \circ \tau)))\\
&\stackrel{def}{=} (f_\beta(\phi_\beta(t_1,\pi_1 \circ \tau),\ldots\phi_\beta(t_n,\pi_1 \circ \tau)), f_\alpha(\phi_\alpha(t_1,\pi_2 \circ \tau),\ldots\phi_\alpha(t_n,\pi_2 \circ \tau)))\\
&\stackrel{def}{=} (\phi_\beta(f(t_1,\ldots,t_n),\pi_1 \circ \tau), \phi_\alpha(f(t_1,\ldots,t_n),\pi_2 \circ \tau)) \ \square
\end{aligned}$$

$\square$

Theorem 7.4 *If $\alpha \in U(R)$ then $\omega^\alpha \in U(R)$.*

Proof Again f_α will denote the interpretation of the function symbols f of R in α. In this proof we identify ω^α with the finite non-empty multisets over α instead of all finite multisets. In terms of ordinals this does not make any difference since $\omega^\alpha - 1 = \omega^\alpha$. Write $[a]$ for the multiset containing only one element a and $\bigsqcup$ for multiset union. We can index multiset union over finite multisets as follows:

$$\bigsqcup_{x\in[a]} \Phi(x) = \Phi(a); \quad \bigsqcup_{x\in X\sqcup Y} \Phi(x) = (\bigsqcup_{x\in X} \Phi(x))\bigsqcup(\bigsqcup_{x\in Y} \Phi(x)),$$

for any function $\Phi : \alpha \to M(\alpha)$. For constants c and function symbols f of arity $n \geq 1$, we define:

- $c_{\omega^\alpha} = [c_\alpha]$.

- $f_{\omega^\alpha}(X_1, \ldots, X_n) = \bigsqcup_{x_1 \in X_1} \cdots \bigsqcup_{x_n \in X_n} [f_\alpha(x_1, \ldots, x_n)]$.

For functions symbols f with arity 1 we additionally define $f([]) = []$. It can be verified that f_{ω^α} is strictly monotone in each argument for all function symbols f; for functions with arity > 1 the non-emptiness restriction is essential.

We remark that for any function symbol f and substitution $\tau : X \to \omega^\alpha$, we have $\phi_{\omega^\alpha}(f(t_1, \ldots, t_n), \tau) = f_{\omega^\alpha}(\phi_{\omega^\alpha}(t_1, \tau), \ldots, \phi_{\omega^\alpha}(t_n, \tau)) = \bigsqcup_{x_1 \in \phi_{\omega^\alpha}(t_1, \tau)} \cdots \bigsqcup_{x_n \in \phi_{\omega^\alpha}} [f_\alpha(x_1, \ldots, x_n)]$

Let $l \to r$ be an arbitrary rule and let $\tau : X \to \omega^\alpha$. We still have to prove that $\phi_{\omega^\alpha}(l, \tau) > \phi_{\omega^\alpha}(r, \tau)$. For any such τ, we define a substitution $\sigma_{max} : X \to \alpha$ by $\sigma_{max}(x) = \max(\tau(x))$ (recall that for every x, $\tau(x) \neq \emptyset$). We see by induction that for any term t, $\max(\phi_{\omega^\alpha}(t, \tau)) = \phi_\alpha(t, \sigma_{max})$. For $t = x \in X$ and $t = c$ the result is a trivial consequence of the definitions of ϕ_α, ϕ_{ω^α} and σ_{max}. If $t = f(t_1, \ldots, t_n)$, then $\phi_{\omega^\alpha}(t, \tau) = f_{\omega^\alpha}(\phi_{\omega^\alpha}(t_1, \tau), \ldots, \phi_{\omega^\alpha}(t_n, \tau)) = \bigsqcup_{x_1 \in \phi_{\omega^\alpha}(t_1, \tau)} \cdots \bigsqcup_{x_n \in \phi_{\omega^\alpha}} [f_\alpha(x_1, \ldots, x_n)]$.
Since f_α is strictly monotone in each argument, the maximum of $\phi_{\omega^\alpha}(t, \tau)$ is obtained when all arguments of f_α equal $\max(\phi_{\omega^\alpha}(t_i, \tau))$, for each $1 \leq i \leq n$. But by induction hypothesis, $\max(\phi_{\omega^\alpha}(t_i, \tau)) = \phi_\alpha(t_i, \sigma_{max})$, for each i, therefore $\max(\phi_{\omega^\alpha}(t, \tau)) = f_\alpha(\phi_\alpha(t_1, \sigma_{max}), \ldots, \phi_\alpha(t_n, \sigma_{max})) = \phi_\alpha(t, \sigma_{max})$.

For all $a \in \phi_{\omega^\alpha}(r, \tau)$, we have

$$a \leq \max(\phi_{\omega^\alpha}(r, \tau)) = \phi_\alpha(r, \sigma_{max}) < \phi_\alpha(l, \sigma_{max}) = \max(\phi_{\omega^\alpha}(l, \tau))$$

Consequently we obtain $\phi_{\omega^\alpha}(l, \tau) > \phi_{\omega^\alpha}(r, \tau)$. We have proven that R is totally terminating in ω^α, so $\omega^\alpha \in U(R)$. □

Now we are ready to prove modularity of total termination under certain conditions.

Theorem 7.5 *Let R_1 and R_2 be totally terminating TRS's, at least one of them not containing duplicating rules. Then $R_1 \oplus R_2$ is totally terminating.*

Proof Let α and β be ordinals in which the proofs of total termination of R_1 and R_2 can respectively be given. Due to theorem 7.4 we may, and shall, assume that $\alpha = \omega^\gamma$ and $\beta = \omega^\eta$, for some $\gamma, \eta \geq 1$. Suppose that R_1 has no duplicating rules (the other case is symmetric). Identify $\beta = \omega^\eta$ with finite multisets over η and define interpretations in β for the functions symbols of R_1 in the following way:

- $c_\beta = []$, for any constant c, where $[] = 0_\beta$ represents the empty multiset.
- $f_\beta(x_1, \ldots, x_n) = \bigsqcup_{i=1}^{n} x_i$, where $\bigsqcup$ represents multiset union.

For a term t let X_t be the multiset of variables occurring in t. For any $\tau : X \to \beta$ we obtain $\phi_\beta(t, \tau) = \bigsqcup_{x \in X_t} \tau(x)$; here the multiset union over an empty index is defined to be $[]$. Since there are no duplicating rules the multiset X_r is contained in X_l for all rewrites rules $l \to r$. Consequently,

$$\phi_\beta(l, \tau) = \bigsqcup_{x \in X_l} \tau(x) \geq \bigsqcup_{x \in X_r} \tau(x) = \phi_\beta(r, \tau).$$

Note that the inequality is not strict in general.

Now in $\alpha.\beta$ (the lexicographic product with weight on β) we define for any n-ary, $n \geq 0$, function symbol f of R_1:

$$f_{\alpha.\beta}((a_1, b_1), \ldots, (a_n, b_n)) = (f_\alpha(a_1, \ldots, a_n), f_\beta(b_1, \ldots, b_n)),$$

where f_α comes from the total termination proof of R_1 in α. Since f_α and f_β are strictly monotone in all coordinates, the same holds for $f_{\alpha.\beta}$.

Let $l \to r$ be a rule in R_1 and let $\tau : X \to \alpha.\beta$. Applying lemma 7.3 and using $\phi_\beta(l, \pi_2 \circ \tau) \geq \phi_\beta(r, \pi_2 \circ \tau)$ and $\phi_\alpha(l, \pi_1 \circ \tau) > \phi_\alpha(r, \pi_1 \circ \tau)$ we conclude

$$\phi_{\alpha.\beta}(l, \tau) = (\phi_\alpha(l, \pi_1 \circ \tau), \phi_\beta(l, \pi_2 \circ \tau)) > (\phi_\alpha(r, \pi_1 \circ \tau), \phi_\beta(r, \pi_2 \circ \tau)) = \phi_{\alpha.\beta}(r, \tau).$$

So we have a proof of total termination of R_1 in $\alpha.\beta$, hence $\alpha.\beta \in U(R_1)$. On the other hand, since $\alpha = \omega^\gamma$, we can apply lemma 7.2 to conclude that $\alpha.\beta \in U(R_2)$. Hence $\alpha.\beta \in U(R_1) \cap U(R_2)$, so $R_1 \oplus R_2$ is totally terminating. □

Note that if both R_1 and R_2 contain duplicating rules, there are particular cases in which we can prove the union is totally terminating.[6] For example, let R_1 and R_2 be totally terminating in α, β, respectively, and assume there are ordinals γ, δ such that $\gamma + \omega^{\cdot^{\cdot^{\omega^{\alpha_0}}}} = \delta + \omega^{\cdot^{\cdot^{\omega^{\beta_0}}}} = A$, for finite exponentiations on both right summands. Then it easily follows from lemma 7.2 and theorem 7.4 that $\omega^A \in U(R_1 \oplus R_2)$, so $R_1 \oplus R_2$ is totally terminating. However, not all α, β satisfy this property; for example $\alpha = 2$ and $\beta = \omega$. The problem boils down to extending functions (of any arity) defined on a certain ordinal, to a given higher one, in such a way that the requirements of total termination are met, that is, in the new ordinal the functions are strictly monotone in all coordinates and for every rule the interpretation of the left hand side is greater than that of the right hand side.

8 String rewriting systems

In the previous sections we saw that when trying to prove total termination of TRS's containing at least a function symbol of arity $n \geq 2$, only ordinals of the form ω^η were relevant. In this section, we discuss whether the same holds for string rewriting systems, i.e., rewriting systems containing only unary function symbols. First we need a lemma.

Lemma 8.1 *Let $\alpha \neq 0$ and $f : \alpha \to \alpha$ be strictly monotone. Then there is a unique ordinal η such that $\omega^\eta \leq \alpha < \omega^{\eta+1}$ and $f(\omega^\eta) \subseteq \omega^\eta$.*

Proof By lemma 4.11, we know that there is $\eta \in Ord$ such that $\omega^\eta \leq \alpha < \omega^{\eta+1}$. If $\alpha = \omega^\eta$ we are done, otherwise we can write $\alpha = \omega^\eta + \delta$, with $\delta > 0$.

We suppose $f(\omega^\eta) \not\subseteq \omega^\eta$ and will derive a contradiction. That means there is $b \in \omega^\eta$ such that $f(b) \geq \omega^\eta$. We now define a function $g : (\omega^\eta + \delta) - b \to \delta$ by $g(x) = f(b + x) - \omega^\eta$. We see that

- since $f(b + x) \geq f(b) \geq \omega^\eta$, g is well-defined.

[6]The obvious case is when the proof of termination is given in the same ordinal for both TRS's.

- g is strictly monotone since for $x' > x$ also $b + x > b + x'$ and we get $\omega^\eta + g(x') = f(b + x') > f(b + x) = \omega^\eta + g(x)$; by left cancellation we get $g(x') > g(x)$.

By lemma 4.10 we obtain $(\omega^\eta + \delta) - b \leq \delta$. Since $b < \omega^\eta$, by lemmas 4.7 and 4.9, we get $(\omega^\eta + \delta) - b = (\omega^\eta - b) + \delta = \omega^\eta + \delta$. So we have $\omega^\eta - b = \omega^\eta$ hence $\omega^\eta + \delta = \delta$. Since $\delta \leq \alpha < \omega^{\eta+1} = \omega^\eta.\omega$, by lemma 4.11 there are uniquely determined ordinals β, γ such that $\beta < \omega$, $\gamma < \omega^\eta$ and $\delta = \omega^\eta.\beta + \gamma$. However, $\delta = \omega^\eta + \delta$, so also $\delta = \omega^\eta.(1 + \beta) + \gamma$. From the uniqueness now follows $\beta = 1 + \beta$, contradicting $\beta < \omega$. □

Remember that for a totally terminating TRS R the ordinal u_R is defined to be the minimum ordinal in which the total termination proof can be given.

Theorem 8.2 *Let R be a totally terminating string rewriting system. Then $u_R = \omega^\eta$ for some $\eta \geq 1$.*

Proof From lemma 8.1 we obtain a unique ordinal η such that $\omega^\eta \leq u_R < \omega^{\eta+1}$ and $f(\omega^\eta) \subseteq \omega^\eta$ for all operation symbols f. By restricting $f : u_R \to u_R$ to ω^η for all operation symbols f, we see that we also have a proof of total termination of R in ω^η, so $\omega^\eta \in U(R)$. Since u_R is the minimum of $U(R)$ and $\omega^\eta \leq u_R$ we obtain $u_R = \omega^\eta$. □

Note that this result is essentially weaker than theorem 6.3 for the case of arity > 1. The fact that $u_R = \omega^\eta$ does not imply that every ordinal in $U(R)$ is of that shape. For example, every proof of total termination of a string rewriting system in ω is easily extended to a similar proof in $\omega + \omega$, which is not of the required shape.

A natural operation on string rewriting systems is reversing: all left hand sides and right hand sides are reversed, considered as strings. For example, the reverse of $f(f(g(x)))$ is $g(f(f(x)))$. Clearly there is a bijective correspondence between reductions in the original system and reductions in the reversed system. As a consequence, a string rewriting system is terminating if and only if the reversed system is terminating. However, a similar observation does not hold for total termination. For example, the system

$$f(f(x)) \to f(g(x)), \quad g(g(x)) \to g(f(x))$$

is not totally terminating since $f(a)$ and $g(a)$ are incomparable for any a in any corresponding monotone algebra. On the other hand, the reversed system

$$f(f(x)) \to g(f(x)), \quad g(g(x)) \to f(g(x))$$

is totally terminating in the natural numbers; a possible interpretation is $f(x) = 4x+2, g(x) = 4x + 1$ for x even, and $f(x) = 4x, g(x) = 4x + 3$ for x odd. Further, if for a totally terminating system the reversed system is totally terminating too, the corresponding ordinal may change. An example is $f(g(x)) \to g(f(f(x)))$; in the next section we shall see that the minimal ordinal of this totally terminating system is ω^2, while termination of the reversed system $g(f(x)) \to f(f(g(x)))$ is proven in the natural numbers by choosing $f(x) = x + 1, g(x) = 3x$.

We conclude this section with some remarks about TRS's that also contain constants, and no function symbols of arity > 1. From theorems 6.3 and 8.2 we know that otherwise total termination implies that $u_R = \omega^\eta$ for some $\eta \geq 0$. However, if there are constants then the proof of theorem 8.2 does not hold any more since the interpretation of the constants can be too great. The simplest example is the TRS R consisting of the rule $a \to b$, where a and b

are constants. It is totally terminating and $u_R = 2$. If we allow infinitely many constants and rewrite rules then for any ordinal α a TRS R can be given with $u_R = \alpha$.
Example 1 The infinite TRS R consisting of the rules

$$n \rightarrow s^i(0) \quad \text{for each } i < \omega$$

where n and 0 are constants, satisfies $u_R = 2$ with 0 interpreted as 0, n as 1 and s as the identity function. If we add the rule

$$s(x) \rightarrow x$$

we see that s can no longer be interpreted as the identity. Furthermore it has to hold $x < s(x)$ for any x. Consequently n has to be interpreted as $\beta \geq \omega$. Also the last rule implies that $\alpha \in Lim$, so $\alpha \geq \omega + \omega$. In $\omega + \omega$ we can indeed prove the system terminating with the interpretations $0 = 0$, $n = \omega$ and $s(x) = x + 1$. □

In general, given α, the minimal ordinal associated with a totally terminating string rewriting system, we know from the results described earlier that any function symbol of arity 1 can be interpreted in ω^{η_1} if $\omega^{\eta_1} + \ldots + \omega^{\eta_k}$ is the Cantor normal form of α. Consequently for any rewrite rule $l \rightarrow r$ involving only unary function symbols and constants that are assigned values smaller that ω^{η_1}, and for any substitution $\tau : X \rightarrow \omega^{\eta_1}$, we have $\phi_\alpha(l, \tau)$, $\phi_\alpha(r, \tau) \in \omega^{\eta_1}$ and $\phi_\alpha(l, \tau) >_{\omega^{\eta_1}} \phi_\alpha(r, \tau)$. And this means that such rules can be interpreted in ω^{η_1}. The problem arises when we have constants that cannot be interpreted in ω^{η_1} as is the case in the second example above. We conjecture that if a TRS R not containing function symbols of arity > 1, is totally terminating and fullfills the conditions:

1. At least one of the rules $l \rightarrow r$ in R contains a variable
2. R contains finitely many rules

then $min\ U(R) = \omega^\eta$, for some $\eta \geq 0$.
From the first example above we see that the first condition is necessary, and the second example shows the necessity of the second condition.

However even if u_R is not of the form ω^η, due to theorem 7.4 we need only consider those ordinals for proving termination.

In the next section we investigate which ordinals actually occur as u_R of such a finite TRS R.

9 Minimal Ordinals

As we have seen previously, when trying to establish total termination of a (finite or infinite) string rewriting system or a TRS containing symbols of arity > 0, we only need to consider algebras with type ω^n for some $n > 0$. Is it the case that all ordinals of that form are important or can we restrict the class even further? Partially answering this question, we have the following theorem.

Theorem 9.1 *For any ordinal* $1 \leq n \leq \omega$ *there is a string rewriting system* R *such that* $u_R = \omega^n$.

Proof For $n = 1$, the string rewriting system $f(x) \to x$ satisfies the requirements by interpreting f as $x + 1$ in ω.

For $1 < n < \omega$, let R_n consist of the $n - 1$ rules

$$f_i(f_{i+1}(x)) \to f_{i+1}(f_i(f_i(x)))$$

for $i = 1, \ldots, n-1$. We will show that $u_{R_n} = \omega^n$ for any n; for $n = 2$ this was already shown in [15] (report version).

For the TRS R defined by

$$\begin{array}{rcl} f(g(x)) & \to & g(f(f(x)) \\ f(h(x)) & \to & h(g(x)) \end{array}$$

we shall prove that $u_R = \omega^\omega$.

According to theorem 8.2 the only ordinals of interest are of the shape ω^n. We need some lemmas; to simplify the treatment we will use the same symbol for a function symbol in a TRS and the corresponding interpretation function in an ordinal.

Lemma 9.2 *Given is a TRS R containing a rule of the form $F(G(x)) \to G(F(F(x)))$ and totally terminating in an ordinal α. Then $\forall k \geq 0\ \forall a \in \alpha :\ G(a) \geq F^k(a)$ (where $F^0 = Id$).*

Proof By induction on k; if $k = 0$ the result follows from lemma 3.1. Assume $G(a) < F^k(a)$ for some $a \in \alpha$. Then

$$\begin{array}{rcl} F^{k+1}(a) = F(F^k(a)) & > & \text{(since } F \text{ is monotone)} \\ F(G(a)) & > & \text{(by total termination)} \\ G(F(F(a))) & \geq & \text{(by induction hypothesis)} \\ F^{k-1}(F(F(a))) & = & F^{k+1}(a), \end{array}$$

contradiction. □

From the above lemma we conclude that $u_{R_n} \geq \omega^2$, since the property does not hold in ω and, from theorem 8.2, $u_{R_n} = \omega^n$, for some ordinal n.

Given a function $F : \omega^m \to \omega^m$, we define

$$O(F) = \min\{k \mid 0 \leq k \leq m \wedge \forall a \in \omega^m : a + \omega^k > F(a)\}$$

Intuitively, $O(F)$ denotes the highest-order coordinate k which may be changed by function F. We remark that $0 \leq O(F) \leq m$; $O(F) = 0 \iff F$ is the identity function. We introduce some needed properties of $O()$.

Lemma 9.3 *Let F, G be monotone functions from ω^m to ω^m, for some $m \geq 1$. Then $\forall x \in \omega^m : F(x) \geq G(x) \Rightarrow O(F) \geq O(G)$.*

Proof Suppose $j = O(F) < O(G) = k$. Then $\exists a \in \omega^m$ such that $F(a) < a + \omega^j \leq a + \omega^{k-1} \leq G(a)$, contradicting the hypothesis. □

Lemma 9.4 *Let F, G be monotone functions from ω^m to ω^m, for some $m \geq 1$. Then $O(F \circ G) = \max(O(F), O(G))$.*

Proof Let $k = \max(O(F), O(G))$. For any $0 \leq j < k$, $\exists a \in \omega^m$ such that either $F(a) \geq a + \omega^j$ or $G(a) \geq a + \omega^j$. In both cases, using monotonicity and lemma 3.1, we conclude $F(G(a)) \geq a + \omega^j$, hence $O(F \circ G) \geq k$.

Since $O(G) \leq k$, we have $G(a) - a < \omega^k$, for any a. Since $O(F) \leq k$ and by lemma 4.5, we have $a + \omega^k = a + (G(a) - a) + \omega^k = G(a) + \omega^k > F(G(a))$. Hence $O(F \circ G) \leq k$. □

Lemma 9.5 *Given is a TRS R containing a rule of the form $F(G(x)) \to G(F(F(x)))$ and totally terminating in an ordinal ω^m for some $m < \omega$. Then $O(G) > O(F)$.*

Proof By the assumption of total termination of R, both F and G are not the identity, so $O(F), O(G) > 0$. Let $O(F) = k$, $1 \leq k \leq m$. Then $\forall x \in \omega^m : x + \omega^k > F(x)$ and $\exists a \in \omega^m : a + \omega^{k-1} \leq F(a)$. Fix this $a \in \omega^m$. Next we prove by induction that $F^j(a) \geq a + \omega^{k-1}.j$, for any $1 \leq j < \omega$. For $j = 1$ it holds by hypothesis. Suppose the property holds for any $i \leq j$. Then

$$\begin{array}{rcl}
F^{j+1}(a) = F(F^j(a)) & \geq & \text{(by monotonicity of } F \text{ and induction hypothesis)} \\
F(a + \omega^{k-1}.j) & \geq & \text{(by lemma 4.8)} \\
F(a) + \omega^{k-1}.j & \geq & \text{(by induction hypothesis)} \\
a + \omega^{k-1} + \omega^{k-1}.j & = & a + \omega^{k-1}.(j+1)
\end{array}$$

But by lemma 9.2, $G(a) \geq F^j(a) \geq a + \omega^{k-1}.j$, for any $j < \omega$. Applying this lemma we also conclude that $G \geq F$ and therefore $O(G) \geq O(F)$ (by lemma 9.3). If it would be $O(F) = O(G)$, then $a + \omega^k > G(a) \geq F^j(a) \geq a + \omega^{k-1}.j$, for any $j < \omega$. But then $G(a) \geq \bigvee_{j<\omega}(a + \omega^{k-1}.j) = a + \omega^{k-1}.(\bigvee_{j<\omega} j) = a + \omega^{k-1}.\omega = a + \omega^k$, which is a contradiction. □

Now going back to our original system R_n and applying the previous results to every rule, we get

$$0 \leq O(f_n) < \ldots < O(f_1) < m,$$

so $m \geq n$, hence, $u_{R_n} \geq \omega^n$. In order to show that u_{R_n} is indeed ω^n we stil have to give an interpretation in ω^n. Identify ω with strictly positive integers and define in ω^n:

$$f_i(\omega^{n-1}.x_n + \ldots + \omega^0.x_1) = \omega^{n-1}.x_n + \ldots + \omega^{i-1}.(xi + 2^{x_{i+1}}) + \ldots + \omega^0.x_1$$

for $i = 1, \ldots, n$, where x_{n+1} is defined to be 1 and where the argument of f_i is represented in its normal form as introduced just after theorem 4.12. With this interpretation, we can easily see that all the requirements of total termination are fulfilled.

For the ordinal ω^ω we consider the TRS R

$$\begin{array}{rcl}
f(g(x)) & \to & g(f(f(x))) \\
f(h(x)) & \to & h(g(x))
\end{array}$$

We shall prove $u_R = \omega^\omega$; first we show that we cannot prove total termination of R in ω^n, for any $n < \omega$. Suppose we can, then there are strictly monotone functions

$f, g, h : \omega^n \to \omega^n$ satisfying $f(g(x)) > g(f(f(x)))$ and $f(h(x)) > h(g(x))$ for all $x \in \omega^n$. Let $O()$ be defined as before. From lemmas 9.3, 9.4 and since left-hand sides are greater than right-hand sides, we have $\max(O(f), O(h)) = O(f \circ h) \geq O(h \circ g) = \max(O(h), O(g))$. Since (by lemma 9.5) $O(f) < O(g)$, this implies $O(h) \geq O(g)$. Let $j = O(f) < O(g)$; then $\exists a \in \omega^n : a + \omega^j \leq g(a) \wedge a + \omega^j > f(a)$. Using lemma 4.8, we derive $h(g(a)) \geq h(a + \omega^j) \geq h(a) + \omega^j > f(h(a))$, contradicting the requirements of total termination.

To prove $u_R = \omega^\omega$ we still need to present an interpretation in ω^ω. Identify ω with natural numbers, including 0, and denote an element $X \in Exp(\omega, \omega)$ by the sequence $(p_0, \ldots, p_k)$ where:

- $X(i) = p_i$, if $0 \leq i \leq k$.
- $X(k) \neq 0$ and $X(i) = 0$ for $i > k$.

We restrict to the part of ω^ω for which $k \geq 1$ in this notation. This means that we skip the first ω elements of ω^ω; since $\omega^\omega - \omega = \omega^\omega$ this does not affect the ordinal. We now define $f, g, h : \mathcal{A} \to \mathcal{A}$ by:

- $f(p_0, \ldots, p_{(k-1)}, p_k) = (p_0 + p_k, \ldots, p_{(k-1)} + p_k, p_k)$
- $g(p_0, \ldots, p_{(k-1)}, p_k) = (p_0, \ldots, p_{(k-1)}, 2.p_k + 1)$
- $h(p_0, \ldots, p_{(k-1)}, p_k) = (p_0, \ldots, p_{(k-1)}, p_k, 0, 1)$

With some easy calculations, it can be shown that the functions are indeed strictly monotonic and that for both rules the left hand side is greater than the right hand side. □

We end this section with an example based on the battle of Hercules and the Hydra (see [8]; another version of this game appears in [4]). For this system we conjecture $u_R = \epsilon_0$. As usual ϵ_0 is defined to be the minimal ϵ-ordinal, i.e., the minimal ordinal α satisfying $\alpha = \omega^\alpha$. It can also be defined as $\lim_{n<\omega} \gamma_n$ where $\gamma_0 = 1$ and $\gamma_{n+1} = \omega^{\gamma_n}$; finally it is the only ordinal satisfying $\alpha < \lambda \Rightarrow \alpha < \omega^\alpha < \lambda$.

The Hydra is a monster with many heads, represented as a finite tree, and the battle proceeds by stages. In stage k

- Hercules chops off a head of the Hydra (a leaf node with associated edge)
- the Hydra answers by growing on the 2^{nd} ancestor of the chopped head, k copies of the subtree that contained the (now missing) head. In the case the head was hanging from the root, it is simply deleted and no copies are made.

The problem is to prove termination of this battle. We generalize the result by removing the rule that in stage k exactly k copies are created; in our version the number of copies n may be chosen randomly at every stage. Also we adopt the strategy that only leftmost heads are chopped.

We code the tree using a binary symbol c: a tree consisting of a root and descendants $t_1, \ldots, t_k$ is represented as $c(t_1, c(t_2, \ldots, c(t_{k-1}, t_k) \ldots))$, that is $c(D, S)$ represents a node whose descendants are coded in the subtree D and whose siblings are coded in subtree S.

Leaves are represented by the constant *nil*. The battle described above is now represented as the infinite TRS H:

$$\begin{array}{rcll} c(nil, x) & \rightarrow & x & \\ c(c(nil, x), y) & \rightarrow & copy(n, x, y) & \\ copy(s(k), x, y) & \rightarrow & copy(k, x, c(x, y)) & \\ copy(0, x, y) & \rightarrow & y & \\ n & \rightarrow & s^i(0) & \text{for each } i \geq 0 \end{array}$$

Termination of this system can be proven by lexicographic path order. The system is totally terminating in ϵ_0 with the following interpretation:

$$\begin{array}{lcl} 0 & = & 0 \\ n & = & \omega \\ nil & = & 2 \\ s(x) & = & x+1 \\ c(x, y) & = & \omega^x \oplus y \\ copy(k, x, y) & = & \omega^{k \oplus x \oplus 1} \oplus x \oplus y. \end{array}$$

Here elements of ϵ_0 are identified with ordinals $< \epsilon_0$; the operation $\oplus$ represents natural addition. Well-definedness of these functions follows from standard properties of ϵ_0. Since natural addition in associative, commutative and strictly monotone in both coordinates, it is not difficult to see that the functions above are strictly monotone in each coordinate. Further it is easy to check that all left hand sides are strictly greater than the corresponding right hand sides. We show it here only for the second rule. So for any substitution $\tau : X \rightarrow \epsilon_0$, we have
$\phi_{\epsilon_0}(l, \tau) = \phi_{\epsilon_0}(c(c(nil, x), y), \tau) = \omega^{\omega^2 \oplus \tau(x)} \oplus \tau(y)$
$\phi_{\epsilon_0}(r, \tau) = \phi_{\epsilon_0}(d(n, x, y), \tau) = \omega^{\omega \oplus \tau(x) \oplus 1} \oplus \tau(x) \oplus \tau(y)$
Since $\omega \oplus 1 < \omega^2$ (because ω^2 is additive principal and $\omega, 1 < \omega^2$), we get $\omega \oplus \tau(x) \oplus 1 < \omega^2 \oplus \tau(x) \Rightarrow \omega^{\omega \oplus \tau(x) \oplus 1} < \omega^{\omega^2 \oplus \tau(x)}$. But also, $\tau(x) < \epsilon_0 \Rightarrow \tau(x) < \omega^{\tau(x)} < \omega^{\omega^2 \oplus \tau(x)}$. Because $\omega^{\omega^2 \oplus \tau(x)}$ is additive principal, we get $\omega^{\omega \oplus \tau(x) \oplus 1} \oplus \tau(x) < \omega^{\omega^2 \oplus \tau(x)} \Rightarrow \omega^{\omega \oplus \tau(x) \oplus 1} \oplus \tau(x) \oplus \tau(y) < \omega^{\omega^2 \oplus \tau(x)} \oplus \tau(y)$.
So $u_H \leq \epsilon_0$. It can be proven (by double induction) that if $f : \alpha \times \alpha \rightarrow \alpha$ is strictly increasing in each argument then $f(x, y) \geq x \oplus y$, for any $x, y \in \alpha$. Using this fact and rules 2, 3 and 4, it can be seen that for any substitution τ, the lhs of rule 2 (l_2) has to fulfil $\phi_{u_H}(l_2, \tau) > \tau(x) \oplus \ldots \oplus \tau(x) \oplus y$, where $\tau(x)$ can appear any finite number of times. Consequently $\phi_{u_H}(l_2, \tau) \geq \omega^{\eta_1 + 1} \oplus y$, where $\tau(x)$ has as Cantor normal form $\omega^{\eta_1} p_1 + \ldots + \omega^{\eta_0} p_0$. With this last inequality it is not difficult to derive $u_H \geq \omega^\omega$. Consequently $\omega^\omega \leq u_H \leq \epsilon_0$ and we conjecture that $u_H = \epsilon_0$.

10 Conclusions

Proving termination of term rewriting systems by interpretation is not easy. We focussed on interpretation in monotone algebras in which the underlying order is total. We have shown that the existence of a function symbol of arity > 1 implies that the underlying order has type ω^η, i. e. is equivalent to finite multisets over some well-order. Furthermore, for any TRS the class of total orders in which it can be shown totally terminating, is closed under multiset construction and lexicographic product. However, it is not clear how to extend a total termination proof in a particular well-order to well-orders that can not be finitely obtained from the original one by these constructions. This problem is closely connected to modularity of total termination, on which we obtained some interesting partial results.

We found examples of TRS's showing that proofs of total termination cannot always be given in well-orders of type smaller than ω^ω. Most of our techniques are based upon ordinal arithmetic; we believe that ordinal arithmetic is a strong and useful tool for proving termination of TRS's. For example, in ϵ_0 we gave an elegant termination proof of a TRS based on the battle of Hydra. We also believe that our framework is a main step towards generalizing and combining existing techniques like recursive path order and Knuth-Bendix order.

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