

# A Semi-Induced Subgraph Characterization of Upper Domination Perfect Graphs

Igor E. Zverovich

Faculty of Mechanics and Mathematics

Belarus State University, Minsk 220050

Belarus

Vadim E. Zverovich\*

Department II of Mathematics

RWTH Aachen, Aachen 52056

Germany

## Abstract

Let  $\beta(G)$  and  $\Gamma(G)$  be the independence number and the upper domination number of a graph  $G$ , respectively. A graph  $G$  is called  $\Gamma$ -perfect if  $\beta(H) = \Gamma(H)$ , for every induced subgraph  $H$  of  $G$ . The class of  $\Gamma$ -perfect graphs generalizes such well-known classes of graphs as strongly perfect graphs, absorbantly perfect graphs, and circular arc graphs. In this article, we present a characterization of  $\Gamma$ -perfect graphs in terms of forbidden semi-induced subgraphs. Key roles in the characterization are played by the odd prism and the even Möbius ladder, where the prism and the Möbius ladder are well-known 3-regular graphs [2]. Using the semi-induced subgraph characterization, we obtain a characterization of  $K_{1,3}$ -free  $\Gamma$ -perfect graphs in terms of forbidden induced subgraphs. *J. Graph Theory* 31 (1999), 29-49

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## 1 Introduction

All graphs will be finite and undirected without multiple edges. Unless otherwise stated, all graphs have no loops. If  $G$  is a graph,  $V(G)$  denotes the set, and  $|G|$  the number, of vertices in  $G$ . Let  $N(x)$  denote the neighborhood of a vertex  $x$ , and let  $\langle X \rangle$  denote the subgraph of  $G$  induced by  $X \subseteq V(G)$ . Also let  $N(X) = \cup_{x \in X} N(x)$  and  $N[X] = N(X) \cup X$ . Denote by  $\delta(G)$  the minimal degree of vertices in  $G$ . A path, a cycle and a complete graph of order  $n$  will be denoted by  $P_n$ ,  $C_n$  and  $K_n$ , respectively.

A set  $I \subseteq V(G)$  is called *independent* if no two vertices of  $I$  are adjacent. A set  $X$  is called a *dominating set* if  $N[X] = V(G)$ . An *independent dominating set* is a vertex subset that is both independent and dominating, or equivalently, is maximal independent. The *independence number*  $\beta(G)$  is the maximum cardinality of a (maximal) independent set of  $G$ , and the *independent domination number*  $i(G)$  is the minimum cardinality taken over all maximal independent sets of  $G$ . The *domination number*  $\gamma(G)$  is the minimum

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cardinality of a (minimal) dominating set of  $G$ , and the *upper domination number*  $\Gamma(G)$  is the maximum cardinality taken over all minimal dominating sets of  $G$ . For  $x \in X$ , the set

$$PN(x, X) = N[x] - N[X - \{x\}]$$

is called the *private neighborhood* of  $x$ . If  $PN(x, X) = \emptyset$ , then  $x$  is said to be *redundant* in  $X$ . A set  $X$  containing no redundant vertex is called *irredundant*. The *irredundance number*  $ir(G)$  is the minimum cardinality taken over all maximal irredundant sets of  $G$ , and the *upper irredundance number*  $IR(G)$  is the maximum cardinality of a (maximal) irredundant set of  $G$ .

The following relationship among the parameters under consideration is well-known [7, 9]:

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G).$$

**Definition 1** A graph  $G$  is called *irredundance perfect* (*ir*-perfect) if  $ir(H) = \gamma(H)$ , for every induced subgraph  $H$  of  $G$ .

**Definition 2** A graph  $G$  is called *domination perfect* ( $\gamma$ -perfect) if  $\gamma(H) = i(H)$ , for every induced subgraph  $H$  of  $G$ .

**Definition 3** A graph  $G$  is called *upper domination perfect* ( $\Gamma$ -perfect) if  $\beta(H) = \Gamma(H)$ , for every induced subgraph  $H$  of  $G$ .

**Definition 4** A graph  $G$  is called *upper irredundance perfect* (*IR*-perfect) if  $\Gamma(H) = IR(H)$ , for every induced subgraph  $H$  of  $G$ .

The classes of upper domination perfect graphs and upper irredundance perfect graphs in a sense are dual to the classes of domination perfect graphs and irredundance perfect graphs, respectively. A lot of interesting results on domination perfect graphs [1, 3, 12, 16, 17, 24, 27, 28, 29, 31, 33] and irredundance perfect graphs [3, 4, 10, 20, 21, 25, 26, 32] are known. A finite induced subgraph characterization of the entire class of domination perfect graphs was recently obtained in [33], while the problems of characterizing the entire class of irredundance perfect graphs and upper irredundance perfect graphs are still open. For a short survey on domination perfect graphs, see also [33].

We summarize the known results on  $\Gamma$ -perfect and *IR*-perfect graphs. The following important theorem gives the relationship between the class of  $\Gamma$ -perfect graphs and the class of *IR*-perfect graphs.

**Theorem A (Gutin and Zverovich [14])** Any  $\Gamma$ -perfect graph is *IR*-perfect.

Thus,  $\Gamma$ -perfect graphs form a subclass of *IR*-perfect graphs. On the other hand, a number of well-known classes of graphs are subclasses of  $\Gamma$ -perfect graphs, and consequently, *IR*-perfect graphs. Cockayne et al. [7] proved that bipartite graphs are  $\Gamma$ -perfect, and Jacobson and Peters [23] showed that chordal graphs are  $\Gamma$ -perfect. The next theorem generalizes these results, since bipartite graphs and chordal graphs are strongly perfect graphs. Recall that a graph  $G$  is called *strongly perfect* if every induced subgraph  $H$  of  $G$  has a stable transversal, where a *stable transversal*  $S$  of  $H$  is a vertex subset of  $H$  such that  $|S \cap C| = 1$  for any maximal clique  $C$  of  $H$ .

**Theorem B (Cheston and Fricke [5], Jacobson and Peters [22])** *A strongly perfect graph is  $\Gamma$ -perfect.*

It may be pointed out that, besides bipartite and chordal graphs, strongly perfect graphs contain comparability graphs, perfectly orderable graphs, peripheral graphs, complements of chordal graphs, Meyniel graphs, parity graphs,  $i$ -triangulated graphs, permutation graphs, cographs, and hence all these classes are subclasses of  $\Gamma$ -perfect graphs. Hammer and Maffray [15] defined a graph  $G$  to be *absorbantly perfect* if every induced subgraph  $H$  of  $G$  contains a minimal dominating set that meets all maximal cliques of  $H$ . It turned out that absorbantly perfect graphs are  $\Gamma$ -perfect (Theorem C). Since every strongly perfect graph is absorbantly perfect, we see that Theorem B follows from the more general Theorem C.

**Theorem C (Gutin and Zverovich [14])** *An absorbantly perfect graph is  $\Gamma$ -perfect.*

A graph is called *circular arc* if it can be represented as the intersection graph of arcs on a circle.

**Theorem D (Golumbic and Laskar [13])** *A circular arc graph is  $\Gamma$ -perfect.*

The following theorem gives a sufficient condition for a graph to be  $IR$ -perfect.

**Theorem E (Cockayne, Favaron, Payan and Thomason [7])** *If a graph  $G$  does not contain  $P_5$ ,  $C_5$ ,  $Pr_3 - v_1$  and  $Pr_3 - v_1 - v_2v_3$  as induced subgraphs, where  $Pr_3$  is shown in Fig. 1, then  $G$  is  $IR$ -perfect.*

Using Theorem A we see that Theorem F improves Theorem E.

**Theorem F (Gutin and Zverovich [14])** *If a graph  $G$  does not contain  $P_5$  and  $Pr_3$  in Fig. 1 as induced subgraphs, then  $G$  is  $\Gamma$ -perfect.*

Other sufficient conditions for a graph to be  $\Gamma$ -perfect or  $IR$ -perfect can be found in [10, 14, 23, 30], one of them is stated in Corollary 2. A number of authors [6, 7, 8, 11] investigated graphs  $G$  having  $\beta(G) = \Gamma(G)$ , i.e., these parameters are not necessarily equal for a proper induced subgraph of  $G$ . Cockayne et al. [8] proved that

$$\beta(G) = \Gamma(G) = IR(G)$$

for the representative graph of any hereditary hypergraph. Cheston et al. [6] showed that this equality is valid for upper bound graphs which extend the class of representative graphs of hereditary hypergraphs, while Fellows et al. [11] proved that the same equality holds for trestled graphs. It was also shown by Cheston et al. [6] that  $\beta(G) = \Gamma(G)$  for simplicial graphs, which generalize the class of upper bound graphs.

In this article, we introduce the concept of a semi-induced subgraph (Definition 8), and we present two characterizations of the entire class of  $\Gamma$ -perfect graphs in terms of forbidden semi-induced subgraphs (Theorems 1 and 2). Key roles in the characterizations are played by the odd prism and the even Möbius ladder, where the prism and the Möbius ladder are well-known 3-regular graphs [2]. Using the semi-induced subgraph characterization of  $\Gamma$ -perfect graphs, we obtain a result of Jacobson and Peters [22] on  $\Gamma$ -perfect graphs (Corollary 1) and also a characterization of  $K_{1,3}$ -free  $\Gamma$ -perfect graphs in terms of forbidden induced

subgraphs. The latter result implies a known sufficient condition for a  $K_{1,3}$ -free graph to be  $\Gamma$ -perfect (Corollary 2). Notice here that  $K_{1,3}$ -free graphs are  $\gamma$ -perfect [1], and that  $K_{1,3}$ -free *ir*-perfect graphs were characterized by Favaron [10], who also found a sufficient condition for a  $K_{1,3}$ -free graph to be *IR*-perfect.

## 2 Basic Definitions

We need the following definitions.

**Definition 5** *Two vertex subsets  $A, B$  of a graph  $G$  independently match each other if  $A \cap B = \emptyset$ ,  $|A| = |B|$ , and all edges between  $A$  and  $B$  form a perfect matching in  $\langle A \cup B \rangle$ .*

**Definition 6** *A graph  $G$  of order  $2k$  is called a  $W$ -graph if there is a partition  $V(G) = A \cup B$  such that  $A$  and  $B$  independently match each other. Clearly,  $|A| = |B| = k$ . The sets  $A$  and  $B$  are called parts, and the graph  $G$  is denoted by  $G(A, B)$ .*

It is not difficult to see that a  $W$ -graph may have several partitions into parts. Hence a  $W$ -graph is considered in Sections 2 and 3 together with a fixed partition into parts.

**Definition 7** *Let  $G$  be a  $W$ -graph,  $G = G(A, B)$ . Edges between the parts  $A$  and  $B$  are called *b*-edges and denoted in our figures by bold lines. Edges which are not *b*-edges are called *l*-edges and denoted by thin lines.*

We can understand the above partition of the edge set as a coloring of the edge set with two colors 'b' and 'l'. Note that if the set  $E_b$  of *b*-edges of a connected  $W$ -graph  $G$  is given, then there is only one partition  $V(G) = A \cup B$  such that  $A$  and  $B$  independently match each other, i.e.,  $G = G(A, B)$  and  $E_b$  is the set of *b*-edges with respect to this partition.

**Definition 8** *Let  $H = H(A, B)$  be a  $W$ -graph with parts  $A$  and  $B$ . The graph  $H$  is called a semi-induced subgraph of a graph  $G$  if  $H$  is a subgraph of  $G$ , and in the graph  $G$  the sets  $A$  and  $B$  independently match each other.*

In other words, let  $A$  and  $B$  independently match each other in  $G$  and let  $P$  be the perfect matching between  $A$  and  $B$  in  $\langle A \cup B \rangle$ . If  $E_1 \subseteq E \langle A \rangle$  and  $E_2 \subseteq E \langle B \rangle$ , then the graph  $H$  having  $V(H) = A \cup B$  and  $E(H) = E_1 \cup E_2 \cup P$  is a semi-induced subgraph of  $G$ . Thus, any semi-induced subgraph of a graph is a  $W$ -graph, and if  $H$  is not a  $W$ -graph, then  $G$  cannot contain  $H$  as a semi-induced subgraph.

**Definition 9** *A graph  $G$  is called a *bl*-graph if a partition of the set  $E(G)$  into the set of *b*-edges (bold) and *l*-edges (thin) is given, provided that the set of *b*-edges forms a matching in  $G$ . If the *b*-edges form a perfect matching, then  $G$  is called a perfect *bl*-graph. For example, any  $W$ -graph is a perfect *bl*-graph. An even (odd) *bl*-graph has the even (odd) number of *b*-edges.*

**Definition 10** *A simple *bl*-chain  $P$  is called alternating if for any two consecutive edges of  $P$  one of them is a *b*-edge and another is an *l*-edge. The alternating simple chain  $P$  is called a *b*-chain (*l*-chain) if the end edges of  $P$  are *b*-edges (*l*-edges). Clearly, *b*-chains and *l*-chains always have even order. If we identify the end vertices  $u_1$  and  $u_{2n}$  in the *l*-chain  $(u_1, u_2, \dots, u_{2n})$ , where  $n \geq 2$ , then we obtain the simple cycle  $(u_1, u_2, \dots, u_{2n-1})$  which is called an *l*-cycle starting with  $u_1$ .*

**Definition 11** For a perfect bl-graph  $G$  we define the operation of W-reducibility as follows. Each vertex  $u \in V(G)$  is labeled by  $c(u) \in \{A, B\}$ . Further, each edge  $e = vw \in E(G)$  is replaced by an alternating bl-chain  $P_e$  with end vertices  $v, w$  in accordance with the next rule:

- If  $e$  is an l-edge and  $c(v) = c(w)$ , then  $P_e$  is an even l-chain.
- If  $e$  is an l-edge and  $c(v) \neq c(w)$ , then  $P_e$  is an odd l-chain.
- If  $e$  is a b-edge and  $c(v) = c(w)$ , then  $P_e$  is an even b-chain.
- If  $e$  is a b-edge and  $c(v) \neq c(w)$ , then  $P_e$  is an odd b-chain.

**Definition 12** The prism  $Pr_n$  ( $n \geq 3$ ) consists of two disjoint cycles

$$C_1 = (u_1, u_2, \dots, u_n), \quad C_2 = (v_1, v_2, \dots, v_n),$$

and the remaining edges are of the form  $u_i v_i$ ,  $1 \leq i \leq n$ . The prism  $Pr_1$  is two loops connected by the edge  $u_1 v_1$ , this is the only case where loops are permitted. If the prism  $Pr_n$  is considered as a perfect bl-graph, then its set of b-edges is  $\{u_i v_i : 1 \leq i \leq n\}$ .

**Definition 13** The Möbius ladder  $Ml_n$  is constructed from the cycle  $C = (u_1, u_2, \dots, u_{2n})$  by adding the edges  $u_i u_{n+i}$  ( $1 \leq i \leq n$ ) joining each pair of opposite vertices of  $C$ . If the Möbius ladder  $Ml_n$  is considered as a perfect bl-graph, then its set of b-edges is  $\{u_i u_{n+i} : 1 \leq i \leq n\}$ .

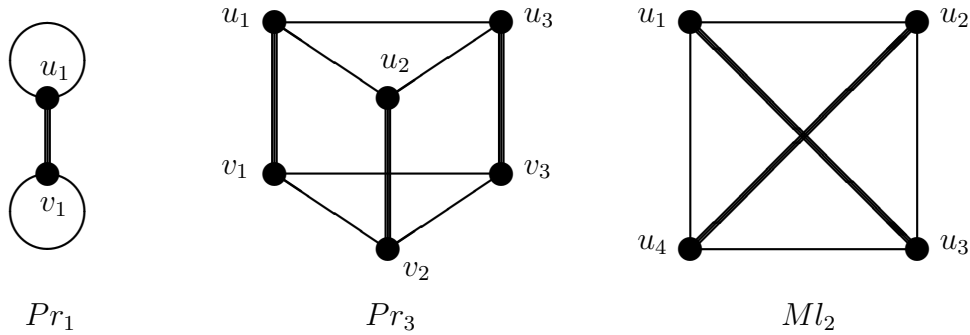


FIGURE 1. Odd prisms  $Pr_1$ ,  $Pr_3$  and even Möbius ladder  $Ml_2$ .

The odd prisms  $Pr_1$  and  $Pr_3$  and the even Möbius ladder  $Ml_2$  are shown in Fig. 1. The odd prisms and the even Möbius ladders play a key role in the definition of basis graphs.

**Definition 14** A graph  $G$  without loops is called a basis if it can be obtained from the odd prism  $Pr_{2n+1}$  ( $n \geq 0$ ) or the even Möbius ladder  $Ml_{2m}$  ( $m \geq 1$ ) by the operation of W-reducibility.

We will prove later that a basis graph is a W-graph whose perfect matching between the parts consists of b-edges determined by the operation of W-reducibility. A basis graph  $G$  cannot have loops. Hence, if  $G$  is obtained from  $Pr_1$ , then every loop (l-edge) of  $Pr_1$  must be replaced in accordance with Definition 11 by an alternating even l-cycle (l-chain with equal end vertices) having at least two b-edges.

### 3 Characterization of $\Gamma$ -Perfect Graphs

The following theorem gives a characterization of upper domination perfect graphs in terms of forbidden semi-induced subgraphs.

**Theorem 1** *A graph  $G$  is a  $\Gamma$ -perfect graph if and only if  $G$  does not contain any basis graph as a semi-induced subgraph.*

**Proof:** The proof of Theorem 1 is based on 11 lemmas.

**Lemma 1** *If  $G$  is a  $W$ -graph of order  $2k$ , then*

$$\Gamma(G) = k \geq \beta(G).$$

**Proof:** Any independent set of  $G$  contains at most one vertex of each b-edges, and hence  $\beta(G) \leq k$ . Since  $A$  is a minimal dominating set, we have  $\Gamma(G) \geq k$ . Let us prove that  $\Gamma(G) \leq k$ . Let  $D$  be a minimal dominating set of  $G$  of cardinality  $\Gamma(G)$ . If  $\deg_{\langle D \rangle} d > 0$  for  $d \in D$ , then there is a vertex  $f \in V(G) - D$  such that  $N(f) \cap D = \{d\}$ . If  $\deg_{\langle D \rangle} d = 0$  for  $d \in D$ , then there is a vertex  $f$  such that  $df$  is a b-edge. Obviously  $f \in V(G) - D$ . Thus, for each vertex  $d \in D$  we can indicate a vertex  $f$  from  $V(G) - D$  and evidently that different vertices of  $D$  result in different vertices of  $V(G) - D$ , i.e.,  $|D| \leq |V(G) - D|$ . We have  $\Gamma(G) = |D| \leq k$ . ■

**Definition 15** *A  $W$ -graph  $G$  of order  $2k$  is called strong if*

$$\beta(G) < k.$$

**Lemma 2** *A graph  $G$  is  $\Gamma$ -perfect if and only if  $G$  does not contain any strong graph as an induced subgraph.*

**Proof:** The necessity follows from the fact that for a strong graph  $H$ ,  $\beta(H) < \frac{1}{2}|H|$ , while  $\Gamma(H) = \frac{1}{2}|H|$  by Lemma 1. To prove the sufficiency, let  $G'$  be an arbitrary induced subgraph of  $G$ , and let  $D$  be a minimal dominating set in  $G'$  of cardinality  $\Gamma(G')$ . Denote by  $J$  the set of all isolated vertices in  $\langle D \rangle$ , i.e.  $J = \{v \in D : \deg_{\langle D \rangle} v = 0\}$ , and let  $A = D - J$ . Since  $D$  is a minimal dominating set, it follows that for each vertex  $a \in A$  there is a vertex  $b \notin D$  such that  $N(b) \cap D = \{a\}$ . Taking such a vertex  $b$  for each  $a \in A$ , we define  $B$  as the union of these vertices. The graph  $H = \langle A \cup B \rangle$  is obviously a  $W$ -graph with parts  $A$  and  $B$ , and  $|A| = |B| = k$ . Since  $H$  cannot be strong, we have  $\beta(H) \geq k$ . Let  $I$  be an independent set of  $H$  of cardinality  $k$ . It is evident that the set  $I \cup J$  is independent in  $G'$ . Consequently,

$$\beta(G') \geq |I| + |J| = |A| + |J| = |D| = \Gamma(G').$$

Since  $\beta(G') \leq \Gamma(G')$ , we have  $\beta(G') = \Gamma(G')$ . Thus, the graph  $G$  is  $\Gamma$ -perfect. ■

**Definition 16** *A connected strong graph  $G(A, B)$  of order  $2k$  is called critical if  $\delta(G) \geq 2$  and for any  $l$ -edge  $e \in E(G)$ ,*

$$\beta(G - e) = k.$$

**Lemma 3** Any strong graph  $G$  with parts  $A$  and  $B$  contains a critical subgraph  $G^*$  with parts  $A^* \subseteq A$  and  $B^* \subseteq B$ .

**Proof:** Let  $E'$  be the maximum set of l-edges in  $G$  such that  $\beta(G') < k$ , where  $V(G') = V(G)$  and  $E(G') = E(G) - E'$ . Since  $E'$  is maximum and deleting all l-edges from  $G$  produces the graph with independence number  $k$ , we obtain  $\beta(G' - e) = k$  for any l-edge  $e \in E(G')$ . Suppose that  $G'$  contains a vertex  $u$  of degree 1, and denote by  $uv$  the b-edge incident to  $u$ . Let us show that  $\deg_{G'} v = 1$ . Suppose to the contrary that there is an l-edge  $vw$  in  $G'$ . Since  $\beta(G' - vw) = k$ , there is an independent set  $I$  in  $G' - vw$  of cardinality  $k$ . We have  $v, w \in I$ , for otherwise  $I$  is independent in  $G'$ , contrary to the fact that  $\beta(G') < k$ . Now the set  $I' = (I - \{v\}) \cup \{u\}$  is independent in  $G'$  and  $|I'| = |I| = k$ , a contradiction again. Consequently,  $\deg_{G'} v = 1$ . Thus, if  $\deg_{G'} u = 1$ , then the b-edge incident to  $u$  is an isolated edge in  $G'$ .

Consider now the connected component  $G^*(A^*, B^*)$  of the graph  $G'$  such that  $A^* \subseteq A$ ,  $B^* \subseteq B$  and  $\beta(G^*) < k^*$ , where  $k^* = |A^*| = |B^*|$ . Such a component does exist, for otherwise  $\beta(H) = \frac{1}{2}|H|$  for each connected component  $H$  of  $G'$  and hence  $\beta(G') = \frac{1}{2}|G'| = k$ , a contradiction. We see that  $G^*$  is a connected strong graph of order  $2k^*$ . If  $\delta(G^*) = 1$ , then  $G^*$  is an isolated b-edge in  $G'$  and so  $\beta(G^*) = k^* = 1$ , a contradiction. Hence  $\delta(G^*) \geq 2$ . If there exists an l-edge  $e$  in  $G^*$  such that  $\beta(G^* - e) < k^*$ , then obviously  $\beta(G' - e) < k$ , contrary to the maximality of  $E'$ . Thus,  $\beta(G^* - e) = k^*$  for any l-edge  $e \in E(G^*)$ . We conclude that  $G^*$  is a critical graph. ■

**Lemma 4** A graph  $G$  is  $\Gamma$ -perfect if and only if  $G$  contains no critical graph as a semi-induced subgraph.

**Proof:** Let  $G$  be a  $\Gamma$ -perfect graph and suppose that  $G$  contains a critical graph  $H(A, B)$  as a semi-induced subgraph. We have  $\beta(H) < k$ , where  $k = |A| = |B|$ . Consider in the graph  $G$  the induced subgraph  $F = \langle A \cup B \rangle$ . This graph is obtained from  $H$  by adding some edges in the parts  $A, B$ . Therefore,  $\beta(F) < k$  and  $F$  is a W-graph, i.e.,  $F$  is a strong graph. This is a contradiction, since, by Lemma 2, the graph  $G$  does not contain any strong graph as an induced subgraph.

Now let  $G$  contain no critical graph as a semi-induced subgraph, and suppose that  $G$  is not  $\Gamma$ -perfect. By Lemma 2, the graph  $G$  contains a strong graph  $H$  as an induced subgraph. Now, by Lemma 3,  $H(A, B)$  contains a critical subgraph  $H^*(A^*, B^*)$  such that  $A^* \subseteq A$  and  $B^* \subseteq B$ , i.e.,  $H^*$  is a semi-induced subgraph of  $H$ . Therefore, the critical graph  $H^*$  is a semi-induced subgraph of  $G$ , a contradiction. ■

In the remaining part of the proof we give a description of the class of critical graphs. In fact we prove that a graph is critical if and only if it is a basis. This result together with Lemma 4 will provide the characterization of  $\Gamma$ -perfect graphs.

**Lemma 5** If  $G'$  is obtained from a perfect bl-graph  $G$  by the operation of W-reducibility, then  $G'$  is a W-graph whose perfect matching between the parts consists of b-edges determined by the operation of W-reducibility.

**Proof:** Let  $G'$  be obtained from a perfect bl-graph  $G$  by the operation of W-reducibility. Note that the graph  $G$  may have loops only if  $G = Pr_1$ . In that case the loops (l-edges) of

$G$  are replaced in accordance with Definition 11 by alternating even  $l$ -cycles. If  $u \in V(G')$  is an old vertex, i.e.  $u \in V(G)$ , then  $u$  is labeled by  $c(u)$  in  $G'$ . If  $u \in V(G')$  is a new vertex, then  $u$  is a non-end vertex of some chain  $P_e$ . We label all vertices from  $V(G') - V(G)$  by the following inductive rule. If  $e = uv$  is an edge of  $G'$  such that  $u$  has a label but  $v$  has no label yet, then we put:

- $c(v) = A$  if  $c(u) = A$  and  $e$  is an  $l$ -edge.
- $c(v) = B$  if  $c(u) = A$  and  $e$  is a  $b$ -edge.
- $c(v) = A$  if  $c(u) = B$  and  $e$  is a  $b$ -edge.
- $c(v) = B$  if  $c(u) = B$  and  $e$  is an  $l$ -edge.

Now, the vertices of  $G'$  with label  $A$  form the part  $A$ , the vertices with label  $B$  form the part  $B$ , and the set of  $b$ -edges of  $G'$  forms a perfect matching between  $A$  and  $B$ , i.e., the sets  $A, B$  independently match each other in  $G'$ . Thus, the graph  $G'$  is a  $W$ -graph. ■

**Lemma 6** *Let  $G$  be a perfect  $bl$ -graph of order  $2k$  and let  $C = (u_1, u_2, \dots, u_{2n+1})$  be an  $l$ -cycle in  $G$  starting with  $u_1$ . If  $\beta(G) = k$ , then the vertex  $u_1$  belongs to no maximum independent set of  $G$ .*

**Proof:** By definition, the edge  $u_{2i}u_{2i+1}$  is a  $b$ -edge for any  $i$ ,  $1 \leq i \leq n$ . Suppose that there is a maximum independent set  $I$  containing  $u_1$ . Since  $\beta(G) = k$ , the set  $I$  contains exactly one vertex of each  $b$ -edge. We have  $u_1 \in I$  and hence  $u_2 \notin I$ . Therefore,  $u_3 \in I$ . If we continue this process, we finally arrive at  $u_{2n+1} \in I$ . This is a contradiction, since the set  $I$  contains two adjacent vertices  $u_1$  and  $u_{2n+1}$ . ■

**Definition 17** *A perfect  $bl$ -graph  $G$  is called a semi-basis if  $G$  consists of two  $l$ -cycles  $C$  and  $C'$  starting with  $u$  and  $u'$  ( $u \neq u'$ ), respectively, and also of a  $b$ -chain  $P$  connecting  $u$  and  $u'$ . Note that  $C, C'$  and  $P$  do not necessarily contain different vertices. However, any of the graphs  $C, C'$  or  $P$  has no self-intersections, since it is simple.*

**Lemma 7** *If a perfect  $bl$ -graph  $G$  of order  $2k$  contains a semi-basis subgraph, then*

$$\beta(G) < k.$$

**Proof:** Suppose to the contrary that  $G$  has an independent set  $I$  of cardinality  $k$ . Then, obviously,  $\beta(G) = k$ . By Lemma 6, the starting vertices  $u, u'$  of the  $l$ -cycles  $C, C'$  do not belong to the set  $I$ . Let  $P = (u_1, u_2, \dots, u_{2m})$  be a  $b$ -chain connecting  $u = u_1$  and  $u' = u_{2m}$ . The edges  $u_{2i-1}u_{2i}$  ( $1 \leq i \leq m$ ) are  $b$ -edges and the set  $I$  contains exactly one vertex of each  $b$ -edge, since  $\beta(G) = k$ . The vertex  $u = u_1$  does not belong to  $I$ , and so  $u_2 \in I$ . Hence  $u_3 \notin I$  and  $u_4 \in I$ . Going on in the same way, we obtain  $u_{2i} \in I$  for all  $i$ ,  $1 \leq i \leq m$ . This is a contradiction, since the vertex  $u' = u_{2m}$  does not belong to the set  $I$ . ■

**Lemma 8** *A critical graph  $G$  is a semi-basis. The graph  $G - e$  does not contain a semi-basis subgraph for any  $l$ -edge  $e \in E(G)$ .*



**Proof:** Let  $v$  be an arbitrary vertex of a critical graph  $G(A, B)$ , say  $v \in A$ . Put  $X_0 = \{v\}$  and  $X'_0 = N(v) \cap A$ . For  $i \geq 0$ , we define the sets  $X_{i+1}$  and  $X'_{i+1}$  as follows:

$$X_{i+1} = \{x \in V(G) : xy \text{ is a b-edge, } y \in X'_i\},$$

$$X'_{i+1} = N(X_{i+1}) - (\cup_{j=0}^i X'_j \cup \{v'\}),$$

where  $v' \in B$  and  $vv'$  is a b-edge. The construction of the sequence

$$X_0, X'_0, X_1, X'_1, \dots, X_n, X'_n$$

is finished for minimal  $n$  such that  $X'_n = \emptyset$ . Clearly, the above sets are pairwise disjoint. Put

$$X = \cup_{j=0}^n X_j,$$

$$X' = \cup_{j=0}^{n-1} X'_j \cup \{v'\}.$$

Let us show that the set  $X$  is not independent. The graph  $G$  is critical, and so  $\delta(G) \geq 2$ . Hence there is a vertex  $w \in A$  adjacent to  $v$ . Moreover,  $\beta(G - vw) = k = |A| = |B|$ . Let  $I$  be an independent set of  $G - vw$  of cardinality  $k$ . Since  $I$  is not independent in  $G$ , we have  $v, w \in I$ . Put

$$A_1 = A - (X \cup X'),$$

$$B_1 = B - (X \cup X'),$$

$$I_1 = A_1 \cap I,$$

$$I_2 = B_1 \cap I.$$

By the definitions, no vertex of  $X$  is adjacent to a vertex of  $A_1 \cup B_1$ . The set  $I' = X \cup I_1 \cup I_2$  has cardinality  $k$ , and hence  $I'$  is not independent in  $G$ . On the other hand,  $I_1 \cup I_2$  is independent in  $G$  and there is no edge between  $I_1 \cup I_2$  and  $X$  in  $G$ . We conclude that  $X$  is not independent, and hence  $x_s \in X_s$  is adjacent to  $y_t \in X_t$ . Clearly,  $s$  and  $t$  have the same parity. If  $s < t$ , then we have a contradiction, since  $y_t$  must belong to  $X'_s$  but  $X'_s \cap X_t = \emptyset$ . Thus,  $s = t$ , i.e.,  $x_s \in X_s$  is adjacent to  $y_s \in X_s$ . Now we construct two alternating simple chains. Put

$$P_1 = (x_s, x'_{s-1}, x_{s-1}, x'_{s-2}, \dots, x'_0, x_0 = v),$$

$$P_2 = (y_s, y'_{s-1}, y_{s-1}, y'_{s-2}, \dots, y'_0, y_0 = v),$$

where  $x_i, y_i \in X_i$  and  $x'_i, y'_i \in X'_i$  ( $0 \leq i \leq s$ ). Let  $z$  be the first common vertex of  $P_1$  and  $P_2$  if we go from  $x_s$  to  $v$  (possibly,  $z = v$ ). Obviously,  $z \in X$ . Now, the edge  $x_s y_s$ , the  $(x_s, z)$ -subchain of  $P_1$  and the  $(y_s, z)$ -subchain of  $P_2$  form the l-cycle  $C$  starting with  $z$ . If  $z \neq v$ , then the  $(z, v)$ -subchain of  $P_1$  is an alternating  $(z, v)$ -chain in which  $z$  is incident to a b-edge and  $v$  is incident to an l-edge.

In fact we proved the following lemma.

**Lemma 9** *For any vertex  $v$  of a critical graph  $G$ , there exists an l-cycle  $C$  starting with  $z$  and such that if  $v \neq z$ , then there is an alternating  $(v, z)$ -chain in which  $v$  is incident to an l-edge, and  $z$  is incident to a b-edge, and moreover,  $z$  is the only common vertex of this chain and  $C$ .*

We go on with the proof of Lemma 8. Denote by  $zz_1$  the b-edge incident to the vertex  $z$ , and apply Lemma 9 to the vertex  $z_1$ . Let  $C'$  be the l-cycle starting with  $z'$ . If  $z_1 \neq z'$ , let  $P$  be the alternating  $(z', z_1)$ -chain in which  $z_1$  is incident to an l-edge and  $z'$  is incident to a b-edge. If  $z_1 = z'$ , then put  $P = \emptyset$ . Let  $P^+ = P \cup z_1z$ , thus  $P^+$  is the b-chain connecting the starting vertices  $z$  and  $z'$  of the l-cycles  $C$  and  $C'$ . The union of the cycles  $C, C'$  and the chain  $P^+$  produces a semi-basis subgraph  $G'$  of the graph  $G$ . The semi-basis graph  $G'$  is shown in Fig. 2 provided that it has no self-intersections.

Let  $e$  be an l-edge of the graph  $G$  and suppose that the graph  $G - e$  contains a semi-basis subgraph. By Lemma 7,  $\beta(G - e) < k$ . On the other hand,  $G$  is critical, and so  $\beta(G - e) = k$ , a contradiction. Thus, the graph  $G - e$  does not contain a semi-basis subgraph for any l-edge  $e \in E(G)$ . Therefore, the semi-basis subgraph  $G'$  of  $G$  contains all l-edges of  $G$ . Since  $G$  is critical, we have  $\delta(G) \geq 2$ . Hence  $V(G') = V(G)$ . Taking into account that any semi-basis graph is a perfect bl-graph, we conclude that  $G'$  must contain all b-edges of  $G$ . Thus,  $G' = G$ . The proof of Lemma 8 is complete.  $\blacksquare$

**Remark 1** *The proof of Lemma 8 implies that the cycle  $C'$  and the chain  $P - \{z_1\}$  may intersect the set  $V(C) - \{z\}$  in the critical graph  $G$ , all other intersections are impossible.*

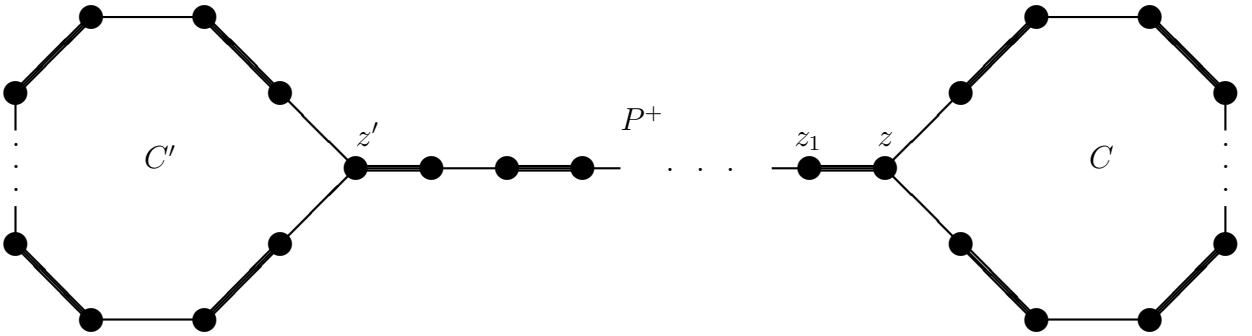


FIGURE 2. Semi-basis graph without self-intersections.

**Lemma 10** *Any critical graph is a basis.*

**Proof:** Let  $G$  be a critical graph. By Lemma 8, the graph  $G$  is a semi-basis. Employing the notation used in Lemma 8 and taking into account Remark 1, we consider all possible intersections of  $V(C') \cup (V(P) - \{z_1\})$  and  $V(C) - \{z\}$ . Suppose that the intersection of these sets is empty. The graph  $G$  is a W-graph, since  $G$  is critical. Hence the cycles  $C$  and  $C'$  have the even number of b-edges. Therefore,  $G$  can be obtained from  $Pr_1$  by the operation of W-reducibility, i.e.,  $G$  is a basis.

Suppose that  $u$  is a common vertex of  $C' \cup P$  and  $C$ . Let  $Q$  be a maximal common subchain of  $C' \cup P$  and  $C$  such that  $Q$  contains  $u$ . Since every vertex in  $G$  is incident to exactly one b-edge, it follows that  $Q$  is a b-chain. Maximal common b-chains will be called *intersection intervals*. Obviously,  $\deg_G v = 2$  for any non-end vertex  $v$  of any intersection interval.

Let us show that the set  $V(P) - \{z_1\}$  does not intersect the set  $V(C) - \{z\}$ . Suppose to the contrary that  $u \in (V(P) - \{z_1\}) \cap (V(C) - \{z\})$  and  $u$  is the nearest vertex to the vertex  $z$  in the chain  $P^+$ . Denote by  $(u, u')$  the corresponding intersection interval. Let  $L$  be the  $(z, u)$ -subchain of  $C$  such that  $u' \in L$ , and let  $y \in L$  be adjacent to  $z$ . Thus,  $zy$  is an l-edge. The  $(z, u)$ -subchain of  $P^+$  and the  $(u, z)$ -subchain of  $C$  not containing  $u'$  form the l-cycle  $C''$  starting with  $u$ . The  $(u, z')$ -subchain  $P'$  of the chain  $P^+$  connects  $C''$  with the l-cycle  $C'$  starting with  $z'$ . Thus, for the l-edge  $zy$  the graph  $G - zy$  contains a semi-basis subgraph formed by  $C'$ ,  $C''$  and  $P'$ , contrary to Lemma 8. Therefore,  $V(P) \cap V(C) = \emptyset$ .

Now consider possible intersections of  $C' - \{z'\}$  and  $C - \{z\}$ . Passing round the cycle  $C'$  from the vertex  $z'$ , denote all intersection intervals by

$$(u_1, u'_1), (u_2, u'_2), \dots, (u_t, u'_t),$$

where  $u_i, u'_i$  ( $1 \leq i \leq t$ ) are end vertices of the intervals. Since  $C' \cap C \neq \emptyset$ , we have  $t \geq 1$ . In what follows it is always supposed that we pass round the cycle  $C$  in the direction from  $u'_1$  to  $u_1$ . We will prove that, passing round the cycle  $C$  in this direction, the end vertices of the above intersection intervals are arranged in the following sequence:

$$u'_1, u_1, u'_2, u_2, \dots, u'_t, u_t, \tag{1}$$

and moreover the vertex  $z$  belongs to the  $(u_t, u'_1)$ -subchain of  $C$ . These statements hold for  $t = 1$ , and so we may assume that  $t \geq 2$ .

Suppose that passing round  $C$  we arrive to the vertex  $u'_1$  from  $u_s$  ( $s > 1$ ), i.e. the  $(u_s, u'_1)$ -subchain  $R$  of  $C$  contains no end vertices of any intersection interval excepting  $u_s$  and  $u'_1$ , and the vertex  $z$  does not belong to the chain  $R$ . We see that  $R$  is an l-chain. Hence the chain  $R$  and the  $(u_s, u'_1)$ -subchain of  $C'$  containing  $z'$  form the l-cycle  $C''$  starting with  $z'$ . Let  $e = vu'_1$  be the l-edge of  $C'$ . Obviously,  $e \notin C''$  and  $e \notin C$ , since  $(u'_1, u_1)$  is a maximal common b-chain of  $C$  and  $C'$ . Thus,  $G - e$  contains a semi-basis subgraph consisting of  $C$ ,  $C''$  and  $P^+$ , contrary to Lemma 8. Now suppose that passing round  $C$  we arrive to  $u'_1$  from  $u'_s$ , and  $z$  does not belong to the  $(u'_s, u'_1)$ -subchain  $R$  of  $C$ . Therefore,  $R$  is an l-chain. The chain  $R$  and the  $(u'_1, u'_s)$ -subchain of  $C'$  not containing  $z'$  form the l-cycle  $C''$  starting with  $u'_1$ . Let  $e = vu'_s$  be the l-edge of  $C'$ . Obviously,  $e \notin C''$  and  $e \notin C$ . Thus,  $G - e$  contains a semi-basis subgraph consisting of  $C$ ,  $C''$ ,  $P^+$  and the  $(z', u'_1)$ -subchain of  $C'$  not containing  $u'_s$ , contrary to Lemma 8. Therefore, passing round  $C$  we arrive to the vertex  $u'_1$  from  $z$ , and the  $(z, u'_1)$ -subchain of  $C$  contains no end vertex of any intersection interval excepting  $u'_1$ .

Suppose now that passing round  $C$  we arrive to the vertex  $u_r$  and the next end vertex of the intersection intervals is  $u_s$ . It is evident that we arrived to the vertex  $u_r$  from the vertex  $u'_r$ . Since  $u_s \neq u'_1$ , the  $(u_r, u_s)$ -subchain of  $C$  does not contain  $z$  and hence it is an l-chain. Let  $s > r$ . Let us define the cycle  $C''$  consisting of the  $(u_r, u_s)$ -subchain of  $C$  and the  $(u_r, u_s)$ -subchain of  $C'$  containing  $u'_r$ . The cycle  $C''$  is an l-cycle starting with  $u_s$ . Let  $P'$  be the  $(u_s, z')$ -subchain of  $C'$  containing  $u'_s$ . Thus,  $P' \cup P^+$  is a b-chain from  $u_s$  to  $z$ . Let  $e = z'w$  be the l-edge of  $C'$  such that  $e \notin P'$ . Since  $z' \notin C$ , we have  $e \notin C$ . Also,  $e \notin C''$ . We conclude that the graph  $G - e$  contains a semi-basis subgraph consisting of  $C$ ,  $C''$  and  $P' \cup P^+$ , contrary to Lemma 8. Now let  $s < r$ . Let us define the cycle  $C''$  consisting of the  $(u_r, u_s)$ -subchain of  $C$  and the  $(u_s, u_r)$ -subchain of  $C'$  containing  $u'_s$ . The cycle  $C''$  is an l-cycle starting with  $u_r$ . Let  $P'$  be the  $(u_r, z')$ -subchain of  $C'$  containing  $u'_r$ . Thus,

$P' \cup P^+$  is a b-chain from  $u_r$  to  $z$ . Let  $e = z'w$  be the l-edge of  $C'$  such that  $e \notin P'$ . We have,  $e \notin C$  and  $e \notin C''$ . We conclude that the graph  $G - e$  contains a semi-basis subgraph consisting of  $C$ ,  $C''$  and  $P' \cup P^+$ , contrary to Lemma 8. Thus, if we arrive to the vertex  $u_r$  passing round  $C$ , then the next end vertex of the intersection intervals must be  $u'_s$ , i.e., passing round the cycle  $C$  the end vertices of the intersection intervals are arranged in the following sequence:

$$u'_1, u_1, u'_{f(2)}, u_{f(2)}, \dots, u'_{f(t)}, u_{f(t)},$$

where  $f : \{2, 3, \dots, t\} \rightarrow \{2, 3, \dots, t\}$  is a bijection. Clearly,  $z$  belongs to the  $(u_{f(t)}, u'_1)$ -subchain of  $C$ . Assume that there is  $j \in \{2, 3, \dots, t\}$  such that  $f(j-1) > f(j)$ . Denote  $r = f(j-1)$  and  $s = f(j)$ . Thus,  $r > s$ . Let  $L$  be the  $(u_r, u'_s)$ -subchain of  $C$ . We see that  $z \notin L$  and hence  $L$  is an l-chain. Let  $L'$  be the  $(u'_s, u_r)$ -subchain of  $C'$  not containing  $z'$ . Obviously,  $L'$  is an l-chain. Replace the chain  $L'$  in  $C'$  by the chain  $L$  and denote the resulting cycle by  $C''$ . The cycle  $C''$  is an l-cycle starting with  $z'$ . Let  $e = u_r w$  be the l-edge of  $L'$ . It is evident that  $e \notin C''$  and  $e \notin C$ . We deduce that the graph  $G - e$  contains a semi-basis subgraph consisting of  $C$ ,  $C''$  and  $P^+$ , contrary to Lemma 8. Consequently, the end vertices of intersection intervals while passing round  $C$  are arranged in accordance with (1).

The graph  $G(A, B)$  is a W-graph, since  $G$  is critical. We label all vertices of  $G$  as follows. Put  $c(v) = A$  if  $v \in A$ , and  $c(v) = B$  if  $v \in B$ . Denote  $u_0 = z$  and  $u'_0 = z'$ . Furthermore, we construct the graph  $G^*$  by the following rule. Let

$$\mathcal{P}_b = \{(u_i, u'_i) : 0 \leq i \leq t\}$$

be the set of b-chains of  $G$  consisting of the chain  $P^+$  and the intersection intervals. Replace each chain  $(u_i, u'_i)$  from  $\mathcal{P}_b$  by the b-edge  $u_i u'_i$ . Note that the chain  $(u_i, u'_i)$  is an even b-chain if  $u_i$  and  $u'_i$  have the same label, and this chain is an odd b-chain otherwise. Now let

$$\mathcal{P}_l = \{(u_t, u_0), (u'_t, u'_0), (u_{i-1}, u'_i), (u'_{i-1}, u_i) : 1 \leq i \leq t\}$$

be the set of l-chains of  $G$ . Replace each chain  $L$  from  $\mathcal{P}_l$  by the l-edge connecting the end vertices of  $L$ . Note that  $L$  is an even l-chain if its end vertices have the same label, and  $L$  is an odd l-chain otherwise. The resulting graph  $G^*$  is the odd prism  $Pr_{t+1}$  whenever  $t \geq 1$  is even, and  $G^*$  is the even Möbius ladder  $Ml_{t+1}$  whenever  $t \geq 1$  is odd. Moreover, using the mapping  $c : V(G^*) \rightarrow \{A, B\}$  constructed above, the graph  $G$  is obtained from  $G^*$  by the operation of W-reducibility. Therefore,  $G$  is a basis graph. The proof of Lemma 10 is complete.  $\blacksquare$

**Lemma 11** *All basis graphs are critical.*

**Proof:** Let  $G$  be a basis graph, i.e.,  $G$  is obtained from  $Pr_{2n+1}$  ( $n \geq 0$ ) or  $Ml_{2n}$  ( $n \geq 1$ ) by the operation of W-reducibility. By Lemma 5,  $G$  is a W-graph,  $G = G(A, B)$ . Obviously,  $\delta(G) \geq 2$  and  $G$  is a connected graph. Let us show that  $\beta(G) < k$ . By Lemma 7, it is sufficient to find a semi-basis subgraph in  $G$ . If  $G$  is obtained from  $Pr_1$ , then  $G$  is evidently a semi-basis subgraph. Now let  $G$  be obtained from  $Pr_{2n+1}$  or  $Ml_{2n}$ ,  $n \geq 1$ . Since the operation of W-reducibility preserves semi-basis subgraphs, it is sufficient to find such a graph in  $Pr_{2n+1}$  or  $Ml_{2n}$ ,  $n \geq 1$ . In fact we will show that both  $Pr_{2n+1}$  and  $Ml_{2n}$  are semi-basis graphs.

Recall that in  $Pr_{2n+1}$ , the cycles  $(u_1, u_2, \dots, u_{2n+1})$  and  $(v_1, v_2, \dots, v_{2n+1})$  consist of l-edges, and  $\{u_i v_i : 1 \leq i \leq 2n + 1\}$  is the set of b-edges. Define two cycles as follows:

$$C = (u_1, u_2, v_2, v_3, u_3, u_4, \dots, u_{2n}, v_{2n}, v_{2n+1}, u_{2n+1}, u_1),$$

and

$$C' = (v_1, v_2, u_2, u_3, v_3, v_4, \dots, v_{2n}, u_{2n}, u_{2n+1}, v_{2n+1}, v_1).$$

The cycle  $C$  starting with  $u_1$  and the cycle  $C'$  starting with  $v_1$  are l-cycles connected by the b-chain  $(u_1, v_1) = u_1 v_1$ , i.e.,  $Pr_{2n+1}$  is a semi-basis graph for  $n \geq 1$ .

Consider now the Möbius ladder  $Ml_{2n}$ . Recall that the cycle  $(u_1, u_2, \dots, u_{4n})$  in  $Ml_{2n}$  consists of l-edges, and  $\{u_i, u_{2n+i} : 1 \leq i \leq 2n\}$  is the set of b-edges of  $Ml_{2n}$ . Define two cycles as follows:

$$C = (u_1, u_2, u_{2n+2}, u_{2n+3}, u_3, u_4, u_{2n+4}, u_{2n+5}, \dots, u_{2n}, u_{4n}, u_1),$$

and

$$C' = (u_{2n+1}, u_{2n+2}, u_2, u_3, u_{2n+3}, u_{2n+4}, u_4, u_5, \dots, u_{4n}, u_{2n}, u_{2n+1}).$$

The cycle  $C$  starting with  $u_1$  and the cycle  $C'$  starting with  $u_{2n+1}$  are l-cycles connected by the b-chain  $(u_1, u_{2n+1}) = u_1 u_{2n+1}$ , i.e.,  $Ml_{2n}$  is a semi-basis graph.

Thus,  $\beta(G) < k$ . It remains to prove that  $\beta(G - e) = k$  for each l-edge  $e \in E(G)$ . Let  $G$  be obtained from  $Pr_1$  by the operation of W-reducibility. Obviously for any l-edge  $e \in E(G)$ , the graph  $G - e$  contains 1 or 2 vertices of degree 1. Starting with a vertex (vertices) of degree 1, it is easily to construct the desired independent set of cardinality  $k$ .

Now let  $G$  be obtained from  $H = \{Pr_{2n+1}, Ml_{2n} : n \geq 1\}$  by the operation of W-reducibility, i.e., b-edges are replaced by alternating b-chains and l-edges are replaced by alternating l-chains. There are two cases to consider.

**Case 1.** The l-edge  $e$  belongs to an l-chain  $P_f$ , where  $f$  is an l-edge of  $H$ . If  $H = Pr_{2n+1}$ , then without loss of generality we may suppose that  $f = u_1 u_{2n+1}$ . Put

$$I = \{u_1, u_{2i+1}, v_{2i} : 1 \leq i \leq n\}.$$

If  $H = Ml_{2n}$ , then we may assume that  $f = u_1 u_2$ . In that case put

$$I = \{u_1, u_{2i}, u_{2n}, u_{2n+2i+1} : 1 \leq i \leq n - 1\}.$$

The set  $I$  is an independent set of  $H - f$  of cardinality  $\frac{1}{2}|H|$ . Now it is not difficult to construct an independent set of  $G - e$  of cardinality  $k = \frac{1}{2}|G|$ . Indeed, let  $uv \in E(H)$  be a b-edge replaced by a b-chain  $P$ . Since  $|I| = \frac{1}{2}|H|$ , we have  $|\{u, v\} \cap I| = 1$ , say  $u \in I$ . From each b-edge of  $P$  we add in  $I$  one vertex which is nearer to the vertex  $u$  in the chain  $P$ . If  $uv \neq f$  is an l-edge of  $H$  replaced by an l-chain  $P$ , then  $|\{u, v\} \cap I| \leq 1$  and we can add vertices in  $I$  in the same way as above. Now suppose that  $uv = f$ , and let  $e = xy$ . Then, from each b-edge of  $P_f - e$  we add in  $I$  one vertex which is nearer to the vertices  $x, y$  in the chain  $P_f$ . The constructed set  $I'$  is an independent set of  $G - e$ . Since  $I'$  contains one vertex of each b-edge in  $G - e$ , we have  $|I'| = k$ . Consequently,  $\beta(G - e) = k$ .

**Case 2.** The l-edge  $e$  belongs to a b-chain  $P_f$ , where  $f$  is a b-edge of  $H$ . If  $H = Pr_{2n+1}$ , then without loss of generality we may suppose that  $f = u_1 v_1$ . Put

$$I = \{u_{2i}, v_{2i+1} : 1 \leq i \leq n\}.$$

If  $H = Ml_{2n}$ , then we may assume that  $f = u_1u_{2n+1}$ . In that case put

$$I = \{u_{2i}, u_{2n}, u_{2n+2i+1} : 1 \leq i \leq n-1\}.$$

The set  $I$  is an independent set of  $H - f$  such that each b-edge of  $H - f$  has one vertex in  $I$  and  $|I| = \frac{1}{2}|H| - 1$ . Note also that the end vertices of  $f$  do not belong to  $I$ . Adding vertices in the set  $I$  in the same way as in Case 1, we obtain the set  $I'$  such that  $I'$  contains one vertex of each b-edge of the graph  $G - e$ . Therefore,  $|I'| = k = \frac{1}{2}|G|$ , i.e.,  $\beta(G - e) = k$ . ■

Thus, Lemmas 10 and 11 imply that a graph is critical if and only if it is a basis. Now the proof of Theorem 1 follows from Lemma 4. ■

## 4 Corollaries

In this section we illustrate some applications of the characterization of  $\Gamma$ -perfect graphs in terms of forbidden semi-induced subgraphs. We say that a graph  $G$  is *2-homeomorphic to  $H$*  if  $G$  can be obtained from  $H$  by replacing edges of  $H$  by chains of even order  $2k$ ,  $k \geq 1$ . Let the *family  $\mathcal{H}$*  consist of graphs 2-homeomorphic to the odd prism  $Pr_{2n+1}$  ( $n \geq 0$ ) or the even Möbius ladder  $Ml_{2m}$  ( $m \geq 1$ ).

**Proposition 1** *If  $H$  belongs to  $\mathcal{H}$ , then  $\beta(H) < \frac{1}{2}|H|$ .*

**Proof:** For the odd prism we have  $\beta(Pr_{2n+1}) = 2n$ , i.e.,  $\beta(Pr_{2n+1}) < \frac{1}{2}|Pr_{2n+1}|$ . For the even Möbius ladder we have  $\beta(Ml_{2m}) = 2m - 1$ , i.e.,  $\beta(Ml_{2m}) < \frac{1}{2}|Ml_{2m}|$ . Let  $F'$  be obtained from a graph  $F$  by the single 2-partition of the edge  $uv$ , i.e.,  $uv$  is replaced by the chain  $P = (u, x, y, v)$ . Let  $U$  be a maximum independent set of  $F'$ . Obviously,  $1 \leq |U \cap P| \leq 2$ . If  $|U \cap P| = 1$ , then  $U - P$  is an independent set of  $F$  of cardinality  $|U| - 1 = \beta(F') - 1$ . If  $|U \cap P| = 2$ , then at least one vertex from  $\{u, v\}$  belongs to  $U$ , say  $u \in U$ . Now  $U - \{x, y, v\}$  is an independent set of  $F$  of cardinality  $|U| - 1 = \beta(F') - 1$ . In any case,  $\beta(F) \geq \beta(F') - 1$ . Thus, if  $\beta(F) < \frac{1}{2}|F|$ , then

$$\beta(F') \leq \beta(F) + 1 < \frac{1}{2}|F| + 1 = \frac{1}{2}|F'|.$$

Since  $H$  is obtained from the odd prism or the even Möbius ladder by applying the operation of 2-partition, we conclude that  $\beta(H) < \frac{1}{2}|H|$ . ■

In our next theorem, the graphs from the family  $\mathcal{H}$  are forbidden as semi-induced subgraphs for a graph to be  $\Gamma$ -perfect. Using the fact that a semi-induced subgraph of a graph is a W-graph, we see that the class of forbidden semi-induced subgraphs of Theorem 2 actually consists of W-graphs from the family  $\mathcal{H}$ . Note that the class of W-graphs from  $\mathcal{H}$  is larger than the class of basis graphs used in Theorem 1. For example,  $Ml_4 = Ml_4(\{u_2, u_3, u_6, u_7\}, \{u_1, u_4, u_5, u_8\})$  is a W-graph from  $\mathcal{H}$  and hence it is forbidden in Theorem 2. On the other hand,  $Ml_4$  is not a basis graph. Another difference between Theorem 1 and Theorem 2 is that a basis graph has a fixed partition into parts determined by the set of its b-edges, while for a W-graph from  $\mathcal{H}$  the partition into parts is not fixed.

**Theorem 2** *A graph  $G$  is  $\Gamma$ -perfect if and only if  $G$  does not contain a semi-induced subgraph 2-homeomorphic to the odd prism  $Pr_{2n+1}$  ( $n \geq 0$ ) or the even Möbius ladder  $Ml_{2m}$  ( $m \geq 1$ ).*

**Proof:** Let  $G$  be a  $\Gamma$ -perfect graph and let  $H$  belong to  $\mathcal{H}$ . If  $H$  is not a W-graph, then  $H$  cannot be a semi-induced subgraph of  $G$ . Suppose now that  $H = H(A, B)$  is a W-graph and  $H$  is a semi-induced subgraph of  $G$ . By Proposition 1,  $\beta(H) < \frac{1}{2}|H|$ . Let  $H' = \langle A \cup B \rangle$ . Evidently,  $\beta(H') \leq \beta(H)$  and  $H'$  is a W-graph. Therefore, by Lemma 1,

$$\Gamma(H') = \frac{1}{2}|H'| = \frac{1}{2}|H| > \beta(H) \geq \beta(H').$$

Thus,  $\Gamma(H') > \beta(H')$ . This is a contradiction, since  $G$  is a  $\Gamma$ -perfect graph.

Suppose that  $G$  does not contain any graph from  $\mathcal{H}$  as a semi-induced subgraph. Any basis graph  $F$  is obtained from the odd prism or the even Möbius ladder by replacing its edges by alternating chains of even order, and the partition into parts of  $F$  is determined by the set of its b-edges. Thus, the graph  $F$  is 2-homeomorphic to the odd prism or the even Möbius ladder and  $F$  has a fixed partition into parts. For a W-graph  $H$  from  $\mathcal{H}$ , the partition into parts of  $H$  is not fixed, and hence we may take any partition  $V(H) = A \cup B$  such that  $A$  and  $B$  independently match each other. Therefore,  $F \in \mathcal{H}$  and the graph  $G$  does not contain any basis graph as a semi-induced subgraph. The result now follows from Theorem 1. ■

Jacobson and Peters [22] considered the class of graphs  $G$  having  $\beta(H) = IR(H)$  for all induced subgraphs  $H$  of  $G$ . By Theorem A, this class is exactly the class of  $\Gamma$ -perfect graphs.

**Corollary 1 (Jacobson and Peters [22])** *A graph  $G$  is  $\Gamma$ -perfect if and only if for any vertex subsets  $A, B \subset V(G)$  that independently match each other, the graph  $\langle A \cup B \rangle$  has an independent set of order  $|A|$ .*

**Proof:** Let  $G$  be a  $\Gamma$ -perfect graph and  $A, B$  independently match each other. The set  $A$  is minimal dominating in  $F = \langle A \cup B \rangle$ . Hence,  $\beta(F) = \Gamma(F) \geq |A|$ . To prove the sufficiency, suppose that  $G$  is not  $\Gamma$ -perfect. By Theorem 2,  $G$  contains a semi-induced subgraph  $H = H(A, B) \in \mathcal{H}$ . By Proposition 1,  $\beta(H) < |A|$ . Thus, the sets  $A, B$  independently match each other in  $G$  and  $\beta\langle A \cup B \rangle < |A|$ , a contradiction. ■

Now we turn to the problem of characterizing  $\Gamma$ -perfect graphs in terms of forbidden induced subgraphs. A graph  $G$  is called *minimal  $\Gamma$ -imperfect* if  $G$  is not  $\Gamma$ -perfect and  $\beta(H) = \Gamma(H)$ , for every proper induced subgraph  $H$  of  $G$ .

**Proposition 2** *If  $G$  is a minimal  $\Gamma$ -imperfect graph, then  $G$  contains a basis graph  $F(A, B)$  of order  $2k$  as a semi-induced subgraph,  $G = G(A, B)$  is a connected W-graph of order  $2k$ ,  $\delta(G) \geq 2$ , and  $\beta(G) = k - 1$ .*

**Proof:** By Theorem 1,  $G$  contains a basis graph  $F$  as a semi-induced subgraph. Since  $G$  is minimal, we have  $V(G) = V(F)$ . By Lemma 11,  $F$  is critical, i.e.,  $F = F(A, B)$  is a connected W-graph of order  $2k$ ,  $\delta(F) \geq 2$  and  $\beta(F) < k$ . The graph  $G$  is obtained from  $F$  by adding edges in the parts  $A, B$ . Therefore,  $G(A, B)$  is a connected W-graph of order  $2k$ ,

$\delta(G) \geq 2$  and  $\beta(G) < k$ . Let  $uv$  be a b-edge of  $G$ . The graph  $G' = G - \{u, v\}$  is  $\Gamma$ -perfect. Hence, using Lemma 1,  $\beta(G') = \Gamma(G') = k - 1$ . We obtain  $\beta(G) \geq \beta(G') = k - 1$ . Thus,  $\beta(G) = k - 1$ .  $\blacksquare$

By Proposition 2, every minimal  $\Gamma$ -imperfect graph has even order  $n \geq 6$ . Let  $\mu_n$  denote the number of nonisomorphic minimal  $\Gamma$ -imperfect graphs of order  $n$ . It was proved in [14] that  $\mu_6 = 1$  and  $\mu_8 = 14$ . Using a computer search, we discovered that  $\mu_{10} = 228$  and the number  $\mu_{12}$  considerably exceeds  $\mu_{10}$ . Therefore, it seems unlikely to obtain an explicit list of all minimal  $\Gamma$ -imperfect graphs, i.e., to provide an induced subgraph characterization of the entire class of  $\Gamma$ -perfect graphs. However, for  $K_{1,3}$ -free  $\Gamma$ -perfect graphs Theorem 1 enables us to obtain such a characterization.

We define the *family*  $\mathcal{S}$  consisting of the following classes  $\mathcal{S}_1$ ,  $\mathcal{S}_2$  and  $\mathcal{S}_3$ . Let  $C = C_{4m}$  and  $C' = C_{4n}$  ( $m, n \geq 1$ ) be two cycles, let  $uv \in E(C)$  and  $xy \in E(C')$ , and let  $(z_1, z_2, \dots, z_{2l})$  ( $l \geq 1$ ) be a chain. Add the edges  $uz_1, vz_1$  and  $xz_{2l}, yz_{2l}$ . The resulting graph belongs to  $\mathcal{S}_1$ . Now let  $(u_1, \dots, u_k)$ ,  $(v_1, \dots, v_l)$  and  $(w_1, \dots, w_m)$  be three chains such that  $k, l, m \geq 2$  and either  $k, l, m \equiv 0 \pmod{4}$  or  $k, l, m \equiv 2 \pmod{4}$ . Adding the edges  $u_1v_1, v_1w_1, w_1u_1$  and  $u_kv_l, v_lw_m, w_mu_k$ , we obtain a graph of the class  $\mathcal{S}_2$ . Lastly, let  $C = C_{4m}$  and  $C' = C_{4n}$  ( $m, n \geq 1$ ) be two cycles and let  $uv \in E(C)$  and  $xy \in E(C')$ . Add the edges  $ux, uy, vx, vy$ . The resulting graph belongs to  $\mathcal{S}_3$ .

**Theorem 3** *A  $K_{1,3}$ -free graph  $G$  is  $\Gamma$ -perfect if and only if  $G$  does not contain any member of  $\mathcal{S}$  as an induced subgraph.*

**Proof:** Any graph  $H$  from the family  $\mathcal{S}$  contains a semi-induced subgraph 2-homeomorphic to  $Pr_1, Pr_3$  or  $Ml_2$ . By Theorem 2,  $H$  is not  $\Gamma$ -perfect. To prove the sufficiency, let  $G$  be a minimal counterexample, i.e.,  $G$  is a  $K_{1,3}$ -free graph not containing any member of  $\mathcal{S}$  as an induced subgraph,  $G$  is not  $\Gamma$ -perfect and  $G$  has minimal order. Obviously,  $G$  is a minimal  $\Gamma$ -imperfect graph. By Proposition 2,  $G$  contains a basis graph  $F = F(A, B)$  as a semi-induced subgraph,  $G = G(A, B)$  is a connected W-graph of order  $2k$ ,  $\delta(G) \geq 2$ , and  $\beta(G) = k - 1$ . If the induced subgraph  $\langle A \rangle$  or  $\langle B \rangle$  of the graph  $G$  contains the induced chain  $P_3$ , then  $G$  has the induced  $K_{1,3}$ , a contradiction. Hence both  $\langle A \rangle$  and  $\langle B \rangle$  are disjoint unions of complete graphs.

**Lemma 12** *Let  $G(A, B)$  be a minimal  $\Gamma$ -imperfect graph of order  $2k$ . If  $\langle A \rangle$  and  $\langle B \rangle$  are disjoint unions of complete graphs, then the following statements hold:*

1. *If  $k$  is odd, then  $\langle A \rangle \cong \langle B \rangle \cong \frac{k-3}{2}K_2 \cup K_3$  ( $k \geq 3$ ).*
2. *If  $k$  is even, then one of the graphs  $\langle A \rangle, \langle B \rangle$  is  $\frac{k}{2}K_2$  and the other is either  $\frac{k-4}{2}K_2 \cup K_4$  ( $k \geq 4$ ) or  $\frac{k-6}{2}K_2 \cup 2K_3$  ( $k \geq 6$ ).*

**Proof:** Let  $\langle A \rangle$  be a disjoint union of the complete graphs  $H_1, \dots, H_p$ . Since  $\delta(G) \geq 2$ , we have  $|H_i| \geq 2$  for any  $i \in \{1, \dots, p\}$ , and hence  $p \leq k/2$ . Let  $I$  be an independent set in  $G$  of cardinality  $k - 1 = \beta(G)$ . Put  $I_A = I \cap A$  and  $I_B = I \cap B$ . The set  $I$  contains at most one vertex of each  $H_i$ , and hence  $|I_A| \leq p \leq k/2$ . Analogously,  $|I_B| \leq k/2$ . Further,  $|I_A| = |I| - |I_B| \geq k - 1 - k/2 = k/2 - 1$ . Thus,

$$k/2 - 1 \leq |I_A| \leq k/2. \quad (2)$$



Analogously,

$$k/2 - 1 \leq |I_B| \leq k/2. \quad (3)$$

Put

$$s = \sum_{i=1}^p (|H_i| - 2) \geq 0.$$

We have,

$$k = |A| = \sum_{i=1}^p |H_i| = s + 2p \geq s + 2|I_A|.$$

Therefore, using (2),

$$s \leq k - 2|I_A| \leq 2.$$

Thus,

$$s \in \{0, 1, 2\}.$$

If  $k = |A|$  is odd, then  $s$  is also odd, since  $s = k - 2p$ . Hence  $s = 1$ ,  $k \geq 3$ , and  $\langle A \rangle \cong \frac{k-3}{2}K_2 \cup K_3$ . Analogously,  $\langle B \rangle \cong \frac{k-3}{2}K_2 \cup K_3$ .

Now let  $k$  be even. Using (2) and (3), we see that one of the sets  $I_A$  and  $I_B$  has cardinality  $k/2 - 1$  and the other has cardinality  $k/2$ . Without loss of generality, let  $|I_A| = k/2 - 1$  and  $|I_B| = k/2$ . Since  $\langle B \rangle$  is a disjoint union of complete graphs and  $\delta(G) \geq 2$ , we have  $\langle B \rangle \cong \frac{k}{2}K_2$ . Further,  $s = k - 2p$  and  $k$  is even. Hence  $s$  is even and  $s = 0$  or  $2$ . If  $s = 0$ , then  $\langle A \rangle \cong \frac{k}{2}K_2$  and therefore  $G$  is a disjoint union of even simple cycles. We obtain  $\beta(G) = k$ , a contradiction. Thus,  $s = 2$ . Hence  $k \geq 4$  and  $\langle A \rangle \cong \frac{k-4}{2}K_2 \cup K_4$  or  $k \geq 6$  and  $\langle A \rangle \cong \frac{k-6}{2}K_2 \cup 2K_3$ . The proof of Lemma 12 is complete.  $\blacksquare$

By Lemma 12,  $G$  has either exactly 6 vertices of degree 3 or exactly 4 vertices of degree 4, and all other vertices have degree 2. The basis graph  $F$  is a spanning subgraph of  $G$ . Therefore, either  $F$  has at most 6 vertices of degree 3 and all other vertices have degree 2, or  $F$  has at most 4 vertices of degree 3 and 4 and all other vertices have degree 2. Consequently,  $F$  is obtained from  $Pr_1$ ,  $Pr_3$  or  $Ml_2$  (see Fig. 1) by the operation of W-reducibility. By Lemma 5, any l-edge of  $F$  belongs to  $A$  or  $B$ .

Suppose that  $F$  is obtained from  $Pr_1$ . Let  $uu_1, u_1u'$  be l-edges of one l-cycle of  $F$  and let  $vv_1, v_1v'$  be l-edges of the other l-cycle of  $F$ . The vertices  $u, u', u_1$  belong to the same part, and  $v, v', v_1$  belong to the same part. We have,  $uu' \in E(G)$  and  $vv' \in E(G)$ , since  $G$  is a  $K_{1,3}$ -free graph. The restrictions on the degrees of vertices of  $G$  imply  $G = F \cup \{uu', vv'\}$ . Since  $G$  is a W-graph, we see that the cycle  $C$  of  $G$  such that  $u, u' \in C$  and  $u_1 \notin C$  has length  $4m$ , and the cycle  $C'$  of  $G$  such that  $v, v' \in C'$  and  $v_1 \notin C'$  has length  $4n$ . Thus,  $G \in \mathcal{S}_1$ , a contradiction.

Assume that  $F$  is obtained from  $Pr_3$ . The prism  $Pr_3$  has 6 vertices of degree 3. Hence,  $G = F$ . Suppose that an l-edge of  $Pr_3$  was replaced by an l-chain having more than 2 vertices. Then  $F$  has the induced  $K_{1,3}$ , a contradiction. Therefore, only b-edges of  $Pr_3$  could be replaced by b-chains to obtain  $F$ . These chains must be odd b-chains if  $C_1 = (u_1, u_2, u_3)$  and  $C_2 = (v_1, v_2, v_3)$  belong to different parts of  $F$ , and they must be even b-chains if  $C_1$  and  $C_2$  belong to the same part of  $F$ . Any odd b-chain has  $4k + 2$  vertices, and any even b-chain has  $4m$  vertices. Therefore,  $G = F \in \mathcal{S}_2$ , a contradiction.

Finally, suppose that  $F$  is obtained from  $Ml_2$  by the operation of W-reducibility. It is easy to see that  $Ml_2$  has 4 different labelings of  $V(Ml_2)$  by  $c(u) \in \{A, B\}$  up to replacing  $A$  by  $B$ . Hence there are 4 cases to consider.

**Case 1:**  $c(u_1) = c(u_4) = A$  and  $c(u_2) = c(u_3) = B$ . By the definition of W-reducibility, the l-edges  $u_1u_2$  and  $u_3u_4$  had to be replaced by the odd l-chains  $(u_1, v_1, \dots, v_k, u_2)$ ,  $k \geq 2$ , and  $(u_3, w_1, \dots, w_m, u_4)$ ,  $m \geq 2$ . Each of the vertices  $v_1, v_k, w_1, w_m$  is an end vertex of  $P_3$  or  $P_4$  consisting of l-edges and hence belonging to  $A$  or  $B$ . Since any part of  $G$  is a disjoint union of complete graphs, we see that each of the above vertices will have degree at least 3 in  $G$ . Thus,  $G$  has at least 8 vertices of degree at least 3, a contradiction.

**Case 2:**  $c(u_1) = c(u_3) = A$  and  $c(u_2) = c(u_4) = B$ . This case is analogous to Case 1, since the l-edges  $u_1u_2$  and  $u_3u_4$  had to be replaced by odd l-chains.

**Case 3:**  $c(u_1) = c(u_2) = c(u_3) = A$  and  $c(u_4) = B$ . The l-edges  $u_1u_4$  and  $u_3u_4$  had to be replaced by the odd l-chains  $(u_1, v_1, \dots, v_k, u_4)$ ,  $k \geq 2$ , and  $(u_3, w_1, \dots, w_m, u_4)$ ,  $m \geq 2$ . Each of the vertices  $v_1, v_k, w_1, w_m$  is an end vertex of  $P_r$  ( $r \geq 3$ ) consisting of l-edges. Hence, each of these vertices has degree at least 3 in  $G$ . Thus,  $G$  has at least 8 vertices of degree at least 3, a contradiction.

**Case 4:**  $c(u_i) = A$ ,  $1 \leq i \leq 4$ . If some two l-edges from  $\{u_1u_2, u_2u_3, u_3u_4, u_4u_1\}$  were replaced by even l-chains having at least two b-edges, then we derive a contradiction in the same way as above. Suppose that only one l-edge, say  $u_1u_2$ , was replaced by the even l-chain  $(u_1, v_1, \dots, v_k, u_2)$ ,  $k \geq 4$ . Then  $\langle v_k, u_2, u_3, u_4, u_1, v_1 \rangle$  is a  $P_6$  in  $F$  consisting of l-edges. Therefore,  $G$  contains  $K_6$ , a contradiction. Thus, only the b-edges  $u_1u_3$  and  $u_2u_4$  of  $Ml_2$  were replaced by even b-chains and  $\langle u_1, u_2, u_3, u_4 \rangle$  is a  $C_4$  in  $F$  consisting of l-edges. Hence  $\langle u_1, u_2, u_3, u_4 \rangle$  is a  $K_4$  in  $G$ , and so all other vertices in  $G$  must have degree 2, i.e.,  $G = F \cup \{u_1u_3, u_2u_4\}$ . Since even b-chains have  $4m$  vertices ( $m \geq 1$ ), we have  $G \in \mathcal{S}_3$ . This contradiction completes the proof of Theorem 3.  $\blacksquare$

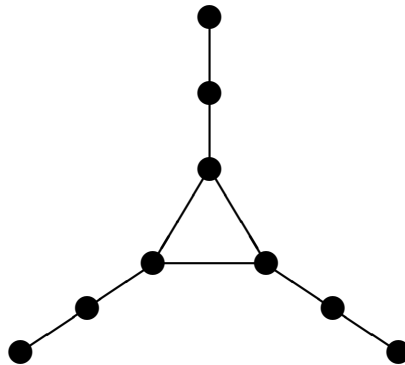


FIGURE 3. Graph  $H$  of Corollary 2.

The next result follows directly from Theorem 3 and Theorem A, since each graph from  $\mathcal{S}$  contains either  $C_4$  or the graph  $H$  of Fig. 3 as an induced subgraph.

**Corollary 2 (Jacobson and Peters [23])** *If a graph  $G$  does not contain either  $K_{1,3}$ ,  $C_4$  or the graph  $H$  of Fig. 3 as an induced subgraph, then  $G$  is  $\Gamma$ -perfect and IR-perfect.*

Note in conclusion that using properties of minimal  $\Gamma$ -imperfect graphs stated in Proposition 2, it is not difficult to prove Theorems B, C, D, E, or F from Section 1.

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