Exact radiative spacetimes: some recent developments

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Abstract

Five classes of radiative solutions of Einstein's field equations are discussed in the light of some new developments. These are plane waves and their collisions, cylindrical waves, Robinson-Trautman and type N spacetimes, boost-rotation symmetric spacetimes and generalized Gowdy-type cosmological models.

1 Introduction

Gernot Franz Sebastian Neugebauer suggested that I should give a fairly broad review on radiative spacetimes. Johann Wolfgang von Goethe believes that this is not easy: "It is extremely difficult to report on the opinions of others ... If the reporter goes into detail, he creates impatience and boredom; if he wants to summarize, he risks giving his own point of view; if he avoids judgements, the reader does not know where to begin, and if he organizes his materials according to principles, the presentation becomes one-sided and arouses opposition, and the history itself creates new histories." (J.W.Goethe, "Materialien zur Geschichte der Farbenlehre".)

My "Materialien to the Exact Gravitational-Waves Lehre" will be: plane waves and their collisions; cylindrical waves and null infinity in (2+1)-dimensional spacetimes; Robinson-Trautman solutions and type N twisting spacetimes; boost-rotation symmetric spacetimes and spinning C-metric; "cosmological" waves (Gowdy models) and the approach to cosmological singularity. Some parts of the "Materialien" are taken from [1]; see also [2] for the more detailed review and references until 1995.

2 Plane waves and their collisions

By the definition (see e.g. [3]) a vacuum spacetime is a *"plane-fronted gravitational wave"* if it contains a shearfree geodesic null congruence (with tangents k^α), and if it admits "plane wave surfaces" (spacelike 2-surfaces orthogonal to k^α). Because of the existence of plane wave surfaces, the expansion and twist must vanish as well. The best known subclass of these waves are *"plane-fronted gravitational waves with parallel rays" (pp-waves)* which are defined by the condition that the null vector k^α is covariantly constant, $k_{\alpha;\beta} = 0$.

In suitable null coordinates with a null coordinate u such that $k_\alpha = u_{,\alpha}$ and $k^\alpha = (\partial/\partial v)^\alpha$, the metric has the form

$$ds^2 = 2d\zeta d\bar{\zeta} - 2dudv - 2H(u,\zeta,\bar{\zeta})du^2, \qquad (1)$$

where H is a real function dependent on u, and on the complex coordinate ζ which spans the wave 2-surfaces $u =$ constant, $v =$ constant. The vacuum field equations imply $2H = f(u,\zeta) + \bar{f}(u,\bar{\zeta})$, where f is an arbitrary function of u, analytic in ζ.

In general the pp-waves have only the single isometry generated by the Killing vector $k^\alpha = (\partial/\partial v)^\alpha$. However, a much larger group of symmetries may exist for various particular choices of the function $H(u,\zeta,\bar{\zeta})$. Jordan, Ehlers and Kundt [4] (see also [3]) gave a complete classification of the pp-waves in terms of their symmetries and corresponding special forms of H. For example, in the best known case of plane waves $H(u,\zeta,\bar{\zeta}) = A(u)\zeta^2 + \bar{A}(u)\bar{\zeta}^2$, with $A(u)$ being an arbitrary function of u. This spacetime admits five Killing vectors.

Recently, Aichelburg and Balasin [5, 6] generalized the classification given in [4] by admitting distribution-valued profile functions and allowing for non-vacuum spacetimes. They have shown that with H in the form of delta-like pulses, $H(u,\zeta,\bar{\zeta}) = f(\zeta,\bar{\zeta})\delta(u)$, new symmetry classes arise even in the vacuum case.

The main motivation to consider impulsive pp-waves stems from the metrics describing a black hole or a "particle" boosted to the speed of light. The simplest metric of this type, given by Aichelburg and Sexl [7], is a Schwarzschild black hole with mass m boosted in such a way that $\mu = m/\sqrt{1-w^2}$ is held constant as $w \to 1$. It reads

$$ds^2 = 2d\zeta d\bar{\zeta} - 2dudv - 4\mu\log(\zeta\bar{\zeta})\delta(u)du^2, \qquad (2)$$

with H clearly in the form of a delta pulse. This is not a vacuum metric: the energy-momentum tensor $T_{\alpha\beta} = \mu\delta(u)\delta(\zeta)k_\alpha k_\beta$ indicates that there is a "point-like particle" moving with the speed of light along $u = 0$.

The interest in impulsive waves generated by boosting a "particle" at rest to the velocity of light by means of an appropriate limiting procedure persists up to the present. The ultrarelativistic limits of Kerr and Kerr-Newman black holes were obtained [8, 9, 10], and recently, boosted static multipole (Weyl) particles were studied [11]. Impulsive gravitational waves were also generated by boosting the Schwarzschild-de Sitter and Schwarzschild-anti de Sitter metrics to the ultrarelativistic limit [12, 13]; see the contribution by J. Griffiths and J. Podolský in these Proceedings.

These types of spacetimes, especially the simple Aichelburg-Sexl metrics, have been employed in current problems of the generation of gravitational radiation from axisymmetric black hole collisions and black hole encounters. The recent monograph by d'Eath [14] gives a comprehensive survey, including the author's new results. There is good reason to believe that spacetime metrics produced in high speed collisions will be simpler than those corresponding to (more realistic) situations in which black holes start to collide with low relative velocities. The spacetimes corresponding to the collisions at exactly the speed of light is an interesting limit which can be treated most easily. Aichelburg-Sexl metrics are used to describe limiting "incoming states" of two black holes, moving one against the other with the speed of light. Great interest has been stimulated by 't Hooft's [15] work on the quantum scattering of two pointlike particles at centre-of-mass energies higher or equal to the Planck energy. This quantum process has been shown to have close connection with classical black hole collisions at the speed of light (see [14, 16] and references therein).

Recently, the Colombeau algebra of generalized functions, which enables one to deal with singular products of distributions, has been brought to general relativity and used in the description of impulsive pp-waves in various coordinate systems [17], and also for a rigorous solution of the geodesic and geodesic deviation equations for impulsive waves [18]. The investigation of the equations of geodesics in non-homogeneous pp-waves (with $f \sim \zeta^3$) has shown that the motion of test particles is chaotic (see [19] and the contribution by J. Podolský in these Proceedings).

Plane-fronted waves have been used as simple metrics in various other contexts, for example, in quantum field theory on a given background (see e.g. [20]). As emphasized very recently by Gibbons [21], since for pp-waves and type N Kundt's class all possible invariants formed from the Weyl tensor and its covariant derivatives vanish [22], these metrics suffer no quantum corrections to all loop orders. Thus they may offer insights into the behaviour of a full quantum theory.

Colliding plane waves

The first detailed study of colliding plane waves was undertaken independently by Khan and Penrose and by Szekeres (see [3, 23] for references). Szekeres formulated the problem as a characteristic initial value problem for a system of hyperbolic equations in two variables (null coordinates) u, v with data specified on the pair of null hypersurfaces, say $u = 0, v = 0$ intersecting in a spacelike 2-surface. Although Szekeres' formulation of a general solution for the problem of colliding parallel-polarized waves is difficult to use for constructing explicit solutions, it has been employed in a general analysis of the structure of the singularities produced by the collision [24].

It has also inspired the work developed at the beginning of the 1990s by Hauser and Ernst [25]. Their new method of analyzing the initial value problem can be used also when the polarization of the approaching waves is not aligned. They formulated the initial value problem in terms of the equivalent matrix Riemann-Hilbert problem. Their techniques are related to those used by Neugebauer and Meinel to construct and analyze the rotating disk solution as a boundary value problem (see their contributions to these Proceedings). Most recently, Hauser and Ernst prepared an extensive treatise [26] in which they give a general description and detailed mathematical proofs of their study of the solutions of the hyperbolic Ernst equation.

The papers on colliding plane waves published until 1991 are reviewed in [23] (see also [2]). New developments have been mostly involved with "non-classical" issues like the inclusion of dilatonic fields or the discussion of the particle production. One of the few exceptions has been the analysis of colliding waves in the expanding backgrounds as e.g. in Friedmann-Robertson-Walker universes filled by stiff fluid [27]. In contrast to the waves propagating and colliding on the "flat backgrounds", no singularities arise in the expanding backgrounds.

3 Cylindrical waves

Despite the fact that cylindrically symmetric waves cannot describe exactly the radiation from bounded sources, they even recently played an important role in clarifying a number of complicated issues, such as testing the quasilocal mass-energy [28], testing codes in numerical relativity [29], investigation of the cosmic censorship [30], and quantum gravity [31] (see [1] for more details and references).

In recent work with Ashtekar and Schmidt [32, 33], we considered gravitational waves with a space-translation Killing field ("generalized Einstein-Rosen waves"). In the (2+1)-dimensional framework the Einstein-Rosen subclass forms a simple instructive example of explicitly given spacetimes which admit a smooth global null (and timelike) infinity even for strong initial data.

4-dimensional vacuum gravity which admits a spacelike hypersurface Killing vector $\partial/\partial z$ is equivalent to 3-dimensional gravity coupled to a scalar field. In 3 dimensions, there is no gravitational radiation. Hence, the local degrees of freedom are all contained in the scalar field. One therefore expects that Cauchy data for the scalar field will suffice to determine the solution. For data which fall off appropriately, we thus expect the 3-dimensional Lorentzian geometry to be asymptotically flat in the sense of Penrose, i.e. that there should exist a 2-dimensional boundary representing null infinity. In general cases, this is analyzed in [32].

Restricting here ourselves to the Einstein-Rosen waves by assuming that there is a further spacelike, hypersurface orthogonal Killing vector $\partial/\partial\varphi$ which commutes with $\partial/\partial z$, we find the 3-metric given by

$$d\sigma^2 = g_{ab}dx^a dx^b = e^{2\gamma}(-dt^2 + d\rho^2) + \rho^2 d\varphi^2, \tag{3}$$

where $\gamma = \gamma(t, \rho)$. The field equations for the scalar field ψ coupled to this metric become

$$-\ddot{\psi} + \psi'' + \rho^{-1}\psi' = 0\ , \quad \gamma' = \rho(\dot{\psi}^2 + \psi'^2)\ , \quad \dot{\gamma} = 2\rho\dot{\psi}\psi'. \tag{4}$$

Thus, we can first solve the axisymmetric wave equation for ψ on Minkowski space and then solve for γ – the only unknown metric coefficient – by quadratures. The "method of descent" from the Kirchhoff formula in 4 dimensions gives the representation of the solution of the wave equation in 3 dimensions in terms of Cauchy data $\Psi_0 = \psi(t = 0, x, y), \Psi_1 = \psi_{,t}(t = 0, x, y)$ (see [32]). We assume that the Cauchy data are axially symmetric and of compact support.

Investigating the behaviour of the solution at future null infinity $\mathcal{J}^+$, one finds

$$\psi(u, \rho) = \frac{f_0(u)}{\sqrt{\rho}} + \frac{1}{\sqrt{\rho}}\sum_{k=1}^{\infty}\frac{f_k(u)}{\rho^k}, \tag{5}$$

where $u = t - \rho$ and the coefficients f's are determined by the Cauchy data. The field equations imply

$$\gamma = \gamma_0 - 2\int_{-\infty}^{u}\left[\dot{f}_0(u)\right]^2 du - \sum_{k=1}^{\infty}\frac{h_k(u)}{(k+1)\rho^{k+1}}. \tag{6}$$

Thus, γ also admits an expansion in ρ^{-1}. It is straightforward to show that the spacetime admits a smooth future null infinity by setting $\tilde{\rho} = \rho^{-1}, \tilde{u} = u, \tilde{\varphi} = \varphi$ and rescaling g_{ab} by a conformal factor $\Omega = \tilde{\rho}$. Hence, the (2+1)-dimensional curved spacetime has a smooth (2-dimensional) null infinity. Penrose's picture works for arbitrarily strong initial data Ψ_0, Ψ_1. We can thus conclude that *cylindrical waves in (2+1)-dimensions give an explicit model of the Bondi-Penrose radiation theory which admits smooth null and timelike infinity for arbitrarily strong initial data.* There is no other such model available. The general results on the existence of $\mathcal{J}$ in 4 dimensions assume weak data.

4 On the Robinson-Trautman and type N twisting solutions

These spacetimes have attracted increased attention in the last decade – most notably in the work by Chruściel, and Chruściel and Singleton [34]. In these studies the Robinson-Trautman spacetimes have been shown to exist globally for all positive "times", and to converge asymptotically to a

Schwarzschild metric. Interestingly, the extension of the spacetimes across the "Schwarzschild-like" event horizon can only be made with a finite degree of smoothness. These studies are based on the derivation and analysis of an asymptotic expansion describing the long-time behaviour of the solutions of the nonlinear parabolic Robinson-Trautman equation.

In our recent work [35, 36], we studied Robinson-Trautman spacetimes with a positive cosmological constant Λ. The results proving the global existence and convergence of the solutions of the Robinson-Trautman equation can be taken over from the previous studies since Λ does not explicitly enter this equation. We have shown that, starting with arbitrary, smooth initial data at $u = u_0$, these cosmological Robinson-Trautman solutions converge exponentially fast to a Schwarzschild-de Sitter solution at large retarded times ($u \to \infty$). The interior of a Schwarzschild-de Sitter black hole can be joined to an "external" cosmological Robinson-Trautman spacetime across the horizon $\mathcal{H}^+$ with a higher degree of smoothness than in the corresponding case with $\Lambda = 0$. In particular, in the extreme case with $9\Lambda m^2 = 1$, in which the black hole and cosmological horizons coincide, the Robinson-Trautman spacetimes can be extended smoothly through $\mathcal{H}^+$ to the extreme Schwarzschild-de Sitter spacetime with the same values of Λ and m. However, such an extension is not analytic (and not unique).

We have also demonstrated that the cosmological Robinson-Trautman solutions are explicit models exhibiting the cosmic no-hair conjecture. As far as we are aware, these models represent the only exact analytic demonstration of the cosmic no-hair conjecture under the presence of gravitational waves. They also appear to be the only exact examples of a black hole formation in nonspherical spacetimes which are not asymptotically flat.

Type N twisting spacetimes

Since diverging, non-twisting Robinson-Trautman spacetimes of type N have singularities, there has been hope that if one admits a nonvanishing twist a more realistic radiative spacetime may exist.

Stephani [37], however, indicated, by constructing a general solution of the linearized equations, that singularities at infinity probably occur. More recently, Finley et al [38] found an approximate twisting type N solution up to the third order of iteration on the basis of which they suggested that it seems that the twisting, type N fields can describe a radiation field outside bounded sources. However, employing the Newman-Penrose formalism and MAPLE we succeeded in discovering a nonvanishing quartic invariant in the 2nd derivatives of the Riemann tensor [22], which shows that solutions of both Stephani and Finley et al contain singularities at large r. Very recently, Mac Alevey [39] argued that an approximate solution at any finite order can be calculated without occurrence of singularities. It is very likely, however, that a corresponding exact solution must contain singularities since Mason [40] proved that the only vacuum algebraically special spacetime that is asymptotically simple is the Minkowski space.

Even if a radiative solution with a complete smooth null infinity may be out of reach, it is of interest to construct radiative solutions which admit at least a global null infinity in the sense that its smooth cross sections exist although this null infinity is not necessarily complete. The only explicit examples of such solutions are spacetimes with boost-rotation symmetry.

5 The boost-rotation symmetric radiative spacetimes

I reviewed these spacetimes representing "uniformly accelerated objects" in various places (see e.g. [1, 2] and references therein); here I shall just mention some new results.

The unique role of the boost-rotation symmetric spacetimes is exhibited by a theorem [41] which roughly states that in axially symmetric, locally asymptotically flat electrovacuum spacetimes (in the sense that a null infinity satisfying Penrose's requirements exists, but it need not necessarily exist globally), the only additional symmetry that does not exclude radiation is the *boost* symmetry.

To prove such a result we start from the metric

$$\begin{aligned} ds^2 \;=\; & -\left(r^{-1}\,V\,e^{2\beta} - r^2 e^{2\gamma} U^2 \cosh 2\delta - r^2 e^{-2\gamma} W^2 \cosh 2\delta - 2r^2 UW \sinh 2\delta\right)\,du^2 \\ & -2e^{2\beta} du\;dr - 2r^2\left(e^{2\gamma} U \cosh 2\delta + W \sinh 2\delta\right)\;du\;d\theta \\ & -2r^2\left(e^{-2\gamma} W \cosh 2\delta + U \sinh 2\delta\right)\sin\theta\;du\;d\phi \\ & +r^2\left[\cosh 2\delta\left(e^{2\gamma} d\theta^2 + e^{-2\gamma}\sin^2\theta\;d\phi^2\right) + 2\sinh 2\delta\,\sin\theta\;d\theta\;d\phi\right]\;, \end{aligned} \tag{7}$$

where all funtions describing the metric and electromagnetic field tensor $F_{\mu\nu}$ are independent of ϕ. Assuming asymptotic expansions of these functions at large r with u, θ, ϕ fixed to guarantee asymptotic flatness, and using the outgoing radiation condition and the field equations, one finds the expansions to have specific forms. For example,

$$\begin{aligned} \gamma = \frac{c}{r} + (C - \tfrac{1}{6}c^3 - \tfrac{3}{2}cd^2)\frac{1}{r^3} + \ldots\;, \qquad & V = r - 2M + \ldots\;, \\ F_{02} = X \;+\; (\epsilon_{,\theta} - e_{,u})\frac{1}{r} + \ldots\;, \qquad & F_{03} = Y \;-\; \frac{f_{,u}}{r} + \ldots\;, \end{aligned} \tag{8}$$

where the 'coefficients' c, d, ... are functions of u and θ. The expansions are needed to further orders – see [41] for their rather lengthy forms. Let us only recall that the decrease of the Bondi mass, $m(u) = \frac{1}{2}\int_0^\pi M(u,\theta)\sin\theta d\theta$, is given by

$$m_{,u} = -\tfrac{1}{2}\int_0^\pi (c_{,u}^2 + d_{,u}^2 + X^2 + Y^2)\sin\theta d\theta \le 0\;, \tag{9}$$

where $c_{,u}$, $d_{,u}$, X, Y are the gravitational and electromagnetic news functions.

Now one writes down the Killing equations and solves them asymptotically in r^{-1}. One arrives at the following theorem [41]: Suppose that an axially symmetric electrovacuum spacetime admits a "piece" of $\mathcal{J}^+$ in the sense that the Bondi-Sachs coordinates can be introduced in which the metric takes the form (7), with the asymptotic form of the metric and electromagnetic field given by (8). If this spacetime admits an additional Killing vector forming with the axial Killing vector a two-dimensional Lie algebra, then the additional Killing vector has asymptotically the form

$$\eta^\alpha = [-ku\cos\theta + \alpha(\theta),\; kr\cos\theta + \mathcal{O}(r^0),\; -k\sin\theta + \mathcal{O}(r^{-1}),\; \mathcal{O}(r^{-1})]\;, \tag{10}$$

where k is a constant. For $k = 0$ it generates asymptotically translations (function α has then a specific form). For $k \neq 0$ it is the boost Killing field.

The case of translations is analyzed in detail in [42]. Theorem 1, precisely formulated and proved there, states that if an asymptotically translational Killing vector is spacelike, then null infinity is singular at some $\theta \neq 0, \pi$; if it is null, null infinity is singular at $\theta = 0$ or π. The first case corresponds to cylindrical waves, the second case to a plane wave propagating along the symmetry axis. We refer to [42] for the case when there is also a cosmic string present along the symmetry axis. The case of timelike Killing vector is described by Theorem 2 (proved also in [42]): If an axisymmetric electrovacuum spacetime with a non-vanishing Bondi mass admits an asymptotically translational Killing vector and a complete cross section of $\mathcal{J}^+$, then the translational Killing vector is timelike and spacetime is thus stationary.

The case of the boost Killing vector ($k \neq 0$) is thoroughly analyzed in [41]. The general functional forms of the news functions (both gravitational and electromagnetic), and of the mass aspect and total Bondi mass of boost-rotation symmetric spacetimes are there given. Very recently these results were obtained [43] by using the Newman-Penrose formalism and under more general assumptions (for example, $\mathcal{J}$ could in principle be polyhomogeneous).

The general structure of the boost-rotation symmetric spacetimes with hypersurface orthogonal Killing vectors was analyzed in detail in [44]. Their radiative properties, including explicit construction of radiation patterns and of Bondi mass for the specific boost-rotation symmetric solutions were investigated in several works – we refer to the reviews [1, 2] and [45] for details. There also the role of the boost-rotation symmetric spacetimes in such diverse fields like numerical relativity and quantum production of black-hole pairs is noticed and references are given.

Here I would like to mention yet a recent progress in understanding specific boost-rotation symmetric spacetimes with Killing vectors which are *not* hypersurface orthogonal. This is the *spinning* C-metric (see e.g. [3]). It was discovered by Plebański and Demaiński as a generalization of the standard C-metric which is known to represent uniformly accelerated non-rotating black holes. In [46] we first transformed the metric into Weyl coordinates, and then found a transformation which brings it into the canonical form of the radiative spacetimes with the boost-rotation symmetry:

$$\begin{aligned} ds^2 &= e^{\lambda} d\rho^2 + \rho^2 e^{-\mu} d\phi^2 \\ &+ (z^2 - t^2)^{-1} \left[(e^{\lambda} z^2 - e^{\mu} t^2) dz^2 - 2zt(e^{\lambda} - e^{\mu}) dz\, dt + (e^{\lambda} t^2 - e^{\mu} z^2) dt^2 \right] \\ &- 2\mathcal{A} e^{\mu} (z dt - t dz) d\phi - \mathcal{A}^2 e^{\mu} (z^2 - t^2) d\phi^2 \ , \end{aligned} \tag{11}$$

where functions e^{μ}, e^{λ} and $\mathcal{A}$ are given in terms of (t, ρ, z) in a somewhat complicated but explicit manner. This metric can represent two uniformly accelerated, spinning black holes, either connected by a conical singularity, or with conical singularities extending from each of them to infinity. The behaviour of the curvature invariants clearly indicates the presence of a non-vanishing radiation field (see Figure 5 in [46]). The spinning C-metric is the only explicitly known example with two Killing vectors which are not hypersurface orthogonal, in which one can give arbitrarily strong initial data on a hyperboloid "above the roof" ($t > |z|$) which evolve into the radiative spacetime with smooth $\mathcal{J}^+$.

6 Inhomogeneous cosmologies and gravitational waves

Among the known vacuum inhomogeneous models, the *Gowdy solutions* (see e.g. [3]) have played the most distinct role. They belong to the class of solutions with two commuting spacelike Killing vectors. Within a cosmological context, they form a subclass of a wider class of G_2 *cosmologies* – as are now commonly denoted models which admit an Abelian group G_2 of isometries with orbits being spacelike 2-surfaces. A 2-surface with a 2-parameter isometry group must be a space of constant curvature, and since neither a 2-sphere nor a 2-hyperboloid possess 2-parameter subgroups, it must be intrinsically flat. If the 2-surface is an Euclidean plane or a cylinder, then one speaks about planar or cylindrical universes. Gowdy universes are compact – the group orbits are 2-tori T^2.

The metrics with two spacelike Killing vectors are often called the generalized Einstein-Rosen metrics as, for example, by Carmeli, Charach and Malin [47] in their comprehensive survey of inhomogeneous cosmological models of this type. In dimensionless coordinates (t, z, x^1, x^2), the line element can be written as $(A, B = 1, 2)$

$$ds^2/L^2 \;=\; e^F(-dt^2 + dz^2) + \gamma_{AB}dx^A dx^B, \tag{12}$$

where L is a constant length, F and γ_{AB} depend on t and z only, and thus the spacelike Killing vectors are ${}^{(1)}\xi^\alpha = (0, 0, 1, 0), {}^{(2)}\xi^\alpha = (0, 0, 0, 1)$.

Let us mention some recent developments in which the Gowdy models have played a role. Gowdy-type models have been used to study the propagation and collision of gravitational waves with toroidal wavefronts in the FRW closed universes with a stiff fluid [27]. In the standard Gowdy spacetimes it is assumed that the "twists" associated with the isometry group on T^2 vanish. In [48] the generalized Gowdy models without this assumption are considered, and their global existence in time is proved.

As both interesting and non-trivial models, the Gowdy spacetimes have recently attracted the attention of mathematical and numerical relativists. Chruściel, Isenberg and Moncrief [49] proved that Gowdy spacetimes developed from a dense subset in the initial data set cannot be extended past their singularities, i.e. in "most" Gowdy models the strong cosmic censorship is satisfied. On cosmic censorship and spacetime singularities, especially in the context of compact cosmologies, we refer to [50]. This review shows how intuition gained from such solutions as the Gowdy models or the Taub-NUT spaces, when combined with new mathematical ideas and techniques, can produce rigorous results with a generality out of reach until recently. To such results belongs also the very recent work of Kichenassamy and Rendall [51] on the sufficiently general class of solutions (containing the maximum number of arbitrary functions) representing unpolarized Gowdy spacetimes. The new mathematical technique, the so called Fuchsian algorithm, enables one to construct singular (and nonsingular) solutions of partial differential equations with a large number of arbitrary functions, and thus provide a description of singularities. Applying the Fuchsian algorithm to Einstein's equations for Gowdy spacetimes with topology T^3, Kichenassamy and Rendall have proved that general solutions behave at the (past) singularity in a Kasner-like manner, i.e. they are asymptotically velocity dominated with a diverging curvature invariant. One needs an additional magnetic field not aligned with the two Killing vectors of the Gowdy unpolarized spacetimes in order to get a general oscillatory (Mixmaster) approach to a singularity, as shown by the numerical calculations [52].

Some metrics can be considered as exact *"gravitational solitons"* propagating on a cosmological background. Verdaguer [53] prepared a very complete review of solitonic cosmological solutions admitting two spacelike Killing vector fields. Recently, differential conservation laws for large perturbations of gravitational field with respect to a given curved background have been fomulated [54]. They should bring more light also on various solitonic models in cosmology.

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