# Bayesian source separation with mixture of Gaussians prior for sources and Gaussian prior for mixture coefficients

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**Abstract.** In this contribution, we present new algorithms to source separation for the case of noisy instantaneous linear mixture, within the Bayesian statistical framework. The source distribution prior is modeled by a mixture of Gaussians [1] and the mixing matrix elements distributions by a Gaussian [2]. We model the mixture of Gaussians hierarchically by mean of hidden variables representing the labels of the mixture. Then, we consider the joint a posteriori distribution of sources, mixing matrix elements, labels of the mixture and other parameters of the mixture with appropriate prior probability laws to eliminate degeneracy of the likelihood function of variance parameters and we propose two iterative algorithms to estimate jointly sources, mixing matrix and hyperparameters: Joint MAP (Maximum *a posteriori*) algorithm and penalized EM algorithm. The illustrative example is taken in [3] to compare with other algorithms proposed in literature.

# **PROBLEM DESCRIPTION**

We consider a linear instantaneous mixture of n sources. Observations could be corrupted by an additive noise. This noise may represent measurement errors or model incertainty:

$$\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{\epsilon}(t), \qquad t = 1, ..., T$$
(1)

where  $\boldsymbol{x}(t)$  is the  $(m \times 1)$  measurement vector,  $\boldsymbol{s}(t)$  is the  $(n \times 1)$  source vector which components have to be separated,  $\boldsymbol{A}$  is the mixing matrix of dimension  $(m \times n)$  and  $\epsilon(t)$ represents noise affecting the measurements. We assume that the  $(m \times T)$  noise matrix  $\epsilon(t)$  is statistically independent of sources, centered, white and Gaussian with known variance  $\sigma_{\epsilon}^2 \boldsymbol{I}$ . We note  $\boldsymbol{s}_{1..T}$  the matrix  $n \times T$  of sources and  $\boldsymbol{x}_{1..T}$  the matrix  $m \times T$  of data.

Source separation problem consists of two sub-problems: Sources restoration and mixing matrix identification. Therefore, three directions can be followed:

- 1. *Supervised learning*: Identify *A* knowing a training sequence of sources *s*, then use it to reconstruct the sources.
- 2. Unsupervised learning: Identify A directly from a part or the whole observations and then use it to recover s.

#### 3. Unsupervised joint estimation: Estimate jointly s and A

In the following, we investigate the third direction. This choice is motivated by practical cases where sources and mixing matrix are unknown.

This paper is organised as follows: We begin in section II by proposing a Bayesian approach to source separation. We set up the notations, present the prior laws of the sources and the mixing matrix elements and present the joint MAP estimation algorithm assuming known hyperparameters. We introduce, in section III, a hierarchical modelisation of the sources by mean of hidden variables representing the labels of the mixture of Gaussians in the prior modeling and present a version of JMAP using the estimation of these hidden variables (classification) as an intermediate step. In both algorithms, we assumed known the hyperparameters which is not realistic in applications. That is why, in section IV, we present an original method for the estimation of hyperparameters which takes advantages of using this hierarchical modeling. Finally, since EM algorithm has been used extensively in source separation [4], we considered this algorithm and propose, in section V, a penalized version of the EM algorithm for source separation. This penalization of the likelihood function is necessary to eliminate its degeneracy when some variances of Gaussian mixture approche zero [5]. Each section is supported by one typical simulation result and partial conclusion. At the end, we compare the two last algorithms.

# **BAYESIAN APPROACH TO SOURCE SEPARATION**

Given the observations  $x_{1..T}$ , the joint *a posteriori* distribution of unknown variables  $s_{1..T}$  and A is:

$$p(\mathbf{A}, \mathbf{s}_{1..T} | \mathbf{x}_{1..T}) \propto p(\mathbf{x}_{1..T} | \mathbf{A}, \mathbf{s}_{1..T}) p(\mathbf{A}) p(\mathbf{s}_{1..T})$$
 (2)

where  $p(\mathbf{A})$  and  $p(\mathbf{s}_{1..T})$  are the prior distributions through which we modelise our *a* priori information about sources *s* and mixing matrix  $\mathbf{A}$ .  $p(\mathbf{x}_{1..T}|\mathbf{A}, \mathbf{s}_{1..T})$  is the joint likelihood distribution. We have, now, three directions:

1. First, integrate (2) with respect to  $s_{1..T}$  to obtain the marginal in A and then estimate A by:

$$\widehat{\boldsymbol{A}} = \operatorname*{arg\,max}_{\boldsymbol{A}} \{ J(\boldsymbol{A}) = \ln p(\boldsymbol{A} | \boldsymbol{x}_{1..T}) \}$$
(3)

2. Second, integrate (2) with respect to A to obtain the marginal in  $s_{1.T}$  and then estimate  $s_{1.T}$  by:

$$\widehat{\boldsymbol{s}}_{1..T} = \underset{\boldsymbol{s}_{1..T}}{\operatorname{arg\,max}} \left\{ J(\boldsymbol{s}_{1..T}) = \ln p(\boldsymbol{s}_{1..T} | \boldsymbol{x}_{1..T}) \right\}$$
(4)

3. Third, estimate jointly  $s_{1..T}$  and A:

$$(\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{s}}_{1..T}) = \underset{(\boldsymbol{A}, \boldsymbol{s}_{1..T})}{\operatorname{arg\,max}} \{ J(\boldsymbol{A}, \boldsymbol{s}_{1..T}) = \ln p(\boldsymbol{A}, \boldsymbol{s}_{1..T} | \boldsymbol{x}_{1..T}) \}$$
(5)

#### Choice of *a priori* distributions

The *a priori* distribution reflects our knowledge concerning the parameter to be estimated. Therefore, it must be neither very specific to a particular problem nor too general (uniform) and non informative. A parametric model for these distributions seems to fit this goal: Its stucture expresses the particularity of the problem and its parameters allow a certain flexibility.

Sources *a priori*: For sources *s*, we choose a mixture of Gaussians [1]:

$$p(s_j) = \sum_{i=1}^{q_j} \alpha_{ji} \mathcal{N}(m_{ji}, \sigma_{ji}^2), \qquad j = 1..n$$
(6)

Hyperparameters  $q_j$  are supposed to be known. This choice was motivated by the following points:

- It represents a general class of distributions and is convenient in many digital communications and image processing applications.
- For a Gaussian likelihood  $p(\mathbf{x}_{1..T}|\mathbf{s}_{1..T}, \mathbf{A})$  (considered as a function of  $\mathbf{s}_{1..T}$ ), the *a posteriori* law remains in the same class (conjugate prior). We then have only to update the parameters of the mixture with the data.

**Mixing matrix** *a priori*: To account for some model uncertainty, we assign a Gaussian prior law to each element of the mixing matrix *A*:

$$p(\boldsymbol{A}_{ij}) = \mathcal{N}(\boldsymbol{M}_{ji}, \sigma_{a,ij}^2)$$
(7)

which can be interpreted as knowing every element  $(M_{ji})$  with some uncertainty  $(\sigma_{a,ij}^2)$ . We underline here the advantage of estimating the mixing matrix A and not a separating matrix B (inverse of A) which is the case of almost all the existing methods for source separation (see for example [6]). This approach has at least two advantages: (i) A does not need to be invertible  $(n \neq m)$ , (ii) naturally, we have some *a priori* information on the mixing matrix not on its inverse which may not exist.

#### JMAP algorithm

We propose an alternating iterative algorithm to estimate jointly  $s_{1..T}$  and A by extremizing the log-posterior distribution:

$$\begin{pmatrix}
\widehat{\boldsymbol{s}}_{1..T}^{(k)} = \arg \max_{\boldsymbol{s}_{1..T}} \ln p\left(\widehat{\boldsymbol{A}}^{(k-1)}, \boldsymbol{s}_{1..T} | \boldsymbol{x}_{1..T}\right) \\
\widehat{\boldsymbol{A}}^{(k)} = \arg \max_{\boldsymbol{A}} \ln p\left(\boldsymbol{A}, \widehat{\boldsymbol{s}}_{1..T}^{(k)} | \boldsymbol{x}_{1..T}\right)
\end{cases}$$
(8)

In the following, we suppose that sources are white and spatially independant. This assumption is not necessary in our approach but we start from here to be able to compare later with other classical methods in which this hypothesis is fundamental.

With this hypothesis, in step (k+1), the criterion to optimize with respect to  $s_{1...T}$  is:

$$J(\boldsymbol{s}_{1..T}) = \sum_{t=1}^{T} \left[ \ln p\left(\boldsymbol{x}(t) | \widehat{\boldsymbol{A}}^{(k)}, \boldsymbol{s}(t)\right) + \sum_{j=1}^{n} \ln p_j\left(\boldsymbol{s}_j(t)\right) \right]$$
(9)

Therefore, the optimisation is done independently at each time t:

$$\widehat{\boldsymbol{s}}(t)^{(k+1)} = \operatorname*{arg\,max}_{\boldsymbol{s}(t)} \left\{ \ln p\left(\boldsymbol{x}(t) | \widehat{\boldsymbol{A}}^{(k)}, \boldsymbol{s}\right) + \sum_{j=1}^{n} \ln p_j\left(\boldsymbol{s}_j(t)\right) \right\}$$
(10)

The *a posteriori* distribution of *s* is a mixture of  $\prod_{j=1}^{n} q_j$  Gaussians. This leads to a high computational cost. To obtain a more reasonable algorithm, we propose an iterative scalar algorithm. The first step consists in estimating each source component knowing the other components estimated in the previous iteration:

$$\widehat{\boldsymbol{s}}_{j}(t)^{(k+1)} = \operatorname*{arg\,max}_{\boldsymbol{s}_{j}(t)} \{ \ln p\left(\boldsymbol{s}_{j}(t) | \boldsymbol{x}(t), \widehat{\boldsymbol{A}}^{(k)}, \widehat{\boldsymbol{s}}_{l \neq j}(t)^{(k)} \right) \}$$
(11)

The *a posteriori* distribution of  $s_j$  is a mixture of  $q_j$  Gaussians:  $\sum_{z=1}^{q_j} \alpha'_{jz} \mathcal{N}(m'_{jz}, {\sigma'_{jz}}^2)$ , with:

$$\begin{cases} m'_{jz} = \frac{\sigma_j^2 m_{jz} + \sigma_{jz}^2 m_j}{\sigma_j^2 + \sigma_{jz}^2} \\ \sigma'_{jz}^2 = \frac{\sigma_j^2 \sigma_{jz}^2}{\sigma_j^2 + \sigma_{jz}^2} \\ \alpha'_{jz} = \alpha_{jz} \sqrt{\frac{1}{\sigma_{jz}^2 + \sigma_j^2}} \exp\left[\frac{-1}{2} \frac{1}{\sigma_{jz}^2 + \sigma_j^2} (m_j - m_{jz})^2\right] \end{cases}$$
(12)

where

$$\begin{cases} \sigma_j^2 = \frac{\sigma_\epsilon^2}{\sum_{i=1}^m \boldsymbol{A}_{ij}^2} \\ m_j = \frac{\sum_{i=1}^n \boldsymbol{A}_{ij} \left( \boldsymbol{x}_i - \widehat{\boldsymbol{x}}_i \right)}{\sum_{i=1}^m \boldsymbol{A}_{ij}^2} \\ \widehat{\boldsymbol{x}}_i = \sum_{l \neq j} \boldsymbol{A}_{il} \boldsymbol{s}_l \end{cases}$$
(13)

If the means  $m'_{jz}$  aren't close to each other, we are in the case of a multi-modal distribution. The algorithm to estimate  $s_j$  is to first compute  $\hat{x}_i$ , i = 1, ..., m,  $m_j$  and

 $\sigma_j^2$  by (13) and then  $\alpha'_{jz}$ ,  ${\sigma'_{jz}}^2$  and  $m'_{jz}$  by (12), and select the  $m'_{jz}$  for which the ratio  $\frac{\alpha'_{jz}}{\sigma'_{jz}}$  is the greatest one.

After a full update of all sources  $s_{1..T}$ , the estimate of A is obtained by optimizing:

$$J(\boldsymbol{A}) = \sum_{t=1}^{T} \ln p\left(\boldsymbol{x}(t) | \boldsymbol{A}, \widehat{\boldsymbol{s}}^{k+1}(t)\right) + \ln p\left(\boldsymbol{A}(t)\right) + cte$$
(14)

which is quadratic in elements of A. The gradient has then a simple expression:

$$\frac{\partial J(\boldsymbol{A})}{\partial \boldsymbol{A}_{i,j}} = \sum_{t=1}^{T} \frac{1}{\sigma_{\epsilon}^2} \widehat{\boldsymbol{s}}_j^{k+1}(t) \left( \boldsymbol{x}_i(t) - \left[ \boldsymbol{A} \, \widehat{\boldsymbol{s}}^{k+1}(t) \right]_i \right) - \frac{1}{\sigma_{a;i,j}^2} \left( \boldsymbol{A}_{i,j} - \boldsymbol{M}_{i,j} \right)$$
(15)

Cancelling the gradient to zero and defining  $\Lambda_{i,j} = \frac{\sigma_{\epsilon}^2}{\sigma_{a;i,j}^2}$ , we obtain the following relation:

$$\left[\sum_{t=1}^{T} \left(\boldsymbol{x}(t) - \boldsymbol{A}\widehat{\boldsymbol{s}}^{k+1}(t)\right)\widehat{\boldsymbol{s}}^{k+1}(t)^{T}\right]_{i,j} - \boldsymbol{\Lambda}_{i,j}\left(\boldsymbol{A}_{i,j} - \boldsymbol{M}_{i,j}\right) = 0$$
(16)

We define the operator **Vect** transforming a matrix to a vector by the concatenation of the transposed rows. Operator **Mat** is the inverse of **Vect**. Applying operator **Vect** to relation (16), we obtain the following expression:

$$Vect\left(\boldsymbol{x}_{1..T}(\widehat{\boldsymbol{s}}_{1..T}^{k+1})^{T}\right) + \mu Vect(M) = (\mu + S^{*}) Vect\boldsymbol{A}$$
(17)

where  $\mu$  is a diagonal matrix  $(nm \times nm)$  which diagonal vector is  $Vect((\Lambda_{i,j})_{i=1..m,j=1..n})$ and  $S^*$  the matrix  $(nm \times nm)$  with block diagonals  $\hat{s}_{1..T}\hat{s}_{1..T}^T$  estimated at iteration (k+1). We have finally the explicit estimation of A:

$$\widehat{\boldsymbol{A}}^{k+1} = Mat\left(\left[\mu + S^*\right]^{-1} \left[\mu Vect(\boldsymbol{M}) + Vect\left(\boldsymbol{x}_{1..T}(\widehat{\boldsymbol{s}}_{1..T}^{k+1})^T\right)\right]\right)$$
(18)

To show the faisability of this algorithm, we consider in the following a telecommunication example. For this, we simulated synthetic data with sources described by a mixture of 4 Gaussians centered at -3, -1, 1 and 3, with the same variance 0.01 and weighted by 0.3, 0.1, 0.4 and 0.2. The unknown mixing matrix is  $\mathbf{A} = \begin{pmatrix} 1 & -0.6 \\ 0.6 & 1 \end{pmatrix}$ . We fixed the *a priori* parameters of  $\mathbf{A}$  to:  $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and  $\mathbf{\Lambda} = \begin{pmatrix} 150 & 0.009 \\ 0.009 & 150 \end{pmatrix}$ , meaning that we are nearly sure of diagonal values but we are very uncertain about the other elements of  $\mathbf{A}$ . Noise of variance  $\sigma_{\epsilon}^2 = 1$  was added to the data. The figure 1 illustrates the ability of the algorithm to perform the separation. However, we note that estimated sources are very centered arround the means. This is because we fixed very low values for the *a priori* variances of Gaussian mixture. Thus, the algorithm is sensitive to the *a priori* parameters and exploitation of data is useful. We will see in section IV how to deal with this issue.



Figure 1- Results of separation with QAM-16 (Quadratic Amplitude Modulation) using JMAP algorithm: (a) phase space distribution of sources, (b) mixed signals, and (c) separated sources

Now, we are going to re-examine closely the expression for the *a posteriori* distribution of sources. It's a multi-modal distribution if the Gaussian means aren't too close. The maximum of this distribution doesn't correspond, in general, to the maximum of the most probable Gaussian. So, we intend to estimate first, at each time t, the *a priori* Gaussian law according to which the source s(t) is generated (classification) and then estimate s(t) as the mean of the *a posteriori* Gaussian. This leads us to the introduction of hidden variables and hierarchical modelization.

# **HIDDEN VARIABLES**

The *a priori* distribution of the component  $s_j$  is  $p(s_j) = \sum_{i=1}^{q_j} \alpha_{ji} \mathcal{N}(m_{ji}, \sigma_{ji}^2)$ . We consider now the hidden variable  $z_j$  taking its values in the discrete set  $\mathcal{Z}_j = (1, \ldots, q_j)$  so each source can belong to one of the  $q_j$  sources, with  $\alpha_{ji} = p(z_j = i)$ . Given  $z_j = i$ ,  $s_j$  is normal  $\mathcal{N}(m_{ji}, \sigma_{ji}^2)$ . We can extend this notion to vectorial case by considering the vector  $\boldsymbol{z} = [z_1, \ldots, z_n]$  taking its values in the set  $\mathcal{Z} = \prod_{j=1}^n \mathcal{Z}_j$ . The *s* distribution given  $\boldsymbol{z}$  is a normal law  $p(\boldsymbol{s}|\boldsymbol{z}) = \mathcal{N}(\boldsymbol{m}_z, \Gamma_z)$  with:

$$\boldsymbol{m}_{z} = [m_{1z_{1}}, m_{2z_{2}}, \dots, m_{nz_{n}}]$$
 (19)

$$\Gamma_{\mathbf{z}} = \mathbf{diag}(\sigma_{1\mathbf{z}_1}^2, \sigma_{2\mathbf{z}_2}^2, \dots, \sigma_{\mathbf{n}\mathbf{z}_n}^2)$$
(20)

The marginal *a priori* law of *s* is the mixture of  $\prod_{j=1}^{n} q_j$  Gaussians:

$$p(\boldsymbol{s}) = \sum_{\boldsymbol{z} \in \mathcal{Z}} p(\boldsymbol{z}) p(\boldsymbol{s} | \boldsymbol{z})$$
(21)

We can re-interpret this mixture by considering it as a discrete set of couples  $(\mathcal{N}_z, p(z))$ (see Figure 2). Sources which belong to this class of distributions are generated as follows: First, generate the hidden variable  $z \in \mathbb{Z}$  according p(z) and then, given this z, generate s according to  $\mathcal{N}_z$ . This model can be extended to include continuous values of z (also continuous distribution f(z)) and then to take account of infinity of distributions in only one class (see Figure 2).



Figure 2- Hierarchical modelization with hidden variables

# a posteriori distribution of sources

In the following, we suppose that mixing matrix is known. The joint law of s, z and x can be factorized in two forms: p(s, z, x) = p(x|s)p(s|z)p(z) or p(s, z, x) = p(s|x, z)p(z|x)p(x). Thus, the marginal *a posteriori* law has two forms:

$$p(\boldsymbol{s}|\boldsymbol{x}) = \sum_{\boldsymbol{z}\in\mathcal{Z}} \frac{p(\boldsymbol{z})p(\boldsymbol{x}|\boldsymbol{s})p(\boldsymbol{s}|\boldsymbol{z})}{p(\boldsymbol{x})}$$
(22)

or

$$p(\boldsymbol{s}|\boldsymbol{x}) = \sum_{\boldsymbol{z}\in\mathcal{Z}} p(\boldsymbol{z}|\boldsymbol{x}) p(\boldsymbol{s}|\boldsymbol{x}, \boldsymbol{z})$$
(23)

We note in the second form that the *a posteriori* is in the same class that of the *a priori* (same expressions but conditionally to  $\boldsymbol{x}$ ). This is due to the fact that mixture of Gaussians is a conjugate prior for Gaussian likelihood. Our strategy of estimation is based on this remark: The sources are modeled hierarchically, we estimate them hierarchically; we begin by estimating the hidden variable using  $p(\boldsymbol{z}|\boldsymbol{x})$  and then estimate sources using  $p(\boldsymbol{s}|\boldsymbol{x}, \boldsymbol{z})$  which is Gaussian of mean  $\boldsymbol{\theta}_{xz}$ :

$$\boldsymbol{\theta}_{\boldsymbol{x}\boldsymbol{z}} = \boldsymbol{m}_{\boldsymbol{z}} + \Gamma_{\mathbf{z}} \boldsymbol{A}^{\mathbf{t}} \mathbf{R}_{\mathbf{z}} (\boldsymbol{x} - \boldsymbol{A} \boldsymbol{m}_{\mathbf{z}}) \tag{24}$$

and variance  $V_{xz}$ :

$$\boldsymbol{V}_{xz} = \boldsymbol{\Gamma}_{\mathbf{z}} - \boldsymbol{\Gamma}_{\mathbf{z}} \boldsymbol{A}^{\mathsf{t}} \mathbf{R}_{\mathbf{z}} \boldsymbol{A} \boldsymbol{\Gamma}_{\mathbf{z}}$$
(25)

where,

$$\boldsymbol{R}_{z} = (\boldsymbol{A}\Gamma_{z}\boldsymbol{A}^{t} + \boldsymbol{R}_{n})^{-1}$$
(26)

and  $R_n$  represent the noise covariance.

Now we have to estimate z by using p(z|x) which is obtained by integrating the joint *a posteriori* of z and s with respect to s:

$$p(\boldsymbol{z}|\boldsymbol{x}) = \int p(\boldsymbol{z}, \boldsymbol{s}|\boldsymbol{x}) d\boldsymbol{s} \propto p(\boldsymbol{z}) \int p(\boldsymbol{x}|\boldsymbol{s}) p(\boldsymbol{s}|\boldsymbol{z}) d\boldsymbol{s}$$
(27)

The expression to integrate is Gaussian in s. The result is immediate:

$$p(\boldsymbol{z}|\boldsymbol{x}) \propto p(\boldsymbol{z}) | \boldsymbol{\Gamma}_{z} |^{-\frac{1}{2}} | \boldsymbol{V}_{xz} |^{\frac{1}{2}} \exp[K_{zx}]$$
(28)

where:

$$\left\{\begin{array}{ll} \boldsymbol{K}_{zx} &= & -\frac{1}{2}(\boldsymbol{A}\boldsymbol{m}_{z}-\boldsymbol{x})^{t}\boldsymbol{Q}_{xz}(\boldsymbol{A}\boldsymbol{m}_{z}-\boldsymbol{x})\\ \boldsymbol{Q}_{xz} &= & (\boldsymbol{I}-\boldsymbol{R}_{z}\boldsymbol{A}\boldsymbol{\Gamma}_{z}\boldsymbol{A}^{t})\boldsymbol{R}_{n}^{-1}(\boldsymbol{I}-\boldsymbol{A}\boldsymbol{\Gamma}_{z}\boldsymbol{A}^{t}\boldsymbol{R}_{z})+\boldsymbol{R}_{z}\boldsymbol{A}\boldsymbol{\Gamma}_{z}\boldsymbol{A}^{t}\boldsymbol{R}_{z}\end{array}\right\}$$
(29)

If now we consider the whole observations, the law of  $z_{1..T}$  is:

$$p(\boldsymbol{z}_{1..T}|\boldsymbol{x}_{1..T}) \propto p(\boldsymbol{z}_{1..T}) \int p(\boldsymbol{x}_{1..T}|\boldsymbol{s}_{1..T}) p(\boldsymbol{s}_{1..T}|\boldsymbol{z}_{1..T}) d\boldsymbol{s}_{1..T}$$
 (30)

Supposing that z(t) are *a priori* independent, the last relation becomes:

$$p(\boldsymbol{z}_{1..T}|\boldsymbol{x}_{1..T}) \propto \Pi_{t=1}^{T} \left\{ p(\boldsymbol{z}(t)) \int p(\boldsymbol{x}(t)|\boldsymbol{s}(t)) p(\boldsymbol{s}(t)|\boldsymbol{z}(t)) d\boldsymbol{s}(t) \right\}$$
(31)

Estimation of  $z_{1..T}$  is then performed observation by observation:

$$\underset{\boldsymbol{z}_{1..T}}{\operatorname{arg\,max}} p(\boldsymbol{z}_{1..T} | \boldsymbol{x}_{1..T}) = \left( \underset{\boldsymbol{z}(t)}{\operatorname{arg\,max}} p(\boldsymbol{z}(t) | \boldsymbol{x}(t)) \right)_{t=1..T}$$
(32)

#### **Hierarchical JMAP algorithm**

Taking into account of this hierarchical model, the JMAP algorithm is implemented in three steps. At iteration (k):

1. First, estimate the hidden variable  $\hat{z}_{MAP}$  (combinatory estimation) given observations and mixing matrix estimated in the previous iteration:

$$\widehat{\boldsymbol{z}}_{MAP}^{(k)}(t) = \operatorname*{arg\,max}_{\boldsymbol{z}(t)} \left\{ p\left(\boldsymbol{z}(t) | \boldsymbol{x}(t), \widehat{\boldsymbol{A}}^{(k-1)}\right) \right\}$$
(33)

- 2. Second, given the estimated  $\hat{z}_{MAP}^{(k)}$ , source vector *s* follows Gaussian law  $\mathcal{N}(\boldsymbol{\theta}_{x\hat{\boldsymbol{z}}_{MAP}^{(k)}}, \boldsymbol{V}_{x\hat{\boldsymbol{z}}_{MAP}^{(k)}})$  and then the source estimate is  $\boldsymbol{\theta}_{x\hat{\boldsymbol{z}}_{MAP}^{(k)}}$ .
- 3. Third, given the estimated sources  $\hat{s}^k$ , mixing matrix is evaluated as in the algorithm of section II.

We evaluated this algorithm using the same synthetic data as in section 2. Separation was robust as shown in Figure 3:



Figure 3- Results of separation with QAM-16 using Hierarchical JMAP algorithm: (a) phase space distribution of sources, (b) mixed signals, and (c) separated sources

The Bayesian approach allows us to express our *a priori* information via parametric prior models. However, in general, we may not know the parameters of the *a priori* distributions. This is the task of the next section where we estimate the unknown hyperparameters always in a Bayesian framework.

## HYPERPARAMETERS ESTIMATION

The hyperparameters considered here are the means and the variances of Gaussian mixture prior of sources:  $s_j \sim \sum_{z=1}^{q_j} \prod_{jz} \mathcal{N}\left(m_{jz}, \frac{1}{\psi_{jz}}\right)$ ,  $j = 1, \ldots, n$ . We develop, in the following, a novel method to extract the hyperparameters from the observations  $x_{1..T}$ . The main idea is: conditioned on the hidden variables  $(z_j)_{1..T} = [z_j(1), \ldots, z_j(T)]$ , hyperparameters  $m_{jz}$  and  $\psi_{jz}$  for  $z \in \mathcal{Z}_j = (1, \ldots, q_j)$  are means and variances of a Gaussian distribution. Thus, given the vector  $(z_j)_{1..T} = [z_j(1), \ldots, z_j(T)]$ , we can perform a partition of the set  $\mathcal{T} = [1, \ldots, T]$  into sub-sets  $\mathcal{T}_z$  as:

$$\mathcal{T}_z = \{ t \,|\, z_j(t) = z \} \,, \, z \in \mathcal{Z}_j \tag{34}$$

This is the classification step.

Suppose now that mixing matrix A and components  $s_{l\neq j}$  are fixed and we are interested in the estimation of  $m_{jz}$  and  $\psi_{jz}$ . Let  $\theta_{jz} = (m_{jz}, \psi_{jz})$ .

The joint *a posteriori* law of  $s_i$  and  $\theta_{iz}$  given  $z_i$  at time t is:

$$p(\boldsymbol{s}_j, \theta_{jz} | \boldsymbol{x}, z_j) \propto p(\boldsymbol{x} | \boldsymbol{s}) p(\boldsymbol{s}_j | \theta_{jz}, z_j) p(\theta_{jz} | z_j)$$
(35)

 $p(\mathbf{s}_j | \theta_{jz}, z_j)$  is Gaussian of mean  $m_{jz}$  and inverted variance  $\psi_{jz}$ .  $p(\theta_{jz} | z_j) = p(\theta_{jz}) = p(m_{jz})p(\psi_{jz})$  is hyperparameters *a priori*. The marginal *a poste*- *riori* distribution of  $\theta_{jz}$  is obtained from previous relation by integration over  $s_j$ :

$$p(\theta_{jz} | \boldsymbol{x}, z_j) \propto p(\theta_{jz}) \int_{\boldsymbol{s}_j} p(\boldsymbol{x} | \boldsymbol{s}) p(\boldsymbol{s}_j | \theta_{jz}, z_j) d\boldsymbol{s}_j.$$
(36)

The expression inside the integral is proportional to the joint *a posteriori* distribution of  $(s_j, z_j)$  given x and  $\theta_{jz}$ , thus:

$$p(\theta_{jz} | \boldsymbol{x}, z_j) \propto p(\theta_{jz}) p(z_j | \boldsymbol{x}, \theta_{jz}).$$
(37)

where  $p(z_j | \boldsymbol{x}, \theta_{jz})$  is proportional to  $\alpha'_{jz}$  as defined in expression (12). Noting  $\phi_j = 1/\sigma_j^2$  and  $\psi_{jz} = 1/\sigma_{jz}^2$ , we have:

$$p(\theta_{jz} | \boldsymbol{x}, z_j) \propto p(\theta_{jz}) \sqrt{\frac{\phi_j \psi_{jz}}{\phi_j + \psi_{jz}}} \exp\left[-\frac{1}{2} \frac{\phi_j \psi_{jz}}{\phi_j + \psi_{jz}} (m_{jz} - m_j)^2\right]$$
(38)

Note that the likelihood is normal for means  $m_{jz}$  and Gamma for  $\lambda_{jz} = (\phi_j \psi_{jz}) / (\phi_j + \psi_{jz})$ .

Choosing a uniform *a priori* for the means, the estimate of  $m_{jz}$  is:

$$\widehat{m}_{jz}^{MAP} = \frac{\sum_{t \in \mathcal{T}_z} m_j(t)}{T_z}$$
(39)

For variances, we can choose (i) an inverted Gamma prior  $\mathcal{G}(\alpha,\beta)$  after developing the expression for  $\lambda_{jz}$  knowing the relative order of  $\psi_{jz}$  and  $\phi_j$  (to make  $\lambda_{jz}$  linear in  $\psi_{jz}$ ) or (ii) an a prior which is Gamma in  $\lambda_{jz}$ . These choices are motivated by two points: First, it is a proper prior which eliminate degenaracy of some variances at zero (It is shown in [5] that hyperparameter likelihood (noiseless case without mixing) is unbounded causing a variance degeneracy at zero). Second, it is a conjugate prior so estimation expressions remain simple to implement. The estimate of inverted variance (first choice when  $\psi_{jz}$  is the same order of  $\phi_j$ ) is:

$$\widehat{\psi}_{jz}^{MAP} = \frac{\alpha^{posteriori} - 1}{\beta^{posteriori}}$$
(40)  
with  $\alpha^{posteriori} = \alpha + \frac{T_z}{2}$  and  $\beta^{posteriori} = \beta + \frac{\sum_{t \in T_z} (m_j(t) - \widehat{m}_{jz}^{MAP})^2}{4}.$ 

#### **Hierarchical JMAP including estimation of hyperparameters**

Including the estimation of hyperparameters, the proposed hierarchical JMAP algorithm is composed of five steps:

1. Estimate hidden variables  $(\hat{z}_j)_{1..T}^{MAP}$  by:

$$(\hat{z}_{j})_{1..T}^{MAP} = (\operatorname*{arg\,max}_{z_{j}} p(z_{j} | \boldsymbol{x}(t), m_{jz}, \psi_{jz}, \boldsymbol{A}, \boldsymbol{s}_{l \neq j}))_{1..T}$$
(41)

which permits to estimate partitions:

$$\widehat{\mathcal{T}}_{z} = \left\{ t \,|\, (\widehat{z}_{j})^{MAP}(t) = z \right\} \tag{42}$$

This corresponds to the classification step in the previous algorithm

- 2. Given the estimate of partitions, hyperparameters  $\widehat{\psi}_{jz}^{MAP}$  and  $\widehat{m}_{jz}^{MAP}$  are updated according to equations (39) and (25). The following steps are the same as those in the previous proposed algorithm
- 3. Re-estimation of hidden variables  $(\hat{z}_j)_{1,T}^{MAP}$  given the estimated hyperparameters.
- 4. Estimation of sources  $(\hat{s})_{1..T}^{MAP}$ .
- 5. Estimation of mixing matrix  $(\widehat{A})^{MAP}$ .

#### Simulation results

To be able to compare the results obtained by this algorithm and the Penalized likelihood algorithm developed in the next section with the results obtained by some other classical methods, we generated data according to the example described in [3].

**Data generation**: 2-D sources, every component *a priori* is mixture of two Gaussians  $(\pm 1), \psi = 100$  for all Gaussians. Original sources are mixed with mixing matrix  $\mathbf{A} = \begin{pmatrix} 1 & -0.6 \\ 0.4 & 1 \end{pmatrix}$ . A noise of variance  $\sigma_{\epsilon}^2 = 0.03$  is added (SNR = 15 dB). Number of observations is 1000.

**Parameters:** 
$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
,  $\Lambda = \begin{pmatrix} 150 & 0.009 \\ 0.009 & 150 \end{pmatrix}$ ,  $\Pi = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$ ,  $\alpha = 200$  and  $\beta = 2$ .

**Initial conditions:**  $\mathbf{A}^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \psi^{(0)} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ m^{(0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \mathbf{s}^{(0)}$  generated according to  $\mathbf{s}_{j}^{(0)} \sim \sum_{z=1}^{q_{j}} \prod_{jz} \mathcal{N}(m_{jz}^{(0)}, \frac{1}{\psi_{iz}^{(0)}}).$ 

Sources are recovered with negligible mean quadratic error:  $MEQ(s_1) = 0.0094$  and  $MEQ(s_2) = 0.0097$ . The following figures illustrate separation results:

The non-negative performance index of [7] is used to chacarterize mixing matrix identification achievement:

$$ind(S = \widehat{\mathbf{A}}^{-1} \mathbf{A}) = \frac{1}{2} \left[ \sum_{i} \left( \sum_{j} \frac{|S_{ij}|^2}{max_l |S_{il}|^2} - 1 \right) + \sum_{j} \left( \sum_{i} \frac{|S_{ij}|^2}{max_l |S_{lj}|^2} - 1 \right) \right]$$

Figure 7*a* represents the index evolution through iterations. Note the convergence of JMAP algorithm since iteration 30 to a satisfactory value of -45 dB. For the same SNR, algorithms PWS, NS [3] and EASI [6] reach a value greater than -35 dB after 6000 observations. Figures 7*b* and 7*c* illustrate the identification of hyperparameters. We note the algorithm convergence to the original values (-1 for  $m_{11}$  and 100 for  $\psi_{11}$ ). In order to validate the idea of data classification before estimating hyperparameters,

we can visualize the evolution of classification error (number of data badly classified). Figure 7d shows that this error converges to zero at iteration 15. Then, after this iteration, hyperparameters identification is performed on the true classified data. Estimation of  $m_{jz}$  and  $\psi_{jz}$  takes into account only data which belong to this class and then it is not corrupted by other data which bring erroneous information on these hyperparameters.

$$\begin{cases} s_1(t) \\ s_2(t) \\ x_1(t) \\ x_2(t) \\ \\ \hat{s}_2(t) \\$$

Figure 4- Separation results with SNR = 15 dB



Figure 5- Separation results with SNR = 15 dB: Phase space distribution of sources, mixed signals and separated sources.



Figure 6- Separation results with SNR = 15 dB: Histograms of sources, mixed signals and separated sources.



Figure 7-a- Evolution of index through iterations





Figure 7-c- Identification of  $\psi_{11}$ 

Figure 7-d- Evolution of classification error

Thus, a joint estimation of sources, mixing matrix and hyperparameters is performed successfully with a JMAP algorithm. The EM algorithm was used in [4] to solve source

separation problem in a maximum likelihood context. We now use the EM algorithm in a Bayesian approach to take into account of our *a priori* information on the mixing matrix.

# PENALIZED EM

The EM algorithm has been used extensively in data analysis to find the maximum likelihood estimation of a set of parameters from given data [8]. Considering both the mixing matrix A and hyperparameters  $\theta$ , at the same level, being unknown parameters and complete data  $x_{1..T}$  and  $s_{1..T}$ . Complete data means jointly observed data  $x_{1..T}$  and unobserved data  $s_{1..T}$ . The EM algorithm is executed in two steps: (i) E-step (expectation) consists in forming the logarithm of the joint distribution of observed data x and hidden data s conditionally to parameters A and  $\theta$  and then compute its expectation conditionally to x and estimated parameters A' and  $\theta'$  (evaluated in the previous iteration), (ii) M-step (maximization) consists of the maximization of the obtained functional with respect to the parameters A and  $\theta$ :

1. E-step :

$$Q(\boldsymbol{A}, \boldsymbol{\theta} | \boldsymbol{A}', \boldsymbol{\theta}') = E_{\boldsymbol{x}, \boldsymbol{s}}[\log p(\boldsymbol{x}, \boldsymbol{s} | \boldsymbol{A}, \boldsymbol{\theta}) | \boldsymbol{x}, \boldsymbol{A}', \boldsymbol{\theta}']$$
(43)

2. M-step :

$$\left(\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{\theta}}\right) = \underset{(\boldsymbol{A}, \boldsymbol{\theta})}{\operatorname{arg\,max}} \left\{ Q\left(\boldsymbol{A}, \boldsymbol{\theta} \,|\, \boldsymbol{A}', \boldsymbol{\theta}'\right) \right\}$$
(44)

Recently, in [4], an EM algorithm has been used in source separation with mixture of Gaussians as sources prior. In this work, we show that:

- 1. This algorithm fails in estimating variances of Gaussian mixture. We proved that this is because the degeneracy of the estimated variance to zero.
- 2. The computational cost of this algorithm is very high.
- 3. The algorithm is very sensitive to initial conditions.
- 4. In [4], there's neither an *a priori* distribution on the mixing matrix A or on the hyperparameters  $\theta$ .

Here, we propose to extend this algorithm in two ways by:

- 1. Introducing an *a priori* distribution for  $\theta$  to eliminate degeneracy and an *a priori* distribution for A to express our previous knowledge on the mixing matrix.
- 2. Taking advantage of our hierarchical model and the idea of classification to reduce the computational cost.

To distinguish the proposed algorithm from the one proposed in [4], we call this algorithm the *Penalized EM*. The two steps become:

1. E-step :

$$Q(\boldsymbol{A}, \boldsymbol{\theta} | \boldsymbol{A}', \boldsymbol{\theta}') = E_{\boldsymbol{x}, \boldsymbol{s}}[\log p(\boldsymbol{x}, \boldsymbol{s} | \boldsymbol{A}, \boldsymbol{\theta}) + \log p(\boldsymbol{A}) + \log p(\boldsymbol{\theta}) | \boldsymbol{x}, \boldsymbol{A}', \boldsymbol{\theta}']$$
(45)

2. M-step :

$$\left(\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{\theta}}\right) = \underset{(\boldsymbol{A}, \boldsymbol{\theta})}{\operatorname{arg\,max}} Q\left(\boldsymbol{A}, \boldsymbol{\theta} \,|\, \boldsymbol{A}', \boldsymbol{\theta}'\right) \tag{46}$$

The joint distribution is factorized as:  $p(x, s, A, \theta) = p(x | A, s) p(A) p(s | \theta) p(\theta)$ . We can remark that  $p(x, s, A, \theta)$  as a function of  $(A, \theta)$  is separable in A and  $\theta$ . Consequently, the functional is separated into two factors: one representing an A functional and the other representing a  $\theta$  functional:

$$Q(\boldsymbol{A},\boldsymbol{\theta} | \boldsymbol{A}',\boldsymbol{\theta}') = Q_a(\boldsymbol{A} | \boldsymbol{A}',\boldsymbol{\theta}') + Q_h(\boldsymbol{\theta} | \boldsymbol{A}',\boldsymbol{\theta}')$$
(47)

with:

$$\begin{cases} Q_a(\boldsymbol{A}|\boldsymbol{A}',\boldsymbol{\theta}') = E[\log p(\boldsymbol{x}|\boldsymbol{A},\boldsymbol{s}) + \log p(\boldsymbol{A})|\boldsymbol{x},\boldsymbol{A}',\boldsymbol{\theta}'] \\ Q_h(\boldsymbol{\theta}|\boldsymbol{A}',\boldsymbol{\theta}') = E[\log p(\boldsymbol{s}|\boldsymbol{\theta}) + \log p(\boldsymbol{\theta})|\boldsymbol{x},\boldsymbol{A}',\boldsymbol{\theta}'] \end{cases}$$
(48)

# - Maximisation with respect to A

The functional  $Q_a$  is:

$$Q_a = \frac{-1}{2\sigma_{\epsilon}^2} \sum_{t=1}^{T} E\left[ \left( \boldsymbol{x}(t) - \boldsymbol{A}\,\boldsymbol{s}(t) \right)^T \left( \boldsymbol{x}(t) - \boldsymbol{A}\,\boldsymbol{s}(t) \right) \, | \, \boldsymbol{x}, \, \boldsymbol{A}', \, \boldsymbol{\theta}' \right] + \log p(\boldsymbol{A}). \tag{49}$$

The gradient of this expression with respect to the elements of A is:

$$\frac{\partial Q_a}{\partial \boldsymbol{A}_{i,j}} = \frac{T}{\sigma_{\epsilon}^2} \left( \widehat{\boldsymbol{R}}_{xs} - \boldsymbol{A} \, \widehat{\boldsymbol{R}}_{ss} \right)_{i,j} - \frac{1}{\sigma_{aij}^2} \left( \boldsymbol{A}_{i,j} - \boldsymbol{M}_{i,j} \right).$$
(50)

where:

$$\begin{cases} \widehat{\boldsymbol{R}}_{xs} = \frac{1}{T} \sum_{t=1}^{T} E\left[\boldsymbol{x}(t) \, \boldsymbol{s}(t)^{T} | \, \boldsymbol{x}, \, \boldsymbol{A}', \, \boldsymbol{\theta}'\right] \\ \widehat{\boldsymbol{R}}_{ss} = \frac{1}{T} \sum_{t=1}^{T} E\left[\boldsymbol{s}(t) \, \boldsymbol{s}(t)^{T} | \, \boldsymbol{x}, \, \boldsymbol{A}', \, \boldsymbol{\theta}'\right] \end{cases}$$
(51)

Evaluation of  $\widehat{R}_{xs}$  and  $\widehat{R}_{ss}$  requires the computation of the expectations of  $\boldsymbol{x}(t)\boldsymbol{s}(t)^T$ and  $\boldsymbol{s}(t)\boldsymbol{s}(t)^T$ . The main computational cost is due to the fact that the expectation of any function  $f(\boldsymbol{s})$  is given by:

$$E[f(\boldsymbol{s}) | \boldsymbol{x}, \boldsymbol{A}', \boldsymbol{\theta}'] = \sum_{\boldsymbol{z}' \in \prod_{i=1}^{n} \mathcal{Z}_{i}} E[f(\boldsymbol{s}) | \boldsymbol{x}, \boldsymbol{z} = \boldsymbol{z}', \boldsymbol{A}', \boldsymbol{\theta}'] p(\boldsymbol{z}' | \boldsymbol{x}, \boldsymbol{A}', \boldsymbol{\theta}').$$
(52)

which involves a sum of  $\prod_{j=1}^{n} q(j)$  terms corresponding to the whole combinations of labels. One way to obtain an approximate but fast estimate of this expression is to limit the summation to only one term corresponding to the MAP estimate of z:

$$E[f(\boldsymbol{s}) | \boldsymbol{x}, \boldsymbol{A}', \boldsymbol{\theta}'] = E[f(\boldsymbol{s}) | \boldsymbol{x}, \boldsymbol{z} = \widehat{\boldsymbol{z}}^{MAP}, \boldsymbol{A}', \boldsymbol{\theta}'].$$
(53)

Then, given estimated labels  $z_{1..T}$ , the source s(t) a posteriori law is Normal with mean  $\theta_{xz}$  and variance  $V_{xz}$  given by (24) and (40).

The source estimate is then  $\theta_{xz}$ .  $\hat{R}_{xs}$  and  $\hat{R}_{ss}$  become:

$$\widehat{R}_{xs} = \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{x}(t) \,\widehat{\boldsymbol{s}}(t)^{T}$$
(54)

and

:

$$\widehat{R}_{ss} = \frac{1}{T} \sum_{t=1}^{T} \widehat{\boldsymbol{s}}(t) \,\widehat{\boldsymbol{s}}(t)^{T} + \frac{1}{T} \sum_{t=1}^{T} (\boldsymbol{A}^{t} \boldsymbol{R}_{n}^{-1} \boldsymbol{A} + \Gamma_{\mathbf{z}}^{-1})^{-1}$$
(55)

When  $S_{1..T}$  estimated and using the matrix operations defined in section II and cancelling the gradient (50) to zero, we obtain the expression of the estimate of A:

$$\widehat{\mathbf{A}}^{k+1} = Mat\left(\left[\mathbf{\Lambda} + T\,\widehat{R}^*_{ss}\right]^{-1}\left[\mathbf{\Lambda}Vect(M) + T\,Vect\left(\widehat{R}_{xs}\right)\right]\right)$$
(56)

# - Maximisation with respect to heta

With a uniform *a priori* for the means, maximisation of  $Q_h$  with respect to  $m_{jz}$  gives

$$\widehat{m}_{jz} = \frac{\sum_{t=1}^{T} \theta_{jz}(t) p(z(t) | \boldsymbol{x}, \boldsymbol{A}', \boldsymbol{\theta}')}{\sum_{t=1}^{T} p(z(t) | \boldsymbol{x}, \boldsymbol{A}', \boldsymbol{\theta}')}$$
(57)

With an Inverted Gamma prior  $\mathcal{G}(\alpha, \beta)$  ( $\alpha > 0$  et  $\beta > 1$ ) for the variances, the maximisation of  $Q_h$  with respect to  $\sigma_{jz}$  gives:

$$\widehat{\sigma}_{jz} = \frac{2\beta + \sum_{t=1}^{T} \left( V_{jz} + \theta_{jz}^2 - 2\,\widehat{m}_{jz}\theta_{jz} + \widehat{m}_{jz}^2 \right) p(z(t) \,|\, \boldsymbol{x}, \, \boldsymbol{A}', \, \boldsymbol{\theta}')}{\sum_{t=1}^{T} p(z(t) \,|\, \boldsymbol{x}, \, \boldsymbol{A}', \, \boldsymbol{\theta}') + 2\,(\alpha - 1)}$$
(58)

#### Summary of the Penalized EM algorithm

Based on the preceeding equations, we propose the following algorithm to estimate sources and parameters using the following five steps:

- 1. Estimate the hyperparameters according to (57) and (58).
- 2. Update of data classification by estimating  $\hat{z}_{1,T}^{MAP}$ .
- 3. Given this classification, sources estimate is the mean of the Gaussian *a posteriori* law (39).
- 4. Update of data classification.
- 5. Estimate the mixing matrix A according to the re-estimation equation (56).

# COMPARISON WITH JMAP ALGORITHM AND ITS SENSITIVITY TO INITIAL CONDITIONS

The Penalized EM algorithm has an optimization cost approximately 2 times higher, per sample, than the JMAP algorithm. However, both algorithms have a reasonable computational complexity, linearly increasing with the number of samples. Sensitivity to initial conditions is inherent to the EM-algorithm even to the penalized version. In order to illustrate this fact, we simulated the algorithm with the same parameters as in section IV. Note that initial conditions for hyperparameters are  $\psi^{(0)} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$  and

 $m^{(0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ . However, the Penalized EM algorithm fails in separating sources (see figure 11). We note then that JMAP algorithm is more robust to initial conditions.



Figure 11- Results of separation with the Penalized EM algorithm: (a) Phase space distribution of sources, (b) mixed signals and (c) separated sources

We modified the initial condition to have means:  $m^{(0)} = \begin{pmatrix} -0.5 & 0.5 \\ -0.5 & 0.5 \end{pmatrix}$ . We noted, in this case, the convergence of the Penalized EM algorithm to the correct solution. Figures 12-16 illustrate the separation results:



Figure 12- Results of separation with the Penalized EM algorithm: (a) Phase space distribution of sources, (b) mixed signals and (c) separated sources



# CONCLUSION

We have proposed solutions to source separation problem using a Bayesian framework. Specific aspects of the described approach include:

- Taking account of errors on model and measurements.
- Introduction of *a priori* distribution for the mixing matrix and hyperparameters. This was motivated by two different reasons: Mixing matrix prior should exploit previous information and variances prior should regularize the log-posterior objective function.

We then consider the problem in terms of a mixture of Gaussian priors to develop a hierarchical strategy for source estimation. This same interpretation leads us to classify data before estimating hyperparameters and to reduce computational cost in the case of the proposed Penalized EM algorithm.

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